Discrete Curvature Flows For Surfaces and 3-Manifolds

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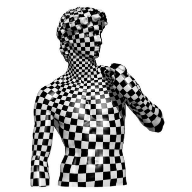
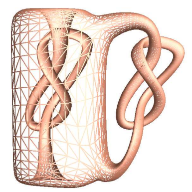
Thanks for the invitation!





Curvature Flow

- A way to deform metric according to prescribed curvature;
- *intrinsic* curvature, *conformal* deformation;
- Solid results from *mathematics*, direct applications in *engineering*;
- *smooth* vs *discrete*.

Our Work

- Discrete Ricci flow and discrete Yamabe flow for surfaces
- Discrete curvature flow for 3-manifolds

Outline

- Curvature flow for surfaces
  - Theories
  - Discretization
  - Applications
- Curvature flow for 3-manifolds

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The Ricci Flow [Hamilton 1982, 1988]

The Riemannian metric **g** is deformed proportional to the curvature K,

$$\frac{d\mathbf{g}}{dt} = -2K\mathbf{g}$$

such that the curvature K evolves like heat diffusion:

$$\frac{dK}{dt} = -\Delta_g K$$

Eventually, the curvature is constant everywhere.

Impact

A powerful tool to design Riemannian metrics by prescribed curvatures.

- In mathematics, it has been applied for proving Poincaré conjecture and Thurston's geometrization conjecture.
- In engineering, it has been applied in shape matching, registration, analysis, classification and etc.

The Big Picture

Geometry

According to Klein, geometry studies the invariants under different transformation groups.

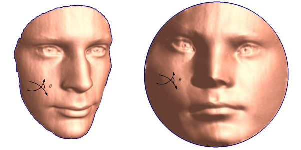
Surface Geometric Structures

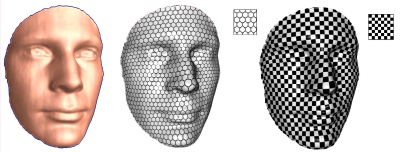
1. Topology : orientability, genus, boundaries, ...
2. Conformal Geometry : measure angles between intersecting curves;
3. Riemannian Geometry : measure lengths of curves;
4. Euclidean Geometry : mean curvature $h : S \to \mathbb{R}$

Conformal Structure

Conformal Mapping

- Angle preserving, locally shape preserving.
- Map infinitesimal circles to infinitesimal circles.
- Möbius transformation group.

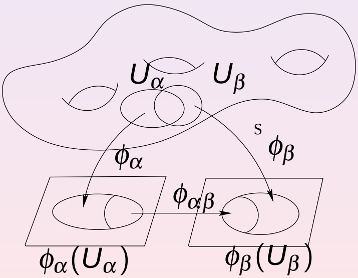




Conformal Structure

Definition (Conformal Structure)

An atlas is conformal, if all its transition maps are conformal (i.e., *biholomorphic*). A conformal structure is the maximal conformal atlas.

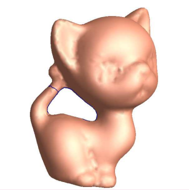


Riemann Geometry

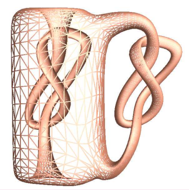
Fact

All metric surfaces are Riemann surfaces.

Spherical



Euclidean



Hyperbolic

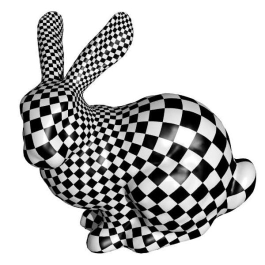
Connecting Conformal and Riemannian

Definition

Let Σ be a surface with a Riemannian metric $\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$.

Let $\lambda : \Sigma \to \mathbb{R}$ be a function defined on the surface. Then

- $e^{2\lambda}\mathbf{g}$ is another Riemannian metric conformal to $\mathbf{g}$;
- $e^{2\lambda}$ is called the conformal factor.



Connecting Conformal and Riemannian

Conformal Structure: Reinterpretation

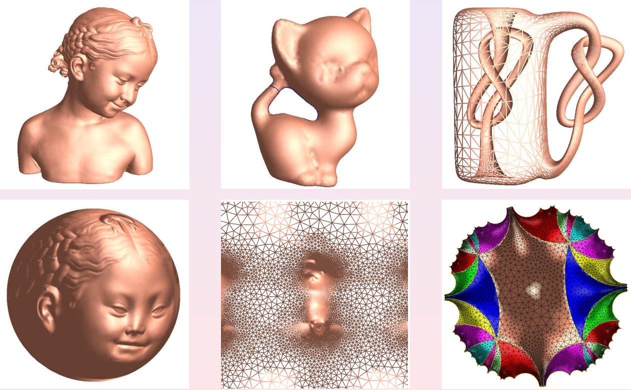
Consider all the Riemannian metrics on a topological surface S. Two Riemannian metrics $\mathbf{g}_1, \mathbf{g}_2$ are conformally equivalent $\mathbf{g}_1 \sim \mathbf{g}_2$, if they differ by a scalar function (i.e. conformal factor), $\mathbf{g}_2 = e^{2u}\mathbf{g}_1, u : S \to \mathbb{R}$. Each conformal equivalence class of Riemannian metrics on S is a *conformal structure* of S.

Theorem (**Poincaré Uniformization Theorem**)

Let $(S, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2u}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.

Uniformization

- All surfaces admit one of three canonical geometries: Spherical, Euclidean or Hyperbolic geometry.
- For any case, Ricci flow can deform metric conformally to be with constant curvature everywhere!



Spherical Euclidean Hyperbolic

Theorem (Hamilton 1988)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Theorem (Chow 1991)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Yamabe Flow

Yamabe Equation

Suppose $\bar{\mathbf{g}} = e^{2u}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2u}(-\Delta_{\mathbf{g}} u + K),$$

geodesic curvature on the boundary

$$\bar{k_g} = e^{-u}(-\partial_n u + k_g).$$

Yamabe Flow

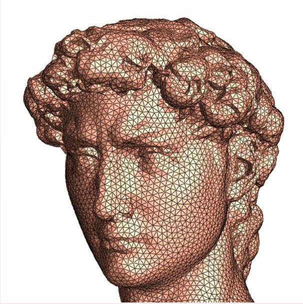
Lead to the solution to the Yamabe equation. Has essentially the same computational power as the Ricci flow for surfaces.

Outline

- Curvature flow for surfaces
  - Theories
  - **Discretization**
  - Applications
- Curvature flow for 3-manifolds

Discrete Representation of Surfaces

- Surfaces are represented as triangular meshes.
- Discrete model can be acquired using 3D scanner in real time.



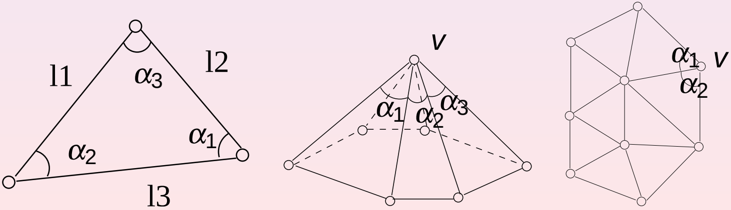
Discrete Metrics and Curvatures

- Discrete Metric: A function defined over edges, $l : E = \{edges\} \to \mathbb{R}^+$, satisfying triangular inequality.
- Discrete Curvature: A function defined over vertices, $K : V = \{vertices\} \to \mathbb{R}^1$,

$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

satisfying the discrete Gauss-Bonnet theorem:

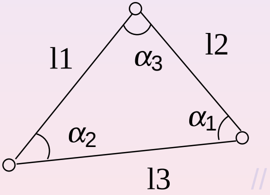
$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$



Discrete Metrics and Curvatures

Discrete metric determines discrete curvature by cosine law:

$$\cos\alpha_i = \frac{l_j^2 + l_k^2 - l_i^2}{2l_jl_k}, l \neq j \neq k \neq i$$



Discrete Curvature Flows

Conformal metric deformation by curvature flow

$$\mathbf{g} \rightarrow e^{2u}\mathbf{g}, u : S \rightarrow \mathbb{R}$$

- analytical approach: discrete Yamabe flow, using typical discrete notations (l_{ij} per edge, u_i per vertex).
- geometric approach: discrete Ricci flow, using so called circle packing metric (γ_i per vertex, Φ_{ij} per edge).

Discrete Yamabe Flow - Analytic Approach

Smooth Conformal Metric Deformation

$$\mathbf{g} \to e^{2u}\mathbf{g}, u : S \to \mathbb{R}$$

Discrete Conformal Metric Deformation

$$l_{ij} \to e^{u_i} e^{u_j} l_{ij}, u : V \to \mathbb{R}$$

Discrete Yamabe flow

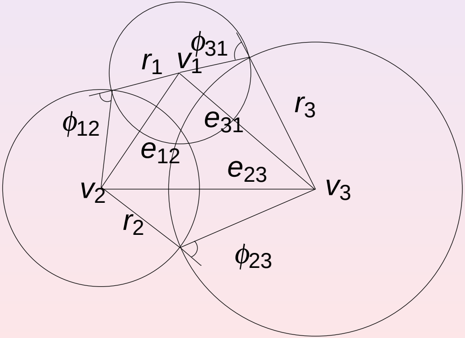
$$\frac{du_i}{dt} = -u_i K_i$$

Discrete Ricci Flow - Geometric Approach

Circle Packing Metric [Thurston 1978]

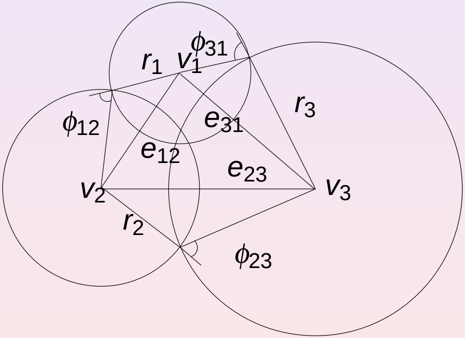
We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j\cos\Phi_{ij}$$



Discrete Ricci flow

$$\frac{d\gamma_i}{dt} = -\gamma_i K_i$$



Discrete Euclidean Ricci Flow

Definition

Let $u_i = ln\gamma_i$, the Entropy Energy is defined as

$$f(\mathbf{u}) = \int_{\mathbf{u}_0}^{\mathbf{u}} \sum_{i=1}^{n} (K_i - \bar{K}_i) du_i,$$

where $\mathbf{u} = (u_1, u_2, \cdots, u_n)$, $\mathbf{u}_0 = (0, 0, \cdots, 0)$.

Theorem (Chow and Luo 2002)

Euclidean Ricci energy is Well defined and convex, namely, there exists a unique global minimum. And the discrete Euclidean Ricci flow converges to the global minimum.

Numerical Methods

Gradient descent Method

Ricci flow is the gradient descent method for minimizing Ricci energy,

$$\nabla f = (K_1 - \bar{K}_1, K_2 - \bar{K}_2, \cdots, K_n - \bar{K}_n).$$

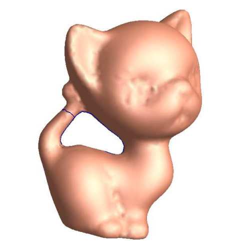
Newton's method

The Hessian matrix of Ricci energy is

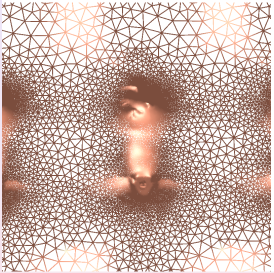
$$\frac{\partial^2 f}{\partial u_i \partial u_j} = \frac{\partial K_i}{\partial u_j}.$$

Newton's method can be applied directly.

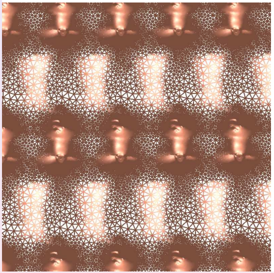
Euclidean Uniform Flat Metric



original surface

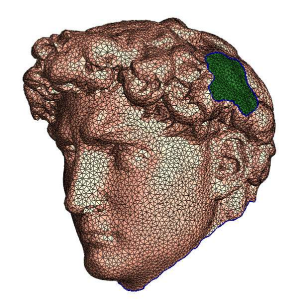


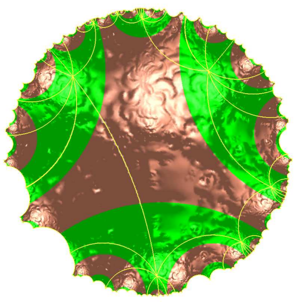
fundamental domain



universal cover

Hyperbolic Uniformization Metric





Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.

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- Curvature flow for surfaces
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  - **Applications**
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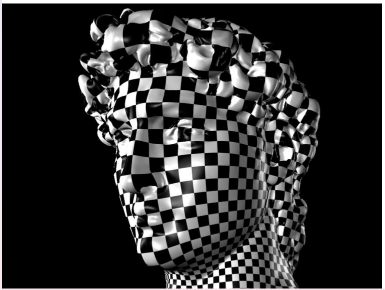
Applications

Many practical problems can be formulated as finding a special metric with certain curvature constraints.

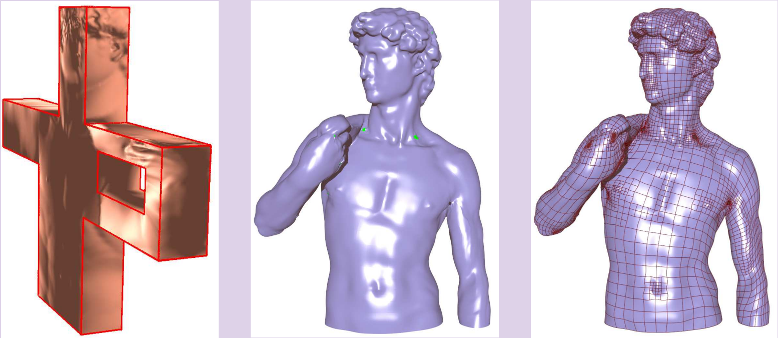
- Parameterization - finding a flat metric
- Polycube map - a flat metric with curvatures concentrated on corners
- Vector Field Design - a flat metric with cone singularities with special holonomy.
- Manifold Spline - constructing affine atlas utilizing flat metrics
- Shape Matching - register and compare two surfaces.
- ......

Surface Parameterization

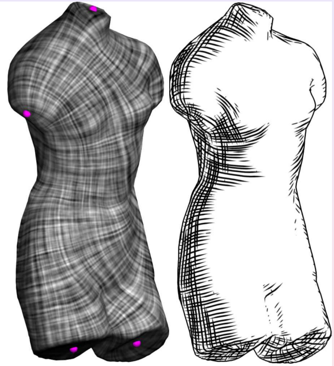


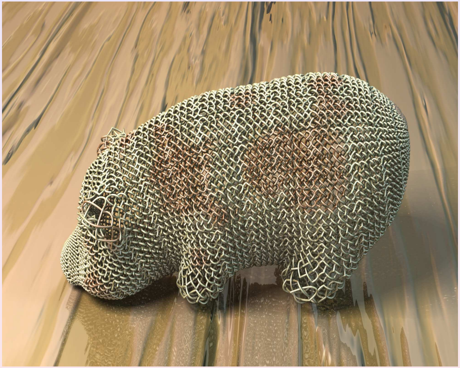
Polycube Map

Polycube Map

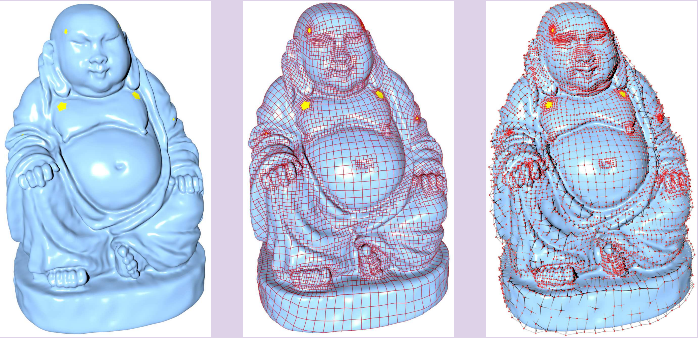


Vector Field Design

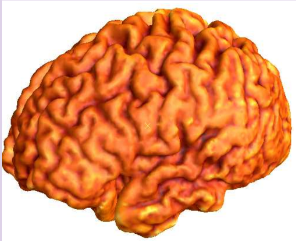


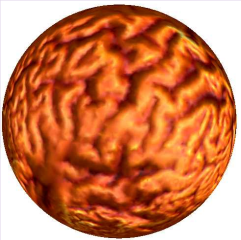


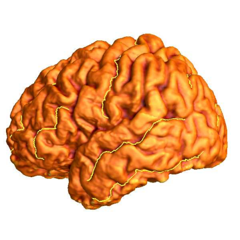
Manifold Spline

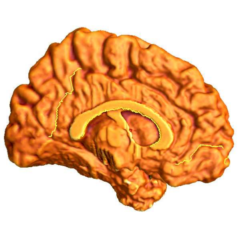


Brain Mapping



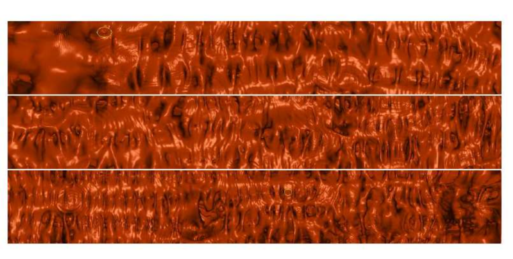




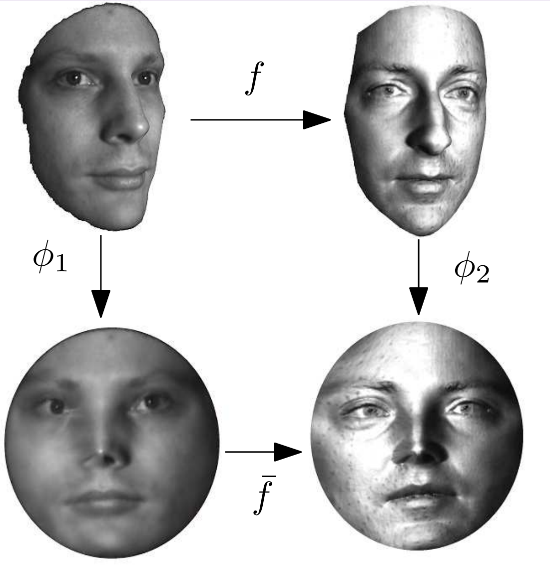


Colon Flattening

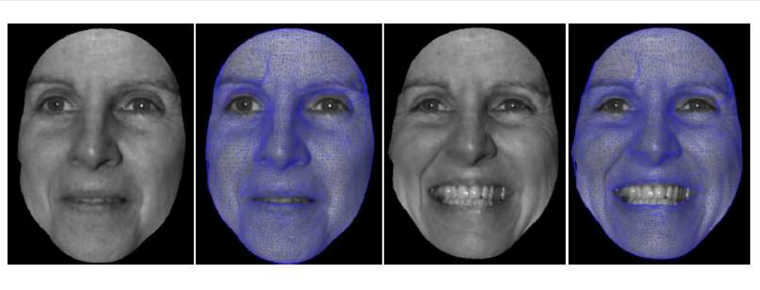
Virtual Colonoscopy



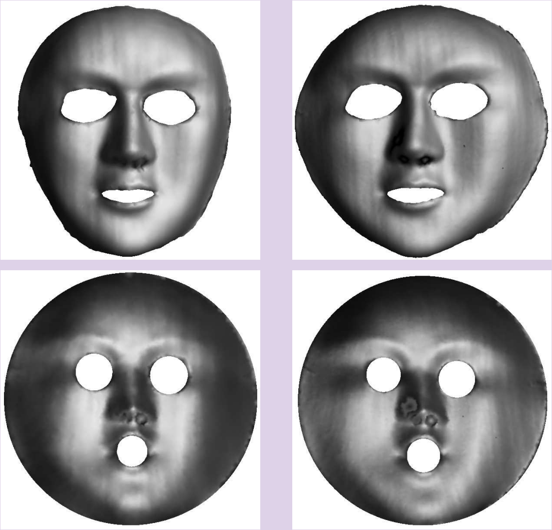
Surface Matching



Surface Tracking

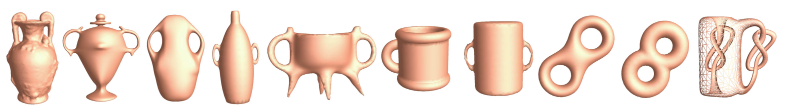


Conformal Invariants



Teichmüller Shape Space

Genus 2 Shapes



Genus 3 Shapes

Teichmüller Shape Space

Teichmüller Space

- In Teichmüller shape space, each point represents a conformal equivalent class of surfaces.
- In Teichmüller shape space, a curve connecting two points represents a deformation from one shape to the other.

Challenges

- Computing the coordinates of a shape in the Teichmüller space.
- Computing the geodesics (i.e. the extremal quasi-conformal map) in the space.

Outline

- Curvature flow for surfaces
  - Theories
  - Discretization
  - Applications
- **Curvature flow for 3-manifolds**

From surfaces to 3-manifolds

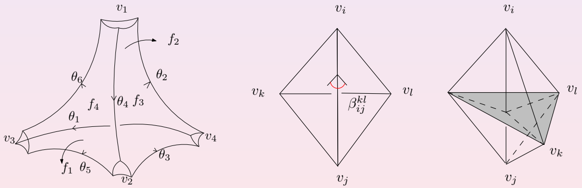
Similar idea

Conformal deformation of metrics driven by intrinsic curvature.

Yet quite different ...

- For 2-manifolds, topology and conformal structure determine a metric; for 3-manifolds: topology determines a metric.
- Surface Ricci flow has no singularities, 3-Manifold Ricci flow has.

- Primitive blocks: truncated tetrahedra (instead of triangles)
- Discrete curvature: edge curvature (instead of vertex curvature)
- Flow equation: $\frac{dx_{ij}}{dt} = -u_i K_{ij}$, x_{ij} edge length, K_{ij} edge curvature.



3-Manifold Curvature Flow

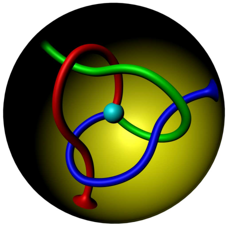
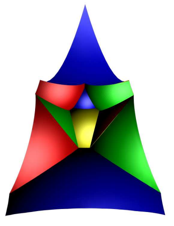
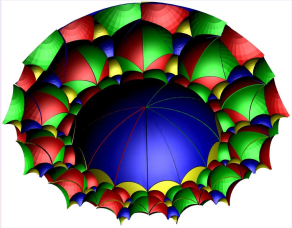
Uniformization

- Thurston's geometrization conjectured all 3-manifolds admit one of eight canonical geometries.
- Poincaré conjecture considers one special case: closed, compact, simply connected 3-manifolds are 3-spheres; proved using the Ricci flow.

Our Work

- A discrete curvature flow [Luo 2005] for hyperbolic 3-manifolds with boundaries.
- Discrete algorithms [Yin et al 2008] to compute this flow and visualize the constant curvature metric.

Discrete curvature flow for hyperbolic 3-manifolds

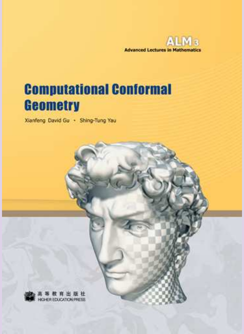
- Would help the study of 3-manifold geometry.
- Expect applications in volumetric parameterization.
- To be generalized to other types of 3-manifolds

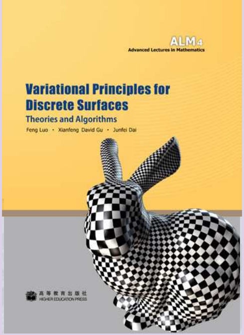
Conclusion

- Curvature flow is a fundamental tool to discover the intrinsic nature of geometry; it can be discretized properly to solve practical problems in engineering fields.
- For surfaces, the discrete curvature flow algorithms are available for all three geometries (spherical, Euclidean and hyperbolic).
- For 3-manifolds, the discrete curvature flow algorithm has been studied for certain class of 3-manifolds; much more to be explored.

References

- More details in *Computational Conformal Geometry* and *Variational Principles of Discrete Surfaces*.
- All the data sets and source codes are available at http://www.cs.sunysb.edu/ gu .





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