

Variational Principles on Triangulated Surfaces

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To Professor S.-T. Yau on his sixtieth birthday

Abstract. We give a brief introduction to some of the recent works on finding geometric structures on triangulated surfaces using variational principles.

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1 Introduction

The main objects of study are geometry of polyhedral surfaces. Take a finite set of points in the Euclidean 3-space \mathbb{E}^3 and the convex hull of these points. We obtain a convex polytope whose vertices are among the given finite set. If the points are generic, then the convex polytope has triangle faces. In this case, the boundary surface is a polyhedral surface. It has two properties. First the surface is triangulated and second the induced metric on the surface is locally flat except at the vertices. Recall that a triangulation of a surface is defined as follows. Take a finite collection of disjoint triangles and identify pairs of edges by homeomorphisms. The quotient space is a surface with a triangulation whose cells are the quotients of triangles, edges and vertices in the disjoint union.

Definition 1.1. A *Euclidean polyhedral surface* is a triple (S, T, d) where S is a closed surface, T is a triangulation of S , and (S, d) is a metric space with metric d so that the restriction of d to each triangle is isometric to a Euclidean triangle. We will call the metric d a *Euclidean polyhedral metric*. The *discrete curvature* (or simply curvature) k_0 of (S, T, d) is a function which assigns each vertex 2π less the sum of inner angles at the vertex, i.e.,

$$k_0(v) = 2\pi - \sum_{i=1}^m \theta_i,$$

where $\theta_1, \dots, \theta_m$ are the inner angles (of triangles) at the vertex v . See Figure 3.1.

We will also call above a \mathbb{E}^2 polyhedral surface. If we use the spherical (or hyperbolic) triangles instead of Euclidean triangles in definition 1.1, the result is called a spherical (or \mathbb{S}^2) polyhedral surface (resp. \mathbb{H}^2 or hyperbolic polyhedral surface). Spherical and hyperbolic polyhedral surfaces have been studied extensively. The discrete curvature k_0 is defined by the same formula for \mathbb{S}^2 and \mathbb{H}^2 polyhedral surfaces.

From the definition, it is clear that the basic unit of discrete curvature is the inner angle. Furthermore, the metric-curvature relation is given by the cosine law. Just like the smooth case, one of the main problems of study in

polyhedral surface is to understand the relationship between the metric and its curvature. Naturally we should study the cosine law carefully.

The goal of the paper is to introduce some of the recent works on finding geometric structures on triangulated surfaces using variational principles. Ever since W. Thuston's work on geometrization of Haken 3-manifolds and circle packing in 1978, there have been many works in this area. The key step in the variational framework is to define an appropriate action functional so that the critical points of the functional are the geometric structures that one is seeking.

The first such action functional was discovered in a seminal work by Colin de Verdière [6] in 1991 for circle packing metrics. Colin de Verdière's functional is derived from the Schläefli formula for volume of tetrahedra. In [6], Colin de Verdière introduced the first variational principle on triangulated surfaces in recent times and gave a proof of Thurston-Andreev's existence and uniqueness theorem for circle packing metrics. In a remarkable paper [18] in 1994, I. Rivin used the 3-dimensional volume of a hyperbolic ideal tetrahedra as the action functional and established a beautiful variational principle for Euclidean polyhedral surfaces. Since then, many other variational principles for polyhedral surfaces have been established. See for instance, [4], [2], [7], [12], [13], [14], [19], [20] and others. Amazingly, almost all action functionals discovered so far are related to the Schläefli formula. The only exception is in the beautiful work of [2]. The action functional is derived from a discrete integrable system. Very recently, we realized [15] that the cosine law and its derivative form are rich sources for action functionals and these include all the previous approaches. These recently discovered action functionals, when view from some perspectives, can be considered as 2-dimensional counterparts of the Schläefli formula. The complete list of all possible 2-dimensional counterparts of the Schläefli formula has been found in [15].

This paper is organized as follows. In section 2, we discuss the construction of action functionals in 2-dimension. In section 3, we discuss various variational principles associated to the action functionals. In section 4, we discuss the problem of moduli space of all curvatures. The last section addresses some open problems.

2 The Schläefli formula and its counterparts in dimension 2

One of the most beautiful identities in low-dimensional geometry is the Schläefli formula. It states that for a tetrahedron in a constant curvature $\lambda = \pm 1$ space, the volume V , the length l_i and the dihedral angle a_i at the i -th edge are related by

$$\frac{\partial V}{\partial a_i} = \frac{\lambda}{2} l_i \quad (2.1)$$

where $V = V(a_1, \dots, a_6)$ is a function of the angles. The formula relates the three most important 3-dimensional geometric quantities: volume, metric (=length), and curvature (=dihedral angles) in a simple identity. One consequence of (2.1) is that the differential 1-forms,

$$\sum_{i=1}^3 l_i da_i \quad \text{and} \quad \sum_{i=1}^3 a_i dl_i \quad \text{are closed.} \quad (2.2)$$

Indeed, $2dV = \lambda \sum_{i=1}^3 l_i da_i$. One can recover (2.1) from (2.2) by integration. By taking the Legendre transformation, one obtains $H(l_1, \dots, l_6) = \sum_{i=1}^6 l_i a_i - 2\lambda V$ so that

$$\frac{\partial H}{\partial l_i} = a_i. \quad (2.3)$$

Identities (2.1) and (2.3) are the starting points of several variational principles for finding constant curvature metrics on triangulated 3-manifolds. They are the basic ingredients in Regge calculus in physics which is a discretized general relativity.

2.1 Regge calculus and Casson's approach in dimension 3

Here is an illustration of the use of (2.3) after A. Casson and others. Fix a triangulated closed 3-manifold (M, T) and consider the space X of all hyperbolic polyhedral metrics on (M, T) . By definition, a hyperbolic polyhedral metric on (M, T) is a metric on M so that the restriction of the metric to each tetrahedron is isometric to a hyperbolic tetrahedron. Let E be the set of edges in the triangulation T . Then a polyhedral metric on (M, T) is determined by the edge length function $l : E \rightarrow \mathbf{R}$ sending an edge to its length. The *discrete curvature* (or simply curvature) K of the metric l is the function $K : E \rightarrow \mathbf{R}$ sending an edge to 2π less the sum of dihedral angles at the edge. If the curvature $K = 0$, then the polyhedral metric is a smooth hyperbolic metric. This is proved as follows. First, the curvature $K = 0$ shows that the metric is smooth at the interior points of each edge. To show that it is also smooth at each vertex, one considers the spherical link at the vertex. The link is isometric to a spherical polyhedral 2-sphere with discrete curvature $k_0 = 0$ at each vertex. This shows the link is isometric to the standard 2-sphere. Thus the metric on the 3-manifold is smooth at each vertex. Now in Casson's approach, one

defines the Einstein-Hilbert action of a polyhedral metric l to be:

$$F(l) = - \sum_{\sigma_i} H(l_{i_1}, \dots, l_{i_6}) + 2\pi \sum_{e_j} l_j$$

where the first sum is over all tetrahedra σ_i with six edge lengths l_{i_1}, \dots, l_{i_6} and the second sum is all edges e_j of length l_j . One easy consequence of (2.3) is that the Euler-Lagrangian equation for F , considered as a function defined on X , is given by

$$\frac{\partial F}{\partial l_i} = K_i \tag{2.4}$$

where l_i and K_i are the length and curvature at the i -th edge. In particular, it follows that the critical points of the Einstein-Hilbert functional are the hyperbolic metrics. To derive the Euler-Lagrangian equation (2.4), let us assume that the dihedral angles at the i -th edge are a_1, \dots, a_n and the tetrahedra adjacent to the i -th edge are $\sigma_1, \dots, \sigma_n$. Then by the definition of F and the Schlaefli formula (2.3) applied to each σ_j , we have

$$\frac{\partial F}{\partial l_i} = (-a_1 - \dots - a_n) + 2\pi = K_i.$$

The above approach is ubiquitous in variational framework on triangulated spaces. The important feature of this approach is that the action functional is local. This means that the value of the action functional on a polyhedral metric is the sum of the functional on its top dimensional simplexes. Thus the main issue for variational framework for triangulated surfaces is to find action functionals for geometric triangles.

2.2 The work of Colin de Verdière, Rivin, Cohen-Kenyon-Propp and Leibon

For a long time, the Gauss-Bonnet formula for area of triangles had been considered as the only counterpart of the Schlaefli formula in dimension 2. This is probably due to the view point that one should emphasize the role of volume in (2.1). The view changed when Colin de Verdière [6] produced the first striking 2-dimensional counterpart of the (1.1) by paying attention to (2.2). In this section, we will introduce briefly the action functionals in the work of [6], [18], [7], and [12].

In the work of [6] which gives a new proof of Andreev-Thurston's existence and uniqueness of circle packing metrics using variational principle, Colin de Verdière considers triangles of edge lengths $r_1 + r_2, r_2 + r_3, r_3 + r_1$ and angles a_1, a_2, a_3 where the angle a_i faces the edge of length $r_j + r_k$, $\{i,j,k\}=\{1,2,3\}$. See Figure 2.1 (a). He proved that the following three 1-forms w are closed. These are the counterparts of (2.2) in 2-dimension.

For a Euclidean triangle, the 1-form

$$w = \sum_{i=1}^3 \frac{a_i}{r_i} dr_i = \sum_{i=1}^3 a_i d \ln r_i \quad (2.5)$$

is closed. For a hyperbolic triangle, the 1-form

$$w = \sum_{i=1}^3 \frac{a_i}{\sinh r_i} dr_i = \sum_{i=1}^3 a_i d \ln \tanh(r_i/2) \quad (2.6)$$

is closed. For a spherical triangle, the 1-form

$$w = \sum_{i=1}^3 \frac{a_i}{\sin r_i} dr_i = \sum_{i=1}^3 a_i d \ln \tan(r_i/2) \quad (2.7)$$

is closed. Furthermore, write the 1-form w as $\sum_{i=1}^3 a_i du_i$. Then he proved that the integration $F(u) = \int^u w$ is concave in $u = (u_1, u_2, u_3)$ in the cases of Euclidean and hyperbolic triangles. We are informed by Colin de Verdière that these 1-forms were discovered by considering the Schlaefli formula.

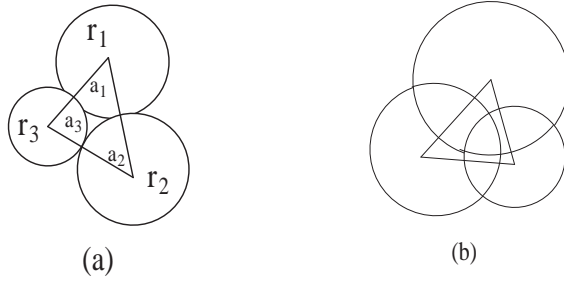


Figure 2.1

The Legendre transformation of $F(u)$ for Euclidean triangle was discussed in the work of Braeagger [4] who identifies it as the volume of a hyperbolic tetrahedron. The geometric meaning of other $F(u)$'s or its Legendre transformation is not known.

Thurston's original work on circle packing allows circles to intersect at angles. See Figure 2.1(b). For these intersecting angle circle packing metrics, Colin de Verdière's 1-form are still closed. This is proved in [8].

In the work of Rivin [18] and Cohen-Kenyon-Propp [7], they consider Euclidean triangles of edge lengths l_1, l_2, l_3 so that the opposite angles are a_1, a_2, a_3 . The action functional considered by Rivin is the Legendre transformation of that of Cohen-Kenyon-Propp. It is proved that the 1-form

$$w = \sum_{i=1}^3 \frac{a_i}{l_i} dl_i = \sum_{i=1}^3 a_i d \ln l_i \quad (2.8)$$

is closed. The integral $F(u) = \int^u w$ is shown to be convex in $u = (\ln l_1, \ln l_2, \ln l_3)$ (see [18]). It is proved in [18] that the Legendre transformation of $F(u)$ is the volume of the hyperbolic ideal tetrahedron with dihedral angles $a_1, a_2, a_3, a_1, a_2, a_3$.

In the work of [12], Leibon considers a hyperbolic triangle of edge lengths l_i so that the opposite angle is $r_j + r_k$ where $\{i, j, k\} = \{1, 2, 3\}$. He proved that the 1-form

$$w = \sum_{i=1}^3 \ln \sinh(l_i/2) dr_i \quad (2.9)$$

is closed and its integration is strictly convex in (r_1, r_2, r_3) . Furthermore, the integration $\int^r w$ is proved in [12] to be the volume of a hyperbolic prism.

Evidently, (2.5)-(2.9) should be considered as 2-dimensional counterparts of the Schlaefli formula (2.2). We will discuss the applications of these action functionals in section 3.

2.3 The Cosine Law and 2-dimensional Schlaefli formulas

The 2-dimensional Schlaefli formulas that we are seeking are some relationship between the lengths and angles of a triangle. Let $\mathbf{E}^2, \mathbf{H}^2$ and \mathbf{S}^2 be the Euclidean plane, the hyperbolic plane and the 2-sphere respectively. Given a triangle in $\mathbf{H}^2, \mathbf{E}^2$ or \mathbf{S}^2 of inner angles $\theta_1, \theta_2, \theta_3$ and edge lengths l_1, l_2, l_3 so that θ_i is facing the l_i -th edge, the cosine law expressing length l_i in terms of the angles θ_r 's is,

$$\cos(\sqrt{\lambda} l_i) = \frac{\cos \theta_i + \cos \theta_j \cos \theta_k}{\sin \theta_j \sin \theta_k}$$

where $\lambda = 1, -1, 0$ is the curvature of the space \mathbf{S}^2 , or \mathbf{H}^2 or \mathbf{E}^2 and $\{i, j, k\} = \{1, 2, 3\}$. Another related cosine law is

$$\cosh(l_i) = \frac{\cosh \theta_i + \cosh \theta_j \cosh \theta_k}{\sinh \theta_j \sinh \theta_k}$$

for a right-angled hyperbolic hexagon with three non-adjacent edge lengths l_1, l_2, l_3 and their opposite edge lengths $\theta_1, \theta_2, \theta_3$.

These formulas suggest that we should consider the following. Suppose a function $y = y(x)$ where $y = (y_1, y_2, y_3) \in \mathbf{C}^3$ and $x = (x_1, x_2, x_3)$ is in some open connected set in \mathbf{C}^3 so that x_i 's and y_i 's are related by

$$\cos(y_i) = \frac{\cos x_i + \cos x_j \cos x_k}{\sin(x_j) \sin(x_k)} \quad (2.10)$$

where $\{i, j, k\} = \{1, 2, 3\}$. We say $y = y(x)$ is the *cosine law function*. Let $r_i = \frac{1}{2}(x_j + x_k - x_i)$. Then $r = (r_1, r_2, r_3)$ is a new parametrization so that $x_i = r_j + r_k$. We will also consider the composition function $y = y(r_1, r_2, r_3)$.

The following proposition establishes the basic properties of the cosine law function. The proof of the proposition is a simple exercise in calculus and will be omitted. See [15] for details.

Proposition 2.1. ([15]) *Suppose the cosine law function $y = y(x)$ is defined on an open connected set in \mathbf{C}^3 which contains a point (a, a, a) so that $y(a, a, a) = (b, b, b)$. Let $A_{ijk} = \sin y_i \sin x_j \sin x_k$ where $\{i, j, k\} = \{1, 2, 3\}$. Then*

$$A_{ijk} = A_{jki}, \quad \text{i.e.,} \quad \frac{\sin(y_i)}{\sin(x_i)} = \frac{\sin(y_j)}{\sin(x_j)} \quad (2.11)$$

$$\frac{\tan(y_i/2)}{\cos(r_i)} = \frac{\tan(y_j/2)}{\cos(r_j)} \quad (2.12)$$

$$A_{ijk}^2 = 1 - \cos^2 x_i - \cos^2 x_j - \cos^2 x_k - 2 \cos x_i \cos x_j \cos x_k.$$

At a point x where $A_{ijk} \neq 0$, then,

$$\frac{\partial y_i}{\partial x_i} = \frac{\sin x_i}{A_{ijk}},$$

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial y_i}{\partial x_i} \cos y_k,$$

$$\cos(x_i) = \frac{\cos y_i - \cos y_j \cos y_k}{\sin y_j \sin y_k}. \quad (2.13)$$

In particular, (2.13) shows that the roles of x and y in the cosine law are essentially symmetric. The identities (2.11) and (2.12) are called the *Sine Law* and the *Tangent Law* of the cosine law function.

The problem of finding all 2-dimensional counterparts of the Schlaefli formula can be rephrased as follows. It corresponds to generalizing Schlaefli identity (2.2).

Problem 2.2. *Suppose the cosine law function $y = y(x)$ is defined on an open connected set in \mathbf{C}^3 .*

(i) *Find all smooth non-constant functions $f(t)$ and $g(t)$ so that the differential 1-form $w = \sum_{i=1}^3 f(y_i) dg(x_i)$ is closed.*

(ii) *Find all smooth non-constant functions $f(t)$ and $g(t)$ so that the differential 1-form $\sum_{i=1}^3 f(y_i) dg(r_i)$ is closed where $r_i = \frac{x_j + x_k - x_i}{2}$, $\{i, j, k\} = \{1, 2, 3\}$.*

If we find these 1-forms w , then the integrals $F = \int w$ will be used as action functionals for variational principles on surfaces. By the construction of the 1-form, the function F satisfies

$$\frac{\partial F}{\partial g(x_i)} = f(y_i)$$

or

$$\frac{\partial F}{\partial g(r_i)} = f(y_i).$$

These are all 2-dimensional counterparts of the Schlaefli formula (2.2).

Problem 2.2 was solved in [15].

Theorem 2.3. ([15]) *The following is the complete list of functions f and g up to scaling and complex conjugation. There exists a complex number h so that, in the case (i),*

$$f(t) = \int^t \sin^h(s) ds \quad \text{and} \quad g(t) = \int^t \sin^{-h-1}(s) ds$$

and in the case (ii),

$$f(t) = \int^t \tan^h(s/2) ds \quad \text{and} \quad g(t) = \int^t \cos^{-h-1}(s) ds.$$

In particular, all closed 1-forms $\sum_{i=1}^3 f(y_i) dg(x_i)$ and $\sum_{i=1}^3 f(y_i) dg(r_i)$ are holomorphic or anti-holomorphic.

The details of the proof the theorem can be found in [15]. We give a sketch of the proof here. To verify that the 1-forms listed above are closed is a straight forward calculation using the sine law $\frac{\sin(y_i)}{\sin(x_i)} = \frac{\sin(y_j)}{\sin(x_j)}$ and the tangent law $\frac{\tan(y_i/2)}{\cos(r_i)} = \frac{\tan(y_j/2)}{\cos(r_j)}$. The proof that these are the complete list of all functions f, g up to scaling and complex conjugation is due to the uniqueness of the sine law and tangent law. To be more precise, in the case of the sine law, it can be shown that ([15], lemma 2.3) if f, g are two smooth non-constant functions so that $\frac{f(y_i)}{g(x_i)} = \frac{f(y_j)}{g(x_j)}$ for all x , then there are constants λ, μ, c_1, c_2 so that, $f(t) = c_1 \sin^\lambda(t) \sin^\mu(\bar{t})$ and $g(t) = c_2 \sin^\lambda(t) \sin^\mu(\bar{t})$.

By specializing theorem 2.3 to triangles in $\mathbf{S}^2, \mathbf{E}^2$ and \mathbf{H}^2 and integrating the 1-forms, we obtain various energy functionals for variational framework on triangulated surfaces. We have identified all those convex or concave energies constructed in this way.

Theorem 2.4 ([15]). *Let $l = (l_1, l_2, l_3)$ and $\theta = (\theta_1, \theta_2, \theta_3)$ be lengths and angles of a triangle in $\mathbf{E}^2, \mathbf{H}^2$ or \mathbf{S}^2 . Let $h \in \mathbf{R}$ and $u = (u_1, u_2, u_3)$.*

The following is the complete list, up to scaling, of all closed real-valued 1-forms of the form $\sum_{i=1}^3 f(\theta_i) dg(l_i)$ for some non-constant smooth functions f, g so that its integral is convex or concave.

(i) *For a Euclidean triangle,*

$$w_h = \sum_{i=1}^3 \frac{\int^{\theta_i} \sin^h(t) dt}{l_i^{h+1}} dl_i.$$

Its integral $\int^u w_h$ is locally convex in variable u where $u_i = \int_1^{l_i} t^{-h-1} dt$.

(ii) For a spherical triangle,

$$w_h = \sum_{i=1}^3 \frac{\int^{\theta_i} \sin^h(t) dt}{\sin^{h+1}(l_i)} dl_i.$$

The integral $\int^u w_h$ is locally strictly convex in u where $u_i = \int_{\pi/2}^{l_i} \sin^{-h-1}(t) dt$.

The following are the complete list, up to scaling, of all closed real-valued 1-forms of the form $\sum_{i=1}^3 f(l_i) dg(r_i)$ (where $\theta_i = r_j + r_k$) or $\sum_{i=1}^3 f(\theta_i) dg(r_i)$ (where $l_i = r_j + r_k$) for some non-constant smooth functions f, g so that its integral is either convex or concave.

(iii) For a Euclidean triangle of angles θ_i and opposite edge lengths $r_j + r_k$,

$$\eta_h = \sum_{i=1}^3 \frac{\int^{\theta_i} \cot^h(t/2) dt}{r_i^{h+1}} dr_i.$$

Its integral $\int^u \eta_h$ is locally concave in $u = (u_1, u_2, u_3)$ where $u_i = \int_1^{r_i} t^{-h-1} dt$.

(iv) For a hyperbolic triangle of angles θ_i and opposite edge lengths $r_j + r_k$,

$$\eta_h = \sum_{i=1}^3 \frac{\int^{\theta_i} \cot^h(t/2) dt}{\sinh^{h+1}(r_i)} dr_i.$$

Its integral $\int^u \eta_h$ is locally strictly concave in u where $u_i = \int_1^{r_i} \sinh^{-h-1}(t) dt$.

(v) For a hyperbolic triangle of edge lengths l_i and opposite angles $r_j + r_k$,

$$\eta_h = \sum_{i=1}^3 \frac{\int^{l_i} \tanh^h(t/2) dt}{\cosh^{h+1}(r_i)} dr_i.$$

Its integral $\int^u \eta_h$ is locally strictly convex in u where $u_i = \int_1^{r_i} \cos^{-h-1}(t) dt$.

(vi) For a hyperbolic right-angled hexagon of three non-pairwise adjacent edge lengths l_i and opposite edge lengths $r_j + r_k$,

$$\eta_h = \sum_{i=1}^3 \frac{\int^{l_i} \coth^h(t/2) dt}{\cosh^{h+1}(r_i)} dr_i.$$

Its integral $\int^u \eta_h$ is locally strictly concave in u where $u_i = \int_1^{r_i} \cosh^{-h-1}(t) dt$.

A sketch of the proof of theorem 2.4 goes as follows. First, by theorem 2.3, these 1-forms are closed. It remains to show that the integrals are convex or concave. This follows by showing that the Hessian matrix of the integral is positive (or negative) definite. We first observe that the Hessian matrices of $\int w_h$ (or $\int \eta_h$) and $\int w_{h'}$ (or $\int \eta_{h'}$) are congruent for different h, h' . Thus, it suffices to check the convexity for one h . This was been achieved in various cases by different authors. Indeed, case (i) for $h = 0$ is proved in [18], case (ii)

for $h = 0$ is proved in [13], (iii) and (iv) for $h = 0$ are proved in [6], case (v) for $h = -1$ is proved in [12], and case (vi) for $h = -1$ is proved in [14].

2.4 The geometric meaning of some action functionals

The geometric meaning of the integrals $\int w_h$ or $\int \eta_h$ are not known except in the following cases. The Legendre transform of $\int w_0$ for Euclidean triangles is the hyperbolic volume of an ideal tetrahedron first discovered by Rivin. Leibon showed that the integral $\int \eta_{-1}$ for hyperbolic triangle is the volume of an ideal hyperbolic prism. We showed in [13] that $\int w_0$ for a spherical triangle is the volume of an ideal hyperbolic octahedron. In these three cases, there is a common way to describe the action functional. Given a Euclidean, or a hyperbolic or a spherical triangle, we consider the triangle to be drawn in the sphere at infinity of the hyperbolic 3-space bounded by three circles. These three circles will intersect at a finite set X of points in \mathbf{S}^2 where $|X| = 4$ for Euclidean triangle and $|X| = 6$ otherwise. Then the action functional associated to the triangle is the hyperbolic volume of the convex hull of X in the hyperbolic 3-space. See Figure 2.2.

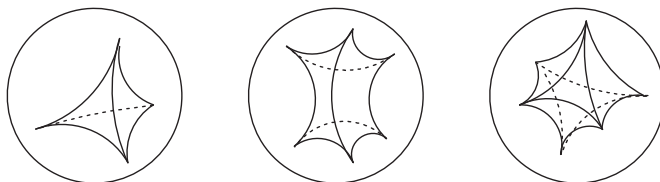


Figure 2.2

3 Variational principles on surfaces

We discuss some applications of the 2-dimensional Schlaefli formulas in this section. The most prominent applications are rigidity theorems for polyhedral surfaces. For simplicity, let us assume that surfaces are closed in this section. All results can be generalized without difficulties to compact surfaces with boundary.

All of these applications to rigidity are based on the following simple lemma.

Lemma 3.1. *Suppose $\Omega \subset \mathbf{R}^n$ is an open convex set and $W : \Omega \rightarrow \mathbf{R}^n$ is a smooth function with positive definite Hessian matrices. Then the gradient $\nabla W : \Omega \rightarrow \mathbf{R}^n$ is a smooth embedding. If Ω is only assumed to be open in \mathbf{R}^n , then $\nabla W : \Omega \rightarrow \mathbf{R}^n$ is a local diffeomorphism.*

Indeed, consider the graph G of the function W in \mathbf{R}^{n+1} . The convexity of W shows that the graph G is a strictly convex hypersurface. Thus normal vectors to the graph G at different points are not parallel. However, normal vectors are of the form $(\nabla W, -1)$. It follows that $\nabla W : \Omega \rightarrow \mathbf{R}^n$ is injective. The Jacob matrix of ∇W is the Hessian of W . Thus the map ∇W is a smooth embedding.

3.1 Colin de Verdière's proof of Thurston-Andreev's rigidity theorem

In his work [21] on constructing hyperbolic metrics on 3-manifolds, Thurston introduced circle packing metrics on a triangulated surface (S, T) . Let V and E be the sets of vertices and edges in the triangulation T . A *circle packing metric* on (S, T) is a polyhedral metric $l : E \rightarrow \mathbf{R}_{>0}$ so that there is a map, called the *radius assignment*, $r : V \rightarrow \mathbf{R}_{>0}$ with $l(vv') = r(v) + r(v')$ whenever the edge vv' has end points v and v' .

Theorem 3.2. (Thurston, Andreev) *Suppose (S, T) is a triangulated closed surface.*

(i) ([21], [1]) *A Euclidean circle packing metric on (S, T) is determined up to isometry and scaling by the discrete curvature k_0 .*

(ii) ([21]) *A hyperbolic circle packing metric on (S, T) is determined up to isometry by the discrete curvature k_0 .*

Colin de Verdière's Proof ([6]). We will consider the case (ii) for simplicity. The same argument also works for (i) with a little care. For a circle packing metric, let $r : V \rightarrow \mathbf{R}_{>0}$ be the radius assignment. Define $u : V \rightarrow \mathbf{R}_{<0}$ by $u(v) = \int_{\infty}^{r(v)} \frac{1}{\sinh(s)} ds$. The set of all circle packing metrics on (S, T) is parameterized by $\mathbf{R}_{<0}^V$ via u .

Recall that for a triangle of edge lengths $r_1 + r_2, r_2 + r_3, r_3 + r_1$ and inner angles a_1, a_2, a_3 , Colin de Verdière defines an action functional $F(r_1, r_2, r_3)$ by integrating the 1-form (2.6) so that

$$\frac{\partial F}{\partial u_i} = a_i \quad (3.1)$$

where $u_i = \int_{\infty}^{r_i} \frac{1}{\sinh(s)} ds$ and proves that F is strictly concave in (u_1, u_2, u_3) in (2.6). We call $F(u_1, u_2, u_3)$ the Colin de Verdière energy of the triangle. Define the energy $W(u)$ of $u \in \mathbf{R}_{<0}^V$ to be the sum of the Colin de Verdière energy of the triangles in the circle packing metric r . Then by the construction, $W(u)$ is strictly concave in u due to the concavity of F . Furthermore, by (3.1),

$$\nabla W = 2\pi(1, \dots, 1) - k_0$$

where k_0 is the curvature of the circle packing metric. Thus, by lemma 3.1, the map from r to its curvature k_0 is injective. This is the statement in (ii).

We remark that Colin de Verdière [6] also gave a very nice proof of the existence of the circle packing metrics.

3.2 The work of Rivin and Leibon

Given a polyhedral surface (S, T) , Rivin [18] introduced the curvature $\phi_0 : E \rightarrow \mathbf{R}$ sending an edge e to $2\pi - a - a'$ where a, a' are the angles facing the edge. Leibon introduced in [12] the $\psi_0 : E \rightarrow \mathbf{R}$ curvature which sends an edge e to $\frac{b+c-a+b'+c'-a'}{2}$ where a, a' are the angles facing the edge e and b, b', c, c' are the angles adjacent to e . The geometric meaning of ϕ_0 and ψ_0 are related to the dihedral angles of the associated 3-dimensional hyperbolic ideal polyhedra. See Figure 3.1.

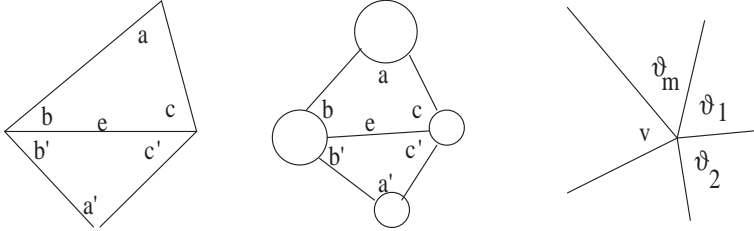


Figure 3.1

Theorem 3.3. (Rivin, Leibon) (i) ([18]). A Euclidean polyhedral metric on (S, T) is determined up to isometry and scaling by the ϕ_0 curvature.

(ii) ([12]). A hyperbolic polyhedral metric on (S, T) is determined up to isometry by the ψ_0 curvature.

The proof in [18] uses the Lagrangian multipliers method. The action functionals in [18] and [12] are the integrations of (2.8) and (2.9) and their Legendre transformations. Due to the convexity of the action functional, the rigidity theorem follows essentially from lemma 3.1.

3.3 New curvatures for polyhedral metrics and some rigidity theorems

Based on theorem 2.4, we introduced families of discrete curvatures in [15]. Recall that (S, T) is a closed triangulated surface so that T is the triangulation, E and V are the sets of all edges and vertices. Let \mathbf{E}^2 , \mathbf{S}^2 and \mathbf{H}^2 be the Euclidean, the spherical and the hyperbolic 2-dimensional geometries.

Definition 3.4. Given a K^2 polyhedral metric l on (S, T) where $K^2 = \mathbf{E}^2$, or \mathbf{S}^2 or \mathbf{H}^2 , the ϕ_h curvature of the polyhedral metric l is the function $\phi_h : E \rightarrow \mathbf{R}$ sending an edge e to:

$$\phi_h(e) = \int_a^{\pi/2} \sin^h(t) dt + \int_{a'}^{\pi/2} \sin^h(t) dt \quad (3.2)$$

where a, a' are the inner angles facing the edge e . See Figure 3.1.

The ψ_h curvature of the metric l is the function $\psi_h : E \rightarrow \mathbf{R}$ sending an edge e to

$$\psi_h(e) = \int_0^{\frac{b+c-a}{2}} \cos^h(t) dt + \int_0^{\frac{b'+c'-a'}{2}} \cos^h(t) dt \quad (3.3)$$

where b, b', c, c' are inner angles adjacent to the edge e and a, a' are the angles facing the edge e .

The h -th discrete curvature k_h of the polyhedral metric l on (S, T) is the function $k_h : V \rightarrow \mathbf{R}$ sending a vertex v to

$$k_h(v) = - \sum_{i=1}^m \int_{\pi/2}^{\theta_i} \tanh^h(t/2) dt + (4 - m)\pi/2$$

where $\theta_1, \dots, \theta_m$ are all inner angles at vertex v and m is the degree of the vertex v . See Figure 3.1.

The curvatures ϕ_0 and ψ_0 were first introduced by I. Rivin [18] and G. Leibon [12] respectively. The positivity of ψ_0 and ϕ_0 is shown in [18] and [12] to be equivalent to the Delaunay condition for polyhedral metrics.

It is shown [15] that the positivity of the curvatures ϕ_h and ψ_h is independent of h . To be more precise, due to $(x + y)(\int_0^x \cos^h(t) dt + \int_0^y \cos^h(t) dt) \geq 0$ for $x, y \in [-\pi/2, \pi/2]$, we have $\psi_h(e) \geq 0$ (or $\phi_h(e) \geq 0$) if and only if $\psi_0(e) \geq 0$ (or $\phi_0(e) \geq 0$). Thus the geometric meaning of positive ψ_h curvature is the same Delaunay condition for polyhedral metrics.

The curvature $\phi_{-2}(e) = \cot(a) + \cot(a')$ has appeared in the finite element method approximation of the Laplace operator. It is called the cotangent formula for discrete Laplacian.

We prove that,

Theorem 3.5. ([15]) *Let $h \in \mathbf{R}$ and (S, T) be a closed triangulated surface.*

(i) *A Euclidean circle packing metric on (S, T) is determined up to isometry and scaling by its k_h -th discrete curvature.*

(ii) *A hyperbolic circle packing metric on (S, T) is determined up to isometry by its k_h -th discrete curvature.*

(iii) *If $h \leq -1$, a Euclidean polyhedral metric on (S, T) is determined up to isometry and scaling by its ϕ_h curvature.*

(iv) If $h \leq -1$ or $h \geq 0$, a spherical polyhedral metric on (S, T) is determined up to isometry by its ϕ_h curvature.

(v) If $h \leq -1$ or $h \geq 0$, a hyperbolic polyhedral surface is determined up to isometry by its ψ_h curvature.

For any $h \in \mathbf{R}$, there are local rigidity theorems in cases (i)-(v) (see theorem 6.2 in [15]). We conjecture that theorem 3.5 holds for all h . To the best of our knowledge, theorem 3.5 for the simplest case of the boundary of a tetrahedron is new.

The ideas of the proof are the same as the ones used in [6] and [18] by applying the action functionals discovered in theorem 2.4. The extra constraints that $h \leq -1$ or $h \geq 0$ in the theorem are caused by the condition on the convexity of the domain Ω in lemma 3.1. To be more precise, these conditions on h guarantee that the corresponding domains of the action functionals are convex.

3.4 Application to Teichmüller theory of surfaces with boundary

The counterpart of theorem 3.5(v) for hyperbolic metrics with totally geodesic boundary on an ideal triangulated compact surface is the following. Recall that an *ideal triangulated compact surface* with boundary (S, T) is obtained by removing a small open regular neighborhood of the vertices of a triangulation of a closed surface. The *edges* of an ideal triangulation T correspond bijectively to the edges of the triangulation of the closed surface. Given a hyperbolic metric l with geodesic boundary on an ideal triangulated surface (S, T) , there is a unique geometric ideal triangulation T^* isotopic to T so that all edges are geodesics orthogonal to the boundary. The edges in T^* decompose the surface into hyperbolic right-angled hexagons. The ψ_h curvature of the hyperbolic metric l is defined to be the map $\psi_h : \{\text{all edges in } T\} \rightarrow \mathbf{R}$ sending each edge e to

$$\psi_h(e) = \int_0^{\frac{b+c-a}{2}} \cosh^h(t) dt + \int_0^{\frac{b'+c'-a'}{2}} \cosh^h(t) dt \quad (3.4)$$

where a, a' are lengths of arcs in the boundary (in the ideal triangulation T^*) facing the edge and b, b', c, c' , are the lengths of arcs in the boundary adjacent to the edge so that a, b, c lie in a hexagon. See Figure 3.1.

Theorem 3.6 ([15]). *A hyperbolic metric with totally geodesic boundary on an ideal triangulated compact surface is determined up to isometry by its ψ_h -curvature. Furthermore, if $h \geq 0$, then the set of all ψ_h curvatures on a fixed*

ideal triangulated surface is an explicit open convex polytope P_h in a Euclidean space so that $P_h = P_0$.

The first part of the theorem is proved by using lemma 3.1 and the action functional in theorem 2.4(v).

The case when $h < 0$ has been recently established by Ren Guo [9]. He proved that,

Theorem 3.7. (Guo) *Under the same assumption as in theorem 3.6, if $h < 0$, the set of all ψ_h curvatures on a fixed ideal triangulated surface is an explicit bounded open convex polytope P_h in a Euclidean space. Furthermore, if $h < \mu$, then $P_h \subset P_\mu$.*

Theorem 3.6 was proved for $h = 0$ in [14] where the open convex polytope P_0 is explicitly described. Evidently for each $h \in \mathbf{R}$, the curvature ψ_h can be taken as a coordinate of the Teichmüller space of the surface. The interesting part of the theorem 3.6 is that the images of the Teichmüller space in these coordinates (for $h \geq 0$) are all the same. Whether these coordinates are related to quantum Teichmüller theory is an interesting topic. See [5], [10], [3], and others for more information. Combining theorem 3.6 with the work of Ushijima [22] and Kojima [11], one obtains for each $h \geq 0$ a cell decomposition of the Teichmüller space invariant under the action of the mapping class group. See corollary 10.6 in [15].

4 The moduli spaces of polyhedral metrics

One of the applications of the rigidity theorems is on the space of all polyhedral surfaces. The most prominent result in the area is the theorem of Andreev-Thurston. It states that the space of all discrete curvatures of all circle packing metrics on a triangulated surface is a convex polyhedron. We will present Marden-Rodin's elegant proof [16] of it in this section. Another proof of it can be found in [6]. Many other results on the space of all curvatures will also be discussed. Most of the results obtained in [15] on the space of all curvatures are modelled on the Marden-Rodin's method.

4.1 Thurston-Andreev's theorem and Marden-Rodin's proof

Theorem 4.1. (Thurston-Andreev) ([1], [21]) *The space of discrete curvatures k_0 of Euclidean or hyperbolic circle packing metrics on a closed triangulated surface is a convex polytope.*

For simplicity, we will present Marden-Rodin’s proof of it for hyperbolic circle packing metrics. Let (S, T) be the closed triangulated surface so that the set of vertices is V . By definition, a circle packing metric is given by the radius parameter $r \in \mathbf{R}_{>0}^V$, its discrete curvature $k_0 \in \mathbf{R}^V$. By Thurston-Andreev’s rigidity theorem 3.2, the curvature map $\Pi : \mathbf{R}_{>0}^V \rightarrow \mathbf{R}^V$ sending a metric r to its curvature k_0 is a smooth embedding. The goal is to show that the image $\Pi(\mathbf{R}_{>0}^V)$ is a convex polytope. To this end, one needs to study how hyperbolic triangles degenerate. Suppose a hyperbolic triangle has edge lengths $r_1 + r_2, r_2 + r_3, r_3 + r_1$ so that the angle opposite to $r_i + r_j$ is θ_k , $\{i, j, k\} = \{1, 2, 3\}$. We say a sequence of hyperbolic metrics degenerates if one of r_i converges to 0 or ∞ .

The following simple lemma summarizes the degenerations. The best proof of it is to draw a picture. See Figure 4.1 and [16] for a proof.

Lemma 4.2. ([16] and [21]). *Under the assumption above,*
 (a) $\lim_{r_i \rightarrow \infty} \theta_i(r_1, r_2, r_3) = 0$ so that the convergence is uniform.
 (b) Suppose f_1, f_2, f_3 are positive real numbers. Then

$$\lim_{(r_i, r_j, r_k) \rightarrow (0, f_j, f_k)} \theta_i(r_1, r_2, r_3) = \pi,$$

$$\lim_{(r_i, r_j, r_k) \rightarrow (0, 0, f_k)} (\theta_i + \theta_j)(r_1, r_2, r_3) = \pi,$$

and

$$\lim_{(r_1, r_2, r_3) \rightarrow 0} (\theta_1 + \theta_2 + \theta_3)(r_1, r_2, r_3) = \pi.$$

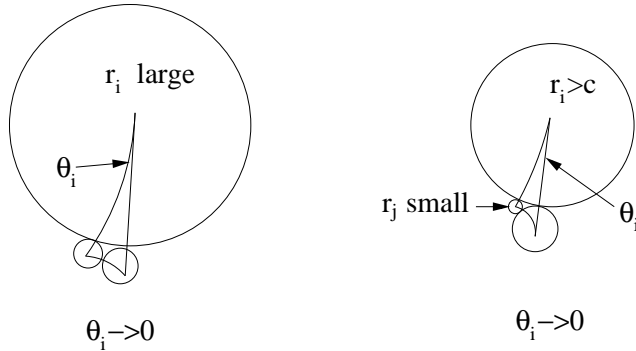


Figure 4.1

Using the lemma, one can now determine the boundary of $\Omega = \Pi(\mathbf{R}_{>0}^V)$ as follows. By the definition of the discrete curvature k_0 , the image Ω lies in the open half-spaces

$$k_0(v) < 2\pi. \tag{4.1}$$

To see that the equality holds in the limit case, take a sequence of circle packing metrics $r^{(n)}$ converging to a point a in the boundary of $\mathbf{R}_{>0}^V$ in the space $[0, \infty]^V$. If one of the coordinate of a , say $a(v)$, is infinite, then by lemma 4.2, its curvature $k_0(v)$ converges to 2π , i.e., (4.1) becomes an equality. Now suppose $a \in [0, \infty)^V$. Let $I \subset V$ be the non-empty set $\{v \in V | a(v) = 0\}$. Let F_I be the set of all triangles having at least one vertex in I . Take a triangle $\sigma \in F_I$. Let the vertices of σ be v_1, v_2, v_3 so that the inner angle at v_i is θ_i . There are three possibilities for σ : (i) there is only one vertex, say v_1 , of σ in I ; (ii) there are two vertices, say v_2, v_3 , of σ in I ; and (iii) all vertices v_1, v_2, v_3 of σ are in I . In the case (i), by lemma 4.1, $\theta_1 = \pi$. In the case (ii), $\theta_2 + \theta_3 = \pi$ (note that the individual value θ_i may not be well defined). In the case (iii), $\theta_1 + \theta_2 + \theta_3 = \pi$. Consider the sum of all discrete curvatures at I ,

$$\sum_{v \in I} k_0(v) = 2\pi|I| - \sum_{\theta \in X} \theta$$

where X is the set of all inner angles having vertices in I . We may group these inner angles in X according to the triangles they lie and sum over each triangle first. The result is that $\sum_{\theta \in X} \theta = \pi|F_I|$ by the discussion above in the three cases (i), (ii) and (iii), i.e.,

$$\sum_{v \in I} k_0(v) = 2\pi|I| - \pi|F_I| \quad (4.2)$$

On the other hand, it is easy to see that,

$$\sum_{v \in I} k_0(v) > 2\pi|I| - \pi|F_I| \quad (4.3)$$

due to the fact that the sum of inner angles of a hyperbolic triangle is less than π . Thus the image $\Pi(\mathbf{R}_{>0}^V)$ is the open bounded convex polytope bounded by linear inequalities (4.1) and (4.3). This ends the proof.

4.2 Some other results on the space of curvatures

Similar results for the moduli spaces of all ϕ_h or ψ_h curvatures, or k_h discrete curvatures on a triangulated surfaces are obtained in [15] using the same method.

Theorem 4.3. ([15]). *Suppose $h \leq -1$ and (S, T) is a closed triangulated surface so that V is the set of all vertices. Then,*

- (i) *The space of all k_h -discrete curvatures of Euclidean circle packing metrics on (S, T) forms a proper codimension-1 smooth submanifold in \mathbf{R}^V .*
- (ii) *The space of all k_h -discrete curvatures of hyperbolic circle packing metrics on (S, T) is an open submanifold in \mathbf{R}^V bounded by the proper codimension-1 submanifold in part (a).*

A related theorem, proved in essentially the same way, is the following. Given a closed triangulated surface (S, T) and $K^2 = \mathbf{S}^2, \mathbf{H}^2, \mathbf{E}^2$, let $P_{K^2}(S, T)$ be the space of all K^2 polyhedral metrics on (S, T) parameterized by the edge length function. Let Φ_h (resp. Ψ_h): $P_{K^2}(S, T) \rightarrow \mathbf{R}^E$ be the map sending a metric to its ϕ_h (resp. ψ_h) curvature.

Theorem 4.4. ([15]) *Suppose (S, T) is a closed triangulated surface so that E is the set of all edges. Let $h \leq -1$.*

(a) *The space $\Phi_h(P_{E^2}(S, T))$ is a proper smooth codimension-1 submanifold in \mathbf{R}^E .*

(b) *The space $\Psi_h(P_{H^2}(S, T))$ is an open set bounded by $\Phi_h(P_{E^2}(S, T))$ and a finite set of linear inequalities.*

Using the same argument, we can give a proof ([15], §9) of the following results of Rivin and Leibon. Recall that a Euclidean or hyperbolic polyhedral surface is said to have Delaunay property if its ψ_0 curvature is non-negative.

Theorem 4.5. (Rivin, Leibon) *Suppose (S, T) is a closed triangulated surface.*

(i) ([18]) *The space of all ϕ_0 -curvatures of Delaunay Euclidean polyhedral metrics on (S, T) is a convex polytope.*

(ii) ([12]) *The space of all ψ_0 -curvatures of Delaunay hyperbolic polyhedral metrics on (S, T) is a convex polytope.*

4.3 A sketch of the proof theorems 3.6 and 3.7

Suppose E is the set of all edges in an ideal triangulation of a compact surface S . Let $\Psi_h : \text{Teich}(S) \rightarrow \mathbf{R}^E$ be the map sending a hyperbolic metric to its ψ_h curvature defined in (3.4). The first part of theorem 3.6 says Ψ_h is a smooth embedding. It is proved using the variational principle associated to the action functional in theorem 2.4(v). To prove the second part of theorem 3.6 and theorem 3.7, we need to show that the image $\Psi_h(\text{Teich}(S))$ is a convex polytope. To this end, one must determine the degenerations of the right-angled hexagons. This was achieved in [15] and [9].

The result corresponding to lemma 4.2 is the following,

Lemma 4.6. ([9], [15]) *Suppose a hyperbolic right-angled hexagon has three non-pairwise adjacent edge lengths l_1, l_2, l_3 and opposite edge lengths $\theta_1, \theta_2, \theta_3$ so that $\theta_i = r_j + r_k$, $\{i, j, k\} = \{1, 2, 3\}$. Then the following holds.*

(a) $\lim_{\theta_i \rightarrow 0} l_j(\theta_1, \theta_2, \theta_3) = \infty$ for $j \neq i$.

(b) $\lim_{l_i \rightarrow 0} r_i(l_1, l_2, l_3) = \infty$.

(c) *Suppose a sequence of hexagons satisfies that $|r_1|, |r_2|, |r_3|$ are uniformly bounded. Then $\lim_{l_i \rightarrow \infty} \theta_j(l)\theta_k(l) = 0$ so that the convergence is uniform in l .*

Using this lemma, by the same analysis of boundary points as in Marden-Rodin's proof, we establish theorem 3.6 that the image of $\Psi_h(T(S))$ is an open

convex polytope in \mathbf{R}^E independent of the choice of $h \geq 0$. In Guo's work [9], he was able to push the analysis further and obtained the result for all $h < 0$.

5 Several open problems

We believe that theorem 3.5 should be true for all parameters h .

The space of all polyhedral metrics on (S, T) carries a natural Poisson structure. It is very interesting to know if the Poisson structure can be expressed explicitly in ϕ_h and ψ_h coordinates for each $h \in \mathbf{R}$. See the recent nice work of Mondello [17] for related works.

By taking a sequence of polyhedral metrics converging to a smooth Riemannian metric, we would like to know if the corresponding discrete curvatures ϕ_h , ψ_h and k_h converge in the sense of measure.

The following question concerning the metric-curvature relationships may have an affirmative answer.

Problem 5.1. ([15]) *Suppose (S, T) is a closed triangulated surface. Let $\Pi : P_{K^2}(S, T) \rightarrow \mathbf{R}^V$ be the curvature map sending a metric to its discrete curvature k_0 . Take $p \in \mathbf{R}^V$.*

(a) *For $K^2 = \mathbf{E}^2$ or \mathbf{H}^2 , show that the space $\Pi^{-1}(p)$ is either the empty set or a smooth manifold diffeomorphic to $\mathbf{R}^{|E|-|V|}$.*

(b) *For $K^2 = \mathbf{S}^2$, show that the space $\Pi^{-1}(p)$ is either the empty set or a smooth manifold diffeomorphic to $\mathbf{R}^{|E|-|V|+\mu}$ where μ is the dimension of the group of conformal automorphisms of a spherical polyhedral metric $l \in \Pi^{-1}(p)$.*

Given a spherical polyhedral metric l on (S, T) , let V' be the set of all vertices so that the discrete curvatures at the vertices are zero. The number μ above is the dimension of the group of all conformal automorphisms of the Riemann surface $S - V'$ where the conformal structure is induced by l . In particular, if the Euler characteristic of $S - V'$ is negative, then $\mu = 0$.

We have shown in [15] that the preimage $\Pi^{-1}(p)$ is either empty or a smooth manifold of dimension $|E| - |V|$ in the Euclidean or hyperbolic cases.

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