CELL DECOMPOSITIONS OF TEICHMÜLLER SPACES OF SURFACES WITH BOUNDARY

REN GUO AND FENG LUO

ABSTRACT. A family of coordinates ψ_h for the Teichmüller space of a compact surface with boundary was introduced in [7]. In the work [8], Mondello showed that the coordinate ψ_0 can be used to produce a natural cell decomposition of the Teichmüller space invariant under the action of the mapping class group. In this paper, we show that the similar result also works for all other coordinate ψ_h for any $h \geq 0$.

1. INTRODUCTION

In this note, we show that each of the coordinate ψ_h $(h \geq 0)$ introduced in [7] can be used to produce a natural cell decomposition of the Teichmüller space of a compact surface with non-empty boundary and negative Euler characteristic. We will show that the underlying point sets of the cells are the same as the one obtained in the previous work of Ushijima [10], Hazel [4], Mondello [8]. However, the coordinates ψ_h for $h \geq 0$ introduce different attaching maps for the cell decomposition. In the sequel, unless mentioned otherwise, we will always assume that the surface S is compact with non-empty boundary so that the Euler characteristic of S is negative.

1.1. **The arc complex.** We begin with a brief recall of the related concepts. An *essential arc* a in S is an embedded arc with boundary in ∂S so that a is not homotopic into ∂S relative to ∂S. The arc complex $A(S)$ of the surface, introduced by J. Harer [3], is the simplicial complex so that each vertex is the homotopy class $[a]$ of an essential arc a, and its simplex is a collection of distinct vertices $[a_1], \ldots, [a_k]$ such that $a_i \cap a_j = \emptyset$ for all $i \neq j$. For instance, the isotopy class of an ideal triangulation corresponds to a simplex of maximal dimension in $A(S)$. The non-fillable subcomplex $A_\infty(S)$ of $A(S)$ consists of those simplexes $([a_1], \ldots, [a_k])$ such that one component of $S - \cup_{i=1}^k a_i$ is not simply connected. The simplices in $A(S) - A_\infty(S)$ are called fillable. The underlying space of $A(S) - A_\infty(S)$ is denoted by $|A(S) - A_\infty(S)|$.

1.2. **The Teichmüller space.** It is well-known that there are hyperbolic metrics with totally geodesic boundary on the surface S. Two hyperbolic metrics with geodesic boundary on S are called isotopic if there is an isometry isotopic to the identity between them. The space of all isotopy classes of hyperbolic metrics with geodesic boundary on S is called the Teichmüller space of the surface S, denoted

by $Teich(S)$. Topologically, $Teich(S)$ is homeomorphic to a ball of dimension $6g - 6 + 3n$ where g is the genus and $n > 0$ is the number of boundary components of S.

Theorem 1 (Ushijima [10], Hazel [4], Mondello [8])**.** *There is a natural cell decomposition of the Teichmüller space $Teich(S)$ invariant under the action of the mapping class group.*

Ushijima [10] proved this theorem by following Penner's convex hall construction [9]. Following Bowditch-Epstein's approach [1], Hazel [4] obtained a cell decomposition of the Teichmüller space of surfaces with geodesic boundary and fixed boundary lengths. In [6], the second named author introduced ψ_0-coordinate to parameterize the Teichmüller space $Teich(S)$ of a surface S with a fixed ideal triangulation. Mondello [8] pointed out that the ψ_0-coordinate produces a natural cell decomposition of $Teich(S)$.

In [7], for each real number h, the second named author introduced the ψ_h-coordinates to parameterize $Teich(S)$ of a surface S with a fixed ideal triangulation. The ψ_0-coordinate is a special case of the ψ_h-coordinates.

The main theorem of the paper is the following.

Theorem 2. *Suppose S is a compact surface with non-empty boundary and negative Euler characteristic. For each $h \geq 0$, there is a homeomorphism*

$$\Pi_h : Teich(S) \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

equivariant under the action of the mapping class group so that the restriction of Π_h on each simplex of maximal dimension is given by the ψ_h-coordinate. In particular, this map produces a natural cell decomposition of the moduli space of surfaces with boundary.

We will show that the underlying cell-structures for various h's are the same.

1.3. **Related results.** For a punctured surface S with weights on each puncture, the classical Teichmüller space of S admits cell decompositions. This was first proved by Harer [3] and Thurston (unpublished) using Strebel's work on quadratic differentials and flat cone metrics. The corresponding result in the context of hyperbolic geometry was proved by Bowditch-Epstein [1] and Penner [9] using complete hyperbolic metrics of finite area on S so that each cusp has an assigned horocycle. The constructions in [1] and [9] are more geometrically oriented. Indeed, the construction of spines and Delaunay decompositions based on a given set of points and horocycles are used in [1]. Our approach is the same as that of [1] using Delaunay decompositions. The existence of such Delaunay decompositions for compact hyperbolic manifolds with geodesic boundary was established in the work of Kojima [5] for 3-manifolds. However, the same method of proof in [5] also works for compact hyperbolic surfaces. Our main observation in this paper is that those ψ_h-coordinates introduced in [7] capture the Delaunay condition well.

1.4. **Plan of the paper.** In section 2, we recall the definition and properties of ψ_h-coordinates which will be used in the proof of Theorem 2. In section 3, we prove a simple lemma which clarifies the geometric meaning of ψ_h-coordinates. In section 4, we review the Delaunay decomposition associated to a hyperbolic metric following Bowditch-Epstein [1] and Kojima [5]. Theorem 2 is proved in section 5.

2. ψ_h-COORDINATES

An ideal triangulated compact surface with boundary (S, T) is obtained by removing a small open regular neighborhood of the vertices of a triangulation of a closed surface. The edges of an ideal triangulation T correspond bijectively to the edges of the triangulation of the closed surface. Given a hyperbolic metric d with geodesic boundary on an ideal triangulated surface (S, T), there is a unique geometric ideal triangulation T^* isotopic to T so that all edges are geodesics orthogonal to the boundary. The edges in T^* decompose the surface into hyperbolic right-angled hexagons.
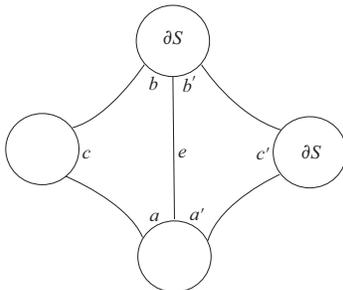

FIGURE 1.

Let E be the set of edges in T. For any real number h, the ψ_h-coordinate of a hyperbolic metric introduced in [7] is defined as $\psi_h : E \to \mathbb{R}$,

$$\psi_h(e) = \int_0^{\frac{a+b-c}{2}} \cosh^h(t)dt + \int_0^{\frac{a'+b'-c'}{2}} \cosh^h(t)dt$$

where e is an edge shared by two hyperbolic right-angled hexagons and c, c' are lengths of arcs in the boundary of S facing e and a, a', b, b' are the lengths of arcs in the boundary of S adjacent to e so that a, b, c lie in a hexagon. See Figure 1.

Now consider the map $\Psi_h : Teich(S) \to \mathbb{R}^E$ sending a hyperbolic metric d to its ψ_h-coordinate. The following two theorems are proved in [7].

Theorem 3 ([7]). *Fix an ideal triangulation of S. For each $h \in \mathbb{R}$, the map $\Psi_h : Teich(S) \to \mathbb{R}^E$ is a smooth embedding.*

An *edge cycle* $(e_1, H_1, ..., e_n, H_n)$ is a collection of hexagons and edges in an ideal triangulation so that two adjacent hexagons H_{i-1} and H_i share the edge e_i for $i = 1, ..., n$ where $H_0 = H_n$.

Theorem 4 ([7]). *Fix an ideal triangulation of S. For each $h \geq 0$, $\Psi_h(Teich(S)) = \{z \in \mathbb{R}^E|$ for each edge cycle $(e_1, H_1, ..., e_n, H_n)$, $\sum_{i=1}^n z(e_i) > 0\}$. Furthermore, the image $\Psi_h(Teich(S))$ is a convex polytope.*

3. HYPERBOLIC RIGHT-ANGLED HEXAGON

We will use the following notations and conventions.

Given two points P, Q in the hyperbolic plan $\mathbb{H}$, the distance between P and Q will be denoted by $|PQ|$. If $P \neq Q$, the complete geodesic in $\mathbb{H}$ containing P and Q will be denoted by $\overline{PQ}$. Suppose H is a hyperbolic right-angled hexagon whose

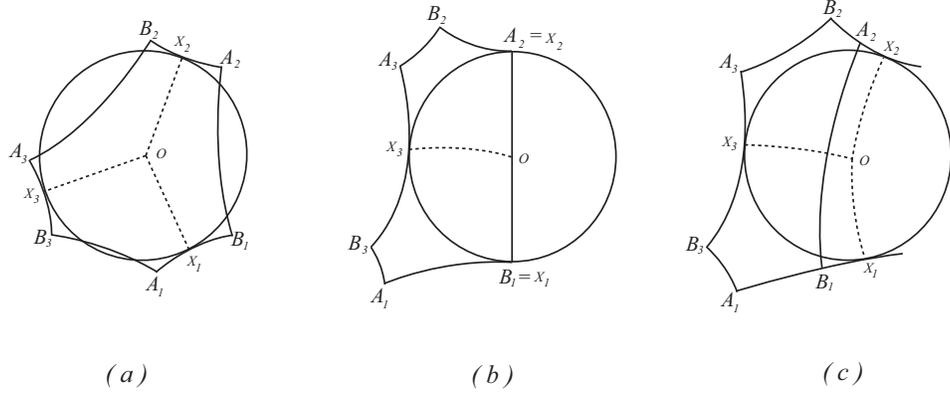(a) (b) (c)

FIGURE 2.

vertices are $A_1, B_1, A_2, B_2, A_3, B_3$ labeled cyclically (see Figure 2). Let C be the circle tangent to the three geodesics $\overline{A_1B_1}$, $\overline{A_2B_2}$ and $\overline{A_3B_3}$. The hyperbolic center of C is denoted by O. Let $X_i = C \cap \overline{A_iB_i}$ be the tangent point for $i = 1, 2, 3$. The geodesic $\overline{B_iA_{i+1}}$ decomposes the hyperbolic plane into two sides. The subindices are counted modulo 3, i.e., $A_4 = A_1$ etc.

Lemma 5. *The following holds for $i = 1, 2, 3$.*

$$|A_iB_i| + |A_{i+1}B_{i+1}| - |A_{i+2}B_{i+2}| =$$

$$\begin{cases} 2|X_iB_i|, & \text{if } O \text{ and } H \text{ are in the same side of } \overline{B_iA_{i+1}} \\ 0, & \text{if } O \in \overline{B_iA_{i+1}} \\ -2|X_iB_i|, & \text{if } O \text{ and } H \text{ are in different sides of } \overline{B_iA_{i+1}} \end{cases}$$

Proof. Since X_j is the tangent point for $j = 1, 2, 3$, we have

$$(1) \qquad\qquad |X_jB_j| = |X_{j+1}A_{j+1}|.$$

According to the location of O with respect to the hexagon, we have three cases to consider.

   Case 1. If O is in the interior of the hexagon, see Figure 2(a). We have, for $j = 1, 2, 3$,

$$|A_jB_j| = |X_jA_j| + |X_jB_j|.$$

Combining with (1), we obtain $|A_jB_j| + |A_{j+1}B_{j+1}| - |A_{j+2}B_{j+2}| = 2|X_jB_j|$. Thus we have verified the lemma in this case since O and H are in the same side of $\overline{B_jA_{j+1}}$ for each $j = 1, 2, 3$.

   Case 2. If O is in the boundary of the hexagon, without of losing generality, we assume $O \in \overline{B_1A_2}$. See Figure 2(b). We have

$$|A_1B_1| = |X_1A_1|,$$
$$|A_2B_2| = |X_2B_2|,$$
$$|A_3B_3| = |X_3A_3| + |X_3B_3|.$$

Combining with (1), we obtain

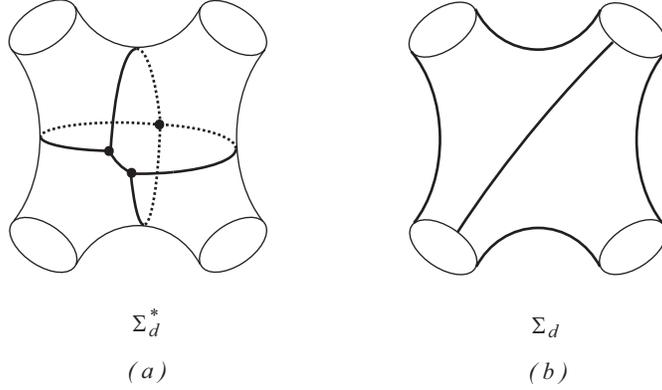$$|A_1B_1| + |A_2B_2| - |A_3B_3| = 0,$$

$$\Sigma_d^*$$
$$(a)$$

$$\Sigma_d$$
$$(b)$$

FIGURE 3.

$$|A_2 B_2| + |A_3 B_3| - |A_1 B_1| = 2|X_2 B_2|,$$
$$|A_3 B_3| + |A_1 B_1| - |A_2 B_2| = 2|X_3 B_3|.$$

Thus we have verified the lemma in this case since $O \in \overline{B_1 A_2}$, O and H are in the same side of $\overline{B_2 A_3}$ and in the same side of $\overline{B_3 A_1}$.

Case 3. If O is outside of the hexagon H, without of losing generality, we may assume O and H are in the same side of $\overline{B_2 A_3}$ and in the same side of $\overline{B_3 A_1}$, but in different sides of $\overline{B_1 A_2}$. See Figure 2(c). We have

$$|A_1 B_1| = |X_1 A_1| - |X_1 B_1|,$$
$$|A_2 B_2| = |X_2 B_2| - |X_2 A_2|,$$
$$|A_3 B_3| = |X_3 A_3| + |X_3 B_3|.$$

Combining with (1), we obtain

$$|A_1 B_1| + |A_2 B_2| - |A_3 B_3| = -2|X_1 B_1|,$$
$$|A_2 B_2| + |A_3 B_3| - |A_1 B_1| = 2|X_2 B_2|,$$
$$|A_3 B_3| + |A_1 B_1| - |A_2 B_2| = 2|X_3 B_3|.$$

Thus we have verify the lemma in this case. $\qquad\square$

4. DELAUNAY DECOMPOSITIONS

Let's recall the construction of the Delaunay decomposition associated to a hyperbolic metric following Bowditch-Epstein [1]. For higher dimensional hyperbolic manifolds, see Epstein-Penner [2] and Kojima [5].

Let (S, d) be a hyperbolic metric with geodesic boundary on the compact surface S. Let d be a hyperbolic metric with geodesic boundary on S. The Delaunay decomposition of (S, d) produces a graph Σ_d^*, called the *spine* of the surface S so that Σ_d^* is the set of points in S which have two or more distinct shortest geodesics to ∂S.

To be more precise, let $n(p)$ be the number of shortest geodesics arcs from p to ∂S. The spine Σ_d^* of (S, d) is the set $\{p \in S | n(p) \geq 2\}$. And the vertex of Σ_d^* is the set $\{p \in S | n(p) \geq 3\}$. The set Σ_d^* is shown (see Bowditch-Epstein [1], Kojima [5]) to be a graph whose edges are geodesic arcs in S. The the edges of Σ_d^* are denoted by $e_1^*, ..., e_N^*$. By the construction, each of point in the interior of
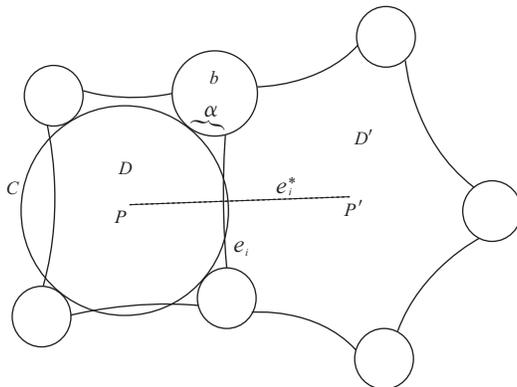
FIGURE 4.

an edge $e_i^*, i = 1, ..., N$, has precisely two distinct shortest geodesics to ∂S. Each edge e_i^* connects the two vertices which are the points having three or more distinct shortest geodesics to ∂S. By [5] or [1], it is known that Σ_d^* is a strong deformation retract of the surface S.

Associated with the spine Σ_d^* is the so called *Delaunay decomposition* of the hyperbolic surface. Here is the construction.

For each edge e^* of the spine, there are two boundary components B_1 and B_2 (may be coincide) of the surface so that points in the interior of e^* have exact two shortest geodesic arcs a_1 and a_2 to B_1 and B_2. Let e be the shortest geodesic from B_1 to B_2. It is known that e is homotopic to $a_1 \cup a_2$ and e intersects e^* perpendicularly. Furthermore, these edges e's are pairwise disjoint. The collection of all such e's decompose the surface S into a collection of right-angled polygons. These are the 2-cell, or the Delaunay domains. We use Σ_d to denote the cell decomposition of the surface S whose 2-cells are the Delaunay domains, whose 1-cells consist of these e's and the arcs in the boundary of S. One can think of Σ_d^* as a dual to Σ_d as follows. For each 2-cell D in Σ_d, there is exactly one vertex v of Σ_d^* so that v lies in the interior of D. Furthermore, by the construction, v is of equal distance to all edges of $D \cap \partial S$. Consider the hyperbolic circle in S centered at v so that it is tangent to all edges in $D \cap \partial S$. We call it the *inscribed circle* of the Delaunay domain D.

Figure 3(a) is an example of the spine of a four-hole sphere, where the spine is the graph of thick lines. In Figure 3(b), the thick lines produce a Denaulay decomposition.

5. PROOF OF THE MAIN THEOREM

5.1. **Construction of the homeomorphism.** To prove Theorem 2, for each $h \geq 0$, we construct the map $\Pi_h : Teich(S) \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$ as follows. Given a hyperbolic metric d with geodesic boundary, we obtain the spine Σ_d^* and the Delaunay decomposition Σ_d of S in the metric d. Let $(e_1^*, ..., e_N^*)$ be the edges of the spine and $(e_1, ..., e_N)$ be the edges of the Delaunay decomposition where e_i is dual to e_i^*. See Figure 4. Suppose e_i is shared by two 2-cells D and D'. The inscribed circle of D is denoted by C. Let b be one of the two edges of D adjacent
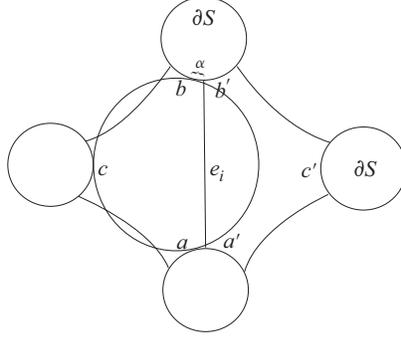
FIGURE 5.

to the edges e_i. Let α be the length of the arc contained in b with end points $C \cap b$ and $e_i \cap b$. Similarly, we find the inscribed circle of D' and the length α'. Now define a function for each $h \geq 0$:

$$(2) \qquad \pi_h(e_i) = \int_0^\alpha \cosh^h(t)dt + \int_0^{\alpha'} \cosh^h(t)dt.$$

Note that, due to Delaunay condition, α, α' are positive for each i. Therefore $\pi_h(e_i) > 0$ for each i.

It is clear from the definition that the Delaunay decomposition and the coordinates $\pi_h(e_i)$ depend only on the isotopy class of the hyperbolic metric. In other words, they are independent of the choice of a representative of a point of the Teichmüller spaces $Teich(S)$. A point of $Teich(S)$ is denoted by $[d]$. We obtain a well-defined map

$$(3) \qquad \Pi_h : Teich(S) \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

$$[d] \mapsto (\sum_{i=1}^N \frac{\pi_h(e_i)}{\sum_{i=1}^N \pi_h(e_i)} \cdot [e_i], \sum_{i=1}^N \pi_h(e_i)),$$

where $(e_1, ..., e_N)$ are the edges of the Delaunay decomposition of (S, d) and $[e_i]$ is a isotopy class. Note that $\sum_{i=1}^N \frac{\pi_h(e_i)}{\sum_{i=1}^N \pi_h(e_i)} \cdot [e_i]$ is a point in the fillable simplex with vertices $[e_1], ..., [e_N]$ of the arc complex, since the sum of the coefficient of the vertices is 1 and $\pi_h(e_i) > 0$ for all i.

In the rest of the section, we will show that Π_h is injective, onto, and is a homeomorphism.

5.2. **One-to-one.** We claim that the map Π_h is one-to-one. Suppose there are two hyperbolic metrics d, d' such that $\Pi_h([d]) = \Pi_h([d'])$. Then their associated Delaunay decompositions are the same by definition. Say $\{e_1, ..., e_N\}$ is the set of edges in $\Sigma_d = \Sigma_{d'}$. If $N = 6g - 6 + 3n$ where g is the genus and n is the number of boundary components of S, then $(e_1, ..., e_N)$ is an ideal triangulation. In this case each 2-cell is a right-angled hexagon. Suppose edge e_i is shared by hexagons D and D'. See Figure 5.

Let c be the length of boundary arc opposite to e_i and a, b be lengths of boundary arcs adjacent to e_i in D. Since the center of the inscribed circle of D and the hexagon

D are in the same side of e_i, by Lemma 5, we have $a + b - c = 2\alpha$. Similarly, for hexagon D', we have $a' + b' - c' = 2\alpha'$. Thus

$$
\begin{aligned}
\pi_h(e_i) &= \int_0^\alpha \cosh^h(t)dt + \int_0^{\alpha'} \cosh^h(t)dt \\
&= \int_0^{\frac{a+b-c}{2}} \cosh^h(t)dt + \int_0^{\frac{a'+b'-c'}{2}} \cosh^h(t)dt \\
&= \psi_h(e_i)
\end{aligned}
$$

where $\psi_h(e_i)$ is exactly the ψ_h-coordinate of a hyperbolic metric evaluated at e_i. Thus from $\Pi_h([d]) = \Pi_h([d'])$ we obtain $\Psi_h([d]) = \Psi_h([d'])$ for the ideal triangulation $(e_1, ..., e_N)$, $N = 6g - 6 + 3n$. By Theorem 3, we see that $[d] = [d'] \in Teich(S)$.

If $N < 6g - 6 + 3n$, we add edges $e_{N+1}, ..., e_{6g-6+3n}$ such that $(e_1, ..., e_N, e_{N+1}, ..., e_{6g-6+3n})$ is an ideal triangulation. More precisely, in a 2-cell of the Delaunay decomposition which is not a hexagon, we add arbitrarily geodesic arcs perpendicular to boundary components bounding the 2-cell which decompose the 2-cell into a union of hexagons.

See Figure 6(a). Suppose edge $e_i, i \leq N$, is shared by two 2-cells D, D'. There is a hyperbolic right-angled hexagon H contained in D having e_i as an edge. Note that H is a component of $S - \cup_{i=1}^{6g-6+3n} e_i$. Recall that the inscribed circle C of D is also the inscribed circle of H. Let c be the length of boundary arc opposite to e_i and a, b be lengths of boundary arcs adjacent to e_i in H. Since the center of C and H are in the same side of e_i, by Lemma 5, we have $a + b - c = 2\alpha$, where α is the length in the definition of $\pi_h(e_i)$. From the 2-cell D', we obtain hexagon H' and $a' + b' - c' = 2\alpha'$. Therefore

$$
\begin{aligned}
\pi_h(e_i) &= \int_0^\alpha \cosh^h(t)dt + \int_0^{\alpha'} \cosh^h(t)dt \\
&= \int_0^{\frac{a+b-c}{2}} \cosh^h(t)dt + \int_0^{\frac{a'+b'-c'}{2}} \cosh^h(t)dt \\
&= \psi_h(e_i)
\end{aligned}
$$

See Figure 6(b). Suppose edge $e_i, i > N$, is shared by two hexagon H, H' in the ideal triangulation $(e_1, ..., e_N, e_{N+1}, ..., e_{6g-6+3n})$, where H and H' are obtained from the same 2-cell. Therefore H and H' have the same inscribed circle C which is also the inscribed circle of the 2-cell containing H, H'. In hexagon H, let c be the length of boundary arc opposite to e_i and a, b be lengths of boundary arcs adjacent to e_i. In hexagon H', we define a', b', c'. There are two possibilities to consider. If the center of C is in e_i. Then by Lemma 5, $a + b - c = 0 = a' + b' - c'$. If the center of C is not in e_i, without of losing generality, we assume the center and H are in the same side of e_i. Denote by A the tangent point of C at a boundary component. Denote by B the intersection point of e_i with the same boundary component. By Lemma 5, we have $a + b - c = 2|AB|$ and $a' + b' - c' = -2|AB|$. The two possibilities give the same conclusion, for $i > N$,

$$
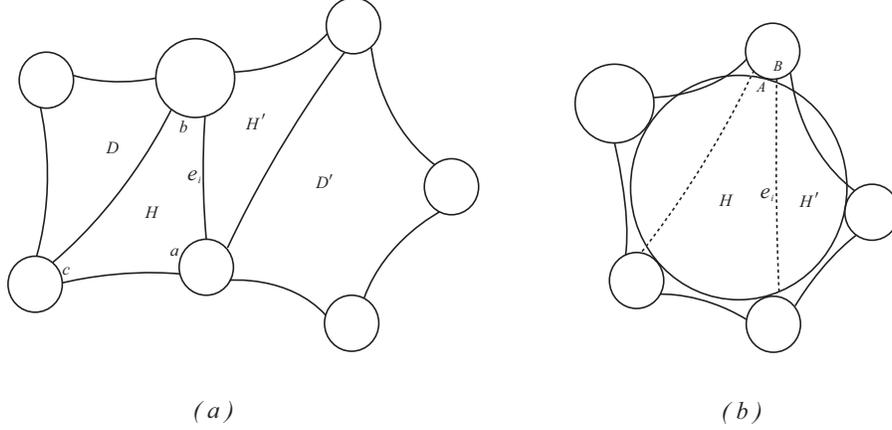\psi_h(e_i) = \int_0^x \cosh^h(t)dt + \int_0^{-x} \cosh^h(t)dt = 0.
$$

(a)   (b)

FIGURE 6.

Thus from $\Pi_h([d]) = \Pi_h([d'])$ we obtain $\Psi_h([d]) = \Psi_h([d'])$ for the ideal triangulation $(e_1, ..., e_N, e_{N+1}, ..., e_{6g-6+3n})$. In fact the i-th entry of $\Psi_h([d]) = \Psi_h([d'])$ is zero as $N + 1 \leq i \leq 6g - 6 + 3n$. By Theorem 3, we see that $[d] = [d'] \in Teich(S)$.

5.3. **Onto.** We claim the map $\Pi_h : Teich(S) \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$ is onto. Given a point $(\sum_{i=1}^N z_i \cdot [e_i], x)$. If $N = 6g - 6 + 3n$, then $(e_1, ..., e_N)$ is an ideal triangulation of S. The vector $(xz_1, ..., xz_N)$ satisfies the condition in Theorem 4 since each entry is positive. By Theorem 4, there is a hyperbolic metric d whose ψ_h-coordinate is $(xz_1, ..., xz_N)$, i.e., $\psi_h(e_i) = xz_i$. Since we have shown in last subsection that $\pi_h(e_i) = \psi_h(e_i)$ in this case. Therefore $\Pi_h([d]) = (\sum_{i=1}^N z_i \cdot [e_i], x)$.

If $N < 6g - 6 + 3n$, then $e_1, ..., e_N$ is a cell decomposition of S. Let T be an ideal triangulation $(e_1, ..., e_N, e_{N+1}, ..., e_{6g-6+3n})$ obtained from the cell decomposition. Then the vector $(xz_1, ..., xz_N, 0, ..., 0)$ (there are $6g - 6 + 3n - N$ zeros) satisfies the condition in Theorem 4 since there does not exists an edge cycle consisting of only the "new" edges $e_i, i > N$. By Theorem 4, there is a hyperbolic metric d whose ψ_h-coordinate is $(xz_1, ..., xz_N, 0, ..., 0)$, i.e., $\psi_h(e_i) = xz_i, i \leq N$ and $\psi_h(e_i) = 0, i > N$.

Suppose edge $e_i, i > N$ is shared by two hexagons H, H'. By the discussion of last subsection, from $\psi_h(e_i) = 0$ we conclude that the inscribed circles of H and H' have the same tangent points at the two boundary components intersecting e_i. Therefore the two circles have the same center. Thus they coincide. If a 2-cell is decomposed into several hexagons, then the inscribed circles of all the hexagons coincide. This shows that the 2-cell has a inscribed circle. Thus the cell decomposition $(e_1, ..., e_N)$ is the Delaunay decomposition of (S, h).

For edge $e_i, i \leq N$, from the discussion of last subsection, we see $\pi_h(e_i) = \psi_h(e_i)$. Therefore $\Pi_h([d]) = (\sum_{i=1}^N z_i \cdot [e_i], x)$.

5.4. **Continuity of Π_h.** We follow the idea in §8 and §9 of Bowditch-Epstein [1] to prove the continuity.

Let $\{d^s\}_{s=1}^\infty$ be a sequence of hyperbolic metrics on S with geodesic boundary converging to a hyperbolic metric d with geodesic boundary. We claim that the sequence of points $\{\Pi_h([d^s])\}_{s=1}^\infty$ converges to the point $\Pi_h([d])$.

Case 1. If, for s sufficiently large, the Delaunay decomposition associated to d has the same combinatorial type as the Delaunay decomposition associated to d^s. Assume that the Delaunay decomposition associated to d has the edges $e_1, ..., e_N$ with $N \leq 6g - 6 + 3n$ and the Delaunay decomposition associated to d^s has the edges $e_1^s, ..., e_N^s$ so that e_i^s is isotopic to e_i for $1 \leq i \leq N$. Since the metrics $\{d^s\}$ converge to the metric d, the geodesic length of edges $\{e_i^s\}$ converge to the geodesic length of the edge e_i.

Assume that the edge e_i is shared by two 2-cells D and D' of (S, d). Correspondingly, the edge e_i^s is shared by two 2-cells D^s and D'^s of (S, d^s). As in §5.1 and Figure 4, let C be the inscribed circle of D and b be one of the two edges of D adjacent to e_i. Let α be the length of the arc contained in b with end points $C \cap b$ and $e_i \cap b$. Let α^s be the length of the corresponding arc in D^s. Assume $e_{D1}, ..., e_{Dt} \in \{e_1, ..., e_N\}$ are the edges of D in the interior of S. By the elementary hyperbolic geometry, the radius of C is a continuous function of the lengths of $e_{D1}, ..., e_{Dt}$. Therefore α is a continuous function of the lengths of $e_{D1}, ..., e_{Dt}$. Thus the sequence $\{\alpha^s\}$ converges to α. By the same argument, for the 2-cell D', we have the length α' and α'^s so that the sequence $\{\alpha'^s\}$ converges to α'. By the definition (2), the sequence $\{\pi_h(e_i^s)\}$ converges to $\pi_h(e_i)$. By the definition (3), the sequence of points $\{\Pi_h([d_s])\}_{s=1}^{\infty}$ converges to the point $\Pi_h([d])$. Geometrically, this is a sequence of interior points in a simplex of the arc complex converging to an interior point in the same simplex.

Case 2. If for s sufficiently large, the Delaunay decomposition associated to d^s have the same combinatorial type with each other but different from that associated to d. Assume that the Delaunay decomposition associated to d has the edges $e_1, ..., e_N$ with $N < 6g - 6 + 3n$ and the Delaunay decomposition associated to d^s has the edges $e_1^s, ..., e_N^s, e_{N+1}^s, ..., e_{N+M}^s$ with $N + M \leq 6g - 6 + 3n$ so that e_i^s is isotopic to e_i for $1 \leq i \leq N$.

Since e_j^s is isotopic to $e_j^{s'}$ for $N + 1 \leq j \leq N + M$ and s, s' sufficiently large, we can add an edge e_j on (S, d) which is isotopic to e_j^s for $N + 1 \leq j \leq N + M$. Now the edges $e_1, ..., e_N, e_{N+1}, ..., e_{N+M}$ produce a cell decomposition of S which has the same combinatorial type with the cell decomposition obtained from the edges $e_1^s, ..., e_N^s, e_{N+1}^s, ..., e_{N+M}^s$.

We get the same situation of Case 1. The convergence of metrics implies the convergence of the edge lengths which implies the convergence of the π_h-coordinates. In Case 2, since the edges $e_{N+1}, ..., e_{N+M}$ are added to a Delaunay decomposition, we know from §5.2 that $\pi_h(e_j) = 0$ as $N + 1 \leq j \leq N + M$. Geometrically, this is a sequence of interior points in the a simplex of the arc complex converging to a point on the boundary of the simplex.

5.5. **Continuity of Π_h^{-1}.** Let $\{p^s\}_{s=1}^{\infty}$ be a sequence of points in $|A(S) - A_{\infty}(S)| \times \mathbb{R}_{>0}$ converging to a point p. We claim that the sequence of hyperbolic metrics $\{\Pi_h^{-1}(p^s)\}$ converges to the hyperbolic metric $\Pi_h^{-1}(p)$.

Case 1. If, for s sufficiently large, $\{p^s\}$ and p are in the same simplex, then the Delaunay decomposition associated to $\Pi_h^{-1}(p^s)$ and $\Pi_h^{-1}(p)$ have the same combinatorial type. If it is needed, by adding edges in the 2-cells which are not hexagons, we obtain a fixed topological ideal triangulation of the surface S. Note that the $\pi_h(e) = 0$ if e is an edge being added. For an edge e_i on $(S, \Pi_h^{-1}(p))$, denote by e_i^s the corresponding edge on $(S, \Pi_h^{-1}(p^s))$. Now we have a fixed ideal triangulation and that the sequence of coordinates $\{\pi_h(e_i^s)\}$ converges to the coordinate $\pi_h(e_i)$

for each edge e_i. By §5.2, $\pi_h(e_i^s) = \psi_h(e_i^s)$ and $\pi_h(e_i) = \psi_h(e_i)$. Therefore the sequence of coordinates $\{\psi_h(e_i^s)\}$ converges to the coordinate $\psi_h(e_i)$ for each edge e_i. By Theorem 3, the sequence of hyperbolic metrics $\{\Pi_h^{-1}(p^s)\}$ converges to the hyperbolic metric $\Pi_h^{-1}(p)$.

Case 2. If, for s sufficiently large, $\{p^s\}$ are in the interior of a simplex and p is on the boundary of the simplex. Assume that the Delaunay decomposition associated to $\Pi_h^{-1}(p)$ has the edges $e_1, ..., e_N$ with $N < 6g - 6 + 3n$ and the Delaunay decomposition associated to $\Pi_h^{-1}(p^s)$ has the edges $e_1^s, ..., e_N^s, e_{N+1}^s, ..., e_{N+M}^s$ with $N + M \leq 6g - 6 + 3n$ so that e_i^s is isotopic to e_i for $1 \leq i \leq N$. We can add an edge e_j on $(S, \Pi_h^{-1}(p))$ which is isotopic to the edge e_j^s for $N + 1 \leq j \leq N + M$. By the assumption that $\{\pi_h(e_i^s)\}$ converges to $\pi_h(e_i)$ for $1 \leq i \leq N$ and $\{\pi_h(e_j^s)\}$ converges to 0 for $N + 1 \leq j \leq N + M$. Since e_j is added to the Delaunay decomposition of $\Pi_h^{-1}(p)$, $\pi_h(e_j) = 0$ as $N + 1 \leq j \leq N + M$. We get the situation of Case 1. We may add more edges to obtain a fixed ideal triangulation. The same arguments of Case 1 can be used to establish the claim.

To sum up, we have proved Theorem 2:

$$\Pi_h : Teich(S) \to |A(S) - A_\infty(S)| \times \mathbb{R}_{>0}$$

is a homeomorphism.

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SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN, 55455
E-mail address: guoxx170@math.umn.edu

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, PISCATAWAY, NJ, 08854
E-mail address: fluo@math.rutgers.edu