

Triangulated 3-Manifolds: from Haken's normal surfaces to Thurston's algebraic equation

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Abstract

We give a brief summary of some of our work and our joint work with Stephan Tillmann on solving Thurston's equation and Haken equation on triangulated 3-manifolds in this paper. Several conjectures on the existence of solutions to Thurston's equation and Haken equation are made. Resolutions of these conjecture will lead to a new proof of the Poincaré conjecture without using the Ricci flow. We approach these conjectures by a finite dimensional variational principle so that its critical points are related to solutions to Thurston's gluing equation and Haken's normal surface equation. The action functional is the volume. This is a generalization of an earlier program by Casson and Rivin for compact 3-manifolds with torus boundary.

1 Introduction

This paper is based on several talks given by the author at the conference "Interactions Between Hyperbolic Geometry, Quantum Topology and Number Theory" at Columbia University in 2009 and a few more places. The goal of the paper is to give a quick summary of some of our work [19] and our joint work with Stephan Tillmann [20], [21] on triangulated 3-manifolds. Our work is an attempt to connect geometry and topology of compact 3-manifolds from the point of view of triangulations. We will recall Haken's normal surface theory, Thurston's work on construction of hyperbolic structures, Neumann-Zagier's work, the notion of angle structures introduced by Casson, Rivin and Lackenby, and the work of several other people. One important point we would like to emphasize is the role that the Neumann-Zagier Poisson structure plays in these theories. It is conceivable that the Neumann-Zagier Poisson structure will play an important role in discretization and quantization of $SL(2, \mathbb{C})$ Chern-Simons theory in dimension three.

A combination of the recent work of Segerman-Tillmann [32], Futer-Guéritaud [8], Luo-Tillmann [21] and [19] has prompted us to make several conjectures on the solutions of Thurston's equation and Haken's normal surface equations. The resolution of some of these conjectures will produce a new proof of the Poincaré conjecture without using the Ricci flow method.

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Let us begin with a recall of closed triangulated pseudo 3-manifolds. Take a disjoint union of tetrahedra. Identify codimension-1 faces of tetrahedra in pairs by affine homeomorphisms. The quotient space is a *triangulated closed pseudo 3-manifold*. (See §2.1 for more details). In particular, closed triangulated 3-manifolds are closed triangulated pseudo 3-manifolds and ideally triangulated 3-manifolds are pseudo 3-manifolds with vertices removed. Given a closed triangulated oriented pseudo 3-manifold, there are linear and algebraic equations associated to the triangulation. Besides the homology theories, the most prominent ones are Haken's equation of normal surfaces [11] and Thurston's algebraic gluing equation for construction of hyperbolic metrics [35] using hyperbolic ideal tetrahedra. Haken's theory is topological and studies surfaces in 3-manifolds and Thurston's equation is geometric and tries to construct hyperbolic metrics from the triangulation. In the most general setting, Thurston's equation tries to find representations of the fundamental group into $PSL(2, \mathbf{C})$ ([39]). Much work has been done on both normal surface theory and Thurston's equation with fantastic consequences in the past fifty years.

Haken's normal surface equation is linear. A basis for the solution space was found recently by Kang-Rubinstein [17]. In particular, there are always solutions to Haken's equation with non-zero quadrilateral coordinates. The situation for solving Thurston's equation is different. The main problem which motivates our investigation is the following.

Main Problem *Given a closed oriented triangulated pseudo 3-manifold (M, \mathbf{T}) , when does there exist a solution to Thurston's gluing equation?*

The most investigated cases in solving Thurston's equation are associated to ideal triangulated 3-manifolds with torus boundary so that the complex numbers z are in the upper-half plane (see for instance [35], [34], [6], [27] and many others). These solutions are closely related to the hyperbolic structures. However, we intend to study Thurston's equation and its solutions in the most general setting of closed oriented triangulated pseudo 3-manifolds, in particular, on closed triangulated 3-manifolds. Even though a solution to Thurston's equation in the general setting does not necessarily produce a hyperbolic structure, one can still obtain important information from it. For instance, it was observed in [39] (see also [25], [32]) that each solution of Thurston's equation produces a representation of the fundamental group of the pseudo 3-manifold with vertices of the triangulation removed to $PSL(2, \mathbf{C})$. A simplified version of a recent theorem of Segerman-Tillmann [32] states that

Theorem 1.1(Segerman-Tillmann) *If (M, \mathbf{T}) is a closed triangulated oriented 3-manifold so that the triangulation supports a solution to Thurston's equation, then each edge in \mathbf{T} either has two distinct end points or is homotopically essential in M .*

In particular, their theorem says any one-vertex triangulation of a simply connected 3-manifold cannot support a solution to Thurston's equation. A combination of theorem 1.1 and a result of [38] gives an interesting solution to the main problem for closed 3-manifold. Namely, a closed triangulated 3-manifold (M, \mathbf{T}) supports a solution to Thurston's equation if and only if there exists a representation $\rho : \pi_1(M) \rightarrow PSL(2, \mathbf{C})$ so that $\rho([e]) \neq 1$ for each edge e having the same end points. The drawback of this solution is that the representation ρ has to be a priori given.

Our recent work [19] suggests another way to resolve the main problem using Haken's normal surface equation. To state the corresponding conjecture, let us recall that a solution to Haken's normal surface equation is said to be of *2-quad-type* if it has exactly one or two non-zero quadrilateral coordinates. A *cluster of three 2-quad-type solutions* to Haken's equation consists of three

2-quad-type solutions x_1, x_2 and x_3 so that there is a tetrahedron containing three distinct quadrilaterals q_1, q_2, q_3 with $x_i(q_i) \neq 0$ for $i = 1, 2, 3$. A triangulation of a 3-manifold is called *minimal* if it has the smallest number of tetrahedra among all triangulations of the 3-manifold.

The main focus of our investigation will be around the following conjecture. We thank Ben Burton and Henry Segerman for providing supporting data which helped us formulating it in the current form.

Conjecture 1 (Haken-Thurston Alternative) *For any closed irreducible orientable minimally triangulated 3-manifold (M, \mathbf{T}) , one of the two holds:*

- (1) *there exists a solution to Thurston's equation associated to the triangulation, or*
- (2) *there exists a cluster of three 2-quad-type solutions to Haken's normal surface equation.*

Using a theorem of Futer-Guéritaud, we proved in [19] the following result which supports conjecture 1.

Theorem 1.2 *Suppose (M, \mathbf{T}) is a closed triangulated oriented pseudo 3-manifold. Then either there exists a solution to the generalized Thurston equation or there exists a cluster of three 2-quad-type solutions to Haken's normal surface equation.*

In our joint work with Tillmann [21], using Jaco-Rubinstein's work [21], we proved the following theorem concerning the topology of 3-manifolds satisfying part (2) of conjecture 1.

Theorem 1.3([21]) *Suppose (M, T) is a minimally triangulated orientable closed 3-manifold so that there exists a cluster of three 2-quad-type solutions to Haken's equation. Then,*

- (a) *M is reducible, or*
- (b) *M is toroidal, or*
- (c) *M is a Seifert fibered space, or*
- (d) *M contains the connected sum $\#_{i=1}^3 RP^2$ of three copies of the projective plane.*

Using theorems 1.1 and 1.3, one can deduce the Poincaré conjecture from conjecture 1 (without using the Ricci flow) as follows. Suppose M is a simply connected closed 3-manifold. By the Kneser-Milnor prime decomposition theorem, we may assume that M is irreducible. Take a minimal triangulation \mathbf{T} of M . By the work of Jaco-Rubinstein on 0-efficient triangulation [14], we may assume that \mathbf{T} has only one vertex, i.e., each edge is a loop. By Segerman-Tillmann's theorem above, we see that (M, \mathbf{T}) cannot support a solution to Thurston's equation. By conjecture 1, there exists a cluster of three 2-quad-type solutions to Haken's equation. By theorem 1.3, the minimality of \mathbf{T} and irreducibility of M , we conclude that $M = \mathbf{S}^3$.

Theorem 1.2 is proved in [19] where we proposed a variational principle associated to the triangulation to approach conjecture 1. In this approach, 2-quad-type solutions to Haken's equation arise naturally from non-smooth maximum points. We generalize the notion of angle structures introduced by Casson, Lackenby [18] and Rivin [28] (for ideally triangulated cusped 3-manifolds) to the circle-valued angle structure (or \mathbf{S}^1 -angle structure or *SAS* for short) and its volume for any closed triangulated pseudo 3-manifold. It is essentially proved in [20] and more specifically in [19] that an *SAS* exists on any closed triangulated pseudo 3-manifold (M, \mathbf{T}) . The space $SAS(\mathbf{T})$ of all circle-valued angle structures on (M, \mathbf{T}) is shown to be a closed smooth manifold. Furthermore, each circle-valued angle structure has a natural volume defined by the Milnor-Lobachevsky

function. The volume defines a continuous but not necessarily smooth volume function vol on the space $SAS(\mathbf{T})$. In particular, the volume function vol achieves a maximum point in $SAS(\mathbf{T})$. The two conclusions in theorem 1.2 correspond to the maximum point being smooth or not for the volume function.

More details of the results obtained so far and our approaches to resolve the conjecture 1 will be discussed in sections 4 and 5. We remark that conjecture 1 itself is independent of the angle structures and there are other ways to approach it.

There are several interesting problems arising from the approach taken here. For instance, how to relate the critical values of the volume function on $SAS(\mathbf{T})$ with the Gromov norm of the 3-manifold. The Gromov norm of a closed 3-manifold is probably the most important topological invariant for 3-manifolds. Yet its computation is not easy. It seems highly likely that for a triangulation without a cluster of three 2-quad-type solutions to Haken's equation, the Gromov norm of the manifold (multiplied by the volume of the regular ideal tetrahedron) is among the critical values of the volume function on $SAS(\mathbf{T})$. In our recent work with Tillmann and Yang [22], we have solved this problem for closed hyperbolic manifolds. An affirmative resolution of this problem for all 3-manifolds may provide insights which help to resolve the Volume Conjecture for closed 3-manifolds.

Futer and Guéritaud have written a very nice paper [7] on volume and angle structures which is closely related to the material covered in this paper.

We remark that this is not a survey paper on the subject of triangulations of 3-manifolds. Important work in the field, in particular the work of Jaco-Rubinstein [14] on efficient triangulations of 3-manifolds, is not discussed in the paper.

The paper is organized as follows. In section 2, we will recall the basic material on triangulations and Haken's normal surface theory. In section 3, we discuss Neumann-Zagier's Poisson structures and Thurston's gluing equation. In section 4, we discuss circle valued angle structures, their volume, some of our work and a theorem of Futer-Guéritaуд. In section 5, we introduce a \mathbf{Z}_2 version of Thurston's equation (\mathbf{Z}_2 -taut structure).

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2 Triangulations and normal surfaces

The normal surface theory, developed by Haken in the 1950's, is a beautiful chapter in 3-manifold topology. In the late 1970's, Thurston introduced the notion of spun normal surfaces and used it to study 3-manifolds.

We will revisit the normal surface theory and follow the expositions in [12] and [33] closely in this section. Some of the notations used in this section are new. The work of Tollefson, Kang-Rubinstein, Tillmann, and Jaco on characterizing the quadrilateral coordinates of normal surfaces

will be discussed.

2.1 Some useful facts about tetrahedra

The following lemma will be used frequently in the sequel. The proof is very simple and will be omitted. To start, suppose $\sigma = [v_1, \dots, v_4]$ is a tetrahedron with vertices v_1, \dots, v_4 and edges $e_{ij} = \{v_i, v_j\}$, $i \neq j$. We call e_{kl} the *opposite edge* of e_{ij} if $\{i, j, k, l\} = \{1, 2, 3, 4\}$.

Lemma 2.1. *Given a tetrahedron σ , assign to each edge e_{ij} a real number $a_{ij} \in \mathbf{R}$, called the weight of e_{ij} . Assume $\{i, j, k, l\} = \{1, 2, 3, 4\}$.*

(a) *If the sum of weights of opposite edges is a constant, i.e., $a_{ij} + a_{kl}$ is independent of indices, then there exist real numbers b_1, \dots, b_4 (weights at vertices) so that*

$$a_{ij} = b_i + b_j.$$

(b) *If the sum of weights of the edges from each vertex is a constant, i.e., $a_{ij} + a_{ik} + a_{il}$ is independent of indices, then weights of opposite edges are the same, i.e.,*

$$a_{ij} = a_{kl}.$$

(c) *If the tetrahedron σ is oriented and edges are labelled by a, b, c so that opposite edges are labelled by the same letter (see figure 1(a)), then the cyclic order $a \rightarrow b \rightarrow c \rightarrow a$ is independent of the choice of the vertices and depends only on the orientation of σ .*

2.2 Triangulated closed pseudo 3-manifolds and Haken's normal surface equation

Let X be a union of finitely many disjoint oriented Euclidean tetrahedra. The collection of all faces of tetrahedra in X is a simplicial complex \mathbf{T}^* which is a triangulation of X . Identify codimension-1 faces in X in pairs by affine orientation-reversing homeomorphisms. The quotient space M is a closed oriented pseudo 3-manifold with a triangulation \mathbf{T} whose simplices are the quotients of simplices in \mathbf{T}^* . Let V, E, F, T (and V^*, E^*, F^* and T^*) be the sets of all vertices, edges, triangles and tetrahedra in \mathbf{T} (in \mathbf{T}^* respectively). The quotient of a simplex $x \in \mathbf{T}^*$ will be denoted by $[x]$ in \mathbf{T} . We call $x \in \mathbf{T}^*$ the unidentified simplex and $[x]$ the quotient simplex. Since the sets of tetrahedra in T^* and T are bijective under the quotient map, we will identify a tetrahedron $\sigma \in T^*$ with its quotient $[\sigma]$, i.e., $\sigma = [\sigma]$ and $T = T^*$.

If $x, y \in V \cup E \cup F \cup T$ (or in \mathbf{T}^*), we use $x > y$ to denote that y is a face of x . We use $|Y|$ to denote the cardinality of a set Y .

Note that in this definition of triangulation, we do not assume that simplices in \mathbf{T} are embedded in M . For instance, it may well be that $|V| = 1$. Furthermore, the non-manifold points in M are contained in the set of vertices.

According to Haken, a normal surface in a triangulated pseudo 3-manifold M is an embedded surface $S \subset M$ so that for each tetrahedron σ , topologically the intersection $S \cap \sigma$ consists of a collection of planar quadrilaterals and planar triangles, i.e., inside each tetrahedron, topologically

the surface S looks like planes cutting through the tetrahedron generically. Haken's theory puts this geometric observation into an algebraic setting. According to [11], a *normal arc* in X is an embedded arc in a triangle face so that its end points are in different edges and a *normal disk* in X is an embedded disk in a tetrahedron so that its boundary consists of 3 or 4 normal arcs. These are called *normal triangles* and *normal quadrilaterals* respectively. A normal isotopy is an isotopy of X leaving each simplex invariant. Haken's normal surface theory deals with normal isotopy classes of normal disks and normal surfaces. For simplicity, we will interchange the use of normal disk with the normal isotopy class of a normal disk.

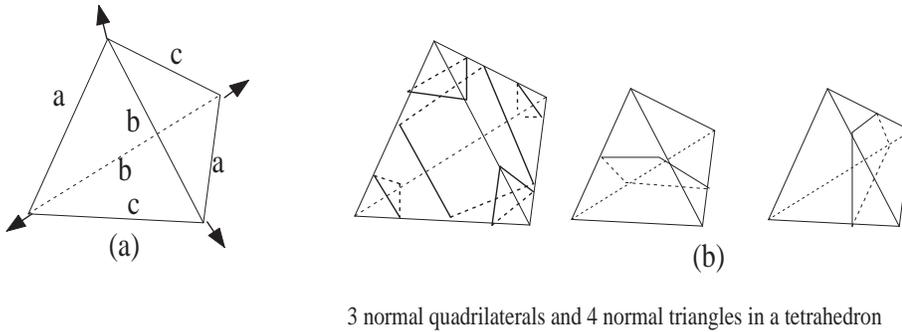


Figure 1: t, t' are normal triangles and q, q' are normal quadrilaterals

The projections of normal arcs and normal disks from X to M constitute normal arcs and normal disks in the triangulated space (M, \mathbf{T}) . For each tetrahedron, there are four normal triangles and three normal quadrilaterals inside it up to normal isotopy. See figure 1(b). Note that there is a natural one-one correspondence between normal disks in T^* and T . In the sequel, we will not distinguish normal disks in \mathbf{T} or \mathbf{T}^* and we will use Δ, \square to denote the sets of all normal isotopy classes of normal triangles and quadrilaterals in the triangulation \mathbf{T} and also \mathbf{T}^* . The set of normal arcs in \mathbf{T}^* and \mathbf{T} are denoted by \mathbf{A}^* and \mathbf{A} respectively.

There are relationships among the sets $V, E, F, T, \Delta, \square, \mathbf{A}$. These incidence relations, which will be recalled below, are the basic ingredients for defining Haken's and Thurston's equations.

Take $t \in \Delta$, $a \in \mathbf{A}$, $q \in \square$, and $\sigma \in T$. The following notations will be used. We use $a < t$ (and $a < q$) if there exist representatives $x \in a$, $y \in t$ (and $z \in q$) so that x is an edge of y (and z). We use $t \subset \sigma$ and $q \subset \sigma$ to denote that representatives of t and q are in the tetrahedron σ . In this case, we say the tetrahedron σ contains t and q .

As a convention, we will always use the letters σ , e and q to denote a tetrahedron, an edge and a quadrilateral in the triangulation \mathbf{T} respectively.

The normal surface equation is a system of linear equations defined in the space $\mathbf{R}^\Delta \times \mathbf{R}^\square$, introduced by W. Haken [11]. It is defined as follows. For each normal arc $a \in \mathbf{A}$, suppose σ, σ' are the two tetrahedra adjacent to the triangular face which contains a . (Note that σ may be σ' .) Then there is a homogeneous linear equation for $x \in \mathbf{R}^\Delta \times \mathbf{R}^\square$ associated to a :

$$x(t) + x(q) = x(q') + x(t') \tag{2.1}$$

where $t, q \subset \sigma$, $t', q' \subset \sigma'$ and $t, t', q, q' > a$. See figure 2(a).

Recall that we identify the set of edges E with the quotient of E^* , i.e., $E = \{[y] | y \in E^*\}$ where $[y] = \{y' \in E^* | y \sim y'\}$. The index $i : E^* \times \square \rightarrow \mathbf{Z}$ is defined as follows: $i(y, q) = 1$ if y, q lie in the same tetrahedron $\sigma \in \mathbf{T}^*$ so that $y \cap q = \emptyset$, and $i(y, q) = 0$ in all other cases. The index $i : E \times \square \rightarrow \mathbf{Z}$ is defined to be $i(e, q) = \sum_{y \in e} i(y, q)$. See figure 2(b) for a picture of $i(e, q) = 1, 2$. For simplicial triangulations, $i(e, q) = 1$ means that the quadrilateral q faces the edge e in a tetrahedron, i.e., $q \cap e = \emptyset$ and $e, q \subset \sigma$. In general, $i(e, q) \in \{0, 1, 2\}$. However, for simplicial triangulations, $i(e, q) = 0, 1$.

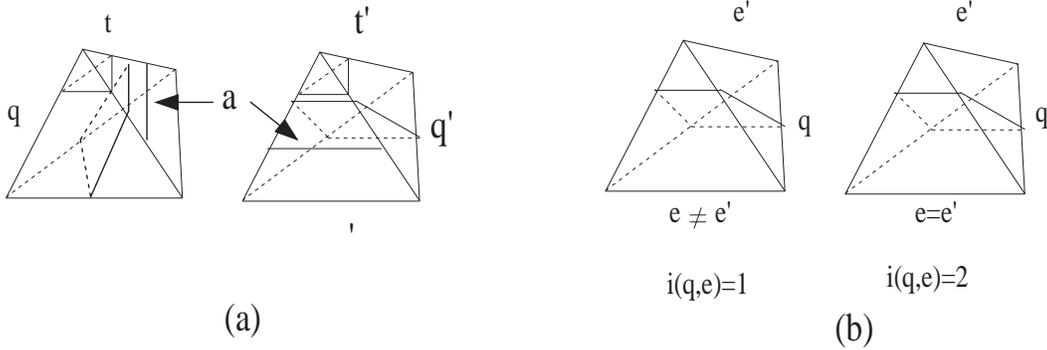


Figure 2: incident indices

2.3 Normal surfaces and tangential angle structures

Given $x \in \mathbf{R}^\Delta \times \mathbf{R}^\square$, we will call $x(t)$ ($t \in \Delta$) and $x(q)$ the t -coordinate and q -coordinate (triangle and quadrilateral coordinates) of x . Haken's normal surface equation addresses the following question. Given a finite set of normal triangles and normal quadrilaterals in a triangulation \mathbf{T} , when can one construct a normal surface with these given triangles and quadrilaterals as its intersections with the tetrahedra? Haken's equation (2.1) is a set of necessary conditions. Spun normal surface theory addresses the following question, first investigated by Thurston [35]. Suppose we are given a finite set of quadrilaterals in each tetrahedron. When can one construct a normal surface whose quadrilateral set is the given one? We can phrase it in terms of the normal coordinates as follows. Given a vector $z \in \mathbf{R}^\square$, when does there exist a solution to Haken's equation (2.1) whose projection to \mathbf{R}^\square is z ? The question was completely solved in [37], [17], [33] and [13]. We will interpret their results in terms of angle structures.

Definition 2.1. A *tangential angle structure* on a triangulated pseudo 3-manifold (M, \mathbf{T}) is a vector $x \in \mathbf{R}^\square$ so that,

for each tetrahedron $\sigma \in T$,

$$\sum_{q \in \square, q \subset \sigma} x(q) = 0, \quad (2.2)$$

and for each edge $e \in E$,

$$\sum_{q \in \square} i(e, q)x(q) = 0. \quad (2.3)$$

The linear space of all tangential angle structures on (M, \mathbf{T}) is denoted by $TAS(\mathbf{T})$ or TAS .

Recall that a (Euclidean type) angle structure, introduced by Casson, Rivin [28] and Lackenby [18], is a vector $x \in \mathbf{R}_{>0}^\square$ so that for each tetrahedron $\sigma \in T$,

$$\sum_{q \in \square, q \subset \sigma} x(q) = \pi, \quad (2.4)$$

and for each $e \in E$,

$$\sum_{q \in \square} i(e, q)x(q) = 2\pi. \quad (2.5)$$

These two conditions (2.4) and (2.5) have very natural geometric meaning. Suppose a hyperbolic manifold admits a geometric triangulation by ideal hyperbolic tetrahedra. The first equation (2.4) says that a normal triangle in a hyperbolic ideal tetrahedron is Euclidean and the second equation (2.5) says that the sum of the dihedral angles around each edge is 2π . By definition, a tangential angle structure is a tangent vector to the space of all angle structures.

The following is a result proved by Tollefson (for closed 3-manifolds), Kang-Rubinstein and Tillmann for all cases. The result was also known to Jaco [13]. Let \mathbf{S}_{ns} be the space of all solutions to Haken's homogeneous linear equations (2.1).

Given a finite set X , the standard basis of \mathbf{R}^X will be denoted by $X^* = \{x^* \in \mathbf{R}^X | x \in X\}$ so that $x^*(t) = 0$ if $t \in X - \{x\}$ and $x^*(x) = 1$. We give \mathbf{R}^X the standard inner product (\cdot, \cdot) so that X^* forms an orthonormal basis.

Theorem 2.2. ([37], [17], [33]) *For a triangulated closed pseudo 3-manifold (M, \mathbf{T}) , let $Proj_\square : \mathbf{R}^\Delta \times \mathbf{R}^\square \rightarrow \mathbf{R}^\square$ be the projection. Then*

$$Proj_\square(\mathbf{S}_{ns}) = TAS(\mathbf{T})^\perp, \quad (2.6)$$

where \mathbf{R}^\square has the standard inner product so that $\{q^* | q \in \square\}$ is an orthonormal basis.

For a short proof of this theorem, see [19]. This result is very important for us to relate normal surfaces to critical points of the volume function on the space of all circle-valued angle structures.

3 Neumann-Zagier Poisson structure and Thurston's gluing equation

The Neumann-Zagier Poisson structure on \mathbf{R}^\square , introduced in [25], is of fundamental importance for studying triangulated 3-manifolds and in particular for Thurston's gluing equation. We will recall its definition and derive some of its properties in this section. See also [3] and [4] for different proofs.

3.1 The Neumann-Zagier Poisson structure

Recall that our triangulated pseudo 3-manifolds (M, \mathbf{T}) are oriented so that each tetrahedron has the induced orientation. Since a pair of opposite edges $\{e, e'\}$ in a tetrahedron σ is the same as

a normal quadrilateral $q \subset \sigma$ with $i(e, q) \neq 0$, by lemma 2.1, for each tetrahedron σ in \mathbf{T} , there exists a natural cyclic order on the three quadrilaterals q_1, q_2, q_3 in σ . We denote the cyclic order by $q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow q_1$, and write $q \rightarrow q'$ in \square if q, q' are in the same tetrahedron and $q \rightarrow q'$ in the cyclic order. Define a map $w : \square \times \square \rightarrow \mathbf{R}$ by $w(q, q') = 1$ if $q \rightarrow q'$, $w(q, q') = -1$ if $q' \rightarrow q$ and $w(x, y) = 0$ otherwise. The Neumann-Zagier skew symmetric bilinear form, still denoted by $w : \mathbf{R}^\square \times \mathbf{R}^\square \rightarrow \mathbf{R}$, is defined to be:

$$w(x, y) = \sum_{q, q' \in \square} w(q, q')x(q)y(q').$$

From the definition, it is evident that $w(x, y) = -w(y, x)$.

The following was proved in [25],

Proposition 3.1 (Neumann-Zagier). *Suppose (M, \mathbf{T}) is a triangulated, oriented closed pseudo 3-manifold. Then*

- (a) for any $q' \in \square$, $\sum_{q \in \square} w(q, q') = 0$,
- (b) for any pair of edges $e, e' \in E$,

$$\sum_{q, q' \in \square} i(e, q)i(e', q')w(q, q') = 0.$$

Let Z be the linear subspace $\{x \in \mathbf{R}^\square \mid \text{for all } \sigma \in T, \sum_{q \subset \sigma} x(q) = 0\}$. Then the *Neumann-Zagier symplectic 2-form* is the restriction of w to Z^2 . It provides an identification between Z and the dual space Z^* .

A simple property of the Neumann-Zagier form is the following identity. For any $q_1, q_2 \in \square$,

$$\sum_{q \in \square} w(q_1, q)w(q, q_2) = \begin{cases} 0, & q_1, q_2 \text{ not in a tetrahedron} \\ -2, & q_1 = q_2 \\ 1, & q_1 \neq q_2 \text{ and } q_1, q_2 \subset \sigma \end{cases} \quad (3.1)$$

If $y \in Z$, then

$$\sum_{q, q_2 \in \square} w(q_1, q)w(q, q_2)y(q_2) = -3y(q_1). \quad (3.2)$$

Indeed, by (3.1), the left-hand-side of (3.2) is equal to $-2y(q_1) + y(q_3) + y(q_4)$ where q_1, q_3, q_4 are three quadrilaterals in a tetrahedron. Since $y(q_1) + y(q_3) + y(q_4) = 0$ by definition of Z , equation (3.2) follows. For any $q' \in \square$, the vector $y(q) = w(q, q')$ is an element in Z by proposition 3.1(a). Putting this $y = y(q_2) = w(q_2, q_4)$ into identity (3.2), we obtain, for any $q_1, q_4 \in \square$,

$$\sum_{q_2, q_3 \in \square} w(q_1, q_2)w(q_2, q_3)w(q_3, q_4) = -3w(q_1, q_4). \quad (3.3)$$

We will identify the dual space $(\mathbf{R}^X)^*$ with \mathbf{R}^X via the standard inner product $(\ , \)$ where X^* is an orthonormal basis.

For a triangulated pseudo 3-manifold (M, \mathbf{T}) , define the linear map $A : Z \rightarrow \mathbf{R}^E$ by

$$A(x)(e) = \sum_q i(e, q)x(q). \quad (3.4)$$

Note that the space of all tangential angle structures TAS is exactly equal to $\ker(A)$.

Lemma 3.2. *Suppose (M, \mathbf{T}) is oriented. The dual map $A^* : \mathbf{R}^E \rightarrow Z$, where the dual spaces of \mathbf{R}^E and Z are identified with themselves via the standard inner product $(,)$ on \mathbf{R}^E and w on Z , is*

$$A^*(x)(q) = \frac{1}{3} \sum_{e \in E} W(e, q)x(e),$$

where

$$W(e, q) = \sum_{q' \in \square} i(e, q')w(q', q).$$

Proof. We need to show for any $x \in \mathbf{R}^E$ and $y \in Z$,

$$(A(y), x) = w(y, A^*(x)).$$

Indeed, the left-hand-side of it is

$$\sum_e A(y)(e)x(e) = \sum_{e, q} x(e)i(e, q)y(q).$$

The right-hand-side of it is

$$\begin{aligned} & \sum_{q', q'' \in \square} y(q')w(q', q'')A^*(x)(q'') \\ &= \frac{1}{3} \sum_{q', q'', e} y(q')w(q', q'')W(e, q'')x(e) \\ &= \frac{1}{3} \sum_{q', q'', e, q} y(q')w(q', q'')i(e, q)w(q, q'')x(e) \\ &= -\frac{1}{3} \sum_{e, q} i(e, q)x(e) \sum_{q', q''} y(q')w(q', q'')w(q'', q) \\ &= \sum_{e, q} i(e, q)x(e)y(q). \end{aligned}$$

Here the last equation comes from (3.3). This ends the proof.

Let $B : \mathbf{R}^E \rightarrow \mathbf{R}^V$ be the map $B(x)(v) = \sum_{e > v} x(e)$. If both end points of e are v , then the edge e is counted twice in the summation $\sum_{e > v} x(e)$. The dual map $B^* : \mathbf{R}^V \rightarrow \mathbf{R}^E$ is given by $B^*(y)(e) = \sum_{v < e} y(v)$.

Theorem 3.3 (Neumann-Zagier) *For any oriented triangulated closed pseudo 3-manifold (M, \mathbf{T}) , the sequences*

$$Z \xrightarrow{A} \mathbf{R}^E \xrightarrow{B} \mathbf{R}^V \rightarrow 0$$

and

$$0 \rightarrow \mathbf{R}^V \xrightarrow{B^*} \mathbf{R}^E \xrightarrow{A^*} Z$$

are exact. Furthermore, if $x, y \in \mathbf{R}^E$, then

$$w(A^*(x), A^*(y)) = 0.$$

Proof. (See also [3].) Since the second sequence is the dual of the first, it suffices to prove that one of them is exact. First, $BA = 0$ follows from the definition of Z . Furthermore, it is easy to see that B^* is injective. Indeed, if $B^*(y) = 0$ for some $y \in \mathbf{R}^V$, then by definition, $y(v) + y(v') = 0$ whenever v, v' form the end points of an edge. Now for any $v \in V$, take a triangle in \mathbf{T} with vertices $v_1 = v, v_2$, and v_3 . Then equations $y(v_i) + y(v_j) = 0$ for $i \neq j$ in $\{1, 2, 3\}$ imply that $y(v_i) = 0$, i.e., $y(v) = 0$. It remains to prove that $\ker(A^*) \subset \text{Im}(B^*)$. Suppose $x \in \mathbf{R}^E$ so that $A^*(x) = 0$, i.e., for all $q \in \square$,

$$A^*(x)(q) = \frac{1}{3} \sum_e W(e, q)x(e) = 0.$$

Spelling out the details of the above equation, we see that it is equivalent to

$$x(e_1) + x(e_2) = x(e_3) + x(e_4)$$

whenever $\{e_1, e_2\}$ and $\{e_3, e_4\}$ are two pairs of opposite edges in a tetrahedron σ in \mathbf{T} . Fix a tetrahedron σ , and consider $x(e)$ as weights on the edges of σ . By lemma 2.1, there exists a map $y : \{(v, \sigma) | v < \sigma, v \in V\} \rightarrow \mathbf{R}$ so that

$$x(e) = \sum_{v < e} y(v, \sigma). \tag{3.5}$$

We claim that the above equation implies that $y(v, \sigma) = y(v, \sigma')$ for any other tetrahedron $\sigma' > v$. Assuming this claim, and taking $y(v) = y(v, \sigma)$, then we have $x(e) = \sum_{v < e} y(v)$, i.e., $x = B^*(y)$, or $x \in \text{Im}(B^*)$.

To see the claim, let us first assume that σ and σ' share a common triangle face which has v as a vertex. Say the three vertices of the triangle face are $v_1 = v, v_2$, and v_3 . Then equation (3.4) says that

$$y(v_i, \sigma) + y(v_j, \sigma) = y(v_i, \sigma') + y(v_j, \sigma'),$$

for $i \neq j$ in $\{1, 2, 3\}$. The common value is $x_{ij} = x(\{v_i, v_j\})$. This system of three equations has a unique solution, namely $y(v_i, \sigma) = y(v_i, \sigma') = (x_{ik} + x_{ij} - x_{jk})/2$ for $\{i, j, k\} = \{1, 2, 3\}$. Now in general, if σ and σ' are two tetrahedra in \mathbf{T} which have a common vertex v , by the definition of pseudo 3-manifolds, there exists a sequence of tetrahedra $\sigma_1 = \sigma, \sigma_2, \dots, \sigma_n = \sigma'$ so that for each index i , σ_i, σ_{i+1} share a common triangle face which has v as a vertex. Thus, by repeating the same argument just given, we see that $y(v, \sigma) = y(v, \sigma')$.

To see the last identity,

$$\begin{aligned}
w(A^*(x), A^*(y)) &= \sum_{q_1, q_2} w(q_1, q_2) A^*(x)(q_1) A^*(y)(q_2) \\
&= \frac{1}{9} \sum_{q_1, q_2, e_1, e_2} w(q_1, q_2) W(e_1, q_1) x(e_1) W(e_2, q_2) y(e_2) \\
&= \frac{1}{9} \sum_{q_1, q_2, q_3, q_4, e_1, e_2} w(q_1, q_2) i(e_1, q_3) w(q_3, q_1) i(e_2, q_4) w(q_4, q_2) x(e_1) y(e_2) \\
&= -\frac{1}{9} \sum_{q_3, q_4, e_1, e_2} i(e_1, q_3) i(e_2, q_4) x(e_1) y(e_2) \sum_{q_1, q_2} w(q_3, q_1) w(q_1, q_2) w(q_2, q_4).
\end{aligned}$$

By (3.3) and proposition 3.1(b), the above is equal to

$$= \frac{1}{3} \sum_{q_3, q_4, e_1, e_2} i(e_1, q_3) i(e_2, q_4) x(e_1) y(e_2) w(q_3, q_4) = 0.$$

This ends the proof.

3.2 Thurston's equation

Let us recall briefly Thurston's gluing equation [35] on a triangulated closed oriented pseudo 3-manifold (M, \mathbf{T}) . Assign each edge in each tetrahedron in the triangulation \mathbf{T} a complex number $z \in \mathbf{C} - \{0, 1\}$. The assignment is said to satisfy the *generalized Thurston algebraic equation* if

- (a) opposite edges of each tetrahedron have the same assignment;
- (b) the three complex numbers assigned to three pairs of opposite edges in each tetrahedron are z , $\frac{1}{1-z}$ and $\frac{z-1}{z}$ subject to an orientation convention; and
- (c) for each edge e in the triangulation, if $\{z_1, \dots, z_k\}$ is the set of all complex numbers assigned to the edge e in the various tetrahedra adjacent to e , then

$$\prod_{i=1}^k z_i = \pm 1. \tag{3.6}$$

If the right-hand-side of (3.6) equals 1 for all edges, we say that the assignment satisfies *Thurston algebraic equation*.

Since a pair of opposite edges in a tetrahedron is the same as the normal isotopy class of a quadrilateral, we see that Thurston's equation is defined on \mathbf{C}^\square . To be more precise, given $z \in \mathbf{C}^\square$, we say z satisfies the generalized Thurston equation, if the following assertions are satisfied:

- (1) if $q \rightarrow q'$ in \square , then $z(q') = \frac{1}{1-z(q)}$, and
- (2) if $e \in E$, then

$$\prod_q z(q)^{i(e,q)} = \pm 1. \quad (3.7)$$

If the right-hand-side of (3.7) equals 1 for all edges, we say z satisfies Thurston's equation.

This equation was introduced by Thurston in [35] in 1978. He used it to construct the complete hyperbolic metric on the figure-eight knot complement. Since then, many authors have studied Thurston's equation. See for instance [25], [26], [3], [10], [39], [34] and others. This equation was originally defined for ideal triangulated 3-manifolds with torus boundary, i.e., closed triangulated pseudo 3-manifolds (M, \mathbf{T}) so that each vertex link is a torus. We would like to point out that Thurston's equation (3.6) is defined on any closed triangulated oriented pseudo 3-manifold. It was first observed by Yoshida [39], a solution to Thurston's equation produces a representation of the fundamental group $\pi_1(M - \mathbf{T}^{(0)})$ to $PSL(2, \mathbf{C})$ where $\mathbf{T}^{(0)}$ is the set of all vertices. Thus, in the broader setting, solving Thurston's equation amounts to find $PSL(2, \mathbf{C})$ representations of the fundamental group. The recent work of [16] seems to have rediscovered equation (3.6) independently while working on TQFT.

Let $\mathbf{D}(\mathbf{T})$ be the space of all solutions to Thurston's equation defined in \mathbf{C}^\square . By definition, $\mathbf{D}(\mathbf{T})$ is an algebraic set. There are several very nice results known for $\mathbf{D}(\mathbf{T})$. Let $\mathbf{H} = \{w \in \mathbf{C} \mid \text{im}(w) > 0\}$ be the upper-half-plane.

Theorem 3.4 (Choi [3]). *The set $\mathbf{D}(\mathbf{T}) \cap \mathbf{H}^\square$ is a smooth complex manifold.*

Her proof makes an essential use of Neumann-Zagier's symplectic form (theorem 3.3).

Another result on Thurston's equation is in the work of Tillmann [34] and Yoshida [39] relating degenerations of solutions of Thurston's equation to normal surface theory. See also the work of [15] and [31]. The geometry behind their construction was first observed by Thurston [36]. Though this work does not address conjecture 1 in the introduction, it does indicate a relationship between Thurston's equation and Haken's equation.

Here is Tillmann's construction. Suppose $z_n \in \mathbf{D}(\mathbf{T})$ is an unbounded sequence of solutions to Thurston's equation (3.7) so that for each $q \in \square$,

$$u(q) = \lim_{n \rightarrow \infty} \frac{\ln |z_n(q)|}{\sqrt{1 + \sum_{q' \in \square} (\ln |z_n(q')|)^2}}$$

exists in \mathbf{R} .

Take the logarithm of equation (3.7) for z_n , divide the resulting equation by $\sqrt{1 + \sum_{q' \in \square} (\ln |z_n(q')|)^2}$, and let $n \rightarrow \infty$. We obtain, for each edge $e \in E$,

$$\sum_q i(e, q)u(q) = 0. \quad (3.8)$$

By definition, $u(q) = 0$ unless $\lim_{n \rightarrow \infty} z_n(q) = 0$, or ∞ . Furthermore, if $\lim_n z_n(q) = 1$ and $q' \rightarrow q \rightarrow q''$, then $\lim_n z_n(q'') = \lim_n \frac{1}{1 - z_n(q)} = \infty$ and $\lim_n z_n(q') = \lim_n \frac{z_n(q) - 1}{z_n(q)} = 0$ so that $\lim_n z_n(q')z_n(q'') = -1$. This implies that $u(q') + u(q'') = 0$ and $u(q'') \geq 0$. Let $I = \{q \in$

$\square \mid \lim_n z_n(q) = 1\}$ and for $q \in I$, let $a_q = u(q'') \geq 0$ where $q \rightarrow q''$. Then

$$u = \sum_{q \in I} a_q \sum_{q'} w(q, q')(q')^* \in \mathbf{R}.$$

Substitute into (3.8), we obtain for each $e \in E$,

$$\sum_{q \in I} v(q)W(e, q) = 0 \tag{3.9}$$

where $v = \sum_{q \in I} a_q q^*$. Equation (3.9) appeared in the work of Tollefson [37] in which he proved that, if (M, \mathbf{T}) is a closed 3-manifold, then (3.9) gives a complete characterization of the quadrilateral coordinates of solutions to Haken's equation. Namely, if M is closed, a vector $v \in \mathbf{R}^\square$ is in $Proj_\square(\mathbf{S}_{ns})$ if and only if (3.9) holds for all $e \in E$. Thus, by Tollefson's theorem, the specific $v = \sum_{q \in I} a_q q^*$ belongs to $Proj_\square(\mathbf{S}_{ne})$. As a consequence, one has,

Theorem 3.5 (Tillmann) *For a closed triangulated 3-manifold (M, \mathbf{T}) , the logarithmic limits of $\mathbf{D}(\mathbf{T})$ correspond to solutions of Haken's normal surface equation.*

We remark that Tillmann's theorem in [34] is more general and works for all pseudo 3-manifolds. We state it in the above form for simplicity. Furthermore, Tillmann observed in [34] that the solution v has the property that there is at most one non-zero quadrilateral coordinate in each tetrahedron. Thus if all coefficients a_q are non-negative integers, then the vector v produces an embedded normal surface in the manifold.

It follows from the definition that for each $e \in E$, the vector

$$u_e = \sum_q w(e, q)q^* \tag{3.10}$$

is in $TAS(\mathbf{T})$. What Tollefson proved, using the language of TAS, is that for a closed triangulated 3-manifold (M, \mathbf{T}) , the set $\{u_e \mid e \in E\}$ generates the linear space $TAS(\mathbf{T})$. A generating set for $TAS(\mathbf{T})$ for all closed triangulated pseudo 3-manifolds (M, \mathbf{T}) was found in the work of Kang-Rubinstein [17] and Tillmann [33].

In the recent work [38], Yang is able to construct many solutions of Thurston's equation on closed triangulated 3-manifolds (M, \mathbf{T}) with the property that each edge has distinct end points.

4 Circle valued angle structures and maximization of volume

Following Casson, Rivin [28] and Lackenby [18], we introduced the following notion in [19],

Definition. An \mathbf{S}^1 -angle structure (or SAS for simplicity) on a closed triangulated pseudo 3-manifold (M, \mathbf{T}) is a function $x : \square(\mathbf{T}) \rightarrow S^1$ so that

- (5) for each tetrahedron σ , $\prod_{q \subset \sigma} x(q) = -1$; and
- (6) for each edge $e \in E$, $\prod_{q \in \square} x(q)^{i(e, q)} = 1$.

Let $SAS(\mathbf{T})$ be the set of all \mathbf{S}^1 -angle structures on the triangulation \mathbf{T} . If $x \in SAS(\mathbf{T})$ and $v \in TAs(\mathbf{T})$, then xe^{iv} , defined by $xe^{iv}(q) = x(q)e^{iv(q)}$, is still in $SAS(\mathbf{T})$. We use this to identify the tangent space of $SAS(\mathbf{T})$ with $TAs(\mathbf{T})$. The *Lobachevsky-Milnor volume* (or simply the *volume*) of an \mathbf{S}^1 -angle structure x is defined to be:

$$vol(x) = \sum_{q \in \square} \Lambda(arg(x(q)))$$

where $arg(w)$ is the argument of a complex number w and $\Lambda(t) = -\int_0^t \ln |2 \sin(s)| ds$. The volume formula is derived from the volume of an ideal hyperbolic tetrahedron. See Milnor [23]. It is well known that $\Lambda(t) : \mathbf{R} \rightarrow \mathbf{R}$ is a continuous function with period π . Thus,

$$vol : SAS(\mathbf{T}) \rightarrow \mathbf{R}$$

is a continuous function. Our goal is to relate the critical points of vol with the topology and geometry of the 3-manifold.

Using volume maximization to find geometric structures based on angle structures for manifolds with cusps was introduced by Casson [2] and Rivin [29]. In a recent work [9], Guéritaud used the tool to prove the existence of hyperbolic metrics on the once-punctured torus bundle over the circle with Anosov holonomy. Our approach follows the same path in a more general setting.

4.1 Existence of SAS and critical points of volume

In [20], we proved a general theorem on the existence of real-valued prescribed-curvature angle structures on a triangulated pseudo 3-manifold. One can check that the proof in [20] implies the following proposition. Also see [19] for a proof.

Proposition 4.1 ([20], [19]) *For a closed triangulated pseudo 3-manifold (M, \mathbf{T}) , the space $SAS(\mathbf{T})$ is non-empty and is a smooth closed manifold of dimension $\chi(M) + |T|$. In particular, the volume $vol : SAS(\mathbf{T}) \rightarrow \mathbf{R}$ has a maximum point.*

Following [20], we give a short proof of it for real-valued angle structures on ideally triangulated 3-manifolds with torus boundary (i.e., closed triangulated pseudo 3-manifolds so that each vertex link is a torus). The main idea of the proof for the general case is the same.

Suppose otherwise that such a manifold (M, \mathbf{T}) does not support a real-valued angle structure. Consider the linear map $h : \mathbf{R}^\square \rightarrow \mathbf{R}^T \times \mathbf{R}^E$ so that $h(x)(\sigma) = \sum_{q \in \sigma} x(q)$ and $h(x)(e) = \sum_q i(e, q)x(q)$. Let $\alpha \in \mathbf{R}^T \times \mathbf{R}^E$ be $\alpha(\sigma) = \pi$ and $\alpha(e) = 2\pi$. Then the assumption that (M, \mathbf{T}) does not support a real-valued angle structure means $\alpha \notin h(\mathbf{R}^\square)$. Therefore, there exists a vector $f \in \mathbf{R}^T \times \mathbf{R}^E$ so that f is perpendicular to the image $h(\mathbf{R}^\square)$ and $(f, \alpha) \neq 0$. This means that

$$\frac{1}{\pi}(f, \alpha) = \sum_{\sigma} f(\sigma) + 2 \sum_e f(e) \neq 0$$

and

$$(h(x), f) = (x, h^*(f)) = 0,$$

for all $x \in \mathbf{R}^\square$, i.e.,

$$h^*(f) = 0,$$

where h^* is the transpose of h .

Since $h^*(f)(q) = \sum_{\sigma, q \subset \sigma} f(\sigma) + \sum_e i(e, q)f(e)$, it follows that if e, e' are two opposite edges in σ , then

$$f(e) + f(e') = -f(\sigma).$$

In particular, the sum of the values of f at opposite edges in σ is independent of the choice of the edge pair. By lemma 2.1 (b), we see that there is a map g defined on the pairs (v, σ) with $v < \sigma$ so that

$$f(e) = g(v, \sigma) + g(v', \sigma)$$

where $v, v' < e$. By the same argument as the one we used in the proof of theorem 3.3, we see that $g(v, \sigma)$ is independent of the choices of tetrahedra, i.e., $g : V \rightarrow \mathbf{R}$ so that

$$f(e) = \sum_{v < e} g(v)$$

and

$$f(\sigma) = - \sum_{v < \sigma} g(v).$$

Now we express

$$\begin{aligned} & \sum_{\sigma} f(\sigma) + 2 \sum_e f(e) \\ &= - \sum_{\sigma, v < \sigma} g(v) + 2 \sum_{e, v < v} g(v) \\ &= - \sum_{v \in V} g(v) \left[\sum_{\sigma > v} 1 - 2 \sum_{e > v} 1 \right] \\ &= \sum_{v \in V} g(v) [-|\{\text{triangles in lk}(v)\}| + 2|\{\text{vertices in lk}(v)\}|] = 0 \end{aligned}$$

The last equality is due to the fact that the number of vertices of a triangulation of the torus is equal to half of the number of triangles in the triangulation. Also, in the summations $\sum_{\sigma > v} 1$ and $\sum_{e > v} 1$, we count σ and e with multiplicities, i.e., if σ (or e) has k vertices which are v , then σ (or e) is counted k times in the sum. This ends the proof for manifolds with torus boundary.

Proposition 4.1 guarantees that critical points for the volume function always exist. Here the concept of critical point of the non-smooth function vol has to be clarified. It can be shown ([19]) that for any point $p \in SAS(\mathbf{T})$ and any tangent vector v of $SAS(\mathbf{T})$ at p , the limit $\lim_{t \rightarrow 0} \frac{vol(pe^{tv}) - vol(p)}{t}$ always exists as an element in $[-\infty, \infty]$. A point $p \in SAS(\mathbf{T})$ is called a critical point of the volume if the above limit is 0 for all tangent vectors v at p .

The main focus of our research is to extract topological and geometric information from critical points of the volume function on $SAS(\mathbf{T})$.

Pursuing in this direction, we have proved the following [19].

Theorem 4.2. *Let (M, \mathbf{T}) be an oriented triangulated closed pseudo 3-manifold. Suppose x is a critical point of the volume function vol on $SAS(\mathbf{T})$.*

(a) *If the critical point x is a non-smooth point for the volume function, so $x(q') = \pm 1$ for some $q' \in \square$, then there exists a solution y to Haken's normal surface equation defined on \mathbf{T} , which has exactly one or two non-zero quadrilateral coordinates with $y(q') \neq 0$;*

(b) *If the critical point x is a smooth point (i.e., $x(q) \neq \pm 1$ for all q), then x produces a solution to the generalized Thurston equation.*

Here are the key steps in the proof of theorem 4.2. Given $x \in SAS(\mathbf{T})$, we say a tetrahedron $\sigma \in T$ is *flat* with respect to x if $x(q) = \pm 1$ for all $q \subset \sigma$ and *partially flat* if $x(q) = \pm 1$ for one $q \subset \sigma$. Let U be the set of all partially flat but not flat tetrahedra and $\mathbf{W} = \{q | x(q) = \pm 1, q \subset \sigma, \sigma \in U\}$. By analyzing the derivative of $\int_0^t \ln |2 \sin(s)| ds$, we obtain the following main identity. For $u \in TAS(\mathbf{T})$,

$$\frac{d}{dt} vol(xe^{iut}) = - \sum_{q \in \mathbf{W}} u(q) \ln |t| - \sum_{q, x(q) = \pm 1} u(q) \ln |u(q)| - \sum_{q, x(q) \neq \pm 1} u(q) \ln |\sin(\arg(x(q)))| + o(t) \quad (4.1)$$

Now at a smooth point x , equation (4.1) becomes

$$\frac{d}{dt} vol(xe^{iu})|_{t=0} = - \sum_q u(q) \ln |\sin(\arg(x(q)))|.$$

By taking $u = u_e$ given by (3.10) and assuming x is a smooth critical point, we obtain a solution $z \in \mathbf{C}^\square$ to the generalized Thurston equation, where

$$z(q) = \frac{\sin(\arg(x(q'))) \sin(\arg(x(q'')))}{\sin(\arg(x(q'')))} x(q),$$

and $q' \rightarrow q \rightarrow q''$. This argument was known to Casson [2] and Rivin. One may find a detailed argument in [19] or [7].

If x is a non-smooth critical point, then we deduce from (4.1) two equations for all $u \in TAS(\mathbf{T})$,

$$\sum_{q \in \mathbf{W}} u(q) = 0, \quad (4.2)$$

and

$$\sum_{q, x(q) = \pm 1} u(q) \ln |u(q)| = g(u), \quad (4.3)$$

where $g(u)$ is a linear function in u . Now we use the following simple lemma.

Lemma 4.3. *Suppose \mathbf{V} is a finite dimensional vector space over \mathbf{R} and f_1, \dots, f_n, g are linear functions on \mathbf{V} so that for all $x \in \mathbf{V}$,*

$$\sum_{i=1}^n f_i(x) \ln |f_i(x)| = g(x).$$

Then for each index i there exists $j \neq i$ and $\lambda_{ij} \in \mathbf{R}$ so that

$$f_i(x) = \lambda_{ij} f_j(x).$$

Using lemma 4.3 for (4.3) where the vector space \mathbf{V} is $TAS(\mathbf{T})$ and the linear functions are $u(q)$ with $x(q) = \pm 1$ and g , we conclude that for each q with $x(q) = \pm 1$, there exist q_1 and $\lambda \in \mathbf{R}$ so that $u(q) = \lambda u(q_1)$ for all $u \in TAS(\mathbf{T})$. This shows that for all $u \in TAS(\mathbf{T})$, the inner product $(u, q^* - \lambda q_1^*) = 0$. By theorem 2.2, $q^* - \lambda q_1^*$ is in $Proj_{\square}(\mathbf{S}_{ns})$. Thus theorem 4.1 (a) follows.

4.2 Futer-Guéritaud's Theorem

In an unpublished work [8], David Futer and Francois Guéritaud proved a very nice theorem concerning the non-smooth maximum points of the volume function. The proof given below is supplied by Futer and Guéritaud. We are grateful to Futer and Guéritaud for allowing us to present their proof in this paper.

Theorem 4.4 (Futer-Guéritaud) *Suppose (M, \mathbf{T}) is an oriented triangulated closed pseudo 3-manifold. If x is a non-smooth maximum point of the volume function on $SAS(\mathbf{T})$, then there exists a non-smooth maximum volume point $y \in SAS(\mathbf{T})$ so that all partially flat tetrahedra in y are flat.*

Proof(Futer-Guéritaud). Suppose x is a non-smooth maximum volume point in $SAS(\mathbf{T})$. Let U be the set of all partially flat but not flat tetrahedra in x and $\mathbf{W} = \{q \subset \sigma \mid \sigma \in U, x(q) = \pm 1\}$ as above. Note that, by assumption, for each tetrahedron σ , there is at most one quadrilateral in \mathbf{W} contained in σ .

Claim. Define the vector $v = \sum_{q' \in \mathbf{W}} \sum_{q \in \square} w(q', q) q^*$, i.e., $v(q) = \sum_{q' \in \mathbf{W}} w(q', q)$. Then $v \in TAS(\mathbf{T})$.

To see the claim, we must verify two conditions for v : (1) for tetrahedron $\sigma \in \mathbf{T}$, $\sum_{q \subset \sigma} v(q) = 0$, and (2) for each edge $e \in E$, $\sum_{q' \in \square} i(e, q') v(q') = 0$.

The first condition follows from the fact that for any $q \in \square$, $\sum_{q' \in \square} w(q', q) = 0$. Indeed, for each tetrahedron σ , $\sum_{q \subset \sigma} v(q) = \sum_{q' \in \mathbf{W}} \sum_{q \subset \sigma} w(q', q) = 0$.

To see the second condition, we use (4.2).

By (3.9), for $e \in E$, the vector

$$u_e = \sum_e W(e, q) q^*,$$

i.e., $u_e(q) = W(e, q)$, is in $TAS(\mathbf{T})$. Taking this u_e to be the vector u in (4.2), we obtain

$$\sum_{q \in \mathbf{W}} W(e, q) = 0$$

i.e.,

$$\sum_{q' \in \square} i(e, q') \left(\sum_{q \in \mathbf{W}} w(q', q) \right) = 0.$$

The last equation says $\sum_{q'} i(e, q') v(q') = 0$. This verifies the claim.

Now back to the proof of the theorem. For each point $p \in SAS(\mathbf{T})$, let $N(p)$ be the number of partially flat but not flat tetrahedra in p . For the maximum point x , we may assume $N(x) > 0$. We will produce a new maximum point y so that $N(y) < N(x)$ as follows. Let v be the tangential angle structure constructed in the claim above. Consider the smooth path

$$r(t) = x e^{itv} \in SAS(\mathbf{T}).$$

Note, by definition, for $|t|$ small, $N(r(t)) = N(x)$. Take $|t_0|$ be the smallest number so that

$$N(r(t)) = N(x)$$

for all $|t| < |t_0|$ and

$$N(r(t_0)) < N(x).$$

Furthermore, due to the basic property of the Lobachevsky function that

$$\Lambda(a) + \Lambda(b) + \Lambda(c) = 0$$

for $a + b + c \in \pi\mathbf{Z}$ and one of a, b, c is in $\pi\mathbf{Z}$, we have

$$vol(r(t)) = vol(x)$$

for $|t| \leq |t_0|$. Take $y = r(t_0)$. Then we have produced a new maximum point y with smaller $N(y)$. Inductively, we produce a new maximum point so that all partially flat tetrahedra are flat. This ends the proof.

Combining theorem 4.2 with the theorem of Futer-Guéritaud, we obtain theorem 1.2,

Theorem 1.2 *Suppose (M, \mathbf{T}) is a closed triangulated oriented pseudo 3-manifold. Then there either exists a solution to the generalized Thurston equation or there exists a cluster of three 2-quad-type solutions to Haken's normal surface equation.*

Indeed, by Futer-Guéritaud's theorem, we can produce a non-smooth maximum point y so that there are three distinct quadrilaterals q_1, q_2, q_3 in a tetrahedron with $y(q_i) = \pm 1$. Now we use theorem 4.2 to produce the corresponding 2-quad-type solutions x_i , one for each q_i with $x_i(q_i) \neq 0$. Note that we do not assume that x_1, x_2, x_3 are pairwise distinct.

A stronger version of conjecture 1 is the following.

Conjecture 2 *Suppose (M, \mathbf{T}) is a minimally triangulated closed irreducible 3-manifold so that all maximum points of the volume function $vol : SAS(\mathbf{T}) \rightarrow \mathbf{R}$ are smooth for vol . Then Thurston's equation on \mathbf{T} has a solution.*

4.3 Minimal triangulations with a cluster of three 2-quad-type solutions

Our recent joint work with Stephan Tillmann shows the following.

Theorem 1.3 ([21]) *Suppose (M, T) is a minimally triangulated orientable closed 3-manifold so that there are three 2-quad-type solutions x_1, x_2, x_3 of Haken's equation with $x_i(q_i) \neq 0$ for three distinct quadrilaterals q_1, q_2, q_3 inside a tetrahedron. Then,*

- (a) M is reducible, or
- (b) M is toroidal, or
- (c) M is a Seifert fibered space, or
- (d) M contains the connected sum $\#_{i=1}^3 RP^2$ of three copies of the projective plane.

By the work of W. Thurston and others, it is known, without using the Ricci flow method, that manifolds in class (d) but not in cases (a), (b), (c) above are either Haken or hyperbolic. See for instance [24]. Indeed, an irreducible, non-Haken, atoroidal, non-Seifert-fibered 3-manifold containing $\#_{i=1}^3 RP^2$ has a two fold cover which is a closed 3-manifold of Heegaard genus at most 2. Such a manifold admits a \mathbf{Z}_2 action with 1-dimensional fixed point set. By Thurston's Orbifold theorem [1], or [5] one concludes that the manifold is hyperbolic.

The proof of theorem 1.3 makes essential uses of Jaco-Rubinstein's work on 0-efficient triangulations. We analyze carefully the cluster of three 2-quad-type solutions of Haken's normal surface equation constructed from theorem 1.2.

Theorem 1.3 takes care of the topology of closed minimally triangulated 3-manifolds which have non-smooth maximum volume points.

We don't know if theorem 1.3 can be improved by using only one 2-quad-type solution instead of a cluster of three 2-quad-type solutions. Such an improvement will help in reproving the Poincaré conjecture. For instance, one can weaken conjecture 1 by replacing the cluster of three 2-quad-type solutions by one 2-quad-type solution. Another related conjecture is the following,

Conjecture 3 Suppose (M, \mathbf{T}) is a minimally triangulated closed orientable 3-manifold so that one edge of \mathbf{T} has the same end points and is null homotopic in M . Then there exists a cluster of three 2-quad-type solutions on \mathbf{T} .

By theorem 1.3, one sees that conjecture 3 implies the Poincaré conjecture without using the Ricci flow.

5 Some open problems

Another potential approach to conjecture 1 is to use volume optimization on a space closely related to $SAS(\mathbf{T})$. Let $W(\mathbf{T})$ be the space $\{z \in \mathbf{C}^\square \mid \text{so that if } q \rightarrow q', \text{ then } z(q') = 1/(1 - z(q)) \text{ and for each edge } e, \text{ the right-hand-side of (3.7) is a positive real number}\}$. The volume function $vol : W(\mathbf{T}) \rightarrow \mathbf{R}$ is still defined. The maximum points of the volume are related to the solutions of Thurston's equation. In fact, a critical point of the volume function in the set $W(\mathbf{T}) \cap (\mathbf{C} - \mathbf{R})^\square$ gives a solution to Thurston's equation.

It is conceivable that the following holds.

Conjecture 4 Suppose (M, \mathbf{T}) is a closed orientable triangulated 3-manifold so that $W(\mathbf{T}) \neq \emptyset$. Then $\sup\{\text{vol}(z) | z \in W(\mathbf{T})\} \leq v_3 \|M\|$ where v_3 is the volume of the regular ideal hyperbolic tetrahedron.

The first step to carry out this approach is to find conditions on the triangulation \mathbf{T} so that $W(\mathbf{T})$ is non-empty. To this end, we consider solving Thurston's equation over the real numbers, i.e., $z \in \mathbf{R}^\square$. Here is a step toward producing a real-valued solution to Thurston's equation.

Definition 5.1. Let \mathbf{Z}_2 be the field of two elements $\{0, 1\}$. A \mathbf{Z}_2 -taut structure on a triangulated closed pseudo 3-manifold (M, \mathbf{T}) is a map $f : \square \rightarrow \{0, 1\}$ so that

(a) if q_1, q_2, q_3 are three quadrilaterals in each tetrahedron σ , then exactly one of $f(q_1), f(q_2), f(q_3)$ is 1, and

(b) for each edge e in \mathbf{T} , $\sum_{q \in \square} i(e, q) f(q) = 0$.

The motivation for the definition comes from taut triangulations and real-valued solutions to Thurston's equation. Indeed, if z is a real-valued solution to Thurston's equation, then there is an associated \mathbf{Z}_2 -taut structure f defined by: $f(q) = 0$ if $z(q) > 0$ and $f(q) = 1$ if $z(q) < 0$. Another motivation comes from taut triangulations. Suppose \mathbf{T} is a taut triangulation, i.e., there is a map $g : \square \rightarrow \{0, \pi\}$ so that for each tetrahedron σ , $\sum_{q \subset \sigma} g(q) = \pi$ and for each edge e , $\sum_q i(e, q) g(q) = 2\pi$. Then one defines a \mathbf{Z}_2 -taut structure by $f(q) = \frac{1}{\pi} g(q)$. A very interesting question is to find condition on \mathbf{T} so that \mathbf{Z}_2 -taut structures exist. Is it possible that the non-existence of \mathbf{Z}_2 -taut structures implies the existence of some special solutions to Haken's normal surface equation?

Tillmann and I observed that the equations for \mathbf{Z}_2 -taut structures are non-linear but quadratic in $f(q)$. Indeed, a vector $f \in \mathbf{Z}_2^\square$ is a \mathbf{Z}_2 -taut structure if and only if condition (b) in definition 5.1 holds and for each tetrahedron σ

$$\sum_{q \subset \sigma} f(q) = 1, \tag{5.1}$$

and

$$\sum_{q \neq q', q, q' \subset \sigma} f(q) f(q') = 0.$$

The condition (b) in definition 5.1 and (5.1) should be considered as the definition of a \mathbf{Z}_2 -angle structure.

We end the paper with several questions.

Question 1. Given a triangulated pseudo 3-manifold (M, \mathbf{T}) , when does there exist a \mathbf{Z}_2 -taut structure? Can one relate the non-existence of \mathbf{Z}_2 -taut structure to some special solutions to Haken's equation?

Question 2. When is a critical point of the volume function of Morse type (i.e., when is the Hessian matrix non-degenerated) and when is the volume function a Morse function?

Let v_3 be the volume of the ideal regular hyperbolic tetrahedron.

Question 3. Is the Gromov norm of a closed 3-manifold multiplied by v_3 among the critical values of the volume function?

Question 4. Is it possible to produce a Floer type homology theory associated to the volume function on $SAS(\mathbf{T})$ which will be a topological invariant of the 3-manifold?

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