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Measured lamination spaces on surfaces and geometric intersection numbers

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Abstract

In this paper, we produce an elementary approach to Thurston's theory of measured laminations on compact surfaces with non-empty boundary. We show that the theory can be derived from a simple inequality for geometric intersection numbers between arcs inside an octagon.

MSC: primary 57M99; secondary 57N05, 58F18

Keywords: Surfaces; Submanifolds; Laminations; Intersection numbers

1. Introduction

1.1. Let us begin with a brief review of Dehn–Thurston's theory of 1-dimensional submanifolds in a surface ([2,7,10,11] and others). Given a compact connected surface Σ with possibly non-empty boundary, a *curve system* on Σ is a proper 1-dimensional submanifold so that each circle component of it is not null homotopic and not homotopic into the boundary and each arc component is not relatively homotopic into the boundary. The space of all isotopy classes of curve systems on Σ is denoted by $CS(\Sigma)$. This space was introduced by Max Dehn in 1938 [2] who called it the *arithmetic field* of the topological surface. Given two isotopy classes α and β of 1-dimensional submanifolds, their *geometric intersection number* $I(\alpha, \beta)$ is defined to be $\min\{|a \cap b|\colon\ a \in \alpha,\ b \in \beta\}$. Thurston observed that, except for the annulus, the Möbius band and the 3-holed sphere, the

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pairing $I(,)$: $CS(\Sigma) \times CS(\Sigma) \to \mathbb{Z}$ behaves like a non-degenerate "bilinear" form in the sense that (1) given any α in $CS(\Sigma)$ there is β in $CS(\Sigma)$ so that their intersection number $I(\alpha, \beta)$ is non-zero, and (2) $I(k_1\alpha_1, k_2\alpha_2) = k_1k_2I(\alpha_1, \alpha_2)$ for $k_i \in \mathbb{Z}_{\geqslant 0}$, $\alpha_i \in CS(\Sigma)$ where $k_i\alpha_i$ is the isotopy class of the collection of k_i parallel copies of $a_i \in \alpha_i$. In linear algebra, given a non-degenerate quadratic form ω on a lattice L of rank r, one can form a completion of (L, ω) by canonically embedding L into R^r so that the form w extends continuously on R^r. Thurston's construction is the exact analogy. Let $CS_0(\Sigma)$ be the subspace consisting of isotopy classes of curve systems whose components are circles. Thurston's space of (compactly supported) measured laminations on the surface Σ, denoted by $ML_0(\Sigma)$, is defined to be the completion of the pair $(CS_0(\Sigma), I)$ in the following sense. Given α in $CS(\Sigma)$, let $\pi(\alpha)$ be the map sending β to $I(\alpha, \beta)$. This gives an embedding $\pi : CS_0(\Sigma) \to \mathbb{R}^{CS(\Sigma)}$ where the target has the product topology. The space $ML_0(\Sigma)$ is define to be the closure of $\mathbb{Q}_{>0} \times \pi(CS_0(\Sigma)) = \{r\pi(x)\colon r \in \mathbb{Q}_{>0},\ x \in CS_0(\Sigma)\}$. The basic results in theory, proved by using the notion of train-tracks, are that the space $ML_0(\Sigma)$ is homeomorphic to a Euclidean space that the intersection pairing $I(,)$ extends to a continuous homogeneous map from $ML_0(\Sigma) \times ML_0(\Sigma)$ to $\mathbb{R}$. See [1,3,6–8, 11] and others for a proof of the first statement and [1] for a proof of the continuity of the extension.

Our goal is to give an elementary proof of the basics results for compact surfaces with non-empty boundary. The continuity of the extension of the intersection number holds the key. It implies that the topology of the space $ML_0(\Sigma)$ is homeomorphic to a Euclidean space. By interpolation, we reduce the continuity of the extension of the intersection number to a simple inequality for geometric intersection number of arcs inside an octagon. Our approach rests on two parts. First, we produce a parametrization of the set of isotopy classes of arcs in polygons. In order to interpolating, we extend the space $CS(\Sigma)$ to a larger space $ES(\Sigma)$ of isotopy classes of essential 1-dimensional submanifolds. We produce a parametrization of the space $ES(\Sigma)$ and establish an inequality concerning geometric intersection numbers between classes in $ES(\Sigma)$. This inequality is essentially a result concerning arcs inside an octagon.

1.2. We give a brief sketch of our approach in this subsection. For simplicity, we assume the surface is compact with non-empty boundary so that its Euler characteristic is negative. A 1-dimensional proper submanifold in a compact surface is called *essential* if each circle component is not null homotopic and each arc component is not hull homotopic relative to the boundary. We denote the set of all isotopy classes of essential 1-dimensional submanifolds in a surface Σ by $ES(\Sigma)$. The space $ES(\Sigma)$ contains $CS(\Sigma)$ as a subset and in general is not equal to $CS(\Sigma)$ if the boundary $\partial\Sigma$ is not empty (i.e., $\partial\Sigma$ is an essential submanifold, but is not a curve system). The reason that we consider $ES(\Sigma)$ instead of $CS(\Sigma)$ (or $CS_0(\Sigma)$) is that $ES(\Sigma)$ satisfies a combinatorial convexity property. To parametrize $ES(\Sigma)$, let us recall that *an ideal triangulation* of a surface Σ is a maximal collection of pairwise disjoint, pairwise non-isotopic essential arcs in Σ (see, for instance, [6]). Fix an ideal triangulation $t = t_1 \cup t_2 \cup \cdots \cup t_N$ of the surface, and a class $[a] \in ES(\Sigma)$. We define the *t-coordinate* of $[a] \in ES(\Sigma)$ with respect to the ideal triangulation to be $(x, \ldots, x_N, x'_1, \ldots, x'_N)$ where $x_i = I([a], [t_i])$ and x'_i is the number of components of a which are parallel to t_i. It can be shown (Lemma 3.1) that this parametrization sends $ES(\Sigma)$ injectively into $\mathbb{Z}^{2N}$. (In the case of $CS_0(\Sigma)$, all coordinates $x_i' = 0$. This was considered in [6].)

Proposition 1.1. *Suppose $t = t_1 \cup \cdots \cup t_N$ is an ideal triangulation of a compact surface. Then for any three classes $\alpha, \beta, \gamma \in ES(\Sigma)$ with t-coordinates $(x_1, \ldots, x_N, x_1', \ldots, x_N')$, $(y_1, \ldots, y_N')$ and $(z_1, \ldots, z_N')$, the following inequality holds*:

$$\left|I(\alpha, \beta) - I(\alpha, \gamma)\right| \leqslant 4|\alpha||\beta - \gamma| \tag{1.1}$$

where $|\alpha| = \sum_{i=1}^N (x_i + x_i')$ and $|\beta - \gamma| = \sum_{i=1}^N (|y_i - z_i| + |y_i' - z_i'|)$.

The key idea in the proof is the following. Given two classes β, γ in $ES(\Sigma)$ so that $|\beta - \gamma| = n$, we produce a sequence of $(n+1)$ essential 1-dimensional submanifolds β_i for $i = 0, 1, \ldots, n$ starting from 2β and ending at 2γ so that $|\beta_i - \beta_{i+1}| = 2$. (This cannot be achieved in $CS(\Sigma)$ nor in $CS_0(\Sigma)$.) Thus, by interpolation, we may assume that $|\beta - \gamma| = 1$ or 2. This reduces inequality (1.1) to a question on intersections of arcs in an octagon. We prove inequality (1.1) in this special case by analyzing the surgery procedure relating β and γ within an octagon. We remark that with a little extra work, one can improve the constant 4 in inequality (1.1) to 2 which is sharp.

We remark that a similar result that $|I(\alpha, \beta) - I(\alpha, \gamma)| \leqslant K|\alpha||\beta - \gamma|$ for $\alpha, \beta \in CS_0(\Sigma)$ was obtained earlier by Rees [9] using train-tracks. The constant K in her theorem depends on the train-tracks. Also, in [4], Hamidi–Tehrani proved $|I(\alpha, \beta) - I(\alpha, \gamma)| \leqslant |\alpha||\beta - \gamma|$ for $\alpha, \beta \in CS_0(\Sigma)$ assuming the theory of measured laminations.

1.3. In this subsection, we give a quick derivation of the continuity of $I(,)$ on the space of compactly supported measured laminations $ML_0(\Sigma)$ which is the closure of $\mathbb{Q}_{\geqslant 0} \times \pi(CS_0(\Sigma))$ using (1.1). One first extends the pairing $I(,)$ to $(\mathbb{Q}_{\geqslant 0} \times CS_0(\Sigma))^2$ by linearity $I(k_1\alpha_1, k_2\alpha_2) = k_1k_2 I(\alpha_1, \alpha_2)$ where $k_1, k_2 \in \mathbb{Q}_{\geqslant 0}$. Thus inequality (1.1) still holds for α, β and γ in $\mathbb{Q}_{\geqslant 0} \times CS_0(\Sigma)$. For simplicity, we will identify $CS_0(\Sigma)$ with a subspace of $ML_0(\Sigma)$ via the map π. Thus the intersection paring $I(,)$ is also defined on the space $(\mathbb{Q}_{\geqslant 0} \times \pi(CS_0(\Sigma)))^2$ by the formula $I(x, y) = I(\pi^{-1}(x), \pi^{-1}(y))$. Our goal is to show that the newly defined paring $I(,)$ extends continuously to $ML_0(\Sigma)^2$. Since the product space $\mathbb{R}^{CS(\Sigma)}$ is metrizable, the continuity of the paring $I(,)$ on $ML_0(\Sigma) \times ML_0(\Sigma)$ follows by showing that if $(\alpha_n, \beta_n) \in (\mathbb{Q}_{\geqslant 0} \times CS_0(\Sigma))^2$ converges, then $I(\alpha_n, \beta_n)$ converges. Now since α_n and β_n converge, both $\lim_n I(\alpha_n, [t_i])$ and $\lim_n I(\beta_n, [t_i])$ exist for all t_i. Thus, $\lim_{n,m} |\alpha_n - \alpha_m| = 0$, $\lim_{n,m} |\beta_n - \beta_m| = 0$ and both $|\beta_n|$ and $|\alpha_n|$ are bounded. (This is where we use the space $CS_0(\Sigma)$ instead of $ES(\Sigma)$.) By inequality (1.1), we have $|I(\alpha_n, \beta_n) - I(\alpha_m, \beta_m)| \leqslant |I(\alpha_n, \beta_n) - I(\alpha_n, \beta_m)| + |I(\alpha_n, \beta_m) - I(\alpha_m, \beta_m)| \leqslant 4|\alpha_n||\beta_n - \beta_m| + 4|\beta_m||\alpha_n - \alpha_m|$ which converges to 0 as m and n tend to infinity.

As a consequence of the continuity, we see that inequality (1.1) still holds for α, β and γ in $ML_0(\Sigma)$. Thus we deduce a result in [6] that each element α in $ML_0(\Sigma)$ is determined by the N-tuple of intersection numbers $T(\alpha) = (I(\alpha, [t_1]), \ldots, I(\alpha, [t_N]))$, i.e., $T : ML_0(\Sigma) \to \mathbb{R}^N_{\geqslant 0}$ is injective and continuous. Furthermore inequality (1.1) implies that the space $ML_0(\Sigma)$ is locally compact and the map $T : ML_0(\Sigma) \to \mathbb{R}^N$ is proper. Indeed, if a sequence α_n in $ML_0(\Sigma)$ is bounded under T, then for any $\beta \in CS(\Sigma)$, inequality (1.1) implies that $I(\alpha_n, \beta) \leqslant |T(\alpha_n)||T(\beta)|$ is bounded in n for each fixed β.

Since there are at most countable β's, by the standard diagonalization argument, there is a subsequence α_{n_i} so that $I(\alpha_{n_i}, \beta)$ converges for all β. This simply says that $\{\alpha_n\}$ contains a convergent subsequence. To see that T is proper, we note that if $T(\alpha_n)$ converges to a point in $\mathbb{R}^N$, then $T(\alpha_n)$ is bounded. Thus α_n contains a convergent subsequence. This shows that T is proper and $T: ML_0(\Sigma) \to \mathbb{R}^N$ is an embedding whose image is a closed subset. The image of $ML_0(\Sigma)$ under T can be identified explicitly as a subspace defined by a finite set of piecewise linear equations. It is shown in Section 4 that the subspace is homeomorphic to a Euclidean space. This shows that the space of measured laminations is homeomorphic to a Euclidean space.

1.4. One can also establish the same results for completions of $ES(\Sigma)$ using Proposition 1.1. Let $\mathcal{ES}(\Sigma)$ be its completions inside the space $\mathbb{R}^{CS(\Sigma)}$ with respect to the same procedure. Using Proposition 1.1, one can show that the intersection paring I extends continuously to $\mathcal{ES}(\Sigma)$ and that $\mathcal{ES}(\Sigma)$ is homeomorphic to $\mathbb{R}^{-3\chi(\Sigma)}$ where $\chi(\Sigma)$ is the Euler characteristic of the surface. These results are proved by showing the following simple fact about the t-coordinates $(x_1, \ldots, x_N, x'_1, \ldots, x'_N)$. Namely, for each i, there is a finite set of elements $c, \ldots, c_k$ in $ES(\Sigma)$ and a universal piecewise linear function f so that $x'_i(\alpha) = f(I(\alpha, c_1), \ldots, I(\alpha, c_k))$ for all $\alpha \in ES(\Sigma)$. The details will be deferred in a future work.

1.5. Notations and conventions

We use $\mathbb{R}_{\geqslant 0}$, $\mathbb{Q}_{\geqslant 0}$ and $\mathbb{Z}_{\geqslant 0}$ to denote the sets of all non-negative real numbers, rational numbers and integers, respectively. All surfaces are connected unless mentioned otherwise. Isotopies of the surface leave the boundary invariant. Given a 1-submanifold s, we denote the isotopy class of s by $[s]$ and a small regular neighborhood of s by $N(s)$. The interior of a manifold X will be denoted by $\mathrm{int}(X)$. The geometric intersection number $I([a],[b])$ will also be denoted by $I(a,b)$ and $I([a],b)$. If X is a finite set, then $|X|$ denotes the number of elements in X.

1.6. We would like to thank Ying-Qing Wu for careful reading of the manuscript and making nice suggestions on improving the exposition. Part of the work is supported by the NSF. The related work for closed surface is in [5].

2. Arcs in polygons

We give a parametrization of the space of all isotopy classes of arc systems in a polygon in this section.

2.1. Let P_n be an n-sided polygon. An *arc* in P_n is a proper embedding of a closed interval into $P_n - \{\text{vertices of } P_n\}$. An arc in P_n is called *trivial* if its end points either lie in one side of P_n or in two adjacent sides of P_n. An *arc system* in P_n is a finite disjoint union of non-trivial arcs in P_n. Let $ES(P_n)$ be the set of all isotopy classes of arc systems in P_n where isotopies leave each side invariant. Given two classes α and β in $ES(P_n)$,

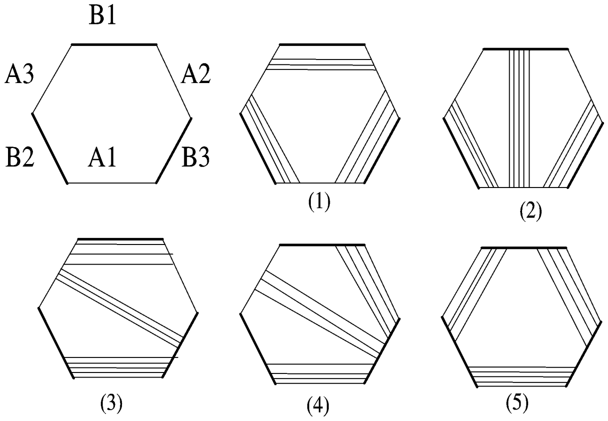

Fig. 1.

we define their intersection number to be $I(\alpha, \beta) = \min\{|a \cap b|\colon a \in \alpha,\ b \in \beta\}$. We say a non-trivial arc s in P_n *parallel* to a side if one of the component of $P_n - s$ is a quadrilateral.

We first give a parametrization of $ES(P_6)$. Let the six sides of the hexagon P_6 be $A_1, B_3, A_2, B_1, A_3, B_3$ labeled cyclically. A parametrization of $ES(P_6)$ using the A-sides is as follows. Take $\alpha = [a]$ in $ES(P_6)$. Let $x_i = I(\alpha, A_i) = |a \cap A_i|$ and x_i' be the number of components of a which are parallel to A_i. Evidently $x_i x_i' = 0$. We call $(x_1, x_2, x_3, x_1', x_2', x_3')$ the *t-coordinate* of α with respect to the A-sides of the hexagon. See Fig. 1.

Lemma 2.1. *Let $\Delta = \{(a_1, a_2, a_3) \in \mathbb{Z}^3_{\geqslant 0}\colon a_i + a_j \geqslant a_k$, for all $i \neq j \neq k \neq i\}$. The map $T : ES(P_6) \to \{(x_1, x_2, x_3, x_1', x_2', x_3') \in \mathbb{Z}^6_{\geqslant 0}\colon x_i x_i' = 0$, if $(x_1, x_2, x_3) \in \Delta$ then $x_1 + x_2 + x_3$ is even$\}$ sending an element to its t-coordinate is a bijection and homogeneous, i.e., $T(k[a]) = kT([a])$ for any $k \in \mathbb{Z}_{\geqslant 0}$.*

Furthermore, if $(x_1, x_2, x_3, x_1', x_2', x_3')$ is the t-coordinate of a class $[a]$, then the number of components of a is at most $x_1 + x_2 + x_3 + x_1' + x_2' + x_3'$.

Proof. Clearly T is well defined. To see that T is onto, we construct the arc system a with a given vector $(x_1, x_2, x_3, x_1', x_2', x_3')$ as the coordinate according to the following five cases:

(1) $(x_1', x_2', x_3') = (0, 0, 0)$, and $(x_1, x_2, x_3) \in \Delta$;
(2) $(x_1', x_2', x_3') = (0, 0, 0)$ and $(x_1, x_2, x_3) \notin \Delta$;
(3) $x_i' = x_j' = 0$, $x_k' > 0$;
(4) $x_i' = 0$ and $x_j' x_k' > 0$, and
(5) $x_1' x_2' x_3' > 0$.

The corresponding arc systems are listed in Fig. 1.

The arc system a can be described as follows. Let a_i (respectively b_i) be an arc parallel to A_i (respectively B_i) and c_i be an arc joining A_i to B_i. We use kx to denote

k parallel copies of an arc x. In the case (1), $a = \bigcup_{k=1}^{3} (\frac{x_i + x_j - x_k}{2}) b_k$; in the case (2) say $x_k > x_i + x_j$, then $a = x_i b_i \cup x_j b_j \cup (x_k - x_i - x_j) c_k$; in the case (3), say $x_i \geqslant x_j$, then $a = x'_k a_k \cup x_j b_j \cup (x_i - x_j) c_i$; in the case (4) $a = x'_j a_j \cup x'_k a_k \cup x_i c_i$ and in the last case (5) $a = \bigcup_{i=1}^{3} x'_i a_i$. Since two non-trivial arcs are isotopic if and only if their end points land on the same set of sides on the polygon, the map T is injective.

The second part of the lemma follows from the definition. □

2.2. Remark

It can be shown that each isotopy class of arc systems in a hexagon is determined by their six geometric intersection numbers with the six edges. It is also interesting to note that if we switch the A-sides and B-sides, then the coordinate change is given by a piecewise linear homogenous continuous function in $(x_1, x_2, x_3, x'_1, x'_2, x'_3)$.

2.3. To parameterize the arc systems on any polygon P_{2n} of an even number of sides, we use disjoint non-trivial arcs to decompose P_{2n} into hexagons. Let the A-sides of the hexagons correspond to the decomposing arcs. Then a parameterization of $ES(P_{2n})$ is given by taking the t-coordinates of the hexagons with respect to the A-sides.

2.4. One of the key ingredients in the proof of Proposition 1.1 is to understand the surgery procedure relating two elements in $ES(P_6)$ whose t-coordinates differ by a basis vector. For simplicity, a class α in $ES(P_6)$ is called *even* if all components of its t-coordinates are even numbers. We shall describe the surgery procedure relating two even arc systems α and β so that their t-coordinates $(x_1, x_2, x_3, x'_1, x'_2, x'_3)$ and $(y_1, y_2, y_2, y'_1, y'_2, y'_3)$ are related by $(x_1, x_2, x_3, x'_1, x'_2, x'_3) = (y_1, y_2, y_2, y'_1, y'_2, y'_3) + (2, 0, \ldots, 0)$.

Note that $x'_1 = y'_1$. Since $x_1 > 0$, it follows that $x'_1 = y'_1 = 0$.

Take a standard representative a for α. To obtain a standard representative b for β, we perform the following surgery operation on a. If a contains arcs b_2 and b_3, we replace a by $(a - b_2 \cup b_3) \cup b_1$ to obtain b; if a contains an arc parallel to c_1, then since α is even, a contains two copies of c_1. We replace a by $a - 2c_1$ to obtain b. In the remaining case, a is disjoint from either c_2 or c_3, say $a \cap c_2 = \emptyset$. Since $x_1 \geqslant 2$, a contains at least 2 copies of b_3. In this case, replace a by $(a - 2b_3) \cup 2c_2$ to obtain b. Note that the arcs created lie in a small regular neighborhood of the boundary and the arcs deleted. See Fig. 2 for the illustration.

To obtain a standard representative of a from b, we perform the following surgery operation on b. If b contains some copies of b_1 but no c_2 or c_3, replace b by $(b - b_1) \cup b_2 \cup b_3$ to obtain a. If b contains no b_1, c_2 and c_3, replace b by $b \cup 2c_1$ to obtain a. If b contains some c_2 or c_3, say $c_2 \subset b$, then b contains even number of copies of c_2. Replace b by $(b - 2c_2) \cup 2b_3$.

To summarize, we have the following,

Lemma 2.2. *Suppose $[a]$ and $[b]$ are two even classes in $ES(P_6)$ whose t-coordinates differ by 2 in one entry. Then a is obtained from b by removing at most two components and adding at most two new components.*

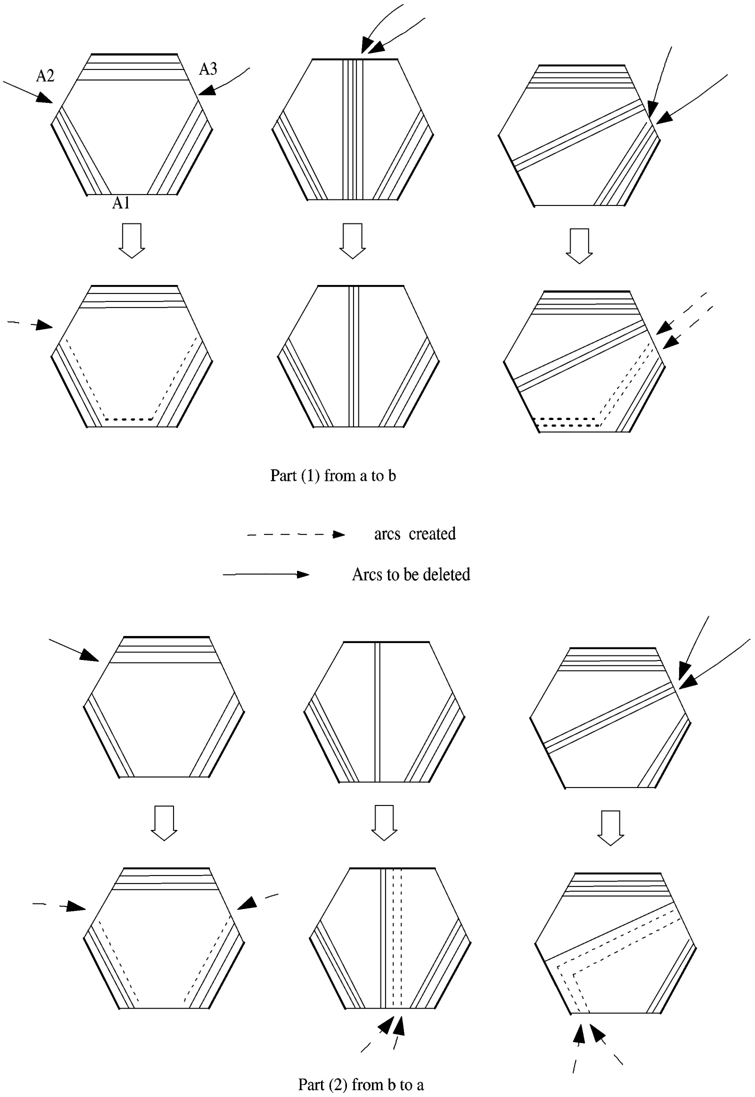

Fig. 2.

3. Geometric intersection numbers on surfaces with boundary

We prove Proposition 1.1 in this section.

3.1. We begin by giving a parameterization of the space $ES(\Sigma)$ as follows. Fix a maximal collection $t = t_1 \cup \cdots \cup t_N$ of pairwise disjoint, non-isotopic essential arcs (an ideal triangulation) of the surface Σ. Thus the components of $\Sigma - \bigcup_{i=1}^{N} \mathrm{int}(N(t_i))$ are hexagons. Let the A-sides of the hexagons correspond to t_i's. Given α in $ES(\Sigma)$, let $t(\alpha)$

be the *t-coordinate* of α which is the collection of t-coordinates of α in each hexagon. Namely, $t(\alpha) = (x_1, \ldots, x_N, x'_1, \ldots, x'_N)$ where $x_i = I(\alpha, t_i)$ and x'_i is the number of components of α equal to $[t_i]$. Clearly $x_i x'_i = 0$.

Lemma 3.1 (See also [6]). *Fix an ideal triangulation t of Σ. Then the map $T : ES(\Sigma) \to X = \{(x_1, \ldots, x_N, x'_1, \ldots, x'_N) \in \mathbb{Z}^N_{\geqslant 0}\colon\ x_i x'_i = 0$, if t_i, t_j and t_k form the A-sides of a hexagon and $(x_i, x_j, x_k) \in \Delta$, then $x_i + x_j + x_k$ is even$\}$ sending an element to its t-coordinate is a bijection and homogeneous, i.e., $T(k[a]) = kT([a])$ for $k \in \mathbb{Z}_{\geqslant 0}$.*

In particular, let $L = \{(x_1, \ldots, x_N, x'_1, \ldots, x'_N) \in (2\mathbb{Z}_{\geqslant 0})^N\colon\ x_i x'_i = 0\}$ be the set of all even vectors. Then L is contained in the image of T.

Proof. To see that the map T is onto, take an element $(x_1, \ldots, x_N, x'_1, \ldots, x'_N)$ in the set X. Let H be a hexagonal component of $\Sigma - \bigcup_{i=1}^N \operatorname{int}(N(t_i))$ with three A-sides parallel to t_i, t_j and t_k (it may occur that $t_i = t_j$). By Lemma 2.1, we construct an arc system in H with the t-coordinate $(x_i, x_j, x_k, x'_i, x'_j, x'_k)$. Now glue these arc systems across $N(t_i) = t_i \times [-1, 1]$ by adding parallel arcs $\{p_1, \ldots, p_n\} \times [-1, 1]$. We obtain a 1-submanifold s properly embedded in Σ. By the construction, there are no Whitney discs in $s \cup t$ and $s \cup \partial\Sigma$. Thus the submanifold s is essential and its t-coordinate is the given vector $(x_1, \ldots, x_N, x'_1, \ldots, x'_N)$. We call s a *standard representative*. To see that the map T is injective, given α in $ES(\Sigma)$, choose a representative $a \in \alpha$ so that $I(\alpha, t) = |a \cap t|$. Thus $a \cap H$ is an arc system in each hexagonal component of $\Sigma - \bigcup \operatorname{int}(N(t_i))$. Since each non-trivial arc in the quadrilateral $N(t_i)$ is parallel to a side, it follows that a is isotopic to a standard representative. It follows that the map T is injective. □

3.2. Now we prove the following:

Proposition 1.1. *Suppose $t = t_1 \cup \cdots \cup t_N$ is an ideal triangulation of a compact surface. Then for any three classes $\alpha, \beta, \gamma \in ES(\Sigma)$ with t-coordinates $(x_1, \ldots, x_N, x'_1, \ldots, x'_N)$, $(y_1, \ldots, y_N, y'_1, \ldots, y'_N)$ and $(z_1, \ldots, z_N, z'_1, \ldots, z'_N)$, the following inequality holds*:

$$\bigl|I(\alpha, \beta) - I(\alpha, \gamma)\bigr| \leqslant 4|\alpha||\beta - \gamma| \tag{1.1}$$

where $|\alpha| = \sum_{i=1}^N (x_i + x'_i)$ and $|\beta - \gamma| = \sum_{i=1}^N (|y_i - z_i| + |y'_i - z'_i|)$.

To begin the proof, first note that since the intersection pairing $I(,)$ is homogeneous, it suffices to prove inequality (1.1) for 2α, 2β and 2γ in $ES(\Sigma)$. The t-coordinate of 2α is an even vector in L. For simplicity, we call a class $\alpha \in ES(\Sigma)$ *even* if $T(\alpha) \in L$. Thus it suffices to prove Proposition 1.1 for even classes.

Given two even vectors $u = (u_1, \ldots, u_{2N})$ and $v = (v_1, \ldots, v_{2N})$ in L so that their distance $|u - v| = \sum_{i=1}^{2N} |u_i - v_i|$ is $2n$, there is a sequence of $n + 1$ even vectors w_j, $j = 0, \ldots, n$ so that $w_0 = u$, $w_n = v$ and $|w_{i+1} - w_i| = 2$. Thus given two even classes β, γ in $ES(\Sigma)$ so that $|\beta - \gamma| = 2n$, by Lemma 3.1, there exists a sequence of $n + 1$ even classes starting from β and ending at γ so that the adjacent elements are of distance-2 apart. Thus it suffices to prove Proposition 1.1 for even classes β and γ so that $|\beta - \gamma| = 2$. Without loss of generality, we may assume that $T(\gamma) = T(\beta) \pm (0, \ldots, 0, 2, 0, \ldots, 0)$, i.e.,

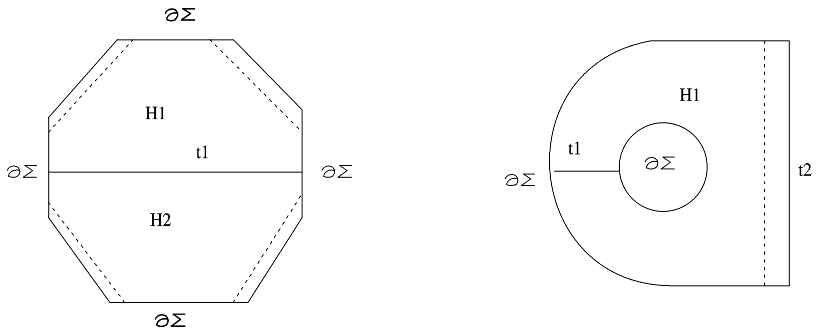

Fig. 3.

$(z_1, \ldots, z_N, z'_1, \ldots, z'_N) = (y_1, \ldots, y_N, y'_1, \ldots, y'_N) \pm (0, \ldots, 0, 2, 0, \ldots, 0)$. We need to consider two cases:

(1) $z'_i = y'_i \pm 2$, and
(2) $z_i = y_i \pm 2$ for some i.

In the first case that $z'_i = y'_i \pm 2$, the class γ is obtained from β by adding or removing two copies of $[t_i]$. Thus $I(\alpha, \beta) = I(\alpha, \gamma) \pm 2x_i$. The inequality follows.

In the second case, let us assume for simplicity that $i = 1$. Let H_1 and H_2 be the closures of the hexagonal components of $\Sigma - (\bigcup_{i=2}^N \mathrm{int}(N(t)) \cup t_1)$ lying on two sides of t_1 (it may occur that $H_1 = H_2$). See Fig. 3. If $H_1 \neq H_2$, then $H_1 \cap H_2 = t_1$ and $H_1 \cup H_2$ is an octagon. In this case we assume that $H_1 \cup H_2$ is a convex octagon. We will consider the cases $H_1 \neq H_2$ and $H_1 = H_2$ separately.

3.3. Suppose $H_1 \neq H_2$. By symmetry, to show (1.1) for $|\beta - \gamma| = 2$, it suffices to prove

$$I(\alpha, \gamma) \leqslant I(\alpha, \beta) + 8|\alpha|. \tag{3.1}$$

Take a standard representative a and b of α and β so that $|a \cap b| = I(\alpha, \beta)$ and $a \cap (H_1 \cup H_2)$ and $b \cap (H_1 \cup H_2)$ consist of straight line segments. Then by the assumption on the t-coordinates of β and γ and Lemmas 3.1 and 2.2, a representative c of γ can be obtained from b by performing the following three surgeries inside the octagon $H_1 \cup H_2$.

Surgery 1. remove at most two components of $b \cap H_1$ and add at most two new arcs.
Surgery 2. remove at most two components of $b \cap H_2$ and add at most two new arcs.
Surgery 3. perform a switching operation inside the neighborhood $N(t_1)$ to join the arcs created or deleted in surgeries 1 and 2. See Fig. 4.

By the surgery construction, we obtain a representative c for γ so that $c \subset b \cup s_1 \cup s_2 \cup s_3$ where s_1, s_2 and s_3 are arcs created in the surgeries 1, 2, and 3.

Thus we have $I(\alpha, \gamma) \leqslant |a \cap c| \leqslant |a \cap b| + |a \cap s_1| + |a \cap s_2| + |a \cap s_3| = I(\alpha, \beta) + |a \cap s_1| + |a \cap s_2| + |a \cap s_3|$. We estimate the last three terms as follows. Note that if u is an arc system in a hexagon with t-coordinate $(v_1, v_2, v_3, v'_1, v'_2, v'_3)$, then u has at most

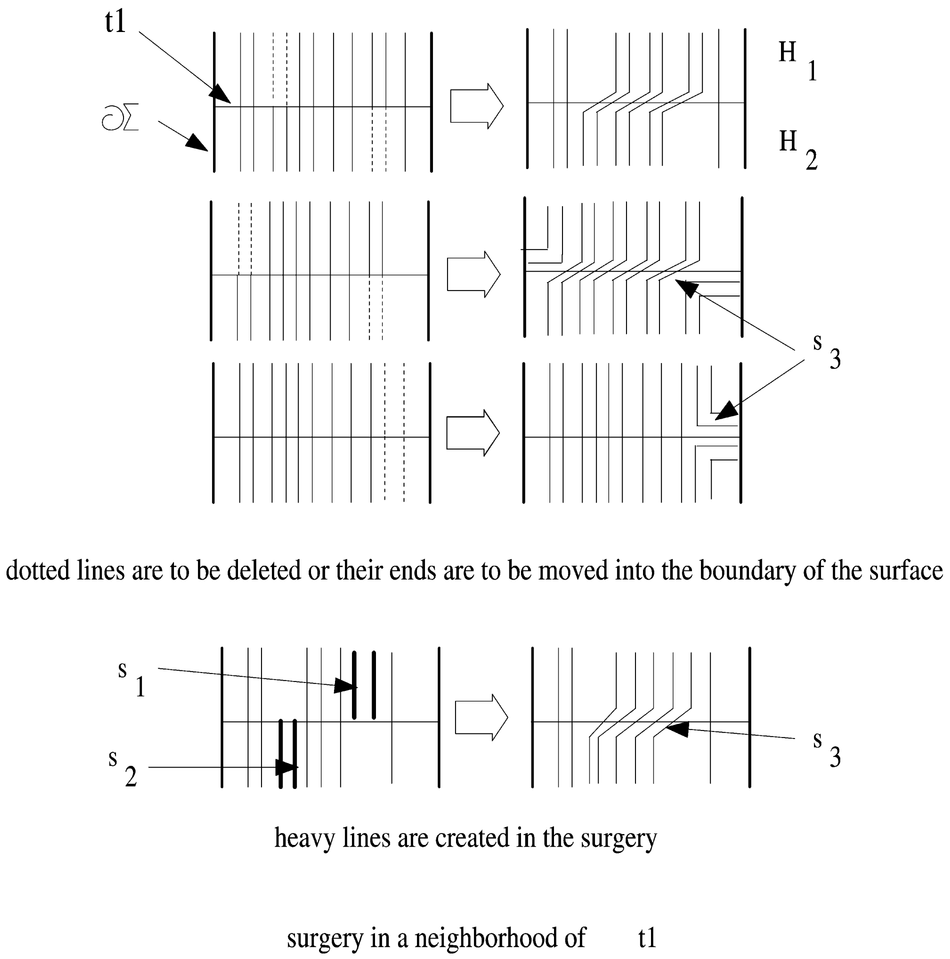

Fig. 4.

$|u| = v_1 + v_2 + v_3 + v_1' + v_2' + v_3'$ components. Now any two straight arcs inside a hexagon intersect in at most one point. Thus for any arc d inside the hexagon, we have $I(d, u) \leqslant |u|$. Since s_i consists of at most two arcs components inside a hexagon, we have $|s_i \cap a| \leqslant 2|a|$. Here the coefficient 2 occurs instead of 1 because two A-sides of the hexagon H_i may be isotopic in the surface. Thus $|a \cap s_1| + |a \cap s_2| \leqslant 4|a|$. On the other hand, by the surgery 3, we see that $|a \cap s_3| \leqslant 4|x_1| \leqslant 4|a|$. This ends the proof of (3.1).

3.4. The second case that $H_1 = H_2$ is an annulus is simple. We simply note that there are three surgeries relating c to b as shown in Fig. 5. The three surgeries depend on the number n of components of arcs jointing t_2 to $\partial\Sigma$ in H_1. In the first case, $n \geqslant 4$, in the second case $n = 0$ and in the last case $n = 2$. In the first case $n \geqslant 4$, we remove four such arcs and replace them by four arcs going around the boundary component of $\partial\Sigma$. In the

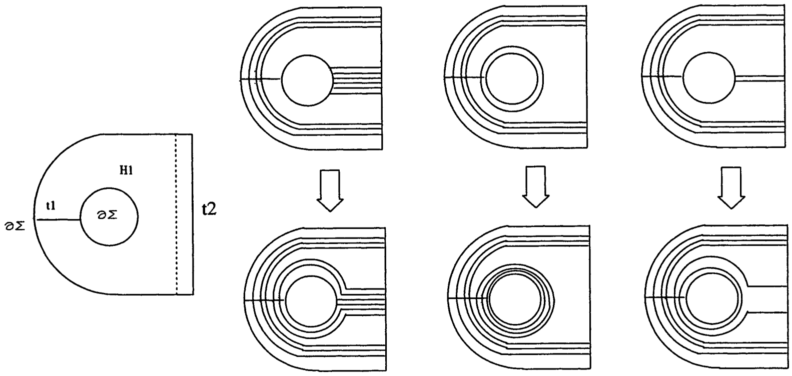

Fig. 5.

second case, we add two parallel copies of the boundary components. In the last case of $n = 2$, we remove these two arcs and replace them by a parallel copy of the boundary component and an arc going around the boundary. In all cases, we see that (3.1) holds.

4. Topology of the space of measured laminations

In this section, we derive the known fact [6,11] that the spaces of all closed measured laminations $ML_0(\Sigma)$ is homeomorphic to a Euclidean space.

4.1. To see that $ML_0(\Sigma)$ is a Euclidean space, we fix an ideal triangulation $t = t_1 \cup \cdots \cup t_N$ of the surface. By Proposition 1.1 and Section 1.3, we see that the map $T : ML_0(\Sigma) \to \mathbb{R}^N_{\geqslant 0}$ sending an element α to its t-coordinate is an embedding into a closed subset. It remains to find the image of the map T. To this end, let us find the images under T of the space of all closed curve systems $CS_0(\Sigma)$. Given a t-coordinate $x = (x_1, \ldots, x_N)$ subject to the condition that when t_i, t_j, and t_j form the A-sides of a hexagonal component of $\Sigma - \bigcup_m t_m$, then $(x_i, x_j, x_k) \in \Delta$, one constructs an essential submanifold s with x as its t-coordinate by Lemma 3.2. This essential submanifold s is a closed curve system if and only if the submanifold s contains no loop parallel to $\partial\Sigma$. This is the same as saying that at least one of the hexagons incident on $\partial_i\Sigma$ does not contain an arc parallel to the B-side corresponding to $\partial_i\Sigma$, i.e., for each boundary component $\partial_i\Sigma$,

$$\min_H \{x_j + x_k - x_l\} = 0 \tag{4.1}$$

where the minimum runs over all hexagons H incident on $\partial_i\Sigma$ and H is formed by the arcs t_j, t_k, and t_l with t_l opposite to a B-side in $\partial_i\Sigma$. Suppose r is the number of boundary components of the surface Σ. There are r Eq. (4.1). Thus we see that $CS_0(\Sigma)$ can be described as a finite union of regions, each of which is described by integer coefficient linear equations (coming from (4.1)) in the x_i and triangle inequalities saying that certain linear combinations of the x_i with integer coefficients are nonnegative. Thus the set of rational solutions to these equations is dense in the set of real solutions. This shows that the image $T(ML_0(\Sigma))$ is equal to the subspace S of $\mathbb{R}^N_{\geqslant 0}$ subject to r Eq. (4.1) and the triangular inequalities:

$$x_j + x_k \geqslant x_l \tag{4.2}$$

where t_j, t_k, t_l form the A-sides of a hexagonal component of $\Sigma - \bigcup_{r=1}^N t_r$.

One may see the topological type of the space defined by Eq. (4.1) and inequalities (4.2) as follows. Let us make a change of variables by letting $y_i = (x_j + x_k - x_l)/2$ in (4.2). Geometrically, y_i is the number of copies of arcs parallel to the B-side of the hexagon corresponding to a boundary component of $\partial\Sigma$. Then (4.2) becomes $(y_1, \ldots, y_M) \in \mathbb{R}^M_{\geqslant 0}$. Eq. (4.1) become

$$\min\{y_i \mid i \in B_j\} = 0, \tag{4.3}$$

where the index set B_j consists of indices i so that y_i is around the jth boundary component of the surface. Finally we have a new set of equation defined on each edge of the form

$$y_i + y_j = y_k + y_l \tag{4.4}$$

for each edge t_n of the ideal triangulation so that y_i, y_j, y_k, y_l are adjacent to t_n. These are exactly the switching equations in the train-track dual to the ideal triangulation t [6,11]. We claim that Eqs. (4.3) and (4.4) define a space S in $\mathbb{R}^M$ homeomorphic to a Euclidean space of dimension $-3\chi(\Sigma) - r$. To this end, consider the linear subspace V of $\mathbb{R}^M$ spanned by the vectors $\sum_{i\in B_j} e_i$ where e_i is the vector with $y_i = 1$ and all other $y_j = 0$. Let W be the linear subspace defined by Eq. (4.4). Let $P : \mathbb{R}^M \to R^M/V$ be the quotient map. We claim that the restriction map $P|_S : S \to P(W)$ is a homeomorphism. Since S is closed and locally compact, it suffices to show that the restriction map $P|_S$ is one-to-one and onto. To see the map is onto, given a vector $y = (y_1, \ldots, y_M)$ in W, by adding the vector $-\sum_{j=1}^r \sum_{i\in B_j} \min\{y_i \mid i \in B_j\} e_i$ to y, we see that the new vector is in the space S. On the other hand, if y and y' are two vectors in S so that $y - y' \in V$, then by looking at the components around each boundary $\partial_j\Sigma$, we conclude that $y = y'$. This shows that the space S and hence $ML_0(\Sigma)$ is homeomorphic to a Euclidean space. To find the dimension of the Euclidean space, we note that the two linear subspaces W and V intersect transversely at 0 in R^M. Assuming this, since the dimension of W is $-3\chi(\Sigma)$ and the dimension of V is r, one finds the dimension of the quotient space to be $-3\chi(\Sigma) - r$.

It remains to show that the subspaces W and V intersect transversely at 0. This follows from a little bit of combinatorics, a linear combination of equations of type (4.3) may be regarded as a linear combination of the duals to the t_i (suitably directed). Suppose a sum of these is a sum of equations of type (4.4). Then duals to consecutive t_i around a boundary component must get weights which differ by a constant. But since the boundary edges cycle, this says that the duals to the t_i incident to a particular boundary component all get the same weight. Since every boundary component is joined in a connected graph by the t_i, we conclude that all duals get the same weight (up to sign for orientation). However looking at a single hexagon shows that the orientations cannot be compatible unless all the weights are zero and therefore all the weights are zero. Thus the only linear combination which vanishes is the trivial one. This establishes the assertion.

4.2. Finally, we remark that the same argument plus the following lemma shows that the closures of $\mathbb{Q}_{\geqslant 0} \times \pi(ES(\Sigma))$ in $\mathbb{R}^{CS(\Sigma)}$ is homeomorphic to a Euclidean space. The image in $\mathbb{R}^{2N}$ is given by $\{(x_1, \ldots, x_N, y_1, \ldots, y_N) \in \mathbb{R}^{2N} \mid x_i \geqslant 0, y_i \geqslant 0$ and $x_i y_i = 0$ for all $i\}$ which can be seen easily to be a Euclidean space.

Lemma 4.1. *Fix an ideal triangulation $t = t_1 \cup \cdots \cup t_N$ of a compact surface t-coordinates. For each index i, there is a finite set of elements $c, \ldots, c_k$ in $ES(\Sigma)$ and a universal piecewise linear function f so that $x_i'(\alpha) = f(I(\alpha, c_1), \ldots, I(\alpha, c_k))$ for all $\alpha \in ES(\Sigma)$.*

The details will be deferred in a future work.

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