

Virtual Class of Zero Loci and Mirror Theorems

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Abstract

Let Y be the zero loci of a regular section of a convex vector bundle E over X . We provide a new proof of a conjecture of Cox, Katz and Lee for the virtual class of the moduli space of genus zero stable maps to Y . This in turn yields the expected relationship between Gromov-Witten theories of Y and X which together with Mirror Theorems allows for the calculation of enumerative invariants of Y inside of X .

1 Introduction

Let X be a smooth, projective variety over \mathbb{C} . A vector bundle

$$E \rightarrow X$$

is called *convex* if $H^1(f^*(E)) = 0$ for any morphism $f : \mathbb{P}^1 \rightarrow X$. Let $Y = Z(s) \subset X$ be the zero locus of a regular section s of a convex vector bundle E and let i denote the embedding of Y in X . It is the relationship between the Gromov-Witten theories of Y and X that we study here.

1.1 Virtual class of the zero loci

Let $\overline{M}_{0,n}(X, d)$ be the \mathbb{Q} -scheme that represents coarsely genus zero, n -pointed stable maps

$$(C, x_1, x_2, \dots, x_n, f : C \rightarrow X)$$

of class $d \in H_2(X, \mathbb{Z})$. Since E is convex the vector spaces $H^0(f^*(E))$ fit into a \mathbb{Q} -vector bundle E_d on $\overline{M}_{0,n}(X, d)$. The section s of E induces a section \tilde{s} of E_d over $\overline{M}_{0,0}(X, d)$ via $\tilde{s}((C, f)) = s \circ f$. If $i_*(\beta) = d$, the map $i : Y \hookrightarrow X$ yields an inclusion $i_\beta : \overline{M}_{0,0}(Y, \beta) \rightarrow \overline{M}_{0,0}(X, d)$. Clearly

$$Z(\tilde{s}) = \coprod_{i_*(\beta)=d} \overline{M}_{0,0}(Y, \beta).$$

The map $i_* : H_2(Y, \mathbb{Z}) \rightarrow (H_2(X, \mathbb{Z}))$ is not injective in general, hence the zero locus $Z(\tilde{s})$ may have more than one connected component. An example is the quadric surface in \mathbb{P}^3 .

The \mathbb{Q} -normal bundle of $Z(\tilde{s})$ in $\overline{M}_{0,0}(X, d)$ is $E_d|_{Z(\tilde{s})}$. Let c_{top} denote the top chern operator. In Section 2.1 of this paper we prove the following theorem

Theorem 1.1.1. *For any $d \in H_2(X, \mathbb{Z})$,*

$$\sum_{i_*\beta=d} (i_\beta)_* [\overline{M}_{0,0}(Y, \beta)]^{\text{vir}} = c_{\text{top}}(E_d) \cap [\overline{M}_{0,0}(X, d)]^{\text{vir}}.$$

Remark 1.1.1. *Theorem 0.1.1 was conjectured in [5] for any n -pointed moduli stack. But one of the functorial properties of the virtual class is that*

$$\pi_n^*([\overline{M}_{0,n-1}(Y, \beta)]^{\text{virt}}) = [\overline{M}_{0,n}(Y, \beta)]^{\text{virt}}.$$

It is also true that $\pi_{n+1}^(E_d) = E_d$ (see Lemma 3.4 in [6].) It suffices then to prove the theorem for 0-pointed stable maps.*

Remark 1.1.2. *The first proof of this theorem appeared in [11]. It is done in the category of stacks and uses the Behrend-Fantechi construction of the virtual fundamental class. The version of this theorem where the virtual classes are those of Li-Tian essentially follows from Proposition 3.9 of [15]. Our approach is a mixture of the two: we use an extension of the Li-Tian construction of the virtual fundamental class in the category of \mathbb{Q} -schemes as well as intersection-theoretic features of a \mathbb{Q} -scheme coming from its stack nature.*

1.2 Mirror Theorems

Theorem 0.1.1 together with the Mirror Theorem provide a complete answer to the relationship between the enumerative invariants of Y and those of X .

Let $\{T_0 = 1, T_1, \dots, T_r, \dots, T_m\}$ be a basis of $H^*(Y, \mathbb{Q})$ such that

$$\text{Span}\{T_1, \dots, T_r\} = H^2(Y, \mathbb{Q}).$$

Let t_1, \dots, t_r be formal variables and $tT = \sum_{i=1}^r t_i T_i$. We denote by c the cotangent line class on $\overline{M}_{0,1}(Y, \beta)$, i.e. the first chern class of the line bundle whose fiber over moduli point (C, x_1, f) is T_{C, x_1}^\vee . Let e be the evaluation map on $\overline{M}_{0,1}(Y, \beta)$. Let \hbar be a formal parameter. Then the quantum \mathcal{D} -module structure of the pure quantum cohomology of Y is determined by the following formal function (see subsection 6 of section 1 in [8]):

$$J_Y := e^{\frac{tT}{\hbar}} \cap \left(1 + \sum_{0 \neq \beta \in H_2(Y, \mathbb{Z})} q^\beta e_* \left(\frac{[\overline{M}_{0,1}(Y, \beta)]^{\text{vir}}}{\hbar(\hbar - c)} \right) \right). \quad (1)$$

Note that $e^{\frac{tT}{\hbar}}$ acts via its power series expansion (which is finite). For our purposes J_Y will be viewed as an element of the Novikov completion $H_* Y[[t_1, \dots, t_r, \hbar^{-1}]][[q^\beta]]$ of the ring $H_* Y[[t_1, \dots, t_r, \hbar^{-1}]]$ along the semigroup of rational curves β in Y . This generator encodes Gromov-Witten invariants and the gravitational descendants of Y .

For each stable map $(C, x_1, f) \in \overline{M}_{0,1}(X, d)$, the sections of $f^*(E)$ that vanish at x_1 form a bundle E'_d that fits into an exact sequence:

$$0 \rightarrow E'_d \rightarrow E_d \rightarrow e^*(E) \rightarrow 0 \quad (2)$$

The quantum \mathcal{D} -module structure of the E -twisted quantum cohomology of X [6][8] is determined by the following formal function:

$$J_E := e^{\frac{tT'}{\hbar}} c_{\text{top}}(E) \cap \left(1 + \sum_{0 \neq d \in H_2(X, \mathbb{Z})} q^d e_* \left(\frac{c_{\text{top}}(E'_d) \cap [\overline{M}_{0,1}(X, d)]^{\text{vir}}}{\hbar(\hbar - c)} \right) \right)$$

where tT' and c denote similar expressions to those in J_Y . The Mirror Theorem states that for a large class of smooth varieties X , the generator J_E is computable via hypergeometric series (for proofs of this theorem and its variations see [2],[6],[8],[13]). While important in itself, this fact is relevant with respect to the Gromov-Witten theory of Y only if one can show that J_E is intrinsically related to Y . The basic example is when X is a

projective bundle and E is a direct sum of positive line bundles. It has been shown in [4] that in this case $i_*(J_Y) = J_E$. Coupled with the fact that J_E is computable this allows for the calculation of (at least some of) the gravitational descendants of Y . This is, for example, one way to compute the enumerative invariants of the quintic threefold. In section 2.2 we use Theorem 0.1.1 to prove the following generalization: Assume that the map $i_* : H_2(Y) \rightarrow H_2(X)$ is surjective. Complete a basis $\{T'_1, T'_2, \dots, T'_r\}$ of H^2X into a basis $\{T_1 = i^*(T'_1), \dots, T_r = i^*(T'_r), \dots, T_m\}$ of H^2Y . We extend the map $i_* : H_*Y \rightarrow H_*X$ to a homomorphism of completions $i_* : H_*Y[[t_1, \dots, t_m, \hbar^{-1}]][[q^\beta]] \rightarrow H_*X[[t_1, \dots, t_r, \hbar^{-1}]][[q^d]]$ via $i_*(t_k) = 0$ for $k > r$ and $i_*(q^\beta) = q^{i_*(\beta)}$.

Theorem 1.2.1. *Assume that $i_* : H_2(Y) \rightarrow H_2(X)$ is surjective. Then $i_*(J_Y) = J_E$.*

Acknowledgements. The author would like to acknowledge helpful discussions with Barbara Fantechi, Tom Graber, and Ravi Vakil. We would also like to thank the referee for providing corrections and many helpful suggestions.

2 Background

2.1 Stable maps and Gromov-Witten invariants

Let g, n be non-negative numbers and $d \in H_2(X, \mathbb{Z})$. A *stable map* of genus g with n -markings consists of a nodal curve C , an n -tuple (x_1, x_2, \dots, x_n) of smooth points of C and a map $f : C \rightarrow X$ that has a finite group of automorphisms. A stable map (C, x_1, \dots, x_n, f) is said to *represent the curve class d* if $f_*[C] = d$. The moduli functor that parameterizes such stable maps is a proper Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}(X, d)$ and is coarsely represented by a \mathbb{Q} -scheme denoted by $\overline{M}_{g,n}(X, d)$ (see section 1.3 for a discussion on the category of \mathbb{Q} -schemes). The expected dimension of this moduli stack is $(\dim X - 3)(1 - g) + n - K_X \cdot \beta$. Let x_k be one of the marked points. The evaluation morphism $e_k : \overline{M}_{g,n}(X, d) \rightarrow X$ sends a closed point (C, x_1, \dots, x_n, f) to $f(x_k)$. The contangent bundle at x_k is denoted by \mathcal{L}_k . Its fiber over the closed point (C, x_1, \dots, x_n, f) is T_{C, x_k}^\vee . The forgetful morphism $\pi_k : \overline{M}_{g, n+1}(X, d) \rightarrow \overline{M}_{g, n}(X, d)$ forgets the k -th marking and stabilizes the

source curve. The universal stable map over $\overline{M}_{g,n}(X, d)$ is

$$\begin{array}{ccc} \overline{M}_{g,n+1}(X, d) & \xrightarrow{e_{n+1}} & X \\ \downarrow \pi_{n+1} & & \\ \overline{M}_{g,n}(X, d) & & \end{array}$$

The bundle E_d from the introduction may be precisely defined as $E_d := \pi_{n+1*} e_{n+1}^*(E)$.

2.2 The category of \mathbb{Q} -schemes

Stable maps have nontrivial automorphisms hence the moduli functor of stable maps is only locally representable, that is the universal family exists only étale locally. It follows that, in dealing with virtual fundamental class and Gromov-Witten invariants, one should work in a category that “remembers” the automorphisms. Stacks and, as we will see shortly, \mathbb{Q} -schemes are two obvious choices.

The key to proving Theorem 0.1.1 is the functoriality of the virtual fundamental class as stated in Proposition 1.3.1. The construction of the virtual class in the category of stacks has been done in [1]. Due to the lack of a good intersection theory for Artin stacks at the time, the authors of that paper were forced to impose a technical hypothesis to carry out that construction. As a result, they were able to prove a stack version of the Proposition 1.3.1 that had a limited scope of applicability (see Proposition 5.10 in [1]). For example, that version could not be used to study the problem we consider here. Since this technical hypothesis was later removed in [10], one obvious approach was to try to prove the functoriality of the Behrend-Fantechi virtual class construction in more generality following the standard framework of the functoriality of the Gysin map (see section 6.5 in [7]). However, we found the technicalities of the normal cone construction and the deformation to the normal cone (see section 5.1 in [7]) in the category of stacks hard to overcome. This functoriality and the subsequent proof of the Cox-Katz-Lee conjecture in the category of stacks were completed later in [11].

The Li-Tian (LT) virtual class construction of [10] is free of any restrictions. Because of this, the authors of that paper were able to prove the functoriality of their construction in a more general form, suitable for our problem.

We found the category of \mathbb{Q} -schemes to be the perfect setting where one can combine the advantages of the LT virtual class construction and the

well established intersection-theoretic constructions of stacks. The precise definition of \mathbb{Q} -schemes is due to Lian-Tian (see definition 5.3 in [15]). It is a straightforward generalization of the notion of \mathbb{Q} -varieties and \mathbb{Q} -stacks introduced by Mumford in [16].

Definition 2.2.1. *A \mathbb{Q} -scheme is a scheme V together with the following data:*

- *A finite collection $(V_\alpha, G_\alpha, q_\alpha)$ where V_α is a quasi-projective scheme, G_α is a finite group acting faithfully on V_α and $q_\alpha : V_\alpha/G_\alpha \rightarrow V$ is an étale map such that $V = \cup \text{Im}(q_\alpha)$.*
- *For each pair of indices (α, β) there is similarly*

$$(V_{(\alpha,\beta)}, G_{(\alpha,\beta)} = G_\alpha \times G_\beta, q_{(\alpha,\beta)} : V_{(\alpha,\beta)} \rightarrow V)$$

together with equivariant finite étale maps

$$p_\alpha : V_{(\alpha,\beta)} \rightarrow V_\alpha, p_\beta : V_{(\alpha,\beta)} \rightarrow V_\beta$$

such that $\text{Im}q_{(\alpha,\beta)} = \text{Im}q_\alpha \cap \text{Im}q_\beta$ and the map $q_{(\alpha,\beta)}$ factors through both q_α and q_β via the above maps.

- *For any triple (α, β, γ) there exists $(V_{(\alpha,\beta,\gamma)}, G_{(\alpha,\beta,\gamma)}, q_{(\alpha,\beta,\gamma)})$ such that $G_{(\alpha,\beta,\gamma)} = G_\alpha \times G_\beta \times G_\gamma$, together with equivariant, finite, étale maps from $V_{(\alpha,\beta,\gamma)}$ to $V_{(\alpha,\beta)}, V_{(\alpha,\gamma)}, V_{(\beta,\gamma)}$ which commute with the maps introduced in the second condition and such that*

$$\text{Im}q_{(\alpha,\beta,\gamma)} = \text{Im}q_\alpha \cap \text{Im}q_\beta \cap \text{Im}q_\gamma.$$

The motivation for using such a category comes from these considerations:

1. Any DM stack is generically a quotient of a scheme by a finite group (see for example Thm. (6.1) of [12]).
2. The LT construction of the virtual fundamental class can be done in the category of \mathbb{Q} -schemes.

If the \mathbb{Q} -scheme V represents a moduli functor, then a point $x \in V_{\alpha,\beta}$ should be thought of as an automorphism between objects corresponding to $p_\alpha(x)$ and $p_\beta(x)$. In fact it is easy to see that the data of a \mathbb{Q} -scheme determines a stack. Namely, let $R = \coprod V_{\alpha,\beta}$ and $U = \coprod V_\alpha$. The morphisms p_α, p_β induce two étale morphisms $p_1, p_2 : R \rightarrow U$ and we get an étale groupoid scheme

$$R \begin{array}{c} \xrightarrow{p_1} \\ \rightrightarrows \\ \xleftarrow{p_2} \end{array} U$$

and in turn a Deligne-Mumford stack with atlas U (see the Appendix of [17]). It is obvious that this stack has generic trivial stabilizers. The underlying space of V does not determine its \mathbb{Q} -scheme structure. For example, in the case of orbifolds (i.e. V_α is smooth for all α), one has to rule out complex reflections.

The definitions of \mathbb{Q} -sheaves and \mathbb{Q} -complexes on a \mathbb{Q} -scheme V follow naturally. A \mathbb{Q} -sheaf is the collection of G_α -equivariant sheaves \mathcal{G}_α on V_α together with isomorphisms $\mathcal{G}_\alpha \otimes_{\mathcal{O}_{V_\alpha}} \mathcal{O}_{V_{\alpha\alpha'}} \simeq \mathcal{G}_{\alpha'} \otimes_{\mathcal{O}_{V_{\alpha'}}} \mathcal{O}_{V_{\alpha\alpha'}}$ that satisfy the usual cocycle condition in triple intersections. The intersection-theoretic machinery that is available for stacks may be used for \mathbb{Q} -schemes. Of particular importance to us is the localized top chern class of a section s of a \mathbb{Q} -vector bundle E on V (see section 14.1 of [7]). Its existence and construction in the category of \mathbb{Q} -schemes follows routinely from the similar construction in the category of stacks (see for example (ix) of [10]). The local description is easily obtained by unwinding the definitions. If s is given locally by s_α then the zero locus $Z(s)$ is a closed \mathbb{Q} -subscheme of V with charts $Z(s_\alpha) \cap V_\alpha$. Let F be a pure n -dimensional subscheme of V . The group G_α acts on the normal cone to $Z(s_\alpha) \cap q_\alpha^{-1}(F)$ in $q_\alpha^{-1}(F)$. The quotients can be patched together to a \mathbb{Q} -cone $C_{Z(s)/F}$ inside of the restriction to $Z(s)$ of the \mathbb{Q} -bundle E . Let i be the zero section of this cone. Then the action of the localized top chern class of (E, s) on F is $i^![C_{Z(s)/F}]$.

2.3 The virtual fundamental class and the associativity of the refined Gysin maps

The moduli spaces of stable maps may behave badly in families and they may have components whose dimension is bigger than the expected dimension. There is, however, a cycle of the expected dimension which is deformation invariant. It is this cycle which is used as the true fundamental class for intersection theory purposes. In this section we review the Li-Tian construction of the virtual fundamental class and a key lemma about the associativity of the refined Gysin maps.

The virtual fundamental class of a moduli functor is constructed using solely a choice of a tangent-obstruction complex. Our interest here is the moduli functor \mathcal{F}_X^d of 0-pointed, genus zero, degree d stable maps to X . We describe the natural tangent-obstruction complex of \mathcal{F}_X^d . Let $\eta \in \mathcal{F}_X^d(S)$ be

represented by the following diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & X \\ \downarrow \pi & & \\ S & & \end{array}$$

The deformations and obstructions of η are described respectively by the global sections of the sheaves $\mathcal{T}^1\mathcal{F}_X^d(\eta) := \mathcal{E}xt_{\mathcal{X}/S}^1([f^*(\Omega_X) \rightarrow \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$ and $\mathcal{T}^2\mathcal{F}_X^d(\eta) := \mathcal{E}xt_{\mathcal{X}/S}^2([f^*(\Omega_X) \rightarrow \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$. The natural tangent obstruction-complex for this moduli problem is $\mathcal{T}_\eta^\bullet := [\mathcal{T}^1\mathcal{F}_X(\eta) \rightarrow \mathcal{T}^2\mathcal{F}_X(\eta)]$ with the zero arrow. This complex is *perfect* in the sense that locally, there is a 2-term complex of locally free sheaves $\mathcal{E}_\eta^\bullet := [\mathcal{E}_{\eta,1} \rightarrow \mathcal{E}_{\eta,2}]$ whose sheaf cohomology yields the tangent-obstruction complex $\mathcal{H}^\bullet\mathcal{E}_\eta^\bullet = \mathcal{T}_\eta^\bullet$.

The *virtual fundamental class* of \mathcal{T}_η^\bullet is denoted here by $[\overline{M}_{g,n}(X, d)]^{\text{vir}}$. It is a Chow class in the Chow group of the coarse moduli space $\overline{M}_{g,n}(X, d)$. If $\overline{M}_{g,n}(X, d)$ is an orbifold, its virtual fundamental class corresponds to $1 \in H^*(\overline{M}_{g,n}(X, d))$ under Poincaré duality .

The key to the proof of Theorem 0.1.1 is a lemma about the associativity of the refined Gysin maps. Let us first formulate it for representable functors. Consider a fibre diagram

$$\begin{array}{ccc} W_0 & \xrightarrow{\delta_0} & W \\ \downarrow \alpha_0 & & \downarrow \alpha \\ T_0 & \xrightarrow{\delta} & T \end{array} \quad (3)$$

where δ is a regular embedding. Let \mathcal{N} be the normal bundle of T_0 in T . Assume that W and W_0 admit perfect tangent obstruction-complexes \mathcal{T}_W^\bullet and $\mathcal{T}_{W_0}^\bullet$. They are said to be *compatible* relative to the fibre diagram (3) if for each affine scheme S and for any morphism $\eta : S \rightarrow W_0 \subset W$ there is an exact sequence

$$0 \rightarrow \mathcal{T}_{W_0}^1(\eta) \rightarrow \mathcal{T}_W^1(\eta) \rightarrow (\alpha_0 \circ \eta)^*\mathcal{N} \rightarrow \mathcal{T}_{W_0}^2(\eta) \rightarrow \mathcal{T}_W^2(\eta) \rightarrow 0. \quad (4)$$

Assume that this compatibility satisfies a *technical condition*. Namely, there exists a short exact sequence of 2-term complexes

$$0 \rightarrow [0 \rightarrow \alpha_0^*\mathcal{N}] \rightarrow \tilde{\mathcal{E}}_\eta^\bullet \rightarrow \mathcal{E}_\eta^\bullet \rightarrow 0 \quad (5)$$

such that

- Its long exact sequence of cohomologies is precisely the exact sequence of the compatibility.
- The cohomologies of $\tilde{\mathcal{E}}_\eta^\bullet$ and \mathcal{E}_η^\bullet yield the tangent-obstruction complex of W_0 .

Proposition 2.3.1. *(Proposition 3.9 of [15]) Assume that \mathcal{T}_W^\bullet and $\mathcal{T}_{W_0}^\bullet$ are compatible and the technical condition (5) is satisfied. Then*

$$\delta^! [W]^{\text{vir}} = [W_0]^{\text{vir}}$$

where the virtual cycles are with respect to \mathcal{T}_W^\bullet and $\mathcal{T}_{W_0}^\bullet$.

The case of interest for us is the localized top chern class [7] of (E, s) , where $E \rightarrow Z$ is a vector bundle and s is a section of E . We use diagram (3) with $T_0 = W = Z$ and T the total space of E . Let $\delta = s_E$ be the zero section of E and $\alpha = s$. It follows that $W_0 = Z(s)$ is the zero locus of s .

Corollary 2.3.1. *With the assumptions of the previous proposition*

$$\alpha_{0*} [Z(s)]^{\text{vir}} = c_{\text{top}}(E) \cap [Z]^{\text{vir}} \tag{6}$$

3 The Proofs

3.1 The associativity of the refined Gysin maps in the category of \mathbb{Q} -schemes.

Let (V_α, G_α) be a \mathbb{Q} -scheme with a perfect tangent-obstruction \mathbb{Q} -complex \mathcal{T}^\bullet which is the cohomology of the \mathbb{Q} -complex \mathcal{E}^\bullet . It has been pointed out in [15] that the virtual fundamental class in the category of \mathbb{Q} -schemes can be constructed as follows: First, one constructs a local virtual cone $C^{\mathcal{E}_\alpha^\bullet}$ using the tangent-obstruction complex $\mathcal{H}^\bullet \mathcal{E}_\alpha^\bullet = \mathcal{T}_\alpha^\bullet$ on V_α . This local virtual cone sits inside the vector bundle $\text{Spec Sym}^\bullet(\mathcal{E}_{2,\alpha}^*)$. The key here is that the local virtual cones do not depend on \mathcal{E}^\bullet but only on \mathcal{T}^\bullet (Lemma 3.2 of [15]). Obviously the restrictions of $\mathcal{E}_\alpha^\bullet$ and $\mathcal{E}_\beta^\bullet$ to $V_{\alpha,\beta}$ yield the same tangent-obstruction complex $\mathcal{T}_{\alpha,\beta}^\bullet$. It follows that the pull backs of $C^{\mathcal{E}_\alpha^\bullet}$ and $C^{\mathcal{E}_\beta^\bullet}$ to $V_{\alpha,\beta}$ are equivariantly isomorphic. The G_α -quotients patch together into the virtual \mathbb{Q} -cone $C^{\mathcal{E}^\bullet}$ inside the \mathbb{Q} -vector bundle $\text{Spec Sym}^\bullet(\mathcal{E}_2)$. Finally the virtual fundamental class is the pull back via the zero section of $[C^{\mathcal{E}^\bullet}]$.

It is known that $\overline{M}_{0,0}(X, d)$ has a \mathbb{Q} -scheme structure so that the functors $\text{Hom}(-, \overline{M}_{0,0}(X, d))$ and \mathcal{F}_X^d are equivalent. Here is a brief sketch. Let

$(C, f) \in \overline{M}_{0,0}(X, d)$ be a stable map. Choose divisors H_1, H_2, \dots, H_r so that f intersects each H_i transversally at y_{i1}, \dots, y_{id_i} (where $d_i := d \cdot H_i$) and (\tilde{C}, y_{ij}) has no automorphisms. Now $\text{Aut}(f)$ acts on $(\tilde{C}, f) \in \overline{M}_{0,dr}(X, d) \cap e_{ij}^*(H_i)$ by permuting the markings. Choose a quasiprojective $\text{Aut}(f)$ -equivariant neighborhood

$$U_f \subset \overline{M}_{0,dr}(X, d) \cap e_{ij}^*(H_i)$$

of (\tilde{C}, f) such that all the stable maps in U_f have no automorphisms and the map that forgets the markings does not change the source curve. There is an action of $\text{Aut}(f)$ on the universal stable map $\eta_f := (\mathcal{C}_f, F_f)$ over U_f . The classifying map $U_f/\text{Aut}(f) \rightarrow \overline{M}_{0,0}(X, d)$ is étale. The neighborhoods U_f satisfy the conditions of a \mathbb{Q} -scheme.

Now, let $E \rightarrow X$ be a convex vector bundle. Recall that the vector spaces $H^0(f^*(E))$ fit into a \mathbb{Q} -vector bundle E_d on $\overline{M}_{0,0}(X, d)$. The section s of E induces a section \tilde{s} of E_d over $\overline{M}_{0,0}(X, d)$ via $\tilde{s}((C, f)) = s \circ f$. Let $(C, f) \in Z(\tilde{s})$. By shrinking U_f if necessary, the group $\text{Aut} f$ acts on the restriction of the universal family (\mathcal{C}_f, F_f) over $U_f^s := Z(\tilde{s}) \cap U_f$ and the classifying map $U_f^s/\text{Aut}(f) \rightarrow Z(\tilde{s})$ is again étale. Just as in the construction of the virtual class of a \mathbb{Q} -scheme, the local Gysin diagrams (3) patch to a global Gysin diagram and that, with the definitions of the section 1.2, Corollary 2.3.1 holds in the category of \mathbb{Q} -schemes. The technical assumption is the same; the only difference is that the complexes become \mathbb{Q} -complexes.

3.2 Virtual class of the zero loci

The proof of theorem 0.1.1 uses corollary 2.3.1. We need to check that the *technical condition* is satisfied. Let η be a 0-pointed, genus zero stable map of class d over an affine scheme S represented by the following diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{f} & Y \subset X \\ \downarrow \pi & & \\ S & & \end{array}$$

The deformations and obstructions of η are described respectively by the global sections of the sheaves $\mathcal{T}^1 \mathcal{F}_X^d(\eta) := \mathcal{E}xt_{\mathcal{X}/S}^1([f^*(\Omega_X) \rightarrow \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$ and $\mathcal{T}^2 \mathcal{F}_X^d(\eta) := \mathcal{E}xt_{\mathcal{X}/S}^2([f^*(\Omega_X) \rightarrow \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$. Since the normal bundle of Y in X is $E|_Y$ the conormal exact sequence writes

$$0 \rightarrow f^*(E^*|_Y) \rightarrow f^*(\Omega_X|_Y) \rightarrow f^*(\Omega_Y) \rightarrow 0 \quad (7)$$

Recall from [15] that there is a short exact sequence of $\mathcal{O}_{\mathcal{X}}$ -sheaves

$$0 \rightarrow W_2 \rightarrow W_1 \rightarrow f^*\Omega_X \rightarrow 0 \quad (8)$$

such that:

- $\mathcal{E}xt_{\mathcal{X}/S}^i([W_1 \rightarrow \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$ and $\mathcal{E}xt^i([W_2 \rightarrow 0], \mathcal{O}_{\mathcal{X}})$ vanish for $i \neq 1$.
- Both $\mathcal{E}_{\eta,1} = \mathcal{E}xt_{\mathcal{X}/S}^1([W_1 \rightarrow \Omega_{\mathcal{X}/S}], \mathcal{O}_{\mathcal{X}})$ and $\mathcal{E}_{\eta,2} = \mathcal{E}xt^1([W_2 \rightarrow 0], \mathcal{O}_{\mathcal{X}})$ are locally free.
- The sheaf cohomology of the complex $\mathcal{E}_{\eta}^{\bullet} = [\mathcal{E}_{\eta,1} \rightarrow \mathcal{E}_{\eta,2}]$ is the tangent-obstruction complex of the stable map η , i.e. there is an exact sequence

$$0 \rightarrow \mathcal{T}^1\mathcal{F}_X^d(\eta) \rightarrow \mathcal{E}_{\eta,1} \rightarrow \mathcal{E}_{\eta,2} \rightarrow T^2\mathcal{F}_X^d(\eta) \rightarrow 0.$$

We pull back (8) exact sequence via (7) and obtain the following diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & W_2 & \longrightarrow & \mathcal{A} & \longrightarrow & f^*(E^*) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & W_2 & \longrightarrow & W_1 & \longrightarrow & f^*(\Omega_X|_Y) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & 0 & \longrightarrow & f^*(\Omega_Y) & \longrightarrow & f^*(\Omega_Y) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Let $\tilde{\mathcal{E}}_{\eta,2} := \mathcal{E}xt_{\mathcal{X}/S}^1([\mathcal{A} \rightarrow 0], \mathcal{O}_{\mathcal{X}})$. We apply the long exact sequence for $\mathcal{E}xt$ to various exact sequences obtained from the above diagram. The middle vertical sequence yields a short exact sequence

$$0 \rightarrow [\mathcal{A} \rightarrow 0] \rightarrow [W_1 \rightarrow \Omega_{\mathcal{X}/S}] \rightarrow [f^*(\Omega_Y) \rightarrow \Omega_{\mathcal{X}/S}] \rightarrow 0.$$

Its long exact sequence for $\mathcal{E}xt$ yields

$$0 \rightarrow \mathcal{T}^1\mathcal{F}_Y^{\beta}(\eta) \rightarrow \mathcal{E}_{\eta,1} \rightarrow \tilde{\mathcal{E}}_{\eta,2} \rightarrow T^2\mathcal{F}_Y^{\beta}(\eta) \rightarrow 0.$$

Let $\tilde{\mathcal{E}}_{\eta}^{\bullet} := [\mathcal{E}_{\eta,1} \rightarrow \tilde{\mathcal{E}}_{\eta,2}]$. The last exact sequence says that the cohomology of $\tilde{\mathcal{E}}_{\eta}^{\bullet}$ is the tangent obstruction complex of η . Next, we apply the long

exact sequence for $\mathcal{E}xt$ to the top horizontal row of the diagram and use the conditions for W_i 's. We obtain

$$\begin{aligned} 0 \rightarrow \mathcal{E}xt_{\mathcal{X}/S}^1([f^*(E^*) \rightarrow 0], \mathcal{O}_{\mathcal{X}}) \rightarrow \tilde{\mathcal{E}}_{\eta,2} \rightarrow \\ \mathcal{E}_{\eta,2} \rightarrow \mathcal{E}xt_{\mathcal{X}/S}^2([f^*(E^*) \rightarrow 0], \mathcal{O}_{\mathcal{X}}). \end{aligned}$$

But one easily sees that

$$\mathcal{E}xt_{\mathcal{X}/S}^2([f^*(E^*) \rightarrow 0], \mathcal{O}_{\mathcal{X}}) \simeq \mathcal{E}xt^1(f^*(E^*), \mathcal{O}_{\mathcal{X}}) = 0$$

and

$$\mathcal{E}xt_{\mathcal{X}/S}^1([f^*(E^*) \rightarrow 0], \mathcal{O}_{\mathcal{X}}) \simeq \pi_* f^*(E)$$

as \mathcal{O}_S -sheaves. It follows that there is an exact sequence

$$0 \rightarrow [0 \rightarrow \pi_*(f^*(E))] \rightarrow \tilde{\mathcal{E}}_{\eta}^{\bullet} \rightarrow \mathcal{E}_{\eta}^{\bullet} \rightarrow 0. \quad (9)$$

Its long exact sequence of sheaf cohomologies is easily seen to be

$$0 \rightarrow \mathcal{T}^1 \mathcal{F}_Y^{\beta}(\eta) \rightarrow \mathcal{T}^1 \mathcal{F}_X^d(\eta) \rightarrow \pi_* f^* E \rightarrow \mathcal{T}^2 \mathcal{F}_Y^{\beta}(\eta) \rightarrow \mathcal{T}^2 \mathcal{F}_X^d(\eta) \rightarrow 0. \quad (10)$$

The technical condition is satisfied.

3.3 Mirror Theorems

We recall the setup. Let i denote the embedding of Y in X . Assume that the map $i_* : H_2(Y) \rightarrow H_2(X)$ is surjective. Complete a basis $\{T'_1, T'_2, \dots, T'_r\}$ of $H^2 X$ to a basis $\{T_1 := i^*(T'_1), \dots, T_r := i^*(T'_r), T_{r+1}, \dots, T_m\}$ of $H^2 Y$. Let $tT = \sum_{i=1}^m t_i T_i$ and $tT' = \sum_{i=1}^r t_i T'_i$ where t_1, \dots, t_n are variables. The map $i_* : H_* Y \rightarrow H_* X$ extends to a homomorphism of completions

$$i_* : H_* Y[[t_1, \dots, t_m, \hbar^{-1}]][[q^{\beta}]] \rightarrow H_* X[[t_1, \dots, t_r, \hbar^{-1}]][[q^d]]$$

via $i_*(t_k) = 0$ for $k > r$ and $i_*(q^{\beta}) = q^{i_*(\beta)}$.

Theorem 3.3.1. *Assume that $i_* : H_2(Y) \rightarrow H_2(X)$ is surjective. Then $i_*(J_Y) = J_E$.*

Proof. Recall that

$$J_Y := e^{\frac{tT}{\hbar}} \cap \left(1 + \sum_{\beta \neq 0} q^{\beta} e_* \left(\frac{[\overline{M}_{0,1}(Y, \beta)]^{\text{vir}}}{\hbar(\hbar - c)} \right) \right)$$

By the definition of i_* and the projection formula we obtain

$$i_*(J_Y) = e^{\frac{iT'}{\hbar}} \cap \left(e_*(1) + \sum_{\beta \neq 0} q^\beta i_* \left(e_* \left(\frac{[\overline{M}_{0,1}(Y, \beta)]^{\text{vir}}}{\hbar(\hbar - c)} \right) \right) \right)$$

As we have said before any $\beta \in H_2(Y, \mathbb{Z})$ such that $i_*(\beta) = d$ induces a morphism $i_\beta : \overline{M}_{0,0}(Y, \beta) \rightarrow \overline{M}_{0,0}(X, d)$. Consider the following commutative diagram:

$$\begin{array}{ccc} \overline{M}_{0,1}(Y, \beta) & \xrightarrow{i_\beta} & \overline{M}_{0,1}(X, d) \\ \downarrow e & & \downarrow e \\ Y & \xrightarrow{i} & X \end{array}$$

The line bundle \mathcal{L}_1 on $\overline{M}_{0,1}(Y, \beta)$ is the pullback via i_β of the bundle on $\overline{M}_{0,1}(X, d)$. By the projection formula and Theorem 2.0.1:

$$\begin{aligned} i_* \left(\sum_{i_*(\beta)=d} e_* \left(\frac{[\overline{M}_{0,1}(Y, \beta)]^{\text{vir}}}{\hbar(\hbar - c)} \right) \right) &= e_* \left(\frac{\sum_{i_*(\beta)=d} (i_\beta)_* [\overline{M}_{0,0}(Y, \beta)]^{\text{vir}}}{\hbar(\hbar - c)} \right) \\ &= e_* \left(\frac{c_{\text{top}}(E_d) \cap [\overline{M}_{0,0}(X, d)]^{\text{vir}}}{\hbar(\hbar - c)} \right) \end{aligned} \quad (11)$$

The exact sequence (2) implies

$$c_{\text{top}}(E_d) = c_{\text{top}}(E'_d) e^*(c_{\text{top}}(E)) \quad (12)$$

hence by the projection formula

$$\begin{aligned} &i_* \left(\sum_{i_*(\beta)=d} e_* \left(\frac{[\overline{M}_{0,1}(Y, \beta)]^{\text{vir}}}{\hbar(\hbar - c)} \right) \right) \\ &= c_{\text{top}}(E) \cap e_* \left(\frac{c_{\text{top}}(E'_d) \cap [\overline{M}_{0,0}(X, d)]^{\text{vir}}}{\hbar(\hbar - c)} \right) \end{aligned}$$

Note also that $i_*(1) = i_*([Y]) = c_{\text{top}}(E) \cap [X]$. Now the theorem follows readily. \square

References

- [1] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math., 128(1):45-88, 1997.

- [2] A. Bertram, *Another way to enumerate rational curves with torus actions*, Invent. Math. **142** (2000) 487-512.
- [3] T.-M. Chiang, A. Klemm, S.-T. Yau, and E. Zaslow, *Local Mirror Symmetry: Calculations and Interpretations*, Adv. Theor. Math. Phys. **3** (1999), 495-565.
- [4] D. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*, Mathematical Surveys and Monographs **68**, AMS, Providence, RI, 1999.
- [5] D. Cox, S. Katz, Y-P. Lee, *Virtual fundamental classes of zero loci*, in *Advances in algebraic geometry motivated by physics (Lowell, MA, 2000)*, Contemp. Math. **276**, Amer. Math. Soc., Providence, RI, 2001, 157-166.
- [6] A. Elezi, *Mirror symmetry for concave vector bundles on projective spaces*, International Journal of Mathematics and Mathematical Sciences **3** (2003), 159-197.
- [7] W. Fulton, *Intersection theory*, Springer-Verlag, New York-Berlin Heidelberg, 1984.
- [8] A. Givental, *A mirror theorem for toric complete intersections*, in *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, Progr. Math., **160**, Birkhäuser, 1998 141-175.
- [9] A. Givental, *Equivariant Gromov-Witten invariants*, Int. Math. Res. Notices **13** (1996), 613-663.
- [10] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. **138** (1999), 495-536.
- [11] B. Kim, A. Kresch and T. Pantev, *Functoriality in intersection theory and a conjecture of Cox, Katz, and Lee*, J. Pure Appl. Algebra **179** (April 2003), no. 1.
- [12] G. Laumon and L. Moret-Bailly, *Champs Algébriques*, Springer Verlag, 2000.
- [13] B. Lian, K. Liu, and S.-T. Yau, *Mirror principle I*, Asian J. Math. Vol. 1, no. 4 (1997), 729-763.
- [14] B. Lian, K. Liu, and S.-T. Yau, *Mirror principle: A Survey*, Current Development in Mathematics, 1998, International Press, Cambridge MA, 1998, 35-65.
- [15] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, Jour. AMS **11** (1998), no. 1, 119-174.

- [16] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and Geometry II, Progress in Mathematics 36 (1983), 271-326.
- [17] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Inventiones Math. 97 (1989), 613-670.