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# *Existence of Multidimensional Traveling Waves in the Transport of Reactive Solutes through Periodic Porous Media*

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**Abstract**

We prove the existence of multidimensional traveling-wave solutions to the scalar equation for the transport of solutes (contaminants) with nonlinear adsorption and spatially periodic convection-diffusion-adsorption coefficients under the assumption that the nonlinear adsorption function satisfies the Lax and Oleinik entropy conditions. In the nondegenerate case, we also prove the uniqueness of the traveling waves. These traveling waves are analogues of viscous shock profiles. They propagate with effective speeds that depend on the periodic porous media only up to their mean states, and are given by an averaged Rankine-Hugoniot relation. This is a direct consequence of the fact that the transport equation is in conservation form. We use the sliding domain method, the continuation method, spectral theory, maximum principles, and a priori estimates. In the degenerate case, the traveling waves are weak solutions of a degenerate parabolic equation and are only Hölder continuous. We obtain them by taking suitable limits on the nondegenerate traveling waves. The uniqueness of the degenerate traveling waves is open.

## 1. Introduction

Reactive solute transport in porous media is a complicated physical-chemical process describing, for instance, the mobility of pollutants in soil and underground water systems. It is of both theoretical and practical interest to understand the basic phenomena of this process in view of its applications in predicting the movement of contaminants in the environment and maintaining groundwater quality. One of the major chemical aspects in the transport of reactive solutes is adsorption or ion exchanges on the surface of the porous skeleton. Nonlinear adsorption makes the solute concentration behave like wave fronts and not spread diffusively, as it does in the linear and zero-adsorption cases. Mathematical models are based on mass conservation and some chemical reaction laws; see BOLT [7], VAN DUIJN & KNABNER [15], GRUNDY, VAN DUIJN, & DAWSON [19], and references therein. If we let $C$ be the concentration of a one-species solute, $S$ the absorbed concentration, $D$ the pore-scale dispersion matrix, $v$ the incompressible water flow velocity, $\theta$ volumetric water content, and $\rho$ the bulk density, then mass conservation yields

$$\frac{\partial}{\partial t}[\theta C + \rho S] = \nabla \cdot [\theta D \nabla C - vC]. \tag{1.1}$$

The reaction laws give

$$S_t = \kappa f(C, S), \tag{1.2}$$

where $\kappa$ is the reaction rate, and $f(C, S) = \psi(C) - S$, where $\psi'(C) > 0$, $\psi$ is smooth on $(0, +\infty)$. The function $\psi$ is called the adsorption isotherm. The isotherms are classified as follows:

(A) $\psi$ is of *Langmuir type* (*type* (L)) if $\Phi$ is strictly concave near $C = 0$, and $\Phi'(0+) < +\infty$.
(B) $\psi$ is of *Freundlich type* (*type* (F)) if $\Phi$ is strictly concave near $C = 0$, $\Phi'(0+) = +\infty$.
(C) $\psi$ is of *convex type* (*type* (S)) if $\psi$ is strictly convex near $C = 0$.

Some well-known isotherms are (A) the Langmuir isotherm: $\psi(C) = \dfrac{\kappa_1 C}{1 + \kappa_2 C}$, $\kappa_1$, $\kappa_2 > 0$, (B) the Freundlich isotherm: $\psi(C) = \kappa_3 C^p$, $0 < p < 1$, $\kappa_3 > 0$, (C) a typical type S isotherm: $\psi(C) = \kappa_4 C^p$, $p > 1$, $\kappa_4 > 0$.

In the case of fast reaction where $\kappa \to +\infty$, (1.1) and (1.2) relax to the scalar equation

$$[\theta C + \rho\psi(C)]_t = \nabla \cdot [\theta D \nabla C - vC] \tag{1.3}$$

up to the leading-order asymptotics. For a convergence study, see VAN DUIJN & KNABNER [16].

Traveling-wave solutions of (1.3) are known to exist in the constant-coefficient case and are asymptotically stable; see OSHER & RALSTON [25]. Existence and uniqueness of traveling-wave solutions in one space dimension to the constant-coefficient system (1.1) and (1.2) are studied in VAN DUIJN & KNABNER [16].

However, in reality, soils and groundwater systems are heterogenous due to the presence of macropores, aggregates, cracks, etc.; see VAN DER ZEE & VAN RIEMSDIJK [34] and references therein. Without loss of generality, we can assume that $\theta$ and $\rho$ are constants and equal to 1, that $D = D(x)$, $v = v(x)$ are stationary ergodic random fields, and that $\psi = k(x)f(c)$, $k(x) > 0$, a stationary ergodic random field. This leads us to

$$u_t + (k(x)f(u))_t = \nabla \cdot (a(x)\nabla u) + b(x) \cdot \nabla u, \tag{1.4}$$

where we have changed notations from $C$ to $u$, $D$ to $a$, and $-v$ to $b$. When $f = 0$, (1.4) reduces to the well-known advection-diffusion equation for nonreacting solute transport. When $a$ = constant positive-definite matrix, $b$ = negative constant, and $x \in R^1$, equation (1.4) becomes that in VAN DER ZEE & VAN RIEMSDIJK [34] and BOSMA & VAN DER ZEE [8] for the transport of reacting solutes in a one-dimensional chemically heterogenous porous medium. For numerical simulations of the traveling-wave solutions and their statistical properties, we refer to [34] and [8].

In this paper, we show the existence and uniqueness of classical traveling-wave solutions to equation (1.4) under the following conditions:

(H1) $a(x)$ is a smooth symmetric positive-definite matrix in $x$, and has period 1 in each component of $x = (x_1, x_2, \ldots, x_n)$,

(H2) $b(x)$ is a smooth divergence-free vector field with $\langle b \rangle \equiv \int_{T^n} b(x)dx \equiv b_0 = (b_0^{(1)}, b_0^{(2)}, \ldots, b_0^{(n)}) \neq 0$, and has period 1 in each component of $x$ (here $T^n$ is the unit $n$-dimensional torus).

(H3) $k(x)$ is a positive smooth function and has period 1 in each component of $x$,

(H4) $f = f(u)$ is of type (L) or (S), $f$ is smooth on $[0, \infty)$, $f(0) = 0$, and $f'(u) > 0$, for all $u > 0$.

Our main results are

**Theorem 1.1** (Existence and Uniqueness). *Suppose that* (H1)–(H4) *hold with $f$ of type* (S) *and that $\langle b \rangle \cdot e < 0$, where $e$ is a unit vector in $R^n$. Let $u_l$ and $u_r$ be any two nonnegative constants such that $0 \leqq u_l < u_r$. If*

(L) $$f'(u_l) < \frac{f(u_r) - f(u_l)}{u_r - u_l} < f'(u_r),$$

(O) $$\frac{f(u) - f(u_l)}{u - u_l} < \frac{f(u_r) - (u_l)}{u_r - u_l}$$

*for all $u \in (u_l, u_r)$, then there exists a classical traveling-wave solution to equation* (1.4) *of the form $u = U(e \cdot x - ct, x) \equiv U(s, y)$, where $s = e \cdot x - ct$, $y = x$, $c$ is the wave speed along direction $e$, $U(-\infty, y) = u_l$, $U(+\infty, y) = u_r$, and $U(s, \cdot)$ has period* 1. *Such solutions are unique up to constant translations in $s$, and satisfy*

$$u_l < U(s, y) < u_r \quad \forall (s, y) \in R^1 \times T^n, \tag{1.5}$$

$$U_s > 0 \quad \forall (s, y) \in R^1 \times T^n, \tag{1.6}$$

$$c = c_{\text{eff}} \equiv \frac{-\langle b \cdot e \rangle (u_l - u_r)}{u_l + \langle k \rangle f(u_l) - (u_r + \langle k \rangle f(u_r))} > 0. \tag{1.7}$$

**Theorem 1.2** (Existence and Uniqueness). *Suppose that* (H1)–(H4) *hold with $f$ of type* (L) *and that $\langle b \rangle \cdot e < 0$, where $e$ is a unit vector in $R^n$. Let $u_l$ and $u_r$ be any two nonnegative numbers such that $0 \leqq u_r < u_l$. If*

(L) $$f'(u_r) > \frac{f(u_r) - f(u_l)}{u_r - u_l} > f'(u_l),$$

(O) $$\frac{f(u) - f(u_r)}{u - u_r} > \frac{f(u_r) - f(u_l)}{u_r - u_l}$$

*for any* $u \in (u_r, u_l)$, *then there exists a classical traveling-wave solution to* (1.4) *of the form* $u = U(e \cdot x - ct, x) \equiv U(s, y)$, $s \equiv e \cdot x - ct, y \equiv x$, $U(-\infty, y) = u_l$, $U(+\infty, y) = u_r$, *and* $U(s, \cdot)$ *has period* 1. *Such solutions are unique up to constant translations in s and satisfy*

$$u_r < U(s, y) < u_l \quad \forall (s, y) \in R^1 \times T^n, \tag{1.8}$$

$$U_s < 0 \quad \forall (s, y) \in R^1 \times T^n, \tag{1.9}$$

$$c = c_{\text{eff}} = \frac{-\langle b \cdot e \rangle (u_l - u_r)}{u_l + \langle k \rangle f(u_l) - (u_r + \langle k \rangle f(u_r))} > 0. \tag{1.10}$$

*Remark 1.1.* The conditions (L) and (O) are exactly the Lax and Oleinik entropy conditions for admissible shock solutions of scalar conservation laws of the form $u_t + f(u)_x = 0$; see LAX [22]. They are also the conditions for the existence of traveling-wave solutions (or viscous shock waves) for the scalar conservation laws $u_t + f(u)_x = u_{xx}$ with viscosity. In fact, equation (1.4) can be formally transformed into this standard conservation form by making the change of variable $v = u + k(x) f(u)$.

*Remark 1.2.* The explicit formulas (1.7) and (1.10) are analogues of the Rankine-Hugoniot relations of conservation laws; see LAX [22]. We notice that the wave speeds depend on the periodic porous media only in terms of their means. The higher moments of the media only affect the wave shapes. In the sorption technology, the wave shapes also play an important role; they are called "breakthrough curves"; see TONDEUR, GORIUS, & BAILLY [28] for the details of their applications. The simple explicit formulas (1.7) and (1.10) are direct consequences of the fact that (1.4) is in conservation form. This is in marked contrast with reaction-diffusion waves, which propagate through inhomogeneous periodic media with effective speeds that depend on all moments of the media. Moreover, there is no explicit formula at present for the effective wave speeds in the bistable and combustion cases; see PAPANICOLAOU & XIN [26], XIN [30–32]. So nonconservative laws, e.g., reaction-diffusion laws, make the wave speeds more complicated in the presence of inhomogeneities.

*Remark 1.3.* Theorems like Theorem 1.1 (or 1.2) can be shown for scalar conservation laws of the form $u_t + \nabla \cdot (f(u, x)) = \Delta u$, where $f$ has period 1 in $x$ and satisfies suitable entropy conditions, and the conservation laws admit constant steady states $u_l$ and $u_r$.

Theorem 1.2 also holds for $f$ of type (F) if $u_r > 0$. If $u_r = 0$, (1.4) becomes a degenerate parabolic equation near $u = u_r = 0$, and classical solutions cease to exist. There is a rich literature on weak solutions of degenerate parabolic equations including equations of porous-medium type such as (1.4), and on the regularity of the solutions; see [1, 9–14], among others. Following the definition for local weak solutions in [14], we show that traveling-wave solutions to equation (1.4) exist as such local weak solutions by taking the limit $\varepsilon \to 0$ on the nondegenerate traveling-wave solutions connecting $u_l$ to $\varepsilon$. Our main results are

**Theorem 1.3** (Existence). *Suppose that* (H1)–(H3) *hold, that* $f(u) = u^p$, $p \in (0, 1)$, *and that* $\langle b \rangle \cdot e < 0$, *where e is a unit vector in* $R^n$. *Let* $u_l$ *be any positive number such that* $u_l > 0$. *Then there exists a local weak traveling-wave solution to* (1.4) *of the form* $u = U(e \cdot x - ct, x) \equiv U(s, y)$, $s \equiv e \cdot x - ct$, $y \equiv x$, $U(-\infty, y) = u_l$, $U(+\infty, y) = 0$, *and* $U(s, \cdot)$ *has period* 1. *Such solutions are Hölder continuous in s and y and satisfy*

$$0 \leqq U(s, y) < u_l \quad \forall (s, y) \in R^1 \times T^n, \tag{1.11}$$

$$U(s_1, y) \leqq U(s_2, y) \quad \forall s_1 \geqq s_2 \in R^1, \; y \in T^n, \tag{1.12}$$

$$U_s < 0 \quad \textit{if } U(s, y) > 0, \tag{1.13}$$

$$c = c_{\text{eff}} = \frac{-\langle b \cdot e \rangle}{1 + \langle k \rangle u_l^{p-1}}. \tag{1.14}$$

*Remark 1.4.* Due to the lack of a strong maximum principle, we do not know the uniqueness (up to constant translations in $s$) of these degenerate traveling waves. In Theorem 1.3, for ease of presentation, we choose to consider a special form $f(u) = u^p$, $p \in (0, 1)$ of type (F) function.

*Remark 1.5.* In the constant-coefficient case, the traveling-wave solutions are finite waves in the sense that $U(s) \equiv 0$ if $s \geqq s_0$ for some $s_0 \in R^1$. We are unable to prove this for the periodic case; instead, we show that $\langle U(s, y) \rangle$, the average of $U$ in $y$, decays to zero exponentially as $s \to +\infty$. The finiteness is due to degeneracy in (1.4). As pointed out in [15, part two, page 211], the difference between finiteness and nonfiniteness in practical terms is not very large, and from the computations the distinction is hard to make.

Although the more realistic assumptions on $a$, $b$, and $k$ in (1.4) are ergodic stationary random fields, the periodic conditions are technically much simpler and still physically relevant. Understanding nonlinear waves in periodic media in a rigorous way is a first step towards approaching problems on nonlinear waves in random media, which will be the topic of my further investigation.

The rest of the paper is organized as follows. In Section 2, we derive the nonlinear eigenvalue problem for the traveling waves, which is a degenerate elliptic (essentially parabolic) problem on the infinite cylinder $R^1 \times T^n$. We study various properties of its solutions, in particular, the monotonicity property as shown in (1.6) and (1.9). Our principal tools are the maximum principle, and the sliding-domain method. The sliding-domain method was developed by Berestycki & Nirenberg, [4–6], and Li [23]. It has been successfully applied to the study of reaction-diffusion waves in periodic media in Xin [30–32]. The Lax entropy condition is crucial for carrying out the sliding-domain method and establishing the monotonicity of solutions. In Section 3, we construct a solution to the elliptic regularization of the problem by using the continuation method. We analyze the spectrum of the linearized operator around an arbitrary traveling-wave solution, and show that it is a Fredholm operator of index zero. The proof of this relies on the spectral theorems of GOHBERG & KREIN [18] and KATO [21]. The openness of the set of continuation parameters for which there is existence then follows. The monotonicity property is essential in proving that the linearized operator has a simple eigenvalue zero corresponding to eigenfunction $U_s$. In Section 4, we show the closedness of the set of continuation parameters for the regularized solutions by examining a sequence of solutions, and using the Oleinik entropy condition prove that the limiting function remains a solution. In Section 5, we remove the regularization and complete the proof of existence for the degenerate problem by using parabolic Schauder estimates and the Oleinik entropy condition. In Section 6, we present the proof of Theorem 1.3 based on results in previous sections, and the proof of Theorem 1.2.

## 2. Uniqueness and Monotonicity of Traveling Waves

Consider the equation for the transport of solutes:

$$u_t + (k(x)f(u))_t = \nabla\cdot(a(x)\nabla u) + b(x)\cdot\nabla u, \tag{2.1}$$

for which (H1)–(H4) hold. Since Theorems 1.1 and 1.2 are similar, we focus on the proof of Theorem 1.1, and so the additional assumptions on $b$ and $f$ in the statement of Theorem 1.1 are valid. We are interested in traveling-wave solutions of the form $u(t, x) = U(e\cdot x - ct, x) \equiv U(s, y)$, where $c$ is the wave speed along direction $e$; $e$ is a unit vector in $R^n$ so that $\langle b\rangle\cdot e < 0$, $U(-\infty, y) = u_l$, $U(+\infty, y) = u_r$, $0 \leqq u_l < u_r < +\infty$. Upon substitution, we have

$$-c(U + k(y)f(U))_s = (e\partial_s + \nabla_y)(a(y)(e\partial_s + \nabla_y)U) + b(y)\cdot(e\partial_s + \nabla_y)U,$$

$$U(-\infty, y) = u_l,\ U(+\infty, y) = u_r,\ U(s,\cdot) \text{ has period } 1. \tag{2.2}$$

The linear terms of (2.2) form a parabolic operator, elliptic in the directions $(e_i, 0, \ldots, 0, y_i, 0, \ldots, 0) \in R^{n+1}$, $i = 1, 2, \ldots, n$, and parabolic in the direction $(1, 0, \ldots, 0) \in R^{n+1}$. Let us first study some basic properties of solutions of (2.2).

**Lemma 2.1.** *Assume that* $(U(s, y), c)$ *is a classical solution to* (2.2) *and that* $U_s$ *decays to zero as* $s \to \infty$ *uniformly in* $y$. *Then* $u_l < U < u_r$ *for all* $(s, y) \in R^1 \times T^n$, *and*

$$c = c_{\text{eff}} = \frac{-\langle b\cdot e\rangle(u_l - u_r)}{u_l + \langle k\rangle f(u_l) - u_r - \langle k\rangle f(u_r)}. \tag{2.3}$$

**Proof.** It follows directly from the maximum principle that $u_l < U < u_r$ for all $(s, y) \in R^1 \times T^n$. The maximum principle for linear operators related to (2.2) is derived in [30]. Averaging (2.2) over $y \in T^n$ gives

$$-c(\langle U\rangle + \langle k(y)f(U)\rangle)_s = e\cdot\langle a(y)(e\partial_s + \nabla_y)U\rangle_s + \langle b(y)\cdot eU\rangle_s. \tag{2.4}$$

Integrating (2.4) over $s$ yields

$$-c(\langle U\rangle + \langle k(y)f(U)\rangle)$$
$$= e\cdot\langle a(y)(e\partial_s + \nabla_y)U\rangle + \langle b(y)\cdot eU\rangle + \text{const.} \tag{2.5}$$
$$= e\cdot\langle a(y)eU_s\rangle - \langle e\cdot(\nabla^T\cdot a(y))U\rangle + \langle b(y)\cdot eU\rangle + \text{const.} \tag{2.6}$$

Letting $s \to \pm\infty$ in (2.6) subtracting the two limiting relations, and using the decay assumption, we get

$$-c(u_l + \langle k\rangle f(u_l) - u_r - \langle k\rangle f(u_r)) = \langle b(y)\cdot e\rangle(u_l - u_r), \tag{2.7}$$

or

$$c = \frac{-\langle b\cdot e\rangle(u_l - u_r)}{u_l + \langle k\rangle f(u_l) - (u_r + \langle k\rangle f(u_r))} > 0.$$

The proof is complete.

**Lemma 2.2.** *Let* $(U, c)$ *satisfy the conditions in Lemma 2.1. Then there exist constants* $s_1 < 0, s_2 > 0, \lambda_1 > 0, \lambda_2 < 0$, *and positive functions* $\Phi_i(y) \in C^2(T^n)$, $i = 1, 2$, *such that*

$$\begin{aligned} U - u_l &\leqq \text{const.}\exp\{\lambda_1 s\}\Phi_1(y), \quad s \leqq s_1, \\ u_r - U &\leqq \text{const. }\exp\{\lambda_2 s\}\Phi_2(y), \quad s \geqq s_2. \end{aligned} \tag{2.8}$$

**Proof.** Write (2.2) as

$$(e\partial_s + \nabla_y)(a(y)(e\partial_s + \nabla_y)U) + b(y)\cdot(e\partial_s + \nabla_y)U + c(U_s + k(y)f'(U)U_s) = 0. \tag{2.9}$$

There exists $s_1$ such that if $s \leqq s_1$ and $U - u_l \leqq \varepsilon$, then $f'(U) - f'(u_l) \leqq L\varepsilon$ for some Lipschitz constant $L$. The function $U - u_l$ satisfies

$$(e\partial_s + \nabla_y)(a(y)(e\partial_s + \nabla_y)V) + b(y)\cdot(e\partial_s + \nabla_y)V + c(1 + k(y)f'(U))V_s = 0, \tag{2.10}$$

where $f'(U)$ is regarded as a part of the coefficients. Let us construct an upper solution for (2.10) on $(-\infty, s_1] \times T^n$. Define

$$L_l(V) \equiv (e\partial_s + \nabla_y)(a(y)(e\partial_s + \nabla_y)V) + b(y)\cdot(e\partial_s + \nabla_y)V$$
$$+ c(1 + k(y)(f'(u_l) + L\varepsilon))V_s. \tag{2.11}$$

Now consider exponential solutions of the form $V_0 \equiv \exp\{\lambda s\}\Phi(y, \lambda), \lambda > 0$, $\Phi(y, \lambda) > 0$, for the equation $L_l(V) = 0$. Upon substitution, we have

$$\nabla_y^T(a\nabla_y\Phi) + 2\lambda(e^T a\nabla_y\Phi) + (b\cdot\nabla_y\Phi) + ((b\cdot e)\lambda$$
$$+ \lambda^2 e^T ae + \lambda\nabla_y^T ae + c\lambda(1 + k(y)(f'(u_l) + L\varepsilon)))\Phi = 0. \tag{2.12}$$

The elliptic operator on the left-hand side of (2.12) has principal eigenvalues $\rho_l = \rho_l(\lambda)$ for positive $\Phi(y, \lambda)$. It is obvious that $\rho_l(\lambda) > 0$ if $|\lambda| \gg 1$ and that $\rho_l(0) = 0$ with $\Phi(y, 0) = 1$. Let us study $\rho_l(\lambda)$ when $0 < \lambda \ll 1$. Replacing the right-hand side of (2.12) by $\rho(\lambda)\Phi$, differentiating the resulting equation with respect to $\lambda$, and setting $\lambda = 0$, we get

$$\nabla_y^T(a\nabla_y\Phi_\lambda)|_{\lambda=0} + 2(e^T a\nabla_y\Phi)|_{\lambda=0} + b\cdot\nabla_y\Phi_\lambda|_{\lambda=0}$$
$$+ [b\cdot e + \nabla_y^T ae + c(1+k(y)(f'(u_l)+L\varepsilon))]\Phi|_{\lambda=0} = \rho_{l,\lambda}\Phi|_{\lambda=0}. \tag{2.13}$$

Since $\Phi(y,\lambda) \to 1$, $\rho_l(\lambda) \to 0$ as $\lambda \to 0$, we simplify (2.13) to find

$$\nabla_y^T(a\nabla_y\Phi_\lambda)|_{\lambda=0} + b\cdot\nabla_y\Phi_\lambda|_{\lambda=0} + b\cdot e$$
$$+ \nabla_y^T ae + c(1+k(y)(f'(u_l)+L\varepsilon)) = \rho_{l,\lambda}|_{\lambda=0}. \tag{2.14}$$

Averaging (2.14) over $y$, we see that

$$\langle b\rangle\cdot e + c(1+\langle k\rangle(f'(u_l)+L\varepsilon)) = \rho_{l,\lambda}|_{\lambda=0}. \tag{2.15}$$

Let us show that if $\varepsilon$ is sufficiently small, then $\rho_{l,\lambda}|_{\lambda=0} < 0$. We use the formula for $c$ in Lemma 2.1 to compute

$$\langle b\cdot e\rangle + c(1+\langle k\rangle f'(u_l))$$
$$= \langle b\cdot e\rangle + \frac{(-\langle b\rangle\cdot e)(u_l-u_r)(1+\langle k\rangle f'(u_l))}{u_l+\langle k\rangle f(u_l)-(u_r+\langle k\rangle f(u_r))}$$
$$= \langle b\cdot e\rangle\langle k\rangle\frac{f(u_l)-f(u_r)+(u_r-u_l)f'(u_l)}{u_l+\langle k\rangle f(u_l)-(u_r+\langle k\rangle f(u_r))}. \tag{2.16}$$

By the Lax entropy condition, $f(u_r)-f(u_l) > f'(u_l)(u_r-u_l)$; thus $f(u_l)-f(u_r)+(u_r-u_l)f'(u_l) < 0$. Now with $u_l+\langle k\rangle f(u_l) < u_r+\langle k\rangle f(u_r)$ and $\langle b\cdot e\rangle < 0$ we see from (2.16) that $\langle b\cdot e\rangle + c(1+\langle k\rangle f'(u_l)) < 0$, and so $\rho_{l,\lambda}|_{\lambda=0} < 0$ if $\varepsilon$ is small enough, depending only on $f(u)$, $\langle k\rangle$, and $\langle b\cdot e\rangle$. By the continuity of $\rho_l$ in $\lambda$, there exists $\lambda_1 > 0$ such that $\rho_l(\lambda_1) = 0$. Denoting the corresponding eigenfunction by $\Phi_1$, we showed that the equation $L_l(V) = 0$ admits positive exponential solutions $V_0 = \exp\{\lambda_1 s\}\Phi_1(y)$. Obviously, $V_{0,s} > 0$. We have

$$(e\partial_s+\nabla_y)(a(y)(e\partial_s+\nabla_y)V_0) + b(y)\cdot(e\partial_s+\nabla_y)V_0 + c(1+k(y)f'(U))V_{0,s}$$
$$= -ck(y)(f'(u_l)+L\varepsilon)V_{0,s} + ck(y)f'(U)V_{0,s}$$
$$= -ck(y)(f'(u_l)+L\varepsilon-f'(U))V_{0,s} \leqq 0. \tag{2.17}$$

Therefore $V_0$ is an upper solution of (2.10), and the first inequality then follows from the maximum principle. The second inequality can be proved similarly by using the other inequality of the Lax entropy condition. The proof is complete.

Lemma 2.2 and the parabolic regularity theory imply

**Corollary 2.1.** *Under the conditions in Lemma 2.2, $U_s$ satisfies*

$$|U_s| \leqq \text{const.}\exp\{-\gamma|s|\} \quad \forall(s,y) \tag{2.18}$$

*for some $\gamma > 0$.*

**Theorem 2.1** (Uniqueness). *Suppose that $(U, c)$ and $(U', C')$ satisfy the conditions in Lemma 2.1. Then $U'(s, y) = U(s + s_0, y)$ for some $s_0 \in R$, and $c' = c$.*

**Proof.** That $c' = c$ follows at once from Lemma 2.1. Let $W(s, y, \lambda) = U(s + \lambda, y) - U'(s, y)$, $\lambda \in R^1$. Lemma 2.2 implies that $W$ decays to zero as $s \to \infty$ exponentially with a rate no less than $\gamma$. The function $W$ satisfies the equation

$$(e\partial_s + \nabla_y)(a(y)(e\partial_s + \nabla_y)W) + b(y)\cdot(e\partial_s + \nabla_y)W \\ + cW_s + ck(y)(f(U) - f(U'))_s = 0,$$

where

$$\begin{aligned}(f(U) - f(U'))_s &= f'(U)U_s - f'(U')U'_s \\ &= f'(U)(U - U')_s + (f'(U) - f'(U'))U'_s \\ &= f'(U)W_s + (f'(U) - f'(U'))U'_s.\end{aligned}$$

Thus $W$ satisfies

$$(e\partial_s + \nabla_y)(a(y)(e\partial_s + \nabla_y)W) + b(y)\cdot(e\partial_s + \nabla_y)W \\ + (c + ck(y)f'(U))W_s + ck(y)(f'(U) - f'(U'))U'_s = 0, \tag{2.19}$$

i.e.,

$$(e\partial_s + \nabla_y)(a(y)(e\partial_s + \nabla_y)W) + b(y)\cdot(e\partial_s + \nabla_y)W \\ + (c + ck(y)f'(U))W_s + ck(y)\beta(s, y, \lambda)U'_sW = 0, \tag{2.20}$$

where

$$\beta(s, y, \lambda) = \int_0^1 f''(\tau U(s + \lambda, y) + (1 - \tau)U'(s, y))d\tau.$$

For any $N_1 > 0$, $N_2 > 0$, there exists $\lambda_0 = \lambda_0(N_1, N_2) > 0$ such that if $\lambda \geqq \lambda_0$, then $W(s, y, \lambda) > 0$ for $(s, y) \in [-N_1, N_2] \times T^n$. Now we choose the sizes of $N_1$ and $N_2$ to prove that $W > 0$ for all $(s, y)$ if $\lambda \geqq \lambda_0$.

Let us set $W \equiv \exp\{\varepsilon s\}\Phi(y)v$, where $\varepsilon > 0$, $\Phi(y) > 0$, are to be chosen. Then $v$ satisfies

$$(e\partial_s + \nabla_y)^T(a(e\partial_s + \nabla_y)v) + \frac{2}{\Phi}\cdot(e\varepsilon\Phi + \nabla_y\Phi)^T a(e\partial_s + \nabla_y)v \\ + (c + ck(y)f'(U))v_s + b\cdot(e\partial_s + \nabla_y)v + (\exp\{\varepsilon s\}\Phi)^{-1}L_2(\exp\{\varepsilon s\}\Phi)W = 0, \tag{2.21}$$

where

$$L_2(\exp\{\varepsilon s\}\Phi) = \exp\{\varepsilon s\}[\nabla_y^T(a\nabla_y\Phi) + 2\varepsilon(\varepsilon^T a\nabla_y\Phi) + b\cdot\nabla_y\Phi \\ + (\varepsilon(b\cdot e) + \varepsilon^2 e^T ae + \varepsilon\nabla_y^T ae \\ + c\varepsilon(1 + k(y)f'(U)) + ck(y)\beta(s, y, \lambda)U'_s)\Phi]. \tag{2.22}$$

Now choose $\exp\{\lambda s\}\Phi$ to be the principal eigenfunction of $L_l$ in (2.11) with $\lambda = \varepsilon$, and $L = 0$, and denote the corresponding principal eigenvalue by $\rho_l = \rho_l(\varepsilon)$. Then (2.22) is just

$$L_2(\exp\{\varepsilon s\}\Phi) = \exp\{\varepsilon s\}\Phi(y)(\rho_l(\varepsilon) + c\varepsilon k(y)(f'(U) - f'(u_l)) + ck(y)\beta(s, y, \lambda)U_s'),$$

and (2.21) becomes

$$(e\partial_s + \nabla_y)^T(a(e\partial_s + \nabla_y)v) + \frac{2}{\Phi}(e\varepsilon\Phi + \nabla_y\Phi)^T a(e\partial_s + \nabla_y)v + (c + ck(y)f'(U))v_s$$
$$+ b\cdot(e\partial_s + \nabla_y)v + (\rho_l(\varepsilon) + c\varepsilon k(y)(f'(U) - f'(u_l)) + ck(y)\beta U_s')v = 0, \quad (2.23)$$

where $\rho_l(\varepsilon) = \rho_l'(0)\varepsilon + O(\varepsilon^2)$, and $\rho_l'(0) < 0$, for $\varepsilon$ small.

Now choose $\varepsilon$ so small that $\rho_l(\varepsilon) \leqq \frac{1}{2}\rho_l'(0)\varepsilon$, and $N_1$ so large that $|U_s'| \leqq \varepsilon^2$, for $(s, y) \in [-\infty, -N_1] \times T^n$, and $|U' - u_l| \leqq \varepsilon^2$ on $(-\infty, -N_1) \times T^n$. This is possible by Corollary 2.1 and Lemma 2.2. Similarly, we can make the transformation $W \equiv \exp\{-\varepsilon s\}\Phi_1(y)v_1$ and choose $\exp\{-\varepsilon s\}\Phi_1$ to be the positive principal eigenfunction of the operator $L_r \equiv L_l$ where $u_r$ replaces $u_l$ and $\rho_r(-\varepsilon)$ is the corresponding eigenvalue. If $\varepsilon$ is small enough (depending only on $f(u)$, $\langle k\rangle$, and $\langle b\cdot e\rangle$), then $\rho_r(-\varepsilon) < -\frac{1}{2}\rho_r'(0)\varepsilon < 0$, with $\rho_r'(0) > 0$. Now choose $N_2 = N_2(\varepsilon)$ so large that $u_r - U(s, y) \leqq \varepsilon^2$, $U_s'(s, y) \leqq \varepsilon^2$ if $s \geqq N_2$. Then $u_r - U(s + \lambda, y) \leqq \varepsilon^2$ for $\lambda \geqq 0$, and $s \geqq N_2$.

Suppose that $\inf_{R^1 \times T^n} W(s, y, \lambda) < 0$. It is clear that

$$\text{either} \quad \inf_{(-\infty, -N_1)\times T^n} W(s, y, \lambda) < 0 \quad \text{or} \quad \inf_{(N_2, +\infty)\times T^n} W(s, y, \lambda) < 0.$$

First assume that $\inf_{(-\infty, -N_1)\times T^n} W(s, y, \lambda) < 0$. Then $\inf_{(-\infty, -N_1)\times T^n} v(s, y, \lambda) < 0$. Reducing the size of $\varepsilon$ if needed, we have that $v \to 0$ as $s \to \infty$, thanks to Lemma 2.2. Thus $\inf_{(-\infty, -N_1)\times T^n} v(s, y, \lambda) = v(s_0, y_0, \lambda) < 0$, for some $s_0 \in (-\infty, -N_1)$ and $y_0 \in T^n$, which implies that $W(s_0, y_0, \lambda) < 0$ or $U(s_0 + \lambda, y_0) < U'(s_0, y_0)$. So $U(s_0 + \lambda, y_0) - u_l \leqq U'(s_0, y_0) - u_l \leqq \varepsilon^2$. The left-hand side of (2.23) evaluated at the point $(s_0, y_0)$ is

$$\geqq v(s_0, y_0)(\rho_l(\varepsilon) + c\varepsilon k(y_0)(f'(U) - f'(u_l)) + ck(y_0)\beta(s_0, y_0)U_s'(s_0, y_0))$$
$$\geqq v(s_0, y_0)(\tfrac{1}{2}\rho_l'(0)\varepsilon + O(\varepsilon^2)) > 0, \quad (2.24)$$

a contradiction.

Assume now that $\inf_{(N_2, +\infty)\times T^n} W(s, y, \lambda) < 0$. Then

$$\inf_{(N_2, +\infty)\times T^n} v_1(s, y, \lambda) = v_1(s_0, y_0) < 0$$

for some $(s_0, y_0) \in (N_2, +\infty)$. The function $v_1$ satisfies

$$(e\partial_s + \nabla_y)^T(a(y)(e\partial_s + \nabla_y)v_1) + \frac{2}{\Phi_1}(-e\varepsilon\Phi_1 + \nabla_y\Phi_1)^T a(e\partial_s + \nabla_y)v_1$$
$$+ b\cdot(e\partial_s + \nabla_y)v_1 + (c + ck(y)f'(U))v_{1,s}$$
$$+ (\rho_r(-\varepsilon) - \varepsilon k(y)(f'(U) - f'(u_r)) + ck(y)\beta U_s')v_1 = 0. \quad (2.25)$$

The left-hand side of (2.25) evaluated at $(s_0, y_0)$ is

$$\begin{aligned}&\geqq (\rho_r(-\varepsilon) - c\varepsilon k(y_0)(f'(U)(s_0, y_0) - f'(u_r))\\&\quad + ck(y_0)\beta(s_0, y_0)U'_s(s_0, y_0))v_1(s_0, y_0)\\&\geqq (-\tfrac{1}{2}\rho'_r(0)\varepsilon + O(\varepsilon^2))v_1(s_0, y_0) > 0,\end{aligned} \tag{2.26}$$

a contradiction. These contradictions imply that $W(s, y, \lambda) \geqq 0$ if $\lambda \geqq \lambda_0$.

We show that $W(s, y, \mu) = 0$ define $\mu = \inf\{\lambda \mid W(s, y, \lambda) \geqq 0\}$. Obviously $W(s, y, \mu) \geqq 0$, $-\infty < \mu \leqq \lambda_0$. If $W(s, y, \mu)$ is zero at any finite point, then the maximum principle says that $W(s, y, \mu) = 0$, and we are done. Otherwise, $W(s, y, \mu) > 0\ \forall (s, y)$. By the minimality of $\mu$, there exists a sequence $\lambda_j \uparrow \mu, j \to \infty$, such that $\inf_{R^1 \times T^n} W(s\ y, \lambda_j) = W(s_j, y_j, \lambda_j) < 0$.

Suppose that the $s_j$'s are unbounded, so $s_j \to +\infty$ or $s_j \to -\infty$, up to a subsequence still denoted by $s_j$. Thus if $j$ is large enough, then $s_j \in [-N_1, N_2]^c \times T^n$. Assume first that $s_j \in (-\infty, -N_1) \times T^n$. Thus $\inf_{(-\infty, -N_1) \times T^n} W(s, y, \lambda_j) < 0$. Letting $W = \exp\{\varepsilon s\}\Phi(y)v$ as before, we see that $\inf_{(-\infty, -N_1) \times T^n} v(s, y, \lambda_j) < 0$. This infimum is attained at a finite point $(\tilde{s}_j, \tilde{y}_j)$. If the $\tilde{s}_j$'s are unbounded (as $j \to \infty$), then $\tilde{s}_j \to -\infty$ up to a subsequence, and evaluating (2.23) at $(\tilde{s}_j, \tilde{y}_j)$ yields a contradiction just as in (2.24). Thus the $\tilde{s}_j$'s are bounded, $\tilde{s}_j \to \tilde{s}_0$, $\tilde{y}_j \to \tilde{y}_0$, $(\tilde{s}_0, \tilde{y}_0) \in (-\infty, -N_1] \times T^n$, up to a subsequence, and

$$v(\tilde{s}_0, \tilde{y}_0, \mu) = \lim_{j \to \infty} v(\tilde{s}_j, \tilde{y}_j, \lambda_j) \leqq 0,$$

which implies that $W(\tilde{s}_0, \tilde{y}_0, \mu) \leqq 0$, a contradiction.

Next assume that $s_j \in (N_2, +\infty) \times T^n$, so $\inf_{(N_2, +\infty) \times T^n} W(s, y, \lambda_j) < 0$. Letting $W = \exp\{-\varepsilon s\}\Phi_1(y)v_1$, we see that $\inf_{(N_2, +\infty) \times T^n} v_1(s, y, \lambda_j) < 0$. This infimum is achieved at a finite point $(\tilde{s}_j, \tilde{y}_j)$. If the $\tilde{s}_j$'s are unbounded, then $\tilde{s}_j \to +\infty$ up to a subsequence, and evaluating equation (2.25) at $(\tilde{s}_j, \tilde{y}_j)$ shows that the left-hand side of (2.25) is strictly larger than zero, a contradiction. If the $\tilde{s}_j$'s are bounded, then $(\tilde{s}_j, \tilde{y}_j) \to (\tilde{s}_0, \tilde{y}_0) \in [N_2, +\infty) \times T^n$ up to a subsequence, and

$$v(\tilde{s}_0, \tilde{y}_0, \mu) = \lim_{j \to \infty} v(\tilde{s}_j, \tilde{y}_j, \lambda_j) \leqq 0,$$

which implies that $W(\tilde{s}_0, \tilde{y}_0, \mu) \leqq 0$, a contradiction.

Hence, the $s_j$'s are bounded, $s_j \to s_0$, $y_j \to y_0$, up to a subsequence, and

$$\lim_{j \to \infty} W(s_j, y_j, \lambda_j) = W(s_0, y_0, \mu) \leqq 0,$$

a contradiction. Finally, we conclude that $W(s, y, \mu)$ cannot be $> 0$. Thus $W(s, y, \mu) \equiv 0$, or $U(s + \mu, y) \equiv U'(s, y)$. Uniqueness of traveling waves is proved.

**Corollary 2.2** (Monotonicity). *If $(U(s, y), c)$ satisfy the conditions in Lemma 2.1, then $U_s(s, y) > 0$, for all $(s, y) \in R^1 \times T^n$.*

**Proof.** Taking $(U', c')$ as $(U, c)$ in Theorem 2.1 and following the same proof, we see that $U(s+\lambda, y) > U(s, y)$ if $\lambda > \mu$, and $U(s+\mu, y) \equiv U(s, y)$. Since $U$ approaches different limits as $s \to \pm\infty$, it follows that $\mu = 0$. This implies that $U_s \geqq 0$. Differentiating (2.2) with respect to $s$, and applying the strong maximum principle [31], we have that $U_s(s, y) > 0$ for all $(s, y) \in R^1 \times T^n$. The proof is complete.

## 3. The Continuation of Regularized Solutions

Due to the degeneracy in (2.2), we first establish existence for its elliptic regularization:

$$\begin{aligned} &\nu U^{\nu}_{ss} + (e\partial_s + \nabla_y)(a(y)(e\partial_s + \nabla_y)U^{\nu}) \\ &\qquad + b(y)\cdot(e\partial_s + \nabla_y)U^{\nu} + c^{\nu}(U^{\nu} + k(y)f(U^{\nu}))_s = 0, \qquad (3.1) \\ &U^{\nu}(-\infty, y) = u_l, \quad U^{\nu}(+\infty, y) = u_r, \quad U^{\nu}(s,\cdot) \text{ has period } 1 \end{aligned}$$

where $\nu$ is a positive number, $\nu \in (0, 1]$. Then we prove the existence of a solution to equation (2.2) by passing to the limit $\nu \to 0$. In the rest of this section, we fix $\nu$, and so omit the $\nu$ dependence of the solution and write $(U^{\nu}, c^{\nu})$ as $(U, c)$. We construct the solution of (3.1) by the continuation method.

Let us consider the family of equations parametrized by $\tau$:

$$\begin{aligned} &\nu U^{\tau}_{ss} + (e\partial_s + \nabla_y)(a^{\tau}(y)(e\partial_s + \nabla_y)U^{\tau}) \\ &\qquad + b^{\tau}(y)\cdot(e\partial_s + \nabla_y)U^{\tau} + c^{\tau}(U^{\tau} + k^{\tau}(y)f(U^{\tau}))_s = 0, \qquad (3.2) \\ &U^{\tau}(-\infty, y) = u_l, \quad U^{\tau}(+\infty, y) = u_r, \quad U^{\tau}(s,\cdot) \text{ has period } 1 \end{aligned}$$

where $a^{\tau} = \tau a(y) + (1-\tau)\mathrm{Id}$, $b^{\tau}(y) = b_0(1-\tau) + b(y)\tau$, $k(y) = \langle k\rangle(1-\tau) + \tau k(y)$. First we show that if (3.2) admits solutions for $\tau = \tau_0$, $\tau_0 \in [0, 1)$, then it has solutions for $\tau = \tau_0 + \delta$ if $\delta$ is sufficiently small. For simplicity, let us write $a^{\tau_0}$, $b^{\tau_0}$, $c^{\tau_0}$, $k^{\tau_0}$, $U^{\tau_0}$ as $a, b, c, k, U$, and write $a^{\tau_0+\delta}$, $b^{\tau_0+\delta}$, $c^{\tau_0+\delta}$, $k^{\tau_0+\delta}$, $U^{\tau_0+\delta}$ as $a^{\delta}$, $b^{\delta}$, $c^{\delta}$, $k^{\delta}$, $U^{\delta}$ in the rest of this section. We can write $a^{\delta} = a + \delta a_1$, $b^{\delta} = b + \delta b_1$, $c^{\delta} = c + \delta c_1^{\delta}$, $k^{\delta} = k + \delta k_1$, $U^{\delta} = U + \delta V^{\delta}$, where $a_1$ is a smooth symmetric matrix, $b_1$ is a smooth vector field, and $c_1$ is a smooth function of $y$; $c_1^{\delta}$ and $V^{\delta}$ are unknowns. Substituting these expressions into (3.2), and simplifying the resulting expression by using the fact that $(U, c)$ is a solution when $\delta = 0$, we have

$$\begin{aligned} LV^{\delta} &\equiv \nu V^{\delta}_{ss} + (\varepsilon\partial_s + \nabla_y)(a(e\partial_s + \nabla_y)V^{\delta}) + b\cdot(e\partial_s + \nabla_y)V^{\delta} \\ &\quad + cV^{\delta}_s + ck(y)(f'(U)V^{\delta})_s \\ &= -(e\partial_s + \nabla_y)(a_1(e\partial_s + \nabla_y)U) + \delta(e\partial_s + \nabla_y)(a_1(e\partial_s + \nabla_y)V^{\delta}) \\ &\quad - b_1\cdot(e\partial_s + \nabla_y)U - \delta b_1\cdot(e\partial_s + \nabla_y)V^{\delta} \\ &\quad - ck_1(f(U + \delta V^{\delta}))_s - \tfrac{1}{2}ck(y)\delta(f''(U + (1-\theta)\delta V^{\delta})(V^{\delta})^2)_s \\ &\quad - c_1^{\delta}(U + \delta V^{\delta} + (k(y) + \delta k_1)f(U + \delta V^{\delta}))_s. \qquad (3.3) \end{aligned}$$

for some $\theta \in (0, 1)$.

To solve (3.3) for $(V^\delta, c^\delta)$, we need to study the invertibility of the linear operator $L$. Let us consider the operator $L$ on $L^2_\rho(R^1 \times T^n)$, where $\rho = \rho(s) = \cosh^2 \varepsilon s, \varepsilon \ll 1$, and

$$L^2_\rho(R^1 \times T^n) \equiv \left\{ h(s, y) \,\middle|\, \int_{R^1 \times T^n} (\cosh^2 \varepsilon s) h^2(s, y) ds\, dy < +\infty \right\} \tag{3.4}$$

with $\varepsilon$ to be chosen. The domain of definition $D(L)$ of $L$ is $H^2_\rho(R^1 \times T^n)$. It is easy to see that $L$ is a closed operator on $L^2_\rho$ (cf. KATO [21, p. 164]). By the properties of $U$ in Section 2, especially Corollary 2.1, we see that $U_s \in L^2_\rho$ if $\varepsilon$ is suitably small and that $U_s$ is in the kernel of $L$.

Our goal is to show that zero is an isolated simple eigenvalue of $L$. Then by the spectral theorem of KATO [21, Theorem 5.28, p. 239], $L$ is a Fredholm operator with index zero. This implies the local continuation of regularized solutions via the contraction mapping theorem.

First, we prove that the essential spectrum of $L$ is bounded away from zero by a positive distance depending on $(U, c)$. Let us make the transformation:

$$V = (\exp\{-\varepsilon s\} \Phi_r(y)\zeta(s) + (1 - \zeta(s))\exp\{\varepsilon s\} \Phi_l(y))w \equiv w_0 w = w_0(s, y)w \tag{3.5}$$

where $\zeta(s)$ is a smooth function of $s$ such that $0 < \zeta(s) < 1$ for all $s \in (-1, 1)$, $\zeta(s) \equiv 0$ for $s \leqq -1$, $\zeta(s) \equiv 1$ for $s \geqq 1$, $\Phi_r = \Phi_r(y) \geqq 1$ and $\Phi_l = \Phi_l(y) \geqq 1$ are in $C^\infty(T^n)$ and are to be determined, $\varepsilon > 0$ is the same as in the weight function $\cosh^2 \varepsilon s$ for $L^2_\rho$. Similarly, let $g = w_0 g_1$. Thus the problem

$$LV = g \quad \text{on } L^2_\rho(R^1 \times T^n) \tag{3.6}$$

becomes

$$L(w_0 w) = w_0 g_1 \tag{3.7}$$

where $w, g_1 \in L^2(R^1 \times T^n)$. Direct calculation shows that

$$L(w) + \frac{2}{w_0}(e\partial_s w_0 + \nabla_y w_0)^T a(e\partial_s + \nabla_y)w + \frac{1}{w_0} L(w_0)w = g_1. \tag{3.8}$$

We compute

$$L(w_0) = L(\exp\{-\varepsilon s\} \Phi_r(y)\zeta(s)) + L((1 - \zeta(s))\exp\{\varepsilon s\} \Phi_l(y)).$$

If $s \leqq -1$, then

$$\begin{aligned} L(w_0) &= L(\exp\{\varepsilon s\} \Phi_l(y)) \\ &= \exp\{\varepsilon s\} [\nabla_y^T(a\nabla_y \Phi_l) + 2\varepsilon(e^T a \nabla_y \Phi_l) + b \cdot \nabla_y \Phi_l + ((b \cdot e)\varepsilon + \varepsilon^2 e^T ae \\ &\quad + \varepsilon \nabla_y^T ae) + c\varepsilon(1 + k(y)f'(U))\Phi_l] + ck(y)f''(U)U_s \Phi_l \exp\{\varepsilon s\}. \end{aligned} \tag{3.9}$$

Let us choose $\Phi_l$ to be the principal eigenfunction (with minimum 1) corresponding to the eigenvalue $\rho_l(\varepsilon)$ of the operator in the bracket of (3.9), but with $f'(U)$ replaced by $f'(u_l)$. Then for $0 < \varepsilon \ll 1$, by the same calculation as in the proof of Lemma 2.2 we have

$$L(w_0) = [\varepsilon ck(y)(f'(U) - f'(u_l)) + \rho_l(\varepsilon)]\Phi_l \exp\{\varepsilon s\} + [ck(y)f''(U)U_s]\Phi_l \exp\{\varepsilon s\}, \tag{3.10}$$

or

$$d = d(s, y) \equiv \frac{L(w_0)}{w_0} = \varepsilon ck(y)(f'(U) - f'(u_l)) + \rho_l(\varepsilon) + ck(y)f''(U)U_s, \tag{3.11}$$

where $\rho_l(\varepsilon) = \rho_l'(0)\varepsilon + O(\varepsilon^2) < 0$. Thus,

$$\lim_{s \to -\infty} \frac{L(w_0)}{w_0} = \rho_l(\varepsilon) < 0. \tag{3.12}$$

Now if $s \geqq 1$, then

$$\begin{aligned} L(w_0) &= L(\exp\{-\varepsilon s\}\Phi_r(y)\zeta(s)) \\ &= L(\exp\{-\varepsilon s\}\Phi_r(y)) \\ &= \exp\{-\varepsilon s\}[\nabla_y^T(a\nabla_y\Phi_r) - 2\varepsilon(e^T a\nabla_y\Phi_r) + b\cdot\nabla_y\Phi_r \\ &\quad + (-(b\cdot e)\varepsilon + \varepsilon^2(e^T ae) - \varepsilon\nabla_y^T ae) \\ &\quad - \varepsilon c(1 + k(y)f'(U))\Phi_r] + ck(y)f''(U)U_s \exp\{-\varepsilon s\}\Phi_r. \end{aligned} \tag{3.13}$$

Choosing $\Phi_r(y)$ to be the principal eigenfunction (with minimum 1) corresponding to the eigenvalue $\rho_r(-\varepsilon)$ of the operator in the bracket of (3.9), but with $f'(u_r)$ replacing $f'(U)$, we have

$$\begin{aligned} L(w_0) &= \exp\{-\varepsilon s\}[-\varepsilon ck(y)(f'(U) - f'(u_r)) + \rho_r(-\varepsilon)]\Phi_r \\ &\quad + ckf''(U)U_s \exp\{-\varepsilon s\}\Phi_r, \end{aligned} \tag{3.14}$$

where $\rho_r(-\varepsilon) = -\rho_r'(0)\varepsilon + O(\varepsilon^2) < 0$, since $\rho_r'(0) > 0$. Then

$$d = \frac{L(w_0)}{w_0} = \varepsilon ck(y)(f'(u_r) - f'(U)) + \rho_r(-\varepsilon) + ck(y)f''(U)U_s, \tag{3.15}$$

where $\rho_r(-\varepsilon) < 0$, and thus

$$\lim_{s \to +\infty} d(s, y) = \rho_r(-\varepsilon) < 0. \tag{3.16}$$

The function $d$ is then smooth in $(s, y)$, approaching $\rho_l(\varepsilon)$ as $s \to -\infty$ and $\rho_r(-\varepsilon)$ as $s \to \infty$.

Now we compute

$$\begin{aligned} ew_{0,s} + \nabla_y w_0 &= e(\exp\{-\varepsilon s\}\zeta)_s \Phi_r(y) + e((1 - \zeta)\exp\{\varepsilon s\})_s \Phi_l(y) \\ &\quad + (\exp\{-\varepsilon s\}\nabla_y\Phi_r)\zeta(s) + (1 - \zeta(s))\exp\{\varepsilon s\}\nabla_y\Phi_l. \end{aligned}$$

If $s \leqq -1$, then

$$d_1 = d_1(s, y) \equiv \frac{2}{w_0}(e\partial_s w_0 + \nabla_y w_0) = 2\varepsilon e + \frac{2\nabla_y \Phi_l}{\Phi_l} \equiv B_l = B_l(y), \qquad (3.17)$$

and if $s \geqq 1$, then

$$d_1(s, y) = -2\varepsilon e + \frac{2\nabla_y \Phi_r}{\Phi_r} \equiv B_r = B_r(y). \qquad (3.18)$$

Thus $d_1(s, y)$ is a bounded smooth vector function of $(s, y) \in R^1 \times T^n$, and is equal to $B_l(y)(B_r(y))$ if $s$ is outside $[-1, 1]$.

Equation (3.8) can be written as

$$L_1 w \equiv Lw + d_1^T a(y)(e\partial_s + \nabla_y)w + dw = g. \qquad (3.19)$$

The spectrum of $L$ on $L_\rho^2$ is same as that of the operator $L_1$ in (3.19) on $L^2$. Let us define the operator

$$\begin{aligned} Qw \equiv\ & vw_{ss} + (e\partial_s + \nabla_y)^T(a(y)(e\partial_s + \nabla_y)w) \\ & + B(s, y)^T \cdot (e\partial_s + \nabla_y)w \\ & + [c + ck(y)(\zeta(\alpha s)f'(u_r) + (1 - \zeta(\alpha s))f'(u_l))]w_s \\ & + (\rho_r(-\varepsilon)\zeta(\alpha s) + (1 - \zeta(\alpha s))\rho_l(\varepsilon))w, \end{aligned} \qquad (3.20)$$

where $\alpha$ is a small positive number to be chosen and where

$$B(s, y) = b(y) + \zeta(\alpha s)a(y)^T B_r(y) + (1 - \zeta(\alpha s))a(y)^T B_l(y).$$

Define

$$Sw \equiv (L_1 - Q)w = B_1(s, y)^T \cdot (e\partial_s + \nabla_y)w + B_2(s, y)w_s + B_3(s, y)w, \qquad (3.21)$$

where

$$B_1(s, y) = a(y)^T d_1(s, y) - (a^T B_r \zeta(\alpha s) + (1 - \zeta(\alpha s))a^T B_l),$$

$$B_2(s, y) = ck(y)[f'(U) - (\zeta(\alpha s)f'(u_r) + (1 - \zeta(\alpha s))f'(u_l))],$$

$$B_3(s, y) = d(s, y) + ck(y)f''(U)U_s - (\rho_r(-\varepsilon)\zeta(\alpha s) + (1 - \zeta(\alpha s))\rho_l(\varepsilon)).$$

We see that $B_i(s, y) \to 0$, $i = 1, 2, 3$, uniformly in $y$ as $s \to \infty$.

Let us show that $Q$ is invertible on $L^2(R^1 \times T^n)$ by the Lax-Milgram Theorem.

**Proposition 3.1.** *There exists a positive number* $\alpha_0 = \alpha_0(\varepsilon) \in (0, 1]$ *such that if* $\alpha \in (0, \alpha_0]$, *then the operator* $Q$ *as defined in* (3.20) *is invertible on* $L^2(R^1 \times T^n)$. *Moreover, there is a positive constant* $M = M(\alpha, \varepsilon)$ *such that*

$$\|Q^{-1}g\|_{H^2} \leqq M\|g\|_{L^2}. \qquad (3.22)$$

**Proof.** First we prove that the equation $Qw = g$ admits a weak solution in $H^1(R^1 \times T^n)$ for $g \in L^2(R^1 \times T^n)$. Consider the following bilinear functional from $H^1 \times H^1$ to $R$:

$$\begin{aligned} D(w, v) = \int_{R^1 \times T^n} & v w_s (mv)_s + (e\partial_s + \nabla_y) w \cdot a(y)(e\partial_s + \nabla_y)(mv) \\ & - B(s, y)^T \cdot (e\partial_s + \nabla_y) w \cdot (mv) \\ & - [(c + ck(y)(\zeta(\alpha s) f'(u_r) + (1 - \zeta(\alpha s)) f'(u_l))] w_s mv \\ & - [\rho_r(-\varepsilon)\zeta(\alpha s) + (1 - \zeta(\alpha s))\rho_l(\varepsilon)] mvw. \end{aligned} \tag{3.23}$$

Here $m = m(s_1, y) \equiv m(\alpha s, y)$, and $m(s_1, y)$ is a smooth function on $R^1 \times T^n$ such that $1 \leqq m \leqq M_1$, uniformly for all $\alpha \in (0, 1]$, where $M_1$ is independent of $\varepsilon$. We choose such a function $m$ later.

Obviously, $D(u, v)$ satisfies

$$|D(w, v)| \leqq M_2 \|w\|_{H^1} \cdot \|v\|_{H^1} \tag{3.24}$$

for some positive constant $M_2$ independent of $\varepsilon$. Now we calculate

$$\begin{aligned} D(v, v) = \int_{R^1 \times T^n} & ds\, dy \{ v v_s^2 m + v v v_s m_s \\ & + m(e\partial_s + \nabla_y) v a (e\partial_s + \nabla_y) v + v(e\partial_s + \nabla_y) v a (e\partial_s + \nabla_y) m \\ & - mB^T \cdot (e\partial_s + \nabla_y)(v^2/2) \\ & - [c + ck(y)(\zeta(\alpha s) f'(u_r) + (1 - \zeta(\alpha s)) f'(u_l))] m (v^2/2)_s \\ & - m(\rho_r(-\varepsilon)\zeta(\alpha s) + (1 - \zeta(\alpha s))\rho_l(\varepsilon)) v^2 \}. \end{aligned}$$

Integration by parts gives

$$\begin{aligned} D(v, v) \geqq \int_{R^1 \times T^n} & ds\, dy \{ v v_s^2 m - v(v^2/2) m_{ss} + m(e\partial_s + \nabla_y) v a (e\partial_s + \nabla_y) v \\ & - (v^2/2)(e\partial_s + \nabla_y)(a(e\partial_s + \nabla_y) m) \\ & + (v^2/2)(e\partial_s + \nabla_y)^T \cdot (mB) \\ & + (v^2/2) m_s [c + ck(\zeta(\alpha s) f'(u_r) + (1 - \zeta(\alpha s)) f'(u_l))] \\ & + m(v^2/2) ck(y) \alpha \zeta'(\alpha s)(f'(u_r) - f'(u_l)) + C_1 \varepsilon v^2 \}, \end{aligned}$$

where $C_1$ is a positive constant independent of $\varepsilon$ such that

$$C_1 \varepsilon \leqq \min(-\rho_r(-\varepsilon), -\rho_l(\varepsilon)).$$

Since $m = m(\alpha s, y)$, all the terms involving $s$ derivatives of $m$ are of order $O(\alpha)$. Similarly, $s$ derivatives of $B$ are of order $O(\alpha)$. It follows that

$$D(v, v) \geqq \int_{R^1 \times T^n} ds\, dy \{ vmv_s^2 + m(e\partial_s + \nabla_y)va(y)(e\partial_s + \nabla_y)v$$
$$+ (v^2/2)[-\nabla_y^T(a\nabla_y m) + \nabla_y^T(mB) + O(\alpha)] + C_1 \varepsilon v^2 \} \tag{3.25}$$

We choose $m$ to satisfy

$$-\nabla_y^T(a\nabla_y m) + \nabla_y^T(mB) = 0, \tag{3.26}$$

or

$$-\nabla_y^T(a\nabla_y m) + \nabla_y^T[m(b + \zeta(\alpha s)a^T B_r(y) + (1 - \zeta(\alpha s))a^T B_l(y))] = 0, \tag{3.27}$$

where $s$ is just a parameter. For any fixed $s$, it is known that (3.27) has a unique positive smooth solution on $T^n$ up to constant multiplication; see [2]. By the theory of elliptic regularity, $m$ depends smoothly on the coefficients, and so $m = m(s', y) \equiv m(\alpha s, y)$ is a bounded smooth function in $(s', y) \in R^1 \times T^n$. If we normalize $m$ so that $m \geqq 1$, then $m = m(\alpha s, y)$ is as desired. We see that there exists a number $\alpha_0 = \alpha_0(\varepsilon)$ such that if $\alpha \in (0, \alpha_0)$, then the $O(\alpha)$ term in (3.25) is no larger than $\frac{1}{2}C_1\varepsilon$ in absolute value. It follows from (3.25), (3.27), and such a choice of $\alpha$ that

$$D(v, v) \geqq C_2 \| v \|^2_{H^1(R^1 \times T^n)} \tag{3.28}$$

for some positive constant $C_2 = C_2(\varepsilon, v)$. Hence, the functional $D(w, v)$ is coercive, and the Lax-Milgram theorem implies the existence of a weak solution to $Qv = g$ in $H^1$. By elliptic regularity (cf. GILBARG & TRUDINGER [17, Theorem 8.8, pp. 183–185]), $v \in H^2$, and estimate (3.22) holds. The proof is complete.

Next, we have

**Lemma 3.1.** *The operator $SQ^{-1}$ is compact on $L^2(R^1 \times T^n)$.*

The proof is similar to that of Lemma 2.7 in [31], and is omitted.

By the Gohberg-Krein theorem (cf. [18] or Theorem A.1, p. 136, of HENRY [20]), $L_1$ and $Q$ differ by a relatively compact operator, so they have the same essential spectrum. Proposition 3.1 says that the essential spectrum is bounded away from zero by a positive distance depending on $\varepsilon$, and hence on $(U, c)$. Thus 0 is an isolated eigenvalue of finite multiplicity of $L_1$ on $L^2$, or of $L$ on $L_\rho^2$.

Summarizing, we have

**Corollary 3.1.** *Zero is an isolated eigenvalue of finite multiplicity of operator $L$ on $L_\rho^2(R^1 \times T^n)$.*

We show next

**Proposition 3.2.** *The kernel of $L$ is one-dimensional, and zero is its algebraically simple eigenvalue.*

The proof is similar to that of Proposition 2.1 in [31] except that near the infinities of $s$, we need to make the change of variable of the form $w = \exp\{\pm\varepsilon_1 s\}\Phi(y)v$, with $0 < \varepsilon_1 \ll 1$ and $\Phi > 0$, for function $v$ in the kernel of operator $L$. For details, we refer to the proof of Theorem 2.1.

By Theorem 5.28, page 239, of KATO [21] $L$ is a Fredholm operator of index zero, and $L^*$, the adjoint operator of $L$, has a simple eigenfunction, denoted by $v^*$, in Ker $(L^*)$. Moreover, the inner product of $U_s$ and $v^*$ can be normalized to 1. See also SATTINGER [27, pp. 320–321]. We have

**Proposition 3.3.** *The equation $Lv = g$, where $g \in L^2_\rho(R^1 \times T^n)$, is solvable in $L^2_\rho(R^1 \times T^n)$ if and only if*

$$\int_{R^1 \times T^n} \rho f v^* \, ds \, dy = 0, \tag{3.29}$$

*where $v^*$ is the simple eigenfunction of $L^*$ corresponding to eigenvalue zero, such that the $L^2_\rho$ inner product of $U_s$ and $v^*$ is equal to 1. When (3.29) holds, the solution space is one-dimensional.*

Applying Proposition 3.3, elliptic-regularity estimates, and the contraction mapping theorem, we get

**Theorem 3.1.** *Suppose that (3.2) with its boundary conditions and the normalization condition $\int_{T^n} U(0, y)dy = u_0$, $u_0 \in (u_l, u_r)$ has a classical solution $(U^\tau, c^\tau)$ where $\tau \in [0, 1)$. Then there exists $\delta_0 = \delta_0(U, c)$ such that if $\delta \in (0, \delta_0)$, then (3.2) admits a unique classical solution $(U^{\tau+\delta}, c^{\tau+\delta})$ satisfying the same boundary conditions and the normalization condition.*

We remark that the solvability condition (3.29) is used to determine the perturbed speed $c^{\tau+\delta}$. For details of the proof, see XIN [29].

## 4. The Limit of Regularized Solutions

Consider the limit of classical solutions $(U^\tau, c^\tau)$ of equation (3.2) satisfying the boundary conditions

$$U^\tau(-\infty, y) = u_l,\ U^\tau(+\infty, y) = u_r,\ U^\tau(s, \cdot) \text{ has period } 1 \tag{4.1}$$

and the normalization condition $\max_{y \in T^n} U^\tau(0, y) = u_0$, as $\tau \to \tau_0 \in (0, 1]$. Due to the uniqueness of solutions (Theorem 2.1), the solutions $(U^\tau, c^\tau)$ are exactly those generated by the continuation method (Theorem 3.1) modulo constant translations in $s$. We have

**Proposition 4.1.** *Let $\{\tau_n\}$ be any sequence tending to $\tau_0 \in (0, 1]$. Then there is a subsequence, still denoted by $\{\tau_n\}$, such that if $u_0 - u_l \leqq \varepsilon_0$, where $\varepsilon_0$ is a small* *positive number depending on the nonlinear function f, then $U^{\tau_n}(s, y)$ converges to $U^{\tau_0}(s, y)$ in $C^1_{\mathrm{loc}}$, and*

$$c^{\tau_n} = c^{\tau_0} = c_{\mathrm{eff}} = \frac{-\langle b\cdot e\rangle(u_l - u_r)}{u_l + \langle k\rangle f(u_l) - (u_r + \langle k\rangle f(u_r))} > 0. \tag{4.2}$$

*Moreover, $(U^{\tau_0}, c^{\tau_0})$ is a classical solution to (3.2) with $\tau = \tau_0$.*

**Proof.** Applying Lemma 2.1, we see that $c^{\tau_n} = c_{\mathrm{eff}}$. Choosing $u_0 - u_l < \varepsilon_0 < 1$, where $\varepsilon_0$ is as small as required in the proof of Lemma 2.2, and following the argument there for constructing upper solutions, we have sequences $\lambda_1^{\tau_n} \to \lambda_1^0 > 0$, $\Phi_1^{\tau_n} \to \Phi_1^0(y)$ in $C(T^n)$, $\min_{y\in T^n}\Phi_1^{\tau_n}(y) = \min_{y\in T^n}\Phi_1^0(y) = 1$, such that

$$U^{\tau_n} - u_l \leqq \exp\{\lambda_1^{\tau_n}s\}\Phi_1^{\tau_n}(y) \quad \forall (s, y)\in(-\infty, 0)\times T^n. \tag{4.3}$$

This follows from

$$U^{\tau_n} - u_l|_{s=0} \leqq u_0 - u_l < 1 \leqq \exp\{\lambda_1^{\tau_n}s\}\Phi_1^{\tau_n}(y)|_{s=0} \tag{4.4}$$

and the maximum principle.

Since $u_l \leqq U_n^\tau \leqq u_r$ and $c^{\tau_n}$ is independent of $\tau_n$, Schauder estimates for elliptic equations imply that $\|U^{\tau_n}\|_{C^1_{\mathrm{loc}}(R^1\times T^n)} \leqq C_1 < +\infty$. As $\tau_n \to \tau_0$, $c^{\tau_n} \to c^{\tau_0}$ and $U^{\tau_n} \to U^{\tau_0}$ in $C^1_{\mathrm{loc}}(R^1\times T^n)$ up to a subsequence of $\{\tau_n\}$. Letting $n \to +\infty$ in (4.3) gives

$$U^{\tau_0} - u_l \leqq \exp\{\lambda_1^0 s\}\Phi_1^0(y), \quad s \leqq 0,$$

which implies that

$$\lim_{s\to-\infty} U^{\tau_0} = u_l. \tag{4.5}$$

Thus $U^{\tau_0}$ satisfies

$$\begin{aligned} &vU^{\tau_0}_{ss} + (e\partial_s + \nabla_y)^T(a^{\tau_0}(y)(e\partial_s + \nabla_y)U^{\tau_0}) \\ &\quad + b^{\tau_0}(y)\cdot(e\partial_s + \nabla_y)U^{\tau_0} + c^{\tau_0}(U^{\tau_0} + k(y)f(U^{\tau_0}))_s = 0 \end{aligned} \tag{4.6}$$

in the weak sense, and by the regularity theory for elliptic equations, $U^{\tau_0}$ is a classical solution of (4.6). Moreover, $U^{\tau_0}(-\infty, y) = u_l$, $\max_{y\in T^n}U^{\tau_0}(0, y) = u_0$, $U_s^{\tau_0} \geqq 0$, and $U^{\tau_0}(s,\cdot)$ has period 1.

We have yet to justify that $U^{\tau_0}(+\infty, y) = u_r$. The limit $\lim_{s\to+\infty}U^{\tau_0}(s, y) \equiv u_+(y)$ exists due to the monotonicity of $U^{\tau_0}$ in $s$. By local regularity estimates and by the fact that $0 < \int_{R^1\times T^n}U_s^{\tau_0}ds dy \leqq u_r - u_l < +\infty$, it follows that $U_s^{\tau_0} \to 0$ as $s \to +\infty$ uniformly in $y$. Differentiating (4.6) with respect to $s$ and applying the elliptic Schauder estimates to $U_s^{\tau_0}$ we find that $U_{ss}^{\tau_0} \to 0$ as $s \to +\infty$.

Multiplying both sides of (4.6) by any smooth test function $\psi(y) \in C^\infty(T^n)$ and by integrating the products by parts with respect to $y$, we get

$$\begin{aligned} &\int_{T^n} vU^{\tau_0}_{ss}\psi\, dy + I + \int_{T^n} dy\{\psi b\cdot eU_s^{\tau_0} - \nabla_y^T\cdot(\psi b)U^{\tau_0}\} \\ &\quad + c^{\tau_0}\int_{T^n} dy\{\psi U_s^{\tau_0} + k(y)\psi(y)f'(U^{\tau_0})U_s^{\tau_0}\} = 0, \end{aligned} \tag{4.7}$$

where

$$
\begin{aligned}
I = & -\int_{T^n} \psi(y)(e\partial_s + \nabla_y)(a(y)(e\partial_s + \nabla_y)U^{\tau 0})dy \\
= & \int_{T^n} dy\psi(y)\{(e\partial_s)^T a(y)(e\partial_s + \nabla_y)U^{\tau 0} + \nabla_y^T(a(y)(e\partial_s + \nabla_y)U^{\tau 0})\} \\
= & \int_{T^n} dy\psi(y)(e^T a(y)e)U_{ss}^{\tau 0} - \int_{T^n} dy\, U_s^{\tau 0}\nabla_y^T \cdot (\psi(y)a(y)e) \\
& - \int_{T^n} dy\, U_s^{\tau 0}((\nabla_y\psi)^T a(y)e) + \int_{T^n} dy\, U^{\tau 0}\nabla_y^T a(y)\nabla_y\psi.
\end{aligned}
$$

Letting $s \to +\infty$ in (4.7) shows that

$$
\int_{T^n} dy\nabla_y^T \cdot (a(y)\nabla_y\psi)\cdot u_+ - \int_{T^n} dy\, \nabla_y(\psi b)u_+ = 0, \tag{4.8}
$$

which implies that $u_+$ is a weak and hence a classical solution of

$$
\nabla_y^T(a(y)\nabla_y u_+) + b\cdot\nabla_y u_+ = 0 \quad \text{on } T^n. \tag{4.9}
$$

The maximum principle implies that $u_+ =$ constant. Thus,

$$
\lim_{s\to+\infty} U^{\tau 0}(s, y) = u_+ = \text{const.} \in [u_0, u_r].
$$

Applying Lemma 2.1, we have then

$$
\begin{aligned}
c^{\tau^0} &= \frac{-\langle b\cdot e\rangle(u_l - u_+)}{u_l + \langle k\rangle f(u_l) - (u_+ + \langle k\rangle f(u_+))} \\
&= \frac{-\langle b\cdot e\rangle}{1 + \langle k\rangle \dfrac{f(u_l) - f(u_+)}{u_l - u_+}}.
\end{aligned} \tag{4.10}
$$

The limit $c^{\tau 0}$ of $c^{\tau_n}$ satisfies (4.2), so we have

$$
\frac{f(u_l) - f(u_+)}{u_l - u_+} = \frac{f(u_l) - f(u_r)}{u_l - u_r}. \tag{4.11}
$$

By the Oleinik entropy condition (O), we see that $u_+ = u_r$. Differentiating (4.6) with respect to $s$ and applying the strong maximum principle yields $U_s^{\tau 0} > 0$, for all $s$ and $y$. The proof of the proposition is complete.

In summary, we have

**Theorem 4.1.** *For any given positive number $\nu > 0$, there exist a classical solution $(U^\nu, c^\nu)$ to equation (3.2) satisfying all the boundary conditions. Moreover, $U_s^\nu > 0$, $u_l < U^\nu < u_r$ for all $(s, y) \in R^1 \times T^n$*, and $c^\nu = c_{\text{eff}}$.

## 5. Proof of Theorem 1.1

We are ready to take the limit $v \to 0$ in (3.2). Since $c^v = c_{\text{eff}} > 0$, and is independent of $v$, and since $f'(U^v) > 0$, we have

$$0 < c_{\text{eff}} \leqq c^v + k(y) f'(U^v) \leqq C_1 < +\infty,$$

where $C_1$ is independent of $v$. Schauder estimates for parabolic equations give

$$\| U^v \|_{C^1_{\text{loc}}} \leqq C_2 < +\infty,$$

with positive constant $C_2$ independent of $v$. We impose,

$$\max_{y \in T^n} U^v(0, y) = u_0, \tag{5.1}$$

where $u_0 \in (u_l, u_r)$. Now choose $u_0$ close to $u_l$ as in Proposition 4.1, and pass to the limit $v \to 0$. All the steps there go through except now we use the Schauder estimates for parabolic equations instead of those for elliptic equations. Justifying the boundary conditions with the Oleinik entropy condition again, we complete the proof of Theorem 1.1.

## 6. Proof of Theorem 1.3

Consider solutions $(U^\varepsilon, c^\varepsilon)$ of

$$(e\partial_s + \nabla_y)(a(y)(e\partial_s + \nabla_y)U^\varepsilon) + b(y)\cdot(e\partial_s + \nabla_y)U^\varepsilon + c^\varepsilon(U^\varepsilon + k(y)(U^\varepsilon)^p)_s = 0, \tag{6.1}$$

$$U^\varepsilon(-\infty, y) = u_l, \quad \min_{y \in T^n} U^\varepsilon(0, y) = u_0, \quad U^\varepsilon(+\infty, y) = \varepsilon, \tag{6.2}$$

where $u_l > u_0 > \varepsilon$, with $u_0$ to be chosen. We have

**Lemma 6.1.** *There exists a positive constant $M$ independent of $\varepsilon$ such that*

$$\| U^\varepsilon_s \|_2(R^1 \times T^n) \leqq M, \tag{6.3}$$

$$\| (e\partial_s + \nabla_y)U^\varepsilon \|_2(R^1 \times T^n) \leqq M. \tag{6.4}$$

**Proof.** Multiply both sides of (6.1) by $U^\varepsilon$ and integrate the product by parts over $R^1 \times T^n$ to get

$$\begin{aligned} &- \int_{R^1 \times T^n} ds\, dy (e\partial_s + \nabla_y) U^\varepsilon a(y)(e\partial_s + \nabla_y) U^\varepsilon \\ &+ \int_{R^1 \times T^n} ds\, dy\, b(y) \cdot e U^\varepsilon_s U^\varepsilon \\ &+ c^\varepsilon \int_{R^1 \times T^n} ds\, dy\, U^\varepsilon (U^\varepsilon + k(y)(U^\varepsilon)^p)_s = 0, \end{aligned} \tag{6.5}$$

or

$$\int_{R^1\times T^n} ds\,dy(e\partial_s+\nabla_y)U^\varepsilon a(y)(e\partial_s+\nabla_y)U^\varepsilon$$
$$=\tfrac{1}{2}\langle b(y)\cdot\varepsilon\rangle(\varepsilon^2-u_l^2)$$
$$+\tfrac{1}{2}c^\varepsilon(\varepsilon^2-u_l^2)+\frac{p}{p+1}c^\varepsilon\langle k(y)\rangle(\varepsilon^{p+1}-u_l^{p+1}). \tag{6.6}$$

By Theorem 1.2, we have

$$c^\varepsilon=\frac{-\langle b(y)\cdot e\rangle(\varepsilon-u_l)}{\varepsilon+\langle k(y)\rangle\varepsilon^p-u_l-\langle k(y)\rangle u_l^p}>0. \tag{6.7}$$

It follows from (6.6) that

$$\|(e\partial_s+\nabla_y)U^\varepsilon\|_2^2(R^1\times T^n)\leqq M^2 \tag{6.8}$$

for $M>0$ independent of $\varepsilon$. Multiply both sides of (6.1) by $U_s^\varepsilon$ and integrate the product by parts over $R^1\times T^n$ to get

$$-\int_{R^1\times T^n} ds\,dy((e\partial_s+\nabla_y)U^\varepsilon)_s a(y)(e\partial_s+\nabla_y)U^\varepsilon$$
$$+\int_{R^1\times T^n} ds\,dy\,U_s^\varepsilon b(y)\cdot(e\partial_s+\nabla_y)U^\varepsilon$$
$$+\int_{R^1\times T^n} ds\,dy\,c^\varepsilon U_s^\varepsilon(U^\varepsilon+k(y)(U^\varepsilon)^p)_s=0. \tag{6.9}$$

Notice that

$$-\int_{R^1\times T^n} ds\,dy((e\partial_s+\nabla_y)U^\varepsilon)_s a(y)(e\partial_s+\nabla_y)U^\varepsilon=0,$$

$$\int_{R^1\times T^n} U_s^\varepsilon b(y)\cdot(e\partial_s+\nabla_y)U^\varepsilon ds\,dy\leqq\|b\|_\infty\|U_s^\varepsilon\|_2\|(e\partial_s+\nabla_y)U^\varepsilon\|_2. \tag{6.10}$$

It follows that

$$c^\varepsilon\int_{R^1\times T^n} ds\,dy(U_s^\varepsilon)^2(1+k(y)(U^\varepsilon)^{p-1})\leqq\|b\|_\infty\|U^\varepsilon\|_2\|(e\partial_s+\nabla_y)U^\varepsilon\|_2, \tag{6.11}$$

which implies that

$$c^\varepsilon\|U_s^\varepsilon\|_2^2\leqq\|b\|_\infty\|U_s^\varepsilon\|_2\|(e\partial_s+\nabla_y)U^\varepsilon\|_2,$$
$$c^\varepsilon\|U_s^\varepsilon\|_2\leqq\|b\|_\infty\|(e\partial_s+\nabla_y)U^\varepsilon\|_2.$$

In view of (6.7) and (6.8), we see that estimates (6.3) and (6.4) both hold. The proof of the lemma is complete.

Now consider nonnegative solutions of equation (1.4), and let $m = \frac{1}{p} > 1$, $\Omega$ be any bounded open set in $R^n$, $T \in (0, +\infty)$, $\Omega_T \equiv \Omega \times (0, T)$. Also let $v = u^p$. Then $u = v^m$, and $v$ satisfies

$$(k(x) + mv^{m-1})v_t = \nabla \cdot (a(x)\nabla v^m) + b(x) \cdot \nabla v^m. \tag{6.12}$$

Following Di Benedetto & Friedman [14], we have

**Definition 6.1.** A *local weak solution* of (6.12) is a measurable function $v: \Omega \to R^+$ such that

$$|||v|||_m \equiv \operatorname{ess\,sup}_{0<t<T} \|v(\cdot, t)\|^2_{2,\Omega} + \|\nabla v^m\|^2_{2,\Omega_T} < +\infty, \tag{6.13}$$

$$\int_{\Omega_T} \{-(k(x)v + v^m)\varphi_t + \nabla v^m \cdot a(x)\nabla\varphi - \varphi b(x) \cdot \nabla v^m\}\, dx\, dt = 0 \tag{6.14}$$

for all $\varphi \in C_0^1(\Omega_T)$.

By Lemma 6.1, $\|\nabla_{s,y} U^\varepsilon\|_2 \leqq M < +\infty$ and $0 < U^\varepsilon \leqq u_l$. Now let $u^\varepsilon = U^\varepsilon(e \cdot x - c^\varepsilon t, x)$. Then

$$v^\varepsilon = (u^\varepsilon)^p \leqq u_l^p,$$

$$\operatorname*{ess\,sup}_{0<t<T} \|v^\varepsilon(\cdot, t)\|^2_{2,\Omega} \leqq (u_l^p)^2 \cdot |\Omega|.$$

Since

$$\nabla_x (v^\varepsilon)^m = \nabla_x u^\varepsilon = (e\partial_s + \nabla_y) U^\varepsilon(s = e \cdot x - c^\varepsilon t, y = x),$$

we have

$$\begin{aligned}
\|\nabla_x (v^\varepsilon)^m\|^2_{2,\Omega_T} &= \int_{\Omega_T} |\nabla_x u^\varepsilon|^2\, dx\, dt \\
&= \int_{\Omega_T} |(e\partial_s + \nabla_y) U^\varepsilon(s = e \cdot x - c^\varepsilon t, y = x)|^2\, dx\, dt \\
&= \int_\Omega dx \int_0^T |(e\partial_s + \nabla_y) U^\varepsilon(s = e \cdot x - c^\varepsilon t, y = x)|^2\, dt \\
&\leqq \frac{1}{c^\varepsilon} \int_\Omega dx \int_{R^1} |(e\partial_s + \nabla_y) U^\varepsilon(s, x)|^2\, ds \\
&\leqq \frac{c_0}{|c^\varepsilon|} \int_{T^n} dx \int_{R^1} |(e\partial_s + \nabla_y) U^\varepsilon|^2\, ds \\
&\leqq \frac{c_0}{c^\varepsilon} M^2,
\end{aligned} \tag{6.15}$$

where $c_0$ is a positive constant depending on the number of base cells of $T^n$ that cover $\Omega$, and $c^\varepsilon \to c \neq 0$, as $\varepsilon \to 0$. Similarly, we have

$$\begin{aligned}
\|(v^\varepsilon)^m_{,t}\|_{2,\Omega_T} &= \int_0^T dt \int_\Omega dx |u^\varepsilon_t|^2 \\
&= c^\varepsilon \int_\Omega dx \int_0^T dt |U_s(s = e\cdot x - c^\varepsilon t, y = x)|^\varepsilon \\
&\leqq (c^\varepsilon)^2 \int_\Omega dx \int_{R^1} U_s^2(s, x) ds \\
&\leqq c_0(c^\varepsilon)^2 \int_{T^n} dx \int_{R^1} U_s^2(s, x) ds \leqq c_0 (c^\varepsilon)^2 M^2. \qquad (6.16)
\end{aligned}$$

Now $(v^\varepsilon)^m \to v^m$ strongly in $L^2_{\text{loc}}(\Omega_T)$, and $\nabla(v^\varepsilon)^m \to \nabla v^m$ weakly in $L^2_{\text{loc}}(\Omega_T)$. It follows that $v^\varepsilon \to v$ a.e. in $\Omega_T$ due to the nonnegativity of $v^\varepsilon$. The functions $v^\varepsilon$ satisfy (6.14) since they are classical solutions of (6.12). Passing to the limit in (6.14) using the Lebesgue dominated convergence theorem, we see that $v$ is a local weak solution of (6.12). By the regularity theory of [14, Theorem 1.2], $v$ is (locally) Hölder continuous in $(x, t)$. Equation (2.11) differs slightly from (1.3) in [14] in that $k(x) + mv^{m-1}$ is replaced by 1 there. However, $k(x) + mv^{m-1}$ is bounded between $\min_{x\in T^n} k(x)$ and $\max_{x\in T^n} k(x) + m\|v\|_\infty^{m-1}$, $m > 1$, and is a regular factor. It is straightforward to see that Theorem 1.2 of [14] still applies. It follows that $u = v^m$ is also (locally) Hölder continuous. On the other hand, $U^\varepsilon(s, y) \to U(s, y)$ strongly in $L^2_{\text{loc}}(R^1 \times T^n)$, and so $U(e\cdot x - ct, x) = u(x, t)$, a.e. in $\Omega_T$. Since $u(x, t)$ is (locally) Hölder continuous, $U(e\cdot x - ct, x) = u(x, t)$ for all $(x, t) \in \Omega_T$ and $U(s, y)$ is (locally) Hölder continuous in $(s, y) \in R^1 \times T^n$, due to the arbitrariness of $T$ and $\Omega$. Moreover, $U(s, y) \in H^1_{\text{loc}}(R^1 \times T^n)$ is a weak solution to equation (6.1) in the sense that for any $\varphi \in C_0^\infty(R^1 \times T^n)$,

$$\begin{aligned}
\int_{R^1\times T^n} ds\, dy [ &-(e\partial_s + \nabla_y) U a(y)(e\partial_s + \nabla_y)\varphi \\
&+ \varphi b(y)\cdot(e\partial_s + \nabla_y) U - c(U + k(y)U^p)\varphi_s] = 0, \qquad (6.17)
\end{aligned}$$

and $0 \leqq U \leqq u_l$. On the open set $G \equiv \{(s, y) \in R^1 \times T^n | U(s, y) > 0\}$, $U$ is a classical solution of (6.12) by the usual parabolic estimates. Obviously, $U(s, y)$ is monotonically decreasing in $s$, i.e., $U(s_1, y) \geqq U(s_2, y)$ if $s_1 \leqq s_2$ for all $y \in T^n$.

If we choose $u_0$ sufficiently close to $u_l$, then by the same argument as in Lemma 2.2 and Proposition 4.1, there exist a positive constant $\lambda > 0$, and a positive smooth function $\Phi(y)$ such that

$$u_l - U(s, y) \leqq \exp\{\lambda s\}\Phi(y) \quad \text{for } s \leqq 0,\ y \in T^n,$$

which implies that $\lim_{s\to-\infty} U(s, y) = u_l$ uniformly in $y$.

Next, we prove that $\lim_{s\to+\infty} U(s, y) = 0$ uniformly in $y$. Thanks to monotonicity, $\lim_{s\to+\infty} U(s, y)$ exists; let us denote it by $U_+(y)$.

Multiplying both sides of (6.1) by $\psi \in C^2(T^n)$ and integrating the product by parts over $[s_1, s_2] \times T^n$, we obtain the following three equivalent equations:

$$\int_{s_1}^{s_2} ds \int_{T^n} dy\{\psi(b\cdot e)U_s^\varepsilon - \nabla_y^T\cdot(\psi b)U^\varepsilon\} + \int_{s_1}^{s_2}\int_{T^n} dy\,\psi e^T a(eU_{ss}^\varepsilon)$$
$$- \int_{s_1}^{s_2} ds \int_{T^n} dy\{\nabla_y^T\cdot(\psi a(y)e)U_s^\varepsilon + U_s^\varepsilon(\nabla_y^T\psi\cdot ae)\}$$
$$+ \int_{s_1}^{s_2} ds \int_{T^n} dy\{U^\varepsilon\nabla_y^T a(y)\nabla_y\psi + c^\varepsilon[U^\varepsilon + k(y)(U^\varepsilon)^p]_s\psi(y)\} = 0,$$

$$\int_{s_1}^{s_2}\int_{T^n} dy\{\psi(b\cdot e)U_s^\varepsilon - \nabla_y^T(\psi b)U^\varepsilon\} + \int_{T^n} dy\,\psi e^T a(eU_s^\varepsilon)\Big|_{s_1}^{s_2}$$
$$- \int_{s_1}^{s_2} ds \int_{T^n} dy\,\nabla_y^T\cdot(\psi ae)U_s^\varepsilon - \int_{T^n} dy\,U^\varepsilon(\nabla_y^T\psi\cdot ae)\Big|_{s_1}^{s_2}$$
$$+ \int_{s_1}^{s_2} ds \int_{T^n} dy\,U^\varepsilon\nabla_y^T a\nabla_y\psi + c^\varepsilon\int_{T^n} dy\{U^\varepsilon + k(y)(U^\varepsilon)^p\}\psi\Big|_{s_1}^{s_2} = 0,$$

$$\int_{T^n}\psi(b(y)\cdot e)U^\varepsilon\Big|_{s_1}^{s_2} - \int_{s_1}^{s_2} ds \int_{T^n} dy\,\nabla_y^T(\psi b)U^\varepsilon$$
$$+ \int_{T^n} dy\,\psi e^T a(eU^\varepsilon)_s\Big|_{s_1}^{s_2} - \int_{T^n} dy\nabla_y^T(\psi ae)U^\varepsilon\Big|_{s_1}^{s_2} - \int_{T^n} dy\,U^\varepsilon(\nabla_y^T\psi\cdot ae)\Big|_{s_1}^{s_2}$$
$$+ \int_{s_1}^{s_2} ds \int_{T^n} dy\,\nabla_y^T(a\nabla_y\psi)U^\varepsilon + c^\varepsilon\int_{T^n} dy\{U^\varepsilon + k(y)(U^\varepsilon)^p\}\psi\Big|_{s_1}^{s_2} = 0.$$

Letting $\varepsilon \to 0$, we get for almost every $(s_1, s_2) \in R^2$ that

$$\int_{T^n} dy\,\psi(b\cdot e)U\Big|_{s_1}^{s_2} - \int_{s_1}^{s_2}\int_{T^n} dy\nabla_y^T(\psi b)U + \int_{T^n} dy\,\psi e^T aeU_s\Big|_{s_1}^{s_2}$$
$$- \int_{T^n} dy\nabla_y^T\cdot(\psi ae)U\Big|_{s_1}^{s_2} - \int_{T^n} dy\,U(\nabla_y^T\psi\cdot ae)\Big|_{s_1}^{s_2}$$
$$+ \int_{s_1}^{s_2}\int_{T^n} dy\nabla_y^T(a\nabla_y\psi)U + c\int_{T^n} dy\{U + k(y)U^p\}\psi\Big|_{s_1}^{s_2} = 0. \tag{6.18}$$

Then by the local Hölder continuity of $U$, we know that (6.18) holds for all $(s_1, s_2) \in R^2$, and that every term is locally Hölder continuous in $s$. Since

$$\int_{R^1}\left(\int_{T^n} dy\,\psi e^T aeU_s\right)^2 ds \leqq \int_{R^1}\left(\int_{T^n}(\psi e^T ae)^2 dy\right)\left(\int_{T^n} U_s^2\,dy\right)ds$$
$$= \int_{T^n}(\psi e^T ae)^2 dy\,\|U_s\|_2^2(R^1\times T^n) < +\infty,$$

there exists a sequence $\{s_l\}$, $l = 1,2,\ldots, s_l \to +\infty$, $s_{l+1} - s_l \to +\infty$, such that

$$\lim_{s=s_l\to+\infty} \int_{T^n} \psi e^T ae U_s dy = 0. \tag{6.19}$$

Setting $s_2 = s_{l+1}$, $s_1 = s_l$ in (6.18), and dividing both sides by $s_{l+1} - s_l$, we have

$$\begin{aligned}
&\frac{1}{s_{l+1}-s_l}\int_{T^n} dy\,\psi(b\cdot e)U\Big|_{s_l}^{s_{l+1}} - \frac{1}{s_{l+1}-s_l}\int_{s_l}^{s_{l+1}} ds \int_{T^n} \nabla_y^T(\psi b)U\,dy \\
&\quad + \frac{1}{s_{l+1}-s_l}\int_{T^n} \psi e^T ae U_s dy\Big|_{s_l}^{s_{l+1}} - \frac{1}{s_{l+1}-s_l}\int_{Tn} dy \nabla_y^T(\psi ae)U\Big|_{s_l}^{s_{l+1}} \\
&\quad - \frac{1}{s_{l+1}-s_l}\int_{T^n} dy\,U(\nabla_y^T\psi\cdot ae)\Big|_{s_l}^{s_{l+1}} \\
&\quad + \frac{1}{s_{l+1}-s_l}\int_{s_l}^{s_{l+1}} ds \int_{T^n} dy \nabla_y^T(a\nabla_y\psi)U \\
&\quad + \frac{c}{s_{l+1}-s_l}\int_{T^n} dy\{U + k(y)U^p\}\psi\Big|_{s_l}^{s_{l+1}} = 0.
\end{aligned} \tag{6.20}$$

Noticing that $0 \leqq U \leqq u_l$ and letting $l \to +\infty$ in (6.20), we obtain

$$-\int_{T^n} \nabla_y^T(\psi b)U_+ dy + \int_{T^n} \nabla_y^T(a\nabla_y\psi)U_+ dy = 0. \tag{6.21}$$

Thus $U_+$ is a weak solution of the elliptic equation

$$\nabla_y^T(a(y)\nabla_y U_+) + b(y)\cdot\nabla_y U_+ = 0 \quad \text{on } T^n, \tag{6.22}$$

and so $U_+$ is a classical solution. By the maximum principle, $U_+ =$ constant, $U_+ \in [0, u_0]$. By Dini's theorem, $U(s, y) \to U_+$ as $s \to +\infty$ uniformly in $y$. Now averaging (6.1) over $T^n$ yields

$$e^T\cdot\langle a(y)(e\partial_s + \nabla_y)U^\varepsilon\rangle_s + \langle b(y)\cdot eU^\varepsilon\rangle_s + c^\varepsilon(\langle U^\varepsilon\rangle + \langle k(y)(U^\varepsilon)^p\rangle)_s = 0. \tag{6.23}$$

Integrating (6.23) over $s$ gives

$$e^T\langle a(y)(e\partial_s + \nabla_y)U^\varepsilon\rangle + \langle b(y)\cdot eU^\varepsilon\rangle + c^\varepsilon(\langle U^\varepsilon\rangle + \langle k(y)(U^\varepsilon)^p\rangle) = c_1^\varepsilon, \tag{6.24}$$

or

$$\langle e^T a(eU^\varepsilon)\rangle_s - \langle e\cdot(\nabla_y^T\cdot a)U^\varepsilon\rangle + \langle b\cdot(eU^\varepsilon)\rangle + c^\varepsilon(\langle U^\varepsilon\rangle + \langle k(y)(U^\varepsilon)^p\rangle) = c_1^\varepsilon. \tag{6.25}$$

Letting $s \to +\infty$ in (6.24) shows that

$$c_1^\varepsilon = c^\varepsilon(\varepsilon + \langle k\rangle\varepsilon^p) + \langle b(y)\cdot e\rangle\varepsilon.$$

Integrating (6.25) over any finite interval $[s', s'']$, we get

$$\langle e^T a e U^\varepsilon \rangle |_{s'}^{s''} = \int_{s'}^{s''} ds (c_1^\varepsilon + \langle e \cdot (\nabla_y^T \cdot a) U^\varepsilon \rangle - \langle b \cdot (e U^\varepsilon) \rangle)$$

$$- \int_{s'}^{s''} c^\varepsilon (\langle U^\varepsilon \rangle + \langle k(y) (U^\varepsilon)^p \rangle) ds. \tag{6.26}$$

Letting $\varepsilon \to 0$ in (6.26) shows that

$$\langle e^T a e U \rangle |_{s'}^{s''} = \int_{s'}^{s''} ds (\langle e \cdot (\nabla_y^T \cdot a) U \rangle - \langle b \cdot e U \rangle) - c \int_{s'}^{s''} (\langle U \rangle + \langle k(y) U^p \rangle) ds. \tag{6.27}$$

Equation (6.27) implies that

$$\langle e^T a e U \rangle_s = \langle e \cdot (\nabla_y^T \cdot a) U \rangle - \langle b(y) \cdot e U \rangle - c(\langle U \rangle + \langle k(y) U^p \rangle). \tag{6.28}$$

Arguing as before, we see that there exists $\{s_l^+\} \to +\infty$ such that

$$\langle e^T a e U \rangle_s |_{s = s_l^+} \to 0$$

as $l \to +\infty$. Setting $s = s_l^+$ in (6.28) and letting $l$ go to $+\infty$, we obtain

$$- \langle b(y) \cdot e \rangle U_+ = c(U_+ + \langle k \rangle U_+^p). \tag{6.29}$$

Similarly, there exists a sequence $\{s_l^-\} \to -\infty$ as $l \to +\infty$ such that

$$\langle e^T a e U \rangle_s |_{s = s_l^-} \to 0,$$

$$- \langle b \cdot e \rangle u_l = c(u_l + \langle k \rangle u_l^p). \tag{6.30}$$

Combining (6.29) and (6.30), we have that if $U_+ > 0$, then

$$c = \frac{- \langle b \cdot e \rangle}{1 + \langle k \rangle U_+^{p-1}} = \frac{- \langle b \cdot e \rangle}{1 + \langle k \rangle u_l^{p-1}}. \tag{6.31}$$

It follows that $U_+ = u_l$, which contradicts $U_+ \leqq u_0 < u_l$. Thus $U_+ = 0$.

To study the decay of $U$ to zero, let us consider (6.28) for $s \geqq s_0$, where $s_0$ is so large that if $s \geqq s_0$, then $U(s, y) \leqq \gamma$ for some $\gamma \ll 1$ to be chosen below. It follows from (6.28) that

$$\begin{aligned}
\langle e^T a e U \rangle_s &\leqq \left( c + \max_{T^n} (|b(y) \cdot e| + |e \cdot (\nabla_y^T \cdot a(y))|) \right) \langle U \rangle - c \langle k(y) U^p \rangle \\
&\leqq \gamma^{1-p} \left( c + \max_{T^n} (|b \cdot e| + |e \cdot (\nabla_y^T \cdot a)|) \right) \frac{1}{\min_{T^n} (e^T a e)} \langle e^T a e U^p \rangle \\
&\quad - c \min_{y \in T^n} \left( \frac{k(y)}{e^T a(y) e} \right) \langle e^T a e U^p \rangle \\
&\leqq - \frac{c}{2} \min_{T^n} \left( \frac{k(y)}{e^T a(y) e} \right) \langle e^T a(y) e U^p \rangle,
\end{aligned} \tag{6.32}$$

which is possible with a small enough $\gamma$ depending only on $a(y)$, $k(y)$, $b(y)$, $p$, and $c$. Inequality (6.32) implies that

$$\langle e^T aeU\rangle_s \leqq -\beta\langle e^T aeU\rangle, \tag{6.33}$$

where $\beta = \frac{c}{2}\min_{T^n}\frac{k}{e^T ae}$. We see right away from (6.33) that

$$\langle U\rangle \leqq \exp\{-\beta s\} B \tag{6.34}$$

for $s \geqq s_0$ and for some positive constant $B$ depending only on $a(y)$. Thus $\langle U\rangle$ decays to zero exponentially as $s \to +\infty$.

On the open set $G = \{(s, y) \in R^1 \times T^n \mid U(s, y) > 0\}$, $U$ is a classical solution and its derivative $U_s \geqq 0$. The function $U_s$ satisfies the strong maximum principle on $G$. If it is zero at any finite point in $G$, then $U_s \equiv 0$ on $G$, which implies that $U \equiv u_l$ on $G$. This contradicts the fact that $U \to 0$ as $s \to +\infty$. Thus $U_s > 0$ on $G$. The proof of Theorem 1.3 is complete.

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