White Noise Perturbation of the Viscous Shock Fronts of the Burgers Equation

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Abstract

We study the front dynamics of solutions of the initial value problem of the Burgers equation with initial data being the viscous shock front plus the white noise perturbation. In the sense of distribution, the solutions propagate with the same speed as the unperturbed front, however, the front location is random and satisfies a central limit theorem with the variance proportional to the time t, as t goes to infinity. With probability arbitrarily close to one, the front width is $O(1)$ for large time.

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1 Introduction

We are concerned with the initial value problem of the Burgers equation:

$$u_t + uu_x = \nu u_{xx}, \quad \nu > 0, x \in R^1, \tag{1.1}$$

with initial data:

$$u(x, 0) = u_s + V_x, \tag{1.2}$$

where $u_s = u_s(x)$ is the profile of the viscous shock front connecting one and zero, V_x is the white noise, or formally the derivative of a two-sided Wiener process W_x starting from zero. Without the white noise perturbation, we have the exact solution:

$$u(x, t) = (1 + \exp\{\frac{1}{2\nu}(x - \frac{1}{2}t)\})^{-1}, \tag{1.3}$$

where x can be shifted by any constant $x_0 \in R^1$, and we choose it to be zero for convenience. It is well known that the viscous shock front (1.3) is asymptotically stable if it is perturbed by an intergrable function at $t = 0$, see Ilin and Oleinik [8]. We are interested here in the behavior of viscous shock fronts under random perturbations. In reality, random perturbations abound in dissipative dynamical systems admitting front solutions. In the case of conservative systems that are derived based on conservation of mass, through suitable asymptotic reductions, one often ends up with a scalar conservation law with either random coefficients or random initial data. Besides in the traditional gas dynamics [15], scalar conservation laws are found in: 1) surface water infiltration into randomly layered soils, see [10], [11], etc, where water concentration obeys the Richards equation, a viscous conservation law whose front solutions are called wetting fronts; 2) transport of contaminants (such as heavy metals) in porous media with randomly distributed sortion sites, [16], [4] etc., where the pollutant concentration satisfies another viscous conservation law whose front solutions are clear indications of underground contaminant movement. In these problems, one faces the difficulty of both randomness and nonlinearity. It is in general hard to pursue detailed analysis unless one specializes to completely solvable cases. On the other hand, what we learn from these special situations helps us gain insight and can serve as guidelines for numerical investigations on nonintegrable cases. This is our motivation to consider Burgers equation with white noise (or Gaussian processes, see later) initial perturbation imposed on fronts. Random initial data problems are related to yet typically simpler than random coefficient ones, hence should be first considered. We hope to extend our work here to random coefficient problems in the future.

Since each realization of the white noise is unbounded and not integrable, classical stability results of [8] do not apply. We remark that various aspects of the Burgers equation with random initial data have been studied in numerous works (see [1], [2], [6], [12], [13],[14]). In particular, in [6] the inviscid Burgers equation with the initial data (1.2) was studied, using different methods and from a somewhat different point of view.

Our main concern is the noise effect on the front behavior, namely, in what sense one still sees a front, and whether the front width increases with time. Intuitively, noise tends to smear out the front structure. We first write down the solution explicitly with Cole-Hopf formula, and identify the different features of various terms as $t \to +\infty$. The initial condition enters the solution formula through its spatial integral. The formula thus makes sense even with singular initial data (1.2) and the randomness appears in it through the Wiener process W_x. We study the distribution of $u(x,t)$ using the scaling and Markov properties of W_x. To extract the asymptotics, the almost sure uniqess of the maximal point of the process $W_x - \frac{x^2}{2}$ is essential for applying the Laplace type method. Our main results are:

Theorem 1.1 *Let $u(x,t)$ be the solution to the problem (1.1)-(1.2), and $f(t)$ be an increasing function of t. Then as $t \to \infty$:*

1). If $t^{-\frac{1}{2}}(f(t) - \frac{1}{2}t) \to -\infty$,

$$u(f(t),t) \xrightarrow{\mathrm{d}} 1;$$

2). If $t^{-\frac{1}{2}}(f(t) - \frac{1}{2}t) \to +\infty$,

$$u(f(t),t) \xrightarrow{\mathrm{d}} 0;$$

3) If $\frac{f(t)-\frac{1}{2}t}{\sqrt{t}} \to c$, $u(f(t),t)$ converges in distribution to a random variable equal to zero with probability $N(c)$ and equal to one with probability $1 - N(c)$, where

$$N(c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{c} \exp\{-\frac{y^2}{2}\} dy,$$

the unit Gaussian distribution function.

Remark 1.1 *Notice that 1) and 2) of Theorem 1.1 are asymptotic cases of 3) when $c \to \pm\infty$.*

Theorem 1.2 (Front Width and Location) *Let $u(x,t)$ be the solution to the problem (1.1)-(1.2). Given any positive number $\epsilon \in (0,1)$, let us define the left and right endpoints of the interval containing the front as:*

$$z_-(t) = \min\{x : u(t,x) = 1 - \epsilon\},$$

$$z_+(t) = \max\{x : u(t,x) = \epsilon\}.$$

Then we have:
1) There exists a constant $t_0 > 0$ such that the random variables $\{z_+(t) - z_-(t)\}$ are tight for $t \geq t_0$, i.e. for any $\delta > 0$ there exists an M, such that $P(z_+(t) - z_-(t) > M) < \delta$ for all $t > t_0$.
2) As $s \to \infty$, we have

$$\frac{z_+(st) - \frac{st}{2}}{\sqrt{s}} \xrightarrow{\text{D}} W_t,$$

and the same is true for z_-.

Remark 1.2 *As part of the proof, we will show that $z_-(t)$, $z_+(t)$ are almost surely finite for $t \geq t_0$. In essence, the Theorem says that the noise does not spread the front width for large time and that the front location as a function of t, when properly centered and rescaled, converges in distribution to the Wiener process. In particular, $\frac{z_+(t)-\frac{t}{2}}{\sqrt{t}}$ converges in distribution to the unit Gaussian.*

Remark 1.3 *The classical result of Ilin and Oleinik [8] says that the system accomodates an integrable initial perturbation of the traveling front by shifting the front by a finite distance. In the case studied here, the perturbations (in addition to being highly irregular) are not integrable and the amount of additional "mass" the system has to "swallow" is infinite. At a finite time t the response of the system consists of shifting the front by a distance of the order $t^{\frac{1}{2}}$ and yet keeping the front width still at $O(1)$.*

The rest of the paper is organized as follows. In section 2, we put the solution formula into a convenient form for later analysis, and show a Laplace-type lemma. In section 3, we perform probabilistic analysis and prove Theorem 1.1. In section 4, we discuss the asymptotic distribution of front locations and the tightness of front widths, and prove Theorem 1.2.

2 Cole-Hopf Formula and a Laplace-type Lemma

By the substitution $u = -2\nu\frac{\varphi_x}{\varphi}$ for the solution of (1.1)-(1.2), we end up with solving the linear heat equation $\varphi_t = \nu\varphi_{xx}$ for φ and the well-known Cole-Hopf formula [15]:

$$u(x,t) = \left(\int_{-\infty}^{\infty} \frac{x-\eta}{t} exp\{-(2\nu)^{-1}G(\eta)\}d\eta\right)\left(\int_{-\infty}^{\infty} exp\{-(2\nu)^{-1}G(\eta)\}d\eta\right)^{-1}, \tag{2.1}$$

where

$$G(\eta) = G(\eta; x, t) = \int_0^\eta u_s(\eta')d\eta' + W(\eta) + (2t)^{-1}(x-\eta)^2. \tag{2.2}$$

Using (1.3), we have:

$$\int_0^\eta u_s(\eta')d\eta' = \int_0^\eta \frac{e^{-x/(2\nu)}}{1+e^{-x/(2\nu)}}dx = -2\nu \log(\frac{1+e^{-\eta/(2\nu)}}{2}),$$

and so:

$$\begin{aligned} u(x,t) &= \frac{\int_{-\infty}^{+\infty} \frac{x-\eta}{t}(\frac{1+e^{-\eta/(2\nu)}}{2}) \exp\{-\frac{W(\eta)}{2\nu} - \frac{(x-\eta)^2}{4\nu t}\}d\eta}{\int_{-\infty}^{+\infty}(\frac{1+e^{-\eta/(2\nu)}}{2}) \exp\{-\frac{W(\eta)}{2\nu} - \frac{(x-\eta)^2}{4\nu t}\}d\eta} \\ &= \frac{t^{-1}\int_{-\infty}^{\infty} \eta(1+e^{(2\nu)^{-1}(\eta-x)}) \exp\{-\frac{W(x-\eta)}{2\nu} - \frac{\eta^2}{4\nu t}\}d\eta}{\int_{-\infty}^{\infty}(1+e^{(2\nu)^{-1}(\eta-x)}) \exp\{-\frac{W(x-\eta)}{2\nu} - \frac{\eta^2}{4\nu t}\}d\eta} \\ &\equiv \frac{Nu}{De}. \end{aligned} \tag{2.3}$$

To study the asymptotic behavior of Nu and De, we rewrite Nu as:

$$\begin{aligned} Nu &= t^{-1}\int_{-\infty}^{\infty} \eta \exp\{-\frac{W(x-\eta)}{2\nu} - \frac{\eta^2}{4\nu t}\}d\eta \\ &+ t^{-1}\int_{-\infty}^{\infty} \eta \exp\{-\frac{W(x-\eta)}{2\nu} - \frac{\eta^2}{4\nu t} + \frac{\eta - x}{2\nu}\}d\eta \\ &= \int_{-\infty}^{\infty} \eta \exp\{-\frac{W(x-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu}\}d\eta \\ &+ t^{-1}\exp\{-\frac{x}{2\nu} + \frac{t}{4\nu}\}\int_{-\infty}^{\infty} \eta \exp\{-\frac{W(x-\eta)}{2\nu} - \frac{(\eta-t)^2}{4\nu t}\}d\eta \\ &= \int_{-\infty}^{\infty} \eta \exp\{-\frac{W(x-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu}\}d\eta \\ &+ \exp\{-(2\nu)^{-1}(x-\frac{t}{2})\}\int_{-\infty}^{\infty} \exp\{-\frac{W(x-\eta)}{2\nu} - \frac{(\eta-t)^2}{4\nu t}\}d\eta \\ &+ t^{-1}\exp\{-(2\nu)^{-1}(x-\frac{t}{2})\}\int_{-\infty}^{\infty} (\eta-t)\exp\{-\frac{W(x-\eta)}{2\nu} - \frac{(\eta-t)^2}{4\nu t}\}d\eta \\ &= \int_{-\infty}^{\infty} \eta \exp\{-\frac{W(x-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu}\}d\eta \\ &+ \exp\{-(2\nu)^{-1}(x-\frac{t}{2})\}\int_{-\infty}^{\infty} \exp\{-\frac{W(x-\eta-t)}{2\nu} - \frac{\eta^2}{4\nu t}\}d\eta \\ &+ t^{-1}\exp\{-(2\nu)^{-1}(x-\frac{t}{2})\}\int_{-\infty}^{\infty} \eta\exp\{-\frac{W(x-\eta-t)}{2\nu} - \frac{\eta^2}{4\nu t}\}d\eta \\ &= \int_{-\infty}^{\infty} \eta \exp\{-\frac{W(x-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu}\}d\eta \end{aligned}$$

$$
\begin{aligned}
&+ \quad \sqrt{t}\exp\{-(2\nu)^{-1}(x-\frac{t}{2})\}\int_{-\infty}^{\infty}\exp\{-\frac{W(x-\sqrt{t}\eta-t)}{2\nu}-\frac{\eta^2}{4\nu}\}d\eta \\
&+ \quad \exp\{-(2\nu)^{-1}(x-\frac{t}{2})\}\int_{-\infty}^{\infty}\eta\exp\{-\frac{W(x-\sqrt{t}\eta-t)}{2\nu}-\frac{\eta^2}{4\nu}\}d\eta, \qquad (2.4)
\end{aligned}
$$

and similarly:

$$
\begin{aligned}
De \quad &= \quad \sqrt{t}\int_{-\infty}^{\infty}\exp\{-\frac{W(x-\sqrt{t}\eta)}{2\nu}-\frac{\eta^2}{4\nu}\}d\eta \\
&+ \quad \sqrt{t}\exp\{-(2\nu)^{-1}(x-\frac{t}{2})\}\int_{-\infty}^{\infty}\exp\{-\frac{W(x-\sqrt{t}\eta-t)}{2\nu}-\frac{\eta^2}{4\nu}\}d\eta. \qquad (2.5)
\end{aligned}
$$

Combining (2.4) and (2.5), we get:

$$
\begin{aligned}
u(x,t) = \quad & (t^{-\frac{1}{2}}\int_{-\infty}^{\infty}\eta\exp\{-\frac{W(x-\sqrt{t}\eta)}{2\nu}-\frac{\eta^2}{4\nu}\}d\eta \\
&+ \quad \exp\{-(2\nu)^{-1}(x-\frac{t}{2})\}\int_{-\infty}^{\infty}\exp\{-\frac{W(x-\sqrt{t}\eta-t)}{2\nu}-\frac{\eta^2}{4\nu}\}d\eta \\
&+ \quad t^{-\frac{1}{2}}\exp\{-(2\nu)^{-1}(x-\frac{t}{2})\}\int_{-\infty}^{\infty}\eta\exp\{-\frac{W(x-\sqrt{t}\eta-t)}{2\nu}-\frac{\eta^2}{4\nu}\}d\eta) \\
&\cdot \quad (\int_{-\infty}^{\infty}\exp\{-\frac{W(x-\sqrt{t}\eta)}{2\nu}-\frac{\eta^2}{4\nu}\}d\eta \\
&+ \quad \exp\{-(2\nu)^{-1}(x-\frac{t}{2})\}\int_{-\infty}^{\infty}\exp\{-\frac{W(x-\sqrt{t}\eta-t)}{2\nu}-\frac{\eta^2}{4\nu}\}d\eta)^{-1} \qquad (2.6)
\end{aligned}
$$

We will analyze the solutions based on (2.6). To this end, we need:

Proposition 2.1 *If $\varphi(u) \in C(R^1)$, $\varphi(u) = O(u^2)$, as $u \to \infty$; $\varphi(u) < \varphi(u_0)$, $\forall u \neq u_0$; then for the probability measures μ_λ with densities $\frac{\exp\{\lambda\varphi(u)\}du}{\int_{R^1}\exp\{\lambda\varphi(u)\}du}$, we have:*

$$
\mu_\lambda \xrightarrow{\text{d}} \delta(u_0), \qquad (2.7)
$$

the unit mass at u_0, as $\lambda \to +\infty$. Moreover, the expected value $E_{\mu_\lambda}(u) \to u_0$, as $\lambda \to +\infty$.

Proof: Let $\psi(u) \in C^\infty(R^1)$, $|\psi(u)| \leq C(1+u^2)^m$, for some $m > 0$. Consider:

$$
\begin{aligned}
&\int\psi(u)d\mu_\lambda - \psi(u_0) = \int(\psi(u)-\psi(u_0))d\mu_\lambda \\
= \quad &\int_{|u-u_0|\leq\delta}(\psi(u)-\psi(u_0))d\mu_\lambda + \int_{|u-u_0|\geq\delta}(\psi(u)-\psi(u_0))d\mu_\lambda \\
= \quad & I + II, \qquad (2.8)
\end{aligned}
$$

where $\delta > 0$ is a small positive number. The first term is bounded as:

$$|I| \leq \sup_{|u-u_0|\leq\delta} |\psi(u) - \psi(u_0)| \equiv \omega_\psi(\delta, u_0). \tag{2.9}$$

The second term can be written as:

$$II = \int_{|u-u_0|\geq\delta} (\psi(u) - \psi(u_0)) \frac{\exp\{\lambda(\varphi(u) - \varphi(u_0))\}du}{\int_{R^1} \exp\{\lambda(\varphi(u) - \varphi(u_0))\}du}. \tag{2.10}$$

For any given $\delta > 0$, there exists a constant $K = K(\delta) > 0$, such that if $u \notin [u_0 - \delta, u_0 + \delta]$,

$$\varphi(u) - \varphi(u_0) \leq -K(\delta)|u - u_0|^2. \tag{2.11}$$

On the other hand, for any $\delta_1 > 0$, we have:

$$\begin{aligned} \int \exp\{\lambda(\varphi(u) - \varphi(u_0))\}du &\geq \int_{u\in[u_0-\delta_1, u_0+\delta_1]} \exp\{\lambda(\varphi(u) - \varphi(u_0))\}du \\ &\geq \int_{u\in[u_0-\delta_1, u_0+\delta_1]} \exp\{-\lambda\omega_\varphi(\delta_1, u_0)\}du = 2\exp\{-\lambda\omega_\varphi(\delta_1, u_0)\}\delta_1, \end{aligned} \tag{2.12}$$

where δ_1 is independent of λ. Combining (2.10), (2.11) and (2.12), we obtain:

$$\begin{aligned} |II| &\leq (2\delta_1)^{-1} \int_{|u-u_0|>\delta} |\psi(u) - \psi(u_0)| \exp\{-\lambda K(\delta)|u - u_0|^2 + \lambda\omega_\varphi(\delta_1, u_0)\}du \\ &\leq C(2\delta_1)^{-1} \exp\{\lambda\omega_\varphi(\delta_1, u_0)\} \int_{|u-u_0|>\delta} (1 + |u - u_0|^2)^m \exp\{-\lambda K(\delta)|u - u_0|^2\}du \\ &\leq C(2\delta_1)^{-1} \exp\{\lambda\omega_\varphi(\delta_1, u_0)\} C_1(m, \delta) \int_{|u-u_0|>\delta} \exp\{-\lambda\frac{\delta}{2}K(\delta)|u - u_0|\}du \\ &\leq C(2\delta_1)^{-1} \exp\{\lambda\omega_\varphi(\delta_1, u_0)\} C_1(m, \delta) \frac{4}{\delta K(\delta)\lambda} \exp\{-\lambda\frac{\delta^2 K(\delta)}{2}\}. \end{aligned}$$

Choosing δ_1 small enough so that $\omega_\varphi(\delta_1, u_0) \leq \delta^2 K(\delta)/4$, and letting $\lambda \to \infty$, we have:

$$\limsup_{\lambda\to\infty} |II| = 0,$$

while

$$\limsup_{\lambda\to\infty} |I| \leq \omega_\psi(\delta; u_0).$$

Finally, sending $\delta \to 0$, we conclude that

$$\int \psi(u)d\mu_\lambda \to \psi(u_0),$$

which implies (2.7) and in particular

$$\int u d\mu_\lambda \to u_0. \tag{2.13}$$

The proof is complete.

3 Proof of Theorem 1.1

In this section, we present probabilistic analysis based on the solution formula of the last section, and complete the proof of Theorem 1.1. The following is of fundamental importance for our analysis:

Proposition 3.1 *Let W be a two-sided Wiener process starting from 0, (i.e. $(W_u : u \geq 0)$ and $(W_u : u \leq 0)$ are two independent Wiener processes). Then with probability one*

$$J \stackrel{\text{def}}{=} sup_u(W_u - \frac{u^2}{2})$$

is finite and strictly positive, and there exists a unique u_0 for which

$$J = W_{u_0} - \frac{u_0^2}{2}.$$

Proof: That J is almost surely finite and positive follows from elementary properties of the Wiener process (see e.g. [9]). It is well-known that with probability one all local maxima of a Wiener path are different, and therefore the same is true for the Wiener process with a drift, thanks to the Girsanov theorem [9]. This implies the uniqueness of u_0. For completeness we present a more elementary proof: let

$$S_u = \sup\{W_v - \frac{v^2}{2} : v \in [-u, u]\}$$

and

$$\tau = \inf\{v > u : W_v - \frac{v^2}{2} = S_u\}.$$

τ is a stopping time. Therefore, by the strong Markov property of the Wiener process [9], on the event $\{\tau < \infty\}$, $B_u = W_{\tau+u} - W_\tau$ has the distribution of the Wiener process. The local behavior of B_u near $u = 0$ implies that $W_v - \frac{v^2}{2}$ takes a value larger than S_u in every neighborhood of τ and, consequently, that S_u can occur only once as a local maximum of the process $\{W_v - \frac{v^2}{2} : v > u\}$. A similar argument for $v < -u$ ends the proof.

Remark 3.1 *A similar argument based on the strong Markov property shows that the distributions of J and u_0 are continuous. The Fourier transform of the density of the joint distribution of J and u_0 was calculated in Groeneboom [7].*

In what follows we shall use a couple of lemmas about the behavior of sequences convergent in distribution under algebraic operations. The first one is known in the probability literature as Slutsky's Theorem (Durret [5]) and the remaining ones are similar in spirit. Since the proofs of all these facts are very short, we include them for completeness.

Lemma 3.1 (Slutsky's Theorem) *Let X_n and Y_n be sequences of random variables on a probability space $(\Omega, \mathcal{F}, P)$, and let X be a random variable such that $X_n \to X$ and $Y_n \to 0$ in distribution. Then $X_n + Y_n \to X$ in distribution.*

Proof: Let x be a continuity point of the distribution function of X. We need to prove that

$$\lim_{n\to\infty} P(X_n + Y_n \leq x) = P(X \leq x).$$

Let us choose ϵ so that $x - \epsilon$ and $x + \epsilon$ are also continuity points of $P(X \leq x)$. We have:

$$P(X_n \leq x - \epsilon) - P(Y_n > \epsilon) \leq P(X_n + Y_n \leq x) \leq P(X_n \leq x + \epsilon) + P(Y_n < -\epsilon).$$

In the limit when $n \to \infty$ this implies

$$P(X \leq x - \epsilon) \leq \liminf_{n\to\infty} P(X_n + Y_n \leq x) \leq \limsup_{n\to\infty} P(X_n + Y_n \leq x) \leq P(X \leq x + \epsilon),$$

and taking ϵ to zero completes the proof.

Lemma 3.2 *Let X_n, Y_n and Z_n be sequences of random variables on $(\Omega, \mathcal{F}, P)$, such that $Y_n > 0$, $Z_n \geq 0$ and $\frac{X_n}{Y_n} \to 0$ in distribution. Then also $\frac{X_n}{Y_n+Z_n} \to 0$ in distribution.*

Proof: Clearly

$$\left|\frac{X_n}{Y_n + Z_n}\right| \leq \left|\frac{X_n}{Y_n}\right|$$

and the last expression converges to zero in distribution. Hence $\left|\frac{X_n}{Y_n+Z_n}\right|$, and therefore also $\frac{X_n}{Y_n+Z_n}$ converges to zero in distribution, as claimed.

Lemma 3.3 *Let p_n and X_n be sequences of random variables on $(\Omega, \mathcal{F}, P)$, such that $p_n \to 0$ and $X_n \to X$ in distribution. Then also $p_n X_n \to 0$ in distribution.*

Proof: Fix $\epsilon > 0$. For any $K > 0$ we have:

$$P(|p_n X_n| > \epsilon) \leq P(|p_n| > \frac{\epsilon}{K}) + P(|X_n| > K),$$

so that for any $K > 0$

$$\limsup_{n\to\infty} P(|p_n X_n| > \epsilon) \leq \limsup_{n\to\infty} P(|X_n| > K)$$

and, consequently,

$$\lim_{n\to\infty} P(|p_n X_n| > \epsilon) = 0,$$

which completes the proof.

Remark 3.2 *$p_n X_n$ does not have to converge almost surely, even if p_n does.*

Let us call the consecutive terms in the numerator of (2.6) A_t, B_t and C_t and the consecutive terms in the denominator D_t and B_t (the second terms in the numerator and in the denominator are identical).

Lemma 3.4 *Let $x = f(t)$, where f is an arbitrary (measurable) function of t. Then, as $t \to \infty$, we have*

$$\frac{A_t + C_t}{D_t + B_t} \to 0$$

in distribution. Thus, by Lemma 3.1, the limiting distribution of $u(x,t)$ (if it exists) is the same as the limiting distribution of $\frac{B_t}{D_t+B_t}$.

Proof: Lemmas 3.1 and 3.2 show that it is enough to prove that

$$\frac{A_t}{D_t} \to 0$$

and

$$\frac{C_t}{B_t} \to 0$$

in distribution.
To prove the first of these two claims, we rewrite $\frac{A_t}{D_t}$ as

$$\frac{\frac{1}{\sqrt{t}} \exp\left(-\frac{W(x)}{2\nu}\right) \int \eta \exp \frac{W(x)-W(x-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu}\, d\eta}{\exp\left(-\frac{W(x)}{2\nu}\right) \int \exp \frac{W(x)-W(x-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu}\, d\eta}.$$

Now, for any fixed x and t, the process $W(x) - W(x - \sqrt{t}\eta)$ has the same distribution as $W(\sqrt{t}\eta)$, by the Markov property of the Wiener process. Changing the variable of integration to $y = t^{-\frac{1}{6}}\eta$ and using the scaling property of the Wiener process, we thus obtain

$$\frac{A_t}{D_t} \overset{\mathrm{d}}{=} t^{-\frac{1}{2}} \frac{t^{\frac{1}{3}} \int y \exp(\frac{1}{2\nu}(W(t^{\frac{2}{3}}y) - \frac{t^{\frac{1}{3}}y^2}{2}))\, dy}{t^{\frac{1}{6}} \int \exp(\frac{1}{2\nu}(W(t^{\frac{2}{3}}y) - \frac{t^{\frac{1}{3}}y^2}{2}))\, dy} \overset{\mathrm{d}}{=} t^{-\frac{1}{3}} \frac{\int y \exp(\frac{t^{\frac{1}{3}}}{2\nu}(W(y) - \frac{y^2}{2}))\, dy}{\int \exp(\frac{t^{\frac{1}{3}}}{2\nu}(W(y) - \frac{y^2}{2}))\, dy}.$$

Using Propositions 2.1 and 3.1, we see that the ratio of the two integrals in the last expression converges almost surely to a finite, nonzero limit and thus, applying Lemma 3.3 twice, we see that $\frac{A_t}{D_t} \overset{\mathrm{d}}{\to} 0$, as claimed. An analogous argument shows that $\frac{C_t}{B_t} \overset{\mathrm{d}}{\to} 0$ and the proof is finished.

To analyze the behavior of $\frac{B_t}{D_t+B_t}$, depending on how x varies with t, let us write, using a similar idea as above:

$$\frac{B_t}{B_t+D_t} = \frac{p(t)\tilde{B}_t}{p(t)\tilde{B}_t+\tilde{D}_t},$$

with $\tilde{B}_t \stackrel{\text{def}}{=} \int \exp(\frac{W(x-t)-W(x-t-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu})\,d\eta$, $\tilde{D}_t \stackrel{\text{def}}{=} \int \exp(\frac{W(x)-W(x-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu})\,d\eta$ and $p(t) = p(x,t) \stackrel{\text{def}}{=} \exp(\frac{-(x-\frac{1}{2}t)+W(x)-W(x-t)}{2\nu})$.

Lemma 3.5 *Let x be an arbitrary (measurable) function of t. Then as $t \to \infty$, both $t^{-\frac{1}{3}}\log\tilde{B}_t$ and $t^{-\frac{1}{3}}\log\tilde{D}_t$ converge in distribution to a.s. positive limits.*

Proof: Proceeding as in the proof of Lemma 3.4, we substitute $y = t^{-\frac{1}{6}}\eta$; and using the Markov property of the Wiener process, we obtain:

$$\tilde{B}_t \stackrel{\text{d}}{=} t^{-\frac{1}{6}} \int \exp(t^{\frac{1}{3}}(W(u) - \frac{u^2}{2}))\,du.$$

By Propositions 2.1 and 3.1, the logarithm of the last expression, multiplied by $t^{-\frac{1}{3}}$ converges with probability one to $\sup(W(u) - \frac{u^2}{2})$. This proves the assertion of the lemma for $\tilde{B}_t$ and the proof for $\tilde{D}_t$ is analogous.

We are now ready to prove Theorem 1.1.

Proof of Thoerem 1.1 1):

We want to show that

$$\frac{p(t)\tilde{B}_t}{p(t)\tilde{B}_t+\tilde{D}_t} \stackrel{\text{d}}{\to} 1,$$

or, equivalently, (by Lemmas 3.1 and 3.3) that

$$\frac{\tilde{D}_t}{p(t)\tilde{B}_t} \stackrel{\text{d}}{\to} 0.$$

Since the above expression is positive, it is enough to show that for any $y > 0$,

$$P(\frac{\tilde{D}_t}{p(t)\tilde{B}_t} \le y) \to 1.$$

We have:

$$P(\frac{\tilde{D}_t}{p(t)\tilde{B}_t} \le y) = P(\log p(t) \ge \log\frac{\tilde{D}_t}{\tilde{B}_t} - \log y) =$$

$$P(\frac{W(x)-W(x-t)}{\sqrt{t}} \ge \frac{f(t)-\frac{1}{2}t}{\sqrt{t}} + \frac{2\nu}{\sqrt{t}}\log\frac{\tilde{D}_t}{\tilde{B}_t} - 2\nu\frac{\log y}{\sqrt{t}}).$$

Thanks to the assumption of the theorem and to Lemma 3.5, the expression on the right hand side of the inequality goes in distribution to $-\infty$. Since for any t and x, $\frac{W(x)-W(x-t)}{\sqrt{t}}$ is a unit Gaussian variable, the probability in the last expression goes to 1 and the proof is finished.

Proof of Theorem 1.1, 2): We proceed exactly as in the proof of Theorem 1.1, 1), except that now in the last expression the direction of the inequality is reversed. As a result, $P(\frac{\tilde{D}_t}{p(t)\tilde{B}_t} \leq y) \to 0$, which implies that $\frac{\tilde{D}_t}{p(t)\tilde{B}_t} \xrightarrow{\text{d}} \infty$, i.e. that $u(x,t) \xrightarrow{\text{d}} 0$ as desired.

Proof of Theorem 1.1, 3): We will prove the theorem in the case $x = \frac{1}{2}t + c\sqrt{t}$ only; essentially the same proof applies to the general case. Let $0 < y < 1$. We have:

$$P(\frac{p(t)\tilde{B}_t}{p(t)\tilde{B}_t + \tilde{D}_t} \leq y) = P(p(t) \leq \frac{\tilde{D}_t y}{\tilde{B}_t(1-y)}) = P(\log p(t) \leq \log \frac{\tilde{D}_t}{\tilde{B}_t} + \log \frac{y}{1-y}).$$

In our case

$$\log p(t) = \frac{1}{2\nu}(-c\sqrt{t} + \chi_t),$$

where $\chi_t = W(x) - W(x-t)$ is a centered Gaussian variable with variance t. Hence

$$P(\frac{p(t)\tilde{B}_t}{p(t)\tilde{B}_t + \tilde{D}_t} \leq y) = P(\chi_t \leq c\sqrt{t} + 2\nu[\log \frac{\tilde{D}_t}{\tilde{B}_t} + \log \frac{y}{1-y}])$$

$$= P(\frac{\chi_t}{\sqrt{t}} \leq c + \frac{2\nu}{\sqrt{t}}[\log \frac{\tilde{D}_t}{\tilde{B}_t} + \log \frac{y}{1-y}]),$$

and, since $\frac{\chi_t}{\sqrt{t}}$ is a centered unit Gaussian variable, the theorem follows from the Lemmas 3.1 and 3.3.

4 Front Width and Asymptotic Distribution of Front Location

So far we have been studying the asymptotic behavior of the distribution of $u(x,t)$, where x was a deterministic function of t. We saw that the solution with randomly perturbed initial condition still in some sense has the front structure. We now want to propose a way to determine the location of the front. This location will be a fuction of t and of the realization of the white noise and we will study its distribution as t goes to infinity. We will first prove Theorem 1.2 for the endpoints defined in a slightly different way (see Theorems 4.1 and 4.2 below). Next, using this result and some techniques developed in its proof, we will prove the Theorem as stated. In fact, in the proofs of Theorems 4.1 and 4.2 we will find it convenient to use yet another modification of the endpoint definition.

As we have seen in Lemma 3.4, the solution $u(x,t)$ to our problem can be written as a sum of two terms, $z(x,t) = \frac{A_t + C_t}{D_t + B_t}$ and $v(x,t) = \frac{B_t}{D_t + B_t}$. It follows from Lemma 3.4 that

$$sup_x|z(x,t)| \xrightarrow{\mathrm{d}} 0.$$

The term $z(x,t)$ is therefore transient in a very strong sense and for the purpose of studying the front structure, we will first analyze the behavior of $v(x,t)$.

Note that $0 < v(x,t) < 1$.

Lemma 4.1 *For every $t > 0$ we have*

$$\lim_{x \to -\infty} v(x,t) = 1$$

and

$$\lim_{x \to \infty} v(x,t) = 0$$

with probability one.

Proof: We will prove the first claim; the second one is proven in a similar way. Since we are studying the behavior of $v(x,t)$ for a typical realization of the randomness and not its distribution, it is of advantage to rewrite it with deterministic exponential prefactors in front of the integrals:

$$v(x,t) = \frac{\exp(-\frac{x-\frac{t}{2}}{2\nu}) \int \exp(-\frac{W(x-t-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu})d\eta}{\exp(-\frac{x-\frac{t}{2}}{2\nu}) \int \exp(-\frac{W(x-t-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu})d\eta + \int \exp(-\frac{W(x-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu})d\eta}.$$

We will first estimate the integral in the numerator from below. For this, it is enough to consider η between -1 and 1. We then have

$$x - t - \sqrt{t} \le x - t - \sqrt{t}\eta \le x - t + \sqrt{t}$$

and hence, for $-x$ large enough (how large, depends on the realization of the Wiener process),

$$W(x - t - \sqrt{t}\eta) \le \frac{1}{2}(t - x)$$

for all η in the interval $[-1, 1]$. We are using here the well-known fact that with probability one $\frac{W_u}{u} \to 0$ (see e.g. [9]). Consequently, for large negative x the numerator is bounded below by const. $\exp(\frac{-x}{4\nu})$ (the constant depends on t). The first term in the denominator is identical to the numerator, so we just need to show that the second term cannot diverge too fast. With probability one there exists a constant c such that for all u

$$W_u > c|u|^{\frac{2}{3}}.$$

This follows, for example, from the Law of the Iterated Logarithm ([9]), although a more elementary proof can be given. Consequently, the second term in the denominator is bounded above by

$$\int \exp(\frac{C|x-\sqrt{t}\eta|^{\frac{2}{3}}}{2\nu} - \frac{\eta^2}{4\nu})d\eta.$$

This diverges slower than exponentially in x and the claim follows.

Definition 4.1 Fix a number $\epsilon \in (0, \frac{1}{2})$. Let

$$\begin{aligned} x_-(t) &= min\{x : v(x,t) = 1-\epsilon\}; \\ x_+(t) &= max\{x : v(x,t) = \epsilon\}. \end{aligned} \tag{4.1}$$

We think of $x_-(t)$ and $x_+(t)$ as the endpoints of the interval containing the front, distorted by the random perturbation of the initial condition. They depend on the choice of ϵ, but as the results below show, this dependence is not changing their behavior in distribution in a significant way.

Theorem 4.1 *The family* $\frac{x_+(t)-x_-(t)}{t^{\frac{1}{3}}}$ *is tight for* $t \geq t_0$*, for some finite* $t_0 > 0$.

Proof: By definition of x_+, we have

$$\frac{p(x_+(t),t)\tilde{B}(x_+(t),t)}{\tilde{D}(x_+(t),t)} = \frac{\epsilon}{1-\epsilon}$$

i.e.

$$\log p(x_+(t),t) + \log\frac{\tilde{B}(x_+(t),t)}{\tilde{D}(x_+(t),t)} = \log\frac{\epsilon}{1-\epsilon}$$

and, similarly,

$$\log p(x_-(t),t) + \log\frac{\tilde{B}(x_-(t),t)}{\tilde{D}(x_-(t),t)} = \log\frac{1-\epsilon}{\epsilon}.$$

Let $\eta > 0$. Since $t^{-\frac{1}{3}}\log\tilde{B}(x_+(t),t)$ and $t^{-\frac{1}{3}}\log\tilde{D}(x_+(t),t)$ converge in distribution, their difference is tight for $t \geq t_0$ for some $t_0 > 0$. We can thus find an $M > 0$ such that

$$P(t^{-\frac{1}{3}}|\log\frac{\tilde{B}(x_+(t),t)}{\tilde{D}(x_+(t),t)}| > M) \leq 2\eta, \tag{4.2}$$

for $t \geq t_0 > 0$. Similarly, M can be chosen to satisfy

$$P(t^{-\frac{1}{3}}|\log\frac{\tilde{B}(x_-(t),t)}{\tilde{D}(x_-(t),t)}| > M) \leq 2\eta. \tag{4.3}$$

Let us introduce the modified front endpoints by

$$\begin{aligned}\tilde{x}_+(t) &= max\{x : \log p(x,t) = \log\frac{\epsilon}{1-\epsilon} - Mt^{\frac{1}{3}}\};\\ \tilde{x}_-(t) &= min\{x : \log p(x,t) = \log\frac{1-\epsilon}{\epsilon} + Mt^{\frac{1}{3}}\}.\end{aligned} \tag{4.4}$$

Of course, on the event $\{|t^{-\frac{1}{3}}\log\frac{\tilde{B}(x_+(t),t)}{\tilde{D}(x_+(t),t)}| \leq M\} \cap \{|t^{-\frac{1}{3}}\log\frac{\tilde{B}(x_-(t),t)}{\tilde{D}(x_-(t),t)}| \leq M\}$, we have

$$\tilde{x}_-(t) \leq x_-(t) \leq x_+(t) \leq \tilde{x}_+(t). \tag{4.5}$$

It is therefore enough to show that the random variables $\frac{\tilde{x}_+(t)-\tilde{x}_-(t)}{t^{\frac{1}{3}}}$ are tight for any choice of M. For u real (positive or negative) let $\mathcal{F}_u = \sigma(\{W_s : s \leq u\})$ (the σ-field generated by the random variables W_s with $s \leq u$). Then $\tilde{x}_-(t)$ is a stopping time relative to the filtration $(\mathcal{F}_u)$. Using the definition of $p(x,t)$, we have

$$P(\tilde{x}_+(t) - \tilde{x}_-(t) > Kt^{\frac{1}{3}}) \leq$$

$$\begin{aligned}P(\exists x > \tilde{x}_-(t) + Kt^{\frac{1}{3}} : W(x) - W(x - \frac{1}{2}t) - W(\tilde{x}_-) + W(\tilde{x}_- - \frac{1}{2}t)\\ = x - \tilde{x}_- - 4\nu Mt^{\frac{1}{3}} - 4\nu\log\frac{1-\epsilon}{\epsilon}).\end{aligned}$$

If the last event happens, then either

$$W(x) - W(\tilde{x}_-) \geq \frac{1}{2}(x - \tilde{x}_-) - 4\nu Mt^{\frac{1}{3}},$$

or

$$W(x - \frac{1}{2}t) - W(\tilde{x}_- - \frac{1}{2}t) \leq -\frac{1}{2}(x - \tilde{x}_-) + 4\nu Mt^{\frac{1}{3}}$$

for t large enough. Let us estimate the probability of the first one of these events. Since $\tilde{x}_-(t)$ is a stopping time, the strong Markov property of the Wiener process [9] implies that the probability is equal to

$$P(\exists u > Kt^{\frac{1}{3}} : W(u) \geq \frac{u}{2} - 4\nu Mt^{\frac{1}{3}}).$$

If we now choose K large enough, this is bounded above by

$$P(\exists u > Kt^{\frac{1}{3}} : W(u) \geq \frac{u}{3}),$$

and this probability can be made as small as desired (uniformly in $t \geq t_0$), by choosing K large. The second probability is estimated in an analogous way and the theorem is proved.

Proof of Theorem 1.2,1): We have

$$P(x_+(t) - x_-(t) \geq M) = P(x_+(t) \geq x_-(t) + M; \frac{p(x_+(t),t)}{p(x_-(t),t)} \frac{\tilde{B}(x_+(t),t)}{\tilde{B}(x_-(t),t)} \frac{\tilde{D}(x_-(t),t)}{\tilde{D}(x_+(t),t)} = \delta),$$

where $\delta \equiv (\frac{\epsilon}{1-\epsilon})^2$. Shifting the variable of integration in the integral defining $\tilde{D}(x_+(t),t)$ by $\frac{x_+(t)-x_-(t)}{\sqrt{t}}$, we obtain:

$$\tilde{D}(x_+(t),t) = \exp(\frac{W(x_+(t)) - W(x_-(t))}{2\nu}) \exp(\frac{-(x_+(t) - x_-(t))^2}{4\nu t})$$
$$\int \exp(\frac{W(x_-(t)) - W(x_-(t) - \sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu} + \frac{\eta(x_+(t) - x_-(t))}{4\nu\sqrt{t}})\, d\eta,$$

and

$$\tilde{B}(x_+(t),t) = \exp(\frac{W(x_+(t)-t) - W(x_-(t)-t)}{2\nu}) \exp(\frac{-(x_+(t) - x_-(t))^2}{4\nu t})$$
$$\int \exp(\frac{W(x_-(t)-t) - W(x_-(t) - t - \sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu} + \frac{\eta(x_+(t) - x_-(t))}{4\nu\sqrt{t}})\, d\eta.$$

Since

$$\frac{p(x_+(t),t)}{p(x_-(t),t)} = \exp(\frac{x_-(t) - x_+(t)}{2\nu}) \exp(\frac{W(x_+(t)) - W(x_-(t))}{2\nu}) \cdot$$
$$\exp(\frac{W(x_-(t)-t) - W(x_+(t)-t)}{2\nu}),$$

we obtain, after a cancellation,

$$P(x_+(t) - x_-(t) \geq M) \leq$$
$$P(x_+ - x_- \geq M; \frac{\int \exp(\frac{W(x_-(t))-W(x_-(t)-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu})\, d\eta,}{\int \exp(\frac{W(x_-(t))-W(x_-(t)-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu} + \frac{\eta(x_+(t)-x_-(t))}{4\nu\sqrt{t}})\, d\eta}$$
$$\geq \frac{\epsilon}{1-\epsilon} \exp(\frac{x_+(t) - x_-(t)}{4\nu}))$$
$$+ \quad P(x_+ - x_- \geq M; \frac{\int \exp(\frac{W(x_-(t)-t)-W(x_-(t)-t-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu} + \frac{\eta(x_+(t)-x_-(t))}{4\nu\sqrt{t}})\, d\eta}{\int \exp(\frac{W(x_-(t)-t)-W(x_-(t)-t-\sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu})\, d\eta}$$
$$\geq \frac{\epsilon}{1-\epsilon} \exp(\frac{x_+(t) - x_-(t)}{4\nu})). \tag{4.6}$$

We shall estimate the first term; the other term is estimated the same way. First, let us show that for any $c > 0$ there is an A large enough, so that the probability

$$P(\forall \eta < -At^{\frac{1}{6}} : W(x_-(t) - W(x_-(t) - \sqrt{t}\eta) \leq c\eta^2)$$

is arbitrarily close to 1. Indeed, using the scaling and strong Markov properties of the Wiener process and the scaling property of the Wiener process [9], we have

$$\begin{aligned} P(\exists \eta < -At^{\frac{1}{6}} : W(x_-(t) - W(x_-(t) - \sqrt{t}\eta) \geq c\eta^2) &= P(\exists \eta < -At^{\frac{1}{6}} : t^{\frac{1}{4}} W(\eta) \geq c\eta^2) \\ &= P(\exists u < -A : W(u) \geq cu^2) \quad (4.7) \end{aligned}$$

and the last quantity can be made arbitrarily small by choosing A large. Consequently, with probability arbitrarily close to 1, we have

$$\int_{\eta \leq -At^{\frac{1}{6}}} \exp(\frac{W(x_-(t) - t) - W(x_-(t) - t - \sqrt{t}\eta)}{2\nu} - \frac{\eta^2}{4\nu} + \frac{\eta(x_+(t) - x_-(t))}{4\nu\sqrt{t}})\, d\eta.$$

$$\leq \int_{\eta \leq -At^{\frac{1}{6}}} \exp(\frac{c\eta^2}{2\nu} - \frac{\eta^2}{4\nu} + \frac{\eta(x_+(t) - x_-(t))}{4\nu\sqrt{t}})\, d\eta.$$

Choosing c so that $\frac{c}{2} < \frac{1}{4}$, we obtain a bound on this part of the integral of the form

$$\exp(C\frac{(x_+(t) - x_-(t))^2}{t})$$

and this is close to 1 with probability close to 1, since by the previous lemma $x_+ - x_-(t) \leq Kt^{\frac{1}{3}}$ with probability close to 1. On the other hand, the whole integral in the denominator is of the order $\exp C_2 t^{\frac{1}{3}}$ with probability close to 1 by Lemma 3.5 and Theorem 4.1. Hence, we do not change the order of the integral in the denominator by restricting the η in the denominator to $\{\eta > -At^{\frac{1}{6}}\}$. A similar (but simpler) argument shows that the integral in the numerator does not change its order of magnitude either, when η is restricted to this set. Finally, we know that with probability close to 1, $|x_+(t) - x_-(t)| \leq Ct^{\frac{1}{3}}$, provided C is large enough. For those realizations for which all the events with probabilities close to 1, mentioned above, hold, we thus pull out a lower bound

$$C_1 \exp(-C_2 \frac{t^{\frac{1}{6}} t^{\frac{1}{3}}}{t^{\frac{1}{2}}}) = C_3$$

on the factor $e^{\frac{\eta(x_+(t) - x_-(t))}{4\nu\sqrt{t}}}$. Then the first term on the right hand side of (4.6) is (up to an arbitrarily small probability) bounded by

$$P(x_+ - x_- \geq M; C_3 \geq \frac{\epsilon}{1-\epsilon} \exp(\frac{x_+(t) - x_-(t)}{4\nu})).$$

The bound on the second term in (4.6) is proven in the same way. The tightness of $x_+(t) - x_-(t)$ is proven.

We now show that the front endpoints as defined in the formulation of Theorem 1.2 satisfy the same tightness statement. To this end it is enough to show that the random variables $x_+(t) - z_+(t)$ are tight. Note that with probability arbitrarily close to one we have

$$\sup |u(x,t) - v(x,t)| < \frac{\epsilon}{2}$$

if we take t large enough. Thus if $z_+(t) > x_+(t) + M$, then except on a set of small probability, we have two points $x_1 = x_+(t)$ and $x_2 = z_+(t)$, for which $x_2 > x_1 + M$, $v(x_1) = \epsilon$ and $v(x_2) > \frac{\epsilon}{2}$. The probability of existence of two such points can be estimated exactly as in the proof of tightness of $x_+(t) - x_-(t)$ above. A similar argument applies to estimate the probability that $z_+(t) < x_+(t) - M$ and the analogous events for $z_-(t)$. This shows that all the properties of $x_+(t)$ and $x_-(t)$ we have proven, also hold for $z_+(t)$ and $z_-(t)$ and we complete the proof of Theorem 1.2, 1).

Theorem 4.2 *As $t \to \infty$,*

$$\frac{x_+(t) - \frac{t}{2}}{\sqrt{t}} \xrightarrow{\text{d}} W_1,$$
$$\frac{x_-(t) - \frac{t}{2}}{\sqrt{t}} \xrightarrow{\text{d}} W_1. \tag{4.8}$$

Proof: As in the above proof, we will use the modified front endpoints $\tilde{x}_+(t)$ and $\tilde{x}_-(t)$. We have:

$$P(\frac{\tilde{x}_+(t) - \frac{t}{2}}{\sqrt{t}} \le y) \le P(W(y\sqrt{t} + \frac{t}{2}) - W(y\sqrt{t} - \frac{t}{2}) \le y\sqrt{t} + 2\nu \log \frac{1-\epsilon}{\epsilon} - 2\nu M t^{\frac{1}{3}})$$
$$= P(\frac{W(y\sqrt{t} + \frac{t}{2}) - W(y\sqrt{t} - \frac{t}{2})}{\sqrt{t}} \le y + 2\nu \frac{\log \frac{1-\epsilon}{\epsilon}}{\sqrt{t}} - 2\nu \frac{M}{t^{\frac{1}{6}}}) \to \mathcal{N}(y).$$

Similarly,

$$P(\frac{\tilde{x}_-(t) - \frac{t}{2}}{\sqrt{t}} \le y) \ge \mathcal{N}(\dagger), \tag{4.9}$$

as t becomes large enough. On the other hand,

$$P(\frac{\tilde{x}_+(t) - \frac{t}{2}}{\sqrt{t}} \le y + \delta) \ge P(\frac{\tilde{x}_-(t) - \frac{t}{2}}{\sqrt{t}} \le y) - P(\frac{\tilde{x}_+(t) - \tilde{x}_-(t)}{\sqrt{t}} > \delta).$$

Since, by the previous theorem, the last term goes to zero, the lim inf of the left hand side is bounded below by $\mathcal{N}(y)$. Taking δ to zero implies that

$$\frac{\tilde{x}_+(t) - \frac{t}{2}}{\sqrt{t}} \xrightarrow{\text{d}} W_1.$$

The previous theorem now implies that also

$$\frac{\tilde{x}_-(t) - \frac{t}{2}}{\sqrt{t}} \xrightarrow{\text{d}} W_1.$$

Convergence to normal for the $x_+(t)$ and $x_-(t)$ follows now easily from the relation (4.5). The proof is complete.

Proof of Theorem 1.2, 2): Step I. The finite-dimensional distributions of the process $\frac{x_+(st)-\frac{st}{2}}{\sqrt{s}}$, $0 \le t \le 1$, converge to those of $W(t)$ as $s \to \infty$. We shall only prove this for the two-dimensional distributions; the proof in the general case is essentially the same, but notationally more complicated. Again, in view of (4.5) it is enough to show this for the modified endpoint process $\frac{\tilde{x}_+(st)-\frac{st}{2}}{\sqrt{s}}$. For $t_1 < t_2$ we have:

$$P(\frac{\tilde{x}_+(st_1) - \frac{st_1}{2}}{\sqrt{s}} \le y_1; \frac{\tilde{x}_+(st_2) - \frac{st_2}{2}}{\sqrt{s}} \le y_2)$$

$$\le P(\frac{W(y_1\sqrt{s} + \frac{st_1}{2}) - W(y_1\sqrt{s} - \frac{st_1}{2})}{\sqrt{s}} \le y_1 + \frac{2\nu \log \frac{1-\epsilon}{\epsilon}}{\sqrt{s}} - \frac{2\nu M t_1^{\frac{1}{3}}}{s^{\frac{1}{6}}};$$
$$\frac{W(y_2\sqrt{s} + \frac{st_2}{2}) - W(y_2\sqrt{s} - \frac{st_2}{2})}{\sqrt{s}} \le y_2 + \frac{2\nu \log \frac{1-\epsilon}{\epsilon}}{\sqrt{s}} - \frac{2\nu M t_2^{\frac{1}{3}}}{s^{\frac{1}{6}}}).$$

Since the difference $y_1\sqrt{s} - y_2\sqrt{s}$ is of order lower than s, we can neglect it and hence when $s \to \infty$, the upper limit of the above probability is bounded by the upper limit of

$$P(\frac{W(\frac{st_1}{2}) - W(-\frac{st_1}{2})}{\sqrt{s}} \le y_1; \frac{W(\frac{st_2}{2}) - W(-\frac{st_2}{2})}{\sqrt{s}} \le y_2),$$

which is the desired two dimensional distribution of the Wiener process, since the process $\frac{W(\frac{st}{2})-W(-\frac{st}{2})}{\sqrt{s}}$ is a Wiener process. Similarly,

$$\limsup P(\frac{\tilde{x}_+(st_1) - \frac{st_1}{2}}{\sqrt{s}} \le y_1; \frac{\tilde{x}_-(st_2) - \frac{st_2}{2}}{\sqrt{s}} \ge y_2) \le P(W(t_1) \le y_1 \ ; \ W(t_2) \ge y_2)$$

and hence, in view of Theorem 4.1., also

$$\limsup P(\frac{\tilde{x}_+(st_1) - \frac{st_1}{2}}{\sqrt{s}} \le y_1; \frac{\tilde{x}_+(st_2) - \frac{st_2}{2}}{\sqrt{s}} \ge y_2) \le P(W(t_1) \le y_1 \ ; \ W(t_2) \ge y_2).$$

In the same way we show that for any combination of the signs of inequalities, probabilities describing the finite-dimensional distributions of $x_+(st) - \frac{st}{2}$ are bounded above by the corresponding probabilities for the Wiener process and this implies, that in the limit, we actually have to have an equality.

Step II. Tightness. In order to prove that the processes $x_+(st) - \frac{st}{2}$ actually converge in distribution to the Wiener process, we need to verify tightness. Note that $x_+(t)$ does not have to be a continuous function of t, so that convergence in distribution and tightness are understood here in the sense of the Skorokhod space [3]. We shall use Theorem 15.5 of [3]. According to this theorem, tightness (and with it, the desired convergence to the Wiener process) will be proven if we show that

1. For each positive α, there exists an a such that

$$P(|x_+(0)| > a\sqrt{s}) \leq \alpha$$

for all s;

2. For each positive α and β, there exist a $\delta \in (0,1)$ and an s_0, such that

$$P(\exists t_1, t_2 \in [0,1] : |t_1 - t_2| \leq \delta; |x_+(st_1) - \frac{st_1}{2} - x_+(st_2) + \frac{st_2}{2}| \geq \alpha\sqrt{s}) \leq \beta, \quad \forall s \geq s_0.$$

The first condition is obviously satisfied; the second one simply says that for $s \geq s_0$ the modulus of continuity of the function $\frac{y_+(st)}{\sqrt{s}}$, where $y_+(t) \stackrel{\text{def}}{=} x_+(t) - \frac{t}{2}$, calculated for the maximum increment δ, is with probability at least $1 - \beta$ bounded above by α. Now, if

$$|y_+(st_1) - y_+(st_2)| \geq \alpha\sqrt{s},$$

then, by the definition of y_+, we have:

$$W(y_+(st_1) + \frac{st_1}{2}) - W(y_+(st_1) - \frac{st_1}{2}) - y_+(st_1) = 2\nu \log\frac{\epsilon}{1-\epsilon} + R_1;$$

$$W(y_+(st_2) + \frac{st_2}{2}) - W(y_+(st_2) - \frac{st_2}{2}) - y_+(st_2) = 2\nu \log\frac{\epsilon}{1-\epsilon} + R_2,$$

where R_1 and R_2 are bounded by a constant times $t^{\frac{1}{3}}$ with probability arbitrarily close to one. This implies, except on a set of measure close to zero, that there exist x_1 and x_2 with $|x_1|, |x_2| < s; |x_1 - x_2| \leq \frac{s\delta}{2}$ and $|W(x_1) - W(x_2)| \geq C\alpha\sqrt{s}$, where C is an absolute constant. The probability of the last event equals, by the scaling property,

$$P(\exists x_1, x_2 : |x_1|, |x_2| \leq 1; |x_1 - x_2| \geq \frac{\delta}{2}; |W(x_1) - W(x_2)| \geq C\alpha)$$

and this goes to zero with $\delta \to 0$, since Wiener paths are continuous. This ends the proof of tightness of the processes $\frac{x_+(st)-\frac{st}{2}}{\sqrt{s}}$. Now as at the end of the proof of Theorem 1.2, 1), we extend the above result to $z_\pm$.

Remark: All the results of the paper can be generalized to the situation in which the viscous shock solution is perturbed by a stationary Gaussian random process with sufficiently fast decay of correlations. Every such process can be represented in the form

$$X_t = \int \phi(t-u) V_u \, du,$$

where V_u is the white noise and f is a square-integrable function. To show that the variables $W_x - \int_0^x dt \int \phi(t-u)V_u \, du$ are tight, it is enough to show that their second moments are uniformly bounded in x, i.e. that

$$\int [I_{[0,x]}(u) - \int_{-u}^{x-u} \phi(v) \, dv]^2 \, du \leq C,$$

where C is a constant, and $I_{[0,x]}$ is the indicator function of the interval $[0,x]$. This is satisfied if $\int_{-\infty}^{u} \phi(v)dv \in L^2(R_-)$ and $\int_u^\infty \phi(v)dv \in L^2(R_+)$. The last condition holds e.g. when $\phi(v)$ decays at infinity as $|v|^{-\alpha}$ with an $\alpha > 2$. This allows to repeat all the calculations, approximating Y_t by cW_t and estimating the probability of the error being large. Here $c = \int \phi(v)dv$; note that all the results of the paper hold for white noise of arbitrary intensity with obvious changes.

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