WETTING FRONTS IN ONE-DIMENSIONAL PERIODICALLY LAYERED SOILS

by

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TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

In this thesis, we study traveling wave solutions to Richards equation in diffusive form which describes wetting fronts in vertical infiltration of water into one-dimensional periodically layered soils. We prove the existence and uniqueness of traveling waves solutions under prescribed flux boundary conditions and certain constitutive conditions on the diffusivity and conductivity functions in the equation. Furthermore, we show the long time stability of these traveling wave solutions under these conditions. The traveling waves are connections between two steady states that form near the ground surface and deep in the soil. We derive an analytical formula for the speed of these traveling waves which depends on the prescribed boundary fluxes and the steady states. Both analytical and numerical examples are found which show that the wave speed in a periodically layered soil may be slower, the same, or faster than the speed in a homogeneous soil. In these examples. if the phases of the diffusivity and conductivity functions are the same, the periodic soils slow down the waves. If the phases differ by half a period, the periodic soils speed up the waves. We also present numerical solutions to Richards equation using a finite difference method to address cases where our constitutive conditions do not hold. Similar stable wetting fronts are observed even in these cases.

1 Introduction

Infiltration and, in particular, the study of wetting and drying fronts is a major focus of hydrology and soil science (see [11], [25], [37], [40], and [53]). These fronts are important since they are the most clear indication of water movement in a soil. By studying the wetting and drying fronts, hydrologists and soil scientists can determine the speed and direction of groundwater movement in soils. They can also use this information to help trace the progress of dissolved solutes and contaminants in a soil since the solute fronts are dictated by water flow ([6] and [52]). This solute front information is critical in determining the viability and safety of various land uses such as irrigated agriculture, waste disposal, and mining. These land uses and others may put the water supply of large regions at risk if their ability to introduce solutes and contaminants to the regions' groundwater is underestimated or misunderstood.

The primary equation used by soil scientists and hydrologists to describe infiltration of water, under the influence of gravity, into a soil is Richards' equation (see [2], [29], [34], and [52]). The water content-based form of this equation in one dimension is

$$u_t = [D(u,x)u_x - K(u,x)]_x \tag{1.1}$$

where $u(x,t)$ is the soil water content, $x \in [0,\infty)$ is the depth below the surface, $t \geq 0$ is time, $K(u,x)$ is the hydraulic conductivity of the soil, and $D(u,x)$ is the soil water diffusivity. The choice of the hydraulic functions, $K(u,x)$ and $D(u,x)$, proves to be quite important both in the physical relevancy of the equation's solutions (see [50]) and in determining these solutions. In this thesis, we will only examine wetting fronts and will therefore neglect any hysterisis or drying effects between water content and hydraulic head, the importance of which is addressed by Serrano [42].

Equation (1.1) was derived by flux balancing using Darcy's Law,

$$J_w = -D(u,x)u_x + K(u.x), \tag{1.2}$$

where J_w is the soil water flux, the volume of flow passing a unit area per unit time. In general, $D(u,x)$ and $K(u,x)$ are dependent not only on u but also on x and are highly nonlinear in both variables. It is these functions' nonlinearity in u and overall dependence on x which makes Richards' equation challenging and interesting to solve in analytical forms.

Most of the past analytical results have been found by ignoring the x dependence of D and K. Physically, this amounts to assuming that the soil is homogeneous. Philip in [34] wrote down a solution to the linearized Richards' equation

$$u_t = Du_{xx} - Ku_x \tag{1.3}$$

where $D, K > 0$ are constants. He applied the initial and boundary conditions

$$u(0,t) = u_{wet}, \quad u(\infty,t) = u_{dry}, \quad u(x,0) = u_{dry} \tag{1.4}$$

where $u_{wet}, u_{dry} \in [0,1]$ are constants. His solution was

$$u(x,t) = u_{dry} + (\frac{u_{wet} - u_{dry}}{2})[erfc(\frac{x - Kt}{2\sqrt{Dt}}) + exp(\frac{Kx}{D})erfc(\frac{x + Kt}{2\sqrt{Dt}})] \tag{1.5}$$

where $erfc(x)$ is the complementary error function. More recently, others have analytically solved specific cases of equation (1.1) by a careful selection of D and K. For example, Srivastava and Yeh [44] took

$$K(h) = K_s exp[\alpha h], \quad u = u_r + (u_s - u_r)exp[\alpha h] \tag{1.6}$$

in the mixed head-water content form of Richards' equation

$$u_t = [K(h)(h_x + 1)]_x \tag{1.7}$$

to produce an analytical solution. K_s is the saturated conductivity, u_s is the saturated water content, u_r is the residual water content, h is the hydraulic head, and α is a parameter. By using (1.6), they effectively linearized Richards' equation and were able to solve for $K(h)$ by using a Laplace transform. This result was also extended to the case of two layer soils by a matching argument along the soils' common boundary. Other analytic work for the two-phase problem in a homogeneous soil has been done by McWhorter and Sunada [25] who were able to write down an integral solution to Richards' equation which showed the position of the region where saturation by an infiltrating fluid had been attained.

Mathematically, wetting fronts in soils are represented by traveling wave solutions to Richards' equation (see [11], [17], [30], [50], and [60]). The existence and uniqueness of traveling wave solutions to the general nonlinear problem (1.1), neglecting the x dependence, has been shown by Khushnytdinova [17] for the fixed boundary conditions

$$u(0,t) = u_1, \quad u(\infty,t) = u_2 \tag{1.8}$$

and by Noren [30] for prescribed fluxes $f_1(t)$ and $f_2(t)$ at the boundaries

$$\begin{aligned} D(u)u_x - K(u)\,|_{x=0} &= -f_1(t), \\ D(u)u_x - K(u)\,|_{x=\infty} &= -f_2(t). \end{aligned} \tag{1.9}$$

Of these two types of boundary conditions, the prescribed flux is more physically relevant, since the soil surface ($x = 0$) is rarely held at a fixed water content while a prescribed flux can be related to rainfall or irrigation. Warrick et al. [50] were able to write down an equation for the position of the wetting front as a function of time for the fixed boundary conditions in homogeneous soils when using Brooks and Corey or van Genuchten hydraulic functions. In addition, many numerical methods for computing solutions to the infiltration problem in homogeneous soils have been devised (see [4], [8], [11], [22], and [39]). The major concern in the methods is mass conservation. The numerical solutions presented in these works are consistent with the theory found in [17] and [30] as they show existence and stability of traveling wave solutions denoting wetting fronts.

While solutions without x dependence produce some useful results, experiments and practical usage of soils show that soils are very heterogeneous and that this spatial variability has a marked effect on infiltration in the soils (see [12]). Therefore. much attention has been paid recently towards how to incorporate spatial variability of soils into infiltration models and equations. This incorporation requires that the x dependence of D and K be included. Ongoing work generating numerical solutions with many kinds of spatial variability, especially those with random x dependence, is currently being pursued (see [9], [10], [27], [40], [47], and [57]). Harter and Yeh [10] and Neuman [27] carry out numerical work by taking the $K(u,x)$ coefficient in Richard's equation to be realizations from random fields. The form of Richards' equation they use is the head based form

$$C(h)h_t = [K(h,x)(h_x - 1)]_x \tag{1.10}$$

where the conductivity function takes the form

$$K(h,x) = K_s(x)exp[\alpha(x)h] \tag{1.11}$$

with $K_s(x)$ and $\alpha(x)$ realizations of three-dimensional. stationary stochastic fields. Unlu et al [47] use a Monte Carlo technique with solutions obtained from many of these realizations to make their models. In contrast, Jury et al [9], [40] and White [57] are not concerned with actual solutions to Richards' equation. Instead, they use transfer functions to determine the flow out of a soil region, given information on the rate and amount of the water added to that region. To this point, analytical results deal mostly with steady state infiltration around localized impermeable obstructions, such as rocks, buried in a homogeneous soil. Philip et al [35] find their solutions by solving the steady state form of Richards' equation with the hydraulic functions given by (1.6). The obstructions are introduced by imposing a no flow boundary condition on the surface of the obstruction. Warrick and Fennemore [51] employ the same equation and hydraulic functions but create their obstructions through the use of source/sink combinations which are analogous to the classical Rankine bodies seen in fluid dynamics.

Since the mathematical work of Khushnytdinova [17] and Noren [30] on wetting fronts in homogeneous soils in the 1960's and 1970's, studies using Richards' equation to model infiltation in heterogeneous soils are mostly found in the soil science and hydrology literature and appear to be either numerical simulations or special cases where closed form analytical solutions are available. In this thesis, we obtain qualitative analytical results on dynamics of wetting fronts in periodically layered soils for a class of heterogeneous conductivity and diffusivity functions in Richards' equation (1.1). Our approach is to establish existence, uniqueness, and long time asymptotic stability of traveling wave solutions. These traveling waves have spatially periodic structures and have been recently studied for other nonlinear parabolic equations (see Xin [60]).

The periodic layering of the soils is manifested in the periodic spatial dependence of D and K in x. These periodically layered soils have been considered in soil science literature by Philip [36] and Hills et al [11]. Hills et al perform a one-dimensional infiltration experiment on a periodically layered soil and then match a numerical model to their results. They observe many of the phenomena that we discuss in this thesis such as wetting fronts and periodic steady states at the boundaries of their apparatus. While assuming that soils act like periodic media is not completely accurate for all soils, it is useful in modeling many laboratory experiments and is more appropriate than assuming that soils are homogeneous. In fact, some numerical models, which use random x dependence for D and K to more accurately model actual soils, assume that these random functions come from a periodic mean function (see [29] p. 160).

The rest of the thesis is organized as follows. Chapter 2 discusses the existence and uniqueness of steady state solutions to Richards' equation with periodic coefficients. Chapters 3 through 6 show the existence and uniqueness of a traveling wave solution to Richards' equation under the following constitutive assumptions on the hydraulic functions:

$$
\begin{aligned}
D(u,x) &= D(u)a(x), \quad (A1)\\
K(u,x) &= K(u)b(x), \quad (A2)\\
D(u) &= K'(u), \quad (A3)\\
K(u), \quad K'(u), \quad K''(u), &> 0, \quad (A4)\\
D(u), \quad D'(u) &> 0. \quad (A5)
\end{aligned}
$$

where $a(x)$ and $b(x)$ are positive, one periodic, and twice continuously differentiable. The following constant prescribed flux boundary conditions will be imposed:

$$
\begin{aligned}
D(u,x)u_x - K(u,x) \mid_{x=0} &= -\bar{c}_l,\\
D(u,x)u_x - K(u,x) \mid_{x=\infty} &= -\bar{c}_r. \qquad (1.12)
\end{aligned}
$$

Chapter 7 shows that time-dependent solutions of Richards' equation (1.1) with front-like initial data (under the assumptions (A1)-(A5)) converge to a traveling wave solution as $t \to \infty$. Chapter 8 discusses the effects of heterogeneity on the speed of the wetting fronts. Chapter 9 shows some numerical solutions which illustrate points discussed in the previous chapters such as the movement of the wetting fronts, the effect of heterogeneity on wave speed, and the significance of (A3) to the solutions. Chapter 10 discusses generalizations, in particular, the possibilities and consequences of relaxing assumptions (A1)-(A5).

While (A1)-(A5) are not general mathematically, they are reasonable assumptions used by the soil science community. (A1) and (A2) are used by Philip (see [36] Equ. 12). Assumption (A3) can be found in Parlange [32] and [33], in Equation 43 of Lockington [24], in Philip [34], Equation 9 of [37] and is even natural for certain selections of D and K such as the Gardner and Russo hydraulic functions (see [50]). Moreover, in cases where diffusivity is much smaller than conductivity, the role of diffusivity is of secondary importance in the solution and assuming (A3) has little bearing (see chapter 9, Example 2). Assumptions (A4) and (A5) are physically justifyable and are used by Neuman [29] and Warrick [52].

The main theoretical results of this thesis are found in Theorem 4.1 (uniqueness), Theorem 6.1 (existence), and Theorem 7.2 (asymptotic stability). These results are briefly stated here:

Theorem 1.1 ***(Existence and Uniqueness)*** *Suppose the hydraulic functions satisfy (A1) - (A5) and the prescribed fluxes, $\bar{c}_l, \bar{c}_r$ satisfy $\max_{x\in[0,1]} K(0,x) < \bar{c}_r < \bar{c}_l < \min_{x\in[0,1]} K(1,x)$. Then Richards' equation (1.1) admits a classical traveling wave solution of the form $u(x,t) = u'(x-ct,x) \equiv u'(s,y)$ where $s = x - ct$ and $y = x$, c is the wave speed; the function $u'(s,y)$ satisfies the traveling wave equation*

$$-cu'_s = (\partial_y + \partial_s)(D(u',y)(u'_s + u'_y) - K(u',y)), \tag{1.13}$$

$u'(-\infty,y) = u_l(y), u'(+\infty,y) = u_r(y)$, and $u'(s,\cdot)$ is 1-periodic in y. The wave speed, c, is given by the formula

$$c = \frac{\bar{c}_l - \bar{c}_r}{\langle u_l(y) - u_r(y)\rangle}, \tag{1.14}$$

where $u_l(y)$ *and* $u_r(y)$ *are solutions to the steady state equation:*

$$-\bar{c}_i = D(u_i, x)u_{i,x} - K(u_i, x) \tag{1.15}$$

and range strictly between zero and one. Furthermore, suppose that (u, c) *and* (u', c') *are classical solutions of this traveling wave equation and* u_s *and* u'_s *decay to zero as* $s \to \pm\infty$ *uniformly in* y. *Then* $u'(s, y) = u(s - s_0, y)$ *for some* $s_0 \in R, c = c'$, *and* $u'_s < 0$ *for any* (s, y).

Theorem 1.2 ***(Asymptotic Stability)*** *Let us consider the initial boundary value problem for Richards' equation (1.1) with flux boundary conditions (1.12) and initial data* $u(x, 0) = u_0(x), x \geq 0$. *Suppose* $u_0(x) \in C^1(R^1_+); u_r(x) \leq u_0(x) \leq u_l(x), |u_{0,x}| \leq M < \infty$ *for a positive constant* $M; u_0 - u_r \in L^1(R^1_+)$, *then a global in time, classical solution* $u(x, t)$ *exists and satisfies*

$$\lim_{t \to \infty} |u(x, t) - u'(x - ct - s_0, x)| = 0 \tag{1.16}$$

uniformly for $x \geq 0$. *Here* $s_0 \in R$ *is a constant translate depending on the initial data.*

Two other topics to note in this thesis are the derivations of an entropy condition (chapter 3 for assumptions (A1)-(A5) and chapter 10 for the general case) and the affects of heterogeneity on wave speed (chapter 8). The entropy condition is similar to the shock condition found in the theory of conservation laws. This condition is a sufficient condition for the existence of traveling waves. Under (A1)-(A5), we show in Theorem 3.1 that the entropy condition is always valid. The periodic heterogeneity of the soils can affect the wave speed by either speeding it up or slowing it down. When the spatial dependences of $K(u, x)$ and $D(u, x)$ are "in phase", the waves are slowed down. When the spatial dependences vary by half a period, the waves are sped up.

Details in many chapters (especially 5 and 6) may be skipped by the reader without loss of understanding. The more important details are found in chapter 2 (steady states), chapter 3 (entropy condition), and chapter 7 (stability). Chapter 8 (heterogeneity and wave speed) and chapter 9 (numerics when (A1)-(A5) fail) are generally lighter reading than the rest of the thesis and discuss this work's application to phenomena of interest to soil scientists and hydrologists.

2 Steady State Solutions

We consider the following diffusive, water content form of the Richards' equation in one-dimension as given in (1.1):

$$u_t = (D(u,x)u_x - K(u,x))_x, \tag{2.1}$$

There are two steady state solutions to (2.1) to be considered; one corresponding to the upper boundary near $x = 0$, $u_l(x)$; the other to the boundary near $x = \infty$, $u_r(x)$. During steady infiltration, $u_l(x)$ is achieved when transients die out and $u_l(x) > u_r(x)$ assuming that the initial water content profile was not too wet near $x = \infty$. We would now like to prove the existence and uniqueness of these steady state solutions.

Proposition 2.1 ***(Existence)*** *Let $D(u,x), K(u,x)$ be positive smooth functions in u and x; 1-periodic in x; $D_u > 0$, $K_u > 0$ for all x. Let $\bar{c}$ satisfy*

$$\sup_x K(0,x) < \bar{c} < \inf_x K(1,x). \tag{2.2}$$

Then there exists a positive, smooth, 1-periodic solution $u(x)$ to the equation

$$D(u,x)u_x - K(u,x) = -\bar{c} \tag{2.3}$$

for $x \in R^1$ such that $0 < u(x) < 1$.

Proof: The steady state solutions, $u_l(x)$ and $u_r(x)$, satisfy the steady state version of equation (2.1)

$$(D(u_i,x)u_{ix} - K(u_i,x))_x = 0, \tag{2.4}$$

or

$$D(u_i,x)u_{ix} - K(u_i,x) = -\bar{c}_i, \tag{2.5}$$

where $i = l, r$ and $\bar{c}_i$ are positive constants which correspond to the flux rates of the infiltration. We will start by rewriting equation (2.5) as

$$-u_x = \frac{\bar{c} - K(u,x)}{D(u,x)} \tag{2.6}$$

for $x \in R^1$. Letting $\tilde{u}(x) = u(-x)$, $\tilde{u}(x)$ satisfies

$$\tilde{u}_x = \frac{\bar{c} - K(\tilde{u}, -x)}{D(\tilde{u}, -x)} = \frac{\bar{c} - \tilde{K}(\tilde{u}, x)}{\tilde{D}(\tilde{u}, x)} \tag{2.7}$$

for $x \in R^1$. For the remainder of the chapter, we will drop the tildes.

It is clear that showing existence of positive periodic solutions to (2.7) is the same as doing so for (2.5). We also want u to range strictly between zero and one since u represents soil water content in unsaturated infiltration which is defined to be the water filled fraction of a soil's total pore space. Let us consider the initial value problem for (2.7) on $[0,1]$ with $u(x=0) = u_b \geq 0$,

$$u_b \leq u^* = \sup\{u \geq 0 | \inf_{x \in [0,1]} K(u,x) \leq \bar{c}\}. \tag{2.8}$$

Assume that

$$\bar{c} > \sup_{x \in [0,1]} K(0,x), \tag{2.9}$$

which implies that u^* is well defined. By our earlier assumptions that $D(u,x) > 0$ and $D_u(u,x) > 0$ for $x \in [0,1]$ and $u_b \in [0,1)$, (2.7) has a local solution on $[0, x^*)$ for some $x^* > 0$ and $u(x) > 0$ for x sufficiently near zero. Now if $u(x_1) = 0$ for some $x_1 > 0$, then $u_x(x_1) \leq 0$. In view of (2.7) and (2.9), we see that such an x_1 does not exist, and $u(x) > 0$ on $[0, x^*)$. Similarly, if $u(x_2) > u^*$ for the first such $x_2 > 0$, then

$$K(u(x_2), x_2) \geq \inf_{x \in [0,1]} K(u(x_2), x) > \bar{c} \tag{2.10}$$

by definition of u^*. So (2.7) says $u_x(x_2) < 0$ which contradicts the existence of such an x_2 where $u_x(x_2) \geq 0$.

Combining the above arguments, we see that $0 < u(x) \leq u^*$, on $(0, x^*)$. This allows the extension of x^* to any value, in particular 1. So, for any $u_b \in [0, u^*]$, (2.7) defines a mapping $T : u_b \rightarrow u(x = 1, u_b)$. T is a C^1 mapping from $[0, u^*]$ into itself. Therefore, T has a fixed point, u_p, such that $u(x, u_p)$ is a non-negative periodic solution to equation (2.7). Since any point x is an interior point, $u(x, u_p)$ is strictly positive.

To insure that $u(x, u_p) \leq 1$ for physical reasons, we also impose

$$\bar{c} < \inf_{x \in [0,1]} K(1, x). \tag{2.11}$$

If $u(x, u_p) \geq 1$ at $x = x_3$ and x_3 is any maximal point, then $u_x(x = x_3, u_p) = 0$ or $\bar{c} = K(u(x_3, u_p), x_3) \geq K(1, x_3)$ by (2.7). But this violates (2.11). Thus, under the assumptions of this proposition, we have the existance of 1-periodic steady state solutions with $0 < u(x) < 1$.

We now turn to a proof of the uniqueness of these steady state solutions.

Proposition 2.2 ***(Uniqueness)*** *Let $D(u, x), K(u, x)$ be positive, smooth functions in u and x; 1-periodic in x; $D_u > 0$, $K_u > 0$ for all x. Also, $D(u, x)$ has the form*

$$D(u, x) = D_0(u) a(x). \tag{2.12}$$

Then, the solution to

$$D(u, x) u_x - K(u, x) = -\bar{c} \tag{2.13}$$

is unique and increases monotonically with $\bar{c}$.

Proof: Suppose that u and v are two positive, periodic, steady states with fluxes $\bar{c}_1 \geq \bar{c}_2$. So

$$-\bar{c}_1 = D(u, x) u_x - K(u.x), \tag{2.14}$$

$$-\bar{c}_2 = D(v,x)v_x - K(v,x). \tag{2.15}$$

Making the change of variable: $U = F(u) = \int D_0(u)du$ and $V = F(v) = \int D_0(v)dv$, we have the following equations for U and V

$$-\bar{c}_1 = a(x)U_x - K(F^{-1}(U), x), \tag{2.16}$$

$$-\bar{c}_2 = a(x)V_x - K(F^{-1}(V), x), \tag{2.17}$$

where $K(F^{-1}(U), x)$ is increasing in U for any fixed x. Now consider the function $W = U - V$ and suppose that W has a negative minimum at x_1, or $U(x_1) < V(x_1)$ while $U_x(x_1) = V_x(x_1)$. Letting $x = x_1$ in (2.16)-(2.17) and subtracting the two equations yields

$$\bar{c}_1 - \bar{c}_2 = K(F^{-1}(U), x) - K(F^{-1}(V), x) < 0, \tag{2.18}$$

which contradicts $\bar{c}_1 \geq \bar{c}_2$. Thus, $U \geq V$ or $u \geq v$ for all x. Additionally, this shows that $u(x)$ increases monotonically with $\bar{c}$. The proof is complete.

Remark 2.1 *A similar argument yields uniqueness of the steady state solutions if (2.12) is replaced by the condition that $\frac{K(u,x)}{D(u,x)}$ is non-decreasing in u for all x. In fact, we can divide (2.14) and (2.15) by $D(u,x)$ and apply a comparison argument as above. This condition is physically valid for infiltration into dry soils. This can be seen if we take $\frac{K(u,x)}{D(u,x)} = (\frac{dh}{du})^{-1}$, a function which increases monotonically for most values of u before decreasing for values very close to saturation $(u = 1)$ (see [5] p.30).*

3 Preliminary Lemmas for Existence of Traveling Waves

In this chapter, we will begin to investigate the existence and uniqueness of traveling wave solutions to (1.1). For our analysis, we would like to know that the steady states at the boundaries, $u_l(y)$ and $u_r(y)$ discussed in chapter 2, bound the traveling wave solution. Physically, this shows that infiltration does not cause the soil to get wetter than its non-transient wet state or get dryer than its non-transient dry state in the long time limit. Also, we would like to have a formula to calculate the speed of the traveling wave. Furthermore, we would like to show that the traveling wave solution goes to the steady states exponentially near the boundaries. This result is useful in showing the existence and uniqueness of the traveling wave solution. These results will be presented in two lemmas. Lemma 3.1 will give a formula for the speed of the traveling wave. Lemma 3.2 uses an entropy condition derived in this section to show that the differences between the traveling wave solutions and the steady states decay exponentially at the boundaries.

We will start with the Richards' equation (1.1) and will be looking for traveling wave solutions of the form $u(t,x) = u'(x - ct, x)$ where c is the wave speed. Thinking of these traveling wave solutions as connections between the two steady states at the boundaries discussed in chapter 2, we will assume that $\bar{c}_l > \bar{c}_r$, where $\bar{c}_l$ and $\bar{c}_r$ are the prescribed fluxes from the boundary conditions in (1.12) for $x = 0$ and $x = \infty$ respectively. By Proposition 2.2, this implies that $u_l(x) > u_r(x)$ or physically, that the infiltrating water generally moves from a region of higher water content at $x = 0$ to a region of lower water content at $x = \infty$.

Changing into some moving frame variables by taking $s = x - ct$ and $y = x$, we have the solution (u', c) to the traveling wave equation:

$$-cu'_s = (\partial_y + \partial_s)(D(u', y)(u'_s + u'_y) - K(u', y)), \tag{3.1}$$

where $u'(-\infty, y) = u_l(y)$, $u'(\infty, y) = u_r(y)$, and u' is 1-periodic in y. For the remainder of this chapter, as well as chapters 4 through 6, we will drop the "prime" on u' and let u represent the traveling wave solution. The "prime" notation for traveling wave solutions will reappear in chapter 7.

In order to prove these lemmas, we need to assume (A1) - (A5). Using our assumptions on the hydraulic functions, we can manipulate equation (3.1) as follows:

$$-cu_s = (\partial_s + \partial_y)[D(u, y)(\partial_s + \partial_y)u - K(u, y)], \tag{3.2}$$

$$-cu_s = (\partial_s + \partial_y)[K_u(u)a(y)(\partial_s + \partial_y)u - K(u)b(y)], \tag{3.3}$$

$$-cu_s = (\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)K(u) - b(y)K(u)]. \tag{3.4}$$

Since $K(u)$ is a monotone, one-to-one function of u, let us use the change of variable $U = K(u)$. Then

$$u = K^{-1}(U), \tag{3.5}$$

$$U_s = K'(u)u_s, \tag{3.6}$$

$$U_s = K'(K^{-1}(U))u_s, \tag{3.7}$$

$$u_s = \frac{U_s}{K'(K^{-1}(U))}. \tag{3.8}$$

Let

$$M(U) = \frac{1}{K'(K^{-1}(U))}. \tag{3.9}$$

Employing the change of variable, (3.1) becomes

$$(\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)U - b(y)U] + cM(U)U_s = 0, \tag{3.10}$$

with

$$U(-\infty,y)=U_l(y),\quad U(\infty,y)=U_r(y), \tag{3.11}$$

and U has period one in y. In proving these lemmas, we would like to invoke a maximum principle. While the solutions of equation (3.10) do not have a maximum principle, we may use a change of variable

$$U(s,y)=e(y)W(s,y), \tag{3.12}$$

where we would like $e(y)$ to be strictly positive, to obtain the equation

$$\begin{aligned} 0 &= e(y)(\partial_s+\partial_y)(a(y)(\partial_s+\partial_y)W)+2a(y)e'(y)(\partial_s+\partial_y)W \\ &- e(y)b(y)(\partial_s+\partial_y)W+ce(y)M(U)W_s \\ &+ ((a(y)e_y(y))_y-(b(y)e(y))_y)W. \end{aligned} \tag{3.13}$$

Removing the lowest order term gives the following equation for $e(y)$:

$$(a(y)e_y(y))_y-(b(y)e(y))_y=0. \tag{3.14}$$

The existence of a positive $e(y)$ is easily seen by directly integrating equation (3.14). Hence, we may divide equation (3.13) by $e(y)$ to get

$$\begin{aligned} 0 &= (\partial_s+\partial_y)(a(y)(\partial_s+\partial_y)W) \\ &+ (2\frac{e'(y)}{e(y)}a(y)-b(y))(\partial_s+\partial_y)W+cM(U)W_s, \end{aligned} \tag{3.15}$$

with

$$W(-\infty,y)=W_l,\quad W(\infty,y)=W_r, \tag{3.16}$$

W 1-periodic in y, and W_l, W_r are now constants. The fact that W_l and W_r are now constants can be seen as $e(y)$ satisfies (3.14) and under the change of variable, $U=K(u)$, the steady state equation (2.4) becomes

$$(a(y)U_{l,ry}(y))_y-(b(y)U_{l,r}(y))_y=0. \tag{3.17}$$

Since (3.14) and (3.17) are the same homogeneous ordinary differential equation,

$$W_l e(y) = U_l(y), \quad W_r e(y) = U_r(y) \tag{3.18}$$

where $W_{l,r}$ are constants. Solutions to (3.15) satisfy a maximum principle, by which we have $W_r \le W(s,y) \le W_l$. To simplify notation, let

$$b_1(y) = 2\frac{e'(y)}{e(y)}a(y) - b(y). \tag{3.19}$$

We are now ready to state our first lemma.

Lemma 3.1 *Assume that $(u(s,y),c)$ is a classical solution of the traveling wave equation (3.1) with assumptions (A1) -(A5) and that u_s decays to zero as $s \to \pm\infty$ uniformly in y. Then we have $u_l(y) > u(s,y) > u_r(y)$ for all $(s,y) \in R \times T$ and the wave speed is*

$$c = c_{eff} = \frac{\bar{c}_l - \bar{c}_r}{\langle u_l(y) - u_r(y)\rangle}. \tag{3.20}$$

where $\langle\cdot\rangle$ denotes the integral average over one period in y.

Proof: Using equation (3.15), it follows directly from the maximum principle that $W_l > W(s,y) > W_r$. Changing back to our original variables we see $\frac{U_l(y)}{e(y)} > \frac{U(s,y)}{e(y)} > \frac{U_r(y)}{e(y)}$. Therefore, $U_l(y) > U(s,y) > U_r(y)$ since $e(y) > 0$ and hence $u_l(y) > u(s,y) > u_r(y)$ for all (s,y) in the original version of the traveling wave equation since $K(u)$ is monotone increasing.

Working now in the original variables and averaging the traveling wave equation (3.1) over one period in y, we obtain

$$-c\langle u\rangle_s = \langle D(u,y)(u_s + u_y)\rangle_s - \langle K(u,y)\rangle_s. \tag{3.21}$$

Integrate once in s to get

$$-c\langle u\rangle = \langle D(u,y)(u_s + u_y)\rangle - \langle K(u,y)\rangle + k_0, \tag{3.22}$$

where k_0 is the constant of integration. Rewriting, we get

$$-c\langle u\rangle = \langle D(u,y)u_s\rangle - \langle D(u,y)u_y\rangle - \langle K(u,y)\rangle + k_0. \tag{3.23}$$

Now, we will let s go to positive and then negative infinity. In the limit, $\langle D(u,y)u_s\rangle$ goes to zero as s goes to infinity due to the assumption that u_s vanishes as s goes to infinity. $\langle D(u,y)u_y\rangle$ goes to $\langle D(u_{l,r},y)u_{l,ry}\rangle$ by the following argument:

$$D(u,y)u_y = (E(u,y))_y - E_y(u,y), \tag{3.24}$$

where $E = \int D(u,y)du$. Averaging over y we get

$$\langle D(u,y)u_y\rangle = -\langle E_y(u,y)\rangle. \tag{3.25}$$

Thus

$$\begin{aligned} \lim_{s\to\pm\infty}\langle D(u,y)u_y\rangle &= \lim_{s\to\pm\infty} -\langle E_y(u,y)\rangle \\ &= -\langle E_y(u_{l,r},y)\rangle = \langle D(u_{l,r},y)u_{l,ry}\rangle. \end{aligned} \tag{3.26}$$

Taking the limits as s goes to positive and negative infinities of (3.26), we end up with

$$-c\langle u_r\rangle = \langle D(u_r,y)u_{ry}\rangle - \langle K(u_r,y)\rangle + k_0, \tag{3.27}$$

$$-c\langle u_l\rangle = \langle D(u_l,y)u_{ly}\rangle - \langle K(u_l,y)\rangle + k_0. \tag{3.28}$$

The right hand sides of (3.27) and (3.28) resemble the steady state equation (2.5). Thus we have

$$-c\langle u_r\rangle = -\bar{c}_r + k_0, \tag{3.29}$$

$$-c\langle u_l\rangle = -\bar{c}_l + k_0. \tag{3.30}$$

Subtracting the second equation from the first, we get

$$c(\langle u_l - u_r\rangle) = \bar{c}_l - \bar{c}_r. \tag{3.31}$$

Solving for the speed c gives

$$c = \frac{\bar{c}_l - \bar{c}_r}{\langle u_l(y) - u_r(y)\rangle}. \tag{3.32}$$

The proof is complete.

We would now like to prove a second lemma which would show that the traveling wave solution goes exponentially to the steady states near positive and negative s-infinities. A piece of information that we will use in proving the second lemma and in proofs throughout the rest of the thesis is an entropy condition. Entropy conditions are analogous to viscous shock conditions and give a criterion for the existence of a traveling wave solution (see Lax [23]). Of particular importance is the question of how inhomogeneity in a medium affects the entropy condition. Below is the derivation of the entropy condition and a theorem which shows the degree of inhomogeneity of a soil with our assumptions (A1)-(A5) will not affect the validity of the entropy condition. Starting with the equation (3.10), we will linearize this equation around the steady state solution, $U_l = K(u_l)$ at $s = -\infty$, by letting

$$M(U) = M(U_l) + M_u(U_l)v + O(v^2), \tag{3.33}$$

where $v = U_l - U$. Substituting and neglecting second order terms, gives

$$-cM(U_l)(U_l - v)_s = (\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)(U_l - v) - b(y)(U_l - v)]. \tag{3.34}$$

Since U_l solves the steady state equation (3.17) and $U_{ls} = 0$, we have

$$-cM(U_l)v_s = (\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)v - b(y)v]. \tag{3.35}$$

Now look for solutions of the form $v(s,y) = e^{\lambda s}\varphi_l(y)$. Substituting, we have

$$-c\lambda M(U_l)\varphi_l = \lambda^2 a\varphi_l + \lambda(2a\varphi_l' + a'\varphi_l - b\varphi_l) + a\varphi_l'' + a'\varphi_l' - b\varphi_l' - b'\varphi_l. \tag{3.36}$$

Use (3.36) to define an operator

$$\begin{aligned} L_l\varphi_l &= \lambda^2 a\varphi_l + \lambda(2a\varphi_l' + a'\varphi_l - b\varphi_l) \\ &+ a\varphi_l'' + a'\varphi_l' - b\varphi_l' - b'\varphi_l + c\lambda M(U_l)\varphi_l. \end{aligned} \tag{3.37}$$

By linearizing (3.10) around the other steady state solution, $U_r = K(u_r)$ at $s = \infty$, letting $v = U - U_r$, and assuming $v(s,y) = e^{-\lambda s}\varphi_r(y)$, we can define an analogous operator

$$\begin{aligned} L_r\varphi_r &= \lambda^2 a\varphi_r - \lambda(2a\varphi_r' + a'\varphi_r - b\varphi_r) \\ &+ a\varphi_r'' + a'\varphi_r' - b\varphi_r' - b'\varphi_r - c\lambda M(U_r)\varphi_r. \end{aligned} \tag{3.38}$$

by the same process. These operators have principal eigenvalues, $\rho_l(\lambda)$ and $\rho_r(\lambda)$ respectively, which are smooth functions of λ. We will examine the eigenvalue problem with the operator in (3.37). Results for the eigenvalue problem in (3.38) will be analogous.

Let $\rho_l(\lambda) = \rho_0 + \lambda\rho_1 + O(\lambda^2)$ and $\varphi_l(y) = \varphi_0(y) + \lambda\varphi_1(y) + O(\lambda^2)$ for small λ and set up the eigenvalue problem:

$$\begin{aligned} \rho_l(\lambda)\varphi_l(y) &= \lambda^2 a\varphi_l + \lambda(2a\varphi_l' + a'\varphi_l - b\varphi_l) \\ &+ a\varphi_l'' + a'\varphi_l' - b\varphi_l' - b'\varphi_l + c\lambda M(U_l)\varphi_l. \end{aligned} \tag{3.39}$$

Substituting the expansions for $\rho_l(\lambda)$ and $\varphi_l(y)$ into (3.39) and separating by powers of λ, we obtain the zeroth order equation

$$\rho_0\varphi_0 = (a\varphi_0')' - (b\varphi_0)'. \tag{3.40}$$

Averaging over one period in y yields $\rho_0 = 0$ due to the 1-periodicity of $a(y), b(y)$, and $\varphi_0(y)$ in y. The first order equation is

$$\rho_1\varphi_0 = (a\varphi_1')' - (b\varphi_1)' + (a\varphi_0)' + a\varphi_0' - b\varphi_0 + cM(U_l)\varphi_0. \tag{3.41}$$

Averaging yields

$$\rho_1\langle\varphi_0\rangle = \langle a\varphi_0' - b\varphi_0\rangle + c\langle M(U_l)\varphi_0\rangle. \tag{3.42}$$

We may scale the solution of (3.40) so that $\langle\varphi_0\rangle = 1$. Then we have

$$\rho_1 = \langle a\varphi_0' - b\varphi_0\rangle + c\langle M(U_l)\varphi_0\rangle. \tag{3.43}$$

From the zeroth order equation, we can solve $\varphi_0(y)$ by integrating (3.40) to get

$$a(y)\varphi_0' - b(y)\varphi_0 = \bar{m}, \tag{3.44}$$

or, dividing by $a(y)$,

$$\varphi_0' - \frac{b(y)}{a(y)}\varphi_0 = \frac{\bar{m}}{a(y)}. \tag{3.45}$$

Using an integrating factor

$$P(y) = \int_0^y \frac{b(y)}{a(y)}dy \tag{3.46}$$

for $y \in [0,1]$, we have

$$(e^{-P(y)}\varphi_0)' = \bar{m}\frac{e^{-P(y)}}{a(y)}. \tag{3.47}$$

Integrate both sides and solve for $\varphi_0(y)$ to get

$$\varphi_0(y) = \bar{m}e^{P(y)}\Big[\int_0^y \frac{e^{-P(s)}}{a(s)}ds + T_0\Big], \tag{3.48}$$

where

$$T_0 = \frac{e^{P(1)}}{1 - e^{P(1)}}\int_0^1 \frac{e^{-P(s)}}{a(s)}ds \tag{3.49}$$

is a constant chosen so that $\varphi_0(y)$ is 1-periodic. To simplify notation, define a $T(y)$ so that

$$\varphi_0(y) = \bar{m}T(y), \tag{3.50}$$

where

$$T(y) = e^{P(y)}\Big[\int_0^y \frac{e^{-P(s)}}{a(s)}ds + T_0\Big]. \tag{3.51}$$

Remark 3.1 *Since equation (3.40) is the same as equations (3.14) and (3.17), we can use the same process to get the solutions* $\epsilon(y) = -kT(y)$ *from (3.14) and* $K(u_l) = U_l = -\bar{c}_l T(y)$ *and* $K(u_r) = U_r = -\bar{c}_r T(y)$ *from (3.17).*

Using the solution in (3.50) and equation (3.43), we can rewrite ρ_1 as

$$\rho_1 = c\langle M(U_l)\varphi_0\rangle + \bar{m}. \tag{3.52}$$

By continuity of $\rho_l(\lambda)$, and since $\rho_l(0) = \rho_0 = 0$ and $\rho_l(\lambda) \to \infty$ as $\lambda \to \infty$, to insure that there is a positive λ_1 such that $\rho_l(\lambda_1) = 0$, we require $\rho_l(\lambda) < 0$ for some small $\lambda > 0$ and hence, we require $\rho_1 < 0$. Thus,

$$-\bar{m} > c\langle M(U_l)\varphi_0\rangle, \tag{3.53}$$

or

$$-\bar{m} > c\langle M(U_l)\bar{m}T(y)\rangle. \tag{3.54}$$

We have $e^{P(y)} > 0$ and

$$\begin{aligned} \int_0^y \frac{e^{-P(s)}}{a(s)}ds \quad &+ \quad T_0 \\ &= \int_0^y \frac{e^{-P(s)}}{a(s)}ds + \frac{e^{P(1)}}{1-e^{P(1)}}\int_0^y \frac{e^{-P(s)}}{a(s)}ds + \frac{e^{P(1)}}{1-e^{P(1)}}\int_y^1 \frac{e^{-P(s)}}{a(s)}ds \\ &= \frac{1}{1-e^{P(1)}}\int_0^y \frac{e^{-P(s)}}{a(s)}ds + \frac{e^{P(1)}}{1-e^{P(1)}}\int_y^1 \frac{e^{-P(s)}}{a(s)}ds < 0, \end{aligned} \tag{3.55}$$

which gives $T(y) < 0$. Having imposed $\langle\varphi_0\rangle = \langle\bar{m}T\rangle = 1$, we see that $\bar{m} < 0$. Therefore, dividing (3.54) by m gives

$$-1 < c\langle M(U_l)T(y)\rangle, \tag{3.56}$$

or

$$-\langle M(U_l)T(y)\rangle^{-1} > c. \tag{3.57}$$

Switching back to the original variables, this is the same as

$$-\langle\frac{T(y)}{K'(u_l)}\rangle^{-1} > \frac{\bar{c}_l - \bar{c}_r}{\langle u_l - u_r\rangle}. \tag{3.58}$$

Note that for the homogeneous case this is simply

$$K'(u_l) > \frac{\bar{c}_l - \bar{c}_r}{u_l - u_r} = \frac{K(u_l) - K(u_r)}{u_l - u_r}, \tag{3.59}$$

which is analogous to the entropy condition used by Noren [30] (Lemma 3, p. 11). Similarly for $s = +\infty$ or u_r, we have

$$-\langle M(U_r)T(y)\rangle^{-1} < c. \tag{3.60}$$

Combining the inequalities (3.57) and (3.60), we have the entropy condition

$$-\langle T(y)M(U_r)\rangle^{-1} < c < -\langle T(y)M(U_l)\rangle^{-1}, \tag{3.61}$$

or

$$-\langle\frac{T(y)}{K'(u_r)}\rangle^{-1} < \frac{\bar{c}_l - \bar{c}_r}{\langle u_l - u_r\rangle} < -\langle\frac{T(y)}{K'(u_l)}\rangle^{-1}. \tag{3.62}$$

Having derived this entropy condition, we will now see that it is valid for all soils whose hydraulic functions satisfy (A1)-(A5) regardless of their degree of heterogeneity.

Theorem 3.1 *Under assumptions (A1)-(A5), the entropy condition holds.*

Proof: Since $K_{uu} > 0$, we have

$$\begin{aligned} K'(u_l) &> \frac{K(u_l) - K(u_r)}{u_l - u_r}, \\ K'(u_r) &< \frac{K(u_l) - K(u_r)}{u_l - u_r}. \end{aligned} \tag{3.63}$$

Therefore, using $K(u_{l,r}) = U_{l,r}(y) = -\bar{c}_{l,r}T(y)$ from Remark 3.1, we see

$$\begin{aligned} -\langle\frac{T(y)}{K'(u_l)}\rangle^{-1} &> -\langle\frac{T(y)(u_l - u_r)}{K(u_l) - K(u_r)}\rangle^{-1} \\ &= \langle\frac{-T(y)(u_l - u_r)}{-T(y)(\bar{c}_l - \bar{c}_r)}\rangle^{-1} = c, \end{aligned} \tag{3.64}$$

and

$$\begin{aligned} -\langle\frac{T(y)}{K'(u_r)}\rangle^{-1} &< -\langle\frac{T(y)(u_l - u_r)}{K(u_l) - K(u_r)}\rangle^{-1} \\ &= \langle\frac{-T(y)(u_l - u_r)}{-T(y)(\bar{c}_l - \bar{c}_r)}\rangle^{-1} = c. \end{aligned} \tag{3.65}$$

The proof is complete.

Corollary 3.1 *For any $\bar{c}_+ \in [\bar{c}_r, \bar{c}_l]$ and its related steady state $u_+(y)$ bounded below by $u_r(y)$ and above by $u_l(y)$ by Proposition 2.2, the entropy condition (3.62) and Theorem 3.1 still hold when $\bar{c}_r$ is replaced by $\bar{c}_+$.*

This corollary follows from the fact that $u_+(y) = -\bar{c}_+ T(y)$. Simply make this substitution in the above arguments to verify Corollary 3.1.

We are now ready to state our second lemma.

Lemma 3.2 *Let (u,c) satisfy the conditions in Lemma 3.1. Then there exist constants $s_1 < 0$, $s_2 > 0$, $\lambda_1 > 0$, $\lambda_2 < 0$, and positive functions $\varphi_i(y) \in C^2(T)$ such that*

$$u_l(y) - u(s,y) \le G_1 e^{\lambda_1 s}\varphi_1(y), \quad s \le s_1, \tag{3.66}$$

$$u(s,y) - u_r(y) \le G_2 e^{\lambda_2 s}\varphi_2(y), \quad s \ge s_2, \tag{3.67}$$

where $G_1, G_2 > 0$ are constants.

Proof: From the equation (3.10), we can write the two equations

$$(\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)U - b(y)U) + cM(U)U_s = 0, \tag{3.68}$$

and

$$(\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)U_l - b(y)U_l) + cM(U_l)U_{ls} = 0, \tag{3.69}$$

where $U_{ls} = 0$ since $U_l = U_l(y)$. Letting $V = U_l - U$, we then have

$$(\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)V - b(y)V) - cM(U)V_s = 0. \tag{3.70}$$

There exists a s_1 such that if $s \le s_1$ then $U_l - U \le \epsilon$ and $M(U) - M(U_l) \le N\epsilon$ for some Lipschitz constant N. Let us construct an upper solution for (3.70) on

$(-\infty, s_1] \times T$. Define the operator

$$\begin{aligned} L_l^\epsilon(V) &\equiv (\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)V) \\ &- (\partial_s + \partial_y)(bV) - c(M(U_l) + N\epsilon)V_s. \end{aligned} \tag{3.71}$$

When exponential solutions to (3.71) of the form

$$V_0(s, y) = e^{\lambda s}\varphi_l(y) \tag{3.72}$$

are considered with $\lambda > 0, \varphi_l(y) > 0$, we have

$$\begin{aligned} L_l^\epsilon(V_0) &= \lambda^2 a\varphi_l + \lambda(2a\varphi_l' + a'\varphi_l - b\varphi_l) \\ &+ a\varphi_l'' + a'\varphi_l' - b\varphi_l' - b'\varphi_l + c\lambda(M(U_l) + N\epsilon)\varphi_l = 0. \end{aligned} \tag{3.73}$$

For the $s \to \infty$ case, letting $V = U - U_r$ would give the analogous operator

$$\begin{aligned} L_r^\epsilon(V) &\equiv (\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)V) \\ &- (\partial_s + \partial_y)(bV) - c(M(U_r) + N\epsilon)V_s, \end{aligned} \tag{3.74}$$

where considering $V_0(s, y) = e^{-\lambda s}\varphi_r(y)$ would give

$$\begin{aligned} L_r^\epsilon(V_0) &= \lambda^2 a\varphi_r - \lambda(2a\varphi_r' + a'\varphi_r - b\varphi_r) \\ &+ a\varphi_r'' + a'\varphi_r' - b\varphi_r' - b'\varphi_r - c\lambda(M(U_r) + N\epsilon)\varphi_r = 0. \end{aligned} \tag{3.75}$$

Notice that (3.73) and (3.75) vary from (3.37) and (3.38) only by the term $N\epsilon$. Recall that the entropy condition from (3.62) was the condition that guaranteed that $\rho_l(\lambda) < 0$ for some positive λ close to zero. where $\rho_l(\lambda)$ is the principle eigenfunction for the operator L_l in (3.37). We can define a principle eigenfunction, ρ_l^ϵ, for L_l^ϵ. If ϵ is small enough, we have $\rho_l^\epsilon < 0$ as can be seen by comparing

$$\rho_l^\epsilon = c\langle (M(U_l) + N\epsilon)\varphi_0\rangle + \bar{m} \tag{3.76}$$

to (3.52). By continuity of ρ_l^ϵ in λ, there exists a $\lambda_1^\epsilon > 0$ such that $\rho_l^\epsilon(\lambda_1^\epsilon) < 0$. Denoting the corresponding eigenfunction by φ_1^ϵ, we showed that $L_l^\epsilon(V) = 0$ admits positive exponential solution $V_1(s,y) = e^{\lambda_1^\epsilon s}\varphi_1^\epsilon(y)$. Obviously, $V_{1s} > 0$. Plugging V_1 into the L.H.S. of (3.73), we have

$$\begin{aligned}(\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)V_1) - (\partial_s + \partial_y)(bV_1) - cM(U)V_{1s} \\ = \rho_l^\epsilon(\lambda_1^\epsilon)V_1 + c(M(U_l) + N\epsilon)V_{1s} - cM(U)V_{1s}.\end{aligned} \tag{3.77}$$

The Lipschitz condition on $M(U)$ implies

$$\begin{aligned}\rho_l^\epsilon(\lambda_1^\epsilon)V_1 + c(M(U_l) + N\epsilon)V_{1s} - cM(U)V_{1s} &\leq \rho_l^\epsilon(\lambda_1^\epsilon)V_1 + 2cN\epsilon V_{1s} \\ = V_1(\rho_l^\epsilon(\lambda_1^\epsilon) + 2cN\epsilon\lambda_1^\epsilon) &\leq 0\end{aligned} \tag{3.78}$$

for ϵ small enough since $\lambda_1^\epsilon > 0, \rho_l^\epsilon(\lambda_1^\epsilon) < 0$, $V_1 > 0$ and the term $2cN\epsilon\lambda_1^\epsilon$ is order ϵ^2. Now, we combine (3.73) and (3.77) to get:

$$\begin{aligned}(\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)(V_1 - V) \\ -b(y)(V_1 - V)) - cM(U)(V_1 - V)_s \\ = \rho_l^\epsilon(\lambda_1)V_1 + c(M(U_l) + N\epsilon)V_{1s} - cM(U)V_{1s}.\end{aligned} \tag{3.79}$$

This form of the equation has no maximum principle so we will make the change of variable $V(s,y) = e(y)W(s,y), V_1(s,y) = e(y)W_1(s,y)$ where $e(y)$ is given by (3.14). Then $W_1 - W$ satisfies

$$\begin{aligned}(\partial_s + \partial_y)(a(y)(\partial_s + \partial_y)(W_1 - W)) \\ +b_1(y)(\partial_s + \partial_y)(W_1 - W) + cM(U)(W_1 - W)_s \\ = \rho_l^\epsilon(\lambda_1^\epsilon)W_1 + c(M(U_l) + N\epsilon)W_{1s} - cM(U)W_{1s} \\ \leq \rho_l^\epsilon(\lambda_1^\epsilon)W_1 + 2cN\epsilon W_{1s} \\ = W_1(\rho_l^\epsilon(\lambda_1^\epsilon) + 2cN\epsilon\lambda_1^\epsilon) \leq 0\end{aligned} \tag{3.80}$$

for ϵ small enough (i.e. $s \leq s_1$). With this sign, the maximum principle implies that there exists a constant, C, depending on $e(y)$ and $\varphi_1(y)$ such that

$$CW_1(s,y) - W(s,y) \geq 0, \tag{3.81}$$

or

$$CW_1(s,y) \geq W(s,y) \tag{3.82}$$

for $s \leq s_1$. Similarly,

$$-W(s,y) \leq CW_1(s,y). \tag{3.83}$$

Therefore,

$$|W(s,y)| \leq CW_1(s,y) = Ce^{\lambda_1 s}\varphi(y). \tag{3.84}$$

So

$$|V(s,y)| = U_l(y) - U(s,y) \leq Ce^{\lambda_1 s}\frac{\varphi_1(y)}{e(y)} \leq C_1 e^{\lambda_1 s}. \tag{3.85}$$

Changing back to our original variables, we see

$$K(u_l) - K(u) \leq C_1 e^{\lambda_1 s} \tag{3.86}$$

for $s \leq s_1$ and, after applying the Mean Value Theorem

$$u_l(y) - u(s,y) \leq C_2 e^{\lambda_1 s}, \tag{3.87}$$

where C_1 and C_2 are constants depending on $\varphi_1(y), e(y), K(u)$, and C. A similar argument will yield

$$u(s,y) - u_r(y) \leq C_3 e^{\lambda_2 s} \tag{3.88}$$

for λ_2 now less than zero and $s \geq s_2$. Therefore, we have exponential decay of the solution to the steady-states at positive and negative infinity. The proof is complete.

4 Uniqueness of Traveling Waves

Having shown Lemmas 3.1 and 3.2, we will now adapt an argument presented by Xin in [60] to show the uniqueness of the traveling wave solution to (3.1). These solutions are unique up to a constant translation, s_0, in the s-variable. The choice of s_0 will prove to be crucial to our proof of stability in chapter 7.

Theorem 4.1 ***(Uniqueness)*** *Suppose that (u, c) and (u', c') all satisfy the conditions in Lemma 3.1 and assume that the hydraulic functions satisfy (A1) - (A5). Then $u'(s, y) = u(s - s_0, y)$ for some s_0 in R and $c = c'$.*

Proof: That $c = c'$ follows from Lemma 3.1.

Recall that we have made the transformation from $u(s, y)$ to $U(s, y) = K(u(s, y))$. Suppose U and U' are two solutions of equation (3.10) with (3.11). Let

$$W(s, y, \lambda) = U(s - \lambda, y) - U'(s, y) \tag{4.1}$$

for $\lambda \in R^1$. We know from Lemma 3.2 that W goes to zero as s goes to infinity. We have the two equations

$$(\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)U - b(y)U] + cM(U)U_s = 0, \tag{4.2}$$

$$(\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)U' - b(y)U'] + cM(U')U'_s = 0. \tag{4.3}$$

Subtract (4.3) from (4.2) to obtain

$$\begin{aligned} 0 &= (\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)(U - U') - b(y)(U - U')] \\ &+ c[M(U)U_s - M(U')U'_s]. \end{aligned} \tag{4.4}$$

$$\begin{aligned} M(U)U_s - M(U')U'_s &= M(U)U_s - M(U)U'_s + M(U)U'_s - M(U')U'_s \\ &= M(U)[U_s - U'_s] + [M(U) - M(U')]U'_s \\ &= M(U)[U_s - U'_s] + M'(Q)(U - U')U'_s \\ &= M(U)[U_s - U'_s] + \beta(U - U')U'_s, \end{aligned} \tag{4.5}$$

where Q is an intermediate value and

$$\beta = \int_0^1 M'(\tau U(s+\lambda, y) + (1-\tau)U'(s,y))d\tau. \tag{4.6}$$

Then we have

$$M(U)U_s - M(U')U_s' = M(U)W_s + \beta W U_s'. \tag{4.7}$$

So we end up with the equation

$$(\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)W - b(y)W] + cM(U)W_s + \beta W U_s' = 0. \tag{4.8}$$

Solutions to the above equation do not have a maximum principle. Therefore, we need to use the change of variables

$$V(s,y,\lambda) = \frac{W(s,y,\lambda)}{\epsilon(y)}, \tag{4.9}$$

where $\epsilon(y) > 0$, to obtain an equation with a maximum principle. Then we have

$$\begin{aligned}
(\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)(\epsilon V) - b(y)\epsilon V] + c(M(U)(\epsilon V)_s + \beta\epsilon V U_s') \\
= \epsilon(\partial_s + \partial_y)[a(\partial_s + \partial_y)V] + 2\epsilon' a(\partial_s + \partial_y)V - \epsilon b(\partial_s + \partial_y)V \\
+[(\epsilon' a)' - (\epsilon b)']V + \epsilon c(M(U)V_s + \beta V U_s').
\end{aligned} \tag{4.10}$$

We will determine $\epsilon(y) > 0$ as in (3.14). Then we can divide by $\epsilon(y)$ to get

$$(\partial_s + \partial_y)[a(\partial_s + \partial_y)V] + (\frac{2\epsilon'}{\epsilon}a - b)(\partial_s + \partial_y)V + c(M(U)V_s + \beta V U_s') = 0. \tag{4.11}$$

Ultimately, we will show that $V(s,y,\lambda) = 0$ for some choice of λ. To this end, we will begin with the following argument:

For any $N_1 > 0, N_2 > 0$ there exists $\lambda_0 = \lambda_0(N_1, N_2)$ such that if $\lambda \geq \lambda_0$, then $V(s,y,\lambda) > 0$ for $(s,y) \in [-N_1, N_2] \times T$. Now we choose the sizes of N_1 and N_2 to prove that $V > 0$ if $\lambda \geq \lambda_0$ for all (s,y).

Let us assume that V is of the form

$$V(s,y,\lambda) = \epsilon^{\epsilon s}\phi(y)w(s,y,\lambda) \tag{4.12}$$

where $\epsilon > 0$ and $\phi(y) > 0$ are to be chosen. Then, w satisfies the equation:

$$\begin{aligned} 0 &= (\partial_s + \partial_y)[a(\partial_s + \partial_y)w] + \frac{2}{\phi}[\epsilon\phi + \phi'](\partial_s + \partial_y)w \\ &+ (\frac{2e'}{e}a - b)(\partial_s + \partial_y)w + cM(U)w_s + K_2 w, \end{aligned} \tag{4.13}$$

where K_2 is defined by

$$\begin{aligned} K_2 &= \frac{1}{\phi}[(a\phi')' + 2\epsilon a\phi' + (\frac{2e'}{e}a - b)\phi' \\ &+ [\epsilon(\frac{2e'}{e}a - b) + \epsilon^2 a + \epsilon a' + \epsilon cM(U) + c\beta U_s']\phi]. \end{aligned} \tag{4.14}$$

Choose $e^{\epsilon s}\phi$ to be the principle eigenfunction of L_l in (3.37) with ϵ replacing λ there and with eigenvalue $\rho_l(\epsilon)$. After performing the change of variable in (4.9), L_l has the form

$$L_l(V) = (\partial_s + \partial_y)[a(\partial_s + \partial_y)V] + (\frac{2e'}{e}a - b)(\partial_s + \partial_y)V + cV_s M(U_l). \tag{4.15}$$

Then,

$$\begin{aligned} L_l(e^{\epsilon s}\phi) &= (\partial_s + \partial_y)[a(\epsilon e^{\epsilon s}\phi + e^{\epsilon s}\phi')] \\ &+ [\frac{2e'}{e}a - b](\epsilon e^{\epsilon s}\phi + e^{\epsilon s}\phi) + c\epsilon e^{\epsilon s}\phi M(U_l) \\ &= e^{\epsilon s}[(a\phi')' + 2\epsilon a\phi' + (\frac{2\epsilon'}{e}a - b)\phi' \\ &+ [\epsilon(\frac{2e'}{e}a - b) + \epsilon^2 a + \epsilon a' + \epsilon cM(U_l)]\phi]. \end{aligned} \tag{4.16}$$

Using the eigenfunction, K_2 becomes

$$K_2 = \rho_l(\epsilon) + \epsilon c[M(U) - M(U_l)] + c\beta U_s'. \tag{4.17}$$

So we obtain the equation

$$\begin{aligned} 0 &= (\partial_s + \partial_y)[a(\partial_s + \partial_y)w] + \frac{2}{\phi}[\epsilon\phi + \phi'](\partial_s + \partial_y)w \\ &+ (\frac{2e'}{e}a - b)(\partial_s + \partial_y)w + cM(U)w_s \\ &+ (\rho_l(\epsilon) + \epsilon c[M(U) - M(U_l)] + c\beta U_s')w. \end{aligned} \tag{4.18}$$

Expand $\rho_l(\epsilon)$ as

$$\rho_l(\epsilon) = \rho_l'(0)\epsilon + O(\epsilon^2), \tag{4.19}$$

where $\rho_l'(0) < 0$ for ϵ small enough. We will choose ϵ small enough that

$$\rho_l(\epsilon) \leq \frac{1}{2}\rho_l'(0)\epsilon \tag{4.20}$$

and N_1 large enough that $|U_s'| \leq \epsilon^2$ and $U_l(y) - U(s,y) \leq \epsilon^2$ for $(s,y) \in [-\infty, N_1] \times T$. This is possible due to Lemma 3.2. Similarly, we can choose $V(s,y,\lambda) = \epsilon^{-\epsilon s}\phi_1(y)w_1(s,y,\lambda)$ with $e^{-\epsilon s}\phi_1(y)$ being the positive principle eigenfunction for the operator L_r in (3.38) and having $\rho_r(-\epsilon)$ as the corresponding eigenvalue. Now. select N_2 so that $U'(s,y) - U_r(y) \leq \epsilon^2$ and $|U_s| \leq \epsilon^2$ for $(s,y) \in [N_2, \infty] \times T$.

Suppose $\inf_{R\times T} V(s,y,\lambda) < 0$. Then $\inf_{(-\infty, N_1)\times T} V(s,y,\lambda) < 0$ or/and $\inf_{(N_2,\infty)\times T} V(s,y,\lambda) < 0$. Assume the first case. This implies that $\inf_{(-\infty,N_1)\times T} w(s,y,\lambda) < 0$. By Lemma 3.2, $w(s,y,\lambda) \to 0$ as $s \to -\infty$. Let $w(s_1, y_1, \lambda)$ be the minimum of $w(s,y,\lambda)$. So $U_l(y) - U(s_1 - \lambda, y_0) \leq \epsilon^2$ since $s_1 - \lambda \leq s_1 < N_1$ and $|U_s'| \leq \epsilon^2$. Evaluate

$$\begin{aligned} 0 &= (\partial_s + \partial_y)[a(\partial_s + \partial_y)w] + \frac{2}{\phi}[\epsilon\phi + \phi'](\partial_s + \partial_y)w \\ &+ (\frac{2e'}{e}a - b)(\partial_s + \partial_y)w + cM(U)w_s \\ &+ (\rho_l(\epsilon) + \epsilon c[M(U) - M(U_l)] + c\beta U_s')w \end{aligned} \tag{4.21}$$

at (s_1, y_1) to obtain

$$\begin{aligned} L.H.S. &\geq w(s_1,y_1)(\rho_l(\epsilon) + \epsilon c[M(U) - M(U_l)] + c\beta U_s') \\ &\geq w(s_1,y_1)(\frac{1}{2}\rho_l'(0)\epsilon + O(\epsilon^2)) > 0. \end{aligned} \tag{4.22}$$

Thus, we have a contradiction to (4.21).

Now assume the second case. This implies $\inf_{(N_2,\infty)\times T} w(s,y,\lambda) < 0$. Let $w(s_1,y_1,\lambda)$ be the minimum. In this case, w solves

$$
\begin{aligned}
0 &= (\partial_s+\partial_y)[a(\partial_s+\partial_y)w] + \frac{2}{\phi}[\epsilon\phi+\phi'](\partial_s+\partial_y)w \\
&+ (\frac{2e'}{e}a-b)(\partial_s+\partial_y)w + cM(U)w_s \\
&+ (\rho_r(-\epsilon)-\epsilon c[M(U)-M(U_r)]+c\beta U_s')w. \qquad (4.23)
\end{aligned}
$$

Evaluating at (s_1,y_1), we get

$$L.H.S. \geq w(s_1,y_1,\lambda)(\rho_r(-\epsilon)-\epsilon c[M(U)-M(U_r)]+c\beta U_s') \qquad (4.24)$$

and since $U'(s_1,y_1)\geq U(s_1-\lambda,y_1)\geq U_r(y_1)$ and $U(s-\lambda,y)-U_r\leq\epsilon^2$ we have

$$L.H.S. \geq w(s_1,y_1,\lambda)(-\frac{1}{2}\rho_r'(0)\epsilon+O(\epsilon^2))>0, \qquad (4.25)$$

a contradiction to (4.23). Therefore, we have shown that $V(s,y,\lambda)\geq 0$ for $\lambda\geq\lambda_0$.

We will now show $V(s,y,\lambda)=0$ for some λ. Let $s_0=inf\{\lambda|V(s,y,\lambda)=0\}$. We know that $V(s,y,s_0)\geq 0$ for all (s,y) and for some $s_0\in[-\infty,\lambda_0]$. If $V(s,y,s_0)=0$ at any finite point, then $V(s,y,\lambda)$ is identically zero by the maximum principle. Otherwise, $V(s,y,s_0)>0$ for all (s,y). By the minimality of s_0, there is a sequence $\lambda_j\to s_0, j\to\infty$, such that $\inf_{R\times T}V(s,y,\lambda_j)=V(s_j,y_j,\lambda_j)<0$.

Suppose that the s_j's are unbounded, so $s_j\to+\infty$ or $s_j\to-\infty$, up to a subsequence still denoted by s_j. Thus, if j is large enough, s_j is in $[-N_1,N_2]^c$. Assume first that $s_j<-N_1$ so $V(s_j,y_j,\lambda_j)<0$. Letting $V=\epsilon^{\epsilon s}\phi(y)w$ as before, we see that $w(s_j,y_j,\lambda_j)<0$. The infimum of $w(s_j,y_j,\lambda_j)$ is obtained at a finite point $(\hat{s_j},\hat{y_j})$. If the $\hat{s_j}$'s are unbounded as $j\to\infty$, then $\hat{s_j}\to-\infty$ up to a subsequence. Evaluating equation (4.21) shows a contradiction just as before.

So the $\hat{s}_j$'s are bounded and $\hat{s}_j \to \hat{s}_1$, $\hat{y}_j \to \hat{y}_1$ with $(\hat{s}_1, \hat{y}_1)$ in $[-\infty, N_1] \times T$ and therefore

$$w(\hat{s}_1, \hat{y}_1, s_0) = \lim_{j \to \infty} w(\hat{s}_j, \hat{y}_j, \lambda_j) \leq 0, \tag{4.26}$$

which implies that $V(\hat{s}_1, \hat{y}_1, s_0) \leq 0$, a contradiction.

Next assume that s_j is in $(N_2, +\infty)$ so that $\inf_{(N_2,\infty)} V(s, y, \lambda_j) < 0$. Letting $V = e^{-\epsilon s}\phi(y)w$, we see that $\inf_{(N_2,\infty)} w(s, y, \lambda_j) < 0$. The infimum of $w(s, y, \lambda_j)$ is achieved at a finite point $(\hat{s}_j, \hat{y}_j)$ since $w(s, y, \lambda_j) \to 0$ as $s \to \infty$. If the $\hat{s}_j$'s are unbounded, then $\hat{s}_j \to +\infty$ up to a subsequence, and evaluating equation (4.23) at $(\hat{s}_j, \hat{y}_j)$ shows that the left hand side is strictly larger than zero, a contradiction. If the $\hat{s}_j$'s are bounded, then $(\hat{s}_j, \hat{y}_j) \to (\hat{s}_1, \hat{y}_1)$ in $[N_2, +\infty] \times T$ up to a subsequence and

$$w(\hat{s}_1, \hat{y}_1, s_0) = \lim_{j \to \infty} w(\hat{s}_j, \hat{y}_j, \lambda_j) \leq 0, \tag{4.27}$$

which implies that $V(\hat{s}_1, \hat{y}_1, s_0) \leq 0$, a contradiction.

Therefore, $V(s_1, y_1, s_0) > 0$ is not possible and $V(s, y, s_0) \equiv 0$ for all (s, y), implying $U(s - s_0, y) \equiv U'(s, y)$. Uniqueness of the traveling wave is proven.

Corollary 4.1 ***(Monotonicity)*** *Suppose that $(U(s, y), c)$ satisfy the conditions in Lemma 3.1, then $U_s(s, y) < 0$, for all $(s, y) \in R^1 \times T$.*

Proof: Taking $(U', c') = (U, c)$ in the uniqueness theorem and following the same proof, we see that $U(s - \lambda, y) > U(s, y)$ if $\lambda > s_0$ and $U(s - s_0, y) = U(s, y)$. Since U approaches different limits at s infinities, $s_0 = 0$. This implies that $U_s \leq 0$. Differentiating the traveling wave equation (3.15) to s and applying the strong maximum principle, we have that $U_s(s, y) < 0$, for all $(s, y) \in R^1 \times T$.

5 The Continuation of Regularized Solutions

Due to the traveling wave equation's (3.10) degeneracy in s and y, we will continue our adaptation of the argument in [60] to establish the existence for the following elliptically regularized equation:

$$\nu U_{ss} + (\partial_s + \partial_y)(a(\partial_s + \partial_y)U) - (\partial_s + \partial_y)(bU) + cM(U)U_s = 0 \qquad (5.1)$$

with

$$U(-\infty, y) = U_l(y), \quad U(\infty, y) = U_r(y), \qquad (5.2)$$

U 1-periodic in y, where $\nu \in (0, 1]$. We will prove the existence of a solution to (3.10) by passing to the limit $\nu \to 0$ in chapter 6. After converting the equation to a form for which the maximum principle applies, we will begin by constructing the solution of the equation by the continuation method. In other words, we know from Noren [30] that a solution exists to the homogeneous case. We can take that solution and perturb it to show the existence of solutions for heterogeneous cases.

Since the above equation does not have a maximum principle, we must introduce the change of variables

$$W(s, y) = \frac{U(s, y)}{e(y)}. \qquad (5.3)$$

Substituting, we obtain

$$\begin{aligned} 0 &= e\nu W_{ss} + e(\partial_s + \partial_y)(a(\partial_s + \partial_y)W) - eb(\partial_s + \partial_y)W \\ &+ cM(e(W))eW_s - eWb_y - bWe_y \\ &+ ae_y(\partial_s + \partial_y)W + Wa_ye_y + aWe_{yy}. \end{aligned} \qquad (5.4)$$

Looking at the lowest order terms of the above equation, we will choose $e(y) > 0$ as in (3.14). After dividing by $e(y)$ we get

$$\nu W_{ss} + (\partial_s + \partial_y)(a(\partial_s + \partial_y)W) + (\frac{2e_y}{e}a - b)(\partial_s + \partial_y)W + cM(eW)W_s = 0. \quad (5.5)$$

To simplify notation, let

$$b_*(y) = \frac{2e_y(y)}{e(y)}a(y) - b(y). \tag{5.6}$$

At this point, our strategy for showing the existence of the traveling wave solution is to start by recognizing that (5.5) has a solution for the homogeneous case (i.e. where $a(y)$ and $b(y)$ are constant). This was shown by Noren [30]. Then, we will perturb the coefficients in (5.5) by a small amount by taking $a(y)$ and $b(y)$ to be small, periodic perturbations of constant functions, then show the existence of this perturbed solution. We will continue this perturbation process until we have the existence of the solution for the equation we are considering.

Let us consider the family of equations parametrized by τ:

$$\nu W^\tau_{ss} + (\partial_s + \partial_y)(a^\tau(\partial_s + \partial_y)W^\tau) + b^\tau_*(\partial_s + \partial_y)W^\tau + c^\tau M(eW^\tau)W^\tau_s = 0, \tag{5.7}$$

where

$$W^\tau(-\infty, y) = W_l, \quad W^\tau(\infty, y) = W_r, \tag{5.8}$$

$W_l > W_r > 0$ are constants, $W^\tau(s,y)$ 1-periodic in y, where $a^\tau = \langle a\rangle(1-\tau)+\tau a(y)$ and $b^\tau = \langle b_*\rangle(1-\tau) + \tau b_*(y)$. First we show that if this equation admits solutions for $\tau = \tau_0, \tau_0 \in [0,1)$, then it has solutions for $\tau = \tau_0 + \delta$ if δ is sufficiently small. For simplicity, let us denote $a^{\tau_0}, b^{\tau_0}, c^{\tau_0}$, and W^{τ_0} as a, b, c, and W and $a^{\tau_0+\delta}, b^{\tau_0+\delta}, c^{\tau_0+\delta}$, and $W^{\tau_0+\delta}$ as $a^\delta, b^\delta, c^\delta$, and W^δ. We can write $a^\delta = a + \delta a_1$, $b^\delta = b + \delta b_1$, $c^\delta = c + \delta c_1$, and $W^\delta = W + \delta V$, where a_1 and b_1 are smooth functions of y and c_1 and V are unknowns. Substituting these expressions into the equation and using the fact that (W,c) is a solution when $\delta = 0$, we find that V satisfies the equation

$$\begin{aligned} LV &= \nu V_{ss} + (\partial_s + \partial_y)(a(\partial_s + \partial_y)V) + b_*(\partial_s + \partial_y)V + cM(eW)V_s \\ &= (\partial_s + \partial_y)(a_1(\partial_s + \partial_y)W) + \delta(\partial_s + \partial_y)(a_1(\partial_s + \partial_y)V) + b_1(\partial_s + \partial_y)W \end{aligned}$$

$$+ \quad \delta b_1(\partial_s + \partial_y)V + c\delta V_s eVM' + c\frac{\delta^2(eV)^2}{2}V_s M''$$
$$+ \quad cW_s eVM' + c\frac{\delta(eV)^2}{2}W_s M'' + c_1(W + \delta V)M(eW + e\delta V). \tag{5.9}$$

To solve for (V, c), we need to study the invertibility of the linear operator L. Let us consider the operator L on $L^2_\rho(R \times T)$ where $\rho(s) = cosh^2 \epsilon s, \epsilon \ll 1$ and

$$L^2_\rho(R \times T) \equiv \{h(s,y)| \int_{R\times T}(cosh^2 \epsilon s)h^2(s,y)dsdy < +\infty\}, \tag{5.10}$$

with ϵ to be chosen. The domain of definition $D(L)$ of L is $H^2_\rho(R \times T)$. It is easy to see that L is a closed operator on L^2_ρ. By the properties of $W(s,y)$ in chapter 4, especially the monotonicity corollary, we see that $W_s \in L^2_\rho$ if ϵ is suitably small, and that W_s is in the kernel of L.

Our goal is to show that zero is an isolated simple eigenvalue of L. Then by the spectral theorem of Kato [16], L is a Fredholm operator with index zero. This will imply the local continuation of regularized solutions via the Contraction Mapping Theorem. First, we will prove that the essential spectrum of L is bounded away from zero by a positive distance depending on (W, c). Let

$$V(s,y) = (e^{-\epsilon s}\phi_r(y)\zeta(s) + (1 - \zeta(s))e^{\epsilon s}\phi_l(y))w \equiv w_0 w, \tag{5.11}$$

where $\zeta(s)$ is a smooth function of s, such that $0 < \zeta(s) < 1$ for all $s \in (-1, 1)$; $\zeta(s) \equiv 0$ for $s \leq -1$; $\zeta(s) \equiv 1$ for $s \geq 1$; $\phi_r(y) \geq 1$; $\phi_l(y) \geq 1$, are in $C^\infty(T)$ to be determined; $\epsilon > 0$ is the same as in the weight function $cosh^2 \epsilon s$ for L^2_ρ. Similarly, let $g = w_0 g_1$. So the problem

$$LV = g, \quad g \in L^2_\rho(R \times T). \tag{5.12}$$

becomes

$$L(w_0 w) = w_0 g_1, \quad w, g \in L^2(R \times T). \tag{5.13}$$

Calculation shows

$$w_0L(w) + 2\nu w_{0s}w_s + 2((\partial_s + \partial_y)w_0)a(\partial_s + \partial_y)w + wL(w_0) = w_0g_1, \tag{5.14}$$

or

$$L(w) + \frac{2}{w_0}(\nu w_{0s}w_s + ((\partial_s + \partial_y)w_0)a(\partial_s + \partial_y)w) + \frac{L(w_0)}{w_0}w = g_1. \tag{5.15}$$

Let us compute $L(w_0)$. If $s \leq -1$, then

$$L(w_0) = e^{\epsilon s}[(a\phi_{ly})_y + 2\epsilon a\phi_{ly} + b\phi_{ly} + (\epsilon b + \epsilon^2 a + \epsilon a_y) + c\epsilon M\phi_l]. \tag{5.16}$$

Choosing ϕ_l to be the eigenfunction of the operator in (3.37) as before, we have

$$L(w_0) = [\rho_l(\epsilon) + c\epsilon(M(U) - M(U_l))]e^{\epsilon s}\phi_l, \tag{5.17}$$

or

$$d(s,y) \equiv \frac{L(w_0)}{w_0} = \rho_l(\epsilon) + c\epsilon(M(U) - M(U_l)), \tag{5.18}$$

where $\rho_l(\epsilon) = \rho_l'(0)\epsilon + O(\epsilon^2) < 0$. Thus

$$\lim_{s\to-\infty} \frac{L(w_0)}{w_0} = \rho_l(\epsilon) < 0. \tag{5.19}$$

Now if $s \geq 1$,

$$L(w_0) = e^{-\epsilon s}[(a\phi_{ry})_y + 2\epsilon a\phi_{ry} + b\phi_{ry} + (-\epsilon b + \epsilon^2 a - \epsilon a_y) - c\epsilon M\phi_r], \tag{5.20}$$

$$L(w_0) = [\rho_r(-\epsilon) + c\epsilon(M(U_r) - M(U))]e^{-\epsilon s}\phi_r, \tag{5.21}$$

or, by choosing ϕ_r from (3.38) in a way analogous to ϕ_l in (5.17),

$$d(s,y) \equiv \frac{L(w_0)}{w_0} = \rho_r(-\epsilon) + c\epsilon(M(U_r) - M(U)), \tag{5.22}$$

where $\rho_r(-\epsilon) = \rho_r'(0)\epsilon + O(\epsilon^2) < 0$. Thus

$$\lim_{s\to+\infty} \frac{L(w_0)}{w_0} = \rho_r(-\epsilon) < 0. \tag{5.23}$$

The function $d(s,y)$ is smooth in (s,y), approaching $\rho_l(\epsilon)(\rho_r(-\epsilon))$ as $s \to -\infty(+\infty)$. Now compute

$$\begin{aligned}(\partial_s + \partial_y)w_0 &= (e^{-\epsilon s}\zeta)_s\phi_r(y) + ((1-\zeta)e^{\epsilon s})_s\phi_l(y) \\ &+ (e^{-\epsilon s}\zeta)\phi_{ry} + (1-\zeta)e^{\epsilon s}\phi_{ly} \end{aligned} \tag{5.24}$$

and

$$w_{0s} = (e^{-\epsilon s}\zeta)_s\phi_r(y) + ((1-\zeta)e^{\epsilon s})_s\phi_l(y). \tag{5.25}$$

If $s \le -1$, then

$$d_1(s,y) = \frac{2}{w_0}(\partial_s + \partial_y)w_0 = 2\epsilon + 2\frac{\phi_{ly}}{\phi_l} \equiv B_l(y), \tag{5.26}$$

$$d_2(s,y) = \frac{2\nu}{w_0}w_{0s} = 2\epsilon\nu. \tag{5.27}$$

If $s \ge 1$, then

$$d_1(s,y) = \frac{2}{w_0}(\partial_s + \partial_y)w_0 = -2\epsilon + 2\frac{\phi_{ry}}{\phi_r} \equiv B_r(y), \tag{5.28}$$

$$d_2(s,y) = \frac{2\nu}{w_0}w_{0s} = -2\epsilon\nu. \tag{5.29}$$

Thus $d_1(s,y)$ and $d_2(s,y)$ are smooth functions of $(s,y) \in R \times T$, and equal to $B_l(B_r)$ and $\pm 2\epsilon\nu$ if s is outside $[-1,1]$.

We can rewrite the operator equation as

$$L_1 w \equiv Lw + d_2 w_s + d_1 a(y)(\partial_s + \partial_y)w + dw = g. \tag{5.30}$$

The spectrum of L on L^2_ρ is the same as that of the operator L_1 on L^2. Define a new operator

$$\begin{aligned} Qw &\equiv \nu w_{ss} + (\partial_s + \partial_y)(a(\partial_s + \partial_y)w) + B(s,y)(\partial_s + \partial_y)w \\ &+ [c(\zeta(\alpha s)(M(U_r) - 2\epsilon\nu) + (1-\zeta(\alpha s))(M(U_l) + 2\epsilon\nu))]w_s \\ &+ (\rho_r(-\epsilon)\zeta(\alpha s) + \rho_l(\epsilon)(1-\zeta(\alpha s)))w, \end{aligned} \tag{5.31}$$

where α is a small, positive number to be chosen and

$$B(s,y) = b_*(y) + \zeta(\alpha s)a(y)B_r(y) + (1-\zeta(\alpha s))a(y)B_l(y). \tag{5.32}$$

Define also

$$Sw \equiv (L_1 - Q)w = B_1(s,y)(\partial_s + \partial_y)w + B_2(s,y)w_s + B_3(s,y)w, \tag{5.33}$$

where

$$B_1 = a(y)d_1(s,y) - (\zeta(\alpha s)a(y)B_r(y) + (1-\zeta(\alpha s))a(y)B_l(y)), \tag{5.34}$$

$$B_2 = c(M(U) - \zeta(\alpha s)(M(U_r) - 2\epsilon\nu) + (1-\zeta(\alpha s))(M(U_l) + 2\epsilon\nu)), \tag{5.35}$$

and

$$B_3 = d(s,y) - (\rho_r(-\epsilon)\zeta(\alpha s) + \rho_l(\epsilon)(1-\zeta(\alpha s))). \tag{5.36}$$

We see that $B_i(s,y) \to 0, i = 1,2,3$, uniformly in y as $s \to \infty$.

Let us show that Q is invertible on $L^2(R \times T)$ by the Lax-Milgram Theorem.

Proposition 5.1 *There exists a positive number $\alpha_0 = \alpha_0(\epsilon) \in (0,1]$, such that if $\alpha \in (0,\alpha_0]$ the operator Q as defined is invertible on $L^2(R \times T)$. Moreover, there is a positive constant $M = M(\alpha,\epsilon)$ such that*

$$\|Q^{-1}g\|_{H^2} \leq M\|g\|_{L^2}. \tag{5.37}$$

Proof: First we prove that the equation $Qw = g$ admits a weak solution in $H^1(R \times T)$ for $g \in L^2(R \times T)$. Consider the following bilinear functional from $H^1 \times H^1$ to R:

$$\begin{aligned} D(w,v) &= \int_{R\times T} \nu w_s(mv)_s + ((\partial_s + \partial_y)w)a(y)(\partial_s + \partial_y)(mv) \\ &- B(s,y)((\partial_s + \partial_y)w)(mv) \\ &- [c(M(U) - \zeta(\alpha s)(M(U_r) - 2\epsilon\nu) \\ &+ (1-\zeta(\alpha s))(M(U_l) + 2\epsilon\nu))]w_s mv \\ &- [\rho_r(-\epsilon)\zeta(\alpha s) + \rho_l(\epsilon)(1-\zeta(\alpha s))]mvw. \end{aligned} \tag{5.38}$$

Here $m = m(s_1, y) \equiv m(\alpha s, y)$, and $m(s_1, y)$ is a smooth function on $R \times T$ such that $1 \le m \le M_1$, uniformly for all $\alpha \in (0, 1]$, where M_1 is independent of ϵ. We will choose such a function m later.

Obviously, $D(w, v)$ satisfies:

$$|D(w,v)| \le M_2 \|w\|_{H^1} \|v\|_{H^1}, \tag{5.39}$$

for some positive constant M_2 independent of ϵ. Now we calculate $D(v, v)$ as follows:

$$\begin{aligned} D(v,v) &= \int_{R\times T} dsdy(\nu v_s^2 m + \nu v v_s m_s \\ &+ m((\partial_s + \partial_y)v)a(\partial_s + \partial_y)v + v((\partial_s + dy)v)a(\partial_s + \partial_y)m \\ &- mB(\partial_s + \partial_y)(v^2/2) \\ &- [c(M(U) - \zeta(\alpha s)(M(U_r) - 2\epsilon\nu) \\ &+ (1 - \zeta(\alpha s))(M(U_l) + 2\epsilon\nu))]m(v^2/2)_s \\ &- [\rho_r(-\epsilon)\zeta(\alpha s) + \rho_l(\epsilon)(1 - \zeta(\alpha s))]mv^2). \end{aligned} \tag{5.40}$$

Integration by parts gives

$$\begin{aligned} D(v,v) &\ge \int_{R\times T} dsdy(\nu v_s^2 m - \nu(v^2/2)m_{ss} + m(\partial_s + \partial_y)v \\ &- (v^2/2)(\partial_s + \partial_y)(a(\partial_s + \partial_y)m) + (v^2/2)(\partial_s + \partial_y)(mB) \\ &+ (v^2/2)m_s[c(M(U) - \zeta(\alpha s)(M(U_r) - 2\epsilon\nu) \\ &+ (1 - \zeta(\alpha s))(M(U_l) + 2\epsilon\nu))] \\ &+ m(v^2/2)[c(M(U) - \zeta(\alpha s)(M(U_r) - 2\epsilon\nu) \\ &+ (1 - \zeta(\alpha s))(M(U_l) + 2\epsilon\nu))]_s + C_1 \epsilon v^2), \end{aligned} \tag{5.41}$$

where C_1 is a positive constant independent of ϵ such that

$$C_1 \epsilon \le min(-\rho_r(-\epsilon), -\rho_l(\epsilon)). \tag{5.42}$$

Noticing that $m = m(\alpha s, y)$, so all the terms involving s derivatives of m are $O(\alpha)$. Similarly, s derivatives of B are $O(\alpha)$. It follows that

$$\begin{aligned} D(v,v) &\geq \int_{R\times T} dsdy(\nu v_s^2 m + m(\partial_s + \partial_y)v \\ &+ (v^2/2)[-(am_y)_y + (mB)_y + O(\alpha)] + C_1\epsilon v^2). \end{aligned} \quad (5.43)$$

We choose m to satisfy the following equation:

$$-(am_y)_y + (mB)_y = 0, \quad (5.44)$$

or

$$-(am_y)_y + (m(b + \zeta(\alpha s)aB_r(y) + (1-\zeta(\alpha s))aB_l(y)))_y = 0 \quad (5.45)$$

where s is just a parameter. For any fixed s, it is known that the above equation has a unique positive smooth solution on T up to constant multiplication as discussed by Bensoussan, et al [3]. By elliptic regularity, m depends smoothly on the coefficients and so $m = m(s', y) \equiv m(\alpha s, y)$ is a bounded smooth function in $(s', y) \in R \times T$. If we normalize m so that $m \geq 1$, then $m = m(\alpha s, y)$ is as desired. We see that there exists a number $\alpha_0 = \alpha_0(\epsilon)$ such that if $\alpha \in (0, \alpha_0)$, the $O(\alpha)$ term is no larger than $\frac{1}{2}C_1\epsilon$ in absolute value. It follows from the bound of $D(v,v)$, the choice of m, and such choice of α that

$$D(v,v) \geq C_2\|v\|^2_{H^1(R\times T)}, \quad (5.46)$$

for some positive constant $C_2 = C_2(\epsilon, \nu)$. Hence, the functional $D(w,v)$ is coercive, and Lax-Milgram theorem implies the existence of a weak solution to $Qv = g$ in H^1. By elliptic regularity (Theorem 8.8, pp.183-185, Gilbarg and Trudinger [7]), $v \in H^2$, and the estimate on $\|Q^{-1}\|$ holds.

Next we have

Lemma 5.1 *The operator SQ^{-1} is compact on $L^2(R \times T)$.*

The proof is similar to that of Lemma 2.7 in [58], and is omitted.

By the Gohberg-Krein Theorem, L_1 and Q differ by a relatively compact operator, so they have the same essential spectrum. Proposition 5.1 says that the essential spectrum is bounded away from zero by a positive distance depending on ϵ, hence (W, c). Thus 0 is an isolated eigenvalue of finite multiplicity of L_1 on L^2 or L on L^2_ρ.

Summarizing, we have

Corollary 5.1 *Zero is an isolated eigenvalue of finite multiplicity of the operator L on $L^2_\rho(R \times T)$.*

We show next:

Proposition 5.2 *The kernel of L is one-dimensional, and zero is its algebraically simple eigenvalue.*

The proof is similar to that of Proposition 2.1 in [58] except that near s infinities, we need to make the change of variable of the form $w = exp(\pm\epsilon_1 s)\phi(y)v$, with $0 < \epsilon_1 \ll 1$ and $\phi > 0$, for function v in the kernel of the operator L. For details, we refer to the proof of Theorem 4.1.

By Kato's theorem [16] (Theorem 5.28, p.239), L is a Fredholm operator of index zero, and L^*, the adjoint operator L, has a simple eigenfunction, denoted by v^*, in Ker(L^*). Moreover, the inner product of W_s and v^* can be normalized to one. See also Sattinger [41], pp. 320-321. We have

Proposition 5.3 *The equation $Lv = g$, where $g \in L^2_\rho(R\times T)$ is solvable in $L^2_\rho(R\times T)$ if and only if*

$$\int_{R\times T} \rho g v^* ds dy = 0, \tag{5.47}$$

where v^* is the simple eigenfunction of L^* corresponding to eigenvalue zero, such that the L^2_ρ inner product of W_s and v^* is equal to one. When the integral condition holds, the solution space is one-dimensional.

Applying Proposition 5.3, elliptic regularity estimates and the contraction mapping theorem, we get:

Theorem 5.1 *Suppose that equation (3.10) with its boundary conditions and the normalization condition $\int_T U(0,y)dy = U_0, U_0 \in (U_r, U_l)$ has a classical solution (U^τ, c^τ), where $\tau \in [0,1)$. Then there exists $\delta_0 = \delta_0(U,c)$ such that if $\delta \in (0,\delta_0)$, the equation admits a unique classical solution $(U^{\tau+\delta}, c^{\tau+\delta})$ satisfying the same boundary conditions and the normalization condition.*

We remark that the solvability condition in the above Proposition is used to determine the perturbed speed $c^{\tau+\delta}$. For details of the proof, we refer to Xin [61] and [62].

6 Limit of Regularized Solutions

Having shown the existence of solutions to (5.7) in chapter 5, we will show that this continuation can be done to construct solutions for any degree of heterogeneity. We will do this by concluding our adaptation of the argument given in [60] by considering the limit as $\nu \longrightarrow 0$ of classical solutions (W^τ, c^τ) of equation (5.7) satisfying the boundary conditions:

$$W^\tau(-\infty, y) = W_l, \quad W^\tau(\infty, y) = W_r, \tag{6.1}$$

$W^\tau(s, y)$ 1-periodic in y and the normalization condition $min_{y\in T} W^\tau(0, y) = W_0$, as $\tau \longrightarrow \tau_0 \in (0, 1]$. Due to uniqueness of solutions (Theorem 4.1), the solutions (W^τ, c^τ) above are just the ones generated by the continuation method (Thereom 5.1) modulo constant translations in s. We have:

Proposition 6.1 *Let τ_n be any sequence tending to $\tau_0 \in (0, 1]$. Then there is a subsequence, still denoted τ_n, such that if $W_l - W_0 \leq \epsilon_0$, ϵ_0 a small positive number depending on the nonlinear function $M(U)$, $W^{\tau_n}(s, y)$ converges to $W^{\tau_0}(s, y)$ in C^1_{loc}, and*

$$c^{\tau_n} = c^{\tau_0} = c_{eff} = \frac{\bar{c}_l - \bar{c}_r}{\langle K^{-1}(e(y)W_l) - K^{-1}(e(y)W_r)\rangle} > 0. \tag{6.2}$$

Moreover, (W^{τ_0}, c^{τ_0}) is a classical solution to equation (5.7) and (3.16) with $\tau = \tau_0$.

Proof: Applying Lemma 3.1, we see that $c^{\tau_n} = c_{eff}$. Choosing $W_l - W_0 < \epsilon_0 < 1$, where ϵ_0 is as small as required in the proof of Lemma 3.2, and following the argument there for constructing upper solutions, we have sequences $\lambda_1^{\tau_n} \longrightarrow \lambda_1^0 > 0$, $\phi_1^{\tau_n} \longrightarrow \phi_1^0(y)$ in $C(T^n)$, $\min_{y\in T^n} \phi_1^{\tau_n}(y) = min_{y\in T^n}\phi_1^0(y) = 1$, such that

$$W_l - W^{\tau_n}(s, y) \leq exp[\lambda_1^{\tau_n} s]\phi_1^{\tau_n}(y), \quad \forall (s, y) \in (-\infty, 0) \times T. \tag{6.3}$$

This follows from

$$W_l - W^{\tau_n}(s, y)\mid_{s=0} \leq W_l - W_0 < 1 \leq exp[\lambda_1^{\tau_n} s]\phi_1^{\tau_n}(y)\mid_{s=0}, \tag{6.4}$$

and the maximum principle.

Since $W_l \geq W^{\tau_n}(s,y) \geq W_r$ and c^{τ_n} is independent of τ_n, $\|W^{\tau_n}\|_{C^1_{loc}(R\times T)} \leq C_1 < +\infty$ due to elliptic Schauder estimates. As $\tau_n \to \tau_0$, $c^{\tau_n} \to c^{\tau_0}$ and $W^{\tau_n} \to W^{\tau_0}$ in $C^1_{loc}(R \times T)$ up to a subsequence of τ_n. Letting $n \to +\infty$ in (6.3) gives:

$$W_l - W^{\tau_0}(s,y) \leq exp[\lambda_1^{\tau_0} s]\phi_1^{\tau_0}(y), \quad s \leq 0 \tag{6.5}$$

which implies that

$$\lim_{s\to-\infty} W^{\tau_0}(s,y) = W_l. \tag{6.6}$$

So W^{τ_0} satisfies:

$$\nu W^{\tau_0}_{ss} + (\partial_s + \partial_y)(a^{\tau_0}(y)(\partial_s + \partial_y)W^{\tau_0})$$

$$+b^{\tau_0}(y)(\partial_s + \partial_y)W^{\tau_0} + c^{\tau_0} M(e(y)W^{\tau_0})W^{\tau_0}_s = 0 \tag{6.7}$$

in the weak sense and, by elliptic regularity, W^{τ_0} is a classical solution of (5.7). Moreover, $W^{\tau_0}(-\infty,y) = W_l$, $\min_{y\in T^n} W^{\tau_0}(0,y) = W_0$, $W^{\tau_0}_s \leq 0$, and $W^{\tau_0}(s,y)$ has period one in y.

We have yet to justify $W^{\tau_0}(+\infty,y) = W_r$. The limit

$$\lim_{s\to+\infty} W^{\tau_0}(s,y) \equiv W_+ \tag{6.8}$$

exists due to monotonicity of $W^{\tau_0}(s,y)$ in s. By local regularity estimates and the fact that $0 < \int_{R\times T} -W^{\tau_0}_s dsdy \leq W_l - W_r < +\infty$, it follows that $W^{\tau_0}_s \to 0$ as $s \to +\infty$ uniformly in y. Differentiating (6.7) to s and applying the elliptic Schauder estimates on $W^{\tau_0}_s$ imply that $W^{\tau_0}_{ss} \to 0$ as $s \to +\infty$.

Multiplying any smooth test function $\psi(y) \in C^\infty(T)$ on both sides of (6.7), integrating over y, we get by integration by parts:

$$\int_T \nu W^{\tau_0}_{ss}\psi dy + I + \int_T dy\{\psi b_1 W^{\tau_0}_s - \partial_y(\psi b_1)W^{\tau_0}\}$$

$$+c^{\tau_0}\int_T dy\psi M(e(y)W^{\tau_0})W_s^{\tau_0}=0, \tag{6.9}$$

where

$$\begin{aligned} I &= -\int_T \psi(y)(\partial_s+\partial_y)(a(y)(\partial_s+\partial_y)W^{\tau_0})dy \\ &= \int_T dy\psi(y)\partial_s(a(y)(\partial_s+\partial_y)W^{\tau_0})+\partial_y(a(y)(\partial_s+\partial_y)W^{\tau_0}) \\ &= \int_T dy\psi(y)a(y)W_{ss}^{\tau_0}-\int_T dyW_s^{\tau_0}\partial_y(\psi(y)a(y)) \\ &- \int_T dyW_s^{\tau_0}(\psi_y a(y))+\int_T dyW^{\tau_0}a_y(y)\psi_y. \end{aligned} \tag{6.10}$$

Letting $s\to+\infty$ in (6.10) shows that:

$$\int_T dy\partial_y(a(y)\psi_y)W_+-\int_T dy\partial_y(\psi b_1)W_+=0 \tag{6.11}$$

which implies that W_+ is a weak and hence a classical solution of the equation:

$$\partial_y(a(y)W_{+y})+b_1W_{+y}=0 \tag{6.12}$$

on T. The maximum principle implies that W_+ =constant. Thus,

$$\lim_{s\to+\infty}W^{\tau_0}(s,y)=W_+=const\in[W_r,W_0]. \tag{6.13}$$

Applying Lemma 3.1, we have then

$$c^{\tau_0}=\frac{\bar{c}_l-\bar{c}_+}{u_l-u_r}=\frac{\bar{c}_l-\bar{c}_+}{\langle K^{-1}(e(y)W_l)-K^{-1}(e(y)W_+)\rangle}. \tag{6.14}$$

The limit c_0^τ of c_n^τ satisfies the speed formula, so we have:

$$\frac{\bar{c}_l-\bar{c}_r}{\langle K^{-1}(e(y)W_l)-K^{-1}(e(y)W_r)\rangle}=\frac{\bar{c}_l-\bar{c}_+}{\langle K^{-1}(e(y)W_l)-K^{-1}(e(y)W_+)\rangle}. \tag{6.15}$$

We see that $W_+=W_r$ is a unique solution by the following argument: Define the right hand side of (6.14) to be the non-linear function

$$\begin{aligned} F(\bar{c}_+) &= \frac{\bar{c}_l-\bar{c}_+}{\langle K^{-1}(e(y)W_l)-K^{-1}(e(y)W_+)\rangle} \\ &= \frac{\bar{c}_l-\bar{c}_+}{\langle K^{-1}(-\bar{c}_lT(y))-K^{-1}(-\bar{c}_+T(y))\rangle} \end{aligned} \tag{6.16}$$

by Remark 3.1, where $T(y)$ is defined in (3.51) to be independent of $\bar{c}_+$. We are interested in showing that the only solution to $F(\bar{c}_+) = c^{T_0}$ is $\bar{c}_+ = \bar{c}_r$ and hence $W_+ = W_r$. The first derivative of $F(\bar{c}_+)$ is

$$\frac{dF}{d\bar{c}_+} = \frac{-\langle K^{-1}(-\bar{c}_l T(y)) - K^{-1}(-\bar{c}_+ T(y))\rangle + (\bar{c}_l - \bar{c}_+)\langle \frac{dK^{-1}(-\bar{c}_+ T(y))}{d\bar{c}_+}\rangle}{\langle K^{-1}(-\bar{c}_l T(y)) - K^{-1}(-\bar{c}_+ T(y))\rangle^2}. \tag{6.17}$$

The following algebraic manipulation of (6.17) will show that its numerator is equivalent to the entropy condition in (3.62). First notice that

$$\frac{dK^{-1}(-\bar{c}_+ T(y))}{d\bar{c}_+} = -T(y)\frac{dK^{-1}(U)}{dU}|_{U=-\bar{c}_+ T(y)}$$

$$= -T(y)\frac{1}{K'(K^{-1}(U))}|_{U=-\bar{c}_+ T(y)} = -T(y)M(U)|_{U=-\bar{c}_+ T(y)}, \tag{6.18}$$

where $M(U)$ is defined in (3.9). Therefore, we can rewrite the numerator as

$$-\langle u_l - u_+\rangle + (\bar{c}_l - \bar{c}_+)\langle -T(y)M(U)|_{U=-\bar{c}_+ T(y)}\rangle. \tag{6.19}$$

We see from Corollary 3.1 and the entropy condition (3.62) that

$$-\langle T(y)M(U_+)\rangle^{-1} = -\langle \frac{T(y)}{K'(u_+)}\rangle^{-1} < \frac{\bar{c}_l - \bar{c}_+}{\langle u_l - u_+\rangle}. \tag{6.20}$$

This implies (6.19) and hence (6.17) are strictly positive. Therefore, $\bar{c}_+ = \bar{c}_r$ is a unique solution.

Now, differentiating equation (6.7) to s and applying the strong maximum principle yields $W_s^{T_0} < 0$, for all s and y. The proof of the proposition is complete.

In summary we have:

Theorem 6.1 *For any given positive number $\nu > 0$, there exist a classical solution (W^ν, c^ν) to (5.7) satisfying all the boundary conditions (3.16). Moreover, $W_s^\nu < 0$, $W_l > W^\nu > W_r$, for all $(s,y) \in R \times T$; $c^\nu = c_{eff}$.*

Theorem 6.2 ***(Existence)*** *Suppose the entropy condition (3.62) holds for all $u(s,y)$ such that $u_l(y) > u(s,y) > u_r(y)$. Then there exists a classical traveling wave solution to equation (3.1) of the form $u = u(x-ct, x) = u(s,y)$ where $s = x - ct, y = x$; c is the wave speed; $u(-\infty, y) = u_l(y), u(+\infty, y) = u_r(y)$, and $u(s,y)$ is 1-periodic in y.*

Proof: We are ready to take the limit $\nu \to 0$ in equation (6.7). Since $K_u > 0$, we have

$$0 < C_1^{-1} \leq c^\nu M(eW^\nu) \leq C_1 < +\infty, \tag{6.21}$$

where C_1 is independent of ν. Parabolic Schauder estimates give:

$$\|W^\nu\|_{C^1_{loc}} \leq C_2 < +\infty, \tag{6.22}$$

with positive C_2 independent of ν. We impose:

$$\min_{y \in T} W^\nu(0,y) = W_0, \tag{6.23}$$

where $W_0 \in (W_r, W_l)$. Now choose W_0 close to W_r as in Proposition 6.1 and pass to the limit $\nu \to 0$. All the steps there go through except now we use the parabolic Schauder estimates instead of the elliptic estimates. Justifying the boundary conditions with the entropy condition again and changing back to the original variables, we complete the proof.

7 Stability Theorems

Having shown that traveling wave solutions exist, in this chapter our goal is use six preliminary lemmas which are proved in Appendix A to demonstrate that a time-dependent solution to (1.1), with front-like initial data, approaches a traveling wave solution asymptotically as $t \to \infty$. This notion of stability is analogous to the better known concept of stability which refers to the dependence of solutions of differential equations on their initial data. In our case, we will show that if the initial data for (1.1) satisfies the conditions (7.6), (7.7), and (7.8), then the solution of (1.1) will tend to a traveling wave. The proofs for the six lemmas are long, tedious, and add no particular insight to the problem. It is suggested that the reader skip the proofs until after the rest of the thesis has been completed at which time they can be read at leisure.

In the course of this stability analysis, we will need to look at three different versions of Richards' equation in diffusive form with some assumptions on the initial data and boundary conditions. We will write down the three forms with their related assumptions.

First, we use the actual Richards' equation from (1.1)

$$u_t = [D(u,x)u_x - K(u,x)]_x. \tag{7.1}$$

Letting $\zeta = x - ct$ where c is the speed of the traveling wave, equation (7.1) becomes

$$(\partial_t - c\partial_\zeta)u = [D(u,\zeta+ct)u_\zeta - K(u,\zeta+ct)]_\zeta, \tag{7.2}$$

where $-ct \leq \zeta \leq \infty$ and $t \geq 0$. Equation (7.2) is considered with the initial condition

$$u(\zeta,0) = u_0(\zeta), \tag{7.3}$$

and boundary conditions

$$D(u, \zeta + ct)u_\zeta - K(u, \zeta + ct)|_{\zeta=-ct} = -\bar{c}_l \tag{7.4}$$

and

$$D(u, \zeta + ct)u_\zeta - K(u, \zeta + ct)|_{\zeta=\infty} = -\bar{c}_r. \tag{7.5}$$

Moreover,

$$0 \le u_r(\zeta) \le u_0(\zeta) \le u_l(\zeta) \le 1, \quad \zeta \in R^1, \tag{7.6}$$

$$|u_{0\zeta}| \le M_2, \quad \zeta \in R^1, \tag{7.7}$$

and

$$\int_0^\infty |u_0(\zeta) - u_r(\zeta)| d\zeta < \infty, \tag{7.8}$$

where u_r is the steady state as $\zeta \to +\infty$. Please note that in this chapter M_1 and M_2 will be used as generic, positive constants depending on $a(x), b(x)$, and initial and boundary data.

Next, under the assumption (A3), that $D = K'$, we make the change of variable $U = K(u)$ to get the equation

$$M(U)U_t = [a(x)U_x - b(x)U]_x. \tag{7.9}$$

Recall from (3.9)

$$M(U) = \frac{1}{K'(K^{-1}(U))}. \tag{7.10}$$

Letting $\zeta = x - ct$ where c is the speed of the travelling wave, this equation becomes

$$M(U)(\partial_t - c\partial_\zeta)U = [a(\zeta + ct)U_\zeta - b(\zeta + ct)U]_\zeta, \tag{7.11}$$

where $U(\zeta, 0) = U_0(\zeta)$, and

$$a(\zeta + ct)U_\zeta - b(\zeta + ct)U|_{\zeta=-ct} = -\bar{c}_l, \tag{7.12}$$

$$a(\zeta+ct)U_\zeta - b(\zeta+ct)U|_{\zeta=\infty} = -\bar{c}_r, \tag{7.13}$$

$K(u_r) = U_r(\zeta) \le U_0(\zeta) \le U_l(\zeta) = K(u_l)$, and $|U_{0\zeta}| \le M_2$. It follows from (7.8) that

$$\int_0^\infty |U_0(\zeta) - U_r(\zeta)| d\zeta < \infty, \tag{7.14}$$

where $U_r = K(u_r)$ is the steady state as $\zeta \to +\infty$.

Finally, when a form of the equation with a maximum principle is required, we make the change of variable $U(x,t) = e(x)W(x,t)$, with $e(x) > 0$, to obtain

$$M(U)W_t = [a(x)W_x]_x + b_1(x)W_x, \tag{7.15}$$

with

$$b_1(x) = 2\frac{e'(x)}{e(x)}a(x) - b(x). \tag{7.16}$$

where the choice of $e(x)$ is determined by (3.14). Letting $\zeta = x - ct$ where c is the speed of the travelling wave, this equation becomes

$$M(U)(\partial_t - c\partial_\zeta)W = [a(\zeta+ct)W_\zeta]_\zeta + b_1(\zeta+ct)W_\zeta, \tag{7.17}$$

where $W(\zeta,0) = W_0(\zeta)$,

$$e(\zeta+ct)a(\zeta+ct)W_\zeta - kW|_{\zeta=-ct} = -\bar{c}_l, \tag{7.18}$$

$$W|_{\zeta=\infty} = W_r = \frac{-\bar{c}_r}{-k}, \tag{7.19}$$

where $k > 0$ is the constant of integration which comes from integrating (3.14):

$$a(x)e'(x) - b(x)e(x) = -k. \tag{7.20}$$

$W_r \le W_0(\zeta) \le W_l = \frac{\bar{c}_l}{k}$ and $|W_{0\zeta}| \le M_2$. We have from (7.8)

$$\int_0^\infty |W_0(\zeta) - W_r| d\zeta < \infty, \tag{7.21}$$

where W_r, a constant as seen in (3.18), is the steady state of (7.15) as $\zeta \to +\infty$.

Since solutions to (7.17) satisfy the parabolic maximum principle, there exists a constant $M_1 > 0$ such that $W(\zeta, t) \leq M_1 < \infty$ for all $-ct \leq \zeta \leq \infty$ and $0 \leq t \leq T$ for $T > 0$ where M depends on the initial data and the coefficients $a(x)$ and $b(x)$ of (7.11). Because of the maximum principle, the largest value of $W(\zeta, t)$ must occur on the boundary. On the $t = 0$ boundary, $W(\zeta, 0) \leq M_1 < \infty$ by assumption. At $\zeta = \infty$, $W = W_r \leq M_1 < \infty$ by the boundary condition (7.19). If the maximum occurs at the $\zeta = -ct$ boundary, assume that the maximum value is attained when $t = t_{max} \leq T$. The boundary condition (7.18) with $\zeta = -ct_{max}$ says:

$$e(0)a(0)W_\zeta(\zeta = -ct_{max}, t_{max}) - kW(\zeta = -ct_{max}, t_{max}) = -\bar{c}_l. \tag{7.22}$$

Since we are at a maximum, $W_\zeta(\zeta = -ct_{max}, t_{max}) \leq 0$ implying

$$-kW(\zeta = -ct_{max}, t_{max}) \geq -\bar{c}_l, \tag{7.23}$$

or

$$W(\zeta = -ct_{max}, t_{max}) \leq \frac{\bar{c}_l}{k} < \infty. \tag{7.24}$$

Since $W(\zeta, t) \leq M_1 < \infty$ on the boundaries, it is bounded by M_1 everywhere in its domain.

For the proofs of Lemma 7.3 and Theorem 7.3, we need the first derivatives in the space variable of solutions to (7.1) and (7.2) to be finite. Therefore, we will cite a theorem by Ladyzenskaja which gives us $|u_\zeta| = |u_x| < \infty$. Since we can make the bound $0 \leq u(\cdot, t) \leq M_1$ and $D(u, \cdot)$ and $K(u, \cdot)$ are both finite for unsaturated flow, we will refer to Theorem 7.2 in Ladyzenskaja [20] (p.486) to show that $|u_\zeta| = |u_x| < \infty$. To employ this theorem, we need to write the initial boundary value problem in the following form:

$$u_t \quad - \quad D(u, x)u_{xx}$$

$$
\begin{aligned}
&- \quad [D_u(u,x)(u_x)^2 + D_x(u,x)u_x - K_u(u,x)u_x - K_x(u,x)] = 0, \\
D(u,x)u_x &- \quad K(u,x) + \bar{c}_l|_{x=0} = 0, \\
D(u,x)u_x &- \quad K(u,x) + \bar{c}_r|_{x=\infty} = 0, \\
u(x,0) &= \quad u_0(x).
\end{aligned} \tag{7.25}
$$

The theorem goes as follows:

Theorem 7.1 *: **(Ladyzenskaja Theorem 7.2)** (Let p denote u_x.) Suppose the functions $D(u,x), [D_u(u,x)p^2 + D_x(u,x)p - K_u(u,x)p - K_x(u,x)]$, $D_u(u,x), D_x(u,x), -K(u,x) + \bar{c}_l, -K(u,x) + \bar{c}_r, K_u(u,x)$, and $K_x(u,x)$ satisfy the conditions*

$$\nu\zeta^2 \leq D(u,x)\zeta^2 \leq \mu\zeta^2, \tag{7.26}$$

$$
\begin{aligned}
&|D_u(u,x), D_x(u,x), -K(u,x) + \bar{c}_l, \\
&-K(u,x) + \bar{c}_r, K_u(u,x), K_x(u,x)| \leq \mu,
\end{aligned} \tag{7.27}
$$

$$|[D_u(u,x)p^2 + D_x(u,x)p - K_u(u,x)p - K_x(u,x)]| \leq \mu(1 + p^2), \tag{7.28}$$

and $|u| \leq M$ for all $(x,t) \in \Omega$, a bounded, open, connected domain in $R \times [0,T]$. Then, any solution $u(x,t)$ to (7.25) has the estimate

$$\max_{(x,t)\in\Omega} |u_x(x,t)| \leq M_2, \tag{7.29}$$

where the constant M_2 depends only on M_1, ν, and μ, constants depending on the coefficients of (7.25).

The lengthy proof will be omitted here. Verifying the conditions (7.26) - (7.28) of the theorem, we have

$$\nu\zeta^2 \leq D(u,x)\zeta^2 \leq \mu\zeta^2, \tag{7.30}$$

since $0 < D(u,x) \leq D(M_1,x) < \infty$ for $0 \leq u \leq M$. Next, by our assumptions on the hydraulic functions, we can bound all the following by some constant μ:

$D_u(u,x), D_x(u,x), -K(u,x)+\bar{c}_l, -K(u,x)+\bar{c}_r, K_u(u,x)$, and $K_x(u,x)$. Finally, we have the bound

$$\begin{aligned}
&|D_u(u,x)p^2 + D_x(u,x)p - K_u(u,x)p - K_x(u,x)| \\
&\le [|D_u(u,x)| + |D_x(u,x)| + |K_u(u,x)|]p^2 + |K_x(u,x)| \\
&\le \mu(p^2+1).
\end{aligned} \tag{7.31}$$

Thus, the conditions of the theorem are satisfied. Still, Ladyzskaja's proof of this theorem considers only bounded domains while we have a semi-infinite domain. However, since our sup-norm on $|u(x,t)|$ does not depend on the value of x, we can extend the bounded domain argument to a semi-infinite domain by covering our domain of interest with bounded domains. Now, we can apply the theorem and bound $|u_x| = |u_\zeta| < M_2$.

Finally, recall that we showed the existence of a traveling wave solution to (1.1) in the form $u'(x-ct-s_0, x)$, $s_0 \in R^1$ where s_0 is the same parameter that appeared in the uniqueness proof in chapter 4. By varying s_0, we can make $u'(x-ct-s_0, x)$ take on any value between $u_l(x)$ and $u_r(x)$ for any particular choice of x and t. Going to the traveling frame variables, we can use this to find the minimum and maximum values of the following functional which is continuous in s_0:

$$\begin{aligned}
H(s_0,t) &= \int_{-ct}^{0} [u'(s-s_0, s+ct) - u_l(s+ct)]ds \\
&+ \int_{0}^{\infty} [u'(s-s_0, s+ct) - u_r(s+ct)]ds.
\end{aligned} \tag{7.32}$$

The minimum of $H(s_0,t)$ for any fixed t is obtained when $s_0 = -\infty$ and

$$H(-\infty,t) = \int_{-ct}^{0} [u_r(s+ct) - u_l(s+ct)]ds \le 0, \tag{7.33}$$

with the actual value depending on t. The maximum is obtained when $s_0 = \infty$ and

$$H(\infty,t) = \int_{0}^{\infty} [u_l(s+ct) - u_r(s+ct)]ds = \infty. \tag{7.34}$$

Thus, the freedom in selecting s_0 allows us to set the value of H to any positive value for all t and some negative values dependent on t. $H(s_0, t)$ will be used in the proof of Theorem 7.2.

We are now ready to state five preliminary lemmas used in chapter 7 which describe the behavior of the time-dependent solution. $u(\zeta, t)$ near the boundaries $\zeta = \infty$ and $\zeta = -ct$.

Lemma 7.1 *If*

$$\lim_{\zeta \to \infty} (u_0(\zeta) - u_r(\zeta)) = 0, \tag{7.35}$$

then

$$\lim_{\zeta \to \infty} (u(\zeta, t) - u_r(\zeta + ct)) = 0 \tag{7.36}$$

uniformly in $0 \leq t \leq T$, *for any* $T > 0$.

Lemma 7.2 *If*

$$\lim_{\zeta \to \infty} \partial_\zeta (u_0(\zeta) - u_r(\zeta)) = 0, \tag{7.37}$$

then

$$\lim_{\zeta \to \infty} \partial_\zeta (u(\zeta, t) - u_r(\zeta + ct)) = 0 \tag{7.38}$$

uniformly in $0 \leq t \leq T$ *for* $T > 0$.

Lemma 7.3

$$\int_\zeta^\infty [u(s, t) - u_r(s + ct)] ds \tag{7.39}$$

exists and is finite for $t \in [0, T]$, *for* $T > 0$ *and any finite* ζ.

Lemma 7.4 *Let*

$$V(\zeta, t) = \int_\zeta^\infty [u(s, t) - u_r(s + ct)] ds. \tag{7.40}$$

If our initial assumptions (7.4)-(7.8) hold, then

$$\lim_{\zeta \to \infty} V(\zeta, t) = 0 \tag{7.41}$$

uniformly for $t \geq 0$.

Lemma 7.5 *For any $\epsilon > 0$, there exists an $X = X(\epsilon) < 0$ such that*

$$|\int_{-ct}^{\zeta}[u_l(s+ct) - u(s,t)]ds| \leq \epsilon \tag{7.42}$$

for $\zeta \leq X$ and $t \geq \frac{X}{-c}$.

Lastly, we require a lemma which is an adaption of some basic theory done by Il'in, Kalashnikov, and Oleinik [13] (Section 12, p. 133) and Il'in and Oleinik [14].

Lemma 7.6 *Suppose that the function $u(x,t)$ satisfies*

$$L(u) = a(x,t)u_{xx} + b(x,t)u_x - u_t = 0 \tag{7.43}$$

in the bounded cylinder $Q = \{x \in [x_l, x_r], t \geq 0\}$, with $u \equiv g_1(t)$ on the left boundary, $x = x_l$, and $u \equiv g_2(t)$ on the right boundary, $x = x_r$, with $g_i(t) \leq \epsilon$ as $t \to \infty$ for $i = 1,2$. Then $limsup_{t\to\infty}|u(x,t)| \leq \epsilon$ uniformly with respect to x.

With these lemmas, we can show the following:

Theorem 7.2 *If $u(x,t)$ is a bounded solution of the initial boundary value problem, then there is a unique $s_0 \in R^1$ depending on the intial data $u_0(x)$ such that*

$$\lim_{t\to\infty}\int_{x}^{\infty}[u(x',t) - v'(x' - ct - s_0, x')]dx' = 0 \tag{7.44}$$

uniformly for all $x \geq 0$. Here, $u'(x - ct - s_0, x)$ is a traveling wave solution.

Proof: Making the change of variable $\zeta = x - ct$, we define

$$Z(\zeta,t) = \int_{\zeta}^{\infty}[u(s,t) - u'(s - s_0, s + ct)]ds. \tag{7.45}$$

As in Lemma 7.4, Z satisfies the equation

$$LZ = 0, \tag{7.46}$$

where

$$
\begin{aligned}
L\cdot &= D(u,\zeta+ct)\partial_\zeta{}^2\cdot+\frac{D(u,\zeta+ct)-D(u',\zeta+ct)}{u(\zeta+ct,t)-u'(\zeta-s_0,\zeta+ct)}u'_\zeta\partial_\zeta\cdot \\
&- \frac{K(u,\zeta+ct)-K(u',\zeta+ct)}{u(\zeta+ct,t)-u'(\zeta-s_0,\zeta+ct)}\partial_\zeta\cdot+c\partial_\zeta\cdot-\partial_t\cdot. \qquad (7.47)
\end{aligned}
$$

Let $\epsilon > 0$. By Lemma 7.4, there exists an $X_1 = X_1(\epsilon, s_0)$ such that $|Z(\zeta,t)| \le \epsilon$ for $\zeta \ge X_1$ and $t \ge 0$. Near the boundary, $\zeta = -ct$, $Z(\zeta,t)$ may be expressed as

$$
\begin{aligned}
Z(\zeta,t) &= \int_{-ct}^{\infty}[u(s,t)-u'(s-s_0,s+ct)]ds \\
&- \int_{-ct}^{\zeta}[u(s,t)-u'(s-s_0,s+ct)]ds. \qquad (7.48)
\end{aligned}
$$

We would like to see $|Z(-ct,t)| \le \epsilon$ for all $t \ge 0$. To see that the second integral in (7.48) is smaller than $\frac{\epsilon}{2}$, rewrite it as

$$
\begin{aligned}
\int_{-ct}^{\zeta}[u(s,t) &- u'(s-s_0,s+ct)]ds \\
&= \int_{-ct}^{\zeta}[u(s,t)-u_l(s+ct)]ds \\
&+ \int_{-ct}^{\zeta}[u_l(s+ct)-u'(s-s_0,s+ct)]ds. \qquad (7.49)
\end{aligned}
$$

The first integral is accounted for in Lemma 7.5, while for the second integral we have

$$u_l(s+ct)-u'(s-s_0,s+ct) < Ce^{\lambda_1 s}, \qquad (7.50)$$

where $\lambda_1 > 0$ and $s < s_1 < 0$ by Lemma 3.2. Now we need to show that

$$\int_{-ct}^{\infty}[u(s,t)-u'(s-s_0,s+ct)]ds < \frac{\epsilon}{2} \qquad (7.51)$$

as $t \to \infty$. Start by rewriting the left hand side of (7.51) as

$$\int_{-ct}^{\infty}[u-u']ds = J(t)-H(s_0,t) \qquad (7.52)$$

where $H(s_0,t)$ was given in (7.32) as

$$
\begin{aligned}
H(s_0,t) &= \int_{-ct}^{0}[u'(s-s_0,s+ct)-u_l(s+ct)]ds \\
&+ \int_{0}^{\infty}[u'(s-s_0,s+ct)-u_r(s+ct)]ds, \qquad (7.53)
\end{aligned}
$$

and we define

$$J(t) = \int_{-ct}^{0} [u(s,t) - u_l(s+ct)]ds + \int_{0}^{\infty} [u(s,t) - u_r(s+ct)]ds. \qquad (7.54)$$

$J(t)$ is finite for $t \geq 0$ by Lemma 7.3. We can see $J(t)$ is oscillating in time by writing it as

$$J(t) = J(0) + \int_0^t \partial_t J(t), \qquad (7.55)$$

where

$$\begin{aligned}
\partial_t J(t) &= \int_{-ct}^{\infty} [u_t(s,t) - cu_{rs}(s+ct)]ds - \int_{-ct}^{0} cu_{ls}(s+ct)ds \\
&+ \int_{-ct}^{0} cu_{rs}(s+ct)ds + c[u(s,t) - u_l(s+ct)]|_{s=-ct} \\
&= \int_{-ct}^{\infty} [D(u,s+ct)u_s - K(u,s+ct) + c(u(s,t) - u_r(s+ct))]_s ds \\
&- \int_{-ct}^{0} cu_{ls}(s+ct)ds + \int_{-ct}^{0} cu_{rs}(s+ct)ds \\
&+ c[u(s,t) - u_l(s+ct)]|_{s=-ct} \\
&= \bar{c}_l - \bar{c}_r + c(u(s,t) - u_r(s+ct))|_{s=\infty} - cu(s,t)|_{s=-ct} - cu_l(s+ct)|_{s=-ct}^{0} \\
&- cu_r(s+ct)|_{s=0} + c(u(s,t)|_{s=-ct} - u_l(s+ct)|_{s=-ct}) \\
&= \bar{c}_l - \bar{c}_r + c(u_r(ct) - u_l(ct)), \qquad (7.56)
\end{aligned}$$

since

$$(u(s,t) - u_r(s+ct))|_{s=\infty} = 0 \qquad (7.57)$$

by Lemma 7.1. Using the formula for speed (3.20), (7.56) becomes

$$\partial_t J(t) = c(\langle u_l - u_r \rangle - (u_l(ct) - u_r(ct))), \qquad (7.58)$$

where u_l and u_r are $\frac{1}{c}$-periodic in t. Since, averaging over $[0, \frac{1}{c}]$, $\langle \partial_t J(t) \rangle = 0$, we see that $J(t)$ is a bounded function which oscillates around $J(0)$.

The strategy at this point is to show that although $J(t)$ and $H(s_0,t)$ are periodic and have no limit as $t \to \infty$ individually, $J(t) - H(s_0,t)$ does have a limit as

$t \to \infty$. Then, we will be able to pick a s_0 value so that this limit is zero.

Take the derivative of (7.52) with respect to t.

$$\begin{aligned}
\partial_t \int_{-ct}^{\infty}[u-u']ds &= \int_{-ct}^{\infty}[u_t-u_t']ds + c[u-u']|_{\zeta=-ct} \\
&= \int_{-ct}^{\infty}[((D(u,s+ct)u_s - K(u,s+ct))_s + cu_s) \\
&- ((D(u',s+ct)u_s' - K(u',s+ct))_s + cu_s')]ds \\
&+ c[u-u']_{\zeta=-ct} \\
&= [(D(u,s+ct)u_\zeta - K(u,s+ct)) \\
&- (D(u',s+ct)u_\zeta' - K(u',s+ct)) \\
&+ c(u-u')]|_{s=-ct}^{\infty} + c[u-u']_{s=-ct} \\
&= (-\bar{c}_r + \bar{c}_l) + c(u-u')|_{s=\infty} \\
&- (-\bar{c}_r - (D(u',s+ct)u_\zeta' - K(u',s+ct))|_{s=-ct}). \quad (7.59)
\end{aligned}$$

Rewriting $u-u'$ as $(u-u_r)-(u'-u_r)$ and applying Lemma 7.1 and Lemma 3.2 gives

$$c(u-u')|_{s=\infty} = 0. \quad (7.60)$$

Then

$$\begin{aligned}
\partial_t \int_{-ct}^{\infty}[u-u']ds &= (D(u',\zeta+ct)u_\zeta' - K(u',\zeta+ct))|_{\zeta=-ct} + \bar{c}_l \\
&= [-(D(u_l,\zeta+ct)u_{l\zeta} - K(u_l,\zeta+ct)) \\
&+ (D(u',\zeta+ct)u_\zeta' - K(u',\zeta+ct))]|_{\zeta=-ct} \\
&= -D(u_l,0)(u_l-u')_\zeta - D_u(\Theta_1,0)u_\zeta'(u_l-u') \\
&+ K_u(\Theta_2,0)(u_l(0) - u'(-ct-s_0,0)), \quad (7.61)
\end{aligned}$$

where Θ_1 and Θ_2 are intermediate values and $|N_u| \leq \bar{M}_2 < \infty$ since $|u_\zeta'| \leq M_2 < \infty$ and $|u_{l\zeta}| \leq M_2 < \infty$. We can then use Lemma 3.2 to make the following bound

$$|\partial_t \int_{-ct}^{\infty}[u-u']ds| \leq |[-N_u D(u_l,0) - D(\Theta_1,0)u_\zeta + K(\Theta_2,0)](u_l-u')|$$

$$\leq \; C_1 e^{-\lambda_{c}t}, \tag{7.62}$$

where $C_1 = C_1(s_0) > 0$. Therefore,

$$|\int_{-ct}^{\infty}[u - u']ds| < C_0 + \int_{t_J}^{t} C_1 e^{-\lambda_{c}\tau}d\tau' \tag{7.63}$$

where $C_0 = C_0(s_0, t_J)$ is a constant and $t_J \geq 0$. By taking t_J sufficiently large depending on ϵ, we can guarantee that

$$\int_{t_J}^{\infty} C_1 \epsilon^{-\lambda_{c}\tau}d\tau < \frac{\epsilon}{4} \tag{7.64}$$

for any $\epsilon_J > 0$. Thus, the quantity, $|J(t) - H(s_0,t)|$ converges to a constant as $t \to \infty$ or

$$\lim_{t\to\infty} J(t) - H(s,t) \equiv \bar{H}(s) \tag{7.65}$$

for all s. Furthermore, $\bar{H}(s)$ is monotonically increasing in s by Corollary 4.1 and the range of $\bar{H}(s)$ is R^1 as seen in (7.33) and (7.34). Now choose s_0 to be the unique value such that $\bar{H}(s_0) = 0$. With this choice, we have our bound (7.51) and $|Z(\zeta,t)| \leq \epsilon$ as $\zeta \leq X_2$. Then, for every $\epsilon > 0$ there exists a $t_0 = t_0(\epsilon)$ such that

$$-\frac{\epsilon}{4} < J(t) - H(s_0,t) < \frac{\epsilon}{4}, \tag{7.66}$$

if $t \geq t_0$. This choice of s_0 is unique by the monotonicity of $\bar{H}(s)$. Now, by our selection of s_0, we have uniquely determined a traveling wave solution which a time-dependent solution may tend to asymptotically as $t \to \infty$, pending the completion of the proof.

To this point we have bounded $|Z(\zeta,t)| \leq \epsilon$ for $\zeta \geq X_1$ and $\zeta \leq X_2$. To make the same bound for the remainder of the ζ domain, we will invoke Lemma 7.6 to show that for some $t = t_1$, $|Z(\zeta,t_1)| \leq \epsilon$ for all ζ. Using what we have done above to show that $Z(\zeta,t) \leq \epsilon$ on the boundaries, we see that the conditions of Lemma 7.6 are met. Applying Lemma 7.6, we can choose a $t = t_1 \geq t_0$ so that

$|Z(\zeta, t_1)| < \epsilon$ for all $\zeta \in [-ct_1, \infty]$.

Now, treating $t = t_1$ as a new initial time value, we can apply the maximum principle argument to show that $|Z(\zeta, t)| \leq \epsilon$ for all ζ and $t \geq t_1$ and hence

$$\lim_{t\to\infty} \int_x^\infty [u(x', t) - u'(x' - ct - s_0, x')]dx' = 0. \tag{7.67}$$

The proof is complete.

Remark 7.1 *The difference between the proof of Theorem 7.2 for the periodic case and for the homogeneous case shown by Noren [30] on p.41 lies in the nature of $J(t)$ and $H(s,t)$. In the periodic case, as shown by (7.58), $J(t)$ and $H(s,t)$ go to bounded periodic functions in t as $t \to \infty$; while in the homogeneous case, $J(t)$ and $H(s,t)$ simply go to constants. Therefore, for the homogeneous case, we can simply choose the s_0 so that $J(\infty) - H(s_0, \infty)$ is identically equal to zero.*

Next we need an interpolation lemma whose proof was given by Noren [30] (p. 43).

Lemma 7.7 *Let $F(x) \in C^2[a, L]$, where $a \geq 0, F(a) = B$, and $|F''(x)| \leq \bar{M}$ where $\bar{M}$ is a positive constant. If*

$$L \geq \sqrt{\frac{2\sup_{x\in[a,L]}|F(x) - B|}{\bar{M}}}, \tag{7.68}$$

then

$$|F'(a)| \leq \sqrt{2\bar{M}sup_{x\in[a,L]}|F(x) - B|}. \tag{7.69}$$

Proof: By Taylor's formula

$$F(x) = F(a) + F'(a)(x - a) + \frac{1}{2}F''(x^*)(x - a)^2. \tag{7.70}$$

Therefore

$$\sup_{x\in[a,L]} |F(x) - B| \geq |F(x) - B| \tag{7.71}$$

$$= |F'(a)x + \frac{1}{2}F''(x^*)(x-a)^2| \geq |F'(a)||x-a| - \frac{\bar{M}}{2}|x-a|^2, \tag{7.72}$$

or, letting $s = |x-a|$ and $m = \sup_{x\in[a,L]} |F(x) - B|$

$$\frac{\bar{M}}{2}s^2 - |F'(a)|s + m \geq 0. \tag{7.73}$$

If s_1 and s_2 are roots of the quadratic expression on the left hand side of (7.73), then $s_1 + s_2 \geq 0$. Thus both roots must be positive. Also, $s_1 s_2 = \frac{2m}{\bar{M}} \leq L^2$. Therefore, s_1 and s_2 can not both be greater than L. Consequently, there can be no real roots of

$$\frac{\bar{M}}{2}s^2 - |F'(a)|s + m = 0, \tag{7.74}$$

and

$$|F'(a)|^2 - 2\bar{M}m \leq 0, \tag{7.75}$$

or

$$|F'(a)| \leq \sqrt{2\bar{M} sup_{x\in[a,L]}|F(x) - B|}. \tag{7.76}$$

The proof is complete.

Now, we are ready to use Lemma 7.7 and Theorem 7.2 to prove the following theorem:

Theorem 7.3 *Under the hypotheses of Theorem 7.2,*

$$\lim_{t\to\infty} |u(x,t) - u'(x-ct-s_0,x)| = 0 \tag{7.77}$$

uniformly for $x \geq 0$.

Proof: Let $\epsilon > 0$ and take T sufficiently large that

$$|\int_x^\infty [u(x',t) - u'(x'-ct-s_0,x')]dx'| \leq \epsilon, \tag{7.78}$$

for $t \geq T$ and $x \geq 0$. Then, we may apply Theorem 7.1 to show

$$\partial_x[u(x,t) - u'(x-ct-s_0,x)] \leq \bar{M}, \tag{7.79}$$

for $x \geq 0$ and $t \geq 0$. Application of Lemma 7.7 by taking $F(x) = |\int_x^\infty [u(x',t) - u'(x' - ct - s_0, x')]dx'|$ and $a = x$ in the notation used in Lemma 7.7 gives

$$|u(x,t) - u'(x - ct - s_0, x)| \leq \sqrt{2\bar{M}\epsilon}, \tag{7.80}$$

for any x. Therefore

$$\lim_{t\to\infty} |u(x,t) - u'(x - ct - s_0, x)| = 0. \tag{7.81}$$

The proof is complete.

8 Soil Heterogeneity and Wave Speed

In regard to the effect of periodic heterogeneity on wave speed, we will show examples where the waves travel at the same rate in a heterogeneous soil as a homogeneous soil, where they travel slower, and where they travel faster. The speed differences depends on the conductivity function selected and, more importantly, the selection of the $a(x)$ and $b(x)$ functions to describe the heterogeneity. The functions $a(x)$ and $b(x)$ were introduced in (A1) and (A2) of the introduction. If we take $a(x)$ and $b(x)$ to be proportional, the wave speed is no faster in a heterogeneous medium than in a homogeneous one, while if we take them completely "out of phase", the wave speed is no slower. We will show five examples to illustrate this. In these examples, we take $a(x) = a(x, \epsilon)$ and $b(x) = b(x, \epsilon)$ where $\epsilon \in [0, 1)$ is the parameter controlling the degree of heterogeneity in the soil (i.e. $\epsilon = 0$ would be a homogeneous soil while $\epsilon = 0.99$ would be a very heterogeneous soil).

Example 1: Recall the formula for the speed was given in (3.12) to be

$$c = \frac{\bar{c}_l - \bar{c}_r}{\langle u_l(x) - u_r(x) \rangle}. \tag{8.1}$$

We would like to show an example in which the partial derivative of c with respect to ϵ is zero,

$$c_\epsilon = 0. \tag{8.2}$$

We consider the conductivity function $K(u) = e^u$. We start by computing the partial with respect to ϵ. Recall from Remark 3.1 that

$$U_{l,r}(x) = K(u_{l,r}(x)) = -\bar{c}_{l,r} T(x), \tag{8.3}$$

or

$$u_{l,r}(x) = K^{-1}(-\bar{c}_{l,r} T(x)), \tag{8.4}$$

where $T(x)$ is defined in (3.51). Notice

$$\begin{aligned}
\langle u_l(x)\rangle_\epsilon &= \langle (K^{-1}(U))' \,|_{U=-\bar{c}_l T}\, \bar{c}_l(-T)_\epsilon\rangle, \\
\langle u_r(x)\rangle_\epsilon &= \langle (K^{-1}(U))' \,|_{U=-\bar{c}_r T}\, \bar{c}_r(-T)_\epsilon\rangle. \qquad (8.5)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
c_\epsilon &= (\frac{\bar{c}_l - \bar{c}_r}{\langle u_l(x) - u_r(x)\rangle})_\epsilon \\
&= -\frac{\bar{c}_l - \bar{c}_r}{\langle u_l(x) - u_r(x)\rangle^2}\langle [(K^{-1}(U))' \,|_{U=-\bar{c}_l T}\, \bar{c}_l \\
&- (K^{-1}(U))' \,|_{U=-\bar{c}_r T}\, \bar{c}_r](-T)_\epsilon\rangle. \qquad (8.6)
\end{aligned}$$

Our choice of conductivity function leads to

$$\begin{aligned}
K^{-1}(U) &= ln(U). \\
(K^{-1}(U))' &= \frac{1}{U}. \qquad (8.7)
\end{aligned}$$

Substituting into the appropriate term in (8.6), we see

$$[(K^{-1}(U))' \,|_{U=-\bar{c}_l T}\, \bar{c}_l - (K^{-1}(U))' \,|_{U=-\bar{c}_r T}\, \bar{c}_r] = \frac{\bar{c}_l}{-\bar{c}_l T} - \frac{\bar{c}_r}{-\bar{c}_r T} = 0. \qquad (8.8)$$

Hence, $c_\epsilon = 0$ and for this choice of hydraulic function, the speed is independent of the amount of heterogeneity in the soil.

Remark 8.1 *Example 1 is of particular interest since it is, in fact, this exponential choice of the conductivity function which is used in most analytical work (see [44] and [51]) and in stochastic numerical work (see [10]). Thus, much of the current work in modeling heterogeneous soils uses a conductivity function for which the front speed in a heterogeneous soil matches the speed in a homogeneous soil with the same mean conductivity.*

Example 2: Take the hydraulic conductivity to be an algebraic function (Brooks and Corey hydraulic function),

$$K(u) = u^n \tag{8.9}$$

for $n > 1$ and

$$\begin{aligned} a(x) &= 1 + \epsilon a_1(x), \quad \langle a_1(x)\rangle = 0, \\ b(x) &= 1 + \epsilon b_1(x), \quad \langle b_1(x)\rangle = 0. \end{aligned} \tag{8.10}$$

We show that the speed in the periodic soil is larger (smaller) than that in the homogenoues soil if $a_1 = -b_1$ ($a_1 = b_1$) and ϵ is small.
We study the inequality:

$$\frac{\bar{c}_l - \bar{c}_r}{\langle u_l(x) - u_r(x)\rangle} \geq \ (\leq) \ \frac{\bar{c}_l - \bar{c}_r}{\bar{u}_l - \bar{u}_r}, \tag{8.11}$$

where $\bar{u}_l$ and $\bar{u}_r$ are constant steady states for the homogeneous case. By the steady state equations and remark 3.1, we have

$$\begin{aligned} \bar{u}_l &= K^{-1}(\frac{\bar{c}_l}{\langle b(x)\rangle}) = K^{-1}(\bar{c}_l)K^{-1}(\langle b(x)\rangle^{-1}), \\ \bar{u}_r &= K^{-1}(\frac{\bar{c}_r}{\langle b(x)\rangle}) = K^{-1}(\bar{c}_r)K^{-1}(\langle b(x)\rangle^{-1}), \\ u_l(x) &= K^{-1}(-\bar{c}_l T(x)) = K^{-1}(\bar{c}_l)K^{-1}(-T(x)), \\ u_r(x) &= K^{-1}(-\bar{c}_r T(x)) = K^{-1}(\bar{c}_r)K^{-1}(-T(x)), \end{aligned} \tag{8.12}$$

since $K^{-1}(U) = U^{\frac{1}{n}}$. For (8.11) to be valid, we need to have

$$\langle u_l(x) - u_r(x)\rangle \leq \ (\geq) \ \bar{u}_l - \bar{u}_r,$$

or

$$\langle K^{-1}(-T(x))\rangle[K^{-1}(\bar{c}_l) - K^{-1}(\bar{c}_r)] \leq \ (\geq) \ K^{-1}(\langle b(x)\rangle^{-1})[K^{-1}(\bar{c}_l) - K^{-1}(\bar{c}_r)].$$

Recall from chapter 3 that T satisfies:

$$a(x)T_x(x) - b(x)T(x) = 1.$$

or:

$$(1+\epsilon a_1)T' - (1+\epsilon b_1)T = 1. \tag{8.13}$$

We express $T(x)$ with the regular perturbation expansion:

$$T = -1 + \epsilon T_1 + \epsilon^2 T_2 + O(\epsilon^3). \tag{8.14}$$

Plugging (8.14) into (8.13), we have:

$$1 = (1+\epsilon a_1)(\epsilon T_1' + \epsilon^2 T_2' + O(\epsilon^3)) - (1+\epsilon b_1)(-1 + \epsilon T_1 + \epsilon^2 T_2 + O(\epsilon^3)). \tag{8.15}$$

Thus, the $O(\epsilon)$ order equation is

$$T_1' - T_1 + b_1 = 0, \tag{8.16}$$

showing that $\langle -T_1 \rangle = 0$. The $O(\epsilon^2)$ order equation is

$$T_2' + a_1 T_1' - T_2 - b_1 T_1 = 0. \tag{8.17}$$

It follows from (8.16) and (8.17) that

$$\begin{aligned} T_2' - T_2 &= -a_1 T_1' + b_1 T_1 = a_1(-T_1 + b_1) + b_1 T_1 \\ &= (b_1 - a_1)T_1 + a_1 b_1, \end{aligned} \tag{8.18}$$

and so:

$$\langle -T_2 \rangle = \langle (b_1 - a_1)T_1 \rangle + \langle a_1 b_1 \rangle. \tag{8.19}$$

Now we compare $\langle K^{-1}(-T) \rangle$ with $K^{-1}(\langle b(x) \rangle^{-1}) = 1$. By (8.14), we have

$$\begin{aligned} \langle K^{-1}(-T) \rangle &= \langle (1 - \epsilon T_1 - \epsilon^2 T_2 + O(\epsilon^3))^{1/n} \rangle \\ &= \langle 1 - \frac{1}{n}(\epsilon T_1 + \epsilon^2 T_2 + O(\epsilon^3)) \\ &- \frac{n-1}{2n^2}(\epsilon T_1 + \epsilon^2 T_2 + O(\epsilon^3))^2 + O(\epsilon^3) \rangle \\ &= 1 - \frac{\epsilon^2}{n}\langle T_2 \rangle - \frac{(n-1)\epsilon^2}{2n^2}\langle T_1^2 \rangle + O(\epsilon^3) \\ &= 1 + \frac{\epsilon^2}{n}(\langle -T_2 \rangle - \frac{n-1}{2n}\langle T_1^2 \rangle) + O(\epsilon^3). \end{aligned} \tag{8.20}$$

We will show that $\langle -T_2\rangle - \frac{n-1}{2n}\langle T_1^2\rangle$ is negative (positive) if $a_1 = -b_1 (a_1 = b_1)$, which implies $\langle K^{-1}(-T)\rangle < (>)\ K^{-1}(\langle b(x)\rangle^{-1}) = 1$. It follows from (8.16) that

$$-(\frac{T_1^2}{2})' + T_1^2 - b_1 T_1 = 0$$

and so $\langle T_1^2\rangle = \langle b_1 T_1\rangle$. By (8.19):

$$\begin{aligned}
-\langle T_2\rangle - \frac{n-1}{2n}\langle T_1^2\rangle &= \langle a_1 b_1\rangle - \langle a_1 T_1\rangle + \frac{n+1}{2n}\langle b_1 T_1\rangle \\
&= -\langle a_1 T_1'\rangle + \frac{n+1}{2n}\langle b_1 T_1\rangle \\
&= \langle (a_1' + \frac{n+1}{2n} b_1) T_1\rangle. \qquad (8.21)
\end{aligned}$$

Let us expand a_1 and b_1 into Fourier series:

$$a_1 = \sum_{l\neq 0} \alpha_l exp\{2\pi i l x\}$$
$$b_1 = \sum_{l\neq 0} \beta_l exp\{2\pi i l x\}.$$

Then (8.16) gives

$$T_1 = \sum_{l\neq 0} \frac{\beta_l}{1-2\pi i l} exp\{2\pi i l x\}.$$

Hence,

$$\begin{aligned}
\langle (a_1' + \frac{n+1}{2n} b_1) T_1\rangle &= \sum_{l\neq 0} (\frac{n+1}{2n}\beta_l + 2\pi i l \alpha_l)\frac{\bar{\beta}_l}{1+2\pi i l} \\
&= \sum_{l>0} (\frac{n+1}{2n}|\beta_l|^2)\cdot 2Re(\frac{1}{1+2\pi i l}) \\
&+ \sum_{l>0} 4\pi l Re(\frac{i\alpha_l\bar{\beta}_l}{1+2\pi i l}), \qquad (8.22)
\end{aligned}$$

where $Re(\frac{1}{1+2\pi i l}) = \frac{1}{1+4\pi^2 l^2}$. Suppose $a_1 = \gamma b_1$, or $\alpha_l = \gamma\beta_l, \gamma \in R$. Then,

$$Re(\frac{i\alpha_l\bar{\beta}_l}{1+2\pi i l}) = \frac{2\pi l\gamma|\beta_l|^2}{1+4\pi^2 l^2}.$$

It follows that

$$\langle (a_1' + \frac{n+1}{2n} b_1) T_1\rangle = \sum_{l>0} |\beta_l|^2 \frac{8\pi^2 l^2\gamma + \frac{n+1}{n}}{1+4\pi^2 l^2}. \qquad (8.23)$$

If $\gamma = -1, \langle (a_1' + \frac{n+1}{2n} b_1) T_1 \rangle < 0$ and the wave speed in periodic soil is faster. On the other hand, if $\gamma = 1$, $\langle (a_1' + \frac{n+1}{2n} b_1) T_1 \rangle > 0$ and the wave speed in periodic soil is slower.

Remark 8.2 *We can think of $a_1(x) = -b_1(x)$ as $a(x)$ and $b(x)$ differing by a phase shift of half the period or completely out of phase. So whether the heterogeneities increase the wave speeds or not in the case of power nonlinearity depends on how correlated the phases of heterogenieties are in diffusivity and conductivity functions.*

Example 3: Numerical examples for four choices of conductivity functions with

$$\begin{aligned} a(x,\epsilon) &= 1 + \epsilon sin(2\pi x), \\ b(x,\epsilon) &= 1 + \epsilon sin(2\pi x), \end{aligned} \tag{8.24}$$

are shown in Figure 8.1. Notice how the speed in non-increasing in ϵ for all the conductivity functions. Assumptions (A1)-(A3) were used and a fourth order Runga-Kutta scheme was used to calculate the steady states which were then incorporated into the speed formula. The selected functions were exponential, cubic, Fujita, and van Genuchten conductivity functions, multiplied by an inhomogeneity term. They had the forms

$$\begin{aligned} A) \quad K(u,x) &= e^{u-1} b(x,\epsilon), \\ B) \quad K(u,x) &= u^3 b(x,\epsilon), \\ C) \quad K(u,x) &= \frac{u^2}{2-u} b(x,\epsilon), \\ D) \quad K(u,x) &= \sqrt{u}[1-\sqrt{1-u^2}]^2 b(x,\epsilon) \end{aligned} \tag{8.25}$$

and

$$D(u,x) = K_u(u) a(x,\epsilon), \tag{8.26}$$

for all four cases.

Example 4: Numerical examples for four choices of conductivity functions with

$$a(\epsilon, x) = 1 - \epsilon sin(2\pi x),$$
$$b(\epsilon, x) = 1 + \epsilon sin(2\pi x), \tag{8.27}$$

are shown in Figure 8.2. Notice how the speed in non-decreasing in epsilon for all the conductivity functions.

Example 5: Numerical examples for four choices of conductivity functions with

$$a(\epsilon, x) = 1 - \epsilon sin(2\pi x) + \epsilon sin^3(2\pi x),$$
$$b(\epsilon, x) = 1 + \epsilon sin(2\pi x) + \epsilon sin^3(2\pi x), \tag{8.28}$$

are shown in Figure 8.3. Notice how the speed is constant for the exponential case, increasing for the cubic and Fujita cases, and decreasing for the van Genuchten case.

Speed calculations based on numerical solutions to the entire problem will also be presented in chapter 9.

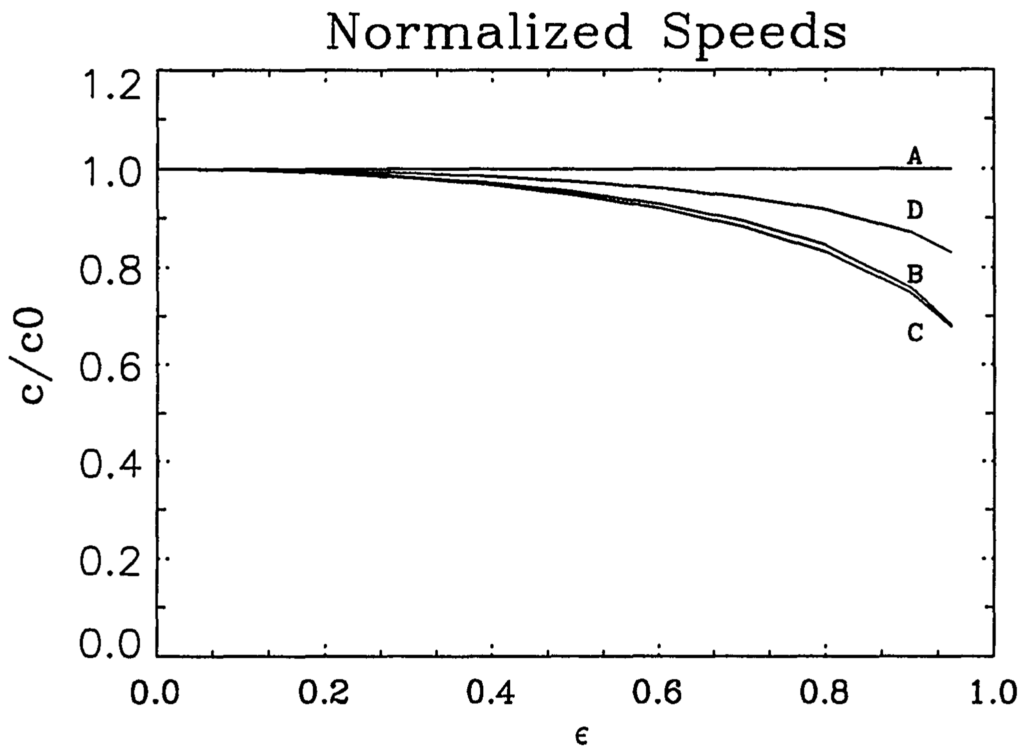

Figure 8.1, Normalized Speeds for A) Exponential. B) Cubic, C) Fujita, D) van Genuchten Conductivity Functions, $a(x,\epsilon) = b(x,\epsilon) = 1 + \epsilon sin(2\pi x)$

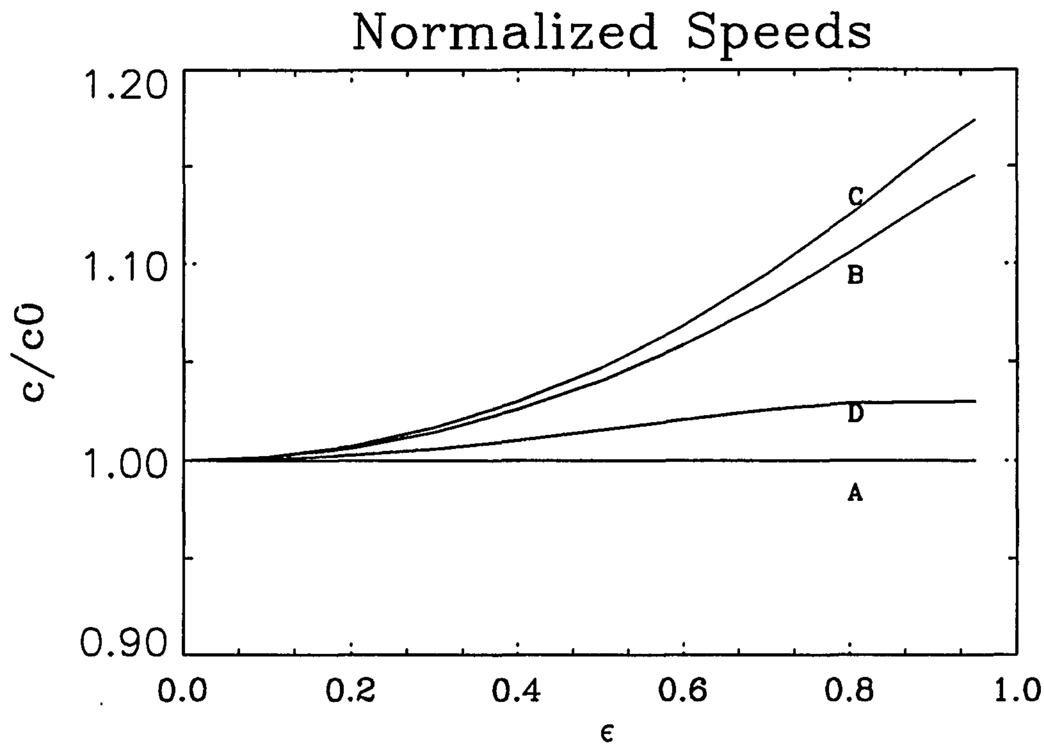

Figure 8.2, Normalized Speeds for A) Exponential. B) Cubic, C) Fujita, D) van Genuchten Conductivity Functions, $a(x, \epsilon) = 1 - \epsilon sin(2\pi x)$, $b(x, \epsilon) = 1 + \epsilon sin(2\pi x)$

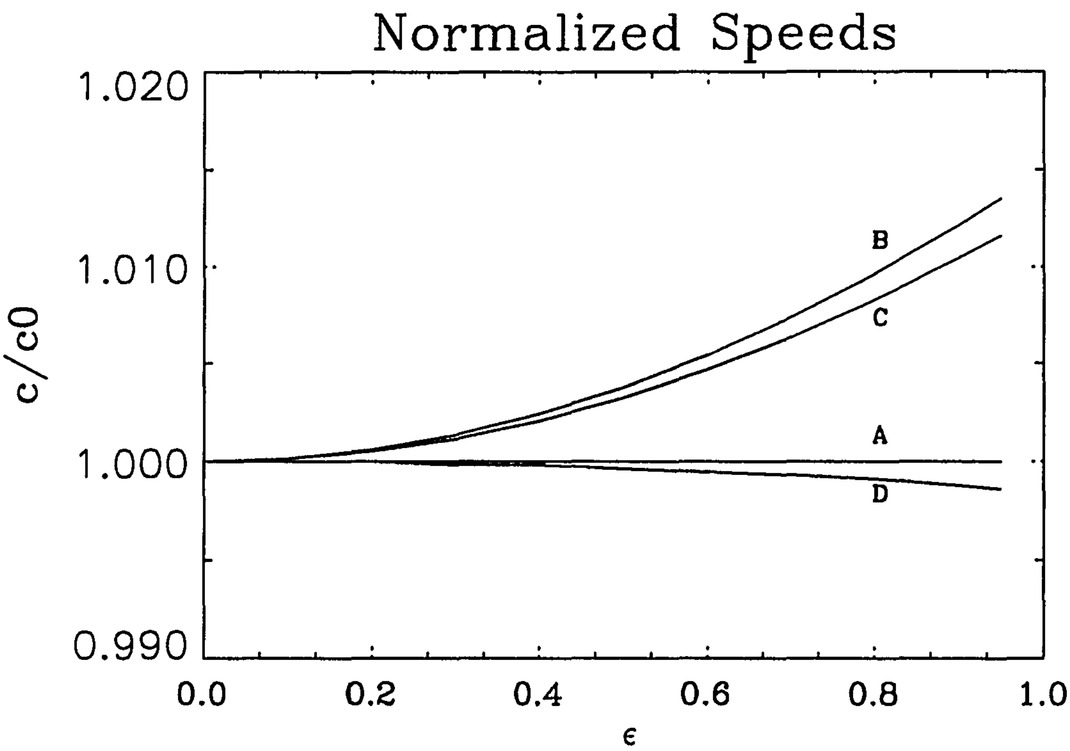

Figure 8.3, Normalized Speeds for A) Exponential. B) Cubic, C) Fujita, D) van Genuchten Conductivity Functions, $a(x,\epsilon) = 1 - \epsilon sin(2\pi x) + \epsilon sin^3(2\pi x)$, $b(x,\epsilon) = 1 + \epsilon sin(2\pi x) + \epsilon sin^3(2\pi x)$

9 Numerical Solutions

A first order in time, second order in space, fully implicit, finite difference scheme was selected. The scheme is the following:

$$
\begin{aligned}
\frac{u_i^{j+1}-u_i^j}{\Delta t} &= \frac{1}{\Delta x}[D_{i+\frac{1}{2}}^{j+1}\frac{u_{i+1}^{j+1}-u_i^{j+1}}{\Delta x} - D_{i-\frac{1}{2}}^{j+1}\frac{u_i^{j+1}-u_{i-1}^{j+1}}{\Delta x}] - \frac{1}{\Delta x}[K_{i+\frac{1}{2}}^{j+1} - K_{i-\frac{1}{2}}^{j+1}],\\
D_{i+\frac{1}{2}}^{j+1} &= [\frac{1}{D(u_{i+1}^{j+1},(i+l)\Delta x)} + \frac{1}{D(u_i^{j+1},i\Delta x)}]^{-1},\\
D_{i-\frac{1}{2}}^{j+1} &= [\frac{1}{D(u_i^{j+1},i\Delta x)} + \frac{1}{D(u_{i-1}^{j+1},(i-1)\Delta x)}]^{-1},\\
K_{i+\frac{1}{2}}^{j+1} &= [\frac{1}{K(u_{i+1}^{j+1},(i+l)\Delta x)} + \frac{1}{K(u_i^{j+1},i\Delta x)}]^{-1},\\
K_{i-\frac{1}{2}}^{j+1} &= [\frac{1}{K(u_i^{j+1},i\Delta x)} + \frac{1}{K(u_{i-1}^{j+1},(i-1)\Delta x)}]^{-1}.
\end{aligned}
\tag{9.1}
$$

For a layered soil, harmonic averages are the most appropriate choices for $D_{i\pm\frac{1}{2}}^{j+1}$ and $K_{i\pm\frac{1}{2}}^{j+1}$ since they satisfy the mass balance condition (see [4]). Intuitively, this choice can be thought of by imagining that we have a layer of soil with conductivity K and an impermeable steal plate of equal depth with conductivity zero. Obviously, no water will flow through the steel plate and the conductivity for the entire system should be zero. However, if we use an arithmetic average to find the conductivity of the system, we will get $K_{ave} = \frac{K}{2} > 0$. Using the harmonic average, we get the correct $K_{ave} = 0$. This use of the harmonic average is also analogous to the difference between electric circuits in parallel as opposed to series in electromagnetic theory.

While higher order accuracy can be attained, the higher order schemes are increasingly more difficult. The second order accuracy proved to be sufficient for the interior of the solutions' domains. Any accuracy problems which occured were observed on the boundaries, particularly the right boundary ($x = L$) which was meant to represent infiltration deep in a soil. A fully implicit scheme was selected due to the nature of the traveling wave solutions. These solutions often have a relatively sharp front. The implicit scheme was able to better resolve the solution at these fronts than explicit or Crank-Nicholson type schemes and it was able to do this with Δt values which were orders of magnitude larger than for the other two types of schemes.

The boundary conditions, especially the right boundary, presented a major challenge in obtaining numerical solutions. As in (1.12), prescribed flux conditions were imposed by balancing fluxes at the boundary nodes with fluxes at interior nodes. The equations for balancing the fluxes look like

$$\begin{aligned} J_2 - 2J_1 + J_0 = 0, \quad left \\ J_N - 2J_{N-1} + J_{N-2} = 0, \quad right \\ J_0 = \bar{c}_l, \\ J_i = D_{i-\frac{1}{2}}^{j+1} \frac{u_{i+1}^{j+1} - u_i^{j+1}}{\Delta x} - K_{i-\frac{1}{2}}^{j+1}, \quad i = 1, 2, N-2, N-1 \\ J_N = \bar{c}_r. \end{aligned} \tag{9.2}$$

A less general way to implement these boundary conditions which avoids the problems mentioned above is to recall from Proposition 2.2 that for any prescribed flux. $\bar{c}$, there is a unique steady state solution. Therefore, if we take our initial condition near the boundary to be the steady state solution which corresponds to the prescribed flux. we can guarantee that the correct flux is being implemented at the boundary. Additionally, since the steady state solution will not change in time, we will be able to use a Dirichlet boundary condition by taking the prescribed value at the boundary from the value of the steady state solution there. This is the real advantage of this handling of the boundary condition since it appears than implementing the prescribed fluxes, especially at the right boundary, is far more difficult that implementing the prescribed values (see [28]). The numerical examples in this section except Example 3 use this implementation of the boundary conditions.

Numerical speed is calculated by selecting

$$u^* = \frac{u_l(0) + u_r(L)}{2} \tag{9.3}$$

to mark the position of the wave front, where $u_l(0)$ is the value of the steady state at the left hand boundary and $u_r(L)$ is the value of the steady state at the right hand boundary. For each timestep, we count the number of nodes, N_t, for which $u_i^j \leq u^*$. As the traveling wave moves from left to right, N_t will decrease. The rate at which N_t decreases is the speed of the wave propogation.

The rest of this section is devoted to showing some numerical examples which illustate properties of these traveling wave solutions to (1.1).

Example 1: In Figures 9.1-9.5, we took the $K(u,x)$ and $D(u,x)$ to be an adaptation of the Fujita hydraulic function (with parameter $m = 2$) where

$$\begin{aligned} K(u,x) &= (m-1)\frac{u^2}{m-u}(1+\epsilon sin(2\pi x)), \\ D(u,x) &= (m-1)\frac{2mu-u^2}{(m-u)^2}(1+\epsilon sin(2\pi x)). \end{aligned} \tag{9.4}$$

The adaption from the actual Fujita function is the multiplication by the term $(1+\epsilon sin(2\pi x))$ which is meant to represent the periodic inhomogeneity of the soil. The parameter $\epsilon \in [0,1)$ controls the degree of inhomogeneity. Additionally, $D(u,x)$ is taken to satisfy assumption (A3) and is not the actual Fujita diffusivity function. Comparison between the $D(u,x)$ in (9.4) and the actual Fujita diffusivity function will be shown in Example 2.

Figures 9.1 through 9.3 show the solutions to Richards' equation with three different ϵ values ($\epsilon = 0, 0.5$, and 0.9) at dimensionless time values of $t = 40, 80,$

and 120. In obtaining these results, we used $\Delta t = 0.1$ and $\Delta x = 0.1$. The prescribed boundary fluxes were $\bar{c}_l = K(0.5, 0)$ and $\bar{c}_r = K(0.3, 0)$. The initial condition was taken to be the discontinuous function

$$\begin{aligned} u(0,x) &= u_l(x) \quad x \leq 16, \\ &= u_r(x) \quad x > 16. \end{aligned} \tag{9.5}$$

where $u_l(x)$ and $u_r(x)$ are the steady states related to $\bar{c}_l$ and $\bar{c}_r$ respectively. With this initial condition, the boundary conditions could be changed to the equivalent forms: $u(t,0) = u_l(0)$ and $u(t,L) = u_r(L)$. Notice how the solutions show periodically oscillating steady states at the boundaries with a connecting front which moves toward the right hand boundary in time.

Figure 9.5 shows the wave speed of these solutions. The wave speed is the negative of the slope of the lines shown. Notice how the speed decreases with increasing ϵ relating to an increased degree of heterogeneity. The roughness of the lines is due to poor spatial resolution and the oscillatory nature of the solutions. Additional comments on the wave speeds were made in chapter 8. The relative mass balance errors for these calculations is less than 2 percent for these three ϵ values with the error generally increasing as ϵ increases.

Example 2: This example is identical to Example 1 except that it compares the choice of hydraulic functions in (9.4) with the actual Fujita Diffusivity function (with $m = 2$)

$$D(u,x) = \frac{m(m-1)}{(m-u)^2}(1 + \epsilon sin(2\pi x)). \tag{9.6}$$

The solution for the diffusivity as given by (9.6) is shown in Figure 9.6. Comparison of Figure 9.6 and Figure 9.2 show that the different choices of $D(u,x)$ do not lead to a qualitative difference in the numerical solutions of (1.1).

Example 3: The hydraulic functions in this example are identical to Example 1 with $\epsilon = 0.5$ except now the prescribed flux boundary conditions are implemented instead of using the equivalent prescribed value boundary conditions. Due to the full matrix inversion needed to use the presecribed flux boundary conditions, we took a much smaller range for our x and calculated fewer time steps. The results are shown in Figure 9.7. The prescribed flux condition tends to lead to slightly larger values on the boundaries. A comparison of the speeds is shown in Figure 9.8. Line A represents the prescribed value boundary condtions; Line B, the prescribed flux. The irregularity in these lines is due to poor spatial resolution. While Line A and Line B are not identical, they have roughly the same average slope indicating the same wave speed.

Example 4: This example is identical to Example 1 except with $\epsilon = 0.05$ and we now take the periodic inhomogeneity function to be

$$\begin{aligned} g(x) &= 1+\epsilon \quad x \leq \frac{1}{2}, \\ &= 1-\epsilon \quad x > \frac{1}{2}, \end{aligned} \tag{9.7}$$

instead of $1 + \epsilon sin(2\pi x)$. Results are shown in Figure 9.9. The solution in Figure 9.9 is a more physical solution than the solutions which use $1 + \epsilon sin(2\pi x)$ to describe there heterogeneity in that it is more comparable to the experimental and numerical results shown in [11]. However, due to the discontinuity of the heterogeneity function $g(x)$, reliable numerical solutions can only be obtained with this scheme when ϵ is very small.

Example 5: This example is identical to Example 2 except that a pulse is added to the initial data. Notice how the transient decays to the familiar traveling wave solution as time increases in Figure 9.10.

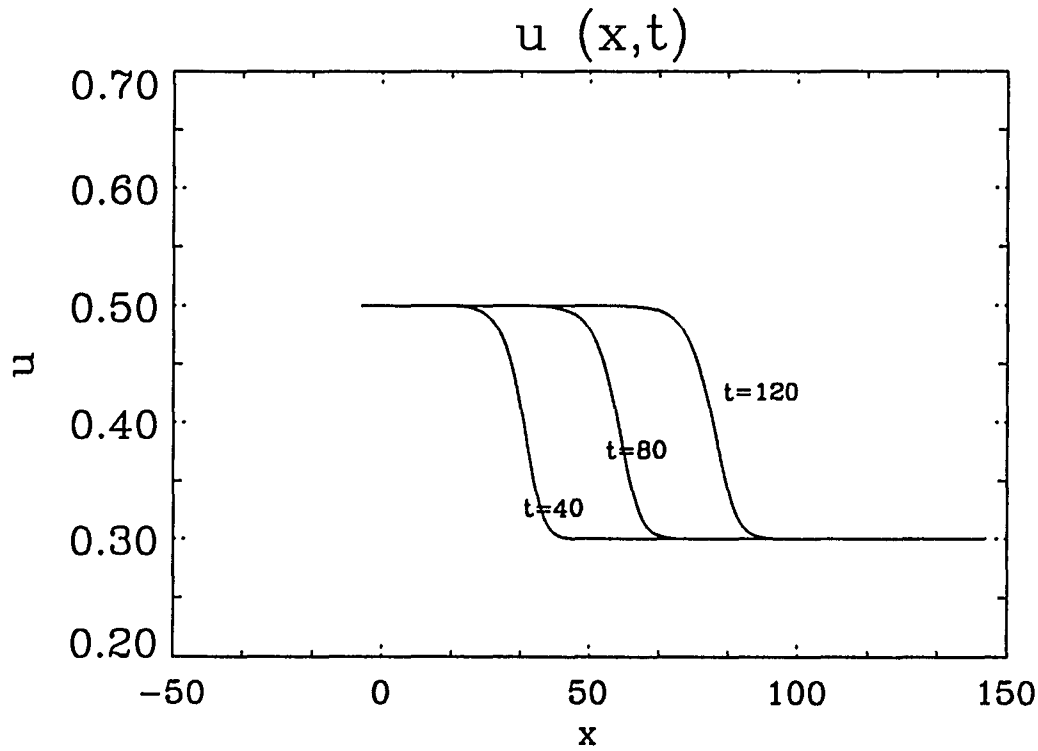

Figure 9.1, $\epsilon = 0.0$ for Three t Values, t = 40,80,120

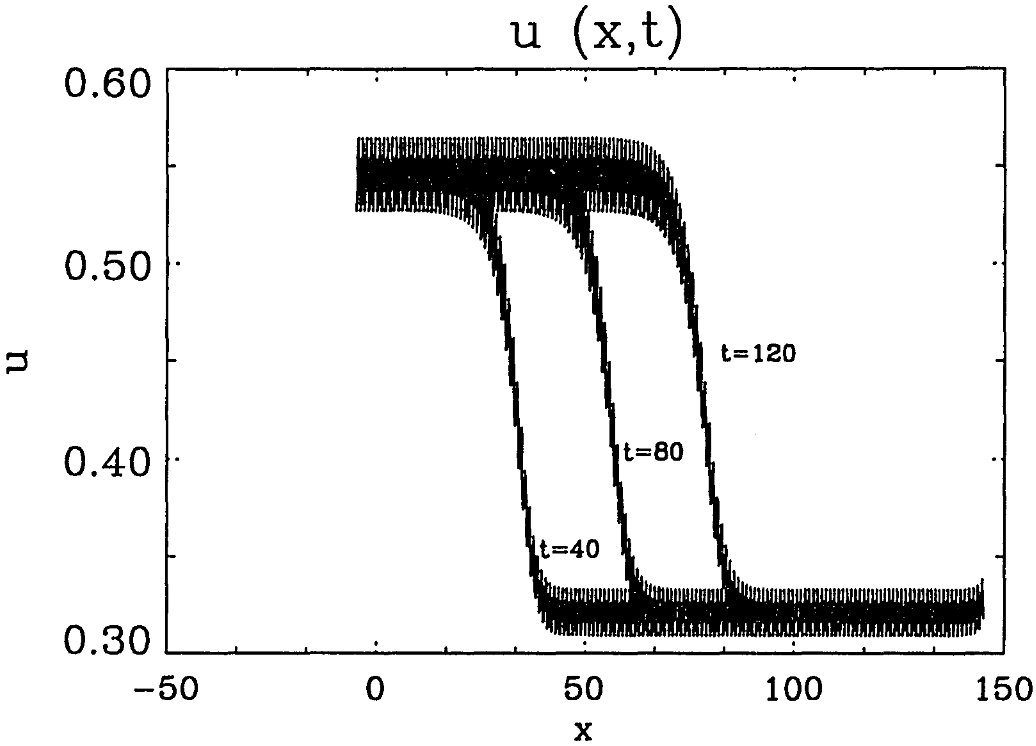

Figure 9.2, $\epsilon = 0.5$ for Three t Values, $t = 40,80,120$

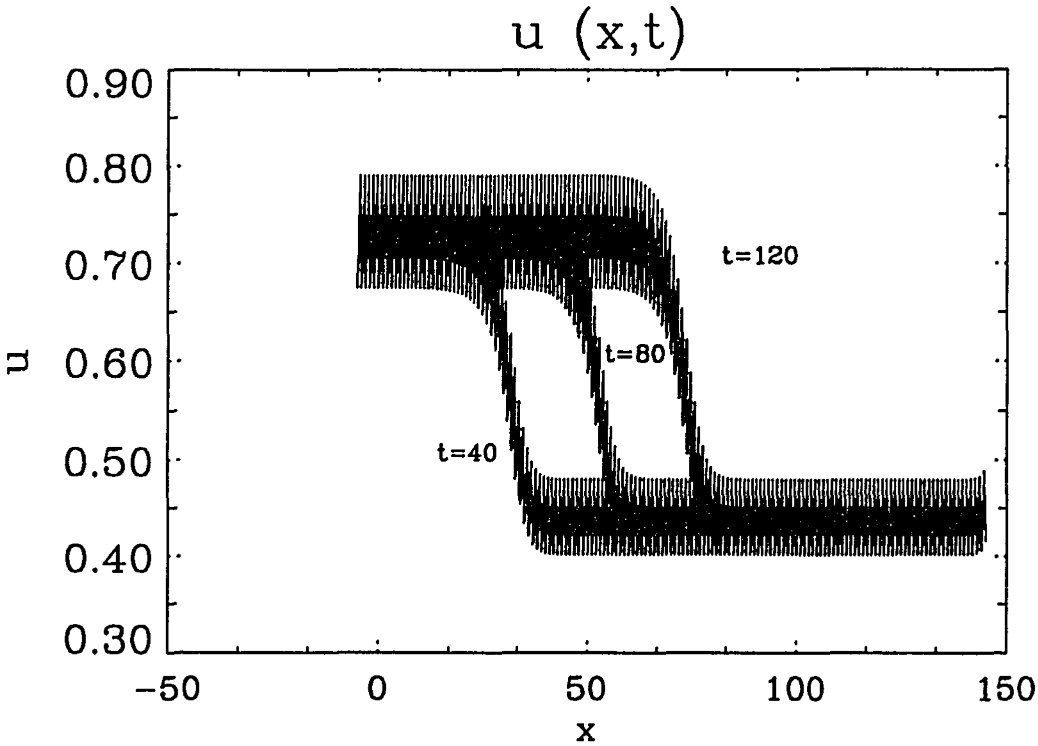

Figure 9.3, $\epsilon = 0.9$ for Three t Values, t = 40,80,120

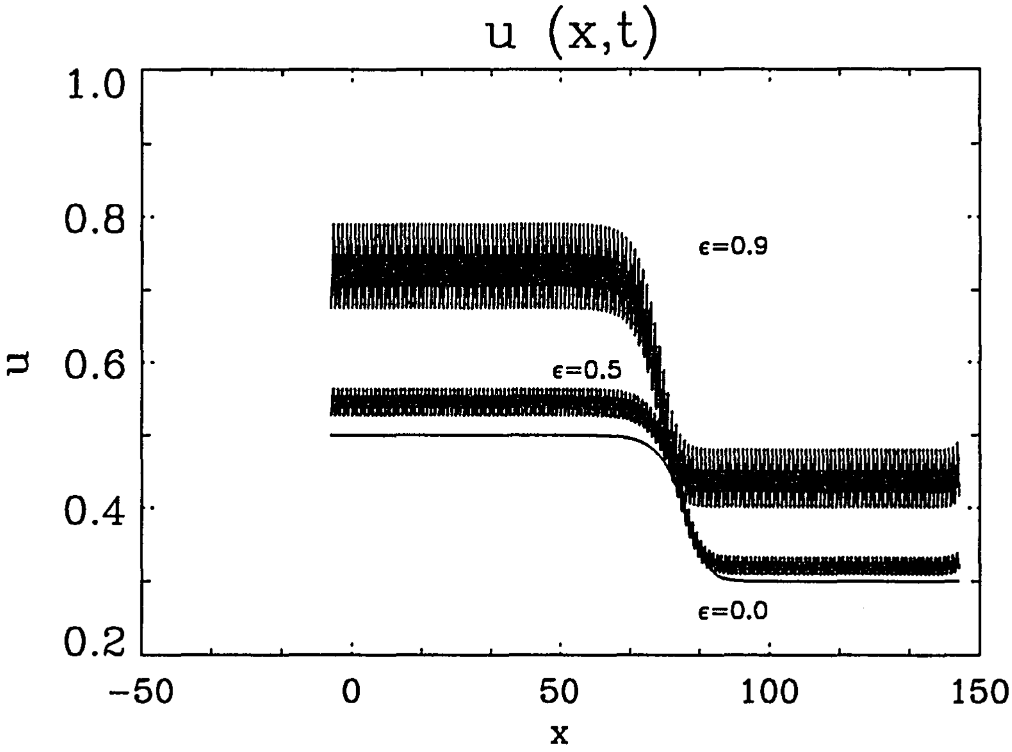

Figure 9.4, $t = 120$ for $\epsilon = 0.0, 0.5, 0.9$

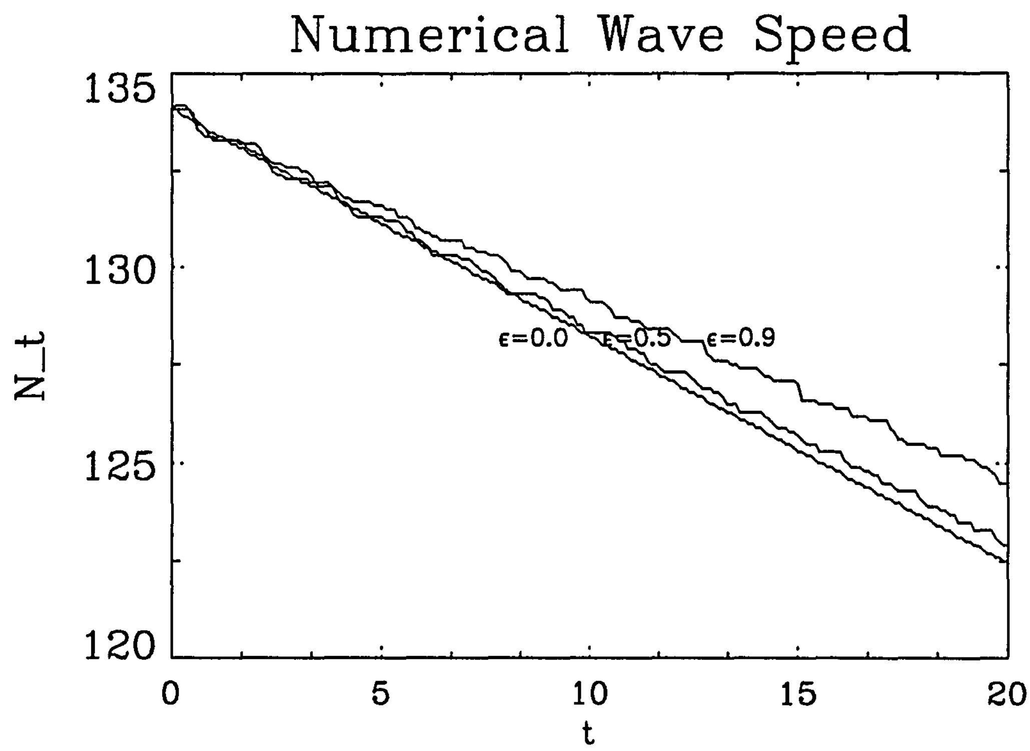

Figure 9.5, Wave Speeds for $\epsilon = 0.0, 0.5, 0.9$

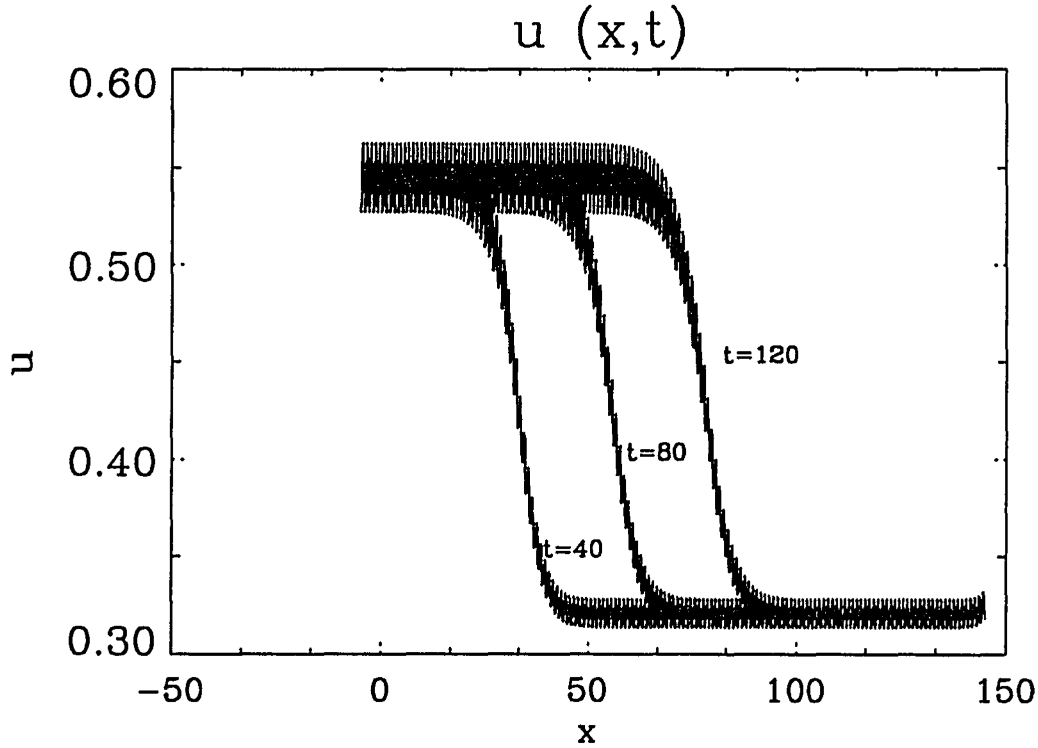

Figure 9.6, Solutions with Diffusivity Given by (9.6)

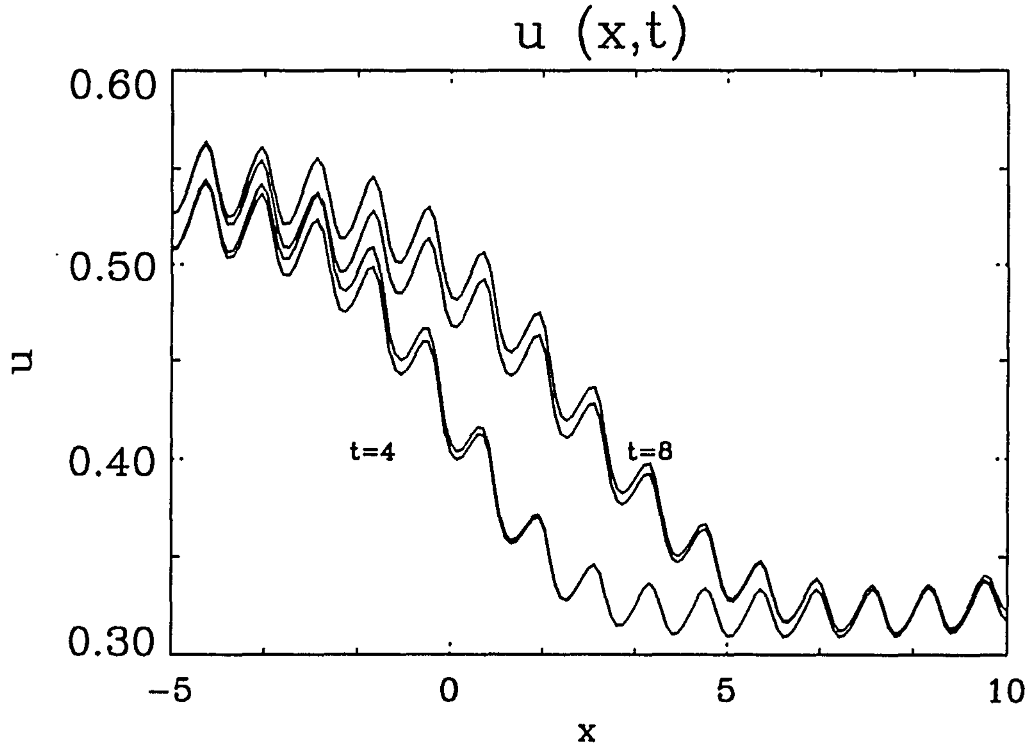

Figure 9.7, Comparison of Boundary Condition Implementation

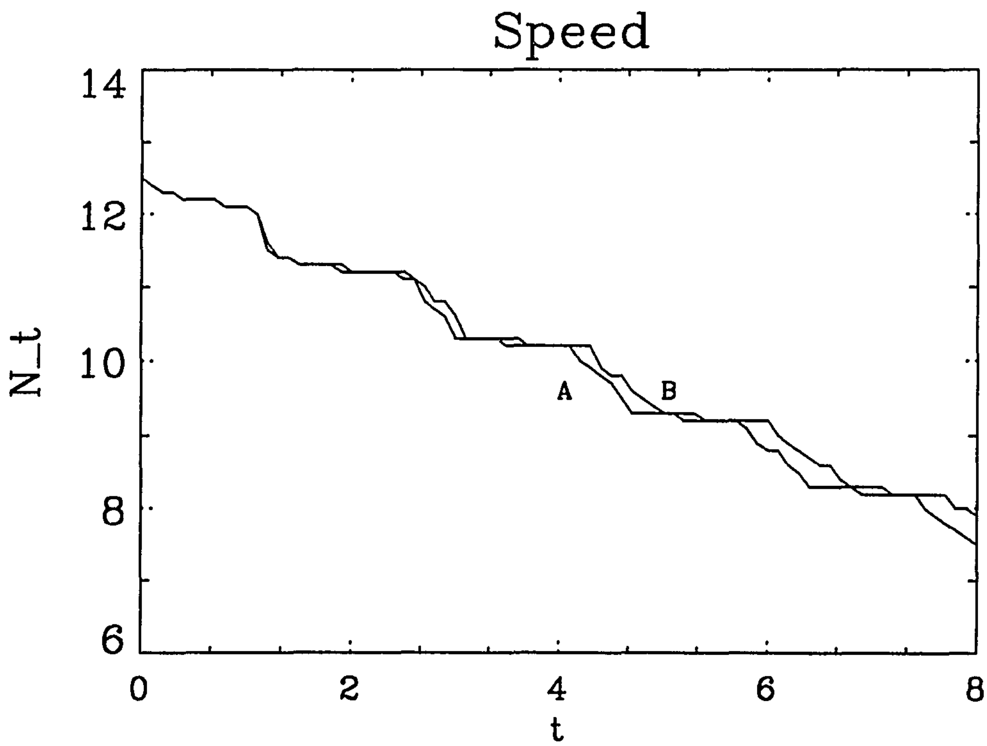

Figure 9.8, Comparison of Speeds for Different Boundary Conditions

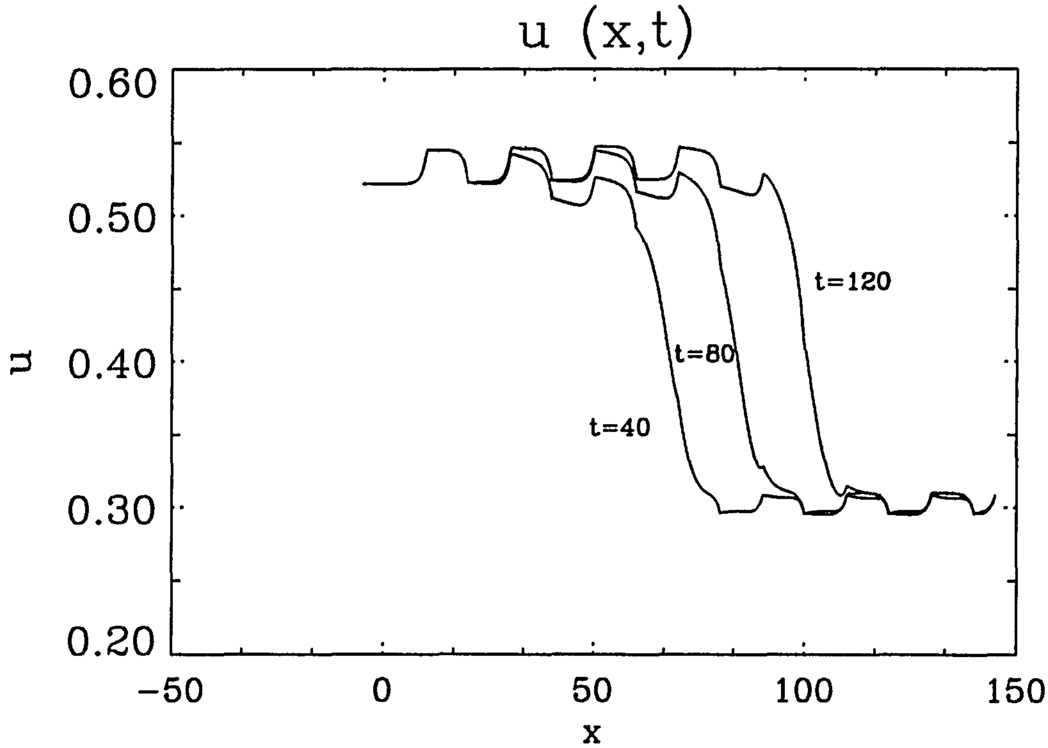

Figure 9.9, Traveling Wave Solutions for Step Function Heterogeneity

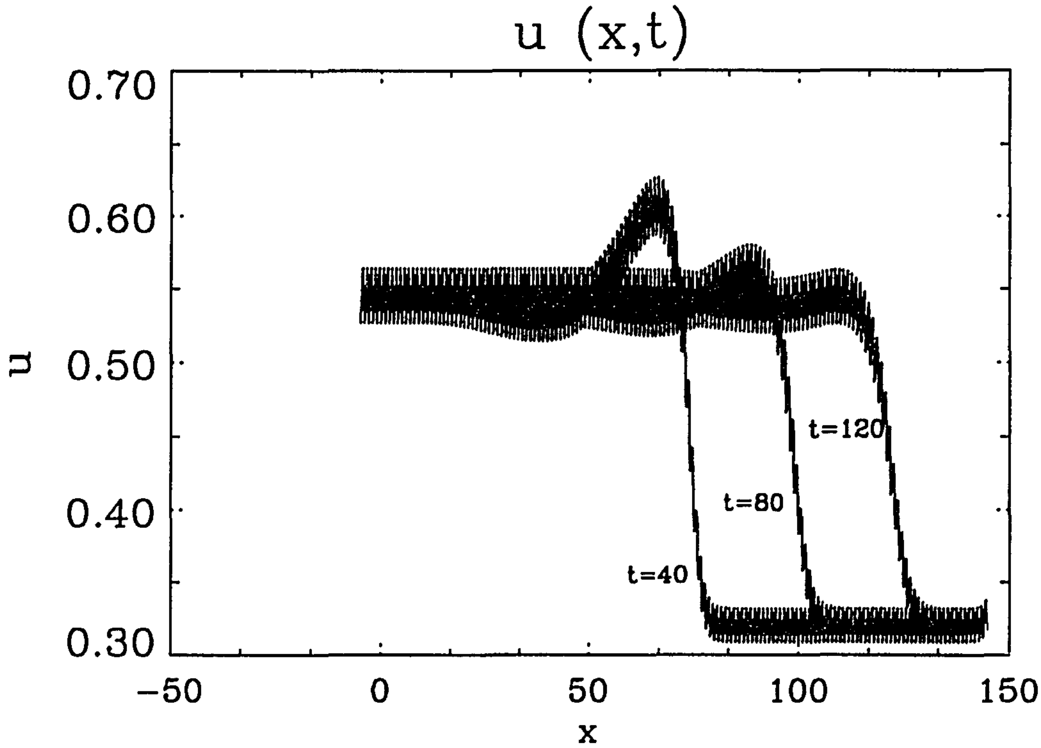

Figure 9.10, Solutions with Diffusivity Given by (9.6) and a Pulse in the Initial Data

10 Analysis with General Hydraulic Functions

As noted in the introduction, during much of the analysis done in Chapters 2 through 9, we use a few assumptions - most notably (A1)-(A5) - on the hydraulic functions, $D(u,x)$ and $K(u,x)$. At this time, we would like to comment on problem (1.1) when we do not make these assumptions. First, recall that Khusnydtinova [17] and Noren [30] showed the existence, uniqueness and asymptotic stability for (1.1) where the hydraulic functions had no x dependence for the Dirichlet and prescribed flux boundary conditions respectively. In their proofs, they only needed assumptions (A4) and (A5). The single most important difference and difficulty which arises when we consider spatial inhomogeneity and do not assume (A1)-(A3) is that we are no longer able to find a change of variable as in (3.14) which produces a form of equation (3.1) with a maximum principle for the entire range of s values. Without a maximum principle, many of the assertions in this thesis are very difficult to prove. We will now briefly outline which parts of the thesis still hold and which parts are lost when we fail to assume some or all of (A1)-(A5).

In chapter 2, Proposition 2.2 uses the assumption that the variable dependence in $D(u,x)$ is separable by taking

$$D(u,x) = D_0(u)a(x), \tag{10.1}$$

in equation (2.12). This is equivalent to (A1). As Remark 2.1 points out, we may even do away with this assumption as long as we are considering infiltration which is not too close to saturation. Using a fourth order Runga-Kutta scheme to numerically solve the steady state equation (2.4), we have never observed a case where a periodic steady state was not achieved for every degree of heterogeneity, even when (10.1) was violated.

In chapter 3, the formula for the speed of the traveling wave given in Lemma 3.1 by equation (3.20) is valid for general $D(u,x)$ and $K(u,x)$. However, we are no longer able to bound the traveling wave solution by the two steady states,

$$u_r(y) \le u'(s,y) \le u_l(y). \tag{10.2}$$

Even without (A1)-(A5), we can still derive an entropy condition similar to the one in (3.62). The derivation goes as follows:
Linearize equation (3.1) around the steady state solution $u_l(y)$ at $s = -\infty$, assuming it exists, by taking

$$\begin{aligned} D(u,y) &= D(u_l(y),y) + vD_u(u_l(y),y) + O(v^2), \\ K(u,y) &= K(u_l(y),y) + vK_u(u_l(y),y) + O(v^2), \end{aligned} \tag{10.3}$$

where $v(s,y) = u_l(y) - u(s,y)$. Ignoring $O(v^2)$ terms and recognizing that $u_l(y)$ satisfies (2.4), (3.1) becomes

$$\begin{aligned} -c(v(s,y))_s &= (\partial_s + \partial_y)[D(u_l(y),y)(\partial_\zeta + \partial_y)v(s,y) \\ &+ v(s,y)(D_u(u_l(y),y)u_{ly}(y) - K_u(u_l(y),y))]. \end{aligned} \tag{10.4}$$

Assume that $v(s,y) = e^{\lambda s}\varphi_l(y)$. Substituting into (10.4) and dividing by $e^{\lambda s}$, we are left with

$$\begin{aligned} 0 &= D\varphi_l\lambda^2 + [(D\varphi_l)' + D\varphi_l' + (D_u u_{ly} - K_u)\varphi_l + c\varphi_l]\lambda \\ &+ [(D\varphi_l')' + ((D_u u_{ly} - K_u)\varphi_l)'], \end{aligned} \tag{10.5}$$

where the "prime" denotes differentiation by y. As we did in chapter 3, we can turn (10.5) into an eigenvalue problem,

$$\begin{aligned} \rho_l(\lambda)\varphi_l &= D\varphi_l\lambda^2 + [(D\varphi_l)' + D\varphi_l' + (D_u u_{ly} - K_u)\varphi_l + c\varphi_l]\lambda \\ &+ [(D\varphi_l')' + ((D_u u_{ly} - K_u)\varphi_l)']. \end{aligned} \tag{10.6}$$

Again, we would like to show the existence of $\lambda_1 > 0$ such that $\rho_l(\lambda_1) = 0$. We can see by inspection that for λ large enough, $\rho_l(\lambda) > 0$. Therefore, we will expand $\rho_l(\lambda) = \rho_0 + \lambda\rho_1 + O(\lambda^2)$ and $\varphi_l(y, \lambda) = \varphi_0(y) + \lambda\varphi_1(y)$ for small λ. Substituting into (10.6), we get the zeroth order equation

$$\rho_0\varphi_0 = (D\varphi_0')' + ((D_u u_{ly} - K_u)\varphi_0)'. \tag{10.7}$$

The periodicity of $\varphi_0(y)$ and the coefficients in y implies that averaging (10.7) over one period in y yields $\rho_0 = 0$. Therefore, $\rho_l(0) = 0$. Thus, by continuity of $\rho_l(\lambda)$, we can show the existence of a λ_1 if we can find a $\lambda > 0$ such that $\rho_l(\lambda) < 0$. The first order equation from (10.6) is

$$\begin{aligned} \rho_1\varphi_0 &= (D\varphi_1')' + ((D_u u_{ly} - K_u)\varphi_1)' \\ &+ (D\varphi_0)' + D\varphi_0' + (D_u u_{ly} - K_u)\varphi_0 + c\varphi_0. \end{aligned} \tag{10.8}$$

Again averaging over one period in y and assumimg that $\langle\varphi_0(y)\rangle \equiv 1$, (10.8) becomes

$$\rho_1 = \langle D\varphi_0' + (D_u u_{ly} - K_u)\varphi_0\rangle + c. \tag{10.9}$$

From the zeroth order equation,

$$0 = (D\varphi_0')' + ((D_u u_{ly} - K_u)\varphi_0)', \tag{10.10}$$

and, integrating once in y, (10.10) becomes

$$\bar{m}_l = D\varphi_0' + (D_u u_{ly} - K_u)\varphi_0, \tag{10.11}$$

where $\bar{m}_l$ is a constant. Substituting (10.11) into (10.9), we get

$$\rho_1 = \bar{m}_l + c. \tag{10.12}$$

In order to have $\rho_l(\lambda) < 0$, we need to have $\rho_1 < 0$ and thus we derive the condition

$$c < -\bar{m}_l. \tag{10.13}$$

Going back to (10.11), we can solve for $\varphi_0(y)$ getting

$$\begin{aligned}
\varphi_0(y) &= \bar{m}_l T_l(y), \\
T_l(y) &= e^{P(y)}\Big[\int_0^y \frac{e^{-P(s)}ds}{D(u_l(s),s)} + \frac{e^{P(1)}}{1-e^{P(1)}}\int_0^1 \frac{e^{-P(s)}ds}{D(u_l(s),s)}\Big], \\
P(y) &= \int_0^y \frac{D_u(u_l(y),y)u_{ly}(y) - K_u(u_l(y),y)}{D(u_l(y),y)}dy. \qquad (10.14)
\end{aligned}$$

As in chapter 3, we can see that $\langle\varphi_0(y)\rangle \equiv 1$ implies that

$$\bar{m}_l = \langle T_l(y)\rangle^{-1} < 0, \qquad (10.15)$$

and thus, (10.13) becomes

$$c < -\langle T_l(y)\rangle^{-1}. \qquad (10.16)$$

Linearizing around the other steady state, $u_r(y)$ at $s = \infty$, and doing the same analysis with $v(s,y) = u(s,y) - u_r(y) = e^{-\lambda s}\varphi_r(y)$, we obtain an analgous condition which we can combine with (10.16) to get the entropy condition for the general case,

$$-\langle T_r(y)\rangle^{-1} < c < -\langle T_l(y)\rangle^{-1}, \qquad (10.17)$$

where $T_r(y)$ is of the same form as $T_l(y)$ with $u_r(y)$ replacing $u_l(y)$ in (10.14).

In chapter 3, we were able to prove Theorem 3.1 which said that the entropy condition (3.62) was never violated for any degree of heterogeneity. We have no similar result for the general case. However, our numerical evaluation of (10.17) for Gardner and Russo, Brooks and Corey, Fujita, and van Genuchten hydraulic functions with various degrees and functional representations of heterogeneity has failed to produce an example where the entropy condition was violated. The following is an example of a numerical calculation.

Example: We evaluated the terms in (10.17) for the Fujita conductivity functions with the heterogeniety term $1 + \epsilon sin(2\pi x)$ for values of $\epsilon \in [0,1)$. A fourth-order Runga-Kutta method was used to solve for the steady states. Figure 10.1 shows the results of this evaluation. If the lines on Figure 10.1 were to intersect, that would give evidence that the entropy condition was violated for some degree of heterogeneity measured by ϵ. However, since the lines did not cross, we can conclude that the entropy condition holds for all ϵ in this example.

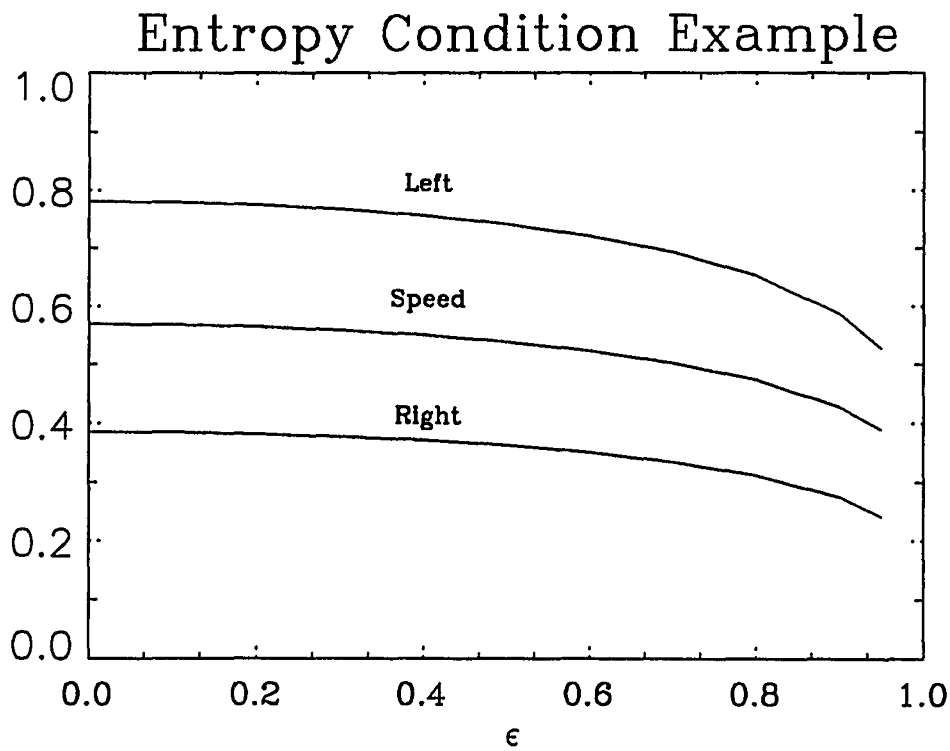

Figure 10.1, Terms of (10.17) as Functions of ϵ

A result like Lemma 3.2 may be proven if we assume (A1) and (A2) without (A3) and if we can establish the solvability of an operator shown below. The argument begins by using (A1) and (A2) to convert (3.1) into an equation of the form

$$(\partial_s + \partial_y)[a(y)D(u)(\partial_s + \partial_y)u - b(y)K(u)] + cu_s = 0. \qquad (10.18)$$

Defining the function $E(u) = \int D(u)du$ and making the change of variables $U(s,y) \equiv E(u(s,y))$, (10.18) becomes

$$(\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)U - b(y)K^*(U)] + cM^*(U)U_s = 0, \tag{10.19}$$

where

$$\begin{aligned} K^*(U) &= K(E^{-1}(U)), \\ M^*(U) &= \frac{1}{E'(E^{-1}(U))} = \frac{1}{D(E^{-1}(U))}. \end{aligned} \tag{10.20}$$

Now assume for $s \geq s_1 >> 1$, the solution takes the form

$$U(s,y) = U_r(y) + \tilde{U}(s,y), \tag{10.21}$$

where $U_r(y) = E(u_r(y))$ satisfies the equation

$$(\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)U_r - b(y)K^*(U_r)] + cM^*(U_r)U_{rs} = 0. \tag{10.22}$$

Then (10.19) becomes

$$\begin{aligned} 0 &= (\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)(U_r + \tilde{U}) - b(y)\frac{K^*(U) - K^*(U_r)}{U - U_r}(U - U_r) \\ &- b(y)K^*(U_r)] + cM^*(U)(U_r + \tilde{U})_s \\ &= (\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)\tilde{U} - b(y)Q(s,y)\tilde{U}] + cR(s,y)\tilde{U}_s, \\ Q(s,y) &= \frac{K^*(U) - K^*(U_r)}{U - U_r} = K^*_U(U_r) + \frac{1}{2}K^*_{UU}(\Theta)(U - U_r), \\ R(s,y) &= M^*(U(s,y)), \end{aligned} \tag{10.23}$$

where Θ is an intermediate value and $Q(s,y) > 0$ since $K_u > 0$. To obtain a form of (10.23) which has a maximum principle, we make the change of variables

$$\tilde{U} = (e(y) + \tilde{e}(s,y))w(s,y). \tag{10.24}$$

Substituting into (10.23), we get

$$0 = (\partial_s + \partial_y)[a(y)(\partial_s + \partial_y)[(e + \tilde{e})w]$$

$$
\begin{aligned}
- &\quad b(y)Q(s,y)(e+\tilde{e})w] + cR(s,y)((\epsilon+\tilde{e})w)_s \\
= &\quad (\epsilon+\tilde{e})(\partial_s+\partial_y)[a(\partial_s+\partial_y)w] \\
+ &\quad (2a(\partial_s+\partial_y)(e+\tilde{e}) - b(e+\tilde{e})Q)(\partial_s+\partial_y)w \\
+ &\quad cR(e+\tilde{e})w_s + [(ae_y)_y - (beQ)_y]w \\
+ &\quad [(\partial_s+\partial_y)[a(\partial_s+\partial_y)\tilde{e}] - (\partial_s+\partial_y)(b\tilde{e}Q) + cR\tilde{e}_s - bQ_s e]w. \qquad (10.25)
\end{aligned}
$$

Notice the similarity between the first lowest order term in (10.25) and equation (3.14) which we used to make a change of variable back in chapter 3. Since $Q(s,y) > 0$, we use the same argument as we used there to show there exists an $\epsilon(y) > 0$ such that

$$(ae_y)_y - (b\epsilon Q)_y = 0. \qquad (10.26)$$

To remove the other lowest order term, it remains to be shown that there exists an $\tilde{e}(s,y)$ such that

$$(\partial_s+\partial_y)[a(\partial_s+\partial_y)\tilde{e}] - (\partial_s+\partial_y)(b\tilde{e}Q) + cR\tilde{e}_s = bQ_s e, \qquad (10.27)$$

and that for s sufficiently large, $\tilde{e}(s,y)$ is small. The solvability of (10.27) is discussed by Ladyzenskaja in [21] (Sec. 3.6, p. 160). If we are able to show that the space which contains $\tilde{e}(s,y)$ is a closed space, then we can establish the existence of a solution. Then, after a similar argument was made for the case where $s \leq s_2 << -1$, we could find a change of variables for which (10.23) has a maximum principle. This would allow us to complete the proof of a result similar to Lemma 3.2.

Without a general maximum principle which is valid for all s, we are unable to show the important results in chapter 4: Theorem 4.1 (Uniqueness) and Corollary 4.1 (Monotonicity in s). However, we can still show the existence of solutions by the same continuation argument given in chapter 5 in Theorem 5.1. Without the maximum principle, we again run into difficulty in chapter 6 when we try to pass the limit of the solutions. This is summarized by the following remark:

Remark 10.1 *Without a general maximum principle, we are only able to prove the existence and uniqueness of traveling wave solutions with a small degree of periodic inhomogeneity. This uniqueness is in the sense of the space we use in our construction of a solution. The proofs of existence and uniqueness are the same as those present in chapter 5. Beyond this point, we have no guarantee of the existence or uniqueness of traveling wave solutions. Additionally, we are unable to show the monotonicity in s of these traveling wave solutions.*

Without (A1)-(A5), the arguments in chapter 7 do not hold again due to the lack of a general maximum principle. Therefore, as of now, we are unable to show rigorously that solutions of (1.1) tend to traveling waves as $t \to \infty$ in general. Never-the-less, numerical solutions indicate that traveling wave solutions exist for all degrees of heterogenity and that time-dependent solutions do tend toward traveling waves even when (A1)-(A5) do not hold.

APPENDIX A Preliminary Lemmas for Stability

In this appendix, we state and prove five preliminary lemmas used in chapter 7 which describe the behavior of the time-dependent solution, $u(\zeta,t)$ near the boundaries $\zeta = \infty$ and $\zeta = -ct$ and a sixth lemma dealing with the long time behavior of $u(\zeta,t)$ for all ζ.

Proof of Lemma 7.1: We will begin by showing that if

$$\lim_{\zeta\to\infty} W_0(\zeta) = W_r, \tag{1.1}$$

then

$$\lim_{\zeta\to\infty} W(\zeta,t) = W_r \tag{1.2}$$

uniformly in $0 \le t \le T$. Then argue that the above statement and the statement of the lemma are equivalent.

Let $\epsilon > 0, T > 0$. Note $W_{0\zeta}$ is bounded for $\zeta \ge -ct, 0 \le t \le T$ by assumption. Therefore, there exists a $\zeta_0 > 0$ such that

$$|W_0(\zeta) - W_r| \le \frac{\epsilon}{2} \tag{1.3}$$

for $\zeta > \zeta_0$ by assumption (7.21). Let

$$V(\zeta,t) = W_r + \frac{\epsilon}{2} + Ne^{\beta t+\zeta_0-\zeta}. \tag{1.4}$$

Define

$$L\cdot = \frac{1}{M(U)}\partial_\zeta[a(\zeta+ct)\partial_\zeta]\cdot + (c + \frac{b_1(\zeta+ct)}{M(U)})\partial_\zeta\cdot - \partial_t\cdot. \tag{1.5}$$

Then

$$LV = (\frac{a(\zeta+ct) - a_\zeta(\zeta+ct) - b_1(\zeta+ct)}{M(U)} - c - \beta)Ne^{\beta t+\zeta_0-\zeta}. \tag{1.6}$$

Since, a, a_ζ, b_1, and $M(U)$ are finite and differentiable, we may choose β large enough that

$$LV \leq 0. \tag{1.7}$$

Then

$$L(V - W) \leq 0, \tag{1.8}$$

since $LW = 0$ by (7.17). Evaluation shows

$$V(\zeta, 0) - W(\zeta, 0) = W_r + \frac{\epsilon}{2} + Ne^{\zeta_0 - \zeta} - W_0(\zeta) \geq 0, \tag{1.9}$$

and

$$V(-ct, t) - W(-ct, t) = W_r + \frac{\epsilon}{2} + Ne^{\beta t + \zeta_0 + ct} - W(-ct, t) \geq 0, \tag{1.10}$$

for a large enough choice of N by (1.9) and the fact that $W(-ct, t)$ is bounded by M_1 as in (7.24). By the maximum principle

$$V(\zeta, t) - W(\zeta, t) \geq 0 \tag{1.11}$$

or

$$W(\zeta, t) \leq W_r + \frac{\epsilon}{2} + Ne^{\beta t + \zeta_0 - \zeta}. \tag{1.12}$$

This implies that

$$W(\zeta + ct, t) \leq W_r + \epsilon \tag{1.13}$$

for $\zeta \geq X$ where X is smallest value of ζ so that

$$Ne^{\beta t + \zeta_0 - \zeta} < \frac{\epsilon}{2}. \tag{1.14}$$

Similarly, if we define

$$V_1(\zeta, t) = W_r - \frac{\epsilon}{2} - N_1 e^{\beta_1 t + \zeta_0 - \zeta}, \tag{1.15}$$

we can show

$$W(\zeta, t) \geq W_r - \epsilon \tag{1.16}$$

for $\zeta \geq \zeta_0$ where ζ_0 is the smallest value of ζ where $N_1 e^{\beta_1 t+\zeta_0-\zeta} < \frac{\epsilon}{2}$. Thus we have

$$\lim_{\zeta\to\infty} W(\zeta,t) = W_r \tag{1.17}$$

uniformly in $0 \leq t \leq T$. This implies

$$\lim_{\zeta\to\infty} W(\zeta,t) - W_r = 0, \tag{1.18}$$

and hence, undoing the change of variables,

$$\lim_{\zeta\to\infty} \frac{U(\zeta,t) - U_r(\zeta+ct)}{e(\zeta+ct)} = 0. \tag{1.19}$$

Since $e(\zeta+ct) > 0$ and is one-periodic in $\zeta+ct$, we have

$$\lim_{\zeta\to\infty} U(\zeta,t) - U_r(\zeta+ct) = 0, \tag{1.20}$$

or going back to our original variables

$$\lim_{\zeta\to\infty} K(u(\zeta,t)) - K(u_r(\zeta+ct)) = 0. \tag{1.21}$$

Since K is continuous and $K' > 0$ we have,

$$\lim_{\zeta\to\infty} K'(\Theta)(u(\zeta,t) - u_r(\zeta+ct)) = 0, \tag{1.22}$$

or

$$\lim_{\zeta\to\infty}(u(\zeta,t) - u_r(\zeta+ct)) = 0. \tag{1.23}$$

The proof is complete.

Proof of Lemma 7.2: We will prove this lemma by a similar strategy to Lemma 7.1. First we will show that if

$$\lim_{\zeta\to\infty} \partial_\zeta(W_0(\zeta) - W_r) = 0, \tag{1.24}$$

then

$$\lim_{\zeta\to\infty} \partial_\zeta(W(\zeta,t) - W_r) = 0 \tag{1.25}$$

uniformly in $0 \le t \le T$. Then we will argue that this statement is equivalent to the lemma by a change of variables.

Start by taking ∂_ζ of the following equation

$$W_t = \frac{1}{M(U)}[a(\zeta+ct)W_\zeta]_\zeta + [c + \frac{b_1(\zeta+ct)}{M(U)}]W_\zeta. \tag{1.26}$$

Letting $w = (W(\zeta,t) - W_r)_\zeta$, this differentiation yields

$$\begin{aligned} w_t &= \frac{a(\zeta+ct)}{M(U)}\partial_\zeta{}^2 w \\ &+ [(\frac{a(\zeta+ct)}{M(U)})_\zeta + \frac{a_\zeta(\zeta+ct)}{M(U)} + c + \frac{b_1(\zeta+ct)}{M(U)}]\partial_\zeta w \\ &+ [(\frac{a_\zeta(\zeta+ct)}{M(U)})_\zeta + (\frac{b_1(\zeta+ct)}{M(U)})_\zeta]w. \end{aligned} \tag{1.27}$$

Define the operator

$$\begin{aligned} L\cdot &= \frac{a(\zeta+ct)}{M(U)}\partial_\zeta{}^2 \cdot + [(\frac{a(\zeta+ct)}{M(U)})_\zeta + \frac{a_\zeta(\zeta+ct)}{M(U)} + c + \frac{b_1(\zeta+ct)}{M(U)}]\partial_\zeta \cdot \\ &+ [(\frac{a_\zeta(\zeta+ct)}{M(U)})_\zeta + (\frac{b_1(\zeta+ct)}{M(U)})_\zeta] \cdot - \partial_t \cdot . \end{aligned} \tag{1.28}$$

Note that this operator does not have a maximum principle in general because its lowest order term does not have a sign. However, since the minimum we are considering is zero, we can still apply a maximum principle by the change of variable argument given by Corollary 9.14 in Smoller [43] (p. 90). Now let

$$V(\zeta,t) = \frac{\epsilon}{2}e^{\beta(t-T)} + Ne^{\beta t + \zeta_0 - \zeta}. \tag{1.29}$$

Then

$$\begin{aligned} L(V-w) &= \{\frac{a(\zeta+ct)}{M(U)} \\ &- [\frac{a_\zeta(\zeta+ct)}{M(U)} + (\frac{a_\zeta(\zeta+ct)}{M(U)})_\zeta + c + \frac{b_1(\zeta+ct)}{M(U)}] \\ &+ [(\frac{a_\zeta(\zeta+ct)}{M(U)})_\zeta + (\frac{b_1(\zeta+ct)}{M(U)})_\zeta] - \beta\}Ne^{\beta t+\zeta_0-\zeta} \\ &+ \{[(\frac{a_\zeta(\zeta+ct)}{M(U)})_\zeta + (\frac{b_1(\zeta+ct)}{M(U)})_\zeta] - \beta\}\frac{\epsilon}{2}e^{\beta(t-T)}. \end{aligned} \tag{1.30}$$

Since a, a_ζ, b_1, and $M(U)$ are finite, we may select β sufficiently large so that

$$L(V - w) \leq 0. \tag{1.31}$$

Now,

$$V(\zeta,0) - W_\zeta(\zeta,0) = \frac{\epsilon}{2}e^{\beta(-T)} + Ne^{\zeta_0 - \zeta} - W_{0\zeta}. \tag{1.32}$$

Since $\lim_{\zeta\to\infty} W_{0\zeta} = 0$ by assumption, there exists a $X_2 = X_2(T) \geq 0$ such that

$$W_{0\zeta} \leq \frac{\epsilon}{2}e^{-\beta T} \tag{1.33}$$

for $\zeta \geq X_2$. Now choosing N sufficiently large to take care of $\zeta \in [-ct, X_2]$ implies

$$V(\zeta,0) - W_\zeta(\zeta,0) \geq 0. \tag{1.34}$$

Likewise,

$$V(-ct,t) - W_\zeta(-ct,t) = \frac{\epsilon}{2}e^{\beta(t-T)} + Ne^{\beta t + \zeta_0 + ct} - W_\zeta(-ct,t). \tag{1.35}$$

We know $|W_\zeta(-ct,t)| \leq M_2 < \infty$ by the boundary condition (7.19) and can thus choose N sufficiently large depending on T so that

$$V(-ct,t) - W_\zeta(-ct,t) \geq 0 \tag{1.36}$$

for $t \in [0,T]$. Thus, we can use the maximum principle to say

$$V(\zeta,t) - W_\zeta(\zeta,t) \geq 0. \tag{1.37}$$

This implies that there exists a $X_3 = X_3(T)$ such that $W_\zeta(\zeta,t) \leq \epsilon$ for $\zeta \geq X_3$. An analogous argument, taking

$$V(\zeta,t) = -\frac{\epsilon}{2}e^{\beta(t-T)} - Ne^{\beta t + \zeta_0 - \zeta} \tag{1.38}$$

will yield $W_\zeta(\zeta,t) \geq -\epsilon$ for $\zeta \geq X_4 = X_4(\epsilon,T)$ for some $X_4 \in R$. Thus

$$\lim_{\zeta\to\infty} \partial_\zeta(W(\zeta,t) - W_r) = 0 \tag{1.39}$$

uniformly in $0 \le t \le T$. Then we have

$$\lim_{\zeta\to\infty} \partial_\zeta \frac{(U(\zeta,t) - U_r(\zeta+ct))}{e(\zeta+ct)} = 0, \tag{1.40}$$

$$\lim_{\zeta\to\infty} \frac{e(\zeta+ct)(U-U_r)_\zeta - e(\zeta+ct)_\zeta(U-U_r)}{e^2(\zeta+ct)} = 0. \tag{1.41}$$

Since $e(\zeta+ct) > 0$ and is one-periodic in ζ, we know

$$\lim_{\zeta\to\infty} e(\zeta+ct)(U-U_r)_\zeta - e(\zeta+ct)_\zeta(U-U_r) = 0. \tag{1.42}$$

Looking at only the second term in the limit and letting $E_1 = \min_\zeta e(\zeta+ct)_\zeta$ and $E_2 = \max_\zeta e(\zeta+ct)_\zeta$, we have

$$\begin{aligned} E_1 \lim_{\zeta\to\infty}(U-U_r) &\le \liminf_{\zeta\to\infty} e(\zeta+ct)_\zeta(U-U_r) \\ &\le \limsup_{\zeta\to\infty} e(\zeta+ct)_\zeta(U-U_r) \le E_2 \lim_{\zeta\to\infty}(U-U_r). \end{aligned} \tag{1.43}$$

By Lemma 7.1, both the bounding limits go to zero. Therefore the middle part of (1.43) also goes to zero. Thus we are left with

$$\lim_{\zeta\to\infty} e(\zeta+ct)(U-U_r)_\zeta = 0, \tag{1.44}$$

and since $e(\zeta+ct) > 0$ and is one-periodic in ζ, we again know

$$\lim_{\zeta\to\infty} (U(s,t) - U_r(s+ct))_\zeta = 0. \tag{1.45}$$

Changing back into our original variables, we see

$$\lim_{\zeta\to\infty} (K(u) - K(u_r))_\zeta = 0, \tag{1.46}$$

or

$$\lim_{\zeta\to\infty} (K'(\Theta)(u-u_r))_\zeta = 0, \tag{1.47}$$

where Θ is an intermediate value of u and u_r. Taking the derivative we have

$$\lim_{\zeta\to\infty} K'(\Theta)(u-u_r)_\zeta + K''(\Theta)\Theta_\zeta(u-u_r) = 0. \tag{1.48}$$

By Lemma 7.1, the second term in (1.48) vanishes and $K' > 0$ implies

$$\lim_{\zeta\to\infty}(u(\zeta,t) - u_r(\zeta + ct))_\zeta = 0. \tag{1.49}$$

The proof is complete.

Proof of Lemma 7.3: Let

$$V(\zeta,t) = \int_\zeta^\infty [u(s,t) - u_r(s+ct)]ds. \tag{1.50}$$

Then by Lemma 7.1 and Richards' equation we have

$$\begin{aligned}
V_t &= \int_\zeta^\infty [(D(u,s+ct)u_\zeta(s,t) - K(u,s+ct))_\zeta \\
&+ cu_\zeta(s,t) - cu_{r\zeta}(s)]ds \\
&= (D(u,s+ct)u_\zeta - K(u,s+ct) + cu(s,t) - cu_r(s+ct))|_{s=\zeta}^\infty \\
&= |(D(u,X+ct)u_\zeta(X,t) - K(u,X+ct)) \\
&- (D(u,\zeta+ct)u_\zeta - K(u,\zeta+ct)) \\
&+ c(u(X,t) - u(\zeta,t))|.
\end{aligned} \tag{1.51}$$

Since $|u_\zeta| \le M_2 < \infty$ by Theorem 7.1, we can bound $|D(u,\cdot)u_\zeta - K(u,\cdot)| < M_3 < \infty$ and, with $|u(\cdot,t)| \le 1$, we obtain

$$|V_t| \le 2M_3 + 2c < \infty. \tag{1.52}$$

Now, think of $V(\zeta,t)$ as having the following form

$$\begin{aligned}
V(\zeta,t) &= V(\zeta,0) + \int_0^t V_t dt \\
&\le \int_\zeta^\infty [u_0(s) - u_r(s+ct)]ds + \int_0^t |V_t|dt,
\end{aligned} \tag{1.53}$$

which is finite for any $t \in [0,T]$ since the first integral on the right hand side is finite by assumption (7.8). Then (1.53) implies

$$\int_\zeta^\infty [u(s,t) - u_r(s+ct)]ds < \infty \tag{1.54}$$

for any $t \in [0,T]$. The proof is complete.

Before stating our next lemma, we will observe the following:

Remark A.1 *Let u_1 and u_2 be any solutions of (7.2) (i.e. full time-dependent, traveling wave, or steady state solutions). Let U_1 and U_2 be any solutions of (7.11); W_1 and W_2, any solutions of (7.17). Then the following are equivalent:*

$$\begin{aligned} \int_{x_1}^{x_2} |u_1 - u_2| ds &< \epsilon, \\ \int_{x_1}^{x_2} |U_1 - U_2| ds &< C'\epsilon, \\ \int_{x_1}^{x_2} |W_1 - W_2| ds &< C''\epsilon, \end{aligned} \tag{1.55}$$

for the same ϵ where $x_1, x_2 \in R^1$ and $C', C'' > 0$ are constants depending on $K(u)$ and $e(\zeta + ct)$.

Proof:

$$\begin{aligned} \int_{x_1}^{x_2} |W_1 - W_2| ds &= \int_{x_1}^{x_2} \frac{1}{e(\zeta + ct)} |U_1 - U_2| ds \\ &= \int_{x_1}^{x_2} \frac{1}{e(\zeta + ct)} |K(u_1) - K(u_2)| ds \\ &= \int_{x_1}^{x_2} \frac{K(\Theta)}{e(\zeta + ct)} |u_1 - u_2| ds, \end{aligned} \tag{1.56}$$

where Θ is an intermediate value between u_1 and u_2. Since $e(\zeta + ct)$ and $K(u)$ are bounded, positive functions, we have

$$\int_{x_1}^{x_2} K(\Theta)|u_1 - u_2| ds = \int_{x_1}^{x_2} |U_1 - U_2| ds \le C' \int_{x_1}^{x_2} |u_1 - u_2| ds, \tag{1.57}$$

where $C' \ge max_u K(u)$. Similarly,

$$\int_{x_1}^{x_2} \frac{K(\Theta)}{e(\zeta + ct)} |u_1 - u_2| ds = \int_{x_1}^{x_2} |W_1 - W_2| ds \le C'' \int_{x_1}^{x_2} |u_1 - u_2| ds, \tag{1.58}$$

where $C'' \ge max_{(u, \zeta + ct)} \frac{K(u)}{e(\zeta + ct)}$. The proof is complete.

Proof of Lemma 7.4: Let

$$V(\zeta,t)=\int_{\zeta}^{\infty}[u(s,t)-u_r(s+ct)]ds. \tag{1.59}$$

Since $W(\zeta,t)\geq W_r$ by the maximum principle, this implies $u(\zeta,t)\geq u_r(\zeta+ct)$ and $V(\zeta,t)\geq 0$ for any (ζ,t). We will now derive an upper bound by defining

$$G(\zeta,t)=\int_{\zeta}^{\infty}[u'(s-s_0,s+ct)-u_r(s+ct)]ds+\frac{\epsilon}{2}, \tag{1.60}$$

where u' is a traveling wave solution with s_0 to be chosen and

$$V_2(\zeta,t)=G(\zeta,t)-V(\zeta,t)=\int_{\zeta}^{\infty}[u'(s-s_0,s+ct)-u(s,t)]ds+\frac{\epsilon}{2}. \tag{1.61}$$

To get our upper bound we will use the version of Richards' equation in (7.2). Define the operator

$$\begin{aligned} L\cdot &= D(u,\zeta+ct)\partial_\zeta{}^2\cdot+\frac{D(u',\zeta+ct)-D(u,\zeta+ct)}{u'(\zeta-s_0,\zeta+ct)-u(\zeta+ct,t)}u'_\zeta\partial_\zeta\cdot \\ &- \frac{K(u',\zeta+ct)-K(u,\zeta+ct)}{u'(\zeta-s_0,\zeta+ct)-u(\zeta+ct,t)}\partial_\zeta\cdot+c\partial_\zeta\cdot-\partial_t\cdot. \end{aligned} \tag{1.62}$$

This operator is derived by writing down (7.2) once with a traveling wave solution and again with a full, time-dependent solution:

$$\begin{aligned} (\partial_t-c\partial_\zeta)u' &= [D(u',\zeta+ct)u'_\zeta-K(u',\zeta+ct)]_\zeta, \\ (\partial_t-c\partial_\zeta)u &= [D(u,\zeta+ct)u_\zeta-K(u,\zeta+ct)]_\zeta. \end{aligned} \tag{1.63}$$

Subtract the two equations to get

$$\begin{aligned} (\partial_t-c\partial_\zeta)(u'-u) &= [D(u',\zeta+ct)u'_\zeta-D(u,\zeta+ct)u_\zeta \\ &- (K(u',\zeta+ct)-K(u,\zeta+ct))]_\zeta. \end{aligned} \tag{1.64}$$

This can be rewritten as

$$\begin{aligned} (\partial_t-c\partial_\zeta)(u'-u) &= [D(u,\zeta+ct)(u'-u)_\zeta \\ &+ \frac{D(u',\zeta+ct)-D(u,\zeta+ct)}{u'(\zeta-s_0,\zeta+ct)-u(\zeta+ct,t)}u'_\zeta(u'-u) \\ &- \frac{K(u',\zeta+ct)-K(u.\zeta+ct)}{u'(\zeta-s_0,\zeta+ct)-u(\zeta+ct,t)}(u'-u)]_\zeta. \end{aligned} \tag{1.65}$$

Integrate (1.65) once from ζ to ∞ and we obtain LV_2. It is easy to verify that $LV_2 = 0$. Now since

$$\begin{aligned} V_2(\zeta, 0) &= \int_\zeta^\infty [u'(s-s_0, s) - u_0(s)]ds + \frac{\epsilon}{2} \\ &= \int_\zeta^\infty [u'(s-s_0, s) - u_r(s)]ds + \int_\zeta^\infty [u_r(s) - u_0(s)]ds + \frac{\epsilon}{2}, \quad (1.66) \end{aligned}$$

where the second integral is finite by assumption (7.8) for $0 \leq \zeta < \infty$. We can choose $s_0 \geq 0$ large enough to make the first integral as large as we need so that the sum of the two integrals is positive. Thus, $V_2(\zeta, 0) \geq 0$ for all $\zeta \geq 0$. When $\zeta = -ct$, we can show $V_2(-ct, t) \geq 0$ by calculating

$$\begin{aligned} V_{2t}(-ct, t) &= \int_{-ct}^\infty (u' - u)_t ds + c[u' - u]|_{\zeta=-ct} \\ &= (D(u', \zeta + ct)u'_\zeta - K(u', \zeta + ct) + cu' \\ &- D(u, \zeta + ct)u_\zeta + K(u, \zeta + ct) - cu(\zeta + ct, t))|_{\zeta=-ct}^\infty \\ &+ c[u' - u]|_{\zeta=-ct}. \quad (1.67) \end{aligned}$$

Evaluating by the boundary conditions (7.4) and (7.5), we see

$$\begin{aligned} &= -\bar{c}_r + \bar{c}_r - (D(u', \zeta + ct)u'_\zeta - K(u', \zeta + ct))|_{\zeta=-ct} - \bar{c}_l \\ &+ c(u' - u)|_{-ct}^\infty + c[u' - u]|_{\zeta=-ct} \\ &= -(D(u', \zeta + ct)u'_\zeta - K(u', \zeta + ct))|_{\zeta=-ct} - \bar{c}_l + c(u' - u)|_{\zeta=\infty}. \quad (1.68) \end{aligned}$$

By Lemma 3.2, we know that $u'(\zeta - s_0, \zeta + ct) - u_r(\zeta + ct) < e^{-\lambda\zeta}$ for $\zeta \geq s_1$, $t \geq 0$ and by Lemma 7.1, we know $\lim_{\zeta\to\infty}(u - u_r) = 0$. Therefore, we have

$$c(u' - u)|_{\zeta=\infty} = c(u_r - u_r) = 0, \quad (1.69)$$

and thus,

$$\begin{aligned} V_2(-ct, t) &= V_2(0, 0) \\ &+ \int_0^t [-(D(u', \zeta + c\tau)u'_\zeta - K(u', \zeta + c\tau))|_{\zeta=-c\tau} - \bar{c}_l]d\tau. \quad (1.70) \end{aligned}$$

We can see that the integral in (1.70) is finite for all t by the following argument. From the steady state equation (2.5), we know $-\bar{c}_l = D(u_l, \zeta+ct)u_{l\zeta} - K(u_l, \zeta+ct)$. Substituting into (1.70), we see that the integrand is equal to

$$\begin{aligned} [(D(u_l, \zeta + c\tau)u_{l\zeta} &- K(u_l, \zeta + c\tau)) \\ -(D(u', \zeta + c\tau)u'_\zeta &- K(u', \zeta + c\tau))]|_{\zeta=-c\tau}. \end{aligned} \tag{1.71}$$

Adding and subtracting $D(u_l, \zeta + c\tau)u'_\zeta$ to expression (1.71) makes the integrand equivalent to

$$\begin{aligned} &|D(u_l, \zeta + c\tau)(u_l - u')_\zeta + D(\Theta_1, \zeta + c\tau)u'_\zeta(u_l - u') - K(\Theta_2, \zeta + c\tau)(u_l - u')| \\ &\leq |[N_u D(u_l, \zeta + c\tau) + D(\Theta_1, \zeta + c\tau)u_\zeta - K(\Theta_2, \zeta + c\tau)](u_l - u')| \end{aligned} \tag{1.72}$$

where Θ_1 and Θ_2 are intermediate values and $|N_u| \leq \bar{M}_2 < \infty$ since $|u'_\zeta| \leq M_2 < \infty$ and $|u_{l\zeta}| \leq M_2 < \infty$. By Lemma 3.2, $(u_l(\zeta + ct) - u'(\zeta - s_0, \zeta + ct)) \leq Ce^{-\lambda c\tau}$ where $\lambda > 0$. Therefore, the integrand can be bounded by $C_1 e^{-\lambda c\tau}$ and hence, the integral in (1.70) exists and is finite for all $t \geq 0$.

Select s_0 such that $V_2(0,0) >> 1$ as in (1.74) and then we have $V_2(-ct, t) \geq 0$. Then by the maximum principle, we have our upper bound

$$G(\zeta, t) \geq V(\zeta, t), \tag{1.73}$$

and therefore, since the limits of both the upper and lower bounds go to zero,

$$\lim_{\zeta \to \infty} V(\zeta, t) = 0. \tag{1.74}$$

The proof is complete.

Proof of Lemma 7.5: Since $W_l \geq W(\zeta, t)$ by the maximum principle, this implies $u_l(\zeta + ct) \geq u(\zeta, t)$ and we have

$$\int_{-ct}^{\zeta} [u_l(s + ct) - u(s, t)]ds \geq 0 \tag{1.75}$$

for any (ζ, t). Now we will show an upper bound. Let

$$\begin{aligned} V_2(\zeta,t) &= \int_{-ct}^{\zeta}[u(s,t)-u'(s-s_0,s+ct)]ds \\ &+ \int_{-\infty}^{-ct}[u_l(s+ct)-u'(s-s_0,s+ct)]ds+\frac{\epsilon}{2}, \end{aligned} \tag{1.76}$$

where $u(s,t)$ is a time-dependent solution of Richards' equation and $u'(s-s_0, s+ct)$ is a traveling wave solution. Now, define the parabolic operator

$$\begin{aligned} L\cdot &= D(u',\zeta+ct)\partial_\zeta{}^2\cdot+\frac{D(u,\zeta+ct)-D(u',\zeta+ct)}{u(\zeta+ct,t)-u'(\zeta-s_0,\zeta+ct)}u_\zeta\partial_\zeta\cdot \\ &- \frac{K(u,\zeta+ct)-K(u',\zeta+ct)}{u(\zeta+ct,t)-u'(\zeta-s_0,\zeta+ct)}\partial_\zeta\cdot+c\partial_\zeta\cdot-\partial_t\cdot. \end{aligned} \tag{1.77}$$

This is the same operator that was derived in the proof of Lemma 7.4 with the roles of u and u' reversed. As before, we can see $LV_2 = 0$. Now

$$\begin{aligned} V_2(\zeta,0) &= \int_0^{\zeta}[u_0-u']ds+\int_{-\infty}^{0}[u_l-u']ds+\frac{\epsilon}{2} \\ &= \int_0^{\zeta}[u_0-u_r]ds+\int_0^{\zeta}[u_r-u']ds+\int_{-\infty}^{0}[u_l-u']ds+\frac{\epsilon}{2} \\ &\geq \int_0^{\zeta}[u_0-u_r]ds+\int_0^{\infty}[u_r-u']ds+\int_{-\infty}^{0}[u_l-u']ds+\frac{\epsilon}{2}, \end{aligned} \tag{1.78}$$

which is non-negative as the sum of the second and third integrals is equal to $-H(s_0, \infty)$ and can be made any positive value by increasing s_0 and, on the other hand, $\int_0^\infty |u_0 - u_r| < \infty$. Additionally,

$$V_2(-ct,t)=\int_{-\infty}^{-ct}[u_l-u']ds+\frac{\epsilon}{2}\geq 0, \tag{1.79}$$

since the integrand is positive by Lemma 3.1. Therefore, the maximum principle for parabolic operators tells us that $V_2 \geq 0$. Thus,

$$\int_{-ct}^{\zeta}[u-u']ds+\int_{-\infty}^{-ct}[u_l-u']ds+\frac{\epsilon}{2}\geq 0, \tag{1.80}$$

and is the same as

$$\int_{-ct}^{\zeta}[u-u_l]ds+\int_{-\infty}^{\zeta}[u_l-u']ds+\frac{\epsilon}{2}\geq 0, \tag{1.81}$$

or

$$\int_{-ct}^{\zeta}[u_l - u]ds \leq \int_{-\infty}^{\zeta}[u_l - u']ds + \frac{\epsilon}{2}. \tag{1.82}$$

By Lemma 3.2, we may choose $X = X(\epsilon)$ such that

$$0 \leq \int_{-\infty}^{\zeta}[u_l - u']ds \leq \frac{\epsilon}{2} \tag{1.83}$$

for $\zeta \leq X$. Then combining (1.86) and (1.87), we have

$$\int_{-ct}^{\zeta}[u_l(s+ct) - u(s,t)]ds \leq \epsilon. \tag{1.84}$$

The proof is complete.

Proof of Lemma 7.6: First we employ a change of variables

$$u(x,t) = v(x,t)q(x) = v(x,t)(1 - \beta e^{\alpha(x-x_l)}), \tag{1.85}$$

where α and β are positive constants to be determined so that $\frac{1}{2} \leq q(x) \leq 1$ for $x \in [x_l, x_r]$. Then

$$L'(v) = a(x,t)v_{xx} + b'(x,t)v_x - c(x,t)v - v_t = 0, \tag{1.86}$$

where

$$\begin{aligned} b'(x,t) &= 2a\frac{q_x}{q} + b, \\ c(x,t) &= a\frac{q_{xx}}{q} + b\frac{q_x}{q} = \frac{\alpha\beta e^{\alpha(x-x_l)}}{1-\beta e^{\alpha(x-x_l)}}(\alpha a + b). \end{aligned} \tag{1.87}$$

Since x comes from a bounded domain, we may choose an α larger than $sup(\frac{1+|b|}{a})$ and then a $\beta = \beta(\alpha, x_r - x_l)$ small enough that $\frac{1}{2} \leq q(x) \leq 1$, for all $x \in [x_l, x_r]$. Then

$$\inf_{(x,t)} c(x,t) \equiv \delta \geq 2\alpha\beta(\alpha a + b) > 0. \tag{1.88}$$

Next, for every $\epsilon > 0$ there exists a $T = T(\epsilon) > 0$ such that $\frac{|g_1(t)|}{q(x)} < \epsilon$ and $\frac{|g_2(t)|}{q(x)} < \epsilon$ for $t > T(\epsilon)$. Next, consider the auxilliary functions

$$w_{\pm}(x,t) = Me^{-c_0(t-T)} + 2\epsilon \pm v(x,t), \tag{1.89}$$

where $M > 0$ and $c_0 > 0$ are constants. By choosing M large enough and using our bounds on the boundary values, we have

$$\begin{aligned} w_{\pm}(x,T) &= M + 2\epsilon \pm v(x,T) \geq 0, \quad x \in Q \\ w_{\pm}(x_l,t) &= Me^{-c_0(t-T)} + 2\epsilon \pm \frac{g_1(t)}{q(x_l)} \geq 0, \quad t \geq T, \\ w_{\pm}(x_r,t) &= Me^{-c_0(t-T)} + 2\epsilon \pm \frac{g_2(t)}{q(x_r)} \geq 0, \quad t \geq T. \end{aligned} \tag{1.90}$$

For $t \geq T$, we have

$$L'(w_{\pm}) = -c(x,t)(Me^{-c_0(t-T)} + 2\epsilon) + c_0Me^{-c_0(t-T)} \leq 0, \tag{1.91}$$

if we select $c_0 = \delta$. Thus, we can apply the maximum principle to show that $w_{\pm}(x,t) \geq 0$ for all $x \in [x_l, x_r]$ and $t \geq T$ or

$$|v(x,t)| \leq \epsilon + Me^{-c_0(t-T)} \tag{1.92}$$

for $t \geq T(\epsilon)$, which implies when sending $t \to \infty$ that

$$\limsup_{t\to\infty} |u(x,t)| \leq \epsilon. \tag{1.93}$$

The proof is complete.

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