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Compressible Euler Equations with Vacuum

Tai-Ping Liu*

Department of Mathematics, Stanford University, Stanford, California 94305-2125

and

Tong Yang†

Department of Mathematics, City University of Hong Kong, Tot Ohee Avenue, Kowloon, Hong Kong

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In this paper, we first show that the regular solutions of compressible Euler equations in R^3 with damping will not be global if the initial density function has compact support. This implies that c^2 cannot be smooth across the boundary Γ separating the gas and the vacuum after a finite time, where c is the speed of sound. Then we study the local existence of solutions for isentropic gas flow in R^1 when c^α, $0<\alpha\leqslant 1$, is smooth across Γ, using the energy method and the characteristics method. © 1997 Academic Press

1. INTRODUCTION

Consider the compressible Euler equations in R^3 with damping,

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$\rho\mathbf{u}^t + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p(\rho) = -\alpha\rho\mathbf{u}, \tag{1.2}$$

$$(\rho s)_t + \nabla\cdot(\rho s\mathbf{u}) = 0. \tag{1.3}$$

When the initial data have compact support, it is interesting to study the general behaviour of solutions near the vacuum boundary Γ, where $\Gamma = \{(\mathbf{x}, t) \mid \rho(\mathbf{x}, t) \geqslant 0\} \cap \{(\mathbf{x}, t) \mid \rho(\mathbf{x}, t) = 0\}$, and $\rho(\mathbf{x}, t)$ is the density.

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The main difficulty in this study is that the system becomes degenerate when vacuum appears; that is, characteristics coincide and become zero in the Lagrangian coordinates. Therefore, even though the system is symmetrizable, in general the coefficients will not satisfy the local existence theories [1, 4].

In [2], the first author constructed a class of spherical symmetric solutions of (1.1) and (1.2) which converge to the self-similar solutions of the porous media equation, and conjectured that the general behaviour near Γ is dc^2/dx bounded away from zero in the Eulerian coordinates after finite time, where c is the speed of sound. This implies that if initial data (ρ, u) are smooth, then there is a waiting time. After that time, the boundary will move due to the effect of the pressure.

Pursuing this direction, in Section 2 we will show that the regular solutions cannot be global if the density function has compact support. Then in Section 3 we study the local existence of solutions when the solutions have the property that dc^{α}/dx, $0<\alpha\leqslant 1$, bounded away from zero across Γ. We show that this kind of phenomina exists and α remains the same locally in time. When the initial data connects to vacuum states discontinuously, local existence for Euler equations without damping was proved in [3]. According to [2], the general behaviours of the solutions could be the one corresponding to $\alpha=2$. But the problem of local existence of solutions for $1<\alpha\leqslant 2$ remains open.

For Euler equations without damping, Sideris [10] gave a sufficient conditions for nonglobal existence of C^1 solutions when $\inf \rho_0(\mathbf{x})>0$, where $\rho_0(\mathbf{x})$ is the initial density. The nonexistence of C^1 solutions in [10] is related to shock formation. In our following discussion, we will study the singularity of the solutions in the vacuum states when the solutions contain no shocks. Thus, the time when the regular solutions blow up in our discussion is before the time when shock forms.

When $\inf \rho_0(\mathbf{x})=0$, Makino [6] proved the nonglobal existence of regular solutions by assuming the initial data $(\rho_0(\mathbf{x}), \mathbf{u}_0(\mathbf{x}))$ to have compact support, where $\mathbf{u}_0(\mathbf{x})$ is the initial velocity. For the Euler–Poisson equations governing gaseous stars, Makino and Perthame [7] proved the nonglobal existence of tame solutions under the condition of spherical symmetry. Local existence of tame or regular solutions for these two systems was proved by Makino [7–9] using Kato's [1] theory for quasilinear symmetric hyperbolic system. Solutions thus obtained correspond to those of $0<\alpha<\frac{2}{3}$ or $\alpha=1$ if the solutions are in H^2. Furthermore, the regular solutions defined below is different from that in [6], where we require $\rho^{\gamma-1}\in C^1([0, T)\times \mathbf{R}^3)$ instead of $\rho^{(\gamma-1)/2}\in C^1([0, T)\times \mathbf{R}^3)$. Since c is a continuous function, the regular solution defined here is more general than the one in [6], and thus our nonexistence theorem, Theorem 2.1, generalizes that of [6].

2. NO GLOBAL EXISTENCE

In this section, we consider the system (1.1)–(1.3) with initial data

$$\rho(\mathbf{x},0)=\rho_0(\mathbf{x})\geqslant 0, \qquad u(\mathbf{x},0)=u_0(\mathbf{x}), \qquad s(\mathbf{x},0)=s_0(\mathbf{x}), \tag{2.1}$$

where ρ, $\mathbf{u}$, $p(\rho)$, and s are the density, velocity, pressure, and entropy, respectively. We consider the regular solution of (1.1)–(1.3) and (2.1) defined as follows.

DEFINITION 2.1. A solution of (1.1)–(1.3) is called a regular solution in $[0,T)\times\mathbf{R}^3$ if

(i) $(\rho,\mathbf{u},s)\in C^1([0,T)\times\mathbf{R}^3)$, $\rho\geqslant 0$ and $s\geqslant 0$,

(ii) $\rho^{\gamma-1}\in C^1([0,T)\times\mathbf{R}^3)$,

and

$$\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}=-\alpha\mathbf{u},$$

holds on the exterior of the support of ρ.

THEOREM 2.1. *Let* $(\rho(\mathbf{x},t),\mathbf{u}(\mathbf{x},t),s(\mathbf{x},t))$ *be a regular solution of* (1.1)–(1.3) *and* (2.1) *on* $0\leqslant t<T$. *If the support of the initial data* $\rho_0(\mathbf{x})$ *has compact support and* $\rho_0(\mathbf{x})\not\equiv 0$, *then there exists a constant* $\alpha_0>0$, *such that when* $\alpha<\alpha_0$, T *is finite. Here* α_0 *is a constant depending only on the initial data.*

Remark 2.1. The following proof is similar to that of Makino for the case when $\alpha=0$ [6]. We generalize his proof to the case when $\alpha>0$ without assuming that $\mathbf{u}$ has compact support. And we assume $\rho^{\gamma-1}\in C^1([0,T)\times\mathbf{R}^3)$ instead of $\rho^{(\gamma-1)/2}\in C^1([0,T)\times\mathbf{R}^3)$.

Proof. Let

$$\Omega(t)=supp\ \rho(x,t), \qquad s(t)=\partial\Omega(t).$$

By Eq. (1.1) and the definition of a regular solution, for any $\mathbf{x}\in s(t_0)$, there exist $\mathbf{x}_0\in s(0)$ and a curve $\mathbf{x}(t)$ connecting $\mathbf{x}_0$ and $\mathbf{x}$ such that

$$\frac{d\mathbf{x}(t)}{dt}=\mathbf{u}(\mathbf{x}(t),t), \qquad \mathbf{x}(t)\in s(t), \qquad 0\leqslant t\leqslant t_0.$$

Since on $s(t)$

$$\frac{d^2\mathbf{x}(t)}{dt^2}=\frac{D}{Dt}\mathbf{u}(t)=-\alpha\mathbf{u}=-\alpha\frac{d\mathbf{x}}{dt},$$

we have

$$\mathbf{x}(t) = \mathbf{x}_0 + \frac{1}{\alpha} \frac{d\mathbf{x}(t)}{dt}\bigg|_{t=0} (1 - e^{-\alpha t}) = \mathbf{x}_0 + \frac{1}{\alpha} u_0(\mathbf{x}_0)(1 - e^{-\alpha t}).$$

Thus there exists a constant d^* depending only on the initial data, such that

$$d(t) = \sup_{\mathbf{x} \in \Omega(t)} |\mathbf{x}| = \sup_{\mathbf{x} \in s(t)} |\mathbf{x}| \leqslant d^*.$$

Now, let

$$H(t) = \tfrac{1}{2} \int_{\Omega(t)} \rho(\mathbf{x}, t)\, |\mathbf{x}|^2 \, d\mathbf{x}.$$

From Eqs. (1.1) and (1.2), we have

$$H'(t) = \int_{\Omega(t)} \rho \mathbf{u} \cdot \mathbf{x} \, d\mathbf{x}$$

and

$$H''(t) = \int_{\Omega(t)} (\rho\, |\mathbf{u}|^2 + 3p(\rho, s)) \, d\mathbf{x} - \alpha \int_{\Omega(t)} \rho \mathbf{u} \cdot \mathbf{x} \, d\mathbf{x}.$$

From $d(t) \leqslant d^*$ and the Cauchy–Schwartz inequality,

$$\left(\int_{\Omega(t)} \rho \mathbf{u} \cdot \mathbf{x} \, d\mathbf{x} \right)^2 \leqslant \left(\int_{\Omega(t)} \rho\, |\mathbf{u}|^2 \, d\mathbf{x} \right) \left(\int_{\Omega(t)} \rho\, |\mathbf{x}|^2 \, d\mathbf{x} \right)$$

$$\leqslant d^{*2} m \int_{\Omega(t)} \rho\, |\mathbf{u}|^2 \, d\mathbf{x},$$

where $m = \int_{\Omega(t)} \rho(\mathbf{x}, t) \, d\mathbf{x}$ is the total mass. Thus

$$H''(t) \geqslant \int_{\Omega(t)} (\rho\, |\mathbf{u}|^2 + 3p(\rho, s)) \, d\mathbf{x} - \alpha d^* \sqrt{m} \left(\int_{\Omega(t)} \rho\, |\mathbf{u}|^2 \, d\mathbf{x} \right)^{1/2}$$

$$\geqslant 3 \int_{\Omega(t)} p(\rho, s) \, d\mathbf{x} - \left(\frac{\alpha d^* \sqrt{m}}{2} \right)^2.$$

Since $p(\rho, s) = k^2\rho^\gamma e^s \geqslant k^2\rho^\gamma$, and $\gamma > 1$, we have from the Hölder inequality that

$$m = \int_{\Omega(t)} \rho \, d\mathbf{x} \leqslant \left(\int_{\Omega(t)} \rho^\gamma \, d\mathbf{x} \right)^{1/\gamma} \left(\int_{\Omega(t)} d\mathbf{x} \right)^{1/\gamma'},$$

where $1/\gamma + 1/\gamma' = 1$. Thus

$$\left(\int_{\Omega(t)} \rho^\gamma \, d\mathbf{x} \right)^{1/\gamma} \geqslant m(V^*)^{-1/\gamma'},$$

where $V^* = \frac{4}{3}\pi(d^*)^3$. Hence

$$H''(t) \geqslant 3k^2 m^\gamma (V^*)^{-\gamma/\gamma'} - \frac{\alpha^2 d^{*2} m}{4}.$$

We let $H''(t) \geqslant \eta > 0$ by assuming

$$\alpha \leqslant \frac{2}{d^*} \left(\frac{3k^2 m^\gamma (V^*)^{-\gamma/\gamma'} - \eta}{m} \right)^{1/2} = \alpha_0,$$

where η is a constant and $0 < \eta < 3k^2 m^\gamma (V^*)^{-\gamma/\gamma'}$. For $\alpha \leqslant \alpha_0$, we have

$$H(t) \geqslant H(0) + H'(0)\, t + \frac{\eta}{2}\, t^2.$$

Since $H(t) \leqslant \frac{1}{2} m d^{*2}$, we have

$$\frac{\eta}{2}\, t^2 + H'(0)\, t + H(0) - \frac{1}{2}\, m d^{*2} \leqslant 0,$$

i.e.,

$$t \leqslant \frac{-H'(0) + \sqrt{H'(0)^2 + \eta(m d^{*2} - 2H(0))}}{\eta}.$$

Therefore T must be finite and this completes the proof of Theorem 2.1. ∎

Remark 2.2. The time in Theorem 2.1 being finite is due to the singularity of the vacuum. After this time, the boundary of the gas will move due to the effect of the pressure. Therefore the behaviour of the solutions near the boundary will be different. Notice that the time here is before the time when the shock forms.

3. LOCAL EXISTENCE

In this section, we are going to use two methods to study the local existence of solutions for isentropic gas flow in R^1 corresponding to the cases when dc^α/dx, $0<\alpha\leqslant 1$, bounded away from zero across the boundary Γ seperating the gas and the vacuum. The first method is the application of Kato's theory for quasi-linear symmetric hyperbolic systems, which is essentially the energy method. In order to rewrite our system into a symmetric hyperbolic system with those corresponding properties near the boundary, we make some change in the variables and the coordinates according to the singularity with respect to different α. We show that if the initial data satisfy dc^α/dx bounded away from zero near Γ when $0<\alpha\leqslant 1$, then the solution remains the same α locally in time. It might be difficult to use this method to study other cases corresponding to $\alpha>1$, Remark 3.3. Next we use the characteristics method to prove the same result. The study for $\alpha>1$ will be our research in the future.

3.1. *Energy Method*

Consider the one dimensional Euler equations with damping for isentropic flow,

$$\begin{cases} \rho_t+(\rho u)_x=0, \\ (\rho u)_t+(\rho u^2+p(\rho))_x=-\rho u, \end{cases} \tag{3.1}$$

where $p(\rho)=k^2\rho^\gamma$ with $\gamma>1$ and the damping constant is normalized to 1. We are going to study the singular behaviour of the solutions near the boundary Γ which is defined in Section 1. In the following, we are going to prove local existence of solutions when dc^α/dx, $0<\alpha\leqslant 1$, is bounded away from zero near Γ, where $c=k\sqrt{\gamma}\,\rho^{(\gamma-1)/2}$ is the speed of sound.

Since away from vacuum the system (3.1) is hyperbolic, the local existence of regular solutions for smooth initial data is well known [1]. Hence in order to study the behaviour of the solutions near Γ, we need only consider the following case. Let the initial density $\rho(x,0)$ be given as

$$\rho(x,0)=\begin{cases} 0, & x\leqslant a, \\ \rho_0(x), & a<x\leqslant b, \end{cases}$$

where $\rho_0(x)>0$ for $a<x\leqslant b$. We also let $x=a(t)$ and $x=b(t)$ be the two particle paths from $x=a$ and $x=b$, respectively. Then we will study the behaviour of solutions between $x=a(t)$ and $x=b(t)$. We will show later that regular solutions exist locally in time, and so these two particle paths are well defined.

Now in order not to consider the free boundaries $x=a(t)$ and $x=b(t)$, we consider the Lagrangian coordinates by the transformation

$$y=\int_{a(t)}^{x} \rho(s,t)\,ds, \qquad \tau=t, \quad a(t)\leqslant x\leqslant b(t).$$

After the total mass between $x=a(t)$ and $x=b(t)$ is normalized to be 1, the system (3.1) can be written (for simplicity in notations, we still use (x, t) for (y, τ)) as

$$\begin{aligned} &v_t-u_x=0,\\ &u_t+p(v)_x=-u, \qquad 0\leqslant x\leqslant 1, \quad t\geqslant 0, \end{aligned} \tag{3.2}$$

where $p(v)=k^2v^{-\gamma}$, $v=1/\rho$ is the specific volume. We consider (3.2) with initial data

$$(u(x,0), v(x,0))=(u_0(x), v_0(x)), \qquad 0<x\leqslant 1, \tag{3.3}$$

where $\lim_{x\to\infty} v_0(x)=\infty$.

Equations (3.2) can be rewritten as a symmetric system in the variables $\phi(v)=\int_v^\infty \sqrt{-p'(s)}\,ds=(2k\sqrt{\gamma}/(\gamma-1))\,v^{-(\gamma-1)/2)}$ and $\mu=\sqrt{-p'(v)}=k\sqrt{\gamma}\,v^{-(\gamma+1)/2}$:

$$\begin{aligned} &\phi_t+\mu u_x=0,\\ &u_t+\mu\phi_x=-u. \end{aligned} \tag{3.4}$$

With $\xi=x^q$, (3.4) becomes

$$\begin{aligned} &\phi_t+q\xi^{(q-1)/q}\mu u_\xi=0,\\ &u_t+q\xi^{(q-1)/q}\mu\phi_\xi=-u. \end{aligned} \tag{3.5}$$

We will require the initial value $\phi(\xi,0)$ to be Hölder continuous at $x=0$. For this we set

$$\phi(\xi,t)=\xi^\beta\eta(\xi,t), \qquad u=c_1e^{-t}+\xi^\beta\zeta(\xi,t),$$

and the system for (η,ζ) is

$$\begin{aligned} &\eta_t+d\xi^{(\gamma+1)/(\gamma-1)\beta-1/q}\eta^{(\gamma+1)/(\gamma-1)}(\beta\zeta+\xi\zeta_\xi)=0,\\ &\zeta_t+d\xi^{(\gamma+1)/(\gamma-1)\beta-1/q}\eta^{(\gamma+1)/(\gamma-1)}(\beta\eta+\xi\eta_\xi)=-\zeta. \end{aligned} \tag{3.6}$$

Here $d=q((\gamma-1)/2)^{(\gamma+1)/(\gamma-1)}\,(k\sqrt{\gamma})^{-2/(\gamma-1)}$, and c_1 is a constant, and β and q are constants to be determined later.

Corresponding to dc^{α}/dx, $0<\alpha\leqslant 1$, bounded away from zero in the Euclerian coordinates, we have $d\phi^{\alpha}/dx$ bounded away from zero in the Lagrangian coordinates, where $2/(\gamma-1)<\bar{\alpha}\leqslant(\gamma+1)/(\gamma-1)$. Depending on the power of ξ in (3.6), we have the following two cases:

Case 1. When $\beta q=(\gamma-1)/(\gamma+1)$, (3.6) becomes

$$\begin{aligned} &\eta_t+d\xi\,\eta^{(\gamma+1)/(\gamma-1)}\zeta_{\xi}=-\beta\,d\eta^{(\gamma+1)/(\gamma-1)}\zeta,\\ &\zeta_t+d\xi\,\eta^{(\gamma+1)/(\gamma-1)}\eta_{\xi}=-\beta\,d\eta^{2\gamma/(\gamma-1)}-\zeta. \end{aligned} \tag{3.7}$$

Without loss of generality, we can choose $q=1$ and $\beta=(\gamma-1)/(\gamma+1)$. Now the system (3.7) is a symmetric quasilinear hyperbolic equations and we can apply Kato's Theorem [1] so long as $\eta(\xi,t)>0$. Therefore, we have the following lemma.

LEMMA 3.1. *Suppose that for a nonnegative integer m and a positive constant $\bar{\eta}$, the initial data $(\eta_0(\xi),\zeta_0(\xi))\in H^{2+m}([0,1])$, and $\eta_0(x)>\bar{\eta}>0$ for $0\leqslant x\leqslant 1$. Then there exists a unique solution of* (3.7) *such that*

$$(\eta(\xi,t),\zeta(\xi,t))\in C^0([0,t_1],H^{2+m}([0,1]))\cap C^1([0,t_1],H^{1+m}([0,1])),$$

$$\eta(\xi,t)>\bar{\bar{\eta}}>0,\qquad 0\leqslant x\leqslant 1,\qquad 0<t\leqslant t_1,$$

for some positive constants t_1 and $\bar{\bar{\eta}}<\bar{\eta}$.

Case 2. When $(\gamma-1)/(\gamma+1)<\beta q<(\gamma-1)/2$, we set $\theta=\beta q$ and

$$q=\frac{(\gamma+1)\,\theta-(\gamma-1)}{\gamma-1}>0,\qquad \beta=\frac{(\gamma-1)\,\theta}{(\gamma+1)\,\theta-(\gamma-1)}.$$

Then the system (3.6) becomes

$$\begin{aligned} &\eta_t+d\xi^2\,\eta^{(\gamma+1)/(\gamma-1)}\zeta_{\xi}=-\beta\,d\xi\,\eta^{(\gamma+1)/(\gamma-1)}\zeta,\\ &\zeta_t+d\xi^2\,\eta^{(\gamma+1)/(\gamma-1)}\zeta_{\xi}=-\beta\,d\xi\,\eta^{2\gamma/(\gamma-1)}-\zeta. \end{aligned} \tag{3.8}$$

Equations (3.8) are again a symmetric quasilinear hyperbolic system satisfying the conditions in [1] provided that $\eta(\xi,t)>0$. Thus by Kato's Theorem [1] Lemma 3.1 also holds for this case.

With Lemma 3.1 and the transformation just made, we can now prove the main theorem.

THEOREM 3.1. *For the system* (3.2) *with initial data* (3.3) *and any* $(\gamma-1)/(\gamma+1) \leqslant \theta < (\gamma-1)/2$, *there exist solutions of the following form locally in time*;

$$\phi(x, t) = x^{\theta}\eta(x^q, t), \quad u(x, t) = c_1 e^{-t} + x^{\theta}\zeta(x_q, t), \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant t_0. \tag{3.9}$$

Here $(\eta(\xi, t), \zeta(\xi, t))$ *satisfies Lemma* 3.1 *and* c_1 *is any constant.*

Proof. By Lemma 3.1, we have

$$|\eta(\xi, t)|_{c^1}, \qquad |\zeta(\xi, t)|_{c^1} \leqslant M,$$
$$\eta(\xi, t) > 0, \qquad 0 \leqslant t \leqslant t_1, \qquad 0 \leqslant \xi \leqslant 1,$$

where M is a positive constant. It remains to show that the particle paths thus obtained will not collapse before $t = t_0$, where $0 < t_0 \leqslant t_1$. Let $y(z, t)$ be a particle path in the Eulerian coordinates with $y(0) = z$, and corresponding to this path $x = \int_{a(t)}^{y(t)} \rho(s, t)\, ds$. By the construction of the solution, we have

$$y(z, t) = z + \int_0^t (c_1 e^{-t} + x^{\theta}\zeta(x^q, s))\, ds,$$

in the $y - t$ plane. Hence

$$\frac{\partial y(z, t)}{\partial z} = 1 + \int_0^t (\zeta + q x^q \zeta_{\xi}) \frac{\partial x}{\partial y} x^{\theta - 1}\, ds$$
$$= 1 + \int_0^t (\zeta + x^q \zeta_{\xi})\, \rho(x^q, t)\, x^{\theta - 1}\, ds.$$

Since $\phi(x^q, t) = 2k\sqrt{\gamma}/(\gamma-1)\, v^{-(\gamma-1)/2} = x^{\theta}\eta(\xi, t)$, we have $\rho(x^q, t) = ((\gamma-1)/2k\sqrt{\gamma})^{2/(\gamma-1)}\, x^{(2/(\gamma-1))\,\theta}\, \eta^{2/(\gamma-1)}(\xi, t)$. Hence

$$\frac{\partial y(z, t)}{\partial z} = 1 + \left(\frac{\gamma - 1}{2k\sqrt{\gamma}}\right)^{2/(\gamma-1)} \int_0^t (\zeta + q x^q \zeta_{\xi})\, x^{((\gamma+1)/(\gamma-1))\,\theta - 1} \eta^{2/(\gamma-1)}(\xi, s)\, ds.$$

Now from $(\gamma-1)/(\gamma+1) \leqslant \theta < (\gamma-1)/2$, $q > 0$ and $0 \leqslant x \leqslant 1$, we have

$$\frac{\partial y(z, t)}{\partial z} > 0 \qquad \text{for} \quad 0 \leqslant t \leqslant t_0, \tag{3.10}$$

where $t_0 = \min\{t_1, (2k\sqrt{\gamma})^{2/(\gamma-1)}/2(\gamma-1)^{2/(\gamma-1)}\, (1+q)\, M^{(\gamma+1)/(\gamma-1)}\}$. (3.10) implies that the particle pathes will not collapse before $t = t_0$. Thus the solutions constructed by (3.9) exist locally in time. ▮

Remark 3.1. In case 1, we have $\phi(x,t)=x^{(\gamma-1)/(\gamma+1)}\eta(x,t)$ since $q=1$. Thus $d\phi^{(\gamma+1)/(\gamma-1)}/dx$ is bounded away from zero near $x=0$ before $t=t_0$ because $\eta(x,t)>0$ there. This case corresponds to the one where dc/dx is bounded away from zero near the boundary Γ in Eulerian coordinates.

Remark 3.2. In Case 2, we have $\phi(x,t)=x^{\theta}\eta(x^q,t)$, and so

$$\frac{d}{dx}((\phi(x,t))^{1/\theta})=\eta^{1/\theta}(x^q,t)+q\eta^{1/\theta-1}\eta_{\xi}\xi.$$

From Lemma 3.1 we know that $\eta(\xi,t)\in H^2([0,1])$, and thus $(d/dx)(\phi(x,t))^{1/\theta}$ is bounded away from zero near $x=0$. When $(\gamma-1)/(\gamma+1)<\theta<(\gamma-1)/2$, this case corresponds to the one where dc^{α}/dx is bounded away from zero near the boundary Γ for $0<\alpha<1$ in Eulerian coordinates.

Therefore, Theorem 3.1 shows that if initially we have dc^{α}/dx bounded away from zero near the boundary Γ for $0<\alpha\leqslant 1$, then this kind of phenomenan will remain for a short time. However, by Theorem 2.1 these are not the general behaviour after finite time; the important problem of how α changes after finite time remains open.

Remark 3.3. The above argument does not hold for the case where $0<\theta<(\gamma-1)/(\gamma+1)$, i.e., $\alpha>1$, since in this case q will become negative if we require $((\gamma+1)/(\gamma-1))\,\beta-1/q$ to be a nonnegative integer. But $q<0$ implies that the interval $(0,1]$ is mapped to $[1,\infty)$, and the coefficients of the system will not be unbounded due to the terms containing ξ to a positive power. Thus the energy method cannot be applied to get a local existence theorem. Furthermore, even when solution (η,ζ) exists locally, the particle paths will collapse immediately near $x=0$, as shown in the proof of Theorem 3.1, because $\theta<(\gamma-1)/(\gamma+1)$. Thus, this implies that the setting (3.9) is not valid for $\alpha>1$.

3.2. *Characteristics Method*

In this subsection, we are going to use the characteristics method to prove a theorem in Eulerian coordinates, which is the same as Theorem 3.1. Firstly, we will show that when dc^{α}/dx, $0<\alpha\leqslant 1$, is smooth across Γ, our system satisfies the condition of Theorem 4.1 of Chap. 1 in [4]. Then the local existence theorem for a class of quasilinear hyperbolic systems in [4] can be applied. Furthermore, for the behaviour of the solutions, we will show that the solutions thus obtained have the property that dc^{α}/dx is bounded away from zero locally in time if initial data has this kind of property.

Now consider system (3.1) with the initial data

$$(c(x,0),u(x,0))=(c_0(x),u_0(x)) \tag{3.11}$$

where $c = k\sqrt{\gamma}\,\rho^{(\gamma-1)/2}$ and $c_0(x) \equiv 0$ for $x \leqslant a$ and $x \geqslant b$, where $a < b$ are two constants. Folllowing is the main theorem in this subsection.

THEOREM 3.2. *Consider the system* (3.1) *and the initial data* (3.11). *Suppose that*

$$(c_0(x), u_0(x)) \in C^1([a, b]),$$
$$|(c_0^\alpha(x))_x|, \quad |c_0^{\alpha-1} u_{0x}(x)| < A, \qquad 0 < \alpha \leqslant 1, \tag{3.12}$$

for some positive constants α, A *and for* $0 < \alpha \leqslant 1$. *Then for some positive constant* T_0 *depending on the initial data, there is a unique solution* $(c(x, t), u(x, t))$ *for* $a(t) \leqslant x \leqslant b(t)$, $0 \leqslant t \leqslant T_0$. *Moreover* $(c(x, t), u(x, t))$ *satisfies*

$$(c(x, t), u(x, t)) \in C^1([a(t), b(t)] \times [0, T_0]), \tag{3.13}$$
$$|(c^\alpha(x, t))_x|, \qquad |c^{\alpha-1}(x, t)\, u_x(x, t)| < G_0 A, \tag{3.14}$$

where G_0 *is a positive constant depending on* T_0 *and where* $x = a(t)$ *and* $x = b(t)$ *are the two particle paths from* $x = a$ *and* $x = b$.

Proof. In [4], a local existence theorem was given for a class of quasilinear hyperbolic systems. Here we show that our system satisfies the conditions of Theorem 4.1 in [4] and the solutions have the property (3.14).

Firstly, we rewrite the system (3.1) as

$$c_t + uc_x + \frac{\gamma-1}{2} cu_x = 0,$$
$$u_t + uu_x + \frac{2}{\gamma-1} cc_x = -u. \tag{3.15}$$

The characteristics of this system are $\lambda_1 = u - c$, $\lambda_2 = u + c$, and the corresponding Riemann invariants are $r = u - 2/(\gamma-1)\, c$, $s = u + 2/(\gamma-1)\, c$. The equations for (r, s) are

$$r_t + \lambda_1 r_x = -u,$$
$$s_t + \lambda_2 s_x = -u. \tag{3.16}$$

Under the condition that the solution $(c(x, t), u(x, t)) \in C^1([a(t), b(t)])$, the particle paths $x = a(t)$ and $x = b(t)$ are well defined once the initial data are given. Thus when we use interation to prove local existence theory, we need not consider the free boundary in this case. But as was shown in Section 2, after a finite time the boundary will move due to the effect of the pressure. Then in the general case, the free boundary problem must be considered.

By assuming $(c(x, t), u(x, t)) \in C^1([a(t), b(t)])$, and $c(a(t), t) = c(b(t), t) = 0$, when $0 \leqslant t \leqslant T_0$, we have

$$\begin{aligned} a(t) &= a + u_0(a)(1 - e^{-t}), \\ b(t) &= b + u_0(b)(1 - e^{-t}). \end{aligned} \tag{3.17}$$

And it is easy to see that

$$\Omega(T_0) = \{(x, t) \mid a(t) \leqslant x \leqslant b(t), 0 \leqslant t \leqslant T_0\}$$

is the strong dependence domain of $\{(x, t) \mid a \leqslant x \leqslant b, t = 0\}$ and $\partial\lambda_1/\partial r = \partial\lambda_2/\partial s = (\gamma+1)/2$, $\partial\lambda_1/\partial s = \partial\lambda_2/\partial r = (3-\gamma)/4$, $\partial u = \partial r = \partial u/\partial s = \frac{1}{2}$. Thus Theorem 4.1 in [4] can be applied to system (3.16) and the conclusion of (3.13) follows.

Now we are going to prove the solution satisfies (3.14). For any point $(\xi, 0)$, $a \leqslant \xi \leqslant b$, we define the two characteristics lines through $(\xi, 0)$ as

$$\begin{aligned} &\frac{\partial x_i(\xi, t)}{\partial t} = \lambda_i(u(x_i(\xi, t), t), c(x_i(\xi, t), t)), \\ &x(\xi, 0) = \xi, \qquad i = 1, 2. \end{aligned} \tag{3.18}$$

Thus

$$\frac{\partial}{\partial t}\left(\frac{\partial x_i(\xi, t)}{\partial \xi}\right) = \frac{\partial \lambda_i(x_i(\xi, t), t)}{\partial x_i}\frac{\partial x_i}{\partial \xi},$$

i.e.,

$$\frac{\partial x_i(\xi, t)}{\partial \xi} = \exp\left(\int_{ol_\xi^i}^{t} \frac{\partial \lambda_i(x_i(\xi, t), t)}{\partial x_i}\, ds\right), \tag{3.19}$$

where $l_\xi^i = \{(x, t) \mid x = x_i(\xi, t), 0 \leqslant t \leqslant T_0\}$, $i = 1, 2$. By integrating Eqs. (3.16), we have

$$\begin{aligned} r(x_1(\xi_1, t), t) &= r_0(\xi_1, 0) - \int_{ol_{\xi_1}^1}^{t} u(x_1(\xi_1, \tau), \tau)\, d\tau, \\ s(x_2(\xi_2, t), t) &= s_0(\xi_2, 0) - \int_{ol_{\xi_2}^2}^{t} u(x_2(\xi_2, \tau), \tau)\, d\tau. \end{aligned} \tag{3.20}$$

Taking derivatives with respect to x on both sides yields

$$\begin{aligned}
\frac{\partial r(x_1(\xi_1, t), t)}{\partial x_1} &= r_\xi(\xi_1, 0) \exp\left(-\int_{ol^1_{\xi_1}}^t \frac{\partial \lambda_1(x_1(\xi_1, s), s)}{\partial x_1}\, ds\right), \\
&\quad -\int_{ol^1_{\xi_1}}^t \frac{\partial u(x_1(\xi_1, \tau), \tau)}{\partial x_1} \exp\left(-\int_{\tau l^1_{\xi_1}}^t \frac{\partial \lambda_1(x_1(\xi, s), s)}{\partial x_1}\, ds\right) d\tau, \\
\frac{\partial s(x_2(\xi_2, t), t)}{\partial x_2} &= s_\xi(\xi_2, 0) \exp\left(-\int_{ol^2_{\xi_2}}^t \frac{\partial \lambda_2(x_2(\xi_2, s), s)}{\partial x_2}\, ds\right), \\
&\quad -\int_{ol^2_{\xi_2}}^t \frac{\partial u(x_2(\xi_2, \tau), \tau)}{\partial x_2} \exp\left(-\int_{\tau l^2_{\xi_2}}^t \frac{\partial \lambda_2(x_2(\xi, s), s)}{\partial x_2}\, ds\right) d\tau.
\end{aligned} \tag{3.21}$$

Now for any $(x, t) \in [a(t), b(t)] \times [0, T_0]$, we let $x = x_1(\xi_1, t) = x_2(\xi, t)$. Then by (3.21), we have

$$\begin{aligned}
u_x(x, t) &= \frac{1}{2}\left\{ s_\xi(\xi_2, 0) \exp\left(-\int_{ol^2_{\xi_2}}^t (u_x + c_x)\, ds\right)\right. \\
&\quad + r_\xi(\xi_1, 0) \exp\left(-\int_{ol^1_{\xi_1}}^t (u_x - c_x)\, ds\right) \\
&\quad - \int_{ol^1_{\xi_1}}^t \frac{\partial u(x_1(\xi_1, \tau), \tau)}{\partial x_1} \exp\left(-\int_{\tau l^1_{\xi_1}}^t (u_x - c_x)\, ds\right) d\tau \\
&\quad \left. - \int_{ol^2_{\xi_2}}^t \frac{\partial u(x_2(\xi_2, \tau), \tau)}{\partial x_2} \exp\left(-\int_{\tau l^2_{\xi_2}}^t (u_x + c_x)\, ds\right) d\tau \right\}, \\
c_x(x, t) &= \frac{\gamma - 1}{4}\left\{ s_\xi(\xi_2, 0) \exp\left(-\int_{ol^2_{\xi_2}}^t (u_x + c_x)\, ds\right)\right. \\
&\quad - r_\xi(\xi_1, 0) \exp\left(-\int_{ol^1_{\xi_1}}^t (u_x - c_x)\, ds\right) \\
&\quad + \int_{ol^1_{\xi_1}}^t \frac{\partial u(x_1(\xi_1, \tau), \tau)}{\partial x_1} \exp\left(-\int_{\tau l^1_{\xi_1}}^t (u_x - c_x)\, ds\right) d\tau \\
&\quad \left. - \int_{ol^2_{\xi_2}}^t \frac{\partial u(x_2(\xi_2, \tau), \tau)}{\partial x_2} \exp\left(-\int_{\tau l^2_{\xi_2}}^t (u_x + c_x)\, ds\right) d\tau \right\}.
\end{aligned} \tag{3.22}$$

Since $c_t + uc_x + ((\gamma - 1)/2)\, cu_x = 0$,

$$\frac{D \ln c}{D_1 t} = -c_x - \frac{\gamma - 1}{2} u_x, \qquad \frac{D \ln c}{D_2 t} = c_x - \frac{\gamma - 1}{2} u_x,$$

where $D/D_1 t = \partial/\partial t + \lambda_1(\partial/\partial x)$, $D/D_2 t = \partial/\partial t + \lambda_2(\partial/\partial x)$. Thus

$$\exp\left(-\int_{\tau l^1_{\xi_1}}^{t} (u_x - c_x)\, ds\right)$$
$$= \frac{c^{\alpha-1}(x_1(\xi_1, \tau), \tau)}{c^{\alpha-1}(x, t)} \exp\left(\int_{\tau l^1_{\xi_1}}^{t} \left((2-\alpha)\, c_x + \left((1-\alpha)\frac{\gamma-1}{2} - 1\right) u_x\right) ds\right),$$
$$\exp\left(-\int_{\tau l^2_{\xi_2}}^{t} (u_x + c_x)\, ds\right)$$
$$= \frac{c^{\alpha-1}(x_2(\xi_2, \tau), \tau)}{c^{\alpha-1}(x, t)} \exp\left(\int_{\tau l^2_{\xi_2}}^{t} \left((\alpha-2)\, c_x + \left((1-\alpha)\frac{\gamma-1}{2} - 1\right) u_x\right) ds\right). \tag{3.23}$$

Let

$$\Theta(t) = \sup_{a(t) \leqslant x \leqslant b(t)} \{|(c^\alpha(x, t))_x|, |c^{\alpha-1} u_x(x, t)|\}, \qquad 0 \leqslant t \leqslant T_0.$$

Combining (3.22) and (3.23), we have

$$\Theta(t) \leqslant G_1 \left(A + \int_0^t \Theta(\tau)\, d\tau\right),$$

i.e.,

$$\Theta(t) \leqslant G_1 A e^{G_1 t}, \qquad 0 \leqslant t \leqslant T_0.$$

Therefore by letting $G_0 = G_1 e^{G_1 T_0}$, we obtain the property (3.13). ∎

Remark 3.4. Actually, the conclusion of Theorem 3.2 can be generalized to the case where $(c_0, u_0) \in C^m([a, b])$, $m \geqslant 1$. In this case we can prove that solutions $(c(x, t), u(x, t))$ exist locally in time and $(c(x, t), u(x, t)) \in C^m([a(t), b(t)])$ under the same conditions as in Theorem 3.2. Therefore, if we consider a solution in $C^2([a(t), b(t)])$ with property (3.14) initially, we have

$$(c^\alpha)_{xt} + u(c^\alpha)_{xx} + \frac{\gamma-1}{2}(c^\alpha)_x\, u_x + \frac{\gamma-1}{2}\, c^\alpha u_{xx} = 0, \qquad 0 < \alpha \leqslant 1.$$

Hence along the boundary $x = a(t)$ and $x = b(t)$, we have

$$(c^\alpha)_{xt} + u(c^\alpha)_{xx} + \frac{\gamma-1}{2}(c^\alpha)_x\, u_x = 0.$$

This implies that if $(c^{\alpha})_x$ is bounded away from zero near $x = a$ and $x = b$ initially, this property will remain locally in time.

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