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Optimal Decay Estimates on the Linearized Boltzmann Equation with Time Dependent Force and their Applications

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Abstract: Although the decay in time estimates of the semi-group generated by the linearized Boltzmann operator without forcing have been well established, there is no corresponding result for the case with general external force. This paper is mainly concerned with the optimal decay estimates on the solution operator in some weighted Sobolev spaces for the linearized Boltzmann equation with a time dependent external force. No time decay assumption is made on the force. The proof is based on both the energy method through the macro-micro decomposition and the L^p-L^q estimates from the spectral analysis. The decay estimates thus obtained are applied to the study on the global existence of the Cauchy problem to the nonlinear Boltzmann equation with time dependent external force and source. Precisely, for space dimension $n \geq 3$, the global existence and decay rates of solutions to the Cauchy problem are obtained under the condition that the force and source decay in time with some rates. This time decay restriction can be removed for space dimension $n \geq 5$. Moreover, the existence and asymptotic stability of the time periodic solution are given for space dimension $n \geq 5$ when the force and source are time periodic with the same period.

Contents

1. Introduction

The Boltzmann equation for the hard-sphere gas in n-dimensional space under the influence of an external force and a source takes the form

$$\partial_t f + \xi \cdot \nabla_x f + F \cdot \nabla_\xi f = Q(f, f) + S. \tag{1.1}$$

Here, the unknown function $f = f(t, x, \xi)$ with $(t, x, \xi) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ is a non-negative function standing for the number density of gas particles which have position $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and velocity $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ at time $t \in \mathbb{R}$. Here, the external force field $F = F(t, x)$ and the source term $S = S(t, x, \xi)$ are assumed to be some given time dependent functions. Q is the usual bilinear collision operator defined by

$$\begin{aligned} Q(f, g) &= \frac{1}{2} \int_{\mathbb{R}^n \times S^{n-1}} (f' g'_* + f'_* g' - f g_* - f_* g) |(\xi - \xi_*) \cdot \omega| d\omega d\xi_*, \\ f &= f(t, x, \xi), \quad f' = f(t, x, \xi'), \quad f_* = f(t, x, \xi_*), \quad f'_* = f(t, x, \xi'_*), \\ \xi' &= \xi - [(\xi - \xi_*) \cdot \omega]\omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega, \quad \omega \in S^{n-1}, \end{aligned}$$

and likewise for g. Although the physical space is three dimensional, in this paper, we consider the general space dimension $n \geq 3$ to show how the space dimension plays in the decay estimates.

Throughout this paper, we consider the perturbative solution near an absolute Maxwellian. Without loss of generality, define the perturbation $u = u(t, x, \xi)$ by

$$f = \mathbf{M} + \mathbf{M}^{1/2} u,$$

where the absolute Maxwellian

$$\mathbf{M} = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{|\xi|^2}{2}\right)$$

is normalized to have zero bulk velocity and unit density and temperature. Then the equation for the perturbation u is:

$$\partial_t u + \xi \cdot \nabla_x u + F \cdot \nabla_\xi u - \frac{1}{2} \xi \cdot F u = \mathbf{L} u + \Gamma(u) + \widetilde{S}, \tag{1.2}$$

where

$$\mathbf{L} u = \mathbf{M}^{-1/2} \Big(Q(\mathbf{M}, \mathbf{M}^{1/2} u) + Q(\mathbf{M}^{1/2} u, \mathbf{M}) \Big), \tag{1.3}$$

$$\Gamma(u, u) = \mathbf{M}^{-1/2} Q \left(\mathbf{M}^{1/2} u, \mathbf{M}^{1/2} u \right), \tag{1.4}$$

$$\widetilde{S} = \mathbf{M}^{-1/2} S + \mathbf{M}^{1/2} \xi \cdot F. \tag{1.5}$$

There are extensive literatures on the existence theory for the Cauchy problem of the Boltzmann equation without external force. The well-known result is the global existence of the renormalized solution with large data proved by DiPerna-Lions [6] where the uniqueness problem remains open. On the other hand, the existence is established in the framework of small perturbation of an absolute Maxwellian [12–14, 17, 19, 21, 23, 24, 29], or an infinite vacuum [2, 9, 15, 16] where uniqueness can be justified. In particular, so far there are two basic methods to deal with solutions near an absolute Maxwellian. One is based on the spectral analysis of the linearized Boltzmann equation and the bootstrap argument for the nonlinear equation initiated by Grad and developed by Ukai, cf. [19,23–25], where the optimal convergence rate to the Maxwellian can be also obtained. Another one is based on the direct energy method for the nonlinear problem through the macro-micro decomposition which was initiated by Liu-Yu and developed by Liu-Yang-Yu [17] and Guo [13] independently in two different ways. The former decomposition is around a local Maxwellian while the latter is around an absolute Maxwellian. Here we use the latter decomposition because we are concerned with the decay structure of the linearized equation around the absolute Maxwellian.

One of the features of the convergence to the equilibrium for the Boltzmann equation is the coupling of the conservative operator for the free transportation and the degenerate dissipative operator on the velocity variables through the celebrated H-theorem. This property can be found in many kinetic equations and it is now called “hypocoercivity” [32]. For the problems in a torus or in a bounded domain, this property is well investigated where an exponential or almost exponential convergence rate in time to the equilibrium for both space and velocity variables can be obtained, cf. [33] and references therein. However, for problems in the whole space, this property is not yet well understood especially under the influence of some enternal force. And this is one of the motivations of this paper to study the convergence to the equilibrium under the influence of the external force in a general form.

To do this, the main part of the paper is concentrated on the decay in time properties of the solution operator for the linearized Boltzmann equation corresponding to (1.2), that is,

$$\partial_t u + \xi \cdot \nabla_x u + F \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot F u = \mathbf{L}u.$$

The decay estimates are obtained in some Sobolev space weighted in velocity variables. Our main result is stated in Theorem 2.2 in Sect. 2, where the obtained decay is optimal in the sense that it is equal to the one for the linearized Boltzmann equation without external force. The proof is a combination of the two methods mentioned above for perturbative solutions. In fact, the energy estimate is first carried out for the linearized Boltzmann equation with an error term determined by the space derivative of the macroscopic component in the perturbation. It is then combined with the L^p-L^q estimates from the spectral analysis to yield the optimal decay in time estimates for the above linear solution operator.

The optimal decay estimates on the solution operator to the linearized equation will then be applied to the study on the existence of solutions to the Cauchy problem for the original nonlinear equation. In particular, we will use it to prove the existence and stability of the time periodic solution for some given time periodic force and source. This problem is related to the generation and propagation of sound waves so that it has its physical importance besides its mathematical interest. In fact, for the time periodic solution, the existence and stability have been studied for the Navier-Stokes equaions, cf. [1,10,30,31] and references therein. Recently, some results on this problem are obtained for the nonlinear Boltzmann equation [26–28] in various function spaces when there is a time periodic external source but no external force, for the space dimension $n \geq 3$. Thus, it is natural to study the problem under the influence of a time periodic external force. We will show that there exists a time periodic solution if the force is small and time periodic when the space dimension $n \geq 5$. The physical case when the space dimension $n = 3$ is still not known and will be pursued by the authors in the future.

A lot of work has been done on the convergence rate estimation of the solutions for the Boltzmann equation to the time asymptotic states. For example, the almost exponential decay in time of the solution for the Cauchy problem was given by Desvillettes-Villani [5] for general cutoff potential cases in either torus or smooth bounded domain under the assumption of the existence of smooth global solutions, and also by Strain-Guo [22] for the cutoff soft potentials in the torus for small pertubation of the absolute Maxwellian. Notice that the convergence rate of the perturbative solution for the cutoff hard potentials is exponential in a torus, [23]. For problems in the whole space, the convergence rate should be algebraic and it depends on the space dimension because the low frequency in the Fourier variable dominates the decay estimate, see [24,25]. For the Boltzmann equation with a time independent potential force, the optimal convergence rate of the solution to a local Maxwellian was obtained in [8], where the proof is motivated by the study of the corresponding problems for the Navier-Stokes equations, cf. [7,18,20].

The rest of this paper is arranged as follows. In Sect. 2, we will first present a decomposition of the linearized Boltzmann equation. Then, some basic estimates on the communicators of the linearized collision operator $\mathbf{L}$ and the differential operator will be derived. Based on these estimates, the optimal decay in time estimates on the linear solution operator are proved in Theorems 2.1 and 2.2. In Sect. 3, we will apply the estimates obtained in Sect. 2 to prove the global existence and convergence rate of the solution to the Cauchy problem for the nonlinear Boltzmann equation. In addition, the existence and asymptotic stability of the time periodic solution are also given. These existence and stability results are summarized in Theorems 3.1, 3.2 and 3.3.

Notation. Throughout this paper, C denotes a general constant. If the dependence needs to be specified, then the notations C_i, $i = 1, 2, \cdots$ are used. In addition, $c > 0$ also denotes a positive constant which may vary from line to line and $\delta > 0$ stands for a small constant. $\langle \cdot, \cdot \rangle$ is the inner product in the space $L^2(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ with the norm denoted by $\|\cdot\|$. Sometimes, $\|\cdot\|$ also denotes the norm of the space $L^2(\mathbb{R}^n_x)$ without any ambiguity. $\|\cdot\|_{L^p_{x,\xi}}$ with $1 \le p \le \infty$ denotes the norm in the Lebesgue space $L^p(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$. The norm in the space $Z_q = L^2_\xi(L^q_x)$ is defined by

$$\|u\|_{Z_q} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |u(x,\xi)|^q dx \right)^{\frac{2}{q}} d\xi \right)^{\frac{1}{2}}, \quad u = u(x,\xi) \in Z_q.$$

For the multiple indices α, β, γ with $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$, and $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, we adopt the usual notations $\partial_x^\beta \partial_\xi^\gamma = \partial_{x_1}^{\beta_1} \partial_{x_2}^{\beta_2} \cdots \partial_{x_n}^{\beta_n} \partial_{\xi_1}^{\gamma_1} \partial_{\xi_2}^{\gamma_2} \cdots \partial_{\xi_n}^{\gamma_n}$, and in particular $\partial^\alpha_{x,\xi} = \partial_x^\beta \partial_\xi^\gamma$ when $\alpha = \beta + \gamma$. The length of α is $|\alpha| = \sum_{i=1}^n \alpha_i$.

2. Decay Estimates on the Linearized Equation

2.1. Preliminaries.

(i) Linearized equation. In this section, we are concerned with the initial value problem for the linearized Boltzmann equation corresponding to (1.1). More generally, for some initial time $s \in \mathbb{R}$, it is in the form

$$\partial_t u + \xi \cdot \nabla_x u + E_1 \cdot \nabla_\xi u = \mathbf{L}u + \xi \cdot E_2 u, \quad t > s, \ x \in \mathbb{R}^n, \ \xi \in \mathbb{R}^n, \tag{2.1}$$

$$u(t,x,\xi)|_{t=s} = u_0(x,\xi), \quad x \in \mathbb{R}^n, \ \xi \in \mathbb{R}^n. \tag{2.2}$$

Here $u_0(x,\xi)$ is given, denoting the same initial data for different initial time, and $E_i = E_i(t,x)$, $i = 1,2$, are given vector-valued functions for generalization. Formally the solution to the initial value problem (2.1)–(2.2) is written as

$$U(t,s)u_0, \quad -\infty < s \le t < \infty,$$

where $U(t,s)$ is called the solution operator for the linear Eq. (2.1). We shall obtain some basic decay in time estimates on $U(t,s)$ in some Sobolev space weighted with velocity functions

$$H^\ell\left(\mathbb{R}^n_x \times \mathbb{R}^n_\xi; (1+|\xi|)^k dx d\xi\right), \quad \ell \ge 2,\ k \ge 1,$$

which enable us to solve the nonlinear problem by the Duhamel formula and the contraction mapping theorem.

(ii) *Known properties of the linearized collision operator.* For the linearized collision operator $\mathbf{L}$ given by (1.3), one has

$$\begin{aligned}
(\mathbf{L}u)(\xi) &= -\nu(\xi)u(\xi) + (Ku)(\xi),\\
\nu(\xi) &= \int_{\mathbb{R}^n\times S^{n-1}} |(\xi-\xi_*)\cdot\omega|\mathbf{M}_*\, d\omega d\xi_*,\\
(Ku)(\xi) &= \int_{\mathbb{R}^n\times S^{n-1}} \left[-\mathbf{M}^{\frac12}u_* + (\mathbf{M}'_*)^{\frac12}u' + (\mathbf{M}')^{\frac12}u'_*\right]|(\xi-\xi_*)\cdot\omega|\mathbf{M}_*^{\frac12}\, d\omega d\xi_*\\
&= \int_{\mathbb{R}^n} K(\xi,\xi_*)u(\xi_*)d\xi_*.
\end{aligned}$$

Moreover, the following well-known properties hold; see [3,4,11].

(a) There exists $\nu_0 > 0$ such that

$$\nu_0(1+|\xi|) \le \nu(\xi) \le \nu_0^{-1}(1+|\xi|);$$

(b) K is a self-adjoint compact operator on $L^2(\mathbb{R}^n_\xi)$ with a real symmetric integral kernel $K(\xi,\xi_*)$ which enjoys the estimate

$$\int_{\mathbb{R}^n} |K(\xi,\xi_*)|(1+|\xi_*|)^{-\beta}d\xi_* \le C(1+|\xi|)^{-\beta-1}, \quad \beta \ge 0; \tag{2.3}$$

(c) the nullspace of the operator $\mathbf{L}$ is the space of collision invariants

$$\mathcal{N} = Ker\mathbf{L} = \text{span}\left\{\mathbf{M}^{1/2};\ \xi_i\mathbf{M}^{1/2}, i = 1,2,\ldots,n;\ |\xi|^2\mathbf{M}^{1/2}\right\};$$

(d) $\mathbf{L}$ is an unbounded, self-adjoint and non-positive operator on $L^2(\mathbb{R}^n_\xi)$ with the domain

$$D(\mathbf{L}) = \left\{u \in L^2(\mathbb{R}^n_\xi)\ \middle|\ \nu(\xi)u \in L^2(\mathbb{R}^n_\xi)\right\}.$$

(iii) *Macro-micro decomposition.* Define $\mathbf{P}$ as a velocity projection operator from $L^2(\mathbb{R}^n_\xi)$ to $\mathcal{N}$. Then any function $u(t,x,\xi)$ for any fixed (t,x) can be uniquely decomposed as the sum of the macroscopic component $\mathbf{P}u$ and microscopic component $\{\mathbf{I}-\mathbf{P}\}u$:

$$u(t,x,\xi) = \mathbf{P}u + \{\mathbf{I}-\mathbf{P}\}u.$$

With this notion, the linearized collision operator $\mathbf{L}$ satisfies

$$-\int_{\mathbb{R}^n} u\mathbf{L}u\, d\xi \geq c_0 \int_{\mathbb{R}^n} \nu(\xi)(\{\mathbf{I}-\mathbf{P}\}u)^2 d\xi, \quad \forall\, u \in D(\mathbf{L}),$$

for some constant $c_0 > 0$. Here for simplicity, throughout this section, one sets

$$u_1 = \mathbf{P}u, \quad u_2 = \{\mathbf{I}-\mathbf{P}\}u.$$

Equation (2.1) is also decomposed as follows. The microscopic equation for u_2 is obtained by applying the microscopic projection $\mathbf{I}-\mathbf{P}$ to (2.1):

$$\partial_t u_2 - \mathbf{L}u_2 = -\{\mathbf{I}-\mathbf{P}\}\,(\xi \cdot \nabla_x u) - \{\mathbf{I}-\mathbf{P}\}\left(E_1 \cdot \nabla_\xi u - \xi \cdot E_2 u\right),$$

or,

$$\begin{aligned}\partial_t u_2 - \mathbf{L}u_2 = & -\xi \cdot \nabla_x u_2 - E_1 \cdot \nabla_\xi u_2 + \xi \cdot E_2 u_2 \\ & - \xi \cdot \nabla_x u_1 - E_1 \cdot \nabla_\xi u_1 + \xi \cdot E_2 u_1 \\ & + \mathbf{P}\left(\xi \cdot \nabla_x u + E_1 \cdot \nabla_\xi u - \xi \cdot E_2 u\right).\end{aligned} \tag{2.4}$$

In order to write the macroscopic equation, as in [13], one first expands $u_1 = \mathbf{P}u$ as

$$u_1 = \left\{ a(t,x) + \sum_{i=1}^{n} b_i(t,x)\xi_i + c(t,x)|\xi|^2 \right\} \mathbf{M}^{1/2}.$$

Putting this expansion into the following equation:

$$\begin{aligned}&\partial_t u_1 + \xi \cdot \nabla_x u_1 + E_1 \cdot \nabla_\xi u_1 - \xi \cdot E_2 u_1 \\ &\quad = -\left\{\partial_t u_2 + \xi \cdot \nabla_x u_2 + E_1 \cdot \nabla_\xi u_2 - \xi \cdot E_2 u_2 - \mathbf{L}u_2\right\} := \Re,\end{aligned} \tag{2.5}$$

and then collecting the coefficients with respect to the basis

$$\mathbf{M}^{1/2},\ \left(\xi_i \mathbf{M}^{1/2}\right)_{1\leq i\leq n},\ \left(|\xi_i|^2 \mathbf{M}^{1/2}\right)_{1\leq i\leq n},\ \left(\xi_i\xi_j \mathbf{M}^{1/2}\right)_{1\leq i<j\leq n},\ \left(|\xi|^2\xi_i \mathbf{M}^{1/2}\right)_{1\leq i\leq n},$$

one has

$$\mathbf{M}^{1/2}:\ \partial_t a + E_1 \cdot b = \Re_0, \tag{2.6}$$

$$\xi_i \mathbf{M}^{1/2}:\ \partial_t b_i + \partial_i a - (a\bar{E}_i - 2cE_{1i}) = \Re_1^i, \tag{2.7}$$

$$|\xi_i|^2 \mathbf{M}^{1/2}:\ \partial_t c + \partial_i b_i - \bar{E}_i b_i = \Re_{21}^i, \tag{2.8}$$

$$\xi_i\xi_j \mathbf{M}^{1/2}:\ \partial_i b_j + \partial_j b_i - (\bar{E}_i b_j + \bar{E}_j b_i) = \Re_{22}^{ij}, \tag{2.9}$$

$$|\xi|^2\xi_i \mathbf{M}^{1/2}:\ \partial_i c - \bar{E}_i c = \Re_3^i, \tag{2.10}$$

where for simplicity, $\partial_i = \partial_{x_i}$, $\partial_j = \partial_{x_j}$, and $\Re_0$, $\Re_1^i$, $\Re_{21}^i$, $\Re_{22}^{ij}$, $\Re_3^i$ with $1 \leq i \neq j \leq n$ are the corresponding coefficients of $\Re$ with respect to the above basis, and $\bar{E}$ is defined by

$$\bar{E} = \frac{1}{2}E_1 + E_2.$$

Finally we list a basic fact for any function $u = u(t,x,\xi)$.

Proposition 2.1. *Let m be a non-negative integer and k be any number. Then for any β and γ, one has $\partial_t^m \partial_x^\beta \mathbf{P} u = \mathbf{P} \partial_t^m \partial_x^\beta u$ with estimates*

$$\frac{1}{C} \left\| \nu^k \partial_t^m \partial_x^\beta \partial_\xi^\gamma \mathbf{P} u \right\| \leq \left\| \partial_t^m \partial_x^\beta a \right\| + \left\| \partial_t^m \partial_x^\beta b \right\| + \left\| \partial_t^m \partial_x^\beta c \right\| \leq C \left\| \partial_t^m \partial_x^\beta \mathbf{P} u \right\|,$$

where $C > 1$ is some constant independent of u.

2.2. Estimates on commutators. In this subsection we study the functional properties of commutators related to $\mathbf{L}$:

$$[\mathbf{L}, \xi_i], \quad \left[\mathbf{L}, \partial_{\xi_i}\right], \quad \left[\left[\mathbf{L}, \partial_{\xi_i}\right], \xi_j\right], \quad \left[\left[\mathbf{L}, \partial_{\xi_i}\right], \partial_{\xi_j}\right], \quad 1 \leq i, j \leq n.$$

Let $\mathcal{L}$ denote this kind of commutator.

Lemma 2.1. *$\mathcal{L}$ is a bounded linear operator from $L^2(\mathbb{R}^n_\xi)$ to itself, i.e., there is some constant C such that*

$$\|\mathcal{L} u\| \leq C \|u\|, \tag{2.11}$$

for any $u = u(\xi) \in L^2(\mathbb{R}^n_\xi)$.

Proof. This lemma is proved by the following steps.

Step 1. The explicit expressions of ν and K are available:

$$\begin{aligned}
\nu(\xi) &= C_n \int_{\mathbb{R}^n} |\xi - \xi_*| \mathbf{M}(\xi_*) d\xi_*, \\
K(\xi, \xi_*) &= K_1(\xi, \xi_*) + K_2(\xi, \xi_*) \\
K_1(\xi, \xi_*) &= -C_n |\xi - \xi_*| \exp\left(-\frac{|\xi|^2 + |\xi_*|^2}{4}\right), \\
K_2(\xi, \xi_*) &= \frac{C_n}{|\xi - \xi_*|^{n-2}} \exp\left(-\frac{1}{8} \frac{(|\xi|^2 - |\xi_*|^2)^2}{|\xi - \xi_*|^2} - \frac{|\xi - \xi_*|^2}{8}\right),
\end{aligned}$$

where for simplicity C_n may be some different positive constants depending only on the space dimension n. The proof for the case $n = 3$ is given in [11]. The general case $n \geq 3$ can be obtained similarly.

Step 2. In this step, some preparations are made for the next step. First, from (2.13), one can easily verify that $\nu(\xi)$ is a smooth function of ξ with bounded derivatives of any order.

Next, for the integral kernels K_1 and K_2, set

$$\begin{aligned}
K_1(\xi, \xi_*) &= K_{11}(|\xi - \xi_*|) K_{12}(\xi, \xi_*), \\
K_2(\xi, \xi_*) &= K_{21}(|\xi - \xi_*|) K_{22}(\xi, \xi_*),
\end{aligned}$$

where

$$\begin{aligned}
K_{11}(|\xi - \xi_*|) &= -C_n |\xi - \xi_*|, \\
K_{21}(|\xi - \xi_*|) &= \frac{C_n}{|\xi - \xi_*|^{n-2}} \exp\left(-\frac{|\xi - \xi_*|^2}{8}\right), \\
K_{12}(\xi, \xi_*) &= \exp(V_1), \quad V_1 = -\frac{|\xi|^2 + |\xi_*|^2}{4}, \\
K_{22}(\xi, \xi_*) &= \exp(V_2), \quad V_2 = -\frac{1}{8} \frac{(|\xi|^2 - |\xi_*|^2)^2}{|\xi - \xi_*|^2}.
\end{aligned}$$

Finally, for the simplicity of notions, we define velocity differential operators $\bar{\partial}_i, i = 1, 2, \ldots, n$ by $\bar{\partial}_i = -\{\partial_{\xi_i} + \partial_{\xi_{i*}}\}$.
Notice that $\bar{\partial}_i h \equiv 0$ for any radial function $h = h(|\xi - \xi_*|)$, and moreover,

$$\begin{aligned}
\bar{\partial}_i V_1 &= V_{1i}, \quad V_{1i} = \frac{\xi_i + \xi_{i*}}{2},\\
\bar{\partial}_i V_2 &= V_{2i}, \quad V_{2i} = \frac{(\xi_i - \xi_{i*})}{2|\xi - \xi_*|^2}(|\xi|^2 - |\xi_*|^2),\\
\bar{\partial}_j V_{1i} &= \bar{\partial}_j \bar{\partial}_i V_1 = V_{1ij}, \quad V_{1ij} = -\delta_{ij},\\
\bar{\partial}_j V_{2i} &= \bar{\partial}_j \bar{\partial}_i V_2 = V_{2ij}, \quad V_{2ij} = \frac{(\xi_i - \xi_{i*})(\xi_j - \xi_{j*})}{|\xi - \xi_*|^2},
\end{aligned}$$

where δ_{ij} is Kronecker's symbol. Then one has

$$\begin{aligned}
\bar{\partial}_i K_{11} &= \bar{\partial}_i K_{21} \equiv 0,\\
\bar{\partial}_i K_{12} &= K_{12} V_{1i}, \quad \bar{\partial}_i K_{22} = K_{22} V_{2i},\\
\bar{\partial}_j (K_{12} V_{1i}) &= K_{12} V_{1i} V_{1j} + K_{12} V_{1ij},\\
\bar{\partial}_j (K_{22} V_{2i}) &= K_{22} V_{2i} V_{2j} + K_{22} V_{2ij}.
\end{aligned}$$

Step 3. This step is concerned with the computation of commutators. Set $V_{0i} = \xi_{i*} - \xi_i$; direct calculations yield

$$\begin{aligned}
[\mathbf{L}, \xi_i] u &= \int_{\mathbb{R}^n} K(\xi, \xi_*) V_{0i} u(\xi_*) d\xi_*,\\
\left[\mathbf{L}, \partial_{\xi_i}\right] u &= \partial_{\xi_i} \nu u + \int_{\mathbb{R}^n} (K_1 V_{1i} + K_2 V_{2i}) u(\xi_*) d\xi_*,\\
\left[[\mathbf{L}, \partial_{\xi_i}], \xi_j\right] &= \int_{\mathbb{R}^n} (K_1 V_{1i} + K_2 V_{2i}) A_j u(\xi_*) d\xi_*,\\
\left[[\mathbf{L}, \partial_{\xi_i}], \partial_{\xi_j}\right] &= -\partial^2_{\xi_i \xi_j} \nu u + \int_{\mathbb{R}^n} [K_1 (V_{1i} V_{1j} + V_{1ij})\\
&\qquad + K_2 (V_{2i} V_{2j} + V_{2ij})] u(\xi_*) d\xi_*.
\end{aligned}$$

Step 4. Write $K_c(\xi, \xi_*)$ as any one of the following integral kernels:

$$\begin{aligned}
&K V_{0i}, \quad K_1 V_{1i} + K_2 V_{2i}, \quad (K_1 V_{1i} + K_2 V_{2i}) V_{0j},\\
&K_1 (V_{1i} V_{1j} + V_{1ij}) + K_2 (V_{2i} V_{2j} + V_{2ij}).
\end{aligned}$$

Direct observations show that K_1 can absorb any finite numbers of velocity functions V_{0i}, V_{1i} and V_{1ij}, while K_2 can absorb any finite number of velocity functions V_{0i}, V_{2i} and V_{2ij}. This means that if one defines

$$\begin{aligned}
\widetilde{K}_1(\xi, \xi_*) &= C_n |\xi - \xi_*| \exp\left(-\frac{|\xi|^2 + |\xi_*|^2}{8}\right),\\
\widetilde{K}_2(\xi, \xi_*) &= \frac{C_n}{|\xi - \xi_*|^{n-2}} \exp\left(-\frac{1}{16} \frac{(|\xi|^2 - |\xi_*|^2)^2}{|\xi - \xi_*|^2} - \frac{|\xi - \xi_*|^2}{16}\right),
\end{aligned}$$

then

$$|K_c(\xi, \xi_*)| \leq \widetilde{K}_1(\xi, \xi_*) + \widetilde{K}_2(\xi, \xi_*) := \widetilde{K}(\xi, \xi_*).$$

Since $\widetilde{K}(\xi, \xi_*)$ satisfies the estimate (2.3) for $\beta = 0$ similar to K, it follows that

$$\int_{\mathbb{R}^n} |K_c(\xi, \xi_*)| d\xi \le C, \quad \int_{\mathbb{R}^n} |K_c(\xi, \xi_*)| d\xi_* \le C,$$

which implies that

$$\left\| \int_{\mathbb{R}^n} K_c(\xi, \xi_*) u(\xi_*) d\xi_* \right\| \le C \|u\|.$$

Thus (2.11) is proved. This completes the proof of the lemma. □

In general, for any positive integer N, define the iterative commutator $\mathcal{L}$ by

$$\mathcal{L} = [\cdots [[\mathbf{L}, \mathbf{X}_1], \mathbf{X}_2] \cdots, \mathbf{X}_N],$$

where for each $k \in \{1, 2, \ldots, N\}$, $\mathbf{X}_k$ denotes the velocity multiplier ξ_{i_k} or the velocity differential operator $\partial_{\xi_{i_k}}$. Write $\mathcal{L}$ as the sum of two parts $\mathcal{L}_I$ and $\mathcal{L}_{II}$:

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_I + \mathcal{L}_{II}, \\ \mathcal{L}_I &= [\cdots [[-\nu(\xi), \mathbf{X}_1], \mathbf{X}_2] \cdots, \mathbf{X}_N], \\ \mathcal{L}_{II} &= [\cdots [[K, \mathbf{X}_1], \mathbf{X}_2] \cdots, \mathbf{X}_N]. \end{aligned}$$

Then $\mathcal{L}$ has the same property as in Lemma 2.1.

Corollary 2.1. *The following properties hold:*

(i) *$\mathcal{L}_I$ is a bounded linear operator on $L^2(\mathbb{R}^n_\xi)$.*

(ii) *$\mathcal{L}_{II}$ is a compact operator on $L^2(\mathbb{R}^n_\xi)$ with the integral kernel $K_c(\xi, \xi_*)$, which satisfies that for any $k \ge 0$, there is some constant C depending on k such that*

$$\|\nu^k \mathcal{L}_{II} u\| \le C \|\nu^{k-1} u\|, \tag{2.12}$$

for any $u = u(\xi)$.

(iii) *$\mathcal{L}$ is a bounded linear operator on $L^2(\mathbb{R}^n_\xi)$.*

Proof. It is obvious that (iii) directly follows from (i) and (ii). Thus it suffices to prove (i) and (ii). For the first part $\mathcal{L}_I$, in fact it is a velocity multiplier generated by $\nu(\xi)$, given by

$$\mathcal{L}_I = \begin{cases} (-1)^{N+1} \left(\prod_{k=1}^{N} \mathbf{X}_k \right) \nu(\xi) & \text{all } \mathbf{X}_k \text{ are } \partial_{\xi_{i_k}}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus (i) holds from the proof of Lemma 2.1. For the second part $\mathcal{L}_{II}$, it can be written as

$$\begin{aligned} (\mathcal{L}_{II} u)(\xi) &= \int_{\mathbb{R}^n} K_c(\xi, \xi_*) u(\xi_*) d\xi_*, \\ K_c(\xi, \xi_*) &= K_1(\xi, \xi_*) V_1 + K_2(\xi, \xi_*) V_2, \end{aligned}$$

where V_1 is the linear combination of products of velocity multipliers V_{0i}, V_{1i} and V_{1ij}, and similarly V_2 is the linear combination of products of velocity multipliers V_{0i}, V_{2i}

and V_{2ij}. Hence, similar to the compact operator K, $\mathcal{L}_{II}$ is also a compact operator on $L^2(\mathbb{R}^n_\xi)$ with the integral kernel K_c satisfying (2.3). Finally we claim that (2.3) implies (2.12). In fact, for any $k \geq 0$ and any $u = u(\xi)$,

$$(\mathcal{L}_{II}u)(\xi) \leq \left\{\int_{\mathbb{R}^n} |K_c(\xi,\xi_*)|\nu^{-2k}(\xi_*)d\xi_*\right\}^{1/2} \left\{\int_{\mathbb{R}^n} |K_c(\xi,\xi_*)|\nu^{2k}(\xi_*)u^2(\xi_*)d\xi_*\right\}^{1/2}$$
$$\leq C\nu^{-(2k+1)/2}(\xi)\left\{\int_{\mathbb{R}^n} |K_c(\xi,\xi_*)|\nu^{2k}(\xi_*)u^2(\xi_*)d\xi_*\right\}^{1/2},$$

which gives

$$\int_{\mathbb{R}^n} \nu^{2k}(\xi)(\mathcal{L}_{II}u)^2(\xi)d\xi \leq C\int_{\mathbb{R}^n} \nu^{2k}(\xi_*)u^2(\xi_*)\int_{\mathbb{R}^n} |K_c(\xi,\xi_*)|\nu^{-1}(\xi)d\xi d\xi_*$$
$$\leq C\int_{\mathbb{R}^n} \nu^{2k-2}(\xi_*)u^2(\xi_*)d\xi_*.$$

That is (2.12). This completes the proof of this lemma. □

Finally, Corollary 2.1 directly gives

Corollary 2.2. *Let γ, k be $|\gamma| \geq 1$ and $k \geq 0$. Then there is some constant C such that*

$$\|[\mathbf{L}, \partial_\xi^\gamma]u\| \leq C \sum_{0\leq|\gamma'|\leq|\gamma|-1} \|\partial_\xi^{\gamma'}u\|,$$
$$\|\nu^k[K, \partial_\xi^\gamma]u\| \leq C\|\nu^{k-1}u\|,$$

for any $u = u(\xi)$.

2.3. Energy estimates. From now on, we use the following notation of the index sets for differentiations: Let ℓ be any positive integer,

$$\Lambda_0(\beta) = \{0 \leq |\beta| \leq \ell\},$$
$$\Lambda_1(\beta) = \{1 \leq |\beta| \leq \ell\},$$
$$\Lambda_2(\beta) = \{0 \leq |\beta| \leq \ell - 1\},$$
$$\Lambda_3^i(\beta,\gamma) = \{|\gamma| = i,\ 0 \leq |\beta| + |\gamma| \leq \ell\},\quad i = 1, 2, \ldots, \ell,$$
$$\Lambda_3(\beta,\gamma) = \{|\gamma| \geq 1,\ 0 \leq |\beta| + |\gamma| \leq \ell\} = \cup_{i=1}^{\ell}\Lambda_3^i(\beta,\gamma),$$
$$\Lambda_4^j(\beta,\gamma) = \{|\gamma| = j,\ 0 \leq |\beta| + |\gamma| \leq \ell - 1\},\quad j = 1, 2, \ldots, \ell - 1,$$
$$\Lambda_4(\beta,\gamma) = \{|\gamma| \geq 1,\ 0 \leq |\beta| + |\gamma| \leq \ell - 1\} = \cup_{i=1}^{\ell-1}\Lambda_4^i(\beta,\gamma).$$

(i) *Assumptions and energy inequality.* Throughout this subsection, the following assumptions are made:
(A1) The integer $\ell \geq 2$;
(A2) For the functions E_1 and E_2, there is $\delta > 0$ such that

$$\sum_{\Lambda_0(\beta)} \left\|(1+|x|)\partial_x^\beta E_i(t,x)\right\|_{L^\infty_{t,x}} + \sum_{\Lambda_2(\beta)} \left\|(1+|x|)\partial_t\partial_x^\beta E_i(t,x)\right\|_{L^\infty_{t,x}} \leq \delta,$$

where $i = 1, 2$.

Under the above assumptions, our final goal of this subsection is to show that if $\delta > 0$ is small enough, then the energy inequality holds:

$$\frac{d}{dt}H(t) + cD(t) \le C\|\nabla_x u_1\|^2, \tag{2.13}$$

where $c > 0$ is some positive constant, C is some constant, $H(t)$ is a nonlinear energy functional and $D(t)$ is the corresponding dissipation rate. For the moment, we would not like to expose the precise forms of $H(t)$ and $D(t)$, see Theorem 2.1, but only point out some important characteristics for them:

- $H(t)$ contains the microscopic component u_2 and its derivatives with respect to t, x, and ξ up to order of $\ell \ge 2$, and also only the derivatives of the macroscopic component u_1 with respect to t and x;
- In $H(t)$, for the time derivatives, the differential order of time is at most one, where there is not any weight function, but for others, the velocity function ν is added.
- $D(t)$ contains those terms corresponding to $H(t)$ but the power of velocity weight function is higher $1/2$.
- There is some constant C such that $H(t) \le CD(t)$ for any $t \ge 0$.

(ii) Energy estimates on the microscopic part. Now we turn to the proof of the energy inequality in the form of (2.13). First consider the estimates on some energy functional $H_1(t)$ which is a linear combination of the following terms:

$$\|u_2\|^2,\ \sum_{\Lambda_1(\beta)}\left\|\partial_x^\beta u\right\|^2,\ \sum_{\Lambda_2(\beta)}\left\|\partial_t\partial_x^\beta u\right\|^2,\ \sum_{\Lambda_3^i(\beta,\gamma)}\left\|\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2,\ \sum_{\Lambda_4^j(\beta,\gamma)}\left\|\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2.$$

For brevity, define the time dependent linear operator $\mathbf{B}(t)$ and $\mathbf{D}(t)$ by

$$\mathbf{B}(t) = \xi\cdot\nabla_x + E_1\cdot\nabla_\xi - \mathbf{L},$$
$$\mathbf{D}(t) = \xi\cdot\nabla_x + E_1\cdot\nabla_\xi - \xi\cdot E_2.$$

Using the above notations, (2.1) and (2.4) can be rewritten as

$$\partial_t u + \mathbf{B}(t)u = \xi\cdot E_2 u, \tag{2.14}$$

and

$$\partial_t u_2 + \mathbf{B}(t)u_2 = \xi\cdot E_2 u_2 + [\mathbf{P},\mathbf{D}(t)]u, \tag{2.15}$$

where $[\mathbf{P},\mathbf{D}(t)]$ is the commutator given by

$$[\mathbf{P},\mathbf{D}(t)] = \mathbf{P}\mathbf{D}(t) - \mathbf{D}(t)\mathbf{P}.$$

In what follows, a series of lemmas are given. The first one is concerned with the $L^2_{x,\xi}$-estimate on the microscopic component u_2. For this purpose, from the properties of the linearized Boltzmann operator $\mathbf{L}$, the smallness assumption we imposed on the external forces E_1, E_2, and by using the Hardy inequality $\left\|\frac{u_1}{|x|}\right\| \le C\|\nabla_x u_1\|$, we have by applying the standard energy method to (2.15) that

Lemma 2.2. *If* $\delta > 0$ *is small enough, then one has*

$$\frac{d}{dt}\|u_2\|^2 + c\left\|\nu^{1/2}u_2\right\|^2 \le C\|\nabla_x u_1\|^2.$$

The next lemma is on the $L^2_{x,\xi}$-estimate on $\partial_x^\beta u$ for $\beta \in \Lambda_1(\beta)$.

Lemma 2.3. *If* $\delta > 0$ *is small enough, then one has*

$$\frac{d}{dt}\sum_{\Lambda_1(\beta)} \left\|\partial_x^\beta u\right\|^2 + c\sum_{\Lambda_1(\beta)} \left\|\nu^{1/2}\partial_x^\beta u_2\right\|^2 \leq C\delta \sum_{\Lambda_1(\beta)} \left\|\partial_x^\beta u_1\right\|^2 + C\delta \sum_{\Lambda_3(\beta,\gamma)} \left\|\partial_x^\beta \partial_\xi^\gamma u_2\right\|^2. \tag{2.16}$$

Proof. Directly applying ∂_x^β with $\beta \in \Lambda_2(\beta)$ to (2.14) gives

$$\partial_t(\partial_x^\beta u) + \mathbf{B}(t)(\partial_x^\beta u) = \partial_x^\beta(\xi \cdot E_2 u) + [\mathbf{B}(t), \partial_x^\beta]u. \tag{2.17}$$

Further multiplying (2.25) by $\partial_x^\beta u$ and then integrating over $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$, one has

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\beta u\|^2 + c_0 \left\|\nu^{1/2}\partial_x^\beta u_2\right\|^2 \leq \sum_{i=1}^2 I_i, \tag{2.18}$$

where we have used the identity

$$\{\mathbf{I}-\mathbf{P}\}\partial_x^\beta u = \partial_x^\beta\{\mathbf{I}-\mathbf{P}\}u = \partial_x^\beta u_2,$$

and $I_i, i = 1, 2$, denote the corresponding terms after taking the inner product with $\partial_x^\beta u$ for ones on the right-hand side of (2.17).

Next we estimate I_1 and I_2. To this end, from the smallness assumption we imposed on E_1 and E_2, the Hardy inequality and the Cauchy-Schwarz inequality, we can deduce that

$$I_1 \leq C\delta \sum_{\Lambda_1(\beta')} \left\|\nu^{1/2}\partial_x^{\beta'} u_2\right\|^2 + C\delta \sum_{\Lambda_1(\beta')} \left\|\partial_x^{\beta'} u_1\right\|^2,$$

and

$$I_2 \leq C\delta \sum_{\Lambda_1(\beta')} \left\|\partial_x^{\beta'} u_1\right\|^2 + C\delta \sum_{\Lambda_3(\beta',\gamma')} \left\|\partial_x^{\beta'}\partial_\xi^{\gamma'} u_2\right\|^2.$$

Thus taking summation over $\beta \in \Lambda_1(\beta)$ for (2.18) and then collecting all estimates, (2.16) follows if $\delta > 0$ is small enough. This completes the proof of the lemma. □

For the $L^2_{x,\xi}$-estimate on $\partial_t\partial_x^\gamma u(\gamma \in \Lambda_2(\beta))$, we have the following result

Lemma 2.4. *If* $\delta > 0$ *is small enough, then one has*

$$\begin{aligned}
&\frac{d}{dt}\sum_{\Lambda_2(\beta)} \left\|\partial_t\partial_x^\gamma u\right\|^2 + c\sum_{\Lambda_2(\beta)} \left\|\nu^{1/2}\partial_t\partial_x^\gamma u_2\right\|^2 \\
&\quad \leq C\delta\left(\sum_{\Lambda_1(\beta)} \|\partial_x^\beta u_1\|^2 + \sum_{\Lambda_2(\beta)} \|\partial_t\partial_x^\beta u_1\|^2\right) \\
&\quad\quad + C\delta\left(\sum_{\Lambda_1(\beta)} \left\|\nu^{1/2}\partial_x^\beta u_2\right\|^2 + \sum_{\Lambda_3(\beta,\gamma)} \left\|\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 + \sum_{\Lambda_4(\beta,\gamma)} \left\|\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2\right).
\end{aligned} \tag{2.19}$$

Proof. First it is easy to see that for $\beta \in \Lambda_2(\beta)$,

$$\partial_t(\partial_t\partial_x^\beta u) + \mathbf{B}(t)(\partial_t\partial_x^\beta u) = \xi \cdot \partial_t\partial_x^\beta(E_2 u) + [\mathbf{B}(t), \partial_t\partial_x^\beta]u,$$

which gives

$$\frac{1}{2}\frac{d}{dt}\|\partial_t\partial_x^\beta u\|^2 + c_0\left\|\nu^{1/2}\partial_t\partial_x^\beta u_2\right\|^2 \leq \sum_{i=1}^{2} I_i. \tag{2.20}$$

For I_1, one has

$$\begin{aligned} I_1 &\leq \delta\left\|\nu^{1/2}\partial_t\partial_x^\beta u\right\|^2 + C\delta\left\|\nu^{1/2}\partial_t\partial_x^\beta(E_2 u)\right\|^2 \\ &\leq C\delta\sum_{\Lambda_2(\beta')}\left\|\nu^{1/2}\partial_t\partial_x^{\beta'}u_2\right\|^2 + C\delta\sum_{\Lambda_2(\beta')}\left\|\partial_t\partial_x^{\beta'}u_1\right\|^2 \\ &\quad + C\delta\sum_{\Lambda_1(\beta')}\left\|\nu^{1/2}\partial_x^{\beta'}u_2\right\|^2 + C\delta\sum_{\Lambda_1(\beta')}\left\|\partial_x^{\beta'}u_1\right\|^2. \end{aligned}$$

For I_2, noticing that

$$\begin{aligned} [\mathbf{B}(t), \partial_t\partial_x^\beta]u = &-\sum_{0\leq|\beta'|\leq|\beta|-1} C_{\beta'}\partial_x^{\beta-\beta'}E_1 \cdot \nabla_\xi\partial_t\partial_x^{\beta'}u \\ &-\sum_{0\leq|\beta'|\leq|\beta|} C_{\beta'}\partial_t\partial_x^{\beta-\beta'}E_1 \cdot \nabla_\xi\partial_x^{\beta'}u, \end{aligned}$$

one also has

$$\begin{aligned} I_2 \leq\ &\delta\left\|\partial_t\partial_x^\beta u_2\right\|^2 + C\delta\sum_{\Lambda_2(\beta')}\left\|\partial_t\partial_x^{\beta'}u_1\right\|^2 + C\delta\sum_{\Lambda_1(\beta')}\left\|\partial_x^{\beta'}u_1\right\|^2 \\ &+ C\delta\sum_{\Lambda_3(\beta,\gamma)}\left\|\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 + C\delta\sum_{\Lambda_4(\beta,\gamma)}\left\|\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2. \end{aligned}$$

Thus taking summation over $\beta \in \Lambda_2(\beta)$ for (2.20) and then collecting all estimates, (2.19) follows if $\delta > 0$ is small enough. This completes the proof of the lemma. □

As to the $L^2_{x,\xi}$-estimate on $\partial_x^\beta\partial_\xi^\gamma u_2$ for $(\beta, \gamma) \in \Lambda_3^i(\beta, \gamma)$, we can conclude that

Lemma 2.5. *If $\delta > 0$ is small enough, then one has*

$$\begin{aligned} &\frac{d}{dt}\sum_{\Lambda_3^i(\beta,\gamma)}\left\|\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 + c\sum_{\Lambda_3^i(\beta,\gamma)}\left\|\nu^{1/2}\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 \\ &\leq C\sum_{\Lambda_1(\beta)}\left\|\partial_x^\beta u_1\right\|^2 + C\sum_{\Lambda_0(\beta)}\left\|\partial_x^\beta u_2\right\|^2 \\ &\quad + C_{i,i-1}\sum_{\Lambda_3^{i-1}(\beta,\gamma)}\left\|\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 + \delta C_{i,i+1}\sum_{\Lambda_3^{i+1}(\beta,\gamma)}\left\|\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2, \end{aligned} \tag{2.21}$$

where $i = 1, 2, \ldots, \ell$, and $C_{i,i-1}$, $C_{i,i+1}$ are some constants with additional conventions:

$$C_{1,0} = C_{\ell,\ell+1} = 0. \tag{2.22}$$

Proof. First apply ∂_ξ^γ with $|\gamma| = i = 1, 2, \ldots, \ell$ to (2.15) to get

$$\begin{aligned}\partial_t(\partial_\xi^\gamma u_2) + B(t)\partial_\xi^\gamma u_2 &= E_2 \cdot \partial_\xi^\gamma(\xi u_2) + \partial_\xi^\gamma[\mathbf{P}, \mathbf{D}(t)]u + [\mathbf{B}(t), \partial_\xi^\gamma]u_2 \\ &= \xi \cdot E_2\partial_\xi^\gamma u_2 + e_\gamma \cdot E_2\partial_\xi^{\gamma-1}u_2 - e_\gamma \cdot \nabla_x\partial_\xi^{\gamma-1}u_2 \\ &\quad +\partial_\xi^\gamma[\mathbf{P}, \mathbf{D}(t)]u - [\mathbf{L}, \partial_\xi^\gamma]u_2,\end{aligned} \tag{2.23}$$

where e_γ denotes a constant vector, and for simplicity we used the notations

$$e_\gamma \cdot E_2\partial_\xi^{\gamma-1}u_2 = \sum_{|\gamma'|=1} \gamma\partial_\xi^{\gamma'}\xi \cdot E_2\partial_\xi^{\gamma-\gamma'}u_2 = \sum_{0\le|\gamma'|\le|\gamma|-1} C_{\gamma'}\partial_\xi^{\gamma-\gamma'}\xi \cdot E_2\partial_\xi^{\gamma'}u_2,$$

and

$$e_\gamma \cdot \nabla_x\partial_\xi^{\gamma-1}u_2 = \sum_{|\gamma'|=1} \gamma\partial_\xi^{\gamma'}\xi \cdot \nabla_x\partial_\xi^{\gamma-\gamma'}u_2 = \sum_{0\le|\gamma'|\le|\gamma|-1} C_{\gamma'}\partial_\xi^{\gamma-\gamma'}\xi \cdot \nabla_x\partial_\xi^{\gamma'}u_2.$$

Further apply ∂_x^β with $(\beta, \gamma) \in \Lambda_3^i(\beta, \gamma)$ to (2.23) to obtain

$$\begin{aligned}&\partial_t(\partial_x^\beta\partial_\xi^\gamma u_2) + \mathbf{B}(t)(\partial_x^\beta\partial_\xi^\gamma u_2) \\ &= \sum_{0\le|\beta'|\le|\beta|} C_{\beta'}\xi \cdot \partial_x^{\beta-\beta'}E_2\partial_x^{\beta'}\partial_\xi^\gamma u_2 + \sum_{0\le|\beta'|\le|\beta|} C_{\beta'}e_\gamma \cdot \partial_x^{\beta-\beta'}E_2\partial_x^{\beta'}\partial_\xi^{\gamma-1}u_2 \\ &\quad - \sum_{0\le|\beta'|\le|\beta|-1} C_{\beta'}\partial_x^{\beta-\beta'}E_1 \cdot \nabla_\xi\partial_x^{\beta'}\partial_\xi^\gamma u_2 - e_\gamma \cdot \nabla_x\partial_x^\beta\partial_x^{\gamma-1}u_2 \\ &\quad +\partial_x^\beta\partial_\xi^\gamma[\mathbf{P}, \mathbf{D}(t)]u - [\mathbf{L}, \partial_\xi^\gamma]\partial_x^\beta u_2.\end{aligned} \tag{2.24}$$

Multiplying (2.24) by $\partial_x^\beta\partial_\xi^\gamma u_2$ and integrating it over $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$, one has

$$\frac{1}{2}\frac{d}{dt}\left\|\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 + c_0\left\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 \le \sum_{i=1}^6 I_i. \tag{2.25}$$

We estimate each term I_i as follows. For I_1, I_2 and I_3, one has

$$\begin{aligned}I_1 &\le \delta\left\|\nu^{1/2}\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 + C\delta\sum_{\Lambda_3^i(\beta',\gamma')}\left\|\nu^{1/2}\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2, \\ I_2 &\le \delta\left\|\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 + \delta C_{i,i-1}\sum_{\Lambda_3^{i-1}(\beta',\gamma')}\left\|\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2 + \delta C\delta_{i1}\sum_{\Lambda_0(\beta')}\left\|\partial_x^{\beta'}u_2\right\|^2, \\ I_3 &\le \delta\left\|\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 + \delta C_{i,i+1}\sum_{\Lambda_3^{i+1}(\beta',\gamma')}\left\|\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2,\end{aligned}$$

where δ_{i1} is the Kroneker symbol and we have set (2.22). In fact, if $i = \ell$, $\Lambda_3^\ell(\beta, \gamma)$ means $\beta = 0$ and $|\gamma| = \ell$, i.e. one has taken only the velocity derivative ∂_ξ^γ with $|\gamma| = \ell$, which implies $I_3 = 0$ for this special case. For I_4, I_5 and I_6, similarly it holds that

$$I_4 \leq \frac{c_0}{6} \left\| \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 + C_{i,i-1} \sum_{\Lambda_3^{i-1}(\beta',\gamma')} \left\| \nu^{1/2} \partial_x^{\beta'} \partial_\xi^{\gamma'} u_2 \right\|^2 + C\delta_{i1} \sum_{\Lambda_0(\beta')} \left\| \partial_x^{\beta'} u_2 \right\|^2,$$

$$I_5 \leq \frac{c_0}{6} \left\| \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 + C \sum_{\Lambda_1(\beta')} \left\| \partial_x^{\beta'} u_1 \right\|^2 + C \sum_{\Lambda_1(\beta')} \left\| \partial_x^{\beta'} u_2 \right\|^2,$$

and

$$\begin{aligned} I_6 &= -\left\langle [\mathbf{L}, \partial_\xi^\gamma] \partial_x^\beta u_2, \partial_x^\beta \partial_\xi^\gamma u_2 \right\rangle \leq \frac{c_0}{6} \left\| \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 + C \left\| [\mathbf{L}, \partial_\xi^\gamma] \partial_x^\beta u_2 \right\|^2 \\ &\leq \frac{c_0}{6} \left\| \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 + C \sum_{\Lambda_0(\beta')} \left\| \partial_x^{\beta'} u_2 \right\|^2 + C_{i,i-1} \sum_{\Lambda_3^{i-1}(\beta',\gamma')} \left\| \partial_x^{\beta'} \partial_\xi^{\gamma'} u_2 \right\|^2, \end{aligned}$$

where Corollary 2.2 was used. Finally it is noticed that

$$\begin{aligned} \left\| \nu^{1/2} \{\mathbf{I} - \mathbf{P}\} \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 &\geq \left\| \nu^{1/2} \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 - \left\| \nu^{1/2} \mathbf{P} \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 \\ &\geq \left\| \nu^{1/2} \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 - C \sum_{\Lambda_0(\beta')} \left\| \partial_x^{\beta'} u_2 \right\|^2. \end{aligned}$$

Putting all the above estimates into (2.25) and then taking summation over $(\beta, \gamma) \in \Lambda_3^i(\beta, \gamma)$ leads to (2.21), provided that $\delta > 0$ is small enough. This completes the proof of the lemma. □

Finally for the $L^2_{x,\xi}$-estimate on $\partial_t \partial_x^\beta \partial_\xi^\gamma u_2$ $((\beta, \gamma) \in \Lambda_4^j(\beta, \gamma),\ j = 1, 2, \ldots, \ell - 1)$, we have

Lemma 2.6. *If $\delta > 0$ is small enough, then one has*

$$\begin{aligned} &\frac{d}{dt} \sum_{\Lambda_4^j(\beta,\gamma)} \left\| \partial_t \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 + c \sum_{\Lambda_4^j(\beta,\gamma)} \left\| \nu^{1/2} \partial_t \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 \\ &\leq C \sum_{\Lambda_2(\beta)} \left\| \partial_t \partial_x^\beta u_1 \right\|^2 + C \sum_{\Lambda_2(\beta)} \left\| \partial_t \partial_x^\beta u_2 \right\|^2 \\ &\quad + C_{j,j-1} \sum_{\Lambda_4^{j-1}(\beta,\gamma)} \left\| \partial_t \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 + \delta C_{j,j+1} \sum_{\Lambda_4^{j+1}(\beta,\gamma)} \left\| \partial_t \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2 \\ &\quad + C\delta \sum_{\Lambda_0(\beta)} \left\| \partial_x^\beta u_2 \right\|^2 + C\delta \sum_{\Lambda_3(\beta,\gamma)} \left\| \partial_x^\beta \partial_\xi^\gamma u_2 \right\|^2, \end{aligned} \tag{2.26}$$

where $j = 1, 2, \ldots, \ell - 1$, and $C_{i,i-1}$, $C_{i,i+1}$ are some constants with additional conventions:

$$C_{1,0} = C_{\ell-1,\ell} = 0.$$

Proof. Notice that (2.24) also holds for $(\beta,\gamma)\in\Lambda_4^j(\beta,\gamma)$ with $j=1,2,\ldots,\ell-1$. Then further applying ∂_t to it, multiplying the resulting identity by $\partial_t\partial_x^\beta\partial_\xi^\gamma u_2$, and integrating the final result over $\mathbb{R}^n\times\mathbb{R}^n$, we have

$$\frac{1}{2}\frac{d}{dt}\left\|\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2+c_0\left\|\nu^{1/2}\{\mathbf{I}-\mathbf{P}\}\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2\le\sum_{i=1}^7 I_i. \tag{2.27}$$

First for I_1, I_2 and I_3, one has

$$\begin{aligned}
I_1&=\sum_{0\le|\beta'|\le|\beta|}C_{\beta'}\left\langle\xi\cdot\partial_x^{\beta-\beta'}E_2\partial_t\partial_x^{\beta'}\partial_\xi^\gamma u_2+\xi\cdot\partial_t\partial_x^{\beta-\beta'}E_2\partial_x^{\beta'}\partial_\xi^\gamma u_2,\,\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\rangle\\
&\le\delta\left\|\nu^{1/2}\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2+C\delta\sum_{\Lambda_4^j(\beta',\gamma')}\left\|\nu^{1/2}\partial_t\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2\\
&\quad+C\delta\sum_{\Lambda_3^j(\beta',\gamma')}\left\|\nu^{1/2}\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2,\\
I_2&=\sum_{0\le|\beta'|\le|\beta|}C_{\beta'}e_\gamma\cdot\left\langle\partial_x^{\beta-\beta'}E_2\partial_t\partial_x^{\beta'}\partial_\xi^{\gamma-1}u_2+\partial_t\partial_x^{\beta-\beta'}E_2\partial_x^{\beta'}\partial_\xi^{\gamma-1}u_2,\,\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\rangle\\
&\le\delta\left\|\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2+\delta C_{j,j-1}\sum_{\Lambda_4^{j-1}(\beta',\gamma')}\left\|\partial_t\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2\\
&\quad+\delta C\delta_{j1}\sum_{\Lambda_2(\beta')}\left\|\partial_t\partial_x^{\beta'}u_2\right\|^2+\delta C_{j,j-1}\sum_{\Lambda_3^{j-1}(\beta',\gamma')}\left\|\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2\\
&\quad+\delta C\delta_{j1}\sum_{\Lambda_0(\beta')}\left\|\partial_x^{\beta'}u_2\right\|^2,
\end{aligned}$$

and

$$\begin{aligned}
I_3&=-\sum_{0\le|\beta'|\le|\beta|-1}C_{\beta'}\left\langle\partial_x^{\beta-\beta'}E_1\cdot\nabla_\xi\partial_t\partial_x^{\beta'}\partial_\xi^\gamma u_2+\partial_t\partial_x^{\beta-\beta'}E_1\cdot\nabla_\xi\partial_x^{\beta'}\partial_\xi^\gamma u_2,\,\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\rangle\\
&\le\delta\left\|\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2+\delta C_{j,j+1}\sum_{\Lambda_4^{j+1}(\beta',\gamma')}\left\|\partial_t\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2+C\delta\sum_{\Lambda_3^{j+1}(\beta',\gamma')}\left\|\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2.
\end{aligned}$$

Furthermore, it holds that

$$\begin{aligned}
I_4&=-e_\gamma\cdot\left\langle\nabla_x\partial_t\partial_x^\beta\partial_x^{\gamma-1}u_2,\,\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\rangle\\
&\le\frac{c_0}{6}\left\|\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2+C_{j,j-1}\sum_{\Lambda_4^{j-1}(\beta',\gamma')}\left\|\partial_t\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2+C\sum_{\Lambda_2(\beta')}\left\|\partial_t\partial_x^{\beta'}u_2\right\|^2,\\
I_5&=-\left\langle\partial_t\partial_x^\beta\partial_\xi^\gamma[\mathbf{P},\mathbf{D}(t)]u,\,\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\rangle\\
&\le\frac{c_0}{6}\left\|\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2+C\sum_{\Lambda_2(\beta')}\left\|\partial_t\partial_x^{\beta'}u_1\right\|^2+C\sum_{\Lambda_2(\beta')}\left\|\partial_t\partial_x^{\beta'}u_2\right\|^2,\\
I_6&=-\left\langle[\mathbf{L},\partial_\xi^\gamma]\partial_t\partial_x^\beta u_2,\,\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\rangle
\end{aligned}$$

$$\leq \frac{c_0}{6}\left\|\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 + C\sum_{\Lambda_2(\beta')}\left\|\partial_t\partial_x^{\beta'}u_2\right\|^2 + C_{j,j-1}\sum_{\Lambda_4^{j-1}(\beta',\gamma')}\left\|\partial_t\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2.$$

Finally,

$$\begin{aligned}I_7 &= -\left\langle\partial_t E_1\cdot\nabla_\xi\partial_x^\beta\partial_\xi^\gamma u_2,\ \partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\rangle\\ &\leq \delta\left\|\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\right\|^2 + C\delta\sum_{\Lambda_3^{j+1}(\beta',\gamma')}\left\|\partial_t\partial_x^{\beta'}\partial_\xi^{\gamma'}u_2\right\|^2.\end{aligned}$$

Inserting all the above estimates into (2.27) and then taking summation over $(\beta,\gamma)\in\Lambda_4^j(\beta,\gamma)$ leads to (2.26), provided that $\delta>0$ is small enough. This completes the proof of the lemma. □

Putting all the above estimates together, we can obtain the following elementary energy estimates, which follow directly from a proper linear combination of all the energy inequalities obtained in Lemma 2.2–Lemma 2.6.

Corollary 2.3. *Under Assumptions (A1)–(A2), if $\delta>0$ is small enough, then there is an energy functional $H_1(t)$ and a corresponding dissipation rate $D_1(t)$ such that*

$$\frac{d}{dt}H_1(t)+cD_1(t)\leq C\left(\sum_{\Lambda_1(\beta)}\|\partial_x^\beta u_1\|^2+\sum_{\Lambda_2(\beta)}\|\partial_t\partial_x^\beta u_1\|^2\right), \tag{2.28}$$

where $H_1(t)$ and $D_1(t)$ is defined by

$$\begin{aligned}H_1(t) &\sim \|u_2\|^2+\sum_{\Lambda_1(\beta)}\|\partial_x^\beta u\|^2+\sum_{\Lambda_2(\beta)}\|\partial_t\partial_x^\beta u\|^2\\ &\quad+\sum_{\Lambda_3(\beta,\gamma)}\|\partial_x^\beta\partial_\xi^\gamma u_2\|^2+\sum_{\Lambda_4(\beta,\gamma)}\|\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\|^2,\\ D_1(t) &\sim \|\nu^{1/2}u_2\|^2+\sum_{\Lambda_1(\beta)}\|\nu^{1/2}\partial_x^\beta u_2\|^2+\sum_{\Lambda_2(\beta)}\|\nu^{1/2}\partial_t\partial_x^\beta u_2\|^2\\ &\quad+\sum_{\Lambda_3(\beta,\gamma)}\|\nu^{1/2}\partial_x^\beta\partial_\xi^\gamma u_2\|^2+\sum_{\Lambda_4(\beta,\gamma)}\|\nu^{1/2}\partial_t\partial_x^\beta\partial_\xi^\gamma u_2\|^2.\end{aligned}$$

(iii) *Estimates on the macroscopic part.* It should be pointed out that $D_1(t)$ is a lack of the macroscopic dissipation rate. Then it is not true that there is a constant C such that $H_1(t)\leq CD_1(t)$ for any $t\geq 0$. However, except for the first order derivatives of the macroscopic component, the higher order derivatives can be bounded by part of the microscopic dissipation rate $D_1(t)$. Thus a proper further linear combination makes the dissipation rate include the derivatives of the macroscopic component of at least first order.

The following estimate is based on the macroscopic equations (2.6)–(2.10) satisfied by a, b, c.

Lemma 2.7. *Under Assumptions (A1) and (A2), if $\delta>0$ is small enough, then it holds that*

$$\sum_{\Lambda_1(\beta)} \left\|\partial_x^\beta u_1\right\|^2 + \sum_{\Lambda_2(\beta)} \left\|\partial_t \partial_x^\beta u_1\right\|^2$$
$$\leq C\frac{d}{dt} \sum_{1\leq|\beta|\leq\ell-1} \left\langle \partial_x^\beta a, \nabla_x \cdot \partial_x^\beta b\right\rangle + C\|\nabla_x u_1\|^2 + C\sum_{\Lambda_2(\beta)} \left\|\partial_x^\beta \Re\right\|^2, \quad (2.29)$$

where for any β, $\|\partial_x^\beta \Re\|^2$ is defined by

$$\|\partial_x^\beta \Re\|^2 = \|\partial_x^\beta \Re_0\|^2 + \|\partial_x^\beta \Re_1\|^2 + \|\partial_x^\beta \Re_{21}\|^2 + \|\partial_x^\beta \Re_{22}\|^2 + \|\partial_x^\beta \Re_3\|^2,$$

with $\|\partial_x^\beta \Re_1\|^2 = \sum_{1\leq i\leq n} \|\partial_x^\beta \Re_1^i\|^2$, and similarly for other terms.

Proof. First consider estimates on the pure space derivatives of a, b, c. We start with b_j, which will satisfy a standard elliptic equation. In fact, for any fixed $j \in \{1, 2, \ldots, n\}$ and $|\beta| \geq 0$, by (2.8) and (2.9), direct calculations yield

$$\Delta\partial_x^\beta b_j = -\partial_{jj}\partial_x^\beta b_j - \sum_{i\neq j} \partial_j \partial_x^\beta (\bar{E}_i b_i) + \sum_{i\neq j} \partial_i \partial_x^\beta (\bar{E}_i b_j + \bar{E}_j b_i) + 2\partial_j \partial_x^\beta (\bar{E}_j b_j)$$
$$- \sum_{i\neq j} \partial_j \partial_x^\beta \Re_{21}^i + \sum_{i\neq j} \partial_i \partial_x^\beta \Re_{22}^{ij} + \partial_j \partial_x^\beta \Re_{21}^j.$$

Thus after multiplying by $\partial_x^\beta b_j$ and taking some integrations by part, it holds that

$$\|\nabla_x \partial_x^\beta b_j\|^2 + \|\partial_j \partial_x^\beta b_j\|^2$$
$$\leq \frac{1}{2}\|\nabla_x \partial_x^\beta b_j\|^2 + \frac{1}{2}\left(\|\partial_x^\beta (\bar{E}\otimes b)\|^2 + \|\partial_x^\beta \Re_{21}\|^2 + \|\partial_x^\beta \Re_{22}\|^2\right)$$
$$\leq \frac{1}{2}\|\nabla_x \partial_x^\beta b_j\|^2 + C\delta^2 \sum_{0\leq|\beta'|\leq|\beta|} \|\nabla_x \partial_x^{\beta'} b\|^2 + C\left(\|\partial_x^\beta \Re_{21}\|^2 + \|\partial_x^\beta \Re_{22}\|^2\right),$$

which implies

$$\|\nabla_x \partial_x^\beta b\|^2 \leq C\delta^2 \sum_{0\leq|\beta'|\leq|\beta|-1} \|\nabla_x \partial_x^{\beta'} b\|^2 + C\left(\|\partial_x^\beta \Re_{21}\|^2 + \|\partial_x^\beta \Re_{22}\|^2\right).$$

Furthermore, since $\delta > 0$ can be small enough, by iteration, one has that for any $|\beta| \geq 0$,

$$\|\nabla_x \partial_x^\beta b\|^2 \leq C \sum_{0\leq|\beta'|\leq|\beta|} \left(\|\partial_x^{\beta'} \Re_{21}\|^2 + \|\partial_x^{\beta'} \Re_{22}\|^2\right), \quad (2.30)$$

which, after taking summation over $0 \leq |\beta| \leq \ell - 1$, gives

$$\sum_{\Lambda_1(\beta)} \|\partial_x^\beta b\|^2 \leq C\sum_{\Lambda_2(\beta)} \left(\|\partial_x^\beta \Re_{21}\|^2 + \|\partial_x^\beta \Re_{22}\|^2\right). \quad (2.31)$$

For the pure space derivatives of c, it follows from (2.10) that for $|\beta| \geq 0$,

$$\|\partial_x^\beta \nabla_x c\|^2 \leq \|\partial_x^\beta (\bar{E}c)\|^2 + \|\partial_x^\beta \Re_3\|^3$$
$$\leq C\delta^2 \sum_{0\leq|\beta'|\leq|\beta|} \|\partial_x^{\beta'} \nabla_x c\|^2 + \|\partial_x^\beta \Re_3\|^2,$$

which, with $\delta > 0$ small enough, implies

$$\|\partial_x^\beta \nabla_x c\|^2 \leq C \sum_{0\leq|\beta'|\leq|\beta|} \|\partial_x^{\beta'} \mathfrak{R}_3\|^2. \tag{2.32}$$

Then, similar to obtaining (2.31), taking summation for (2.32) over $0 \leq |\beta| \leq \ell - 1$ gives

$$\sum_{\Lambda_1(\beta)} \|\partial_x^\beta c\|^2 \leq C \sum_{\Lambda_2(\beta)} \|\partial_x^\beta \mathfrak{R}_3\|^2. \tag{2.33}$$

For the pure space derivatives of a, one has from (2.7) that for any $|\beta| \geq 0$,

$$\begin{aligned}
\|\nabla_x \partial_x^\beta a\|^2 = & \frac{d}{dt}\left\langle \partial_x^\beta a, \nabla_x \cdot \partial_x^\beta b\right\rangle - \left\langle \partial_x^\beta \partial_t a, \nabla_x \cdot \partial_x^\beta b\right\rangle \\
& + \sum_{i=1}^n \left\langle \partial_i \partial_x^\beta a, \partial_x^\beta (a\bar{E}_i - 2cE_1 i) + \partial_x^\beta \mathfrak{R}_1^i \right\rangle \\
\leq & \frac{d}{dt}\left\langle \partial_x^\beta a, \nabla_x \cdot \partial_x^\beta b\right\rangle + \frac{1}{2}\|\partial_x^\beta \partial_t a\|^2 + \frac{1}{2}\|\nabla_x \partial_x^\beta b\|^2 + \frac{1}{2}\|\nabla_x \partial_x^\beta a\|^2 \\
& + C\delta^2 \sum_{0\leq|\beta'|\leq|\beta|} \left(\|\nabla_x \partial_x^{\beta'} a\|^2 + \|\nabla_x \partial_x^{\beta'} c\|^2\right) + \frac{1}{2}\|\partial_x^\beta \mathfrak{R}_1\|^2.
\end{aligned} \tag{2.34}$$

Notice that (2.6) together with (2.30) gives that for any $|\beta| \geq 0$,

$$\begin{aligned}
\|\partial_x^\beta \partial_t a\|^2 & \leq \|\partial_x^\beta (E_1 \cdot b)\|^2 + \|\partial_x^\beta \mathfrak{R}_0\|^2 \\
& \leq C\delta^2 \sum_{0\leq|\beta'|\leq|\beta|} \left(\|\partial_x^{\beta'} \mathfrak{R}_{21}\|^2 + \|\partial_x^{\beta'} \mathfrak{R}_{22}\|^2\right) + \|\partial_x^\beta \mathfrak{R}_0\|^2.
\end{aligned} \tag{2.35}$$

Putting (2.30), (2.32) and (2.35) into (2.34) and taking summation over $1 \leq |\beta| \leq \ell - 1$, one has

$$\begin{aligned}
\sum_{1\leq|\beta|\leq\ell-1} \|\nabla_x \partial_x^\beta a\|^2 \leq & C \sum_{1\leq|\beta|\leq\ell-1} \frac{d}{dt}\langle \partial_x^\beta a, \nabla_x \cdot \partial_x^\beta b\rangle + C\delta^2 \|\nabla_x a\|^2 \\
& + C \sum_{\Lambda_2(\beta)} \|\partial_x^\beta \mathfrak{R}\|^2.
\end{aligned} \tag{2.36}$$

Next we estimate $\|\partial_t \partial_x^\beta u_1\|$ with $\beta \in \Lambda_2(\beta)$. It directly follows from (2.35) that

$$\sum_{\Lambda_2(\beta)} \|\partial_t \partial_x^\beta a\|^2 \leq C \sum_{\Lambda_2(\beta)} \|\partial_x^\beta \mathfrak{R}\|^2. \tag{2.37}$$

In addition, (2.8) gives that for any $|\beta| \geq 0$,

$$\begin{aligned}
\|\partial_x^\beta \partial_t c\|^2 & \leq C\left\{\|\nabla_x \partial_x^\beta b\|^2 + \|\partial_x^\beta (\bar{E} \cdot b)\|^2 + \|\partial_x^\beta \mathfrak{R}_{21}\|^2\right\} \\
& \leq C \sum_{0\leq|\beta'|\leq|\beta|} \left(\|\partial_x^{\beta'} \mathfrak{R}_{21}\|^2 + \|\partial_x^{\beta'} \mathfrak{R}_{22}\|^2\right),
\end{aligned}$$

which implies that

$$\sum_{\Lambda_2(\beta)} \|\partial_t \partial_x^\beta c\|^2 \leq C \sum_{\Lambda_2(\beta)} \|\partial_x^\beta \mathfrak{R}\|^2. \tag{2.38}$$

Similarly (2.7) together with (2.33) and (2.36) gives

$$\sum_{\Lambda_2(\beta)} \|\partial_t \partial_x^\beta b\|^2 \le C \sum_{1\le|\beta|\le\ell} \left(\|\partial_x^\beta a\|^2 + \|\partial_x^\beta c\|^2\right) + C \sum_{\Lambda_2(\beta)} \|\partial_x^\beta \Re\|^2$$
$$\le C \sum_{1\le|\beta|\le\ell-1} \frac{d}{dt}\langle \partial_x^\beta a, \nabla_x \cdot \partial_x^\beta b\rangle + C\|\nabla_x a\|^2 + C \sum_{\Lambda_2(\beta)} \|\partial_x^\beta \Re\|^2.$$

Finally, collecting all estimates (2.31), (2.33), (2.36), (2.37), (2.38) and (2.39) yields (2.29). This completes the proof of the lemma. □

(iv) *Combination of estimates on the macro-micro components.* As in [13], from the representation (2.5) of $\Re$, we can prove the following lemma.

Lemma 2.8. *It holds that*

$$\sum_{\Lambda_2(\beta)} \|\partial_x^\beta \Re\|^2 \le C \sum_{\Lambda_0(\beta)} \left\|\partial_x^\beta u_2\right\|^2 + C \sum_{\Lambda_2(\beta)} \left\|\partial_x^\beta \partial_t u_2\right\|^2. \tag{2.39}$$

Thus the further linear combination of (2.28), (2.29) and (2.39) gives the following result.

Corollary 2.4. *Under Assumptions (A1)–(A2), if $\delta > 0$ is small enough, then there is an energy functional $H_2(t)$ and a corresponding dissipation rate $D_2(t)$ such that for any $t \ge 0$,*

$$\frac{d}{dt} H_2(t) + c D_2(t) \le C\|\nabla_x u_1\|^2, \tag{2.40}$$

and

$$H_2(t) \le C D_2(t),$$

where

$$H_2(t) \sim \|u_2\|^2 + \sum_{\Lambda_1(\beta)} \|\partial_x^\beta u\|^2 + \sum_{\Lambda_2(\beta)} \|\partial_t \partial_x^\beta u\|^2$$
$$+ \sum_{\Lambda_3(\beta,\gamma)} \|\partial_x^\beta \partial_\xi^\gamma u_2\|^2 + \sum_{\Lambda_4(\beta,\gamma)} \|\partial_t \partial_x^\beta \partial_\xi^\gamma u_2\|^2,$$
$$D_2(t) \sim \|\nu^{1/2} u_2\|^2 + \sum_{\Lambda_1(\beta)} \|\nu^{1/2}\partial_x^\beta u_2\|^2 + \sum_{\Lambda_2(\beta)} \|\nu^{1/2}\partial_t \partial_x^\beta u_2\|^2$$
$$+ \sum_{\Lambda_3(\beta,\gamma)} \|\nu^{1/2}\partial_x^\beta \partial_\xi^\gamma u_2\|^2 + \sum_{\Lambda_4(\beta,\gamma)} \|\nu^{1/2}\partial_t \partial_x^\beta \partial_\xi^\gamma u_2\|^2,$$
$$+ \sum_{\Lambda_1(\beta)} \|\partial_x^\beta u_1\|^2 + \sum_{\Lambda_2(\beta)} \|\partial_t \partial_x^\beta u_1\|^2.$$

(v) *Further energy estimates on the microscopic part with velocity weight functions.* For later use, we shall make further energy estimates on the microscopic component weighted by velocity functions $\nu(\xi)$. We remark that it is necessary to introduce this velocity weight function to eliminate the time derivatives so that one can make use of the decay in time estimates for the linearized equation to deal with the nonlinear problems in terms of the contraction mapping theorem.

For generality, we shall make the weighted energy estimates on $w = w(t, x, \xi)$, which is the solution to the following nonhomogeneous linear equation:

$$\partial_t w + \nu w + \xi \cdot \nabla_x w + E_1 \cdot \nabla_\xi w = \phi + \xi \cdot E_2 w, \tag{2.41}$$

where $\phi = \phi(t, x, \xi)$ is a given function.

Lemma 2.9. *Under Assumptions (A1)–(A2), if $\delta > 0$ is small enough, then for any k, the solution w to Eq. (2.41) enjoys the following estimates:*

$$\frac{d}{dt}\|\nu^k w\|^2 + c\|\nu^{k+1/2} w\|^2 \le C\|\nu^{k-1/2}\phi\|^2, \tag{2.42}$$

$$\begin{aligned}&\frac{d}{dt}\sum_{\Lambda_1(\beta)}\|\nu^k \partial_x^\beta w\|^2 + c\sum_{\Lambda_1(\beta)}\|\nu^{k+1/2}\partial_x^\beta w\|^2\\ &\le C\sum_{\Lambda_1(\beta)}\|\nu^{k-1/2}\partial_x^\beta \phi\|^2 + C\delta\sum_{\substack{\Lambda_3(\beta,\gamma)\\ |\beta|\ge 1}}\|\nu^{k-1/2}\partial_x^\beta\partial_\xi^\gamma w\|^2,\end{aligned} \tag{2.43}$$

and

$$\begin{aligned}&\frac{d}{dt}\sum_{\Lambda_3(\beta,\gamma)} C_{\beta,\gamma}\|\nu^k \partial_x^\beta\partial_\xi^\gamma w\|^2 + c\sum_{\Lambda_3(\beta,\gamma)}\|\nu^{k+1/2}\partial_x^\beta\partial_\xi^\gamma w\|^2\\ &\le C\sum_{\Lambda_3(\beta,\gamma)}\|\nu^{k-1/2}\partial_x^\beta\partial_\xi^\gamma \phi\|^2 + C\sum_{\Lambda_0(\beta)}\|\nu^{k-1/2}\partial_x^\beta w\|^2,\end{aligned} \tag{2.44}$$

where $C_{\beta,\gamma}$ with $(\beta,\gamma) \in \Lambda_3(\beta,\gamma)$ are some positive constants, and positive constants c and C may depend on k. Furthermore, it holds that

$$\begin{aligned}&\frac{d}{dt}\sum_{0\le|\alpha|\le\ell} C_\alpha\|\nu^k \partial_{x,\xi}^\alpha w\|^2 + c\sum_{0\le|\alpha|\le\ell}\|\nu^{k+1/2}\partial_{x,\xi}^\alpha w\|^2\\ &\le C\sum_{0\le|\alpha|\le\ell}\|\nu^{k-1/2}\partial_{x,\xi}^\alpha \phi\|^2,\end{aligned} \tag{2.45}$$

where C_α are also some positive constants.

Proof. For simplicity of presentation, denote the time dependent linear operator $\mathbf{A}(t)$ by

$$\mathbf{A}(t) = \nu + \xi \cdot \nabla_x + E_1(t, x) \cdot \nabla_\xi.$$

Then (2.41) is rewritten as

$$\partial_t w + \mathbf{A}(t) w = \phi + \xi \cdot E_2 w.$$

Since for each multi-index β and γ, one has

$$
\begin{aligned}
&\partial_t(\nu^k\partial_x^\beta\partial_\xi^\gamma w)+\mathbf{A}(t)(\nu^k\partial_x^\beta\partial_\xi^\gamma w)\\
&\quad=\nu^k\partial_x^\beta\partial_\xi^\gamma\phi+\nu^k\xi\cdot\partial_x^\beta\partial_\xi^\gamma(E_2w)+e_\gamma\cdot\nu^k\partial_x^\beta\partial_\xi^{\gamma-1}(E_2w)-e_\gamma\cdot\nu^k\nabla_x\partial_x^\beta\partial_\xi^{\gamma-1}w\\
&\qquad-\sum_{0\le|\gamma'|\le|\gamma|-1}\partial_\xi^{\gamma-\gamma'}\nu\nu^k\partial_x^\beta\partial_\xi^{\gamma'}w-\sum_{0\le|\beta'|\le|\beta|-1}C_{\beta'}\partial_x^{\beta-\beta'}E_1\cdot\nu^k\nabla_\xi\partial_x^{\beta'}\partial_\xi^\gamma w\\
&\qquad+E_1\cdot\nabla_\xi\nu^k\partial_x^\beta\partial_\xi^\gamma w,
\end{aligned}
$$

and (2.42)–(2.44) can be proved by mimicking the arguments used in the proof of Lemma 2.5.

Finally (2.45) follows from the linear combination of (2.42)–(2.44). This completes the proof of the lemma. □

By applying the above result to the solutions of Eqs. (2.21) and (2.22), one has

Corollary 2.5. *Under Assumptions (A1)–(A2), if* $\delta > 0$ *is small enough, then for any* k*, it holds that*

$$
\begin{aligned}
&\frac{d}{dt}\left\|\nu^k u_2\right\|^2+c\left\|\nu^{k+1/2}u_2\right\|^2\\
&\quad\le C\|\nabla_x u_1\|^2+C\left(\left\|\nu^{(k-1/2)^+-1}u_2\right\|^2+\left\|\nu^{(k-1/2)^+-1}\nabla_x u_2\right\|^2\right),
\end{aligned}\tag{2.46}
$$

$$
\begin{aligned}
&\frac{d}{dt}\sum_{\Lambda_1(\beta)}\|\nu^k\partial_x^\beta u\|^2+c\sum_{\Lambda_1(\beta)}\|\nu^{k+1/2}\partial_x^\beta u\|^2\\
&\quad\le C\sum_{\Lambda_1(\beta)}\|\partial_x^\beta u_1\|^2+C\sum_{\Lambda_1(\beta)}\|\nu^{(k-1/2)^+-1}\partial_x^\beta u_2\|^2+C\delta\sum_{\Lambda_3(\beta,\gamma)}\|\nu^{k-1/2}\partial_x^\beta\partial_\xi^\gamma u_2\|^2,
\end{aligned}\tag{2.47}
$$

and

$$
\begin{aligned}
&\frac{d}{dt}\sum_{\Lambda_3(\beta,\gamma)}C_{\beta,\gamma}\|\nu^k\partial_x^\beta\partial_\xi^\gamma u_2\|^2+c\sum_{\Lambda_3(\beta,\gamma)}\|\nu^{k+1/2}\partial_x^\beta\partial_\xi^\gamma u_2\|^2\\
&\quad\le C\sum_{\Lambda_1(\beta)}\|\partial_x^\beta u_1\|^2+C\sum_{\Lambda_0(\beta)}\|\nu^{k-1/2}\partial_x^\beta u_2\|^2+C\sum_{\Lambda_3(\beta,\gamma)}\|\nu^{(k-1/2)^+-1}\partial_x^\beta\partial_\xi^\gamma u_2\|^2,
\end{aligned}\tag{2.48}
$$

where $(\cdot)^+$ *means that* $(m)^+ = m$ *if* $m \ge 0$ *and* 0 *otherwise. Furthermore, for any* k*, there is an energy functional* $H_{3,k}(t)$ *and a corresponding dissipation rate* $D_{3,k}(t)$ *such that for any* $t \ge 0$*,*

$$
\begin{aligned}
\frac{d}{dt}H_{3,k}(t)+cD_{3,k}(t)\le{}&C\sum_{\Lambda_1(\beta)}\|\partial_x^\beta u_1\|^2+C\sum_{\Lambda_0(\beta)}\|\nu^{(k-1/2)^--1}\partial_x^\beta u_2\|^2\\
&+C\sum_{\Lambda_3(\beta,\gamma)}\|\nu^{(k-1/2)^+-1}\partial_x^\beta\partial_\xi^\gamma u_2\|^2,
\end{aligned}\tag{2.49}
$$

and

$$H_{3,k}(t) \le C D_{3,k}(t), \tag{2.50}$$

where

$$H_{3,k}(t) \sim \|\nu^k u_2\|^2 + \sum_{\Lambda_1(\beta)} \|\nu^k \partial_x^\beta u\|^2 + \sum_{\Lambda_3(\beta,\gamma)} \|\nu^k \partial_x^\beta \partial_\xi^\gamma u_2\|^2, \tag{2.51}$$

$$D_{3,k}(t) \sim \|\nu^{k+1/2} u_2\|^2 + \sum_{\Lambda_1(\beta)} \|\nu^{k+1/2} \partial_x^\beta u\|^2 + \sum_{\Lambda_3(\beta,\gamma)} \|\nu^{k+1/2} \partial_x^\beta \partial_\xi^\gamma u_2\|^2. \tag{2.52}$$

Proof. Notice that (2.14) and (2.15) can be rewritten as

$$\partial_t u + \mathbf{A}(t)u = Ku + \xi \cdot E_2 u, \tag{2.53}$$

and

$$\partial_t u_2 + \mathbf{A}(t)u_2 = Ku_2 + [\mathbf{P}, \mathbf{D}(t)]u + \xi \cdot E_2 u_2. \tag{2.54}$$

Thus one can apply the estimate (2.43) to Eq. (2.53) with $\phi = Ku$ to obtain (2.47), where (2.12) was used. Similarly by applying the estimates (2.42) and (2.44) to Eq. (2.54) with

$$\phi = Ku_2 + [\mathbf{P}, \mathbf{D}(t)]u = Ku_2 + \mathbf{PD}(t)u - \mathbf{D}(t)u_1,$$

one can obtain (2.46) and (2.48). Here we have used the following identities:

$$\partial_x^\beta \partial_\xi^\beta K u_2 = K \partial_x^\beta \partial_\xi^\beta u_2 - [K, \partial_\xi^\beta] \partial_x^\beta u_2,$$

and

$$\mathbf{PD}(t)u = \mathbf{PD}(t)u_1 + \mathbf{PD}(t)\nu^{1-(k-1/2)^+} \left\{ \nu^{(k-1/2)^+ - 1} u_2 \right\}.$$

Finally (2.49) follows from the linear combination of (2.46)–(2.48). It is obvious that (2.50) holds from the equivalent forms (2.51) and (2.52) of $H_{3,k}(t)$ and $D_{3,k}(t)$. This completes the proof of the corollary. □

So far, based on the energy estimates on the linearized Eq. (2.1) only, we can obtain a standard energy inequality only with the first order derivatives of the macroscopic component u_1 as an error term. In fact, by a proper linear combination of (2.40) and (2.49) with $k = 1$ yields

Theorem 2.1. *Under Assumptions (A1)–(A2), if $\delta > 0$ is small enough, then there is an energy functional $H(t)$ and a corresponding dissipation rate $D(t)$ such that for any $t \ge 0$,*

$$\frac{d}{dt} H(t) + cD(t) \le C \|\nabla_x u_1\|^2, \tag{2.55}$$

and

$$H(t) \le C D(t), \tag{2.56}$$

where

$$
\begin{aligned}
H(t) \sim \|\nu u_2\|^2 &+ \sum_{\Lambda_1(\beta)} \|\nu \partial_x^\beta u\|^2 + \sum_{\Lambda_2(\beta)} \|\partial_t \partial_x^\beta u\|^2 \\
&+ \sum_{\Lambda_3(\beta,\gamma)} \|\nu \partial_x^\beta \partial_\xi^\gamma u_2\|^2 + \sum_{\Lambda_4(\beta,\gamma)} \|\partial_t \partial_x^\beta \partial_\xi^\gamma u_2\|^2, \\
D(t) \sim \|\nu^{3/2} u_2\|^2 &+ \sum_{\Lambda_1(\beta)} \|\nu^{3/2} \partial_x^\beta u_2\|^2 + \sum_{\Lambda_2(\beta)} \|\nu^{1/2} \partial_t \partial_x^\beta u_2\|^2 \\
&+ \sum_{\Lambda_3(\beta,\gamma)} \|\nu^{3/2} \partial_x^\beta \partial_\xi^\gamma u_2\|^2 + \sum_{\Lambda_4(\beta,\gamma)} \|\nu^{1/2} \partial_t \partial_x^\beta \partial_\xi^\gamma u_2\|^2 \\
&+ \sum_{\Lambda_1(\beta)} \|\partial_x^\beta u_1\|^2 + \sum_{\Lambda_2(\beta)} \|\partial_t \partial_x^\beta u_1\|^2.
\end{aligned}
$$

It is noticed that in $H(t)$, the power of the velocity weight function for the time derivatives is one less than that for others. Thus one can eliminate those terms involving the time derivatives by the equation. In fact, at first by $u_2 = u - u_1$, it holds that

$$
\sum_{\Lambda_4(\beta,\gamma)} \|\partial_t \partial_x^\beta \partial_\xi^\gamma u_2\|^2 \leq \sum_{\Lambda_4(\beta,\gamma)} \|\partial_t \partial_x^\beta \partial_\xi^\gamma u\|^2 + \sum_{\Lambda_4(\beta,\gamma)} \|\partial_t \partial_x^\beta \partial_\xi^\gamma u_1\|^2,
$$

where it further follows that

$$
\sum_{\Lambda_4(\beta,\gamma)} \|\partial_t \partial_x^\beta \partial_\xi^\gamma u_1\|^2 \leq \sum_{\Lambda_2(\beta)} \|\partial_t \partial_x^\beta u_1\|^2 \leq \sum_{\Lambda_2(\beta)} \|\partial_t \partial_x^\beta u\|^2.
$$

Then by Eq. (2.1), one has

$$
\partial_t u = -\xi \cdot \nabla_x u - E_1 \cdot \nabla_\xi u - \nu u_2 + K u_2 + \xi \cdot E_2 u,
$$

which implies that

$$
\begin{aligned}
\sum_{\Lambda_2(\beta)} \|\partial_t \partial_x^\beta u\|^2 &\leq C\|\nu u_2\|^2 + \sum_{\Lambda_1(\beta)} \|\nu \partial_x^\beta u\|^2, \\
\sum_{\Lambda_4(\beta,\gamma)} \|\partial_t \partial_x^\beta \partial_\xi^\gamma u\|^2 &\leq C\|\nu u_2\|^2 + \sum_{\Lambda_1(\beta)} \|\nu \partial_x^\beta u\|^2 + \sum_{\Lambda_3(\beta,\gamma)} \|\nu \partial_x^\beta \partial_\xi^\gamma u_2\|^2.
\end{aligned}
$$

Thus we have proved the following proposition.

Proposition 2.2. *Under the assumptions of Theorem 2.1, $H(t)$ has the equivalent form:*

$$
\begin{aligned}
H(t) &\sim \|\nu u_2\|^2 + \sum_{\Lambda_1(\beta)} \|\nu \partial_x^\beta u\|^2 + \sum_{\Lambda_3(\beta,\gamma)} \|\nu \partial_x^\beta \partial_\xi^\gamma u_2\|^2 \\
&\sim \sum_{1 \leq |\beta| \leq \ell} \|\partial_x^\beta u_1\|^2 + \sum_{0 \leq |\alpha| \leq \ell} \|\nu \partial_{x,\xi}^\alpha u_2\|^2.
\end{aligned}
$$

2.4. Optimal decay rates. (i) Estimates based on the spectral analysis. Set

$$\mathbf{B} = -\xi \cdot \nabla_x + \mathbf{L}.$$

Then from [27], one has

Proposition 2.3. *The linear operator* $\mathbf{B}$ *generates a semigroup* $e^{\mathbf{B}t}$ *which enjoys the decay in time estimates*

$$\|\nabla_x^m e^{\mathbf{B}t} g\| \leq C(1+t)^{-\sigma_{q,m}} \left(\|g\|_{Z_q} + \|\nabla_x^m g\|\right), \tag{2.57}$$

for any integer $m \geq 0$ *and any function* $g = g(x, \xi)$*, where* $q \in [1, 2]$ *and the decay rate is measured by*

$$\sigma_{q,m} = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{m}{2}. \tag{2.58}$$

Note that in terms of the linear operator $\mathbf{B}$, (2.1) can be rewritten as

$$\partial_t u = \mathbf{B}u - E_1 \cdot \nabla_\xi u + \xi \cdot E_2 u.$$

Then the solution to the initial value problem (2.1) and (2.2), with $s = 0$ for brevity, can be written in the mild form

$$u(t) = e^{\mathbf{B}t} u_0 + \int_0^t e^{\mathbf{B}(t-s)} \left\{-E_1 \cdot \nabla_\xi u + \xi \cdot E_2 u\right\}(s) ds. \tag{2.59}$$

Based on the above mild form and Proposition 2.3, one has the following lemma.

Lemma 2.10. *Assume that there is a constant* $\delta > 0$ *such that*

$$\|(1+|x|)E_i(t,x)\|_{L^\infty_{t,x}} + \||x|E_i(t,x)\|_{L^\infty_t\left(L_x^{2q/(2-q)}\right)} \leq \delta,$$

where $i = 1, 2$ *and* $1 \leq q \leq 2$*. Then it holds that*

$$\begin{aligned}\|\nabla_x u(t)\| \leq{}& C\lambda_0 (1+t)^{-\sigma_{q,1}} \\ &+ C\delta \int_0^t (1+t-s)^{-\sigma_{q,1}} (\|\nabla_x u_1(s)\| + \|\nu \nabla_x u_2(s)\| + \|\nabla_\xi \nabla_x u_2(s)\|) ds,\end{aligned} \tag{2.60}$$

where λ_0 *is given by*

$$\lambda_0 = \|u_0\|_{Z_q} + \|\nabla_x u_0\|. \tag{2.61}$$

Proof. For simplicity, set

$$G = -E_1 \cdot \nabla_\xi u + \xi \cdot E_2 u.$$

Then applying (2.57) to (2.59) yields

$$\|\nabla_x u(t)\| \leq C\lambda_0 (1+t)^{-\sigma_{q,1}} + C\delta \int_0^t (1+t-s)^{-\sigma_{q,1}} \left(\|G(s)\|_{Z_q} + \|\nabla_x G(s)\|\right) ds.$$

Furthermore, one has

$$\begin{aligned}\|G(s)\|_{Z_q} &\le \left\| \||x|E_1\|_{L_x^{2q/(2-q)}} \left\|\frac{\nabla_\xi u}{|x|}\right\|_{L_x^2} + C\nu\, \||x|E_2\|_{L_x^{2q/(2-q)}} \left\|\frac{u}{|x|}\right\|_{L_x^2} \right\|_{L_\xi^2}\\ &\le C\delta\left(\left\|\nabla_\xi\nabla_x u(s)\right\|_{L_\xi^2(L_x^2)} + \|\nu\nabla_x u(s)\|_{L_\xi^2(L_x^2)}\right)\\ &\le C\delta\left(\|\nabla_x u_1(s)\| + \|\nu\nabla_x u_2(s)\| + \|\nabla_\xi\nabla_x u_2(s)\|\right).\end{aligned}$$

Similarly it holds that

$$\begin{aligned}\|\nabla_x G(s)\| \le \Bigg\| &\||x|\nabla_x E_1\|_{L_x^\infty} \left\|\frac{\nabla_\xi u}{|x|}\right\|_{L_x^2} + \|E_1\|_{L_x^\infty} \|\nabla_x\nabla_\xi u\|_{L_x^2}\\ &+C\nu\, \||x|\nabla_x E_2\|_{L_x^\infty} \left\|\frac{u}{|x|}\right\|_{L_x^2} + C\nu\,\|E_2\|_{L_x^\infty}\|\nabla_x u\|_{L_x^2}\Bigg\|_{L_\xi^2}\\ \le{}& C\delta\left(\|\nabla_x u_1(s)\| + \|\nu\nabla_x u_2(s)\| + \|\nabla_\xi\nabla_x u_2(s)\|\right).\end{aligned}$$

Thus (2.60) is proved. This completes the proof of the lemma. □

(ii) Optimal decay rates. Combining Theorem 2.1 and Lemma 2.10 gives the optimal decay rates.

Lemma 2.11. *Assume*

$$n \ge 3,\quad 1 \le q < \frac{2n}{n+2}. \tag{2.62}$$

Under the assumptions of Theorem 2.1 and Lemma 2.10, if $\delta > 0$ is small enough, then it holds that

$$\sqrt{H(t)} \le C(1+t)^{-\sigma_{q,1}}\left\{\sqrt{H(0)} + \|u_0\|_{Z_q}\right\}, \tag{2.63}$$

and

$$\|u(t)\| \le C(1+t)^{-\sigma_{q,0}}\left\{\sqrt{H(0)} + \|u_0\|_{Z_q\cap L^2}\right\}. \tag{2.64}$$

Proof. Define

$$M(t) = \sup_{0\le s\le t}\left\{(1+s)^{2\sigma_{q,1}}H(s)\right\}. \tag{2.65}$$

Notice that $M(t)$ is non-decreasing and

$$\|\nabla_x u_1(s)\| + \|\nu\nabla_x u_2(s)\| + \|\nabla_\xi\nabla_x u_2(s)\| \le C\sqrt{H(s)} \le C(1+s)^{-\sigma_{q,1}}\sqrt{M(t)} \tag{2.66}$$

for any $0 \le s \le t$. Then (2.60) with (2.66) implies that for any $t \ge 0$,

$$\begin{aligned}\|\nabla_x u_1(t)\| &\le \|\nabla_x u(t)\|\\ &\le C\lambda_0(1+t)^{-\sigma_{q,1}} + C\delta\int_0^t (1+t-s)^{-\sigma_{q,1}}(1+s)^{-\sigma_{q,0}}ds\sqrt{M(t)}\\ &\le C(1+t)^{-\sigma_{q,1}}\left(\lambda_0 + \delta\sqrt{M(t)}\right),\end{aligned} \tag{2.67}$$

since $\sigma_{q,1} > 1$ from (2.58) and (2.62).

On the other hand, by the Gronwall inequality, (2.55) together with (2.56) gives

$$H(t) \leq e^{-ct} H(0) + C \int_0^t e^{-c(t-s)} \|\nabla_x u_1(s)\|^2 ds,$$

for some constant $c > 0$. Then, further using (2.67) yields

$$\begin{aligned} H(t) &\leq e^{-ct} H(0) + C \int_0^t e^{-c(t-s)} (1+s)^{-2\sigma_{q,1}} ds \left(\lambda_0^2 + \delta^2 M(t)\right) \\ &\leq C(1+t)^{-2\sigma_{q,1}} \left(H(0) + \lambda_0^2 + \delta^2 M(t)\right). \end{aligned}$$

Hence for any $t \geq 0$,

$$\sup_{0 \leq s \leq t} \left\{(1+s)^{2\sigma_{q,1}} H(s)\right\} \leq C \left(H(0) + \lambda_0^2 + \delta^2 M(t)\right),$$

i.e.,

$$M(t) \leq C \left(H(0) + \lambda_0^2 + \delta^2 M(t)\right).$$

Then if $\delta > 0$ is small enough, one has

$$M(t) \leq C \left(H(0) + \lambda_0^2\right). \tag{2.68}$$

Recalling the definitions (2.61) and (2.65) of λ_0 and $M(t)$, (2.68) gives (2.63).

Finally it follows from (2.57) and (2.63) that

$$\begin{aligned} \|u(t)\| &\leq C(1+t)^{-\sigma_{q,0}} \|u_0\|_{Z_q \cap L^2} + C \int_0^t (1+t-s)^{-\sigma_{q,0}} \|G(s)\|_{Z_q \cap L^2} ds \\ &\leq C(1+t)^{-\sigma_{q,0}} \|u_0\|_{Z_q \cap L^2} + C\delta \int_0^t (1+t-s)^{-\sigma_{q,1}} \sqrt{H(s)} ds \\ &\leq C(1+t)^{-\sigma_{q,0}} \|u_0\|_{Z_q \cap L^2} \\ &\quad + C\delta \int_0^t (1+t-s)^{-\sigma_{q,0}} (1+s)^{-\sigma_{q,1}} ds \left(\sqrt{H(0)} + \|u_0\|_{Z_q}\right) \\ &\leq C(1+t)^{-\sigma_{q,0}} \left(\sqrt{H(0)} + \|u_0\|_{Z_q \cap L^2}\right). \end{aligned}$$

Thus (2.64) is proved. This completes the proof of the lemma. □

(iii) Decay estimates on the solution operator $U(t,s)$. For any number k, define a norm $[[\cdot]]_{0,k}$ and a seminorm $[[\cdot]]_{1,k}$ over the Sobolev space $H^\ell(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ by

$$[[u]]_{0,k} = \sum_{0 \leq |\alpha| \leq \ell} \|\nu^k \partial^\alpha_{x,\xi} u\|, \tag{2.69}$$

$$[[u]]_{1,k} = \sum_{1 \leq |\beta| \leq \ell} \|\partial^\beta_x \mathbf{P} u\| + \sum_{0 \leq |\alpha| \leq \ell} \|\nu^k \partial^\alpha_{x,\xi} \{\mathbf{I} - \mathbf{P}\} u\|, \tag{2.70}$$

where $u = u(x, \xi)$. Notice that

$$[[u]]_{0,k} \sim [[u]]_{1,k} + \|u\|. \tag{2.71}$$

Theorem 2.2. *Suppose that*

(i) *the integers* $n \geq 3$, $\ell \geq 2$ *and the number* $1 \leq q < \frac{2n}{n+2}$*;*
(ii) *there a constant* $\delta > 0$ *such that*

$$\sum_{0\leq|\beta|\leq\ell} \left\|(1+|x|)\partial_x^\beta E_i(t,x)\right\|_{L^\infty_{t,x}} + \sum_{0\leq|\beta|\leq\ell-1} \left\|(1+|x|)\partial_t\partial_x^\beta E_i(t,x)\right\|_{L^\infty_{t,x}} \leq \delta,$$

and

$$\||x|E_i(t,x)\|_{L^\infty_t\left(L_x^{2q/(2-q)}\right)} \leq \delta,$$

where $i = 1, 2$.

Then for any $k \geq 1$, *there exist constants* $\delta_0 > 0$ *and* $C_0 > 0$ *such that for any* $\delta \leq \delta_0$, *the linear solution operator* $U(t,s)$, $-\infty < s \leq t < \infty$, *corresponding to the linear Eq. (2.1) satisfies the decay in time estimates*

$$[[U(t,s)u_0]]_{m,k} \leq C_0(1+t-s)^{-\sigma_{q,m}}([[u_0]]_{m,k} + \|u_0\|_{Z_q}), \quad m = 0, 1, \tag{2.72}$$

for any $u_0 = u_0(x,\xi)$, *where the constant* C_0 *depends only on* n, ℓ, q, k *and* δ_0.

Proof. It suffices to consider the case when $s = 0$. We now prove (2.87) by induction for $k \geq 1$. When $k = 1$, (2.72) follows from Proposition 2.2, Lemma 2.11 and (2.71).

Now suppose that (2.72) holds for some $k \geq 1$. We claim that it also holds for $k + \epsilon$ with any $0 \leq \epsilon \leq 3/2$. First consider the case of $m = 0$. Notice that $u = U(t,0)u_0$ satisfies

$$\partial_t u + \nu u + \xi \cdot \nabla_x u + E_1 \cdot \nabla_\xi u = Ku + \xi \cdot E_2 u.$$

Then recalling Eq. (2.41) and then applying the estimate (2.45) with $\phi = Ku$, one has

$$\begin{aligned}\frac{d}{dt}\sum_{0\leq|\alpha|\leq\ell} C_\alpha\|\nu^{k+\epsilon}\partial_{x,\xi}^\alpha u\|^2 + c\sum_{0\leq|\alpha|\leq\ell}\|\nu^{k+\epsilon+1/2}\partial_{x,\xi}^\alpha u\|^2 \\ \leq C\sum_{0\leq|\alpha|\leq\ell}\|\nu^{k+\epsilon-1/2}\partial_{x,\xi}^\alpha Ku\|^2,\end{aligned} \tag{2.73}$$

where by Lemma 2.9 and the inductive assumption, it holds that

$$\sum_{0\leq|\alpha|\leq\ell}\|\nu^{k+\epsilon-1/2}\partial_{x,\xi}^\alpha Ku\|^2 \leq C[[u]]_{0,k}^2 \leq C(1+t)^{-2\sigma_{q,0}}\left([[u_0]]_{0,k} + \|u_0\|_{Z_q}\right)^2. \tag{2.74}$$

Thus by the Gronwall inequality, (2.73) and (2.74) imply (2.72) with $m = 0$ for $k + \epsilon$.

Next consider the case of $m = 1$. Notice that the following equivalent property also holds:

$$[[u]]_{1,k} \sim \sum_{\Lambda_1(\beta)}\left\|\nu^k\partial_x^\beta u\right\| + \left\|\nu^k\{\mathbf{I}-\mathbf{P}\}u\right\| + \sum_{\Lambda_3(\beta,\gamma)}\left\|\nu^k\partial_x^\beta\partial_\xi^\gamma\{\mathbf{I}-\mathbf{P}\}u\right\|.$$

Thus from Corollary 2.5, similarly (2.72) with $m = 1$ holds for $k + \epsilon$. The details of the proof are omitted for brevity. Hence (2.72) with $m = 0$ or 1 holds for any $k \geq 1$. This completes the proof of the theorem. □

Remark 2.1. In the above theorem, the external force needs not to have time decay. Rather, it may be time independent, time periodic, or even bounded in time, though it should be small. In the case when the force is a small perturbation of some stationary potential force, i.e. in the form

$$F(t, x) = -\nabla_x \phi(x) + E(t, x),$$

where $\phi(x) \to 0$ as $|x| \to \infty$, we can have the same optimal decay estimates as (2.72) for the linearized equation derived by setting

$$f = \widetilde{\mathbf{M}} + \mathbf{M}^{1/2} u,$$

where

$$\widetilde{\mathbf{M}} = \tilde{\rho}(x)\mathbf{M}, \quad \tilde{\rho}(x) = e^{-\phi(x)}.$$

In this case, the linear equation is

$$\partial_t u + \xi \cdot \nabla_x u + F \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot F u = \tilde{\rho}(x)\mathbf{L}u. \tag{2.75}$$

If the same assumptions of Theorem 2.2 hold for $F(t, x)$ and $\phi(x)$ itself is also small in some Sobolev space, then the energy estimate similar to (2.13) still holds. For the estimates on the macroscopic component u_1, we consider Eq. (2.75) which can be rewritten as

$$\partial_t u - \mathbf{B}u = -F \cdot \nabla_\xi u + \frac{1}{2}\xi \cdot F u + (\tilde{\rho} - 1)\mathbf{L}u,$$

where the right-hand side can be regarded as a source term. Thus the decay estimate (2.72) is valid for the solution operator corresponding to (2.75) and can be used for the nonlinear problem considered in Sect. 3.

3. Applications to the Nonlinear Equation

3.1. Basic estimates. First from the definition (2.69) of the norm $[[\cdot]]_{0,k}$, Corollary 2.2 and $\partial_x^\beta \partial_\xi^\beta K u = K \partial_x^\beta \partial_\xi^\beta u - [K, \partial_\xi^\beta]\partial_x^\beta u$, we have

Lemma 3.1. *Let k be any number. For any $u = u(x, \xi)$, it holds that*

$$[[Ku]]_{0,k} \le C[[u]]_{0,(k-1)^+},$$

where C is some constant.

Lemma 3.2. *For any $u = u(x, \xi)$ and $v = v(x, \xi)$, it holds that*

$$\|\Gamma(u, v)\|_{Z_1} \le C\left(\|\nu u\|\|v\| + \|u\|\|\nu v\|\right),$$

where C is some constant.

The proof of the above lemma can be found in [28]. Finally we give a lemma on the estimates on the nonlinear term Γ in the norm $[[\cdot]]_{0,k}$.

Lemma 3.3. *Let $k \geq 0$ and $k_0 \leq 1$. Suppose that $\ell \geq [n/2]+2$. Then for any $u = u(x,\xi)$ and $v = v(x,\xi)$, it holds that*

$$[[\Gamma(u,v)]]_{0,k-k_0} \leq C([[u]]_{0,k+1-k_0}[[v]]_{0,k} + [[u]]_{0,k}[[v]]_{0,k+1-k_0}), \tag{3.1}$$

where C is some constant.

Proof. Write

$$\Gamma(u,v) = \frac{1}{2}\left\{\Gamma_1(u,v) + \Gamma_1(v,u) - \Gamma_2(u,v) - \Gamma_2(v,u)\right\},$$

with

$$\Gamma_1(u,v) = \int_{\mathbb{R}^n \times S^{n-1}} |(\xi - \xi_*)\cdot\omega|\mathbf{M}_*^{1/2} u(\xi')v(\xi_*')d\xi_* d\omega,$$
$$\Gamma_2(u,v) = \int_{\mathbb{R}^n \times S^{n-1}} |(\xi - \xi_*)\cdot\omega|\mathbf{M}_*^{1/2} u(\xi)v(\xi_*)d\xi_* d\omega.$$

It is obvious that (3.1) holds if it does for each Γ_j, $j = 1, 2$.

First consider Γ_1. As in [14], after taking a change of variable $z = \xi - \xi_*$, Γ_1 can be rewritten as

$$\Gamma_1(u,v)(\xi) = \int_{\mathbb{R}^n \times S^{n-1}} |z\cdot\omega|\mathbf{M}^{1/2}(\xi - z)u(\xi')v(z')dz d\omega, \tag{3.2}$$

where

$$\xi' = \xi - z_\parallel, \quad z' = \xi - z_\perp,$$

with $z_\parallel = (z\cdot\omega)\omega$, $z_\perp = z - z_\parallel$. Applying $\partial_{x,\xi}^\alpha = \partial_x^\beta \partial_\xi^\gamma$ with $0 \leq |\alpha| \leq \ell$ and $\alpha = \beta + \gamma$ to (3.2) yields

$$\begin{aligned}
\partial_{x,\xi}^\alpha \Gamma_1(u,v)(\xi) &= \sum_{\beta_1+\beta_2=\beta} C_{\beta_1}^\beta \partial_\xi^\gamma \int_{\mathbb{R}^n \times S^{n-1}} |z\cdot\omega|\mathbf{M}^{1/2}(\xi - z)(\partial_x^{\beta_1}u)(\xi')(\partial_x^{\beta_2}v)(z')dz d\omega \\
&= \sum_{\substack{\beta_1+\beta_2=\beta \\ \gamma_1+\gamma_{21}+\gamma_{22}=\gamma}} C_{\beta_1}^\beta C_{\gamma_1}^\gamma C_{\gamma_{21}}^{\gamma-\gamma_1} \\
&\quad \times \int_{\mathbb{R}^n \times S^{n-1}} |z\cdot\omega|\partial_\xi^{\gamma_1}\mathbf{M}^{1/2}(\xi - z)(\partial_x^{\beta_1}\partial_\xi^{\gamma_{21}}u)(\xi')(\partial_x^{\beta_2}\partial_\xi^{\gamma_{22}}v)(z')dz d\omega.
\end{aligned}$$

Notice that for any γ_1,

$$\left|\partial_\xi^{\gamma_1}\mathbf{M}^{1/2}(\xi - z)\right| \leq C\mathbf{M}^{1/4}(\xi - z).$$

Then

$$|\partial_{x,\xi}^\alpha \Gamma_1(u,v)(\xi)| \leq C \sum_{\alpha_1+\alpha_2\leq\alpha} \int_{\mathbb{R}^n \times S^{n-1}} |z\cdot\omega|\mathbf{M}^{1/4}(\xi - z)|\partial_{x,\xi}^{\alpha_1}u(\xi')|\,|\partial_{x,\xi}^{\alpha_2}v(z')|dz d\omega. \tag{3.3}$$

Without loss of generality, suppose $|\alpha_1| \leq |\alpha|/2$ in (3.3). Then by integrating (3.3) over $\mathbb{R}^n_x$ with respect to the space variable and using the Sobolev inequality, one has

$$\|\partial^{\alpha}_{x,\xi}\Gamma_1(u,v)(\xi)\|_{L^2_x} \leq C \sum_{|\alpha_1|\leq|\alpha|/2} \Gamma_{\alpha_1}(\xi), \tag{3.4}$$

where

$$\Gamma_{\alpha_1}(\xi) = \int_{\mathbb{R}^n\times S^{n-1}} |z\cdot\omega|\mathbf{M}^{1/4}(\xi - z)\|\nabla_x\partial^{\alpha_1}_{x,\xi}u(\xi')\|_{H^1_x}\|\partial^{\alpha_2}_{x,\xi}v(z')\|_{L^2_x}dzd\omega.$$

Noting that for any $k \geq 0$,

$$\nu^k(\xi')\nu^k(z') = \nu^k(\xi - z_{\|})\nu^k(\xi - z_{\perp}) \geq C\nu^k(\xi), \tag{3.5}$$

where the constant $C > 0$, then for each α_1, one has

$$\begin{aligned}
\nu^k\Gamma_{\alpha_1}(\xi) &\leq C\int_{\mathbb{R}^n\times S^{n-1}} |z\cdot\omega|\mathbf{M}^{1/4}(\xi - z)\|\nu^k\nabla_x\partial^{\alpha_1}_{x,\xi}u(\xi')\|_{H^1_x}\|\nu^k\partial^{\alpha_2}_{x,\xi}v(z')\|_{L^2_x}dzd\omega\\
&\leq C\left\{\int_{\mathbb{R}^n\times S^{n-1}} |z|^2\mathbf{M}^{1/2}(\xi - z)dzd\omega\right\}^{1/2}\\
&\quad\times\left\{\int_{\mathbb{R}^n\times S^{n-1}}\left[\|\nu^k\nabla_x\partial^{\alpha_1}_{x,\xi}u(\xi')\|_{H^1_x}\|\nu^k\partial^{\alpha_2}_{x,\xi}v(z')\|_{L^2_x}\right]^2 dzd\omega\right\}^{1/2}\\
&\leq C\nu(\xi)\left\{\int_{\mathbb{R}^n\times S^{n-1}}\left[\|\nu^k\nabla_x\partial^{\alpha_1}_{x,\xi}u(\xi')\|_{H^1_x}\|\nu^k\partial^{\alpha_2}_{x,\xi}v(z')\|_{L^2_x}\right]^2 dzd\omega\right\}^{1/2}.
\end{aligned}$$

Taking further integration over $\mathbb{R}^n_\xi$ with respect to the velocity variable gives

$$\begin{aligned}
\|\nu^{k-k_0}\Gamma_{\alpha_1}\|^2_{L^2_\xi} &\leq C\int_{\mathbb{R}^n\times S^{n-1}} \nu^{2-2k_0}(\xi)\|\nu^k\nabla_x\partial^{\alpha_1}_{x,\xi}u(\xi')\|^2_{H^1_x}\|\nu^k\partial^{\alpha_2}_{x,\xi}v(z')\|^2_{L^2_x}d\xi dzd\omega\\
&\leq C\int_{\mathbb{R}^n\times S^{n-1}}\left[\nu^{2-2k_0}(\xi') + \nu^{2-2k_0}(z')\right]\\
&\qquad\times\|\nu^k\nabla_x\partial^{\alpha_1}_{x,\xi}u(\xi')\|^2_{H^1_x}\|\nu^k\partial^{\alpha_2}_{x,\xi}v(z')\|^2_{L^2_x}d\xi'dz'd\omega,
\end{aligned}$$

where we have used the inequality (3.5) since $2-2k_0 \geq 0$ and taken change of variables $(\xi, z) \to (\xi', z')$, whose Jacobian is unity. Hence

$$\left\|\nu^{k-k_0}\Gamma_{\alpha_1}\right\|^2_{L^2_\xi} \leq C\Big([[u]]^2_{0,k+1-k_0}[[v]]^2_{0,k} + [[u]]^2_{0,k}[[v]]^2_{0,k+1-k_0}\Big). \tag{3.6}$$

Thus combining (3.4) and (3.6) implies that (3.1) holds for Γ_1.

Finally it is more straightforward to carry out the estimates on $\Gamma_2(u,v)$ in a similar way. The details are omitted. This completes the proof of the lemma. □

3.2. Global existence for the Cauchy problem. In this subsection, we consider the global existence and decay rates of the solution to the Cauchy problem for the non-linear Boltzmann equation:

$$\partial_t u + \xi \cdot \nabla_x u + F \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot F u = \mathbf{L}u + \Gamma(u) + \widetilde{S}, \tag{3.7}$$

$$u(t, x, \xi)|_{t=0} = u_0(x, \xi), \tag{3.8}$$

where $u = u(t, x, \xi)$, $(t, x, \xi) \in \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n$, and $\widetilde{S}$ is given by (1.5).

The main result is stated as follows.

Theorem 3.1. *Suppose that*

(B1) The integers $n \geq 3$, $\ell \geq [n/2] + 2$.
(B2) The functions $F = F(t, x)$, $S = S(t, x, \xi)$ and $u_0 = u_0(x, \xi)$ satisfy

$$F \in C_b^i\left(\mathbb{R}_t^+; H^{\ell-i}(\mathbb{R}_x^n)\right), \;\; i = 0, 1, \;\; S \in C_b^0\left(\mathbb{R}_t^+; H^\ell(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)\right),$$
$$u_0 \in H^\ell(\mathbb{R}_x^n \times \mathbb{R}_\xi^n).$$

(B3) There are constants $\delta > 0$, $k \geq 1$ and $\kappa > 1$ such that F and u_0 are bounded in the sense that

$$\sum_{0\leq|\beta|\leq\ell} \left\|(1 + |x|)\partial_x^\beta F(t, x)\right\|_{L^\infty_{t,x}} + \sum_{0\leq|\beta|\leq\ell-1} \left\|(1 + |x|)\partial_t\partial_x^\beta F(t, x)\right\|_{L^\infty_{t,x}} + \||x|F(t, x)\|_{L_t^\infty(L_x^2)} \leq \delta, \tag{3.9}$$

$$[[u_0]]_{0,k+1/2} + \|u_0\|_{Z_1} \leq \delta, \tag{3.10}$$

and moreover, F and S decay in time in the sense that

$$\|F(t)\|_{H_x^\ell \cap L_x^1} \leq \delta(1 + t)^{-\kappa}, \tag{3.11}$$

$$[[\mathbf{M}^{-1/2}S(t)]]_{0,k-1/2} + \left\|\mathbf{M}^{-1/2}S(t)\right\|_{Z_1} \leq \delta(1 + t)^{-\kappa}. \tag{3.12}$$

Then there are constants $\delta_1 > 0$ and $C_1 > 0$ such that for any $\delta \leq \delta_1$, the Cauchy problem (3.7)–(3.8) corresponding to (1.1) has a unique global classical solution

$$u \in C_b^i\left(\mathbb{R}_t^+; H^{\ell-i}(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)\right), \;\; i = 0, 1, \tag{3.13}$$

which satisfies

$$\sup_{t\geq 0}(1 + t)^{2\kappa_0}[[u(t)]]_{0,k}^2 + \int_0^\infty [[u(s)]]_{0,k+1/2}^2 ds \leq C_1^2, \tag{3.14}$$

where C_1 can be also taken as $C_1 = C_1'\delta$ for another constant C_1' independent of δ, and κ_0 is given by

$$\begin{cases} \frac{1}{2} < \kappa_0 < \kappa - \frac{1}{2} & \text{if } \sigma_{1,0} \geq \kappa - \frac{1}{2}, \\ \kappa_0 = \sigma_{1,0} & \text{if } \sigma_{1,0} < \kappa - \frac{1}{2}. \end{cases} \tag{3.15}$$

Furthermore, it holds that

$$\sum_{0\le|\alpha|\le\ell-1}\left\|\nu^{k-1}\partial_t\partial_{x,\xi}^{\alpha}u(t)\right\|\le C\delta(1+t)^{-\kappa_0}, \tag{3.16}$$

for some constant C.

In order to prove the above theorem, we introduce a function set $\mathbb{S}(C_1)$ by

$$\mathbb{S}(C_1)=\left\{u=u(t,x,\xi)\;\middle|\;u\in C_b^0\left(\mathbb{R}_t^+;H^\ell(\mathbb{R}_x^n\times\mathbb{R}_\xi^n)\right),\ |||u|||_{k,\kappa_0}\le C_1\right\},$$

where $C_1>0$ is some constant to be determined later, and the norm $|||\cdot|||_{k,\kappa_0}$ is defined by

$$|||u|||_{k,\kappa_0}^2=\sup_{t\ge0}(1+t)^{2\kappa_0}[[u(t)]]_{0,k}^2+\int_0^\infty[[u(s)]]_{0,k+1/2}^2ds.$$

Clearly, $\mathbb{S}(C_1)$ is a complete metric space with the metric induced by the norm $|||\cdot|||_{k,\kappa_0}$. Under some conditions, the solution to (3.7)–(3.8) will be obtained by applying the contraction mapping theorem to find a fixed point in $\mathbb{S}(C_1)$ for some nonlinear mapping $\mathbf{\Psi}$, where $\mathbf{\Psi}$ is defined by

$$\mathbf{\Psi}(u)=U(t,0)u_0+\int_0^t U(t,s)\{\Gamma(u(s),u(s))+\widetilde{S}(s)\}ds. \tag{3.17}$$

Thus one has to estimate the time integral in (3.17) in terms of the norm $|||\cdot|||_{k,\kappa_0}$. For this, in what follows, given a function $\phi=\phi(t,x,\xi)$, we will first consider the estimate on the general time integral

$$(\mathbf{T}\phi)(t,x,\xi)=\int_0^t U(t,s)\phi(s,x,\xi)ds.$$

This time integral can be written as two parts again by Duhamel's formula. In fact, define the solution operator $U_1(t,s)$ for any $0\le s\le t$ in the sense that for any $v_0=v_0(x,\xi)$, $v=v(t,x,\xi)=U_1(t,s)v_0$ denotes the solution to the following initial value problem:

$$\partial_t v+\nu v+\xi\cdot\nabla_x v+F\cdot\nabla_\xi v-\frac{1}{2}\xi\cdot Fv=0,$$
$$v(t,x,\xi)|_{t=s}=v_0(x,\xi).$$

Note that $\mathbf{L}=-\nu+K$. Then again by Duhamel's formula, the solution operator $U(t,s)$ can be rewritten as

$$U(t,s)=U_1(t,s)+U_2(t,s),\quad 0\le s\le t,$$

where

$$U_2(t,s)=\int_s^t U(t,\tau)KU_1(\tau,s)d\tau.$$

Thus we further define

$$(\mathbf{T}_j\phi)(t,x,\xi)=\int_0^t U_j(t,s)\phi(s,x,\xi)ds,\quad j=1,2.$$

Then

$$\mathbf{T}\phi=\mathbf{T}_1\phi+\mathbf{T}_2\phi.$$

The following estimates follow.

Lemma 3.4. *Suppose (3.9). If $\delta > 0$ is small enough, then one has*

$$(1+t)^{2m}[[\mathbf{T}_1\phi(t)]]_{0,k}^2 + \int_0^t (1+s)^{2m}[[\mathbf{T}_1\phi(s)]]_{0,k+1/2}^2 ds \leq C\int_0^t (1+s)^{2m}[[\phi(s)]]_{0,k-1/2}^2 ds, \tag{3.18}$$

for any $m \geq 0$ and any k, and

$$(1+t)^{2m}\|\mathbf{T}_1\phi(t)\|_{Z_1}^2 + \int_0^t (1+s)^{2m}\|\mathbf{T}_1\phi(s)\|_{Z_1}^2 ds \leq C\int_0^t (1+s)^{2m}\left([[\phi(s)]]_{0,k-1/2}^2 + \|\phi(s)\|_{Z_1}^2\right) ds, \tag{3.19}$$

for any $m \geq 0$ and any $k \geq 1/2$.

Proof. For simplicity, write $w = \mathbf{T}_1\phi$, which by the definitions of $\mathbf{T}_1$ and $U_1(t,s)$, satisfies the following Cauchy problem with zero initial data:

$$\partial_t w + \nu w + \xi \cdot \nabla_x w + F \cdot \nabla_\xi w - \frac{1}{2}\xi \cdot F w = \phi, \tag{3.20}$$

$$w(t,x,\xi)|_{t=0} = 0. \tag{3.21}$$

By (2.45), one has the energy inequality

$$\frac{d}{dt}J_{0,k}[w(t)] + cJ_{0,k+1/2}[w(t)] \leq C[[\phi(t)]]_{0,k-1/2}^2, \tag{3.22}$$

for any k, where to the end, the nonlinear functional $J_{0,k}[\cdot]$ is defined by

$$J_{0,k}[w(t)] \sim [[w(t)]]_{0,k}. \tag{3.23}$$

After integration, (3.22) implies

$$J_{0,k}[w(t)] + \int_0^t J_{0,k+1/2}[w(s)]ds \leq C\int_0^t [[\phi(s)]]_{0,k-1/2}^2 ds. \tag{3.24}$$

On the other hand, multiplying (3.22) by $(1+t)^{2m}$ with $m \geq 0$ and further integrating it gives

$$\begin{aligned}
&(1+t)^{2m}J_{0,k}[w(t)] + c\int_0^t (1+s)^{2m}J_{0,k+1/2}[w(s)]ds \\
&\leq 2m\int_0^t (1+s)^{m-1}J_{0,k}[w(s)]ds + C\int_0^t (1+s)^{2m}[[\phi(s)]]_{0,k-1/2}^2 ds \\
&\leq \frac{c}{2}\int_0^t (1+s)^m J_{0,k+1/2}[w(s)]ds + C\int_0^t J_{0,k+1/2}[w(s)]ds \\
&\quad + C\int_0^t (1+s)^{2m}[[\phi(s)]]_{0,k-1/2}^2 ds.
\end{aligned} \tag{3.25}$$

Then (3.25) together with (3.23) and (3.24) yields (3.18).

Next consider the estimate (3.19) in the norm $\|\cdot\|_{Z_1}$. It can be based on the explicit form for the solution w from (3.20)–(3.21):

$$w(t,x,\xi)=\int_0^t e^{-\nu(\xi)(t-s)}\left\{F\cdot\nabla_\xi w-\xi/2\cdot Fw+\phi\right\}(s,x-(t-s)\xi,\xi)ds,$$

which implies

$$\|w(t,\xi)\|_{L^1(\mathbb{R}^n_x)}\leq C\int_0^t e^{-\nu_0(t-s)}\Big(\|\nabla_\xi\nabla_x w(s,\xi)\|_{L^2(\mathbb{R}^n_x)} \\ +\nu\|\nabla_x w(s,\xi)\|_{L^2(\mathbb{R}^n_x)}+\|\phi(s,\xi)\|_{L^1(\mathbb{R}^n_x)}\Big)ds.$$

Further taking the norm $\|\cdot\|_{L^2(\mathbb{R}^n_\xi)}$ gives

$$\|w(t)\|_{Z_1}\leq C\int_0^t e^{-\nu_0(t-s)}G(s)ds, \tag{3.26}$$

where for simplicity, we used the notion

$$G(s)=\|\nabla_\xi\nabla_x w(s)\|+\|\nu\nabla_x w(s)\|+\|\phi(s)\|_{Z_1}. \tag{3.27}$$

From (3.26), we claim that for any $m\geq 0$,

$$(1+t)^{2m}\|w(t)\|_{Z_1}^2+\int_0^t(1+s)^{2m}\|w(s)\|_{Z_1}^2ds\leq C\int_0^t(1+s)^{2m}G(s)^2ds. \tag{3.28}$$

In fact, on one hand, by the Hölder inequality, it is easy to see from (3.26) that

$$\begin{aligned}\|w(t)\|_{Z_1}^2&\leq C\int_0^t e^{-2\nu_0(t-s)}(1+s)^{-2m}ds\int_0^t(1+s)^{2m}G(s)^2ds\\&\leq C(1+t)^{-2m}\int_0^t(1+s)^{2m}G(s)^2ds.\end{aligned} \tag{3.29}$$

On the other hand, again by (3.26), one has

$$\int_0^t(1+s)^{2m}\|w(s)\|_{Z_1}^2ds\leq\int_0^t(1+s)^{2m}\left[\int_0^s e^{-\nu_0(s-\tau)}G(\tau)d\tau\right]^2ds. \tag{3.30}$$

By the Schwarz inequality, it holds that

$$\begin{aligned}&\left[\int_0^s e^{-\nu_0(s-\tau)}G(\tau)d\tau\right]^2\\&\leq\int_0^s e^{-\nu_0(s-\tau)}(1+\tau)^{-2m}d\tau\int_0^s e^{-\nu_0(s-\tau)}(1+\tau)^{2m}G(\tau)^2d\tau\\&\leq C(1+s)^{-2m}\int_0^s e^{-\nu_0(s-\tau)}(1+\tau)^{2m}G(\tau)^2d\tau,\end{aligned}$$

which together with (3.30) gives

$$\begin{aligned}\int_0^t (1+s)^{2m}\|w(s)\|_{Z_1}^2 ds &\le C\int_0^t\int_0^s e^{-\nu_0(s-\tau)}(1+\tau)^{2m}G(\tau)^2 d\tau ds\\ &= C\int_0^t d\tau (1+\tau)^{2m}G(\tau)^2\int_\tau^t e^{-\nu_0(s-\tau)}ds\\ &\le C\int_0^t (1+\tau)^{2m}G(\tau)^2 d\tau. \end{aligned}\tag{3.31}$$

Thus (3.28) follows from (3.29) and (3.31). Furthermore, notice from (3.27) and $k \ge 1/2$ that

$$\begin{aligned}G(s)^2 &\le C\left(\|\nabla_\xi\nabla_x w(s)\|^2 + \|\nu\nabla_x w(s)\|^2 + \|\phi(s)\|_{Z_1}^2\right)\\ &\le C\left([[w(t)]]_{0,k+1/2}^2 + \|\phi(s)\|_{Z_1}^2\right),\end{aligned}$$

which by (3.18), implies

$$\begin{aligned}\int_0^t (1+s)^{2m}G(s)^2 ds &\le C\int_0^t (1+s)^{2m}\left([[w(t)]]_{0,k+1/2}^2 + \|\phi(s)\|_{Z_1}^2\right)ds\\ &\le C\int_0^t (1+s)^{2m}\left([[\phi(t)]]_{0,k-1/2}^2 + \|\phi(s)\|_{Z_1}^2\right)ds. \end{aligned}\tag{3.32}$$

With the notion $w = \mathbf{T}_1\phi$, combining (3.28) and (3.32) leads to (3.19). This completes the proof of the lemma. □

Lemma 3.5. *Suppose (3.9). If* $\delta > 0$ *is small enough, then one has*

$$\begin{aligned}&(1+t)^{2m}[[\mathbf{T}_2\phi(t)]]_{0,k}^2 + \int_0^t [[\mathbf{T}_2\phi(s)]]_{0,k+1/2}^2 ds\\ &\le C\int_0^t (1+s)^{2m}\left([[\phi(s)]]_{0,k-1/2}^2 + \|\phi(s)\|_{Z_1}^2\right)ds, \end{aligned}\tag{3.33}$$

for any $1/2 < m \le \sigma_{1,0}$ *and any* $k \ge 1$.

Proof. First fix some m and k with $1/2 < m \le \sigma_{1,0}$ and $k \ge 1$. Set $z = \mathbf{T}_2\phi$ for simplicity. By the definitions of $\mathbf{T}_i$ and $U_i(t,s)$, $i = 1, 2$, note that

$$z(t) = \mathbf{T}_2\phi(t) = \int_0^t U_2(t,s)\phi(s)ds = \int_0^t U(t,s)K\mathbf{T}_1\phi(s)ds.$$

Then by Theorem 2.2 and Lemma 3.4, it holds that

$$\begin{aligned}
[[z(t)]]_{0,k}^2 &\leq C\left|\int_0^t (1+t-s)^{-\sigma_{1,0}}\left([[K\mathbf{T}_1\phi(s)]]_{0,k}+\|K\mathbf{T}_1\phi(s)\|_{Z_1}\right)ds\right|^2\\
&\leq C\left|\int_0^t (1+t-s)^{-\sigma_{1,0}}([[\mathbf{T}_1\phi(s)]]_{0,k-1}+\|\mathbf{T}_1\phi(s)\|_{Z_1})ds\right|^2\\
&\leq C\int_0^t (1+t-s)^{-2\sigma_{1,0}}(1+s)^{-2m}ds\\
&\quad\times\int_0^t (1+s)^{2m}\left([[\mathbf{T}_1\phi(s)]]_{0,k+1/2}^2+\|\mathbf{T}_1\phi(s)\|_{Z_1}\right)^2ds\\
&\leq C(1+t)^{-2m}\int_0^t (1+s)^{2m}\left([[\phi(s)]]_{0,k-1/2}^2+\|\phi(s)\|_{Z_1}^2\right)ds. \qquad (3.34)
\end{aligned}$$

On the other hand, $z=z(t,x,\xi)$ is the solution to the following initial value problem with zero initial data:

$$\partial_t z+\nu z+\xi\cdot\nabla_x z+F\cdot\nabla_\xi z-\frac{1}{2}\xi\cdot Fz=Kz+KT_1\phi,$$
$$z(t,x,\xi)|_{t=0}=0.$$

This means that

$$z=T_1(Kz+KT_1\phi).$$

Use (3.18) with $m=0$ to deduce

$$\begin{aligned}
\int_0^t [[z(s)]]_{0,k+1/2}^2ds &\leq C\int_0^t [[Kz+KT_1\phi]]_{0,k-1/2}^2ds\\
&\leq C\int_0^t [[z(s)]]_{0,k-3/2}^2ds+C\int_0^t [[T_1\phi(s)]]_{0,k-3/2}^2ds,
\end{aligned}$$

where further, it holds from (3.34) that

$$\begin{aligned}
&\int_0^t [[z(s)]]_{0,k-3/2}^2ds\leq\int_0^t [[z(s)]]_{0,k}^2ds\\
&\leq C\int_0^t (1+s)^{-2m}ds\sup_{0\leq s\leq t}\int_0^s (1+\tau)^{2m}\left([[\phi(\tau)]]_{0,k-1/2}^2+\|\phi(\tau)\|_{Z_1}^2\right)d\tau\\
&\leq C\int_0^t (1+\tau)^{2m}\left([[\phi(\tau)]]_{0,k-1/2}^2+\|\phi(\tau)\|_{Z_1}^2\right)d\tau,
\end{aligned}$$

and again from (3.18) with $m=0$ that

$$\int_0^t [[T_1\phi(s)]]_{0,k-3/2}^2ds\leq\int_0^t [[T_1\phi(s)]]_{0,k+1/2}^2ds\leq C\int_0^t [[\phi(s)]]_{0,k-1/2}^2ds.$$

Then,

$$\int_0^t [[z(s)]]_{0,k+1/2}^2ds\leq C\int_0^t (1+s)^{2m}\left([[\phi(s)]]_{0,k-1/2}^2+\|\phi(s)\|_{Z_1}^2\right)ds. \qquad (3.35)$$

Thus (3.33) follows from (3.34) and (3.35). This completes the proof of the lemma. □

Corollary 3.1. *Suppose (3.9). If* $\delta > 0$ *is small enough, then one has*

$$\begin{aligned}(1+t)^{2m}[[\mathbf{T}\phi(t)]]_{0,k}^2 + \int_0^t [[\mathbf{T}\phi(s)]]_{0,k+1/2}^2 ds \\ \leq C\int_0^t (1+s)^{2m}\left([[\phi(s)]]_{0,k-1/2}^2 + \|\phi(s)\|_{Z_1}^2\right) ds,\end{aligned}$$

for any $1/2 < m \leq \sigma_{1,0}$ *and any* $k \geq 1$.

Now we are in a position to prove the global existence of the solution to the Cauchy problem for the nonlinear Boltzmann equation.

Proof of Theorem 3.1. First we prove that there is a proper constant $C_1 > 0$ such that $\mathbf{\Psi}$ is a contraction mapping from $\mathbb{S}(C_1)$ to itself, and thus it has a fixed point in $\mathbb{S}(C_1)$ which is a unique solution to the Cauchy problem (3.7)–(3.8). For this purpose, we start with a claim that there is a constant C such that for any $u, v \in \mathbb{S}(C_1)$,

$$|||\mathbf{\Psi}(u)|||_{k,\kappa_0} \leq C\delta + C|||u|||_{k,\kappa_0}^2, \tag{3.36}$$

$$|||\mathbf{\Psi}(u) - \mathbf{\Psi}(v)|||_{k,\kappa_0} \leq C|||u+v|||_{k,\kappa_0}|||u-v|||_{k,\kappa_0}. \tag{3.37}$$

In fact, recall the definition (3.17) of $\mathbf{\Psi}$, and then it is straightforward to compute

$$\begin{aligned}|||U(t,0)u_0|||_{k,\kappa_0}^2 &\leq \sup_{t\geq 0}(1+t)^{2\kappa_0}[[U(t,0)u_0]]_{0,k}^2 + \int_0^\infty [[U(s,0)u_0]]_{0,k+1/2}^2 ds \\ &\leq C\sup_{t\geq 0}(1+t)^{2\kappa_0-2\sigma_{1,0}}[[u_0]]_{0,k}^2 + C\int_0^\infty (1+s)^{-2\sigma_{1,0}} ds[[u_0]]_{0,k+1/2}^2 \\ &\leq C[[u_0]]_{0,k+1/2}^2 \leq C\delta^2,\end{aligned} \tag{3.38}$$

where we used (3.10), and the inequalities $\kappa_0 \leq \sigma_{1,0}$ and $2\sigma_{1,0} > 1$ since $n \geq 3$. Furthermore, noticing from (3.15) and $n \geq 3$ that $1/2 < \kappa_0 \leq \sigma_{1,0}$, one can apply Corollary 3.1 with $m = \kappa_0$ to obtain

$$\begin{aligned}&\left|\left|\left|\int_0^t U(t,s)\Gamma(u(s),u(s))ds\right|\right|\right|_k^2 \\ &\leq C\int_0^\infty (1+s)^{2\kappa_0}\left([[\Gamma(u(s),u(s))]]_{0,k-1/2}^2 + \|\Gamma(u(s),u(s))\|_{Z_1}^2\right) ds \\ &\leq C\int_0^\infty (1+s)^{2\kappa_0}[[u(s)]]_{0,k+1/2}^2[[u(s)]]_{0,k}^2 ds \\ &\leq C\int_0^\infty [[u(s)]]_{0,k+1/2}^2 ds \sup_{s\geq 0}(1+s)^{2\kappa_0}[[u(s)]]_{0,k}^2 \\ &\leq C|||u|||_{k,\kappa_0}^2,\end{aligned} \tag{3.39}$$

where Lemma 3.3 was used. Since (3.11) and (3.12) together with (1.5) imply

$$[[\widetilde{S}(s)]]_{0,k-1/2} + \|\widetilde{S}(s)\|_{Z_1} \leq C\delta(1+s)^{-\kappa},$$

similarly applying Corollary 3.1 with $m = \kappa_0$ yields

$$
\begin{aligned}
\left|\left|\left|\int_0^t U(t,s)\widetilde{S}(s)ds\right|\right|\right|_k^2 &\leq C\int_0^\infty (1+s)^{2\kappa_0}\left([[\widetilde{S}(s)]]_{0,k-1/2}^2 + \|\widetilde{S}(s)\|_{Z_1}^2\right)ds \\
&\leq C\delta^2\int_0^\infty (1+s)^{2\kappa_0-2\kappa}ds \\
&\leq C\delta^2, \qquad (3.40)
\end{aligned}
$$

where by (3.15), $\kappa_0 < \kappa - 1/2$ was used. Thus by (3.17), combining (3.38), (3.39) and (3.40) proves (3.36). For (3.37), notice that since Γ is bilinear,

$$
\Gamma(u,u) - \Gamma(v,v) = \Gamma(u+v, u-v).
$$

Then it holds that

$$
\mathbf{\Psi}(u) - \mathbf{\Psi}(v) = \int_0^t U(t,s)\Gamma(u+v, u-v)(s)ds,
$$

which similar to the proof of (3.39), implies (3.37).

Now suppose $u, v \in \mathbb{S}(C_1)$. Then based on (3.36) and (3.37), it is easy to see that

$$
\mathbf{\Psi}(u), \mathbf{\Psi}(v) \in C_b^0\left(\mathbb{R}_t^+; H^\ell(\mathbb{R}_x^n)\right),
$$

with estimates

$$
\begin{aligned}
|||\mathbf{\Psi}(u)|||_{k,\kappa_0} &\leq C\delta + CC_1^2, \\
|||\mathbf{\Psi}(u) - \mathbf{\Psi}(v)|||_{k,\kappa_0} &\leq 2CC_1|||u-v|||_{k,\kappa_0}.
\end{aligned}
$$

If $\delta \leq \delta_1$ with $\delta_1 > 0$ small enough, then there is a constant $C_1 > 0$ depending only on δ_1 and C such that

$$
C\delta + CC_1^2 \leq C_1, \quad 2CC_1 < 1.
$$

Thus $\mathbf{\Psi}(u), \mathbf{\Psi}(v) \in \mathbb{S}(C_1)$ and

$$
|||\mathbf{\Psi}(u) - \mathbf{\Psi}(v)|||_{k,\kappa_0} \leq \mu|||u-v|||_{k,\kappa_0}, \quad \mu = 2CC_1 < 1.
$$

Therefore $\mathbf{\Psi}$ is a contraction mapping over $\mathbb{S}(C_1)$. Thus there is a unique fixed point u in $\mathbb{S}(C_1)$ as a mild solution to the Cauchy problem (3.7)–(3.8). Then (3.13) with $i = 0$ and (3.14) are proved. In addition, it is obvious that C_1 can be also taken as $C_1 = C_1'\delta$ for another constant C_1' independent of δ.

Finally the time-differentiability (3.13) with $i = 1$ of the solution u and the estimate (3.16) directly follow from the equation. This completes the proof of the theorem.

3.3. Existence of time periodic solution. In this subsection, we are concerned with the existence of the time periodic solution to the nonlinear Boltzmann equation

$$\partial_t u + \xi \cdot \nabla_x u + F \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot F u = \mathbf{L}u + \Gamma(u) + \widetilde{S}, \tag{3.41}$$

where $u = u(t, x, \xi)$, $(t, x, \xi) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, and $\widetilde{S}$ is given by (1.5).

Roughly speaking, our goal is to show that if the time dependent external force F and source S are time periodic with period T, then Eq. (3.41) should have a time periodic solution with the same period under some additional assumptions. When the space dimension $n \geq 5$, this can be achieved by making use of the decay in time property of the linearized equation which is established in Sect. 2.

Precisely, the main result is stated as follows.

Theorem 3.2. *Suppose that*

(C1) the integers $n \geq 5$, $\ell \geq [n/2] + 2$*;*
(C2) the functions $F = F(t, x)$ *and* $S = S(t, x, \xi)$ *are time periodic with period* T*, satisfying*

$$F \in C_b^i\left(\mathbb{R}_t; H^{\ell-i}(\mathbb{R}_x^n)\right), \quad i = 0, 1, \quad S \in C_b^0\left(\mathbb{R}_t; H^{\ell}(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)\right);$$

(C3) there are constants $\delta > 0$ *and* $k \geq 1$ *such that* F *and* S *are bounded in the sense that*

$$\sum_{0\leq|\beta|\leq\ell} \left\|(1+|x|)\partial_x^\beta F(t,x)\right\|_{L^\infty_{t,x}} + \sum_{0\leq|\beta|\leq\ell-1} \left\|(1+|x|)\partial_t\partial_x^\beta F(t,x)\right\|_{L^\infty_{t,x}} + \||x|F(t,x)\|_{L^\infty_t(L^2_x)} \leq \delta, \tag{3.42}$$

$$\sup_{t\in\mathbb{R}}\left\{\|F(t)\|_{H^\ell_x\cap L^1_x} + [[\mathbf{M}^{-1/2}S(t)]]_{0,k-1/2} + \left\|\mathbf{M}^{-1/2}S(t)\right\|_{Z_1}\right\} \leq \delta. \tag{3.43}$$

Then there are constants $\delta_2 > 0$ *and* $C_2 > 0$ *such that for any* $\delta \leq \delta_2$*, Eq. (3.41) corresponding to (1.1) has a unique time periodic solution*

$$u^* \in C_b^i\left(\mathbb{R}_t; H^{\ell-i}(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)\right), \quad i = 0, 1,$$

with the same period T*, which satisfies*

$$\sup_{0\leq t\leq T} [[u^*(t)]]_{0,k}^2 + \int_0^T [[u^*(t)]]_{0,k+1}^2 dt \leq C_2^2, \tag{3.44}$$

where precisely, C_2 *can be chosen as* $C_2 = C_2'\delta$ *with* C_2' *independent of* δ*. Furthermore, it holds that*

$$\sup_{0\leq t\leq T} [[u^*(t)]]_{0,k+1/2} + \sup_{0\leq t\leq T} \sum_{0\leq|\alpha|\leq\ell-1} \left\|\nu^{k-1}\partial_t\partial_{x,\xi}^\alpha u^*(t)\right\| \leq C\delta, \tag{3.45}$$

for some constant C*.*

In order to prove Theorem 3.2, we shall use the arguments developed in [26] to deal with the existence of the periodic solution. Define

$$\mathbf{\Phi}(u) = \int_{-\infty}^{t} U(t,s)\{\Gamma(u(s),u(s)) + \widetilde{S}(s)\}ds.$$

Suppose that $\mathbf{\Phi}$ has a unique fixed point $\bar{u}(t)$. Then if $\widetilde{S}(t)$ is time periodic with period T, so is $\bar{u}(t)$ as in [26]. Furthermore, $\bar{u}(t)$ is a desired time periodic solution provided that it is differentiable with respect to time t. Thus it suffices to find the fixed point of $\mathbf{\Phi}$ in a proper complete metric space. We choose it as $\mathbb{S}(C_2)$ defined by

$$\mathbb{S}(C_2) = \left\{ u = u(t,x,\xi) \,\middle|\, \begin{array}{l} u \text{ is time periodic with period } T, \\ u \in C_b^0\left(\mathbb{R}_t; H^\ell(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)\right),\ |||u|||_{k,*} \le C_2 \end{array} \right\},$$

where $C_2 > 0$ is some constant to be determined later, and

$$|||u|||_{k,*}^2 = \sup_{0\le t\le T} [[u(t)]]_{0,k}^2 + \int_0^T [[u(s)]]_{0,k+1}^2 ds.$$

As before, we first consider some general estimates on a linear operator $\mathbf{T}_*$ given by

$$\mathbf{T}_*\phi(t) = \int_{-\infty}^{t} U(t,s)\phi(s)ds,$$

for any $\phi = \phi(t,x,\xi)$.

Lemma 3.6. *Suppose that ϕ is time periodic with period T and*

$$\phi_0 = \int_0^T \left([[\phi(t)]]_{0,k}^2 + \|\phi(s)\|_{Z_1}^2\right) dt < \infty.$$

Under the assumptions of Theorem 3.2, if $\delta > 0$ is small enough, then $\mathbf{T}_*\phi$ is well-defined, time periodic with the same period T, and the following estimate holds

$$\sup_{0\le t\le T} [[\mathbf{T}_*\phi(t)]]_{0,k+1/2}^2 + \int_0^T [[\mathbf{T}_*\phi(t)]]_{0,k+1}^2 dt \le C\phi_0. \tag{3.46}$$

Proof. For simplicity, set $w = \mathbf{T}_*\phi$. By Theorem 2.2, it holds that

$$[[w(t)]]_{0,k} \le C\int_{-\infty}^{t} (1+t-s)^{-\sigma_{1,0}} G(s)ds = C\sum_{j=0}^{\infty} I_j(t), \tag{3.47}$$

where

$$G(s) = [[\phi(s)]]_{0,k} + \|\phi(s)\|_{Z_1}, \tag{3.48}$$

$$I_j(t) = \int_{t-(j+1)T}^{t-jT} (1+t-s)^{-\sigma_{1,0}} G(s)ds. \tag{3.49}$$

Since ϕ is time periodic with period T and so is $G(s)$, one has from (3.49) that

$$\begin{aligned} I_j^2(t) &\leq \int_{t-(j+1)T}^{t-jT} (1+t-s)^{-2\sigma_{1,0}} ds \int_{t-(j+1)T}^{t-jT} G^2(s) ds \\ &= \int_0^T (1+(j+1)T-r)^{-2\sigma_{1,0}}\, dr \int_0^T G^2(r) dr \\ &\leq C(1+jT)^{-2\sigma_{1,0}} \|G\|^2_{L^2(0,T)}, \end{aligned}$$

which implies

$$\sum_{j=0}^{\infty} I_j(t) \leq C \sum_{j=0}^{\infty} (1+jT)^{-\sigma_{1,0}} \|G\|_{L^2(0,T)} \leq C\|G\|_{L^2(0,T)}, \tag{3.50}$$

where $\sigma_{1,0} = n/4 > 1$ was used because $n \geq 5$. Then it follows from (3.47), (3.48) and (3.50) that

$$[[w(t)]]^2_{0,k} \leq C\|G\|^2_{L^2(0,T)} \leq C \int_0^T \left([[\phi(t)]]^2_{0,k} + \|\phi(s)\|^2_{Z_1} \right) dt \leq C\phi_0. \tag{3.51}$$

Next, the periodicity of w directly follows from

$$\begin{aligned} w(t+T) &= \int_{-\infty}^{t+T} U(t+T,s)\phi(s) ds \\ &= \int_{-\infty}^{t} U(t+T,s+T)\phi(s+T) ds \\ &= \int_{-\infty}^{t} U(t,s)\phi(s) ds, \end{aligned}$$

where we have used that for any $-\infty < s \leq t < \infty$,

$$\phi(s+T) = \phi(s), \quad U(t+T,s+T) = U(t,s).$$

Finally consider the estimate (3.46). Notice that w satisfies the initial value problem

$$\partial_t + \nu w + \xi \cdot \nabla_x w + F \cdot \nabla_\xi w - \frac{1}{2}\xi \cdot F w = K w + \phi,$$
$$w(t,x,\xi)|_{t=0} = 0.$$

Recalling Eq. (2.41) and the corresponding estimate (2.45), one has

$$\begin{aligned} &[[w(t)]]^2_{0,k+1/2} + c \int_0^T [[w(t)]]^2_{0,k+1} dt \leq C \int_0^T [[K w(t) + \phi(t)]]^2_{0,k} dt \\ &\leq C \int_0^T [[K w(t)]]^2_{0,k} dt + C\phi_0, \end{aligned}$$

where further by Lemma 3.1 and (3.51), it holds that

$$\int_0^T [[K w(t)]]^2_{0,k} dt \leq \int_0^T [[w(t)]]^2_{0,k-1} dt \leq CT \sup_{0\leq t\leq T} [[w(t)]]^2_{0,k-1} \leq C\phi_0.$$

Thus (3.46) holds. This completes the proof of the lemma. □

Proof of Theorem 3.2. Similar to the proof of Theorem 3.1, we first prove that there is a constant C such that for any $u, v \in \mathbb{S}(C_2)$ with some constant C_2 to be determined later,

$$|||\mathbf{\Phi}(u)|||_{k,*} \leq C\delta + C|||u|||^2_{k,*}, \tag{3.52}$$

$$|||\mathbf{\Phi}(u) - \mathbf{\Phi}(v)|||_{k,*} \leq C|||u + v|||_{k,*}|||u - v|||_{k,*}. \tag{3.53}$$

Notice that (3.43) implies

$$[[\widetilde{S}(t)]]_{0,k} + \|\widetilde{S}(t)\|_{Z_1} \leq \delta,$$

for any $t \in \mathbb{R}$. Thus based on Lemma 3.6, (3.52) and (3.53) are proved similarly as before and the details are omitted for brevity. □

Hence the contraction mapping theorem can be applied over the complete metric space $\mathbb{S}(C_2)$ for a proper constant $C_2 > 0$, provided that $\delta \leq \delta_2$ with $\delta_2 > 0$ small enough. Then there is a unique fixed point u^* in $\mathbb{S}(C_2)$ for the nonlinear mapping $\mathbf{\Phi}$. Notice that it is obvious that C_2 can be also chosen as $C_2'\delta$ for some constant C_2' independent of δ.

Finally by $u^* = \mathbf{\Phi}(u^*)$, it follows from (3.46) and (3.52) that

$$\sup_{0\leq t\leq T} [[u^*(t)]]_{0,k+1/2} \leq C\delta + C(C_2'\delta)^2 \leq C\delta,$$

since $\delta \leq \delta_2$ with δ_2 small enough. Further by the equation, the estimate (3.44) holds. Thus this complete the proof of the theorem.

3.4. Asymptotic stability of time periodic solution. In order to study the stability of the time periodic solution u^*, we shall consider the Cauchy problem

$$\partial_t u + \xi \cdot \nabla_x u + F \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot Fu = \mathbf{L}u + \Gamma(u) + \widetilde{S}, \tag{3.54}$$

$$u(t, x, \xi)|_{t=t_0} = u_0(x, \xi), \tag{3.55}$$

for some $t_0 \in \mathbb{R}$, where $u = u(t, x, \xi)$, $(t, x, \xi) \in (t_0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n$. It it noticed that the initial time t_0 can be chosen arbitrarily. By putting

$$v = u - u^*,$$

the initial value problem (3.54) and (3.55) can be rewritten as

$$\partial_t v + \xi \cdot \nabla_x v + F \cdot \nabla_\xi v - \frac{1}{2}\xi \cdot Fv = \mathbf{L}v + \Gamma(v, v) + 2\Gamma(u^*, v), \tag{3.56}$$

$$v(t, x, \xi)|_{t=t_0} = v_0(x, \xi), \tag{3.57}$$

where

$$v_0(x, \xi) \equiv u_0(x, \xi) - u^*(t_0, x, \xi).$$

Then we have the following result.

Theorem 3.3. *Let all assumptions in Theorem 3.2 hold and u^* be the corresponding time periodic solution obtained. Moreover, suppose that $u_0 \in H^\ell(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and there are constants $\delta > 0$ and $k \geq 2$ such that*

$$[[v_0]]_{0,k} + \|v_0\|_{Z_1} \leq \delta.$$

Then there are constants $\delta_3 > 0$ and $C_3 > 0$ such that for any $\delta \leq \delta_3$, the Cauchy problem (3.56)–(3.57) has a unique global solution

$$v \in C^i_b\left([t_0, \infty); H^{\ell-i}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)\right), \quad i = 0, 1, \tag{3.58}$$

with bounds

$$\sup_{t\geq t_0}(1 + t - t_0)^{2\kappa_1}[[v(t)]]^2_{0,k} + \int_{t_0}^{\infty} (1 + s)^{2\kappa_1}[[v(s)]]^2_{0,k+1/2} ds \leq C_3^2, \tag{3.59}$$

where κ_1 is some constant with

$$\sigma_{1,0}/2 \leq \kappa_1 < \sigma_{1,0} - 1/2, \tag{3.60}$$

and C_3 can be also chosen as $C_2 = C_3'\delta$ with C_3' independent of δ. Furthermore it holds that

$$[[v(t)]]_{0,k} \leq C\delta(1 + t - t_0)^{-\sigma_{1,0}}, \tag{3.61}$$

for some constant C.

To prove the above theorem, as before we first consider the decay in time estimates on the linear solution operator $\widetilde{U}(t, t_0)$, $-\infty < t_0 \leq t < \infty$ corresponding to the nonlinear equation (3.56). Here $\widetilde{U}(t, t_0)$ is defined in the sense that for any $w_0 = w_0(x, \xi)$, then $w = \widetilde{U}(t, t_0)w_0$ denotes the solution to the following initial value problem:

$$\partial_t w + \xi \cdot \nabla_x w + F \cdot \nabla_\xi w - \frac{1}{2}\xi \cdot F w = \mathbf{L}w + 2\Gamma(u^*, w), \tag{3.62}$$

$$w(t, x, \xi)|_{t=t_0} = w_0(x, \xi). \tag{3.63}$$

Lemma 3.7. *Let all assumptions in Theorem 3.2 hold and u^* be the corresponding time periodic solution obtained. Moreover, let $k \geq 2$. Then there exist constants $\delta_4 > 0$ and C_4 such that for any $\delta \leq \delta_4$, the linear solution operator $\widetilde{U}(t, t_0)$, $-\infty < t_0 \leq t < \infty$ satisfies the following decay estimates:*

$$[[\widetilde{U}(t, t_0)w_0]]_{0,k} \leq C_4(1 + t - t_0)^{-\sigma_{1,0}}\left([[w_0]]_{0,k} + \|w_0\|_{Z_1}\right), \tag{3.64}$$

for any $w_0 = w_0(x, \xi)$, where the constant C_4 depends only on n, ℓ, k and δ_4.

Proof. Without loss of generality, it suffices to prove this lemma for $t_0 = 0$. By (2.45) and (3.45), for Eq. (3.62) one has

$$\begin{aligned}
&\frac{d}{dt}J_{0,k}[w(t)] + cJ_{0,k+1/2}[w(t)] \leq C[[Kw(t) + 2\Gamma(u^*(t), w(t))]]_{0,k-1/2}\\
&\quad\leq C[[w(t)]]^2_{0,k-3/2} + C[[u^*(t)]]^2_{0,k+1/2}[[w(t)]]^2_{0,k+1/2}\\
&\quad\leq C[[w(t)]]^2_{0,k-1} + C\delta^2 J_{0,k+1/2}[w(t)],
\end{aligned}$$

where the nonlinear functional $J_{0,k}[\cdot]$ is given by (3.23). Thus if $\delta > 0$ is small enough, then

$$\frac{d}{dt}J_{0,k}[w(t)] + cJ_{0,k+1/2}[w(t)] \le C[[w(t)]]_{0,k-1}^2. \tag{3.65}$$

On the other hand, by the Duhamel's principle, w can be written as the mild form

$$w(t) = U(t,0)w_0 + \int_0^t U(t,s)\{2\Gamma(u^*(s), w(s))\}ds,$$

which from Theorem 2.2, (3.45) and $k \ge 2$, implies

$$\begin{aligned}[[w(t)]]_{0,k-1} \le{}& C\left([[w_0]]_{0,k-1} + \|w_0\|_{Z_1}\right)(1+t)^{-\sigma_{1,0}}\\ &+C\delta\int_0^t (1+t-s)^{-\sigma_{1,0}}[[w(s)]]_k ds.\end{aligned} \tag{3.66}$$

Since $\sigma_{1,0} > 1$ from $n \ge 5$, then similar to the proof of Lemma 2.11, combining (3.65) and (3.66) yields (3.64) with $t_0 = 0$. This completes the proof of the lemma. □

Furthermore, define the linear mapping $\widetilde{\mathbf{T}}$ by

$$\widetilde{\mathbf{T}}\phi(t) = \int_0^t \widetilde{U}(t,s)\phi(s)ds, \tag{3.67}$$

for any $\phi = \phi(t,x,\xi)$. Then similar to Corollary 3.1, we have the following estimates.

Lemma 3.8. *Under the assumptions of Lemma 3.7, if further $\delta > 0$ is small enough, then one has*

$$\begin{aligned}&(1+t)^{2m}[[\widetilde{\mathbf{T}}\phi(t)]]_{0,k}^2 + \int_0^t (1+s)^{2m}[[\widetilde{\mathbf{T}}\phi(s)]]_{0,k+1/2}^2 ds\\ &\le \int_0^t (1+s)^{2m}\left([[\phi(s)]]_{0,k-1/2}^2 + \|\phi(s)\|_{Z_1}^2\right)ds,\end{aligned} \tag{3.68}$$

for any $0 \le m < \sigma_{1,0} - 1/2$.

Proof. For simplicity, set $z(t) = \widetilde{\mathbf{T}}\phi(t)$. Fix some $0 \le m < \sigma_{1,0} - 1/2$. Then similar to the proof of (3.65) in Lemma 3.7, one has

$$\frac{d}{dt}J_{0,k}[z(t)] + cJ_{0,k+1/2}[z(t)] \le C[[z(t)]]_{0,k-1/2}^2 + C[[\phi(t)]]_{0,k-1/2}^2. \tag{3.69}$$

Further applying Lemma 3.7 to (3.67) gives

$$[[z(t)]]_{0,k-1/2} \le C\int_0^t (1+t-s)^{-\sigma_{1,0}}\left([[\phi(s)]]_{0,k-1/2} + \|\phi(s)\|_{Z_1}\right)ds. \tag{3.70}$$

Since $\sigma_{1,0>1}$ and $0 \le m < \sigma_{1,0} - 1/2$, then similar to the proof of (3.28), it follows from (3.70) that

$$\begin{aligned}&(1+t)^{2m}[[z(t)]]_{0,k-1/2}^2 + \int_0^t (1+t)^{2m}[[z(t)]]_{0,k-1/2}^2\\ &\le C\int_0^t (1+s)^{2m}\left([[\phi(s)]]_{0,k-1/2}^2 + \|\phi(s)\|_{Z_1}^2\right)ds.\end{aligned} \tag{3.71}$$

Finally similar to the proof of (3.25), combining (3.69) and (3.71) gives (3.68). This completes the proof of the lemma. □

Now we are in a position to prove the asymptotical stability of the time periodic solution.

Proof of Theorem 3.3. The proof is almost the same as that for Theorem 3.1. In fact, Without loss of generality, it suffices to prove Theorem 3.3 for $t_0 = 0$. The corresponding integral equation to solve is $v(t) = \mathbf{\Upsilon}(v)(t)$ for any $t \geq 0$, where the nonlinear mapping $\mathbf{\Upsilon}$ is given by

$$\mathbf{\Upsilon}(v)(t) = \widetilde{U}(t,0)v_0 + \int_0^t \widetilde{U}(t,s)\Gamma(v(s),v(s))ds.$$

By the contraction mapping theorem, the solution v will be obtained as a fixed point of $\mathbf{\Upsilon}$ on the complete metric space

$$\mathbb{S}(C_3) = \left\{ v = v(t,x,\xi) | v \in C_b^0\left(\mathbb{R}_t^+; H^\ell(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)\right),\ |||v|||_{k,\kappa_1} \leq C_3 \right\},$$

where κ_1 is given by (3.60) and the norm $|||\cdot|||_{k,\kappa_1}$ is defined by

$$|||v|||_{k,\kappa_1} = \sup_{t\geq 0}(1+t)^{2\kappa_1}[[v(t)]]_{0,k}^2 + \int_0^\infty (1+s)^{2\kappa_1}[[v(s)]]_{0,k+1/2}^2 ds.$$

In fact, based on Lemma 3.7 and Lemma 3.8 with $m = \kappa_1$, as before it is easy to show that there is a constant C such that for any $u, v \in \mathbb{S}(C_3)$ with some constant C_3 to be determined later,

$$\begin{aligned} |||\mathbf{\Upsilon}(u)|||_{k,\kappa_1} &\leq C\delta + C|||u|||_{k,\kappa_1}^2, \\ |||\mathbf{\Upsilon}(u) - \mathbf{\Upsilon}(v)|||_{k,\kappa_1} &\leq C|||u+v|||_{k,\kappa_1}|||u-v|||_{k,\kappa_1}, \end{aligned}$$

where $\kappa_1 < \sigma_{1,0} - 1/2$ was used. Thus if $\delta \leq \delta_3$ with $\delta_3 > 0$ small enough and C_3 is chosen properly, the unique fixed point v in $\mathbb{S}(C_3)$ as a solution is found. Hence (3.58) with $i = 0$ and (3.59) are proved. In addition, it is easy to see that the constant C_3 can be chosen as $C_3'\delta$ for another constant C_3', and (3.58), and $i = 1$ follows from the equation.

Finally we consider the improved decay rate (3.61). From the mild form $v = \mathbf{\Upsilon}(v)$ of the solution v, it follows that

$$\begin{aligned} [[v(t)]]_{0,k-1/2} &\leq C\delta(1+t)^{-\sigma_{1,0}} + C\int_0^t (1+t-s)^{-\sigma_{1,0}}[[v(s)]]_{0,k+1/2}[[v(s)]]_{0,k-1/2}ds \\ &\leq C\delta(1+t)^{-\sigma_{1,0}} + C\left\{\int_0^t (1+t-s)^{-2\sigma_{1,0}}(1+s)^{-4\kappa_1}ds\right\}^{1/2} \\ &\quad \times \left\{\int_0^t (1+s)^{2\kappa_1}[[v(s)]]_{0,k+1/2}^2 ds\right\}^{1/2} \sup_{s\geq 0}(1+s)^{\kappa_1}[[v(s)]]_{0,k} \\ &\leq C\delta(1+t)^{-\sigma_{1,0}}, \end{aligned}$$

since $4\kappa_1 \geq 2\sigma_{1,0} > 1$. Furthermore, in terms of Eq. (3.56) satisfied by v, then similar to the proof of (3.69), one has

$$\begin{aligned} \frac{d}{dt}J_{0,k}[v(t)] + c[[v(t)]]_{0,k+1/2}^2 &\leq C[[v(t)]]_{0,k-1/2}^2 + C[[\Gamma(v(t),v(t))]]_{0,k-1/2}^2 \\ &\leq C\delta^2(1+t)^{-2\sigma_{1,0}} + C[[v(t)]]_{k+1/2}^2[[v(t)]]_{k-1/2}^2 \\ &\leq C\delta^2(1+t)^{-2\sigma_{1,0}} + C\delta^2[[v(t)]]_{k+1/2}^2, \end{aligned}$$

which implies

$$\frac{d}{dt}J_{0,k}[v(t)] + cJ_{0,k+1/2}[v(t)] \leq C\delta^2(1+t)^{-2\sigma_{1,0}},$$

since $\delta \leq \delta_3$ with $\delta_3 > 0$ small enough. Thus by the Gronwall's inequality, it holds that

$$[[v(t)]]_{0,k}^2 \leq CJ_{0,k}[v(t)] \leq C\delta^2(1+t)^{-2\sigma_{1,0}}.$$

Hence (3.61) is proved. This completes the proof of the theorem.

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