# THE BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF IN THE WHOLE SPACE: II, GLOBAL EXISTENCE FOR HARD POTENTIAL

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ABSTRACT. As a continuation of our series works on the Boltzmann equation without angular cutoff assumption, in this part, the global existence of solution to the Cauchy problem in the whole space is proved in some suitable weighted Sobolev spaces for hard potential when the solution is a small perturbation of a global equilibrium.

## 1. INTRODUCTION

This paper is among the series works on the Boltzmann equation with non-angular cutoff cross-section and it follows the paper [7] (herein referred as Part I), extending our initial work [5, 6] on the same problem for Maxwellian molecule. Consider

$$(1.1) \qquad f_t + v \cdot \nabla_x f = Q(f, f), \quad f|_{t=0} = f_0.$$

Recall that the right hand side of (1.1) is the Boltzmann bilinear collision operator, which is given in the classical $\sigma-$representation by

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{g'_* f' - g_* f\} \, d\sigma dv_* \,,$$

where $f'_* = f(t, x, v'_*), f' = f(t, x, v'), f_* = f(t, x, v_*), f = f(t, x, v)$, and for $\sigma \in \mathbb{S}^2$,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma.$$

As in our previous papers, we assume that the cross-section takes the form

$$(1.2) \qquad B(v - v_*, \cos\theta) = \Phi(|v - v_*|)b(\cos\theta), \quad \cos\theta = \frac{v - v_*}{|v - v_*|} \cdot \sigma, \ 0 \leq \theta \leq \frac{\pi}{2},$$

in which it contains a kinetic factor given by

$$\Phi(|v - v_*|) = \Phi_\gamma(|v - v_*|) = |v - v_*|^\gamma,$$

and a factor related to the collision angle with singularity,

$$b(\cos\theta) \approx K\theta^{-2-2s} \text{ when } \theta \to 0+,$$

for some constant $K > 0$ and a parameter $0 < s < 1$. Notice that this includes the potential of inverse power law as a special example.

And the setting of the problem is for perturbation of an equilibrium state, without loss of generality, that can be normalized as

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

In order to avoid the unnecessary repetition, readers can refer to Part I, comments and references. Here, we just refer the references [9, 10, 11, 17, 18, 19] for the general background of the Boltzmann equation and the recent progress on the mathematical theories for the case without angular cutoff, [1, 2, 3, 4, 5, 6, 7, 8, 14, 15, 16]. Hence, we now directly go to the Cauchy problem for the perturbation denoted by $g = \mu^{-\frac{1}{2}}(f - \mu)$

$$(1.3) \qquad \begin{cases} g_t + v \cdot \nabla_x g + \mathcal{L}g = \Gamma(g, g), \quad t > 0, \\ g|_{t=0} = g_0. \end{cases}$$

In the following discussion, we will show that this equation can be solved in some weighted Sobolev spaces defined by: for $k, \ \ell \in \mathbb{R}$, set

$$H_\ell^k(\mathbb{R}^6_{x,v}) = \left\{ f \in \mathcal{S}'(\mathbb{R}^6_{x,v}) ; \ W_\ell f \in H^k(\mathbb{R}^6_{x,v}) \right\},$$

where $\mathbb{R}^6_{x,v} = \mathbb{R}^3_x \times \mathbb{R}^3_v$ and $W_\ell(v) = \langle v \rangle^\ell = (1 + |v|^2)^{\ell/2}$ is the weight with respect to the velocity variable $v \in \mathbb{R}^3$.

Note that in Part I, we introduced a new norm for the description of the dissipation and coercivity of the linearized collisional operator. Some properties of this norm together with some estimations on the upper bounds for the nonlinear collision operator were also given there. And we studied the Cauchy problem for the soft potential case, that is, the case when $\gamma + 2s \leq 0$ (recall that this terminology is an extension of the cutoff case, which loosely speaking corresponds to the case when $s = 0$).

Along this direction, this paper is for the hard potential case, that is, when $\gamma + 2s > 0$. Note that in particular this includes the case of the Maxwellian molecule. But this latter case was already considered in [5, 6].

The main result of this paper can be stated as follows.

**Theorem 1.1.** *Assume that the cross-section satisfies* (1.2) *with* $0 < s < 1$ *and* $\gamma + 2s > 0$. *Let* $g_0 \in H_\ell^k(\mathbb{R}^6)$ *for some* $k \geq 6, \ell > 3/2 + 2s + \gamma$. *There exists* $\varepsilon_0 > 0$, *such that if* $\|g_0\|_{H_\ell^k(\mathbb{R}^6)} \leq \varepsilon_0$, *then the Cauchy problem* (1.3) *admits a global solution*

$$g \in L^\infty([0, +\infty[ \ ; \ H_\ell^k(\mathbb{R}^6)).$$

**Remark 1.2.** *The uniqueness of the solution obtained in Theorem 1.1 will be proved in* [8] *in the general setting, together with the non-negativity of* $f = \mu + \mu^{\frac{1}{2}}g$. *Recently, a similar global existence result for the inverse power law was proved in* [14, 15] *by using different method in the setting of torus. The method used here is simpler by using the newly introduced non-isotropic norm which essentially captures the coercivity property of the linearized operator. Note that this method can be applied to the Landau equation that leads to the same global existence result obtained in* [12]. *Therefore, it is expected that this method can also be used for other kinetic equations.*

The rest of the paper will be organized as follows. In Section 2, we recall some basic lower and upper bound estimates on both the linearized and nonlinear operators from Parts I. With these estimates and some others valid for hard potential, the local and global existences will be proved in Sections 3 and 4, respectively.

## 2. Functional estimates of collision operators

First of all, let us recall the non-isotropic norm introduced in Part I associated with $\mathcal{L}$. Corresponding to the cross-section $\Phi(|v - v_*|)b(\cos\theta)$, it is defined by

$$\||g\||_{\Phi_\gamma}^2 = \iiint \Phi(|v - v_*|)b(\cos\theta)\mu_* \left(g' - g\right)^2$$
$$+ \iiint \Phi(|v - v_*|)b(\cos\theta)g_*^2\left(\sqrt{\mu'} - \sqrt{\mu}\right)^2,$$

where the integration is over $\mathbb{R}_v^3 \times \mathbb{R}_{v_*}^3 \times SS_\sigma^2$. Without any ambiguity, sometimes we simply use $\||\cdot\||$ for $\||\cdot\||_{\Phi_\gamma}$.

This norm was shown to be useful for the study on the soft potential. And here, we will show that it works well for the study on the hard potential. In the later discussion, we need the following propositions proved in the previous two parts in this series.

**Proposition 2.1.** *(Prop. 2.1, [7]) Assume that the cross-section satisfies* (1.2) *with* $0 < s < 1$ *and* $\gamma > -3$. *Then for* $g \in \mathcal{N}^\perp$

$$\||g\||_{\Phi_\gamma}^2 \lesssim \left(\mathcal{L}g, g\right)_{L^2(\mathbb{R}_v^3)} \leq 2\left(\mathcal{L}_1 g, g\right)_{L^2(\mathbb{R}_v^3)} \lesssim \||g\||_{\Phi_\gamma}^2,$$

*where* $\mathcal{N}$ *is the null space of* $\mathcal{L}$ *defined in Part I.*

**Proposition 2.2.** *(Prop. 2.2, [7]) Assume that the cross-section satisfies* (1.2) *with* $0 < s < 1$ *and* $\gamma > -3$. *Then*

$$\|g\|_{H^s_{\frac{\gamma}{2}}(\mathbb{R}_v^3)}^2 + \|g\|_{L^2_{s+\frac{\gamma}{2}}(\mathbb{R}_v^3)}^2 \lesssim \||g\||_{\Phi_\gamma}^2 \lesssim \|g\|_{H^s_{s+\frac{\gamma}{2}}(\mathbb{R}_v^3)}^2.$$

**Proposition 2.3.** *(Theorem 1.2, [7]) Assume that* $0 < s < 1, \gamma + 2s > 0$. *Then*

$$\left|(\Gamma(f, g), h)_{L^2(\mathbb{R}^3)}\right| \lesssim \left\{\|f\|_{L^2_{s+\gamma/2}}\||g\||_{\Phi_\gamma} + \|g\|_{L^2_{s+\gamma/2}}\||f\||_{\Phi_\gamma}\right\}\||h\||_{\Phi_\gamma}.$$

Note that the above estimate on the nonlinear collision operator is not enough for the proof of global existence because of the weight in $L^2_{s+\gamma/2}$. For this, we need to combine this with the following proposition. For the proof of the following proposition, we first recall an upper bound estimate for a modified kernel $\tilde{\Phi}_\gamma(z) = (1 + |z|^2)^{\gamma/2}$, cf. Theorem 2.1 in [4]. That is, for any $0 < s < 1, \gamma \in \mathbb{R}$ and any $m, \alpha \in \mathbb{R}$, we have

$$(2.1) \qquad \|Q_{\tilde{\Phi}_\gamma}(f, g)\|_{H^m_\alpha(\mathbb{R}_v^3)} \lesssim \|f\|_{L^1_{\alpha^+ + (\gamma+2s)^+}(\mathbb{R}_v^3)}\|g\|_{H^{m+2s}_{(\alpha+\gamma+2s)^+}(\mathbb{R}_v^3)}.$$

**Proposition 2.4.** *Let* $0 < s < 1, \gamma + 2s > 0$. *Then*

$$\left|\left(\Gamma(f, g), h\right)\right| \lesssim \||h\||_{\Phi_\gamma}\left\{\|f\|_{L^2_{3/2+2s+\gamma+\epsilon}}\|g\|_{H^{\max\{2s,1\}}_{s+\gamma/2}} + \||f\||_{\Phi_\gamma}\|g\|_{L^\infty}\right\}.$$

*Proof.* As in Part I, we apply the decomposition on the kinetic factor in the cross-section: Let $0 \leq \phi(z) \leq 1$ be a smooth radial function with 1 for $z$ close to 0, and 0 for large value of $z$. Set

$$\Phi_\gamma(z) = \Phi_\gamma(z)\phi(z) + \Phi_\gamma(z)(1 - \phi(z)) = \Phi_c(z) + \Phi_{\bar{c}}(z).$$

We denote by $\Gamma_c(\cdot, \cdot), \Gamma_{\bar{c}}(\cdot, \cdot)$ the collision operators with the kinetic factors in the cross-section given by $\Phi_c$ and $\Phi_{\bar{c}}$ respectively. Note that

$$\left(\Gamma(f, g), h\right) = \left(\Gamma_c(f, g), h\right) + \left(\Gamma_{\bar{c}}(f, g), h\right).$$

Note that

$$\left(\Gamma_{\bar{c}}(f, g), \ h\right)_{L^2(\mathbb{R}^3_v)} = \left(Q_{\bar{c}}(\sqrt{\mu} f, g), \ h\right)_{L^2(\mathbb{R}^3_v)}$$
$$+ \iiint \Phi_{\bar{c}}(|v - v_*|)b(\cos\theta)\left(\sqrt{\mu_*} - \sqrt{\mu'_*}\right)f'_* g' h dv_* d\sigma dv.$$

Since $\Phi_{\bar{c}} \lesssim \tilde{\Phi}_\gamma$, as shown in the Proposition 3.5 of [5], we use (2.1) with $m = 0, \alpha = -s - \gamma/2$ to have

$$\left|\left(Q_{\bar{c}}(\sqrt{\mu} f, g), \ h\right)_{L^2(\mathbb{R}^3_v)}\right| \lesssim \|h\|_{L^2_{s+\gamma/2}(\mathbb{R}^3)}\|\sqrt{\mu} f\|_{L^1_{2s+\gamma}(\mathbb{R}^3_v)}\|g\|_{H^{2s}_{s+\gamma/2}(\mathbb{R}^3_v)}$$
$$\lesssim \|f\|_{L^2(\mathbb{R}^3)}\|g\|_{H^{2s}_{s+\gamma/2}(\mathbb{R}^3)}\|h\|_{L^2_{s+\gamma/2}(\mathbb{R}^3)}.$$

On the other hand, we can write

$$\iiint \Phi_{\bar{c}}(|v - v_*|)b(\cos\theta)\left(\sqrt{\mu_*} - \sqrt{\mu'_*}\right)f'_* g' h dv_* d\sigma dv$$
$$= \iiint \Phi_{\bar{c}}(|v - v_*|)b(\cos\theta)\left(\sqrt{\mu_*} - \sqrt{\mu'_*}\right)f'_* g'\left(h - h'\right)dv_* d\sigma dv$$
$$+ \iiint \Phi_{\bar{c}}(|v - v_*|)b(\cos\theta)\left(\sqrt{\mu_*} - \sqrt{\mu'_*}\right)f'_* g' h' dv_* d\sigma dv$$
$$= D_1 + D_2.$$

By the Cauchy-Schwarz inequality, one has

$$|D_1| \le \left(\iiint \Phi_{\bar{c}}(|v - v_*|)b(\cos\theta)|f'_*|^2|g'|^2\left((\mu_*)^{1/4} - (\mu'_*)^{1/4}\right)^2 dv_* d\sigma dv\right)^{1/2}$$
$$\times \left(\iiint \Phi_{\bar{c}}(|v - v_*|)b(\cos\theta)\left(\mu_*^{1/4} + (\mu'_*)^{1/4}\right)^2(h - h')^2 dv_* d\sigma dv\right)^{1/2}.$$

As Lemma 2.6 in [7], for $\gamma + 2s > 0$, we have

$$\iiint \Phi_{\bar{c}}(|v - v_*|)b(\cos\theta)|f'_*|^2|g'|^2\left((\mu_*)^{1/4} - (\mu'_*)^{1/4}\right)^2 dv_* d\sigma dv$$
$$\lesssim \iint |v - v_*|^{\gamma+2s}|f_*|^2|g|^2 dv_* dv$$
$$\lesssim \iint |f_*|^2|g|^2\langle v\rangle^{2s+\gamma}\langle v_*\rangle^{2s+\gamma} dv_* ddv$$
$$\lesssim \|f\|^2_{L^2_{s+\gamma/2}(\mathbb{R}^3)}\|g\|^2_{L^2_{s+\gamma/2}(\mathbb{R}^3)},$$

and

$$\iiint \Phi_{\bar{c}} b(\cos\theta)\left(\mu_*^{1/4} + (\mu'_*)^{1/4}\right)^2(h - h')^2 dv_* d\sigma dv$$
$$\le 4 \iiint \Phi_{\bar{c}} b(\cos\theta)\mu_*^{1/2}(h - h')^2 dv_* d\sigma dv \lesssim \||h\||^2_{\Phi_\gamma}.$$

Therefore, we obtain

$$|D_1| \lesssim \|f\|_{L^2_{s+\gamma/2}(\mathbb{R}^3)}\|g\|_{H^{2s}_{s+\gamma/2}(\mathbb{R}^3)}\||h\||_{\Phi_\gamma}.$$

For the term $D_2$, we have by using the symmetry in the integral to have

$$
\left| \iiint \Phi_{\bar{c}} b(\cos\theta) \left( \sqrt{\mu_*} - \sqrt{\mu'_*} \right) f'_* \, g' h' \, dv_* d\sigma dv \right|
$$
$$
= \left| \iiint \Phi_{\bar{c}} b(\cos\theta) \left( \sqrt{\mu'_*} - \sqrt{\mu_*} \right) f_* \, gh \, dv_* d\sigma dv \right|
$$
$$
\lesssim \int_{\mathbb{R}^6_{v,v_*}} |f_*| \, |g| \, |h| \langle v \rangle^{2s+\gamma} \langle v_* \rangle^{2s+\gamma} dv_* d d v
$$
$$
\lesssim \|f\|_{L^1_{2s+\gamma}(\mathbb{R}^3)} \| g\|_{L^2_{s+\gamma/2}(\mathbb{R}^3)} \|h\|_{L^2_{s+\gamma/2}(\mathbb{R}^3)}
$$
$$
\lesssim \|f\|_{L^2_{3/2+2s+\gamma+\epsilon}(\mathbb{R}^3)} \|g\|_{L^2_{s+\gamma/2}(\mathbb{R}^3)} \|h\|_{L^2_{s+\gamma/2}(\mathbb{R}^3)}.
$$

Hence

$$
|D_2| \lesssim \|f\|_{L^2_{3/2+2s+\gamma+\epsilon}(\mathbb{R}^3)} \|g\|_{L^2_{s+\gamma/2}(\mathbb{R}^3)} \|h\|_{L^2_{s+\gamma/2}(\mathbb{R}^3)}.
$$

Therefore, it follows that

$$
\left| \left( \Gamma_{\bar{c}}(f, g), \, h \right)_{L^2(\mathbb{R}^3)} \right| \lesssim \|f\|_{L^2_{3/2+2s+\gamma+\epsilon}(\mathbb{R}^3)} \|g\|_{H^{2s}_{s+\gamma/2}} \|h\|_{\Phi_\gamma}.
$$

Next, similar to the arguments used in Part II, we have

$$
\left| \left( \Gamma_c(f, g), \, , h \right) \right| \lesssim \left( \iiint b\Phi_c \mu_*^{1/2} \left( f'_* g' - f_* g \right)^2 d\sigma dv dv_* \right)^{1/2} \|h\|_{\Phi_c}
$$
$$
= A^{1/2} \|h\|_{\Phi_c}.
$$

Since

$$
\Phi_c \leq \Phi_\gamma,
$$

we have $\|h\|_{\Phi_c} \leq \|h\|_{\Phi_\gamma}$, and

$$
A \lesssim \left\{ \|f\|_{\Phi_\gamma}^2 \|g\|_{L^\infty}^2 + \|f\|_{L^2}^2 \|g\|_{H^{\max(-\gamma/2,1)}}^2 \right\}.
$$

Then, for $\gamma > -2s > -2$, we have $-\gamma/2 < 1$ and

$$
\left| \left( \Gamma_c(f, g), \, h \right)_{L^2(\mathbb{R}^3)} \right| \lesssim \left\{ \|f\|_{\Phi_\gamma} \|g\|_{L^\infty} + \|f\|_{L^2} \|g\|_{H^1} \right\} \|h\|_{\Phi_\gamma}.
$$

And this completes the proof of the proposition. □

In the following, we also need the following estimate on the commutator of the weight function $W_\ell$ and the nonlinear collisional operator $\Gamma(\cdot, \cdot)$ that follows from Proposition 2.17 in [7].

**Proposition 2.5.** *Assume that $0 < s < 1$ and $\gamma + 2s > 0$. Then, for any $\ell \geq 0$, one has*

$$
\left| (W_\ell \Gamma(f, g) - \Gamma(f, W_\ell g), \, h)_{L^2(\mathbb{R}^3_v)} \right| \lesssim \left\{ \|f\|_{L^2_{s+\gamma/2}} \|g\|_{L^2_{\ell+\gamma/2}} + \|g\|_{L^2_{s+\gamma/2}} \|f\|_{L^2_{\ell+\gamma/2}} \right\} \|h\|_{\Phi_\gamma}.
$$

## 3. Local existence

First of all, the Leibniz formula gives

$$
\partial^\beta \Gamma(f, g) = \sum_{\beta_1+\beta_2+\beta_3=\beta} C_{\beta_1,\beta_2,\beta_3} \mathcal{T}(\partial^{\beta_1} f, \, \partial^{\beta_2} g, \, \mu_{\beta_3}),
$$

with

$$
\mathcal{T}(F, G, \mu_{\beta_3}) = Q(\mu_{\beta_3} F, G) + \iint \Phi(|v - v_*|) b(\cos\theta) \left( (\mu_{\beta_3})_* - (\mu_{\beta_3})'_* \right) F'_* G' dv_* d\sigma,
$$

where $\mu_{\beta_3} = p_{\beta_3}(v)\sqrt{\mu(v)} = \partial^{\beta_3}(\sqrt{\mu})$ is a Maxwellian type function of the variable $v$ in the sense that it is a product of a polynomial and a Gaussian. As noted in the previous parts in this series, one can check that $\mathcal{T}(F, G, \mu_{\beta_3})$ enjoys the same properties as $\Gamma(F, G)$ stated above. Therefore, we will apply those estimates obtained for $\Gamma(F, G)$ to $\mathcal{T}(F, G, \mu_{\beta_3})$.

Define the norm associated with the collision operator in the variables $(x, v)$ by setting for $m \in \mathbb{N}, \ell \in \mathbb{R}$,

$$\mathcal{B}_\ell^m(\mathbb{R}_{x,v}^6) = \left\{ g \in \mathcal{S}'(\mathbb{R}_{x,v}^6); \ \|g\|_{\mathcal{B}_\ell^m(\mathbb{R}^6)}^2 = \sum_{|\beta|\leq m} \int_{\mathbb{R}_x^3} \||W_\ell \partial_{x,v}^\beta g(x, \cdot)\||_{\Phi_\gamma}^2 dx < +\infty \right\}.$$

First of all, recall

**Lemma 3.1.** *(Lemma 4.1, [7]) For any $\ell \geq 0, \alpha, \beta \in \mathbb{N}^3$,*

$$\|W_\ell \partial_x^\alpha \partial_v^\beta \, \mathbf{P}g\|_{\mathcal{B}_0^0(\mathbb{R}^6)} + \|\mathbf{P}(W_\ell \partial_x^\alpha \partial_v^\beta \, g)\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \leq C_{\ell,\beta} \|\partial_x^\alpha g\|_{L^2(\mathbb{R}^6)}^2,$$

$$C_0\|g\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 - C_1\|g\|_{L^2(\mathbb{R}^6)}^2 \leq \left(\mathcal{L}g, \, g\right)_{L^2(\mathbb{R}^6)} \lesssim \||g\||_{\mathcal{B}_0^0(\mathbb{R}^6)}^2,$$

*where $C_0$ and $C_1$ are some positive constants, and*

$$\|g\|_{L^2_{l+s+\frac{\gamma}{2}}(\mathbb{R}^6)}^2 + \|g\|_{L^2(\mathbb{R}_x^3; H^s_{l+\frac{\gamma}{2}}(\mathbb{R}_v^3))}^2 \lesssim \|g\|_{\mathcal{B}_l^0(\mathbb{R}^6)}^2 \lesssim \|g\|_{L^2(\mathbb{R}_x^3; H^s_{l+s+\frac{\gamma}{2}}(\mathbb{R}_v^3))}^2.$$

*Here $\mathbf{P}$ is the projection to the null space $\mathcal{N}$.*

We are now ready to prove the following estimate.

**Proposition 3.2.** *Let $\gamma + 2s > 0, N \geq 6, \ell > 3/2 + 2s + \gamma$. Then, for any $\beta \in \mathbb{N}^6, |\beta| \leq N$,*

$$\left|\left(W_\ell \partial_{x,v}^\beta \Gamma(f, g), h\right)_{L^2(\mathbb{R}^6)}\right| \lesssim \left\{\|f\|_{H_\ell^N(\mathbb{R}^6)} \|g\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)} + \|f\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)} \|g\|_{H_\ell^N(\mathbb{R}^6)}\right\} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}.$$

*Proof.* By using the Leibniz formula, we have

$$\left(W_\ell \partial^\beta \Gamma(f, g), g\right)_{L^2(\mathbb{R}^6)} = \sum_{\beta_1+\beta_2+\beta_3=\beta} C_{\beta_1,\beta_2,\beta_3}\left\{\left(\mathcal{T}(\partial^{\beta_1}f, W_\ell \partial^{\beta_2}g, \mu_{\beta_3}), h\right)\right.$$

$$\left. + \left(W_\ell \mathcal{T}(\partial^{\beta_1}f, \partial^{\beta_2}g, \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1}f, W_\ell \partial^{\beta_2}g, \mu_{\beta_3}), h\right)\right\}.$$

If $|\beta_1| \leq N - 3$, we get from Proposition 2.3 that

$$\left|\left(\mathcal{T}(\partial^{\beta_1}f, W_\ell \partial^{\beta_2}g, \mu_{\beta_3}), h\right)_{L^2(\mathbb{R}^6)}\right|$$

$$\lesssim \left(\int_{\mathbb{R}_x^3} \left(\|\partial^{\beta_1}f\|_{L^2_{\gamma/2+s}(\mathbb{R}_v^3)}^2 \||W_\ell \partial^{\beta_2}g\||_{\Phi_\gamma}^2 + \|\partial^{\beta_1}f\|_{\Phi_\gamma}^2 \|W_\ell \partial^{\beta_2}g\|_{L^2_{s+\gamma/2}(\mathbb{R}_v^3)}^2\right)dx\right)^{1/2} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}$$

$$\lesssim \left(\|\partial^{\beta_1}f\|_{L^\infty(\mathbb{R}_x^3; L^2_{\gamma/2+s}(\mathbb{R}_v^3))}^2 \|W_\ell \partial^{\beta_2}g\|_{\mathcal{B}_0^0(\mathbb{R}^6)}^2 + \|\partial^{\beta_1}f\|_{L^\infty(\mathbb{R}_x^3; H^s_{\gamma+2s}(\mathbb{R}_v^3))}^2 \|W_\ell \partial^{\beta_2}g\|_{L^2_{s+\gamma/2}(\mathbb{R}^6)}^2\right)^{1/2} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}$$

$$\lesssim \|f\|_{H_{\gamma+2s}^N(\mathbb{R}^6)} \||g\||_{\mathcal{B}_\ell^N(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}.$$

On the other hand, if $|\beta_1| > N - 3$, then $|\beta_2| \leq 2 \leq N - 4$. In this case, Proposition 2.4 implies,

$$\left|\left(\mathcal{T}(\partial^{\beta_1}f, W_\ell \partial^{\beta_2}g, \mu_{\beta_3}), h\right)_{L^2(\mathbb{R}^6)}\right|$$

$$\lesssim \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}\left(\|f\|_{H_{3/2+2s+\gamma+\epsilon}^{|\beta_1|}(\mathbb{R}^6)}\|W_\ell \partial^{\beta_2}g\|_{L^\infty(\mathbb{R}_x^3; H^{\max\{2s,1\}}_{s+\gamma/2}(\mathbb{R}_v^3))} + \|W_\ell \partial^{\beta_2}g\|_{L^\infty(\mathbb{R}^6)}\|\partial^{\beta_1}f\|_{\mathcal{B}_0^0}\right)$$

$$\lesssim \left(\|f\|_{H_{3/2+2s+\gamma+\epsilon}^N(\mathbb{R}^6)}\|W_\ell \partial^{\beta_2}g\|_{H_{s+\gamma/2}^{\max\{2s,1\}+3/2+\epsilon}(\mathbb{R}^6)} + \|f\|_{\mathcal{B}_0^{|\beta_1|}}\|W_\ell \partial^{\beta_2}g\|_{H^{3+\epsilon}(\mathbb{R}^6)}\right)\|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}$$

$$\lesssim \left(\|f\|_{H_\ell^N(\mathbb{R}^6)}\|g\|_{\mathcal{B}_\ell^N} + \|f\|_{\mathcal{B}_0^{|\beta_1|}}\|g\|_{H_\ell^N(\mathbb{R}^6)}\right)\|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}.$$

Finally, Proposition 2.5 yields

$$\left|\left(W_\ell \mathcal{T}(\partial^{\beta_1} f, \partial^{\beta_2} g, \mu_{\beta_3}) - \mathcal{T}(\partial^{\beta_1} f, W_\ell \partial^{\beta_2} g, \mu_{\beta_3}), h\right)_{L^2(\mathbb{R}^6)}\right|$$

$$\lesssim \left(\|f\|_{H_\ell^N(\mathbb{R}^6)} \|g\|_{\mathcal{B}_\ell^N} + \|f\|_{\mathcal{B}_\ell^N} \|g\|_{H_\ell^N(\mathbb{R}^6)}\right) \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}.$$

The combination of the above estimates completes the proof of the proposition. □

For the linear operator $\mathcal{L}_2$, Proposition 4.5 of [7] and the commutator estimate give

**Proposition 3.3.** *We have for any $\beta \in \mathbb{N}^6$,*

$$\left|\left(W_\ell \partial_{x,v}^\beta \mathcal{L}_2(f), h\right)_{L^2(\mathbb{R}^6)}\right| \le C_{\ell,|\beta|} \|f\|_{H_\ell^{|\beta|}(\mathbb{R}^6)} \|\mu^{1/10^4} h\|_{L^2(\mathbb{R}^6)}.$$

By using the interpolation inequalities

$$\|g\|_{H_{\ell+\gamma/2}^N} \le \epsilon \|g\|_{H_{\ell+\gamma/2+s}^N} + C_\epsilon \|g\|_{H_\ell^N},$$

for any small constant $\epsilon$, the following proposition follows from the same argument given in Proposition 4.8 of [7].

**Proposition 3.4.** *Let $\gamma + 2s > 0, \beta \in \mathbb{N}^6, |\beta| > 0, \ell \ge 0$. Then*

$$\left|\left(\mathcal{L}_1(W_\ell \partial_{x,v}^\beta g) - W_\ell \partial_{x,v}^\beta \mathcal{L}_1(g), h\right)_{L^2(\mathbb{R}^6)}\right|$$

$$\lesssim \left(\|g\|_{H_\ell^{|\beta|}(\mathbb{R}^6)} + \|g\|_{\mathcal{B}_\ell^{|\beta|-1}(\mathbb{R}^6)}\right) \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)},$$

*and for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that*

$$\left|\left(\mathcal{L}_1(W_\ell g) - W_\ell \mathcal{L}_1(g), h\right)_{L^2(\mathbb{R}^6)}\right| \lesssim \|g\|_{L_{\ell+\gamma/2}^2(\mathbb{R}^6)} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}$$

$$\le \left\{\epsilon \|g\|_{L_{\ell+\gamma/2+s}^2(\mathbb{R}^6)} + C_\epsilon \|g\|_{L_\ell^2(\mathbb{R}^6)}\right\} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}$$

$$\le \left\{\epsilon \|g\|_{\mathcal{B}_\ell^0(\mathbb{R}^6)} + C_\epsilon \|g\|_{L_\ell^2(\mathbb{R}^6)}\right\} \|h\|_{\mathcal{B}_0^0(\mathbb{R}^6)}.$$

We are now ready to show the local existence of solutions in some weighted Sobolev spaces. Consider the following Cauchy problem for a linear Boltzmann equation with a given function $f$,

$$(3.1) \qquad \partial_t g + v \cdot \nabla_x g + \mathcal{L}_1 g = \Gamma(f, g) - \mathcal{L}_2 f, \qquad g|_{t=0} = g_0,$$

which is equivalent to the problem:

$$\partial_t G + v \cdot \nabla_x G = Q(F, G), \qquad G|_{t=0} = G_0,$$

with $F = \mu + \sqrt{\mu} f$ and $G = \mu + \sqrt{\mu} g$.

We shall now study the energy estimates on (3.1) in the function space $H_\ell^N(\mathbb{R}^6)$ for a smooth function $g$. For $N \ge 6, \ell > 3/2 + 2s + \gamma$ and $\beta \in \mathbb{N}^6, |\beta| \le N$, taking

$$\varphi(t, x, v) = (-1)^{|\beta|} \partial_{x,v}^\beta W_{2\ell} \partial_{x,v}^\beta g(t, x, v),$$

as a test function on $\mathbb{R}_x^3 \times \mathbb{R}_v^3$, we get

$$\frac{1}{2} \frac{d}{dt} \|\partial^\beta g\|_{L_\ell^2(\mathbb{R}^6)}^2 + \left(W_\ell [\partial^\beta, v] \cdot \nabla_x g, W_\ell \partial^\beta g\right)_{L^2(\mathbb{R}^6)} + \left(W_\ell \partial^\beta \mathcal{L}_1(g), W_\ell \partial^\beta g\right)_{L^2(\mathbb{R}^6)}$$

$$= \left(W_\ell \partial^\beta \Gamma(f, g), W_\ell \partial^\beta g\right)_{L^2(\mathbb{R}^6)} - \left(\partial^\beta \mathcal{L}_2(f), W_{2\ell} \partial^\beta g\right)_{L^2(\mathbb{R}^6)},$$

where we have used the fact that

$$\left(v \cdot \nabla_x (W_\ell \partial^\beta g), W_\ell \partial^\beta g\right)_{L^2(\mathbb{R}^6)} = 0.$$

Applying now Propositions 3.2, 3.3 and 3.4, we get for $N \geq 6, \ell > 3/2 + 2s + \gamma$ and $|\beta| \leq N$,

$$\frac{1}{2}\frac{d}{dt}\|\partial^\beta g\|^2_{L^2_\ell(\mathbb{R}^6)} + \left(\mathcal{L}_1\big(W_\ell \, \partial^\beta_{x,v}g\big), \, W_\ell \, \partial^\beta_{x,v}g\right)_{L^2(\mathbb{R}^6)}$$

$$\lesssim \Big\{ \|f\|_{H^N_\ell(\mathbb{R}^6)} \|g\|^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} + \|f\|_{\mathcal{B}^N_\ell(\mathbb{R}^6)} \|g\|_{H^N_\ell(\mathbb{R}^6)}\|g\|_{\mathcal{B}^N_\ell(\mathbb{R}^6)}$$

$$+ \|g\|^2_{H^N_\ell(\mathbb{R}^6)} + \|f\|^2_{H^N_\ell(\mathbb{R}^6)} + \|g\|_{\mathcal{B}^{N-1}_\ell(\mathbb{R}^6)}\|g\|_{\mathcal{B}^N_\ell(\mathbb{R}^6)}\Big\}.$$

By induction on $\beta$ from $|\beta| = 1$ to $N$, the Cauchy-Schwarz inequality implies that

$$\frac{d}{dt}\|g\|^2_{H^N_\ell(\mathbb{R}^6)} + \frac{C_0}{4}\|g\|^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} \lesssim \Big\{ \|f\|_{H^N_\ell(\mathbb{R}^6)} \|g\|^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} + \|g\|^2_{H^N_\ell(\mathbb{R}^6)} \|f\|^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)}$$

$$+ \|g\|^2_{H^N_\ell(\mathbb{R}^6)} + \|f\|^2_{H^N_\ell(\mathbb{R}^6)} + \|g\|^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)}\Big\}.$$

On the other hand, taking $\beta = 0$, we have

$$\frac{1}{2}\frac{d}{dt}\|g\|^2_{L^2_\ell(\mathbb{R}^6)} + \left(\mathcal{L}_1\big(W_\ell \, g\big), \, W_\ell \, g\right)_{L^2(\mathbb{R}^6)}$$

$$\lesssim \Big\{ \|f\|_{H^3_\ell(\mathbb{R}^6)} \|g\|^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} + \|f\|_{\mathcal{B}^3_\ell(\mathbb{R}^6)} \|g\|_{L^2_\ell(\mathbb{R}^6)}\|g\|_{\mathcal{B}^0_\ell(\mathbb{R}^6)}$$

$$+ \|g\|^2_{L^2_\ell(\mathbb{R}^6)} + \|f\|^2_{L^2_\ell(\mathbb{R}^6)} + \epsilon \, \|g\|^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)}\Big\},$$

which together with the coercivity estimate implies that

$$\frac{1}{2}\frac{d}{dt}\|g\|^2_{L^2_\ell(\mathbb{R}^6)} + \frac{C_0}{4}\|g\|^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)} \lesssim \Big\{ \|f\|_{H^3_\ell(\mathbb{R}^6)} \|g\|^2_{\mathcal{B}^0_\ell(\mathbb{R}^6)}$$

$$+ \|f\|^2_{\mathcal{B}^3_\ell(\mathbb{R}^6)} \|g\|^2_{L^2_\ell(\mathbb{R}^6)} + \|g\|^2_{L^2_\ell(\mathbb{R}^6)} + \|f\|^2_{L^2_\ell(\mathbb{R}^6)}\Big\}.$$

In summary, we have shown that there exists a constant $\eta_0 > 0$ such that for $N \geq 6, \ell > 3/2 + 2s + \gamma$,

$$\frac{d}{dt}\|g\|^2_{H^N_\ell(\mathbb{R}^6)} + \eta_0\|g\|^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} \lesssim \Big\{ \|f\|_{H^N_\ell(\mathbb{R}^6)} \|g\|^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)}$$

$$+ \|g\|^2_{H^N_\ell(\mathbb{R}^6)} \|f\|^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} + \|g\|^2_{H^N_\ell(\mathbb{R}^6)} + \|f\|^2_{H^N_\ell(\mathbb{R}^6)}\Big\}.$$

With the above differential inequality, the same argument for the soft potential applies and it leads to the following theorem.

**Theorem 3.5.** *Let $0 < s < 1, \gamma + 2s > 0, N \geq 6, \ell > 3/2 + 2s + \gamma$. Assume that $g_0 \in H^N_\ell(\mathbb{R}^6)$ and $f \in L^\infty([0,T]; H^N_\ell(\mathbb{R}^6)) \bigcap L^2([0,T]; \mathcal{B}^N_\ell(\mathbb{R}^6))$. If $g \in L^\infty([0,T]; H^N_\ell(\mathbb{R}^6)) \bigcap L^2([0,T]; \mathcal{B}^N_\ell(\mathbb{R}^6))$ is a solution of the Cauchy problem (3.1), then there exists $\epsilon_0 > 0$ such that if*

$$\|f\|^2_{L^\infty([0,T]; H^N_\ell(\mathbb{R}^6))} + \|f\|^2_{L^2([0,T]; \mathcal{B}^N_\ell(\mathbb{R}^6))} \leq \epsilon^2_0,$$

*we have*

$$\|g\|^2_{L^\infty([0,T]; H^N_\ell(\mathbb{R}^6))} + \|g\|^2_{L^2([0,T]; \mathcal{B}^N_\ell(\mathbb{R}^6))} \leq Ce^{CT}\Big(\|g_0\|^2_{H^N_\ell(\mathbb{R}^6)} + \epsilon^2_0 T\Big),$$

*for a constant $C > 0$ depending only on $N, \ell$.*

And this yields the local existence of solution by the contraction mapping theorem through the standard argument. Therefore, we omit the details for the brevity of the paper.

## 4. Global Existence

In this section, we derive a global energy estimate for the solution in the weighted function spaces. In the soft potential case considered in Part I, we could obtain two types of global energy estimates, one for only $x$ derivatives without requiring any weight in the variable $v$ and one for both $x$ and $v$ derivatives with weight in $v$ whose order varies with the order of $v$ derivative. On the other hand, in the hard potential case, the energy estimate can be closed only when both $x$ and $v$ derivatives are taken into account together with weight in $v$. This is due to the upper bound estimate on the nonlinear collision operator given in Section 2 where some weighted norms are used. However, the order of weight can be fixed in contrast to the case of soft potential.

Set

$$\mathcal{E}_{N,\ell} = \|g\|^2_{H^N_\ell(\mathbb{R}^6)} \sim \|g_1\|^2_{H^N_\ell(\mathbb{R}^6)} + \|g_2\|^2_{H^N_\ell(\mathbb{R}^6)} \sim \|\mathcal{A}\|^2_{H^N(\mathbb{R}^3)} + \|g_2\|^2_{H^N_\ell(\mathbb{R}^6)},$$

$$\mathcal{D}_{N,\ell} = \|\nabla_x g_1\|^2_{H^{N-1}(\mathbb{R}^6)} + \|g_2\|^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)} \sim \|\nabla_x \mathcal{A}\|^2_{H^{N-1}(\mathbb{R}^3)} + \|g_2\|^2_{\mathcal{B}^N_\ell(\mathbb{R}^6)},$$

where $\mathcal{A} = (a, b, c)$ with $\mathbf{P}g = g_1 = (a + v \cdot b + |v|^2 c)\mu^{\frac{1}{2}}$, and $g_2 = g - \mathbf{P}g$.

Let $g = g(t, x, v)$ be a smooth solution to the Cauchy problem (1.3). The main goal of this section is to establish

**Proposition 4.1. (Energy Estimate)** *Assume $0 < s < 1$ and $2s + \gamma > 0$, and let $N \geq 6$, $\ell > 3/2 + 2s + \gamma$. Then,*

$$\frac{d}{dt}\mathcal{E}_{N,\ell} + \mathcal{D}_{N,\ell} \lesssim \mathcal{E}_{N,\ell}^{1/2}\mathcal{D}_{N,\ell},$$

*holds as long as the solution $g$ exists.*

With this proposition, the standard continuity argument and the local existence assure the global existence of solution when the intial data $g_0$ satisfies that $\mathcal{E}_{N,\ell}(0)$ is sufficiently small. And the above energy estimate will be obtained by using the coercivity, upper bound and commutator estimates through the macro-micro decomposition introduced in [13] as follows.

4.1. **Macroscopic energy estimate.** As in [13], the macroscopic component $\mathcal{A} = (a, b, c)$ satisfies

$$(4.2) \quad \begin{cases} v_i|v|^2\mu^{1/2} : & \nabla_x c = -\partial_t r_c + l_c + h_c, \\ v_i^2\mu^{1/2} : & \partial_t c + \partial_i b_i = -\partial_t r_i + l_i + h_i, \\ v_i v_j \mu^{1/2} : & \partial_i b_j + \partial_j b_i = -\partial_t r_{ij} + l_{ij} + h_{ij}, \quad i \neq j, \\ v_i \mu^{1/2} : & \partial_t b_i + \partial_i a = -\partial_t r_{bi} + l_{bi} + h_{bi}, \\ \mu^{1/2} : & \partial_t a = -\partial_t r_a + l_a + h_a, \end{cases}$$

where

$$r = (g_2, e)_{L^2_v}, \qquad l = -(v \cdot \nabla_x g_2 + \mathcal{L}g_2, e)_{L^2_v}, \qquad h = (\Gamma(g, g), e)_{L^2(\mathbb{R}^3_v)},$$

stand for $r_c, \cdots, h_a$, while

$$e \in \text{span}\{v_i|v|^2\mu^{1/2}, v_i^2\mu^{1/2}, v_i v_j \mu^{1/2}, v_i \mu^{1/2}, \mu^{1/2}\}.$$

Same as Lemma 7.2 in [5], we have the following property on $\mathcal{A}$.

**Lemma 4.2.** *Let $\partial^\alpha = \partial_x^\alpha$, $\alpha = \alpha_1 + \alpha_2 \in \mathbb{N}^3$, $|\alpha| \leq N$, $N \geq 3$. Then,*

$$\|(\partial^{\alpha_1}\mathcal{A})(\partial^{\alpha_2}\mathcal{A})\|_{L^2_x} \leq \|\nabla_x \mathcal{A}\|_{H^{N-1}_x}\|\mathcal{A}\|_{H^{N-1}_x}.$$

The following lemma is similar to Lemma 7.3 in [5] with some modification regarding to the hard potential assumption. Here, we include its proof for the completeness.

**Lemma 4.3.** *Let* $\partial^\alpha = \partial_x^\alpha, \partial_i = \partial_{x_i}, |\alpha| \le N - 1, N \ge 3$. *Then, one has*

$$(4.3) \qquad \|\partial_i \partial^\alpha r\|_{L_x^2} + \|\partial^\alpha l\|_{L_x^2} \lesssim \|g_2\|_{H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))} \equiv A_1,$$

$$(4.4) \qquad \|\partial^\alpha h\|_{L_x^2} \lesssim \|\nabla_x \mathcal{A}\|_{H_x^{N-2}} \|\mathcal{A}\|_{H_x^{N-1}} + \|\mathcal{A}\|_{H_x^N} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)}$$
$$+ \|g_2\|_{H_{s+\gamma/2}^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)} \equiv A_2.$$

*Proof.* By the Cauchy-Schwarz inequality,

$$\|\partial_i \partial^\alpha r\|_{L^2(\mathbb{R}_x^3)} = \|(\partial^\alpha \nabla_x g_2, e)_{L^2(\mathbb{R}_v^3)}\|_{L^2(\mathbb{R}_x^3)} \le \|\partial^\alpha \nabla_x g_2\|_{L^2(\mathbb{R}^6)} \le \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))},$$

and

$$\|\partial^\alpha l\|_{L_x^2} \le \|(\nabla_x \partial^\alpha g_2, ve)_{L^2(\mathbb{R}_v^3)}\|_{L^2(\mathbb{R}_x^3)} + \|(\partial^\alpha g_2, \mathcal{L}^* e)_{L^2(\mathbb{R}_v^3)}\|_{L^2(\mathbb{R}_x^3)}$$
$$\le \|\partial^\alpha \nabla_x g_2\|_{L^2(\mathbb{R}^6)} + \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)} \le \|g_2\|_{H^N(\mathbb{R}_x^3, L^2(\mathbb{R}_v^3))}.$$

Then (4.3) follows because $H_{\gamma/2+s}^N(\mathbb{R}^6) \subset \mathcal{B}_0^N(\mathbb{R}^6)$ holds by virtue of Proposition 2.2. We shall prove (4.4) as follows. By Proposition 2.3,

$$|\partial_x^\alpha h| \le \sum_{\alpha_1 + \alpha_2 = \alpha} |(\Gamma(\partial_x^{\alpha_1} g, \partial_x^{\alpha_2} g), e)_{L_v^2}| \lesssim \sum_{\alpha_1 + \alpha_2 = \alpha} \|\partial_x^{\alpha_1} g\|_{L_{s+\gamma/2}^2} \|\|\partial_x^{\alpha_2} g\|\| \|\|e\|\|$$
$$\lesssim \sum_{\alpha_1 + \alpha_2 = \alpha} (|\partial_x^{\alpha_1} \mathcal{A}| + \|\partial_x^{\alpha_1} g_2\|_{L_{s+\gamma/2}^2})(|\partial_x^{\alpha_2} \mathcal{A}| + \|\|\partial_x^{\alpha_2} g_2\|\|).$$

Hence

$$\|\partial_x^\alpha h\|_{L_x^2} \lesssim \| |\partial_x^{\alpha_1} \mathcal{A}|^2 |\partial_x^{\alpha_1} \mathcal{A}| \|_{L_x^2} + H,$$

where

$$H = \| |\partial_x^{\alpha_1} \mathcal{A}| \|\|\partial_x^{\alpha_2} g_2\|\| \|_{L_x^2} + \| \|\partial_x^{\alpha_1} g_2\|_{L_{s+\gamma/2}^2} |\partial^{\alpha_2} \mathcal{A}| \|_{L_x^2}$$
$$+ \| \|\partial_x^{\alpha_1} g_2\|_{L_{s+\gamma/2}^2} \|\|\partial_x^{\alpha_2} g_2\|\| \|_{L_x^2}.$$

The first term on the right hand of the above inequality can be evaluated by using Lemma 4.2. As for $H$, when $|\alpha_1| = 0, 1$, by using Proposition 2.2, we have

$$H \lesssim \|\partial_x^{\alpha_1} \mathcal{A}\|_{H^2} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)} + \|\partial_x^{\alpha_1} g_2\|_{H_{s+\gamma/2}^2(\mathbb{R}^6)} \|\mathcal{A}\|_{H_x^N} + \|\partial_x^{\alpha_1} g_2\|_{H_{s+\gamma/2}^2(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)}$$
$$\lesssim \|\mathcal{A}\|_{H_x^N} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)} + \|g_2\|_{H_{s+\gamma/2}^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)}.$$

Similarly, when $2 \le |\alpha_1| \le N$, we have

$$H \lesssim \|\partial_x^{\alpha_1} \mathcal{A}\|_{L^2} \|\partial_x^{\alpha_2} g_2\|_{H^2(\mathbb{R}^3; \mathcal{B}_0^0(\mathbb{R}^3))} + \|\partial_x^{\alpha_1} g_2\|_{H_{s+\gamma/2}^0(\mathbb{R}^6)} (\|\mathcal{A}\|_{H_x^N} + \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)})$$
$$\lesssim \|\mathcal{A}\|_{H_x^N} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)} + \|g_2\|_{H_{s+\gamma/2}^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_0^N(\mathbb{R}^6)}.$$

Thus, the proof of the lemma is completed. $\qquad \square$

The following lemma about the energy estimate on the macroscopic component which is also similar to the corresponding one in [5] for the Maxwellian molecule.

**Lemma 4.4.** *For* $|\alpha| \le N - 1$, *we have*

$$(4.5) \qquad \|\nabla_x \partial^\alpha \mathcal{A}\|_{L^2(\mathbb{R}_x^3)}^2 \lesssim -\frac{d}{dt} \{(\partial^\alpha r, \nabla_x \partial^\alpha(a, -b, c))_{L^2(\mathbb{R}_x^3)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}_x^3)}\}$$
$$+ \|g_2\|_{H^N(\mathbb{R}_x^3; L^2(\mathbb{R}_v^3))}^2 + E_{N,1} D_{N,0},$$

*where*

$$E_{N,\ell} = \|\mathcal{A}\|^2_{H^N(\mathbb{R}^3)} + \|g_2\|^2_{H^N(\mathbb{R}^3_x; L^2_\ell(\mathbb{R}^3_v))}, \quad D_{N,\ell} = \|\nabla_x \mathcal{A}\|^2_{H^{N-1}(\mathbb{R}^3)} + \sum_{|\alpha| \le N} \|\partial^\alpha_x g_2\|_{\mathcal{B}^0_\ell(\mathbb{R}^6)}.$$

*Proof.* (a) Estimate on $\nabla_x \partial^\alpha a$. Let $A_1, A_2$ be those defined in Lemma 4.3. From (4.2) (iv),

$$\begin{aligned}
\|\nabla_x \partial^\alpha a\|^2_{L^2(\mathbb{R}^3_x)} &= (\nabla_x \partial^\alpha a, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} \\
&= (\partial^\alpha(-\partial_t b - \partial_t r + l + h), \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} \\
&\le R_1 + C_\eta(A_1^2 + A_2^2) + \eta \|\nabla_x \partial^\alpha a\|^2_{L^2(\mathbb{R}^3_x)}.
\end{aligned}$$

Here,

$$\begin{aligned}
R_1 &= -(\partial^\alpha \partial_t b + \partial^\alpha \partial_t r, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} \\
&= -\frac{d}{dt}(\partial^\alpha(b+r), \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} + (\nabla_x \partial^\alpha(b+r), \partial_t \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} \\
&\le -\frac{d}{dt}(\partial^\alpha(b+r), \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)} + C_\eta(\|\nabla_x \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)} + A_1^2) + \eta \|\partial_t \partial^\alpha a\|^2_{L^2(\mathbb{R}^3_x)}.
\end{aligned}$$

(b) Estimate on $\nabla_x \partial^\alpha b$. From (4.2) (iii) and (ii),

$$\begin{aligned}
\Delta_x \partial^\alpha b_i + \partial_i^2 \partial^\alpha b_i &= \sum_{j \ne i} \partial_j \partial^\alpha(\partial_j b_i + \partial_i b_j) + \partial_i \partial^\alpha(2\partial_i b_i - \sum_{j \ne i} \partial_j b_j) \\
&= \partial_i \partial^\alpha(-\partial_t r + l + h),
\end{aligned}$$

$$\|\nabla_x \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)} + \|\partial_i \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)} = -(\Delta_x \partial^\alpha b_i + \partial_i^2 \partial^\alpha b_i, \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} = R_2 + R_3 + R_4,$$

where

$$\begin{aligned}
R_2 &= (\partial_t \partial^\alpha r, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} = \frac{d}{dt}(\partial^\alpha r, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} + (\partial_i \partial^\alpha r, \partial_t \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} \\
&\le \frac{d}{dt}(\partial^\alpha r, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} + C_\eta A_1^2 + \eta \|\partial_t \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)}, \\
R_3 &= -(\partial^\alpha l, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} \le C_\eta A_1^2 + \eta \|\partial_i \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)}, \\
R_4 &= -(\partial^\alpha h, \partial_i \partial^\alpha b)_{L^2(\mathbb{R}^3_x)} \le C_\eta A_2^2 + \eta \|\partial_i \partial^\alpha b\|^2_{L^2(\mathbb{R}^3_x)}.
\end{aligned}$$

(c) Estimate on $\nabla_x \partial^\alpha c$. From (4.2) (i),

$$\begin{aligned}
\|\nabla_x \partial^\alpha c\|^2_{L^2(\mathbb{R}^3_x)} &= (\nabla_x \partial^\alpha c, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}^3_x)} = (\partial^\alpha(-\partial_t r + l + h), \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}^3_x)} \\
&\le R_5 + C_\eta(A_1^2 + A_2^2) + \eta \|\nabla_x \partial^\alpha c\|^2_{L^2(\mathbb{R}^3_x)},
\end{aligned}$$

where

$$\begin{aligned}
R_5 &= -(\partial^\alpha \partial_t r, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}^3_x)} = -\frac{d}{dt}(\partial^\alpha r, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}^3_x)} + (\nabla_x \partial^\alpha r, \partial_t \partial^\alpha c)_{L^2(\mathbb{R}^3_x)} \\
&\le -\frac{d}{dt}(\partial^\alpha r, \nabla_x \partial^\alpha c)_{L^2(\mathbb{R}^3_x)} + C_\eta A_1^2 + \eta \|\partial_t \partial^\alpha c\|^2_{L^2(\mathbb{R}^3_x)}.
\end{aligned}$$

(d) Estimate on $\partial_t \partial^\alpha \mathcal{A}$. We directly have

$$\begin{aligned}
\|\partial_t \partial^\alpha \mathcal{A}\|_{L^2(\mathbb{R}^3_x)} &= \|\partial^\alpha \partial_t \mathbf{P} g\|_{L^2(\mathbb{R}^6_{x,v})} \\
&= \|\partial^\alpha \mathbf{P}\big(-v \cdot \nabla_x g - \mathcal{L}g + \Gamma(g,g)\big)\|_{L^2(\mathbb{R}^6_{x,v})} \\
&= \|\partial^\alpha \mathbf{P}(v \cdot \nabla_x g)\|_{L^2(\mathbb{R}^6_{x,v})} \le \|\nabla_x \partial^\alpha \mathcal{A}\|_{L^2(\mathbb{R}^3_x)} + \|\nabla_x \partial^\alpha g_2\|_{L^2(\mathbb{R}^6_{x,v})}.
\end{aligned}$$

Combining all the above estimates and taking $\eta > 0$ sufficiently small, we deduce

$$\|\nabla_x \partial^\alpha \mathcal{A}\|^2_{L^2(\mathbb{R}^3_x)} \lesssim -\frac{d}{dt}\Big\{(\partial^\alpha r, \nabla_x \partial^\alpha(a, -b, c))_{L^2(\mathbb{R}^3_x)} + (\partial^\alpha b, \nabla_x \partial^\alpha a)_{L^2(\mathbb{R}^3_x)}\Big\}$$
$$+ A_1^2 + A_2^2 + \|\nabla_x \partial^\alpha g_2\|^2_{L^2(\mathbb{R}^6_{x,v})}.$$

Finally, by choosing $|\alpha| \le N - 1$ and using Lemma 4.3, we obtain

$$A_1^2 + A_2^2 + \|\nabla_x \partial^\alpha g_2\|^2_{L^2(\mathbb{R}^6_{x,v})} \lesssim \|g_2\|^2_{H^N(\mathbb{R}^3_x; L^2(\mathbb{R}^3_v))} + E_{N,1} D_{N,0},$$

which completes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.2. **Microscopic energy.** The energy estimate on the microscopic component will be given in two parts, that is, one without weight and another one with weight as follows.

**Microscopic energy estimate without weight.** In this subsection, we shall prove the following estimate with only $x$-derivatives of the solution.

**Lemma 4.5.** *Let $N \ge 3$. Then,*

$$(4.6) \qquad\qquad\qquad\qquad \frac{d}{dt}E_{N,0} + D_{N,0} \lesssim E_{N,s+\gamma/2}^{1/2} D_{N,0}.$$

Notice that this is not a closed estimate because of the presence of $E_{N,s+\gamma/2}$ on the right hand side. Since $s + \gamma/2 > 0$, this is exactly why we can not prove global existence with only differentiation in $x$ variable.

For the proof, let $\alpha \in \mathbb{N}^3$, $|\alpha| \le N$, and apply $\partial^\alpha = \partial_x^\alpha$ to (1.3) to have,

$$\partial_t(\partial^\alpha g) + v \cdot \nabla_x(\partial^\alpha g) + \mathcal{L}(\partial^\alpha g) = \partial^\alpha \Gamma(g, g).$$

Then take the $L^2(\mathbb{R}^6_{x,v})$ inner product of the above equation with $\partial^\alpha g$. By Proposition 2.1, we have

$$(4.7) \qquad\qquad\qquad \frac{d}{dt}\|\partial^\alpha g\|^2_{L^2(\mathbb{R}^6_{x,v})} + |||\partial^\alpha g_2|||^2_{\mathcal{B}_0^0(\mathbb{R}^6)} \lesssim J_\alpha,$$

where

$$J_\alpha = (\partial^\alpha \Gamma(g, g), \partial^\alpha g)_{L^2(\mathbb{R}^6)} = \sum_{i,j=1}^2 (\partial^\alpha \Gamma(g_i, g_j), \partial^\alpha g_2)_{L^2(\mathbb{R}^6)}$$

$$= \sum_{i,j=1}^2 J^{(ij)}.$$

Remember that we are dealing with the case when $0 < s < 1, \gamma + 2s > 0$. Firstly, consider $J^{(11)}$. For $\psi_j \in \mathcal{N}$,

$$|J^{(11)}| \lesssim \int_{\mathbb{R}^3} |\partial^\alpha \mathcal{A}^2| \, |(\Gamma(\psi_j, \psi_k), \partial^\alpha g_2)_{L^2(\mathbb{R}^3)}| dx.$$

According to Theorem 1.2 of [7], we have,

$$|(\Gamma(\psi_j, \psi_k), \partial^\alpha g_2)_{L^2(\mathbb{R}^3)}| \lesssim \left(\|\psi_j\|_{L^2_{s+\gamma/2}} |||\psi_k||| + \|\psi_k\|_{L^2_{s+\gamma/2}} |||\psi_j|||\right) |||\partial^\alpha g_2|||$$
$$\lesssim |||\partial^\alpha g_2|||,$$

and hence

$$|J^{(11)}| \lesssim \|\partial^\alpha \mathcal{A}^2\|_{L^2(\mathbb{R}^3)} \|\partial^\alpha g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)} \lesssim \|\nabla \mathcal{A}\|_{H_x^{N-1}(\mathbb{R}^3)} \|\mathcal{A}\|_{H_x^{N-1}(\mathbb{R}^3)} \|\partial^\alpha g_2\|_{\mathcal{B}_0^0(\mathbb{R}^6)}$$
$$\lesssim E_{N,0}^{1/2} D_{N,0}.$$

On the other hand,

$$J^{(12)} \sim \int_{\mathbb{R}_x^3} (\partial_x^{\alpha_1} \mathcal{A})(\Gamma(\psi_k, \partial_x^{\alpha_2} g_2), \partial_x^{\alpha} g_2)_{L^2(\mathbb{R}_v^3)} dx.$$

In view of Theorem 1.2 of [7], we obtain

$$
\begin{aligned}
|(\Gamma(\psi_k, \partial_x^{\alpha_2} g_2), \partial_x^{\alpha} g_2)_{L^2(\mathbb{R}_v^3)}| &\lesssim \left\{ \|\psi_k\|_{L^2_{s+\gamma/2}} \|\!|\partial_x^{\alpha_2} g_2|\!\| + \|\partial_x^{\alpha_2} g_2\|_{L^2_{s+\gamma/2}} \|\!|\psi_k|\!\| \right. \\
&\quad \left. + \min\left(\|\psi_k\|_{L^2} \|\partial_x^{\alpha_2} g_2\|_{L^2_{s+\gamma/2}}, \|\psi_k\|_{L^2_{s+\gamma/2}} \|\partial_x^{\alpha_2} g_2\|_{L^2}\right) \right\} \|\!|\partial_x^{\alpha} g_2|\!\| \\
&\lesssim \left(\|\!|\partial_x^{\alpha_2} g_2|\!\| + \|\partial_x^{\alpha_2} g_2\|_{L^2_{s+\gamma/2}}\right) \|\!|\partial_x^{\alpha} g_2|\!\| \\
&\lesssim \|\!|\partial_x^{\alpha_2} g_2|\!\| \, \|\!|\partial_x^{\alpha} g_2|\!\|.
\end{aligned}
$$

Here and hereafter, we will use freely that

$$\|g\|_{L^2} \lesssim \|g\|_{L^2_{s+\gamma/2}} \lesssim \|\!|g|\!\|.$$

The first inequality above holds since we assume $s + \gamma/2 > 0$ and the second one follows from Theorem 1.2 of [7]. By using Proposition 2.3,

$$|J^{(12)}| \lesssim \int_{\mathbb{R}_x^3} \|\partial_x^{\alpha_1} \mathcal{A}\| \, \|\!|\partial_x^{\alpha_2} g_2|\!\| \, \|\!|\partial_x^{\alpha} g_2|\!\| dx \lesssim E_{N,0}^{1/2} D_{N,0}.$$

A similar argument applies to

$$J^{(21)} \sim \int_{\mathbb{R}_x^3} (\partial_x^{\alpha_2} \mathcal{A})(\Gamma(\partial_x^{\alpha_1} g_2, \varphi_k), \partial_x^{\alpha} g_2)_{L^2(\mathbb{R}_v^3)} dx.$$

In fact, Theorem 1.2 of [7] gives,

$$
\begin{aligned}
\left| \left(\Gamma(\partial_x^{\alpha_1} g_2, \varphi_k), \partial_x^{\alpha} g_2\right)_{L^2(\mathbb{R}_v^3)} \right| &\lesssim \left(\|\partial_x^{\alpha_1} g_2\|_{L^2_{s+\gamma/2}} \|\!|\psi_k|\!\| + \|\psi_k\|_{L^2_{s+\gamma/2}} \|\!|\partial_x^{\alpha_1} g_2|\!\| \right. \\
&\quad \left. + \min\left(\|\partial_x^{\alpha_1} g_2\|_{L^2} \|\psi_k\|_{L^2_{s+\gamma/2}}, \|\partial_x^{\alpha_1} g_2\|_{L^2_{s+\gamma/2}} \|\psi_k\|_{L^2}\right)\right) \|\!|\partial_x^{\alpha} g_2|\!\| \\
&\lesssim (\|\partial_x^{\alpha_1} g_2\|_{L^2_{s+\gamma/2}} + \|\!|\partial_x^{\alpha_1} g_2|\!\|) \|\!|\partial_x^{\alpha} g_2|\!\| \\
&\lesssim \|\!|\partial_x^{\alpha_1} g_2|\!\| \, \|\!|\partial_x^{\alpha} g_2|\!\|.
\end{aligned}
$$

Therefore, similar to $J^{(12)}$, we have

$$|J^{(21)}| \lesssim E_{N,0}^{1/2} D_{N,0}.$$

For the estimation on $J^{(22)}$, from Theorem 1.2 of [7] again, we have

$$
\begin{aligned}
|(\Gamma(\partial_x^{\alpha_1} g_2, \partial_x^{\alpha_2} g_2), \partial_x^{\alpha} g_2)_{L^2(\mathbb{R}^3)}| &\lesssim \left\{ \|\partial_x^{\alpha_1} g_2\|_{L^2_{s+\gamma/2}} \|\!|\partial_x^{\alpha_2} g_2|\!\| + \|\!|\partial_x^{\alpha_1} g_2|\!\| \, \|\partial_x^{\alpha_2} g_2\|_{L^2_{s+\gamma/2}} \right. \\
&\quad \left. + \min\left(\|\partial_x^{\alpha_1} g_2\|_{L^2} \|\partial_x^{\alpha_2} g_2\|_{L^2_{s+\gamma/2}}, \|\partial_x^{\alpha_1} g_2\|_{L^2_{s+\gamma/2}} \|\partial_x^{\alpha_2} g_2\|_{L^2}\right) \right\} \|\!|\partial_x^{\alpha} g_2|\!\| \\
&\lesssim \left(\|\partial_x^{\alpha_1} g_2\|_{L^2_{s+\gamma/2}} \|\!|\partial_x^{\alpha_2} g_2|\!\| + \|\!|\partial_x^{\alpha_1} g_2|\!\| \, \|\partial_x^{\alpha_2} g_2\|_{L^2_{s+\gamma/2}}\right) \|\!|\partial_x^{\alpha} g_2|\!\|.
\end{aligned}
$$

Firstly, suppose that $|\alpha_1| \leq N - 2$. Then

$$
\begin{aligned}
|J^{(22)}| &= |(\Gamma(\partial_x^{\alpha_1} g_2, \partial_x^{\alpha_2} g_2), \partial_x^{\alpha} g_2)_{L^2(\mathbb{R}^6)}| \\
&\lesssim \|\partial_x^{\alpha_1} g_2\|_{L_x^{\infty}(L^2_{s+\gamma/2})} \int_{\mathbb{R}^3} \|\!|\partial_x^{\alpha_2} g_2|\!\| \, \|\!|\partial_x^{\alpha} g_2|\!\| dx + \| \, \|\!|\partial_x^{\alpha_1} g_2|\!\| \, \|_{L_x^{\infty}} \int_{\mathbb{R}^3} \|\partial_x^{\alpha_2} g_2\|_{L^2_{s+\gamma/2}} \|\!|\partial_x^{\alpha} g_2|\!\| dx \\
&\lesssim E_{N,s+\gamma/2}^{1/2} D_{N,0}.
\end{aligned}
$$

Similary, when $|\alpha_1| > N - 2$, we have $|\alpha_2| \leq 1$ so that

$$|J^{(22)}| \lesssim \|\partial_x^{\alpha_2} g_2\|_{L_x^\infty(L_{s+\gamma/2}^2)} \int_{\mathbb{R}^3} \||\partial_x^{\alpha_1} g_2\|\| \||\partial_x^\alpha g_2\||dx + \| \||\partial_x^{\alpha_2} g_2\|\| \|_{L_x^\infty} \int_{\mathbb{R}^3} \|\partial_x^{\alpha_1} g_2\|_{L_{s+\gamma/2}^2} \||\partial_x^\alpha g_2\||dx$$

$$\lesssim E_{N,s+\gamma/2}^{1/2} D_{N,0}.$$

Taking the summation of (4.7) over $|\alpha| \leq N, N \geq 3$ gives (4.6).

**Microscopic energy estimate with weight.** In order to close the estimate (4.6), we need the estimates of $x$-$v$ derivatives of the solution with weight. Firstly, let $\partial_\beta^\alpha = \partial_x^\alpha \partial_v^\beta, |\alpha + \beta| \leq N, N \geq 6$, and apply $W_\ell \partial_\beta^\alpha (\mathbf{I} - \mathbf{P})$ to (1.3). We have

$$\partial_t (W_\ell \partial_\beta^\alpha g_2) + v \cdot \nabla_x (W_\ell \partial_\beta^\alpha g_2) + \mathcal{L}_1 (W_\ell \partial_\beta^\alpha g_2) = \mathcal{M}^{\alpha,\beta},$$

where, with $e_i \in \mathbb{N}^n, |e_i| = 1$,

$$\mathcal{M}^{\alpha,\beta} = W_\ell \partial_\beta^\alpha \Gamma(g, g) + W_\ell \partial_\beta^\alpha [\mathbf{P}, v \cdot \nabla_x] g - \sum_{i=1}^N \partial_{x_i} (W_\ell \partial_{\beta-e_i}^\alpha g_2) - [W_\ell \partial_\beta^\alpha, \mathcal{L}_1] g_2 - W_\ell \partial_\beta^\alpha \mathcal{L}_2 g_2$$

$$= \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \mathcal{M}_4 + \mathcal{M}_5.$$

Take the $L^2(\mathbb{R}_{x,v}^6)$ inner product of this equation with $W_\ell \partial_\beta^\alpha g_2$ to deduce

(4.8) $$\frac{d}{dt} \|W_\ell \partial_\beta^\alpha g_2\|_{L^2(\mathbb{R}^6)}^2 + D \lesssim M,$$

where $D$ is the dissipation rate given by

$$D = \int_{\mathbb{R}^3} \||(\mathbf{I} - \mathbf{P}) W_\ell \partial_\beta^\alpha g_2\||^2 dx \geq \|W_\ell \partial_\beta^\alpha g_2\|_{\mathcal{B}_0^0(\mathbb{R}^3)}^2 - C \|g_2\|_{H_0^N(\mathbb{R}^6)}^2.$$

Here, we have used

$$\||\mathbf{P} W_\ell \partial_\beta^\alpha g_2\|| \lesssim \|\partial^\alpha g_2\|_{L_v^2}.$$

And $M$ is defined by

$$M = \sum_{j=1}^5 (\mathcal{M}_j, W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^6)} = \sum_{j=1}^5 M_j.$$

Firstly, note that from Propositions 2.3 and 2.5, we have

(4.9) $$|(W_\ell \Gamma(\partial_{\beta_1} f, \partial_{\beta_2} g), h)_{L^2(\mathbb{R}^3)}| \lesssim \Big( \|W_{s+\gamma/2} \partial_{\beta_1} f\|_{L^2(\mathbb{R}^3)} \||W_\ell \partial_{\beta_2} g\||$$

$$+ \|W_{s+\gamma/2} \partial_{\beta_2} g\|_{L^2(\mathbb{R}^3)} \||W_\ell \partial_{\beta_1} f\|| \Big) \||h\||.$$

Write

$$M_1 = \sum_{i,j=1,2} (W_\ell \partial_\beta^\alpha \Gamma(g_i, g_j), W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^6)} = \sum_{i,j=1,2} M_{1ij}.$$

We have

$$M_{111} = (W_\ell \partial_\beta^\alpha \Gamma(g_1, g_1), W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^6)}$$

$$\sim \int_{\mathbb{R}^3} (\partial^{\alpha_1} \mathcal{A})(\partial^{\alpha_2} \mathcal{A})(W_\ell \partial_\beta \Gamma(\psi_j, \psi_k), W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^3)} dx.$$

Recall that by Leibnitz formula, the differentiation on $\Gamma$ involves the nonlinear operators $\Gamma$ and $\mathcal{T}$. Since these two operators share the same upper bound and commutator properties,

for brevity, we only consider the nonlinear operator $\Gamma$. By using (4.9), since $\psi$ is a function with an exponential decay factor, we obtain for $|\beta_1| + |\beta_2| \leq |\beta|$,

$$|(W_\ell \Gamma(\partial_{\beta_1}\psi_j, \partial_{\beta_2}\psi_k), W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^3)}| \lesssim |||W_\ell \partial_\beta^\alpha g_2|||.$$

Therefore,

$$M_{111} \sim \int_{\mathbb{R}^3} (\partial^{\alpha_1}\mathcal{A})(\partial^{\alpha_2}\mathcal{A})|||W_\ell \partial_\beta^\alpha g_2|||dx$$
$$\lesssim \|\nabla_x \mathcal{A}\|_{H^{N-1}(\mathbb{R}^3)} \|\mathcal{A}\|_{H^N(\mathbb{R}^3)} \|g_2\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}.$$

Now consider

$$M_{112} = (W_\ell \partial_\beta^\alpha \Gamma(g_1, g_2), W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^6)}$$
$$\sim \int_{\mathbb{R}^3} (\partial^{\alpha_1}\mathcal{A})(W_\ell \partial_\beta \Gamma(\psi_j, \partial^{\alpha_2} g_2), W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^3)}dx.$$

By (4.9),

$$|(W_\ell \Gamma(\partial_{\beta_1}\psi_j, \partial_{\beta_2}^{\alpha_2} g_2), W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^3)}|$$
$$\lesssim \left(\|W_{s+\gamma/2}\partial_{\beta_1}\psi_j\|_{L^2(\mathbb{R}^3)}|||W_\ell \partial_{\beta_2}^{\alpha_2} g_2||| + \|W_{s+\gamma/2}\partial_{\beta_2}^{\alpha_2} g_2\|_{L^2(\mathbb{R}^3)}|||W_\ell \partial_{\beta_1}\psi_j|||\right)|||W_\ell \partial_\beta^\alpha g_2|||$$
$$\lesssim |||W_\ell \partial_{\beta_2}^{\alpha_2} g_2||| \; |||W_\ell \partial_\beta^\alpha g_2|||,$$

where we have used

$$\|W_{s+\gamma/2}\partial_{\beta_2}^{\alpha_2} g_2\|_{L^2(\mathbb{R}^3)} \lesssim |||W_\ell \partial_{\beta_2}^{\alpha_2} g_2|||.$$

Thus,

$$M_{112} \lesssim \|\mathcal{A}\|_{H^N(\mathbb{R}^3)} \|g_2\|_{\mathcal{B}_\ell^N}^2.$$

Next, notice that

$$M_{121} = (W_\ell \partial_\beta^\alpha \Gamma(g_2, g_1), W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^6)}$$
$$\sim \int_{\mathbb{R}^3} (\partial^{\alpha_2}\mathcal{A})(W_\ell \partial_\beta \Gamma(\partial^{\alpha_1} g_2, \psi_j), W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^3)}dx.$$

As above, (4.9) yields

$$|(W_\ell \Gamma(\partial_{\beta_1}^{\alpha_1} g_2, \partial_{\beta_2}\psi_j), W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^3)}|$$
$$\lesssim \left(\|W_{s+\gamma/2}\partial_{\beta_1}^{\alpha_1} g_2\|_{L^2(\mathbb{R}^3)}|||W_\ell \partial_{\beta_2}\psi_j||| + \|W_{s+\gamma/2}\partial_{\beta_2}\psi_j\|_{L^2(\mathbb{R}^3)}|||W_\ell \partial_{\beta_1}^{\alpha_1} g_2|||\right)|||W_\ell \partial_\beta^\alpha g_2|||$$
$$\lesssim \left(\|W_{s+\gamma/2}\partial_{\beta_1}^{\alpha_1} g_2\|_{L^2(\mathbb{R}^3)} + |||W_\ell \partial_{\beta_1}^{\alpha_1} g_2|||\right)|||W_\ell \partial_\beta^\alpha g_2|||$$
$$\lesssim |||W_\ell \partial_{\beta_1}^{\alpha_1} g_2||| \; |||W_\ell \partial_\beta^\alpha g_2|||,$$

where we have used $\|W_{s+\gamma/2}g\|_{L^2(\mathbb{R}^3)} \lesssim |||W_\ell g|||$. Consequently,

$$M_{121} \lesssim \|\mathcal{A}\|_{H^N(\mathbb{R}^3)} \|g_2\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}^2.$$

It remains to evaluate

$$M_{122} = (W_\ell \partial_\beta^\alpha \Gamma(g_2, g_2), W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^6)}.$$

For this, we can apply Proposition 3.2 to have

$$M_{122} \lesssim \|g_2\|_{H_\ell^N(\mathbb{R}^6)} \|g_2\|_{\mathcal{B}_\ell^N(\mathbb{R}^6)}^2.$$

In conclusion, we have proved

$$M_1 \lesssim \mathcal{E}_{N,\ell}^{1/2} \mathcal{D}_{N,\ell}.$$

By using integration by parts and taking into account that $s + \gamma/2 > 0$, we get

$$\begin{aligned}
M_2 &\lesssim |(W_\ell \partial_\beta^\alpha [\mathbf{P}, v \cdot \nabla_x] g, W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^6)}| \\
&\lesssim (\|\partial^\alpha \nabla_x \mathcal{A}\|_{L^2(\mathbb{R}^3)} + \|\partial^\alpha \nabla_x g_2\|_{L^2(\mathbb{R}^6)}) \|\partial^\alpha g_2\|_{L^2(\mathbb{R}^6)} \\
&\lesssim \delta_0 \|\nabla \mathcal{A}\|^2_{H^{N-1}(\mathbb{R}^3)} + C_{\delta_0} \|g_2\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))},
\end{aligned}$$

where $\delta_0 > 0$ is a small constant.

Similarly, For $1 \le |\beta| \le N$,

$$M_3 \lesssim \|W_\ell \partial_{\beta-1}^{\alpha+1} g_2\|_{L^2(\mathbb{R}^6)} \|W_\ell \partial_\beta^\alpha g_2\|_{L^2(\mathbb{R}^6)} \le C_\delta \|\partial_{\beta-1}^{\alpha+1} g_2\|^2_{\mathcal{B}_\ell^0(\mathbb{R}^6)} + \delta \|g_2\|^2_{\mathcal{B}_\ell^N(\mathbb{R}^6)},$$

where $\delta > 0$ is another small constant.

The main ingredients of the estimation on $M_4$ are the commutator estimates $I$ and $II$ which are defined in the proof of Proposition 4.8 in [7]. Note that they are valid in general for $\gamma > -3$ so that the estimates there can be used here. That is,

$$|I| = |([W_\ell, \mathcal{L}_1] g, W_\ell g)_{L^2(\mathbb{R}^3)}| \lesssim \|W_\ell g\|^2_{L^2_{\gamma/2}(\mathbb{R}^3)}.$$

$$\begin{aligned}
|II| &= |(W_\ell [\partial_\beta, \mathcal{L}_1] g, W_\ell \partial_\beta g)_{L^2(\mathbb{R}^3)} \\
&\lesssim \sum_{\beta_1+\beta_2+\beta_3=\beta, \, \beta_2 \neq 0} |(W_\ell \mathcal{T}(\partial_{\beta_1} \mu^{1/2}, \partial_{\beta_2} g, \partial_{\beta_3} \mu^{1/2}), W_\ell \partial_\beta g)_{L^2(\mathbb{R}^3)}| \\
&\lesssim \Big( \sum_{\beta_1+\beta_2=\beta, \, \beta_2 \neq 0} |||W_\ell \partial_{\beta_1} g|||_{\Phi_\gamma} \Big) |||W_\ell \partial_\beta g|||_{\Phi_\gamma}.
\end{aligned}$$

With this, later we also need the following interpolation inequality

$$\|W_\ell \partial_\beta h\|_{L^2_{\gamma/2}} \lesssim C_\delta \|\partial_\beta h\|_{L^2_{s+\gamma/2}} + \delta \|W_\ell \partial_\beta h\|_{L^2_{s+\gamma/2}} \lesssim C_\delta |||\partial_\beta h|||_{\Phi_\gamma} + \delta |||W_\ell \partial_\beta h|||_{\Phi_\gamma}.$$

For the term $M_4$, using Proposition 3.4

$$\begin{aligned}
M_4 &\le |([W_\ell \partial_\beta^\alpha, \mathcal{L}_1] g_2, W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^6)}| \\
&\lesssim \Big( \|g_2\|_{H_\ell^{|\alpha|+|\beta|}} + \sum_{|\alpha'|=|\alpha|, |\beta'|=|\beta|-1} \|\partial_{\beta'}^{\alpha'} g_2\|_{\mathcal{B}_\ell^0(\mathbb{R}^3)} \Big) \|g_2\|_{\mathcal{B}_\ell^{|\alpha|+|\beta|}}
\end{aligned}$$

Finally, in view of Proposition 3.3, we have

$$\begin{aligned}
M_5 &= |(W_\ell \partial_\beta^\alpha \mathcal{L}_2 g_2, W_\ell \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^6)}| \le |(\partial_\beta^\alpha \mathcal{L}_2 g_2, W_{2\ell} \partial_\beta^\alpha g_2)_{L^2(\mathbb{R}^6)}| \\
&\le C \|g_2\|_{H^{|\alpha|+|\beta|}(\mathbb{R}^6)} \|\mu^{1/10^3} W_{2\ell} \partial_\beta^\alpha g_2\|_{L^2(\mathbb{R}^6)} \lesssim \|g_2\|^2_{H^{|\alpha|+|\beta|}(\mathbb{R}^6)}.
\end{aligned}$$

Now interpolation inequality

$$\|g_2\|_{H_\ell^{|\alpha|+|\beta|}} + \|g_2\|_{H^{|\alpha|+|\beta|}(\mathbb{R}^6)} \le \delta \|g_2\|_{\mathcal{B}_\ell^{|\alpha|+|\beta|}} + C_\delta \sum_{|\alpha'|=|\alpha|+1, |\beta'|=|\beta|-1} \|\partial_{\beta'}^{\alpha'} g_2\|_{\mathcal{B}_\ell^0}.$$

Plugging all these estimates to (4.8) and fix the small constant $\delta$ with respect to the coefficient in front of the dissipation rate, we have for $|\beta| \neq 0$,

$$\begin{aligned}
\frac{d}{dt} \|W_\ell \partial_\beta^\alpha g_2\|^2_{L^2(\mathbb{R}^6)} &+ \|W_\ell \partial_\beta^\alpha g_2\|_{\mathcal{B}_0^0} \\
&\lesssim \|g_2\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))} + \mathcal{E}_{N,\ell}^{1/2} \mathcal{D}_{N,\ell} \\
&+ \delta_0 \|\nabla_x \mathcal{A}\|^2_{H^{N-1}(\mathbb{R}^3)} + \sum_{|\alpha'|=|\alpha|+1, |\beta'|=|\beta|-1} \|\partial_{\beta'}^{\alpha'} g_2\|^2_{\mathcal{B}_\ell^0}.
\end{aligned}$$

By induction on $|\beta|$ and $|\alpha| + |\beta|$, we have

$$(4.10) \qquad \frac{d}{dt} \mathcal{E}_{N,\ell} + \mathcal{D}_{N,\ell} \lesssim \|g_2\|^2_{H^N(\mathbb{R}^3_x, L^2(\mathbb{R}^3_v))} + \delta_0 \|\nabla_x \mathcal{A}\|^2_{H^{N-1}(\mathbb{R}^3)} + \mathcal{E}_{N,\ell}^{1/2} \mathcal{D}_{N,\ell}.$$

Taking a suitable linear combination of the estimates (4.5), (4.6), and (4.10), we then conclude Proposition 4.1. And this completes the energy estimate on the solution so that the global existence follows in the standard way for small perturbation.

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