Commun. Math. Phys. 230, 289–327 (2002)
Digital Object Identifier (DOI) 10.1007/s00220-002-0705-4

Communications in
Mathematical Physics

Weak Solutions of General Systems of Hyperbolic Conservation Laws

Tai-Ping Liu[1,2,⋆], **Tong Yang**[3,⋆⋆]

[1] Department of Mathematics, Stanford University, Stanford, CA 94305, USA
[2] Institute of Mathematics, Academia Sinica, Nankang, Taipei, Taiwan, R.O.C.
[3] Department of Mathematics, City University of Hong Kong, Hong Kong, P.R. China

Received: 16 October 2001 / Accepted: 8 May 2002
Published online: 4 September 2002 –

Abstract: In this paper, we establish the existence theory for general system of hyperbolic conservation laws and obtain the uniform L_1 boundness for the solutions. The existence theory generalizes the classical Glimm theory for systems, for which each characteristic field is either genuinely nonlinear or linearly degenerate in the sense of Lax. We construct the solutions by the Glimm scheme through the wave tracing method. One of the key elements is a new way of measuring the potential interaction of the waves of the same characteristic family involving the angle between waves. A new analysis is introduced to verify the consistency of the wave tracing procedure. The entropy functional is used to study the L_1 boundedness.

1. Introduction

Consider the Cauchy problem for a general system of hyperbolic conservation laws

$$u_t + f(u)_x = 0, \tag{1.1}$$
$$u(x,0) = u_0(x), \tag{1.2}$$

here $u = u(x,t) = (u^1(x,t), \dots, u^n(x,t))$ and $f(u)$ are n-vectors.

The system is assumed to be strictly hyperbolic, that is, the eigenvalues of the $n \times n$ matrix $f'(u)$ are real and distinct:

$$\begin{gathered} f'(u)r_i(u) = \lambda_i(u)r_i(u), \\ l_i(u)f'(u) = \lambda_i(u)l_i(u), \\ l_i(u)\cdot r_j(u) = \delta_{ij}, \quad i,j = 1,2,\dots,n, \\ \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \end{gathered} \tag{1.3}$$

⋆ The research was supported in part by NSF Grant DMS-9803323.
⋆⋆ The research was supported in part by the RGC Competitive Earmarked Research Grant CityU 1032/98P.

By a linear transformation, if necessary, we may assume that the i^{th} component u^i of the vector u is strictly increasing in the direction of r_i. This can be done at least for a small neighborhood of a given state. In the following we will use u^i to measure the wave strength of an i-wave.

It is well-known that, because of the dependence of the characteristics $\lambda_i(u)$ on the dependent variables u, waves may compress and smooth solutions in general do not exist globally in time. One therefore considers the weak solution:

Definition 1.1. *A bounded measurable function $u(x,t)$ is a weak solution of (1.1), (1.2) if and only if*

$$\int_0^\infty \int_{-\infty}^\infty [\phi_t u + \phi_x f(u)](x,t)dx\,dt + \int_{-\infty}^\infty \phi(x,0)u_0(x)dx = 0 \tag{1.4}$$

for any smooth function $\phi(x,t)$ of compact support in $\left\{(x,t)|(x,t) \in \mathbf{R}^2\right\}$.

As a consequence of the weak formulation, a discontinuity (u_-, u_+) in the weak solution with speed s satisfies the Rankine-Hugoniot (jump) condition

$$s(u_+ - u_-) = f(u_+) - f(u_-), \tag{1.5}$$

where u_- and u_+ are the left and right states of the discontinuity respectively.

This prompts the introduction of the Hugoniot curves $H(u_0)$ passing through a given state u_0 as follows:

$$H(u_0) \equiv \{u : \sigma(u_0 - u) = f(u_0) - f(u)\}, \tag{1.6}$$

for some scalar $\sigma = \sigma(u_0, u)$.

The Rankine-Hugoniot condition says that $u_+ \in H(u_-)$ and that $s = \sigma(u_-, u_+)$. It follows easily from the strict-hyperbolicity of the system that in a small neighborhood of a given state u_0, the set $H(u_0)$ consists of n smooth curves $H_i(u_0)$, $i = 1, 2, \ldots, n$, through u_0, such that $\sigma_i(u_0, u)$ tends to $\lambda_i(u_0)$ as u moves along $H_i(u_0)$ toward u_0. Here we use the notation $\sigma_i(u_0, u)$ to denote the scalar $\sigma(u_0, u)$ in $H_i(u_0)$. A discontinuity (u_-, u_+), $u_+ \in H_i(u_-)$, is called an i-discontinuity.

In general, weak solutions to the initial value problem (1.1) and (1.2) are not unique. A certain admissibility condition, the entropy condition, needs to be imposed on the weak solution to rule out non-physical discontinuities as follows.

Definition 1.2 (Liu, [20]). *A discontinuity (u_-, u_+) is admissible if*

$$\sigma(u_-, u_+) \le \sigma(u_-, u), \tag{1.7}$$

for any state u on the Hugoniot curve $H(u_-)$ between u_- and u_+.

If a characteristic field of the system (1.1) is genuinely nonlinear, [14], in the sense that

$$\nabla\lambda_i(u) \cdot r_i(u) \neq 0. \quad (g.nl.), \tag{1.8}$$

then the entropy condition is reduced to Lax's entropy condition

$$\lambda_i(u_+) < \sigma_i(u_-, u_+) < \lambda_i(u_-). \tag{1.9}$$

If a characteristic field of the system (1.1) is linearly degenerate, i.e.

$$\nabla\lambda_i(u) \cdot r_i(u) \equiv 0. \quad (l.dg.), \tag{1.10}$$

then the entropy condition is reduced to the one for linear waves

$$\lambda_i(u_+) = \sigma_i(u_-, u_+) = \lambda_i(u_-). \tag{1.11}$$

When each characteristic field is either genuinely nonlinear or linearly degenerate, there is the classical existence theory of James Glimm, [12]. An important physical example of such a system is the Euler equations in gas dynamics. Other physical systems, such as those in elasticity and magneto-hydrodynamics, for instance, are not necessarily genuinely nonlinear or linearly degenerate.

The goal of the present paper is to study, particularly to establish the existence theory, for such a general system. Thus for a given characteristic field $\lambda_i(u)$, we allow the linearly degenerate manifold $LG_i \equiv \{u : \nabla\lambda_i(u) \cdot r_i(u) = 0\}$ to be neither the empty space, as in the case of genuine nonlinearity, nor the whole space, as in the case of linear degeneracy.

Theorem 1.1. *Suppose that system (1.1) is strictly hyperbolic with flux function $f(u) \in C^3$, and that for each characteristic field $\lambda_i(u)$ the linear degeneracy manifold LD_i either is the whole space or consists of a finite number of smooth manifolds of codimension one, each transversal to the characteristic vector $r_i(u)$. Then for the initial data (1.2) with sufficiently small total variation $T.V.$, there exists a global weak admissible solution $u(x,t)$ to the Cauchy problem (1.1) and (1.2) satisfying $total\ variation\ u(\cdot,t) = O(1)T.V.$.*

Remark 1.1. In this paper, we only prove the existence of the weak solution to (1.1) and (1.2). The admissibility of the weak solution has been established in [19]. It is shown, cf. Theorem 15.1 in [19], that there exist subsets Λ_1 and Λ_2 of $\{(x,t) : -\infty < x < \infty, \quad t \geq 0\}$ with the following properties. Λ_1 consists of countable Lipschitz continuous curves and Λ_2 consists of countable points. Each curve Γ in Λ_1 represents a curve of jump discontinuity in the weak solution satisfying the entropy condition (1.7) except for countable points. Each point in Λ_2 represents a point of interaction in the weak solution. And outside $\Lambda_1 \cup \Lambda_2$, the weak solution is continuous. In fact, for each shock wave in the weak solution, there exists a corresponding approximate shock wave in the approximate solution when the mesh sizes are sufficiently small. Consequently, the admissibility of the shock waves in the weak solution follows from the admissibility of the shock waves in the approximate solutions as the consequence of the design of the scheme.

The Glimm theory for systems with genuinely nonlinear or linearly degenerate fields is based on the study of the interactions of elementary waves in the solutions of the Riemann problems solved by Peter Lax, [14]. The random choice method, the Glimm scheme, is introduced to construct the general solutions using the Riemann solutions as building blocks. A nonlinear functional, the Glimm functional $F[u]$, is constructed to bound the total variation of the approximate solutions. The functional yields a global measure of the total wave interactions, [13], and allows for the consistency study of the wave tracing method, [19].

For systems, which are not necessarily genuinely nonlinear or linearly degenerate, there are richer phenomena for nonlinear wave interactions, [19]. We adopt the

Glimm quadratic functional for the interaction of waves in different families. However, for the interaction between waves of the same family, the quadratic functional in general does not exist, and a cubic functional is needed. A cubic functional was introduced in [19], which, however, fails to take into account some aspects of wave interactions. Here we revise the cubic functional in [19] so that it depends globally on the wave patterns in the solution. It is defined by the product of the strengths of two waves times the angle between them, when that angle is negative. This is so that such a pair of waves of the same family will interact in general at a later time. This new cubic functional is an effective measure of the wave interactions in that the functional decreases only due to the interaction of the waves next to each other and that the decrease is exactly of the same order of the waves produced by the interaction.

With the present existence theory and the qualitative theory of regularity and large-time behavior of solutions in [19], there is the open problem of the L_1 stability of the solutions with respect to the initial data. We study the stability problem here, but only for the stability of the constant solutions. For the stability analysis we make use of the classical entropy functional, which is shown to yield the estimate to control the bifurcation of the Hugoniot curve from the rarefaction wave curve in the general setting. To construct a generalized entropy functional, as for the case when each characteristic field is either genuinely nonlinear or linearly degenerate, to control the estimates of the same cubic order as mentioned above would be the main task to study the stability of the weak solution to this general system.

In the next section we sketch the construction of the solution to the Riemann problem and some basic estimates on the Hugoniot curves. These estimates allow us to study the local wave interactions in Sect. 3. In Sect. 4 we study the nonlinear functional and thereby establish the convergence of the approximate solutions. The wave tracing mechanism of [17, 19] is refined here. Previous consistency analysis, [17], requires the boundedness of the quadratic functional. For a non-genuinely nonlinear system, a quadratic functional for interactions of waves of the same characteristic family does not exist in general. Our consistency analysis for the wave tracing method here uses only the estimates resulting from the cubic functionals. The cubic estimates are weaker and there is a new, interesting consistency analysis here.

In the last section of this paper, we study the L_1 stability of constant state solutions to the system (1.1). There has been much progress on the well-posedness, in L_1 topology, problem when each characteristic field in the system is either genuinely nonlinear or linearly degenerate. There are two approaches. One starts with [4] on the comparison of infinitesimally close solutions, see [5] and [6]; the other approach [22] is based on the construction of the robust functional, see also [7, 23]. For the more general system (1.1), there is the recent result for the case of one reflection point in [1]. To our knowledge, there is no general well-posedness theory without assuming genuine nonlinearity or linear degeneracy on characteristic fields, beyond that of [1].

The purpose of Sect. 6 is to study the L_1 stability of the constant state solutions of the general systems. We adopt the general approach of [22] and construct a new time-decreasing nonlinear functional $H(t) = H[u(\cdot, t)]$, which is equivalent to $||u||_{L_1}$ of a weak solution $u(x, t)$. It also depends explicitly on the wave pattern of this solution. The functional $H[u(\cdot, t)]$ consists of three parts: the first part is the product of the Glimm's functional and a linear functional $L(t)$; the second part is a quadratic functional $Q_d(t)$; and the third part is the convex entropy functional. Here $L(t)$ represents the L_1-norm of $u(x, t)$. $Q_d(t)$ registers the effect of nonlinear coupling of waves in different families on $||u(x, t)||_{L_1(x)}$ by making use of the strict hyperbolicity of the system, and $E(t)$

captures the nonlinearity of the characteristic fields. The existence of such a functional immediately yields the following theorem.

Theorem 1.2. *Suppose that the total variation of the initial data is sufficiently small and is in L_1, then the L_1 norm of the weak admissible solution to the Cauchy problem (1.1) and (1.2) constructed by the Glimm scheme is bounded by a constant times the L_1 norm of the initial data.*

2. Riemann Problem

The solution to the Riemann problem

$$u(x,0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases} \tag{2.1}$$

for the general system (1.1) was solved in [16, 19]. We enclose the following lemmas on the properties on wave curves in [19] for the self-containedness of the paper.

The i-rarefaction wave curve from a state u_0, denoted by $R_i(u_0)$, is the integral curve of the right eigenvector r_i passing through u_0, $i = 1, 2, \ldots, n$. In general the Hugoniot curve $H_i(u_0)$ and the rarefaction wave curve $R_i(u_0)$ have second order contact at the initial state u_0, [14]. In general no higher-order contact is expected when the characteristic field is genuinely nonlinear. However, as we will see in the following lemmas, the situation is more interesting for non-genuinely nonlinear characteristic fields. The following lemmas are needed for the construction of the wave curve $W_i(u_0)$ through the state u_0. As mentioned before, the strength of the i-wave is measured by the difference of the parameter u^i between the right and left states.

Lemma 2.1. *For any $u \in H_i(u_0)$ in a small neighborhood of u_0, we have*
(i) $\lambda_i(u) > \sigma(u_0, u)$ (or $\lambda_i(u) < \sigma(u_0, u)$) if and only if

$$\frac{d}{du^i}\sigma(u_0, u) > 0, \qquad (or \quad \frac{d}{du^i}\sigma(u_0, u) < 0);$$

(ii) $H_i(u_0)$ is tangent to $R_i(u)$ at u on $H_i(u_0)$ if $\sigma(u_0, u) = \lambda_i(u)$.

Proof. Let

$$u - u_0 = \sum_{j=1}^{n} \alpha_j r_j(u),$$

$$\frac{du}{du^i} = \sum_{j=1}^{n} \beta_j r_j(u).$$

Then for weak waves the second order contact between $H_i(u_0)$ and $R_i(u_0)$ implies

$$\begin{aligned} &\alpha_i \beta_i > 0 \quad for\ u \neq u_0, \\ &\tfrac{|\alpha_j|}{|u-u_0|^2} \quad is\ bounded\ for \quad j \neq i. \end{aligned} \tag{2.2}$$

By differentiating

$$\sigma(u_0, u)(u_0 - u) = f(u_0) - f(u),$$

with respect to u^i, we have

$$\alpha_j \frac{d}{du^i}\sigma(u_0, u) = (\lambda_j(u) - \sigma(u_0, u))\beta_j, \qquad j = 1, 2, \dots, n. \tag{2.3}$$

Thus (i) follows from (2.2) and $(2.3)_i$. Since $\sigma(u_0, u)$ is close to λ_i, by strict hyperbolicity and $(2.3)_j$, we have (ii). $\square$

The following lemma gives an estimate on the interaction of two shock waves in the same direction and shows that the interaction of two admissible shock waves yields an admissible shock plus a cubic order error term.

Lemma 2.2. *Suppose that (u_0, u_1) and (u_1, u_2) with $u_2^i > u_1^i > u_0^i$ are two admissible i-shocks with strengths α_1 and α_2 and speeds σ_1 and σ_2 respectively, cf. Definition 1.2. Let $u_* \in H_i(u_0)$ be the state with $u_2^i = u_*^i$, then*
(i) (u_0, u_*) is admissible;
(ii) $|u_2 - u_*| = 0(1)\alpha_1\alpha_2(\sigma_1 - \sigma_2)$;
(iii) $\sigma\alpha = \sigma_1\alpha_1 + \sigma_2\alpha_2 + 0(1)\alpha_1\alpha_2(\sigma_1 - \sigma_2)$, where α and σ are the strength and speed of the admissible shock (u_0, u_*) respectively. The same estimate holds for the case when $u_0^i > u_1^i > u_2^i$.

Proof. Set

$$\tilde{\sigma}\alpha \equiv \sigma_1\alpha_1 + \sigma_2\alpha_2,$$

where $\alpha = \alpha_1 + \alpha_2$. Then by using the Hugoniot conditions for (u_0, u_1), (u_1, u_2) and (u_0, u_*), we have

$$\begin{aligned}
&\tilde{\sigma}(u_2^j - u_0^j) - [f^j(u_2) - f^j(u_0)] \\
&\quad = \tilde{\sigma}(u_2^j - u_0^j) - [\sigma_2(u_2^j - u_1^j) + \sigma_1(u_1^j - u_0^j)] \\
&\quad = (\tilde{\sigma} - \sigma_1)(u_2^j - u_1^j) + (\tilde{\sigma} - \sigma_2)(u_1^j - u_0^j).
\end{aligned} \tag{2.4}$$

Choose $\tilde{u}_0$ and $\tilde{u}_2$ on the straight line through u_1 with tangent $r_i(u_1)$ such that

$$\tilde{u}_0^i = u_0^i, \qquad \tilde{u}_2^i = u_2^i.$$

Then we have

$$\begin{aligned}
&|\tilde{u}_0 - u_0| = 0(1)\alpha_1^2, \qquad |\tilde{u}_2 - u_2| = 0(1)\alpha_2^2, \\
&(u_2^i - u_1^i)(u_1^j - \tilde{u}_0^j) = (\tilde{u}_2^j - u_1^j)(u_1^i - u_0^i).
\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5) yields

$$\begin{aligned}
&\tilde{\sigma}(u_2^j - u_0^j) - [f^j(u_2) - f^j(u_0)] \\
&\quad = (\tilde{\sigma} - \sigma_2)(\alpha)^{-1}[-(u_2^i - u_1^i)(u_1^j - u_0^j) + (u_2^j - u_1^j)(u_1^i - u_0^i)] \\
&\quad = 0(1)(\tilde{\sigma} - \sigma_2)\alpha_2(\alpha_1 + \alpha_2) \\
&\quad = 0(1)(\sigma_1 - \tilde{\sigma})\alpha_1(\alpha_1 + \alpha_2).
\end{aligned}$$

By (2.5) again, we have

$$\tilde{\sigma}(u_2 - u_0) - [f(u_2) - f(u_0)] = 0(1)\alpha_1\alpha_2(\sigma_1 - \sigma_2). \tag{2.6}$$

By comparing the jump condition of the i^{th} components for (u_0, u_2) and (u_0, u_*), we have

$$\sigma - \tilde{\sigma} = 0(1)\alpha_1\alpha_2(\alpha_1 + \alpha_2)^{-1}(\sigma_1 - \sigma_2),$$

and (iii) follows. From (2.4), (2.6) and the Hugoniot condition for (u_0, u_*), we have

$$\tilde{\sigma}(u_* - u_2) = f(u_*) - f(u_2) + 0(1)\alpha_1\alpha_2(\sigma_1 - \sigma_2). \tag{2.7}$$

Notice that $\tilde{\sigma}$ is close to λ_i. By considering $u_* - u_2$ in the r_j direction, $j \neq i$, strict hyperbolicity implies (ii).

Finally we prove that the discontinuity (u_0, u_*) is admissible. If $\sigma_1 = \sigma_2$, then clearly $u_* = u_2$ and (u_0, u_2) is admissible. If $\sigma_1 > \sigma_2$, then the admissibility is proved by contradiction as follows. Since

$$\sigma_2 \le \sigma \le \sigma_1,$$

we assume, without loss of generality, that $\sigma - \sigma_2 \ge \sigma_1 - \sigma$. Under the condition that (u_0, u_1) is admissible, and assuming that (u_0, u_*) is not admissible, then there exists a state $\tilde{u}$ with $u_1^i \le \tilde{u}^i < u_*^i = u_2^i$, such that $\sigma(u_0, \tilde{u}) = \sigma$. Thus $\tilde{u} \in H_i(u_*)$. By (ii), for the state $\bar{u} \in H_i(u_2)$ with $\tilde{u}^i = \bar{u}^i$, we have

$$\sigma(u_2, \bar{u}) - \sigma(u_*, \tilde{u}) = 0(1)\alpha_1\alpha_2(\sigma_1 - \sigma_2). \tag{2.8}$$

Since (u_1, u_2) is admissible, we have

$$\sigma_2 \ge \sigma(u_2, \bar{u}). \tag{2.9}$$

Combining (2.8) and (2.9) yields

$$\sigma_2 - \sigma \ge 0(1)\alpha_1\alpha_2(\sigma_1 - \sigma_2). \tag{2.10}$$

Since all the wave strengths are small, (2.10) contradicts the assumption that $\sigma - \sigma_2 \ge \frac{1}{2}(\sigma_1 - \sigma_2)$. Hence (u_0, u_*) is admissible and this completes the proof. □

For the interaction of a rarefaction wave and a shock of the same family, we have the following lemma. To obtain the precise interaction estimates, we introduce a new infinite step approach by replacing the rarefaction wave by small rarefaction shocks. By doing this, we can apply Lemma 2.2 and show that the limit exists. Without any ambiguity, for any discontinuity wave $\gamma = (u_-, u_+)$, we denote its speed by $\sigma(\gamma) = \sigma(u_-, u_+)$ from now on.

Lemma 2.3. *Suppose that (u_l, u_1) is an i-rarefaction wave, (u_1, u_r) is an admissible i-discontinuity, and $u_l^i < u_1^i < u_r^i$. Then there exists $u_* \in R_i(u_l)$ with $u_l^i \le u_*^i \le u_1^i$, and $\tilde{u}_* \in H_i(u_*)$ with $\tilde{u}_*^i = u_r^i$ such that*

(i) $|\tilde{u}_* - u_r| = 0(1)\alpha_2 \int_{u_*^i}^{u_1^i} (\lambda_i(u) - \sigma(u_1, u_r))du^i$, where the integral is along the $R_i(u_*)$ curve, and $\alpha_2 = u_r^i - u_1^i$.

(ii) $\sigma(u_*, \tilde{u}_*)\beta = \hat{\lambda}\alpha_1 + \sigma(\alpha_2)\alpha_2 + 0(1)\alpha_2 \int_{u_*^i}^{u_1^i} (\lambda_i(u) - \sigma(u_1, u_r))du^i$, where $\alpha_1 = u_1^i - u_*^i$, $\beta = \alpha_1 + \alpha_2$, and $\hat{\lambda}$ is the average speed of the centered rarefaction wave (u_*, u_1):

$$\hat{\lambda} \equiv \hat{\lambda}(u_*, u_1) \equiv \frac{1}{u_1^i - u_*^i} \int_{u_*^i}^{u_1^i} \lambda_i(u)du^i.$$

(iii) $(u_*, \tilde{u}_*)$ is admissible.

Proof. If $\sigma(u_1, u_r) = \lambda_i(u_1)$, then the lemma holds trivially because the linear superposition of the two Riemann solutions yields the solution to the Riemann data (u_l, u_r). When $\sigma(u_1, u_r) < \lambda_i(u_1)$, for any state $u \in R_i(u_l)$ between u_l and u_1, let $\tilde{u} \in H_i(u)$ with $\tilde{u}^i = u_r^i$. Set $\theta(u) \equiv \lambda_i(u) - \sigma(u, \tilde{u})$. Then we have $\theta(u_1) > 0$.

Suppose that $(u, \tilde{u})$ is admissible and $\theta(u) > 0$. We claim that for $w \in R_i(u_l)$ with $u^i - w^i$ positive and sufficiently small, then $\theta(w) < \theta(u)$ and $(w, \tilde{w})$ is also admissible. In fact, by Lemmas 2.1 and 2.2, we know that when $\epsilon = u^i - w^i$ is sufficiently small,

$$(\alpha + \epsilon)\sigma(w, \tilde{w}) = \alpha\sigma(u, \tilde{u}) + \epsilon\lambda_i(u) + 0(1)\epsilon\alpha\theta(u) + 0(1)\epsilon^2, \tag{2.11}$$

where $\alpha \equiv \tilde{u}^i - u^i > 0$. By using $\lambda_i(w) < \lambda_i(u)$ and the entropy condition $\lambda_i(u) > \sigma(u, \tilde{u})$, (2.11) implies $\theta(w) < \theta(u)$.

That $(w, \tilde{w})$ is admissible can be proved by contradiction. Suppose that $(w, \tilde{w})$ is not admissible, then there exists $\hat{w} \in H_i(\tilde{w})$ such that $\sigma(\hat{w}, \tilde{w}) = \sigma(w, \tilde{w})$. If $\hat{w}^i < u^i$, then we let $w \equiv \hat{w}$ and $\epsilon \equiv u^i - \hat{w}^i$. Otherwise, we choose $\hat{u} \in H_i(\tilde{u})$ with $\hat{w}^i = \hat{u}^i$. When ϵ is sufficiently small, we have

$$\sigma(\hat{u}, \tilde{u}) = \sigma(\hat{w}, \tilde{w}) + 0(1)|u^i - \tilde{u}^i|\theta(u)\epsilon. \tag{2.12}$$

Since $(u, \tilde{u})$ is admissible, the entropy condition yields

$$\sigma(\tilde{u}, \hat{u}) \le \sigma(u, \tilde{u}). \tag{2.13}$$

Combining (2.11), (2.12) and (2.13) yields

$$0(1)\theta(u)\epsilon\alpha \ge \epsilon[\theta(u) + \epsilon],$$

which is a contradiction to the assumption that $\theta(u) > 0$, given that ϵ and the wave strength are weak.

Now we are ready to prove (i) and (ii). We first divide the rarefaction wave α_1 into small rarefaction waves with each strength less than ϵ, ϵ a given small positive number. And then we replace each small rarefaction wave by a small rarefaction shock. Denote all these rarefaction shocks from left to right by $\alpha_{1,k} \equiv u_{1,k-1} - u_{1,k}$, $k = 1, 2, \ldots, m$, with speed

$$\sigma(\alpha_{1,k}) = \frac{1}{2}(\lambda_i(u_{1,k-1}) + \lambda_i(u_{1,k})), \qquad k = 1, 2, \ldots, m.$$

Obviously, $u_{1,k} \in R_i(u_1)$, $u_{1,0} = u_1$ and $u_{1,m} = u_*$. Now we consider the sequence of interactions between $\beta_k \equiv (u_{1,k-1}, \tilde{u}_{1,k-1})$ and $\alpha_{1,k}$. By using the fact that the Hugoniot curve and the rarefaction curve have second order contact, an application of Lemma 2.2 yields the following estimate for the interaction of β_k and $\alpha_{1,k}$:

$$\begin{aligned} \sigma(u_{1,k}, \tilde{u}_{1,k})\beta_{k+1} &= \sigma(u_{1,k-1}, \tilde{u}_{1,k-1})\beta_k + \alpha_{1,k}\sigma(\alpha_{1,k}) \\ &\quad + 0(1)\beta_k\alpha_{1,k}\theta(u_{1,k-1}) + 0(1)\epsilon^2, \\ |\tilde{u}_{1,k} - \tilde{u}_{1,k-1}| &= 0(1)\beta_k\alpha_{1,k}\theta(u_{1,k-1}) + 0(1)\epsilon^2. \end{aligned} \tag{2.14}$$

By summing up (2.14) with respect to k from $k = 1$ to m, we have

$$\sigma\beta = \sigma_2\alpha_2 + \sum_{k=1}^{m} \alpha_{1,k}\sigma(\alpha_{1,k}) + 0(1)\sum_{k=1}^{m} \beta_k\alpha_{1,k}\theta(u_{1,k-1}) + 0(1)\alpha_1\epsilon,$$

$$|\tilde{u}_* - \tilde{u}| = 0(1)\sum_{k=1}^{m} \beta_k \alpha_{1,k}\theta(u_{1,k-1}) + 0(1)\alpha_1\epsilon, \tag{2.15}$$

where $\beta = \beta_{k+1} = (u_*, \tilde{u}_*)$.

Now we estimate the term $\sum_{k=1}^{m} \beta_k \alpha_{1,k}\theta(u_{1,k-1})$.

We denote

$$\mathcal{E}_k^l = \beta_l \alpha_{1,k}\theta(u_{1,l-1}), \qquad 1 \le l \le k,$$
$$\mathcal{E}_k = \mathcal{E}_k^k = \beta_k \alpha_{1,k}\theta(u_{1,k-1}).$$

For $\mathcal{E}_k^l$, noticing that each $|\alpha_{1,k}| \le \epsilon$, we have the following estimate:

$$\begin{aligned}\mathcal{E}_k^l &= \mathcal{E}_k^{l-1} + \alpha_{1,k}[\beta_{l-1}(\theta(u_{1,l-1}) - \theta(u_{1,l-2})) + (\beta_l - \beta_{l-1})\theta(u_{1,l-1})] \\ &= \mathcal{E}_k^{l-1} + \alpha_{1,k}\beta_{l-1}(\lambda_i(u_{1,l-1}) - \lambda_i(u_{1,l-2})) \\ &\quad - \alpha_{1,k}[\beta_{l-1}(\sigma(\beta_l) - \sigma(\beta_{l-1})) + \alpha_{1,l-1}(\sigma(\beta_l) - \lambda_i(u_{1,l-1}))] \\ &\le \mathcal{E}_k^{l-1} + 0(1)\alpha_{1,k}\mathcal{E}_l + 0(1)\epsilon^3. \end{aligned} \tag{2.16}$$

Hence we have

$$\mathcal{E}_l \le \mathcal{E}_l^0 + 0(1)\alpha_{1,k}\sum_{i=1}^{l} \mathcal{E}_i + 0(1)\epsilon^2.$$

Using the fact that $\sum_{k=1}^{m} \alpha_{1,k} \le \alpha_1$ is small, we have

$$\sum_{l=1}^{m} \mathcal{E}_l \le 0(1)\sum_{l=1}^{m} \mathcal{E}_l^0 + 0(1)\epsilon.$$

Therefore, by letting ϵ tending to zero, (2.15) and (2.17) imply (i) and (ii). □

We next construct the i-wave curve from a state $u_l, i = 1, 2, \dots, n$, with the property that any state $u \in W_i(u_l)$ can be connected to u_l on the left by i-waves. That is, we will construct a curve $W_i(u_l)$ through u_l such that it passes through a single state u on each hyperplane with fixed u^i in a small neighborhood of u_l. For definiteness, we consider the case $u_l^i < u^i$. The case when $u_l^i > u^i$ can be discussed similarly. First we find a unique state u_1 with the following properties:

(i) $u_l^i \le u_1^i \le u^i$;
(ii) (u_l, u_1) is an admissible discontinuity such that $u_1^i - u_l^i$ is maximum.

If $u_1^i = u^i$, then we are done with $u = u_1$. If not, by Lemma 2.2, there is no admissible discontinuity with left state u_1 and the u^i component of the right state lies in $(u_1^i, u^i]$. Therefore, according to Lemma 2.1, we have $\nabla\lambda_i \cdot r_i(u_1) \ge 0$, and $\nabla\lambda_i \cdot r_i(u) > 0$ for states $u \in R_i(u_1)$ near u_1 with the i^{th} component larger than u_1^i. Thus, there exists a unique state $u_2 \in R_i(u_1)$ with the following properties:

(i) u_1 and u_2 are connected by i-rarefaction wave and $u_1^i < u_2^i \le u^i$.
(ii) u_2^i is the maximum in the sense that there is no state $u_* \in R_i(u_1)$ with the property that there exists admissible discontinuity (u_*, u_{**}) with $u_1^i < u_*^i < u_2^i$ and $u_*^i < u_{**}^i \le u^i$.

If $u_2^i = u^i$, then $u = u_2$ and we are done. If not, the above procedure can be continued until we finally reach the state u on the curve $W_i(u_l)$ with the given u^i. Thus (u_l, u)

forms an elementary i-wave described above when $u \in W_i(u_l)$. The wave curves are Lipschtz continuous, but have the following basic stability property:

Lemma 2.4. *Wave curves $W_i(\bar{u}_0)$ and $W_i(\tilde{u}_0)$ through different initial states have the following C^2-like property: Given a state $\bar{u}$ on $W_i(\bar{u}_0)$, there exists a state $\tilde{u}$ on $W_i(\tilde{u}_0)$ such that*

$$\bar{u} - \tilde{u} = \bar{u}_0 - \tilde{u}_0 + O(1)|\bar{u}_0 - \tilde{u}_0||\bar{u} - \bar{u}_0|.$$

Proof. We first remark that for a genuinely nonlinear field, the wave curve W_i consists of Hugoniot and rarefaction curves. For linearly degenerate field the wave curve is the rarefaction curve, which is the same as the Hugoniot curve, [12]. In either case, the dependence of a wave curve on its initial state is C^2, [12], and the lemma follows immediately by mean value theorem. However, this may not be the case when the i^{th} characterisitic field in not genuinely nonlinear or linearly degenerate, as in the case we are interested in. For the general case, an i-wave in the Riemann solution may contain both shock and rarefaction waves of the same i^{th} characteristic family, called a composite i-wave. From the above description, the i-wave curve consists of a finite number of Hugoniot, rarefaction, and a new type of "mixed" curves. A mixed curve $M_i(u_0)$ is a collection of states u_*, which is related to a fixed rarefaction curve $R_i(u_0)$ with the following properties: (i) $u_* \in H_i(u)$ for a state u on $R_i(u_0)$; (ii) $\sigma(u, u_*) = \lambda_i(u)$; (iii) (u, u_*) satisfies the entropy condition (E); (iv) at the initial state u_0 where the mixed curve and the rarefaction wave meet the characteristic is linearly degenerate $\nabla\lambda_i \cdot r_i(u_0) = 0$; and (v) the wave curve contains $R_i(u_0)$ and $M_i(u_0)$, which meet at u_0. These properties are used to construct a wave pattern which contains the rarefaction wave followed by a one-sided contact discontinuity (u, u_*). As the one-sided contact discontinuity (u, u_*) grows in strength, the rarefaction waves weaken as its end state u moves away from u_0 along $R_i(u_0)$.

We first show that the aforementioned two curves $M_i(u_0)$ and $R_i(u_0)$ are of second order tangency at u_0. Differentiate the jump condition $\sigma(u_* - u) = f(u_*) - f(u)$ with respect to the arc length of $R_i(u_0)$:

$$\sigma'(u_* - u) = (f'(u_*) - \sigma)u'_* - (f'(u) - \sigma)u'.$$

Note that $u' = r(u)$. Evaluating the above at $u = u_* = u_0$, using (iv) above, which implies that $\sigma' = 0$, and that $\sigma = \lambda_i$ there, we have

$$(f' - \lambda_i)u'_* = (f' - \lambda_i)r_i = 0.$$

Thus u'_* is parallel to $r_i(u)$ at $u_* = u = u_0$. Set $u_*' = cr_i(u)$. Next, we differentiate the jump condition twice to yield

$$\begin{aligned}\sigma''(u_* - u) + 2\sigma'(u'_* - u') = {} & f''(u_*)u_*'u_*' - f''(u)u'u' \\ & +(f'(u_*) - \sigma)u''_* - (f'(u) - \sigma)u''.\end{aligned}$$

And we evaluate this, again at $u = u_* = u_0$,

$$(c^2 - 1)f''r_ir_i + (f' - \lambda_i)u_*'' = (f' - \lambda_i)r_i'.$$

We now differentiate $f'r_i = \lambda_i r_i$ along $r_i(u)$ at $u = u_0$ and use (iv) above, $\lambda_i' = 0$,

$$f''r_ir_i = (\lambda_i - f')r_i'.$$

Since $r_i' = u''$, the last two identities yield

$$(f' - \lambda_i){u_*}'' = c^2(f' - \lambda_i)u''$$

at $u = u_* = u_0$. Recall that ${u_*}' = cr_i$ and $c \neq 1$ in general. Thus we need to renormalize the differentiation along $M_i(u_0)$ to be with respect to the arc length as follows:

$$\dot{u}_* \equiv \frac{1}{c}{u_*}'.$$

We have

$$\ddot{u}_* = c^2{u_*}'' + \frac{1}{c}\left(\frac{1}{c}\right)'{u_*}',$$

and so from the previous identity we have, at $u = u_* = u_0$,

$$(f' - \lambda_i)\ddot{u}_* = (f' - \lambda_i)u''.$$

Thus $\ddot{u}_* = u''$ except for a multiple of r_i. On the other hand, both $\dot{u}_*$ and u' are unit vectors and so $\ddot{u}_*$ and u'' are perpendicular to $\dot{u}_* = u' = r_i$. We therefore conclude that

$$\ddot{u}_* = u''$$

at $u = u_* = u_0$. Thus $M_i(u_0)$ and $R_i(u_0)$ are of second order contact at u_0.

On the other hand, a wave curve is in general only Lipschitz when two mixed curves meet. This corresponds to the vanishing of the rarefaction wave between two discontinuities to form a single discontinuity. We now concentrate on proving our lemma for this key case. Thus we assume that two wave curves are very close and one of them is a single Hugoniot curve $H_i(u_0)$ corresponding to an admissible shock $\beta = (u_0, u_1)$ with one-sided contact discontinuity $\sigma(\beta) = \lambda(u_1),\ \lambda \equiv \lambda_i$. The other nearby wave curve correspond to a shock $\alpha = (u_2, u_3)$ followed by a rarefaction wave $\delta = (u_3, u_4)$. Let u_5 and u_6 be states on $H_i(u_3)$ and $H_i(u_2)$, respectively, and denote by $\bar{\beta} = (u_2, u_6)$, see the picture below. We assume that the states $u_1,\ u_4,\ u_5$ and u_6 are on the same hyperplane transversal to the i^{th} curves, as do the states u_0 and u_2. Thus we have

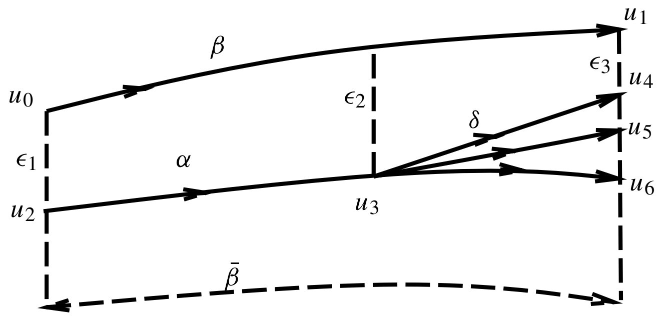
$$u_1 \in H(u_0); \quad u_3 \in H(u_2); \quad u_6 \in H(u_2);$$
$$u_5 \in H(u_3); \quad u_4 \in R(u_3),$$
$$\sigma(\beta) = \lambda(u_1), \quad \sigma(\alpha) = \lambda(u_3).$$

We want to show that

$$\epsilon_3 \leq \epsilon_1 + O(1)\epsilon_1(\alpha + \delta),$$

where $\epsilon_3 \equiv |u_1 - u_4|,\ \epsilon_1 \equiv |u_1 - u_2|$. We have

$$\epsilon_3 \leq |u_1 - u_6| + |u_6 - u_5| + |u_5 - u_4|.$$



For simplicity in notation we denote by $f''(u) \equiv \nabla\lambda \cdot r(u)$ the change of $\lambda \equiv \lambda_i$ along the characteristic direction $r \equiv r_i$. This measures the degree of genuine nonlinearity at the state u. The Hugoniot and rarefaction curves are close to each other if f'' is small:

$$|u_5 - u_4| = 0(1)|f''(u_3)|\delta^3.$$

From the second-order contact of Hugoniot and rarefaction curves and our analysis of Hugoniot curves before,

$$|u_1 - u_6| = \epsilon_1 + 0(1)\epsilon_1(\alpha + \delta),$$

$$|u_6 - u_5| = 0(1)\delta\alpha|\sigma(\alpha) - \sigma(\delta)| = 0(1)\delta\alpha|\lambda(u_3) - \sigma(\delta)| = 0(1)|f''(u_3)|\delta^2\alpha.$$

Thus we have

$$\epsilon_3 \leq \epsilon_1 + 0(1)\epsilon_1(\alpha + \delta) + 0(1)|f''(u_3)|\delta^2(\alpha + \delta). \tag{2.17}$$

With this clearly our estimate $\epsilon_3 \leq \epsilon_1 + 0(1)\epsilon_1(\alpha + \delta)$ follows if we can show that *Claim.*

$$|f''(u_3)|\delta = 0(1)(\epsilon_1 + \epsilon_3).$$

To prove the Claim we have from above that

$$\begin{aligned} |f''(u_3)|\delta &\equiv 0(1)|\lambda(u_4) - \lambda(u_3)| \\ &\leq 0(1)(|\lambda(u_4) - \lambda(u_1)| + |\lambda(u_1) - \lambda(u_3)|) \\ &= 0(1)\epsilon_3 + 0(1)|\sigma(\beta) - \sigma(\alpha)| \\ &\leq 0(1)(\epsilon_3 + |\sigma(\beta) - \sigma(\bar{\beta})| + |\sigma(\bar{\beta}) - \sigma(\alpha)|) \\ &= 0(1)[(\epsilon_3 + \epsilon_1) + |\sigma(\bar{\beta}) - \sigma(\alpha)|]. \end{aligned}$$

To finish the proof of the Claim we note from simple scalar consideration that

$$|\sigma(\bar{\beta}) - \sigma(\alpha)| = 0(1)\frac{\delta^2}{\alpha + \delta}|f''(u_3)| \ll |f''(u_3)|\delta.$$

This completes the proof of the lemma. □

Theorem 2.1 (Liu [16]). *Under the same hypotheses as in Theorem 1.1, the Riemann problem (1.1) and (2.1) has a unique solution in the class of elementary waves satisfying the entropy condition, cf. Definition 1.2, provided that the states are in a small neighborhood of a given state.*

Proof. The i-waves, $i = 1, 2, \ldots, n$, are the building blocks for the solution of the Riemann problem. The i-waves take values along the wave curves W_i. Since the wave curves W_i have tangent r_i at the initial state, it follows from the independency of the vectors r_i, $i = 1, 2, \ldots, n$, and the inverse function theorem that the Riemann problem can be solved uniquely in the class of elementary waves. □

3. Wave Interaction

In this section the relation of the waves before the interaction and the scattering data for the completed interaction is studied for the interaction of two sets of solutions of the Riemann problem.

For an i-wave α to the left of an i-wave β, we define $\Theta(\alpha, \beta)$ to represent the effective angle between them:

$$\Theta(\alpha, \beta) \equiv \theta_\alpha^+ + \theta_\beta^- + \sum \theta_\gamma. \tag{3.1}$$

Here θ_α^+ represents the value of λ_i at the right state of α minus its wave speed. It is negative if α is a shock and is set zero if it is a i-rarefaction wave. Similarly the term θ_β^- denotes the difference between the speed of β and the value of λ_i at its left end state. θ_γ is the value of λ_i at the right state of the wave γ minus that at the left state. It is positive if γ is a rarefaction wave and is negative if it is a shock. The sum $\sum \theta_\gamma$ is over the i-waves γ between α and β. Subject to wave interactions of distinct families, $-\Theta(\alpha, \beta)$ represents the angle between α and β when waves of other characteristic families between them propagate away. When $\Theta(\alpha, \beta)$ is positive, the two waves will not be likely to meet and should not be included in the potential wave interaction functional. When $\Theta(\alpha, \beta)$ is negative, the two waves may eventually meet and interact. In this case $|\alpha||\beta||\Theta(\alpha, \beta)|$ reflects accurately the potential interactions of waves of the same characteristic family.

To obtain the estimate for the interaction of two Riemann solutions, we need the following lemmas from [19]. Let (u_l, u_r) be an i-discontinuity, set

$$\begin{aligned} D_i(u_l, u_r) \equiv \{u : (u - u_l)\sigma(u_l, u_r) - (f(u) - f(u_l)) \\ = c(u)r_i(u) \quad for\ some\ scalar\ c(u)\}. \end{aligned}$$

The following lemma is similar to Lemma 2.2, its proof is therefore omitted.

Lemma 3.1. *For u_l and u_r close, $D_i(u_l, u_r)$ is a smooth curve through u_l and u_r in a small neighborhood of u_r. Moreover, if a state $\tilde{u}$ satisfies*

$$(\tilde{u} - u_l)\sigma - (f(\tilde{u}) - f(u_l)) = \tilde{c}r_i(\tilde{u}) + K,$$

for some scalar $\tilde{c}$ and some vector K, then there exists a vector u on $D_i(u_l, u_r)$ such that

$$|u - \tilde{u}| = 0(1)|K|.$$

To express the stability of a wave pattern in the next lemma, we need the following definition on partition of waves.

Definition 3.1. *Let $u_r \in W_i(u_l)$ so that u_l is related to u_r by i-discontinuities (u_{j-1}, u_j), and i-rarefaction waves (u_j, u_{j+1}), j odd, $1 \le j \le m-1$, $u_0 = u_l$ and $u_m = u_r$. A set of vectors $\{v_0, v_1, \dots, v_p\}$ is a partition of (u_l, u_r) if*

(i) $v_0 = u_l$, $v_p = u_r$, $v_{k-1}^i \le v_k^i$, $k = 1, 2, \dots, p$,
(ii) $\{u_0, u_1, \dots, u_m\} \subset \{v_0, v_1, \dots, v_p\}$,
(iii) $v_k \in R_i(u_j)$, j odd, if $u_j^i < v_k^i < u_{j+1}^i$,
(iv) $v_k \in D_i(u_{j-1}, u_j)$, j odd, if $u_{j-1}^i < v_k^i < u_j^i$.

We set

(v) $y_k \equiv v_k - v_{k-1}$,
(vi) $\lambda_{i,k} \equiv \lambda_i(v_{k-1})$ *and*

$$[\lambda_i]_k \equiv [\lambda_i](v_{k-1}, v_k) \equiv \lambda_i(v_k) - \lambda_i(v_{k-1}) > 0$$

if (iii) holds, and
(vii) $\lambda_{i,k} \equiv \sigma(u_{j-1}, u_j)$ *and* $[\lambda_i]_k \equiv [\lambda_i](v_{k-1}, v_k) \equiv 0$ *if (iv) holds.*

A partition $\{w_r\}$ is finer than another partition $\{v_k\}$ if $\{v_k\}$ is a subset of $\{w_k\}$. The important factor of interaction between waves of the same family is the angle between these waves, that is, the difference of the wave speeds. In the following discussion, we will partition rarefaction waves into small rarefaction shocks, hence both rarefaction wave and shock are treated similarly.

Lemma 3.2. *Suppose that* $u_r \in W_i(u_l)$, $\bar{u}_r \in W_i(\bar{u}_l)$, *with* $u_r^i - u_l^i = \bar{u}_r^i - \bar{u}_l^i \equiv \alpha > 0$, *and* $|u_l - \bar{u}_l| \equiv \beta$. *Then there exist partitions* $\{v_0, v_1, \dots, v_p\}$ *and* $\{\bar{v}_0, \bar{v}_1, \dots, \bar{v}_p\}$ *for the i-waves* (u_l, u_r) *and* $(\bar{u}_l, \bar{u}_r)$ *respectively such that* $\bar{v}_k^i - \bar{v}_0^i = v_k^i - v_0^i$, $k = 1, 2, \dots, p$, *and the following holds:*
(i) $\sum_{k=1}^{p} |y_k - \bar{y}_k| = 0(1)\alpha\beta$,
(ii) $|\lambda_{i,k} - \bar{\lambda}_{i,k}| = 0(1)\beta$, $k = 1, 2, \dots, p$,
(iii) Let $\Theta^+(u_l, u_r)$ *represent the value of* λ_i *at the right state* u_r *minus the wave speed of the right-most i-wave in* (u_l, u_r). *A similar definition holds for* $\Theta^-(u_l, u_r)$. $|\Theta^-(u_l, u_r) - \Theta^-(\bar{u}_l, \bar{u}_r)| + |\Theta^+(u_l, u_r) - \Theta^+(\bar{u}_l, \bar{u}_r)| = 0(1)\alpha\beta$.

Moreover, $\{1, 2, \dots, p\}$ *can be written as a disjoint union of subsets I, II and III such that*
(iv) for $k \in I$ *corresponding to rarefaction waves, both* v_k *and* $\bar{v}_k$ *are of type (iii) of Definition 3.1 and*

$$\sum_{k \in I} |[\lambda_i]_k - [\bar{\lambda}_i]_k| = 0(1)\alpha\beta,$$

(v) for $k \in II$ *corresponding to discontinuities, both* v_k *and* $\bar{v}_k$ *are of the type (iv) of Definition 3.1,*
(vi) for $k \in III$ *corresponding to mixed types,* v_k *and* $\bar{v}_k$ *are of different types and*

$$\sum_{k \in III} |[\lambda_i]_k + [\bar{\lambda}_i]_k| = 0(1)\alpha\beta.$$

Here $\Theta^+(u_l, u_r)$ *represents the value of* λ_i *at the right state* u_r *minus the wave speed of the righmost i-wave in* (u_l, u_r). *Similar definition holds for* $\Theta^-(u_l, u_r)$.

Proof. The proof is based on a continuity argument and the partition of the waves (u_l, u_r) and $(\bar{u}_l, \bar{u}_r)$. For definiteness, we consider the partitions of an i^{th} discontinuity (u_{j-1}, u_j) in (u_l, u_r) and its corresponding discontinuity $(\bar{u}_{j-1}, \bar{u}_j)$ in $(\bar{u}_l, \bar{u}_r)$. For β sufficiently small, (u_{j-1}, u_j) and $(\bar{u}_{j-1}, \bar{u}_j)$ are isolated in the sense that there exist w_k and $\bar{w}_k$, $k = 1, 2, 3, 4$, which form part of the partitions for (u_l, u_r) and $(\bar{u}_l, \bar{u}_r)$ respectively, $\{w_k^i - w_l^i, k = 1, 2, 3, 4\} = \{\bar{w}_k^i - \bar{w}_l^i, k = 1, 2, 3, 4\} = \{\bar{u}_{j-1}^i - u_l^i, u_{j-1}^i - u_l^i, \bar{u}_j^i - u_l^i, u_j^i - u_l^i\}$, and (u_{j-1}, u_j) and $(\bar{u}_{j-1}, \bar{u}_j)$ are the only discontinuities in (w_1, w_4) and $(\bar{w}_1, \bar{w}_4)$ respectively. We will prove the lemma by induction. We first assume that it holds for the partitions for (u_l, w_1) and $(\bar{u}_l, \bar{w}_1)$. Now we show it also holds for the partitions for (u_l, w_4) and $(\bar{u}_l, \bar{w}_4)$. Thus by the induction hypothesis, we have

$$|w_1 - \bar{w}_1| = 0(1)\beta. \tag{3.2}$$

For definiteness, we consider the case when $\sigma(w_2, w_4) = \lambda_i(w_2)$ and $\sigma(\bar{w}_1, \bar{w}_3) = \lambda_i(\bar{w}_1) = \lambda_i(\bar{w}_3)$. Choose two states $w_* \in H_i(w_1)$ and $\bar{w}_* \in H_i(\bar{w}_1)$ with $w_*^i = \bar{w}_*^i = w_4^i$. Using the argument in the proof of Lemma 2.3, we have

$$\begin{aligned}(w_4^i - w_1^i)\sigma(w_1, w_*) &= (w_4^i - w_2^i)\sigma(w_2, w_4) + (w_2^i - w_1^i)\hat{\lambda}_i(w_1, w_2)\\ &\quad + 0(1)|w_4^i - w_1^i||w_2^i - w_1^i|(\lambda_i(w_2) - \lambda_i(w_1)), \quad (3.3)\\ (\bar{w}_4^i - \bar{w}_1^i)\sigma(\bar{w}_1, \bar{w}_*) &= (\bar{w}_3^i - \bar{w}_1^i)\sigma(\bar{w}_3, \bar{w}_1) + (\bar{w}_4^i - \bar{w}_3^i)\hat{\lambda}_i(\bar{w}_3, \bar{w}_4)\\ &\quad + 0(1)|w_4^i - w_1^i||w_4^i - w_3^i|(\lambda_i(\bar{w}_4) - \lambda_i(\bar{w}_3)). \quad (3.4)\end{aligned}$$

By continuity, we have

$$\begin{aligned}\sigma(w_1, w_*) - \lambda_i(w_1) &= \sigma(\bar{w}_1, \bar{w}_*) - \lambda_i(\bar{w}_1) + 0(1)\beta|w_4^i - w_1^i|,\\ \sigma(w_1, w_*) - \lambda_i(w_4) &= \sigma(\bar{w}_1, \bar{w}_*) - \lambda_i(\bar{w}_4) + 0(1)\beta|w_4^i - w_1^i|.\end{aligned} \quad (3.5)$$

Summing up (3.3) and (3.4) and using (3.5), we have from direct calculations,

$$(w_4^i - w_2^i)(\lambda_i(w_2) - \lambda_i(w_1)) + (\bar{w}_3^i - \bar{w}_1^i)(\lambda_i(\bar{w}_4) - \lambda_i(\bar{w}_3)) = 0(1)|w_4^i - w_1^i|^2\beta.$$

For small β, we may assume that $|w_4^i - w_1^i| \leq 0(1)|w_3^i - w_2^i|$. Thus

$$\lambda_i(w_2) - \lambda_i(w_1) + \lambda_i(\bar{w}_4) - \lambda_i(\bar{w}_3) = 0(1)|w_4^i - w_1^i|\beta, \quad (3.6)$$

which implies (iv). As a consequence of (3.3), (3.4) and (3.6), we have

$$\begin{aligned}\sigma(w_1, w_*) &= \sigma(w_2, w_4) + 0(1)|w_4^i - w_1^i|\beta,\\ \sigma(\bar{w}_1, \bar{w}_*) &= \sigma(\bar{w}_1, \bar{w}_3) + 0(1)|w_4^i - w_1^i|\beta,\\ |w_* - w_4| + |\bar{w}_* - \bar{w}_4| &= 0(1)|w_4^i - w_1^i|\beta.\end{aligned} \quad (3.7)$$

(iii) follows from (3.6) and (3.7) by the continuity argument:

$$\sigma(u_{j-1}, u_j) - \lambda_i(u_j) = \sigma(\bar{u}_{j-1}, \bar{u}_j) - \lambda_i(\bar{u}_j) + 0(1)|w_4^i - w_1^i|\beta. \quad (3.8)$$

For (i) and (ii), we only need to show that for w on $D_i(w_2, w_4)$ and $\bar{w}$ on $D_i(\bar{w}_1, \bar{w}_3)$, with $w^i - w_l^i = \bar{w}^i - \bar{w}_l^i$ and w taking values between $w_2^i - w_l^i$ and $w_3^i - w_l^i$, then

$$w - w_1 = \bar{w} - \bar{w}_1 + 0(1)|w^i - w_1^i|\beta. \quad (3.9)$$

Other cases can be discussed similarly. To verify (3.9), we consider the state $\hat{w}$ on $D_i(w_1, \hat{w}_3)$ and $\bar{w}$ on $D_i(\bar{w}_1, \bar{w}_3)$, with $\hat{w}^i = \bar{w}^i$ and $\hat{w}_3 \in H_i(w_1)$. By continuity, we have

$$\hat{w} - w_1 = \bar{w} - \bar{w}_1 + 0(1)|\bar{w}^i - \bar{w}_1^i|\beta. \quad (3.10)$$

Thus it remains to estimate $|w - \hat{w}|$. By Lemma 3.1 this is can be done by estimating

$$(\hat{w} - w_2)\sigma(w_2, w_4) - (f(\hat{w}) - f(w_2)).$$

By (3.6), (3.7) and continuity, we have

$$\begin{aligned}&(\hat{w} - w_2)\sigma(w_2, w_4) - (f(\hat{w}) - f(w_2))\\ &\quad = cr_i(\hat{w}) + 0(1)|w^i - w_1^i||w_4^i - w_1^i|\beta,\end{aligned} \quad (3.11)$$

where c is a scalar. Hence (3.9) follows from (3.11) and Lemma 3.1.

A similar argument applies to other cases for sufficiently small β. If β is not small enough, we divide it into the sum of small increments and apply the above procedure repeatedly to each increment. This completes the proof of the lemma. □

For any two functions, u and v, we set

$$(u-v)_+ = \begin{cases} u-v, & u \geq v, \\ 0, & u < v, \end{cases} \qquad (u-v)_- = \begin{cases} 0, & u \geq v, \\ v-u, & u < v. \end{cases}$$

If α_i is a composite of i-subwaves $\alpha_{i,k} \equiv (u_{i,k-1}, u_{i,k})$, $k = 1, 2, \ldots, m$ from left to right, then we set

$$\eta(\alpha_i) = \sum_{k=1}^{m} \eta(\alpha_{i,k}),$$

where $\eta(\alpha_{i,k}) = \alpha_{i,k}\sigma(\alpha_{i,k})$ if $\alpha_{i,k}$ is a shock; and

$$\eta(\alpha_{i,k}) = \int_{u_{i,k-1}}^{u_{i,k}} \lambda_i(u)du^i,$$

if $\alpha_{i,k}$ is a rarefaction wave and $\lambda_i(u)$ takes value along $R_i(u_{i,k-1})$.

The Glimm scheme through the wave tracing method is based on the study of the wave interaction between two Riemann solutions. By using Lemmas 2.2, 2.3 and 3.2, the interaction estimate can be summarized as follows.

Theorem 3.1 (Liu [16]). *Let u_l, u_m, and u_r be three nearby states and the elementary i-waves in the solutions of the Riemann problems (u_l, u_m) and (u_m, u_r) be (u_{i-1}, u_i) and (v_{i-1}, v_i), $i = 1, 2, \ldots, n$, respectively. Then the solution (w_{i-1}, w_i), $i = 1, 2, \ldots, n$, of the Riemann problem (u_l, u_r) is the linear superposition of the above two solutions modulo the nonlinear effect of the order $Q(u_l, u_m, u_r) = Q(\mathbf{W})$, the degree of interaction for the wave pattern $\mathbf{W}$ consisting of the solutions of the Riemann problems (u_l, u_m) on the left and (u_m, u_r) on the right, and $\delta C(u_l, u_m, u_r) = \delta C(\mathbf{W})$, the product of the variation $\delta = |u_m - u_l| + |u_r - u_m|$ and the cancellation. In other words,*

$$\gamma_i = \alpha_i + \beta_i + O(1)(\delta C(u_l, u_m, u_r) + Q(u_l, u_m, u_r)), \tag{3.12}$$

$$\eta(\gamma_i) = \eta(\alpha_i) + \eta(\beta_i) + O(1)(\delta C(u_l, u_m, u_r) + Q(u_l, u_m, u_r)), \tag{3.13}$$

$$\alpha_i = \sum_{k=1}^{n_{\alpha_i}} \alpha_{i,k}, \quad \text{and} \quad \beta_i = \sum_{k=1}^{n_{\beta_i}} \beta_{i,k},$$

$$\alpha_i \equiv u_i - u_{i-1}, \quad \beta_i \equiv v_i - v_{i-1}, \quad \gamma_i \equiv w_i - w_{i-1},$$

$$C(u_l, u_m, u_r) \equiv \sum_{i=1}^{n} C^i(u_l, u_m, u_r) = \frac{1}{2}||\gamma_i| - |\alpha_i| - |\beta_i||,$$

for some constants n_{α_i} and n_{β_i}, $i = 1, 2, \ldots, n$. Each $\alpha_{i,k} = (u_{i,k-1}, u_{i,k})$ and $\beta_{i,k} = (v_{i,k-1}, v_{i,k})$ is either a shock or a rarefaction wave. $C(u_l, u_m, u_r)$ measures the amount of cancellation. The measure of wave interaction is

$$Q(u_l, u_m, u_r) = Q_s(u_l, u_m, u_r) + Q_h(u_l, u_m, u_r), \tag{3.14}$$

where Q_h measures the coupling of different characteristic families:

$$Q_h(u_l, u_m, u_r) \equiv \sum_{i>j} |\alpha_i||\beta_j|,$$

and

$$Q_s(u_l, u_m, u_r) \equiv \sum_{i=1}^{n} Q_s^i(u_l, u_m, u_r), \qquad Q_s^i(u_l, u_m, u_r) \equiv \sum_{k=1}^{n_{\alpha_i}} \sum_{l=1}^{n_{\beta_i}} Q_s^i(\alpha_{i,k}, \beta_{i,l}),$$

measures the interaction potential of the same characteristic family and is defined as follows:
(i) Both $\alpha_{i,k}$ and $\beta_{i,l}$ are shocks. Set

$$Q_s^i(\alpha_{i,k}, \beta_{i,l}) = |\alpha_{i,k}||\beta_{i,l}| \max\{0, -\Theta(\alpha_{i,k}, \beta_{i,l})\};$$

(ii) One of $\alpha_{i,k}$ and $\beta_{i,l}$ is a shock and the other is a rarefaction wave. For definiteness, we let $\alpha_{i,k}$ be a shock and $\beta_{i,l}$ be a rarefaction wave and set

$$Q_s^i(\alpha_{i,k}, \beta_{i,l}) = |\alpha_{i,k}| \int_{v_{i,l-1}^i}^{v_{i,l}^i} (\lambda_i(v) - \lambda_i(v_{i-1}) + \lambda_i(u_i) - \sigma(\alpha_{i,k}))_- dv^i,$$

where $\lambda_i(v)$ takes value along $R_i(v_{i,l-1})$.
(iii) Both $\alpha_{i,k}$ and $\beta_{i,l}$ are rarefaction waves. Set

$$Q_s^i(\alpha_{i,k}, \beta_{i,l}) = \int_{u_{i,k-1}}^{u_{i,k}} \int_{v_{i,l-1}}^{v_{i,l}} (\lambda_i(v) - \lambda_i(v_{i-1}) + \lambda_i(u_i) - \lambda_i(u))_- du^i dv^i,$$

where $\lambda_i(u)$ and $\lambda_i(v)$ take values along the curves $R_i(u_{i-1})$ and $R_i(v_{i-1})$ respectively. In (3.12) and (3.13) $O(1)$ is a bounded function which depends only on the flux $f(u)$.

Proof. The proof is done in steps. We first consider the data with $Q_s(u_l, u_m, u_r) = 0$. Thus we assume that there exists k, $1 \le k \le n$, such that there is no wave faster (or slower) than k-waves to the right (or left): $\alpha_1 = \cdots = \alpha_k = 0$, $\beta_{k+1} = \cdots = \beta_n = 0$. We have

$$Q_h(u_l, u_m, u_r) = |\alpha||\beta|, \quad |\alpha| \equiv \sum_i |\alpha_i|, \quad |\beta| \equiv \sum_i |\beta_i|.$$

If α (or β) is zero then $w_i = v_i$ (or $w_i = u_i$) and the solution of the Riemann problem (u_l, u_r) is the same as that of (u_m, u_r) (or (u_l, u_m)) and the theorem holds trivially with $Q(u_l, u_m, u_r) = 0$. We need to show that the deviation from the linear superposition is $O(1)\alpha\beta = O(1)Q_h(u_l, u_m, u_r)$. This follows from the C^2-like property of Lemma 2.4.

We next consider the interaction of waves of the same family. Let $|\alpha_j| = |\beta_j| = 0$ for all $j \neq i$ and both $|\alpha_i|$ and $|\beta_i|$ are not zero. Then we have the following cases.

Case 1. $(u_i^i - u_{i-1}^i)(v_i^i - v_{i-1}^i) > 0$. For definiteness, we assume $u_i^i - u_{i-1}^i > 0$, $v_i^i - v_{i-1}^i > 0$. The proof is based on a limiting process and Lemma 3.2 on the stability of the wave pattern. First we divide each rarefaction wave into small rarefaction shocks, each with strength less than ϵ, where ϵ is chosen to be a small positive number. We consider the interaction of α_i and β_i as follows: We consider the interaction of the first right wave of α_i and the first left wave of β_i, and it gives an i-wave and a cubic error term. By Lemma 3.2, we can shift the error term to the left of the wave patterns of the i-waves and consider the interaction of the produced shock wave and the nearest i-wave with the largest interaction angle. The total error thus caused is the sum of the cubic error in the interaction which is therefore summable.

The above procedure can be continued and we claim that the interaction potential Q_s of the wave patterns of the remaining i-waves will approach zero as the number of the interaction times j tends to infinity. Furthermore, the estimates (3.12) and (3.13) hold. Therefore the final result is the solution to the Riemann problem (u_l, u_r) with small perturbation Q_s. To prove this claim, we construct the following functionals:

$$\begin{aligned} F^j &\equiv \sum_{\gamma^j} |\gamma^j| + k_1 Q_s^{i,j}, \\ G^j &\equiv \sum_{\gamma^j} \eta(\gamma^j) + k_1 Q_s^{i,j}, \end{aligned} \tag{3.16}$$

where $\{\gamma^j\}$ is the family of i-waves after the j^{th} interaction. We will show that both F^j and G^j are decreasing functions of j when k_1 is appropriately chosen, and $\lim_{j\to\infty} Q_s^{i,j} = 0$.

Let ξ_l and ξ_r be the two waves interacting at the j^{th} step. And let ξ be any remaining wave in the i-wave pattern before interaction. By Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned} \bar{\xi} &= \xi_l + \xi_r + 0(1)\xi_l\xi_r(\sigma(\xi_l) - \sigma(\xi_r)), \\ \eta(\bar{\xi}) &= \eta(\xi_l) + \eta(\xi_r) + 0(1)\xi_l\xi_r(\sigma(\xi_l) - \sigma(\xi_r)), \end{aligned} \tag{3.17}$$

where $\bar{\xi}$ is the i-wave after interaction. Since $Q_s^{i,j}$ has a term $\xi_l\xi_r(\sigma(\xi_l)-\sigma(\xi_r))$, which is not in $Q_s^{i,j+1}$. By using Lemma 3.2, there exists a constant $0(1)$ K_1 such that $F^{j+1} < F^j$ and $G^{j+1} < G^j$.

It remains to show that $\lim_{j\to\infty} Q_s^{i,j} = 0$. According to our assumption and the construction of $\{\gamma^j\}$ in (3.16), we know that the strength of the shock wave increases after interaction at each step. Since in each step we choose the interaction of the two i-waves with largest interaction angle, under the assumption that each rarefaction shock is of order ϵ except those at the edge of the centered rarefaction waves, it takes at most two more steps so that the decrease of $Q_s^{i,j}$ is of the order of $\epsilon^2 Q_s^{i,j}$ after the j^{th} step, i.e.

$$Q_s^{i,j+2} - Q_s^{i,j} \leq -0(1)\epsilon^2 Q_s^{i,j},$$

for some positive constant $0(1)$. Furthermore, since $Q_s^{i,j}$ is decreasing in j, it will approach zero as j tends to infinity. By letting ϵ tend to zero, we have the estimate (3.12) and (3.13).

Case 2. $(u_i^i - u_{i-1}^i)(v_i^i - v_{i-1}^i) < 0$. For this case, we also divide the rarefaction wave into small rarefaction shocks with each strength less than ϵ.

The main difference between this case and Case 1 is that new i^{th} rarefaction waves may be created after wave interaction. From the construction of the wave curves W_i, W_i is tangent at each point to either an H_i or R_i curve. By a continuity argument, the strength of the new created i^{th} rarefaction waves can be controlled by the cancellation of the i-wave. The strength of the new created j-wave, $j \neq i$, and the change of the product of the wave strength and its speed can be controlled by the product of the cancellation of the i-wave and the total strength of the waves. Notice that the term $\delta C(u_l, u_m, u_r)$ appears on the right hand sides of (3.12) and (3.13). Thus by the continuity argument both (3.12) and (3.13) hold.

The general interaction is reduced to a series of interactions of the above two types plus interaction with waves of strength of the order $Q(u_l, u_m, u_r)$, cf. [12, 19, 23]. □

4. Glimm-Type Functional

Nonlinear interaction of weak waves can be controlled globally and solutions of general initial-value problems can be constructed using as building blocks the elementary waves studied in the last section. This has been done in the fundamental paper of Glimm, [12], for the system (1.1) under the assumption that each characteristic field is either genuinely nonlinear or linearly degenerate, and that the initial data have small total variation TV:

$$u(x, 0) = u_0(x), \ TV \equiv \text{variation}_{-\infty<x<\infty} u_0(x). \tag{4.1}$$

The Glimm scheme is a finite difference scheme involving a random sequence $a_i, \ i = 0, 1, \dots, \ 0 < a_i < 1$. Let $r = \Delta x, \ s = \Delta t$ be the mesh sizes satisfying the (C-F-L) condition

$$\frac{r}{s} > 2|\lambda_i(u)|, \quad 1 \le i \le n, \tag{4.2}$$

for all states u under consideration. The approximate solutions $u(x, t) = u_r(x, t)$ depend on the random sequence $\{a_k\}$ and is defined inductively in time as follows:

$$u(x, 0) = u_0((h + a_0)r), \quad hr < x < (h + 1)r, \tag{4.3}$$

$$\begin{gathered} u(x, ks) = u((h + a_i)r - 0, ks - 0), \ \ hr < x < (h + 1)r, \\ k = 0, \pm 1, \pm 2, \dots . \end{gathered} \tag{4.4}$$

Thus the approximate solution is a step function for each layer $t = ks, \ k = 1, 2, \dots$. Between the layers it consists of elementary waves by solving the Riemann problems at grid points $x = hr, \ h = 0, \pm 1, \dots$. Due to the C-F-L condition (4.2) these elementary waves do not interact within the layer. Thus the approximate solution is an exact solution except at the interfaces $t = ks, \ k = 1, 2, \dots$. The numerical error depends on the random sequence. In fact, as shown by Glimm [12] for the case when each characteristic field is either genuinely nonlinear or linearly degenerate, we have the following theorem.

Theorem 4.1. *Suppose that the initial data $u_0(x)$ is of small total variation TV. Then the approximate solutions $u(x, t)$ are of small total variation $O(1)TV$ in x for all time t. Moreover, for almost all random sequences $\{a_k\}_{k=1}^{\infty}$, the approximate solutions tend to an exact solution for a sequence of the mesh sizes r, s tending to zero with r/s fixed and r, s satisfying the C-F-L condition. The exact solution $u(x, t)$ is of bounded variation in x for any time $t \ge 0$:*

$$\text{variation}_{-\infty<x<\infty} u(x, t) = O(1)TV,$$

and is continuous in $L_1(x)$-norm:

$$\int_{-\infty}^{\infty} |u(x, t_1) - u(x, t_2)| dx = O(1)|t_1 - t_2|, \ t_1, t_2 \ge 0.$$

Proof. The proof is done in three steps:

The first step is to prove the uniform boundedness of the total variation of the approximate solutions. The main idea is to generalize the Glimm functional to approximate solutions for the general system. The functional $F(J)$ is defined on spacelike curves J. It consists of a linear part $L(J)$, measuring the total variation, a quadratic part $Q_h(J)$ and a cubic part $Q_s(J)$, measuring the potential wave interaction. The curve J incorporates the scheme and consists of line segments connecting points $((h \pm a_k)r, ks)$ and $(hr, (k \pm 1/2)s)$. The elementary waves issued from the grid points (hr, ks) will cross the line segments. These functionals are defined as follows:

$$\begin{aligned} L(J) &\equiv \sum\{|\alpha| : \ \alpha \text{ any wave crossing } J\}, \\ Q_h(J) &\equiv \sum\{|\alpha||\beta| : \ \alpha \text{ and } \beta \text{ interacting waves of distinct} \\ &\qquad\qquad \text{characteristic families crossing } J\}, \\ Q_s(J) &\equiv \sum_{i=1}^n Q_s^i, \\ Q_s^i &\equiv \sum\{|\alpha||\beta| \max\{-\Theta(\alpha,\beta), 0\} : \ \alpha \text{ and } \beta \ i-\text{waves crossing } J, \\ &\qquad \alpha \text{ to the left of } \beta\}, \\ Q(J) &\equiv Q_h(J) + Q_s(J), \\ F(J) &\equiv L(J) + MQ(J). \end{aligned} \tag{4.5}$$

Here M is a sufficiently large constant to be chosen later. In the above definition of $Q_h(J)$, an i-wave to the left of a j-wave is interacting if $i > j$.

The main estimate is that, for any curves J_1 and J_2, J_2 lies toward larger time than J_1,

$$F(J_2) \leq F(J_1), \tag{4.6}$$

provided that the total variation TV of the initial data is small and that M is chosen sufficiently large. It suffices to prove (4.6) when J_2 is an immediate successor of J_1, meaning that J_1 and J_2 differ only at one grid point, say J_1 goes through $(hr, (k-1/2)s)$; while J_2 goes through $(hr, (k+1/2)s)$ and they sandwich a diamond $\Delta = \Delta_{h,k}$ with vertices $((h-1+a_k)r, ks)$, $((h+a_k)r, ks)$, $(hr, (k-1/2)s)$ and $(hr, (k+1/2)s)$. The waves entering Δ are part of the solutions of the Riemann problems issued from $(hr, (k-1)s)$ and either from $((h-1)r, (k-1)s)$ or $(h+1)r, (k-1)s)$, depending on whether $a_{k-1} \leq 1/2$ or $a_{k-1} > 1/2$. The wave leaving Δ is the solution of the Riemann problem issued from (hr, ks). Thus the situation is the same as that dealt with in the last section. We denote by (u_l, u_m), (u_m, u_r) the Riemann problems corresponding to the waves entering Δ and (u_l, u_r) that leaving Δ. The amount of interaction within Δ is

$$Q(\Delta) \equiv Q_h(u_l, u_m, u_r) + Q_s(u_l, u_m, u_r), \tag{4.7}$$

and, for later use, the amount of interaction in a region Λ is denoted by

$$Q(\Lambda) \equiv \sum\{Q(\Delta_{i,j}) : \ (ir, js) \ \in \ \Lambda\}. \tag{4.8}$$

The same notations apply to

$$C(\Delta) \equiv C(u_l, u_m, u_r), \qquad D(\Delta) \equiv C(\Delta) + Q(\Delta).$$

For the first curve J_0 between $t = 0$ and $t = s$, the functional is dominated by the total variation TV of the initial data:

$$F(J_0) = O(1)TV,$$

which is assumed to be small. To prove (4.6) by induction we assume

$$F(J_1) \leq F(J_0) = O(1)TV.$$

The waves crossing J_1 and J_2 are the same outside Δ and, around Δ, waves crossing J_1 are the solution of the Riemann problems (u_l, u_m) and (u_m, u_r); while those crossing J_2 are the solution of the Riemann problem (u_l, u_r). These waves are related according to Theorem 3.1, whence we have

$$L(J_2) \leq L(J_1) - 2C(\Delta) + O(1)(Q(\Delta) + TVC(\Delta)).$$

There are two considerations for the difference of the wave interaction functionals $Q(J_1)$ and $Q(J_2)$: Due to the changes of wave strengths and speeds after interaction, there is a change in the nonlinear functional of the order $O(1)D(\Delta)$ times the total strength, which is $O(1)TV$, of waves crossing the common part of J_1 and J_2. On the other hand, and this is the key point, waves entering Δ are interacting with the measure of interaction $Q(\Delta)$; while those leaving Δ are the solution of a Riemann problem and are therefore non-interacting. For the quadratic wave interaction measure Q_h the above two considerations yield

$$Q_h(J_2) - Q_h(J_1) \leq O(1)TV \cdot D(\Delta) - Q_h(\Delta).$$

The cubic measure Q_s requires some computations, which differs in a basic way from the genuinely nonlinear case of Glimm [9], where the measure Q_s can be chosen to be quadratic. We consider Q_s^k when the two k-waves before interaction are shocks, denoted by α_k and β_k, with speeds σ_1 and σ_2, respectively. The k-wave after interaction is denoted γ_k with speed σ. Since the case of cancellation can be discussed easily, we assume that α_k and β_k are in the same direction. We now study the interaction potential of these waves with a k-wave δ_k, which is located to the right of the diamond Δ. Assume that these waves are interacting in the sense that the angle Θ between them is negative. The other case when some of them are interacting and the others are not can be discussed similarly. Then the potential interaction measure between δ_k and the k-waves entering Δ is

$$|\alpha_k||\beta_k|(\sigma_1 - \sigma_2) + (|\alpha_k|(\sigma_1 - \sigma_2) + (|\alpha_k| + |\beta_k|)|\Theta(\beta_k, \delta_k)|)|\delta_k|.$$

From Theorem 3.1, this equals

$$\begin{aligned} &|\alpha_k||\beta_k|(\sigma_1 - \sigma_2) + (|\gamma_k|\sigma - (|\alpha_k| + |\beta_k|)\sigma_2 \\ &\quad + (|\alpha_k| + \beta_k|)|\Theta(\beta_k, \delta_k)|)|\delta_k| + O(1)D(\Delta)|\delta_k|. \end{aligned}$$

The interaction measure between δ_k and the k-waves γ_k leaving Δ is

$$|\gamma_k||\Theta(\gamma_k, \delta_k)||\delta_k| = |\gamma_k|(\sigma - \sigma_2 + |\Theta(\beta_k, \delta_k)|)|\delta_k|.$$

Since $|\gamma_k| = |\alpha_k| + |\beta_k| + O(1)D(\Delta)$ the difference of interaction measures after and before the interaction is:

$$-|\alpha_k||\beta_k||\sigma_1 - \sigma_2| + O(1)D(\Delta)|\delta_k|.$$

With the above analysis, we have

$$Q(J_2) - Q(J_1) = -Q(\Delta) + O(1)TV \cdot D(\Delta). \tag{4.9}$$

We conclude from the above estimates that, for TV sufficiently small and M chosen suitably large,

$$F(J_2) - F(J_1) \leq \left(O(1) - \frac{M}{2} \right) Q(\Delta) + (0(1)MTV - 2)C(\Delta) \leq -D(\Delta), \tag{4.10}$$

whence we have (4.6). For later uses we have, by summing up these estimates over a region Λ bounded by two curves J_- and J_+,

$$D(\Lambda) \leq F(J_-) - F(J_+). \tag{4.11}$$

The second step, the convergence of the approximate solutions follows easily from the boundedness of the total variation of the approximate solutions already shown in the first step. In fact, it follows easily from Helly's theorem that there exists a sequence of mesh sizes tending to zero such that the approximate solutions tends to a limit function $u_*(x,t)$. This is done first for rational times and then we use the fact that the approximate solutions are continuous in t in the $L_1(x)$ topology:

$$\int_{-\infty}^{\infty} |u(x,t_2) - u(x,t_1)| dx = O(1)|t_2 - t_1|. \tag{4.12}$$

This is a consequence of the finite speed of propagation of the scheme and that the solution changes due to the wave interactions, which are bounded, (4.11). Thus

$$|u(x,t_2) - u(x,t_1)| = O(1) variation_y \{u(y,t_1) : \ |x-y| \leq L|t_2 - t_1|\}.$$

Equation (4.12) follows from integrating this in x and the change of the order of integrations.

The final step is to show that the limit function $u(x,t)$ is a weak solution of the initial value problem (1.1), (1.2). This can be done as in [12]. □

As in [17], we are going to make the Glimm scheme deterministic, i.e., to show that the scheme is consistent if and only if the random sequence is equidistributed.

To illustrate that the scheme is consistent only if the random sequence is equidistributed, we consider the example of the propagation of a single shock with positive speed σ:

$$u_0(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases}$$

The shock is located at $x = \sigma s$ at $t = s - 0$. According to (4.4), at $t = s$ it is located at

$$x = \begin{cases} 0, & \text{if } a_1 r > \sigma s, \\ r, & \text{if } a_1 r \leq \sigma s. \end{cases}$$

Given a fixed time $T = Ks$ the location of the shock in the approximate solution is

$$x = A(K,I)r, \qquad I \ \equiv \ (0, \sigma \frac{s}{r}).$$

Here, for a given subinterval I of $(0,1)$ and positive integer N, $A(N,I)$ denotes the number of k, $1 \leq k \leq N$, such that $a_k \in I$. When the meshes $r, s, \ r/s$ fixed, are refined we have $K \to \infty$ and the shock location becomes exact at $x = \sigma T$ if

$$A(K,I)r \ \to \sigma T; \quad \text{or,} \quad \frac{A(K,I)}{K|I|} \ \to \ 1, \qquad \text{as } K \ \to \ \infty.$$

Here $|I| = \sigma s/r$ is the length of the interval I. In other words, the shock location is exact in the limit if the random sequence is equidistributed:

Definition 4.1. *A sequence* $\{a_k\}_{k=1}^{\infty}$ *in* $(0, 1)$ *is equidistributed if*

$$B(N, I) \equiv \left|\frac{A(N,I)}{N} - |I|\right| \to 0, \quad \text{as } N \to \infty,$$

for any subinterval I *of* $(0, 1)$. *Here* $A(N, I)$ *denotes the number of* k, $1 \leq k \leq N$, *such that* $a_k \in I$, *and* $|I|$ *is the length of* I.

To show that equidistributedness is sufficient, we need the wave tracing method to be discussed in the next section. We will therefore put off the consistency analysis in deterministic version till then.

5. Wave Tracing Method

The local nonlinear superposition of waves has been expressed in Theorem 3.1. We now describe a bookkeeping scheme of subdividing the elementary waves in the approximate solution to obtain global nonlinear superposition. This is the idea of wave tracing, [17]. Here, however, we introduce a new analysis of consistency, in the L_1 topology, of the method. New analysis is needed because we have only the cubic measure of Q_s, rather than the quadratic measure, which exists for the genuinely nonlinear fields.

We illustrate the basic notion of the partitioning of waves by considering first the scalar equation. Take the example of two shocks (u_1, u_2), (u_2, u_3), $u_1 > u_2 > u_3$, of speed σ_1, σ_2, respectively, combining into a single shock (u_1, u_3) of speed σ_3. We divide (u_1, u_3) into the superposition of the original two shocks. The result of the interaction is then viewed as that both of the original shocks keep their identities but only with a change of their speed. This is compared with the linear superposition of two shocks with their original speeds kept. The time change of the error in $L_1(x)$ after the interaction is the product of the wave jump and the change in the wave speed:

$$\begin{aligned}&|\alpha_1||\sigma_1 - \sigma_3| + |\alpha_2||\sigma_2 - \sigma_3| = |\alpha_1|[\sigma(\alpha_1)] + |\alpha_2|[\sigma(\alpha_2)],\\ &\alpha_1 \equiv u_2 - u_1, \qquad \alpha_2 \equiv u_3 - u_2.\end{aligned}$$

Consequently the time change of the error is $\sum_\alpha |\alpha|[\sigma(\alpha)]$, where $[\sigma(\alpha)]$ denotes the variation of speed $\sigma(\alpha)$ at that time.

Consider next the cancellation of a wave (u_1, u_2) and another stronger wave (u_2, u_3), $u_2 > u_1 > u_3$. After the interaction, (u_1, u_2) is cancelled, so does a portion of (u_2, u_3). We divide the wave (u_2, u_3) into subwaves (u_2, u_1) and (u_1, u_3). The nonlinear interaction is then viewed as the wave (u_1, u_2) and the subwave (u_2, u_1) cancelling each other; while the subwave (u_1, u_3) surviving. Notice here that all the waves may be composite waves. Denote by $C \equiv |u_1 - u_2|$ the amount of wave cancellation. In terms of the time change of the $L_1(x)$ norm again, the error is bounded by the amount of wave cancellation $O(1)C = O(1)|u_1 - u_2|$.

We may perform this partition of waves in an approximate solution as follows: Fix a small time $t_1 = N\Delta t$ and consider the approximate solution $u(x, t) = u_r(x, t)$ in the time zone $0 < t < t_1$. Waves interact and cancel in the time zone in a way that is not easily foreseen because of the nonlinearity and the randomness of the scheme. The wave partition is a posteriori bookkeeping scheme. Given a shock at time $t = 0$ we partition it into subshocks sufficiently fine that each subshock is either cancelled completely or surviving as a shock or a rarefaction wave with strength unchanged in the zone. The situation is the same for a rarefaction wave: In addition to the cancellation, a rarefaction wave could become part of a shock wave, or it could be split when the random number a_k times $r = \Delta x$ equals Δt times one of the characteristic speeds of the rarefaction wave. Nevertheless we may keep refining a partition of a wave so that each subwave is either completely cancelled or propagating intact as a single wave, either shock or rarefaction wave, in the zone. Furthermore, since we divide each rarefaction wave into small rarefaction shocks with each strength less than ϵ, from now on we treat both rarefaction wave and shock in the same way.

Notice that how fine a given wave needs to be partitioned and which subwaves survive depends on the random sequence as well as the time zone. This is expected as the waves behave nonlinearly.

Next we turn to the system. In addition to wave combining and cancelling, wave interaction may alter the wave states and produce new waves. Thus we have three categories of waves, surviving ones, cancelled ones, and those produced by interactions. We have the following theorem on wave partition.

Theorem 5.1. *Let δ be a constant with $0 < \delta < 1$. The waves in an approximate solution in a given a time zone $\Lambda_l = \{(x,t) : -\infty < x < \infty, (l-1)Ns \leq t < lNs\}$, can be partitioned into subwaves of categories I, II or III with the following properties:*

(i) The subwaves in I are surviving. Given a subwave $\alpha(t)$, $(l-1)Ns \leq t < lNs$, in I, write $\alpha \equiv \alpha((l-1)Ns)$ and denote by $|\alpha(t)|$ its strength at time t, by $[\sigma(\alpha)]$ the variation of its speed and by $[\alpha]$ the variation of the jump of the states across it over the time interval $(l-1)Ns \leq t < lNs$. Then

$$\sum_{\alpha\in I}([\alpha] + |\alpha((l-1)Ns)|[\sigma(\alpha)]) = O(1)(D(\Lambda_l)(Ns)^{-\delta} + T.V.N^{1+\delta}s^{\delta} + \epsilon).$$

(ii) A subwave $\alpha(t)$ in II has positive initial strength $|\alpha((l-1)Ns)| > 0$, but is cancelled in the zone Λ_l, $|\alpha(lNs)| = 0$. Moreover, the total strength and variation of the wave speed satisfy

$$\sum_{\alpha\in II}([\alpha] + |\alpha((l-1)Ns)|[\sigma(\alpha)]) \leq 0(1)(D(\Lambda_l)(Ns)^{-\delta} + T.V.N^{1+\delta}s^{\delta} + \epsilon).$$

(iii) A subwave in III has zero initial strength $|\alpha((l-1)Ns)| = 0$, and is created in the zone Λ_l, $|\alpha(lNs)| > 0$. Moreover, the total variation satisfies

$$\sum_{\alpha\in III}([\alpha] + |\alpha(t)|) = O(1)(D(\Lambda) + \epsilon), \quad (l-1)Ns \leq t < lNs.$$

Remark 5.1. The theorem differs from the similar ones in previous works in that the error estimate is made for the interaction of relatively strong and weak waves separately. This accounts for the errors $O(1)(D(\Lambda_l)(Ns)^{-\delta}$ for strong waves and $T.V.N^{1+\delta}s^{\delta}$ for weak waves in (i) and (ii) above. This analytical refinement is necessary because of the third order estimate Q_s, and not quadratic estimate, that is available for general non-genuinely nonlinear systems.

Proof. For the case when each characteristic field is either genuinely nonlinear or linearly degenerate, the summation of the wave strength time the variation of its wave speed in each time zone Λ_l is of the order of $D(\Lambda_l)+\epsilon$, where Q_s is quadratic. This is no longer true for the general system because Q_s is cubic and waves may split due to cancellation. To overcome this difficulty we consider the wave interaction in more detail to use the cubic wave interaction potential to control the variation of the wave speed.

Now we choose a positive constant $\delta < 1$ and consider a wave $\alpha(t)$ in Λ_l. It is obvious that $|\alpha((l-1)Ns)|[\sigma(\alpha)]$ at time t can be controlled by $D(\Lambda_l)(t)$ if wave interaction is between waves of different families or cancellation. Thus we only need to consider the interaction of $\alpha(t)$ with waves of the same family and direction, denoted by α_i with speed σ_i. For simplicity of presentation, we can assume that all the waves are from a scalar conservation law and that $\alpha(t)$ is on the left of α_i for all i. If we denote the wave after interaction by $\bar{\alpha}$ with wave speed $\bar{\sigma}$, then we have

$$\alpha(t)(\sigma(\alpha(t))-\bar{\sigma})=\sum_i \alpha_i(\bar{\sigma}-\sigma_i)\le \sum_i \alpha_i(\sigma-\sigma_i),$$

where $\sigma(\alpha(t))>\bar{\sigma}$ and $\bar{\sigma}>\sigma_i$ for all i. Hence if $\sum_i|\alpha_i|>(Ns)^\delta$, then

$$\sum_i |\alpha(t)||\alpha_i|\Theta(\alpha(t),\alpha_i)\ge \sum_i |\alpha_i||\alpha(t)|[\sigma(\alpha)](t)\ge (Ns)^\delta|\alpha(t)|[\sigma(\alpha)](t),$$

where $[\sigma(\alpha)](t)$ represents the variation of the speed $\sigma(\alpha)$ at time t. Otherwise, we have

$$|\alpha(t)|[\sigma(\alpha)](t)\le 0(1)|\alpha(t)|(Ns)^\delta.$$

Since waves propagate at finite speeds, the number of times of wave interactions with $\alpha(t)$ in Λ_l is of order N. Therefore by summing all the above terms over $\alpha(t)$ and the time steps in Λ_p, we have

$$\sum_\alpha |\alpha((l-1)Ns)|[\sigma(\alpha)]\le 0(1)(Q_s(\Lambda_l)(Ns)^{-\delta}+T.V.N(Ns)^\delta).$$

Thus Theorem 5.1 is true for the waves of a scalar conservation law. As for the system in general, we just add to the above estimate the term Q_h for interaction between waves in different families, cancellation C and the error $0(1)\epsilon$ due to dividing rarefaction waves into rarefaction shocks. The rest of the proof is similar to the case when each characteristic field is either genuinely nonlinear or linearly degenerate, [23]. This completes the proof of the theorem. □

As an application of the theorem, we prove the "consistency" part of Theorem 4.1 as follows.

Proof. Theorem 4.1. Consistency. As we have seen for the propagation of a single shock that the limit function $u_*(x,t)$ can not be a weak solution of (1.1) and (1.2) for any choice of the random sequence, which is not equidistributed. The error is accumulated at $t=ks$, $k=0,1,\dots$:

$$\int_{-\infty}^{\infty}\int_0^{\infty}(u\phi_t+f(u)\phi_x)(x,t)dx\,dt+\int_{-\infty}^{\infty}(u\phi)(x,0)dx$$
$$=\sum_{k=0}^{MN}\int_{-\infty}^{\infty}(u(x,ks+0)-u(x,ks-0))\phi(x,ks)dx. \tag{5.1}$$

Here $\phi(x,t)$ is the test function with compact support, $\phi(x,t) = 0,\ t > T = MNs$. (The choice of the form MNs is for later convenience when we let $M, N \to \infty$ as $s \to 0$.) We will show that this error term will approach zero as the mesh sizes tend to zero due to the equidistributedness of the random sequence.

For illustration, we now calculate out the measure of consistency (5.1) for the simple example of one shock studied in the paragraph immediately before Definition 4.1. By our study of shock location then, we know that the limiting function in this case is a weak solution if the random sequence is equidistributed. Denote by $x = x(k)r$ the location of the shock at time $t = ks$. We have

$$\int_{-\infty}^{\infty} (u(x,ks+0) - u(x,ks-0))\phi(x,ks)dx$$
$$= \begin{cases} \int_{x(k)r}^{x(k)r+\sigma s} (u_+ - u_-)\phi(x,ks)dx, & \text{if } a_k r > \sigma s, \\ \int_{x(k)r+\sigma s}^{(x(k)+1)r} (u_- - u_+)\phi(x,ks)dx, & \text{if } a_k r < \sigma s. \end{cases}$$

If we simplify the situation by assuming that the test function is a constant ϕ_0 then the (5.1) becomes, for the interval $I = (0, \sigma s/r)$,

$$\begin{aligned} &\textstyle\sum_{k=0}^{MN} \int_{-\infty}^{\infty} (u(x,ks+0) - u(x,ks-0))\phi(x,ks)dx \\ &= \phi_0(u_+ - u_-)(A(MN,I)(r-\sigma s) - A(MN,I^c)\sigma s \\ &= \phi_0(u_+ - u_-)T(A(MN,I)(\tfrac{r}{s} - \sigma) - (MN - A(MN,I))\sigma)\tfrac{1}{MN} \\ &= \left(\tfrac{A(MN,I)}{MN} - \sigma\tfrac{s}{r}\right)\tfrac{r}{s}, \end{aligned}$$

which tends to zero as $MN \to \infty$ when the random sequence is equidistributed, Definition 4.1. To deal with the non-constancy of the test function $\phi(x,t)$, we divide the time zone $0 \le t < T = MNS$ into small time zones $N(l-1)s \le t < Nls,\ l = 1, 2, \ldots, M$. The test function is close to a constant in each time zone. The closeness is of the order $O(1)LNs = O(1)LT/M$, L the Lipschitz constant of $\phi(x,t)$, and tends to zero as $M \to \infty$. In each time zone the random sequence becomes increasingly equidistributed as N becomes large. Thus the above analysis applies and we have established the consistency for the propagation of a singe shock as $M, N \to \infty$.

For a general solution, we first partition each wave according to Theorem 5.1. Since a subwave α has varying speed, the analysis given above for a single shock does not apply directly. Nevertheless, the scheme (4.4) can determine the location of α up to the variation of its speed in the time zone. The variation of the speed has been discussed in Theorem 5.1, which shows that for a surviving wave α, its strength $|\alpha|$ times the variation of its speed in a time zone Λ_l is of the order of $D(\Lambda_l)(Ns)^{-\delta} + N^{1+\delta}s^\delta|\alpha|$ up to the error caused by dividing the rarefaction waves into rarefaction shocks. Thus the new error contributed by surviving subwaves to the measure of consistency in a given time zone Λ_l is $O(1)(Q(\Lambda_l)(Ns)^{-\delta} + T.V.N^{1+\delta}s^\delta + \epsilon)Ns$. The total new error of this kind over $0 \le t < T$ is then $E_1 = O(1)(Q(t \ge 0)(Ns)^{1-\delta} + T.V.N^{1+\delta}s^\delta T + \epsilon T)$.

Now if we choose $M = N^2$ and $\frac{1}{2} < \delta < 1$, then

$$E_1 = 0(1)\left(Q(t \ge 0)\left(\frac{T}{M}\right)^{1-\delta} + T.V.T^{\frac{4+\delta}{3}}s^{\frac{2\delta-1}{3}} + \epsilon T\right),$$

which tends to zero as $M \to \infty$ and $\epsilon \to 0$.

The error contributed by the cancelled subwaves in $0 \le t \le T$ is

$$E_2 = O(1)\left(C(t \ge 0)\frac{T}{M} + Q(t \ge 0)(Ns)^{1-\delta} + T.V.N^{1+\delta}s^{\delta}T + \epsilon T\right),$$

and is dealt with similarly. Thus the total error is of the form

$$\begin{aligned} E &= E_1 + E_2 \\ &= O(1)\Bigg[(A(N,I)/N - |I|)T + \epsilon T + C(t \ge 0)\frac{T}{M} \\ &\qquad + Q(t \ge 0)\left(\frac{T}{M}\right)^{1-\delta} + T.V.T^{\frac{4+\delta}{3}}s^{\frac{2\delta-1}{3}}\Bigg], \end{aligned}$$

which tends to zero as $M, N \to \infty$ and $\epsilon \to 0$.

Notice that in the above we have made use of the boundedness of the total cancellations and interactions in $\{(x,t) : x \in R, t \ge 0\}$, (4.11), and also that the wave partition is done independently for each time zone $\Lambda_l, l = 1, 2, \ldots, M$. An one-time partition for the entire region $\{(x,t) : x \in R, t \ge 0\}$ would be too crude and does not yield the vanishing factor T/M to a positive power in the above error estimate.

This completes the proof of the Theorem 4.1. □

Another application of the wave tracing is that it is useful for the study of the evolution of the $L_1(x)$-norm of a weak solution, Sect. 6. The approximate solutions, and thereby the exact solution, can be approximated locally in time with a wave pattern $\bar{u}(x,t)$ of linear superposition of nonlinear waves constructed as follows, [23]: First, since the solution is of bounded variation, within any degree of accuracy, e.g. $T.V.\epsilon$, in the L_1-norm and wave strength, we may ignore the waves near $x = \pm\infty$, say $|x| > E$, and consider only a finite number of subwaves in a given time zone $\Lambda_p \equiv \{(x,t) : -\infty < x < \infty,\ (p-1)Ns \le t < pNs\}$. We number the surviving i-waves by $\alpha^i_1, \alpha^i_2, \ldots, \alpha^i_K$. Associated with each i-wave α^i_k we assign an approximate i-wave $\bar{\alpha}^k_i$ with the same states as $\alpha^i_k((p-1)Ns)$ at time $t = (p-1)Ns$ and propagating along the straight line joining the end positions of the wave $(x_k(\alpha^i_k((p-1)Ns)), (p-1)Ns)$ and $(x_k(\alpha^i_k(pNs)), pNs)$ of α^i_k. The resulting speed is denoted by $\lambda^*(\bar{\alpha}^i_k)$. The non-surviving waves also propagate along lines with end states unchanged in the wave pattern $\bar{u}(x,t)$. Their speeds are defined arbitrarily so long as they are finite and no waves of the same family in $\bar{u}$ interact in the time zone Λ_p. Each surviving rarefaction wave is assumed to be partitioned into subwaves of strength less than ϵ, and viewed as a rarefaction shock. This introduces another error of the order ϵ. We can summarize this in the following theorem.

Theorem 5.2. *There exists a wave pattern $\bar{u}(x,t)$ consisting of linear superposition of a finite number of nonlinear waves $K = \{\bar{\alpha}\}$ and a large constant E such that:*

(i) There exists a one-to-one correspondence $\alpha \to \bar{\alpha}$ between the surviving waves I in $|x| < E$ of Theorem 5.1 and a subset L of K such that:

$$\sum_{\alpha} |\alpha - \bar{\alpha}| = O(1)D(\Lambda_p) + \epsilon,$$

$$\sum_{\alpha} |\alpha||\lambda(\alpha) - \lambda^*(\bar{\alpha})| = O(1)(D(\Lambda_p)(Ns)^{-\delta} + T.V.N^{1+\delta}s^{\delta} + e + \epsilon).$$

Here the function e measures the equidistributedness of the random sequence $\{a_i\}$*, where* $(p-1)N \le i < pN$ *for* $1 \le p \le M$*:*

$$e = \sup_{I,p} \left| \frac{A_p(N,I)}{N} - |I| \right|,$$

for any sub-interval I of $(0,1)$*. As in Definition 4.1,* $A_p(N,I)$ *denotes the number of k,* $(p-1)N \le k < pN$ *such that* $a_k \in I$*.*

(ii) $\sum_{\bar{\alpha}\in K-L} |\bar{\alpha}| = O(1)D(\Lambda_p)$.

Moreover,

(iii) $\bar{u}(x,(p-1)Ms) - u(x,(p-1)Ms) = 0$ *for* $|x| < E$.

(iv) $\int_{|x|>K} |\bar{u}(x,(p-1)Ms) - u(x,(p-1)Ms)|dx + \sum\{|\alpha| : \alpha \in u(x,(p-1)Ms),\ |x| > E\} < T.V.\epsilon$.

(v)

$$\int_{-\infty}^{\infty} |u_r(x,t)-\bar{u}(x,t)|dx = O(1)(D(\Lambda_p)(Ns)^{1-\delta}+T.V.N^{2+\delta}s^{1+\delta}+(e+\epsilon)Ns),$$
$$(p-1)Ns \le t < pNs, \quad p = 1,2,\ldots,M. \quad (5.2)$$

6. L_{bf1} Stability of Constant States

In this section, we are going to study the uniform boundedness of the L_1 norm for weak solutions to the general hyperbolic system (1.1), that is, the L_1 stability of constant state solutions. Without loss of generality, we take the constant state to be zero.

As for the system studied in [22, 23] in which each characteristic field is either genuinely nonlinear or linearly degenerate, there are two natural ways, i.e., by using rarefaction wave curves and Hugoniot curves, to measure the distance between two states in the phase plane. In the following, we are going to use rarefaction wave curves to measure the distance. The advantage of this measurement is that we only need to control the error caused by the bifurcation of the Hugoniot curve from the rarefaction wave curve besides the nonlinear coupling of waves in different families. But the disadvantage is that the relation between this kind of bifurcation needs to be considered in two weak solutions. When one solution is a constant state, this kind of error can be controlled by the time derivative of the integral of any convex entropy when the characteristic fields in the system are either genuinely nonlinear or linear degenerate, cf. [21]. In this section, we are going to show that this is also true in the general case.

Consider a general scalar conservation law

$$u_t + f(u)_x = 0. \quad (6.1)$$

By choosing the particular convex entropy $\eta(u) = \frac{u^2}{2}$ with entropy flux $q(u) = \int^u sf'(s)ds$, we have the following entropy estimate.

Lemma 6.1. *Let* $u(x,t)$ *be a weak solution to the scalar conservation law (6.1) consisting of countable many admissible shocks, denoted by* $\{\alpha_i\}$*. Then we have*

$$\frac{d}{dt}\int u^2(x,t)dx = -2\sum_{\alpha_i} A(\alpha_i),$$

where the integral is over $\mathbf{R}$. *Here, for any admissible shock* $\alpha = (u^-, u^+)$, $A(\alpha)$ *denotes the area bounded by the curve* $y = f(u)$ *and the straight line segment connecting the end points* $(u^-, f(u^-))$ *and* $(u^+, f(u^+))$ *in the* $u - y$ *plane.*

Proof. If the solution is smooth, then it is obvious that $\frac{d}{dt}\int u^2(x,t)dx = 0$. Without loss of generality, we consider the contribution of a single shock to this derivative. Let $\alpha_i = (u^-, u^+)$ be an admissible shock located at $x = x(t)$. We have

$$\frac{d}{dt}\int \frac{u^2}{2}(x,t)dx = \frac{1}{2}\dot{x}(t)(u^{-2} - u^{+2}) - q(u^-) + q(u^+)$$
$$+ \mathit{other\ terms\ not\ related\ to} \quad (u^-, u^+),$$

where $q' = uf'$ is the corresponding entropy flux. The term on the right hand side of the above equality can be calculated as follows:

$$\begin{aligned}
&\frac{1}{2}\dot{x}(t)(u^{-2} - u^{+2}) - q(u^-) + q(u^+)\\
&= \frac{1}{2}(f(u^-) - f(u^+))(u^- + u^+) - f(u^-)u^- + f(u^+)u^+ + \int_{u^+}^{u^-} f(t)dt\\
&= \frac{1}{2}(u^+ - u^-)(f(u^+) + f(u^-)) - \int_{u^-}^{u^+} f(t)dt\\
&= -A(\alpha_i).
\end{aligned}$$

Summing the terms for all shocks gives the proof of this lemma. □

Now we are going to show that this entropy estimate is closely related to the error caused by the bifurcation of the Hugoniot curve from the rarefaction wave curve in the general system.

Consider the general system of conservation laws (1.1). For illustration, we consider the wave of the first family and waves of the other families can be dealt with similarly. As before, we assume that u^1 is a non-singular parameter along the 1-wave curve. For simplicity, we choose the right eigenvector corresponding to λ_1 as $r_1(u) = (1, \xi_2, \xi_3, \ldots, \xi_n)$.

For any state $u = (u^1, u^2, \ldots, u^n)$ along the rarefaction wave curve $R_1(u^-)$ through u^-, we write

$$u = u^- + \int_{u^{-,1}}^{u^1} r_1(s)ds = \begin{pmatrix} u^1 \\ g(u^1) \end{pmatrix}, \tag{6.2}$$

for a smooth $(n-1)$-vector function $g(u^1)$. Similarly, for any state $u = (u^1, u^2, \ldots, u^n) \in H_1(u^-)$, we write $u = \binom{u^1}{h(u^1)}$, for a smooth function $h(u^1)$.

Let $\alpha = (u^-, u^+)$ be an admissible 1-shock to the system, and, without loss of generality, we assume that $u^{-,1} < u^{+,1}$. Then we have $s(u^+ - u^-) = f(u^+) - f(u^-)$ for some scalar $s = \sigma(u^-, u^+)$. For any $u = (u^1, u^2, \ldots, u^n) \in H_1(u^-)$, we denote $s^l(u^1) = \sigma(u^-, u)$. Then by the entropy condition for the system, we have

$$s^l(u^1) > s \quad \text{for} \quad u^{-,1} < u^1 < u^{+,1}. \tag{6.3}$$

If we consider the scalar conservation law

$$u_t^1 + f_x^1(u^1, h(u^1)) = 0, \tag{6.4}$$

then both the Rankine-Hugoniot condition and the Oleinik entropy condition [24] hold for the discontinuity $(u^{-,1}, u^{+,1})$. That is, $\alpha^1 = (u^{-,1}, u^{+,1})$ is an admissible shock of (6.4). In the following lemma, we will compare the values $\max_{u^{-,1}\le u^1\le u^{+,1}} |g(u^1) - h(u^1)|$ and $A(\alpha^1)$ defined for α^1 as an admissible shock to the scalar conservation law (6.4) and show that they are in fact of the same order.

Lemma 6.2. *Based on the above notations, we have*

$$\max_{u^{-,1}\le u^1\le u^{+,1}} |g(u^1) - h(u^1)| = 0(1)A(\alpha^1). \tag{6.5}$$

Proof. First we have the following expression for $A(\alpha^1)$:

$$\begin{aligned} A(\alpha^1) &= \int_{u^{-,1}}^{u^{+,1}} (f^1(t, h(t)) - f^1(u^{+,1}, h(u^{+,1})) - s(t - u^{+,1}))dt \\ &= \int_{u^{-,1}}^{u^{+,1}} (s - s^r(t))(u^{+,1} - t)dt, \end{aligned} \tag{6.6}$$

where $s^r(t)$ satisfies

$$s^r(t)(t - u_1^+) = f^1(t, h(t)) - f^1(u_1^+, h(u_1^+)).$$

By the entropy condition for the scalar conservation law (6.4), we have

$$s^r(t) < s \quad \text{for} \quad u_1^- < t < u_1^+.$$

Similarly, we have

$$A(\alpha^1) = \int_{u^{-,1}}^{u^{+,1}} (s^l(t) - s)(t - u^{-,1})dt. \tag{6.7}$$

For any state $\bar{u} = (u^1, u^2, \dots, u^n) \in R(u^-)$, we now denote the $(n-1)$-vector function along the Hugoniot curve through $\bar{u}$ by $h_{\bar{u}}(u^1)$. Then, we let

$$d = \max_{u^{-,1}\le \bar{u}^1\le u^{+,1}, \bar{u}^1\le u^1\le u^{+,1}} |h_{\bar{u}}(u^1) - g(u^1)|.$$

We are going to prove that $d = 0(1)A(\alpha^1)$, which immediately implies the lemma.

Let $\bar{u} \in R(u^-)$ with $u^{-,1} < \bar{u}^1 < u^{+,1}$ and du be a small variation along $R(u^-)$. We consider the Hugoniot curves through $\bar{u}$ and $\tilde{u} = \bar{u} + du$. By Lemma 2.2, we have

$$\begin{aligned} |h_{\bar{u}}(u^{+,1}) - h_{\tilde{u}}(u^{+,1})| &= c(u^{+,1} - u^1)(\lambda_1(\bar{u}) - \sigma_1)du^1 \\ &= c((u^{+,1} - u^1)(\lambda_1(\bar{u}) - s^r(u^1)) + d)du^1, \end{aligned}$$

where c is a $0(1)$ constant, $\sigma_1 = \sigma(\bar{u}, \bar{u}^+)$ with $\bar{u}^+ \in H_1(\bar{u})$ and $\bar{u}^{+,1} = u^{+,1}$. For simplicity, we denote $\lambda_1(u^1) = \lambda_1(\bar{u})$. Therefore, we have

$$\begin{aligned} d &= c\int_{u^{-,1}}^{u^{+,1}} (u^{+,1} - u^1)(\lambda_1(u^1) - s^r(u^1))du^1 + cd\int_{u^{-,1}}^{u^{+,1}} du^1 \\ &= c\int_{u^{-,1}}^{u^{+,1}} (u^{+,1} - t)(\lambda_1(t) - s^r(t))dt + c|u^{+,1} - u^{-,1}|d, \end{aligned} \tag{6.8}$$

where $\lambda_1(t)$ is taken value along $R_1(u^-)$.

Now we let $B(\alpha^1) = \int_{u^{-,1}}^{u^{+,1}} (u^{+,1} - t)(\lambda_1(t) - s^r(t))dt$, where $\lambda_1(t)$ is evaluated along $R_1(u^-)$. Then by (6.6) we have

$$A(\alpha^1) - B(\alpha^1) = \int_{u^{-,1}}^{u^{+,1}} (u^{+,1} - t)(s - \lambda_1(t))dt.$$

To study the right-hand side of the above equality, we introduce a notation, $D(u^1) = \int_{u^{-,1}}^{u^1} (s - \lambda_1(t))dt$, where $\lambda_1(t)$ again is evaluated along $R_1(u^-)$. From the choice of $r_1(u)$, we have

$$\begin{aligned} D(u^1) &= (s - s^l(u^1))(u^1 - u^{-,1}) + s^l(u^1)(u^1 - u^{-,1}) - \int_{u^{-,1}}^{u^1} \lambda_1(t)dt \\ &= (s - s^l(u^1))(u^1 - u^{-,1}) + f^1(u^1, h(u^1)) - f^1(u^{-,1}, h(u^{-,1})) \\ &\quad - \int_{u^{-,1}}^{u^{+,1}} \nabla f^1(t) \cdot r_1(t)dt \\ &= (s - s^l(u^1))(u^1 - u^{-,1}) + f^1(u^1, h(u^1)) - f^1(u^1, g(u^1)) \\ &= (s - s^l(u^1))(u^1 - u^{-,1}) + 0(1)|h(u^1) - g(u^1)| \\ &= (s - s^l(u^1))(u^1 - u^{-,1}) + 0(1)d. \end{aligned}$$

Now we can estimate $A(\alpha^1) - B(\alpha^1)$ as follows,

$$\begin{aligned} A(\alpha^1) - B(\alpha^1) &= \int_{u^{-,1}}^{u^{+,1}} (u^{+,1} - t)dD(t) \\ &= (u^{+,1} - t)D(t)\Big|_{u^{-,1}}^{u^{+,1}} + \int_{u^{-,1}}^{u^{+,1}} D(t)dt \\ &= \int_{u^{-,1}}^{u^{+,1}} (s - s^l(t))(t - u^{-,1})dt + 0(1)\int_{u^{-,1}}^{u^{+,1}} ddt \\ &= -A(\alpha^1) + 0(1)|u^{+,1} - u^{-,1}|d. \end{aligned}$$

Thus

$$A(\alpha^1) = \frac{B(\alpha^1)}{2} + 0(1)|u^{+,1} - u^{-,1}|d. \tag{6.9}$$

By combining (6.8) and (6.9), we get

$$d = cB(\alpha^1) + 0(1)|u^{+,1} - u^{-,1}|d = 2cA(\alpha^1) + 0(1)|u^{+,1} - u^{-,1}|d.$$

Therefore, when $|u^+ - u^-|$ is sufficiently small, we have $d = 0(1)A(\alpha^1)$ which completes the proof of the lemma. □

We are now ready to define the time-decreasing nonlinear functional $H[u(x,t)]$. Given a solution $u(x,t)$ of the system (1.1), we define the pointwise distance along the rarefaction wave curves: solve the Riemann problem $(u(x,t), 0)$ by waves:

$$u_0 = u(x,t), \qquad u_i \in R_i(u_{i-1}), \quad i = 0, 1, \ldots, n, \;\; u_n = 0.$$

We set

$$q_i(x,t) \equiv (u_i - u_{i-1})^i, \qquad \lambda_i(q_i(x,t)) \equiv \mu_i(u^i_{i-1}, u^i_i), \tag{6.10}$$

where $\mu(u^i_{i-1}, u^i_i)$ is the average speed of waves in the Riemann problem with states (u^i_{i-1}, u^i_i) to the scalar conservation law defined along $R_i(u_{i-1})$, i.e.,

$$u^i_t + f^i_x(u) = 0, \quad u \in R_i(u_{i-1}).$$

This way of assigning the distance is convenient in that u^i is a conservative quantity and so it satisfies simple wave interaction estimates. For an i-wave α^i in the solution $u(x,t)$, we denote by $x(\alpha^i) = x(\alpha^i(t))$ its location at time t, and $q^\pm_j(\alpha^i)$ for $q_j(x(\alpha^i)\pm, t)$, $1 \le j \le n$. For $j = i$ we also use the abbreviated notations $q^\pm(\alpha^i) = q^\pm_i(\alpha^i)$. The linear part $L[u]$ of the nonlinear functional $H[u]$ is equivalent to the $L_1(x)$-norm of the solution:

$$\begin{aligned} L[u(\cdot,t)] &\equiv \textstyle\sum_{i=1}^n L_i[u(\cdot,t)], \\ L_i[u(\cdot,t)] &\equiv \textstyle\int_{-\infty}^{\infty} |q_i(x,t)|dx. \end{aligned} \tag{6.11}$$

Without any ambiguity, we will use u to denote the approximate solutions in the Glimm scheme and also the corresponding weak solution when the mesh sizes tend to zero. We will use the notations J to denote the waves in the solution u at a given time. Moreover, α^i denotes an i-wave in J. The other two components of the nonlinear functional $H[u]$, the quadratic $Q_d(t)$ and the convex entropy $E(t)$, are defined as follows:

$$\begin{aligned} Q_d(t) &\equiv Q_d[u(\cdot,t)] = \textstyle\sum_{\alpha^i \in J} Q_d(\alpha^i), \\ Q_d(\alpha^i) &= |\alpha^i|\big(\textstyle\sum_{j>i} \int_{-\infty}^{x(\alpha^i)} |q_j(x,t)|dx + \sum_{j<i} \int_{x(\alpha^i)}^{\infty} |q_j(x,t)|dx\big), \end{aligned} \tag{6.12}$$

$$E(t) \equiv E[u(\cdot,t)] = \sum_{i=1}^n E_i(t) = \int_{-\infty}^{\infty} |q_i(x,t)|^2 dx. \tag{6.13}$$

For any given time $T = MNs$ in the Glimm scheme through the wave tracing method, we define the main nonlinear functional $H(t)$ as follows:

$$H(t) \equiv H[u(\cdot,t)] \equiv (1 + K_1 F(p-1)Ns))L(t) + K_2(Q_d(t) + E(t)),$$

for $t \in ((p-1)Ns, pNs)$, $p = 1, \ldots, M$. Notice here that the Glimm functional $F = F(u)$ is valued at the end time $t = (p-1)Ns$. The jump of the functionals $L(t)$, $Q_d(t)$ and $E(t)$ at each time step $t = pNs$, $p = 1, 2, \ldots, M$ due to wave interaction can be controlled by $0(1)[F(pNs) - F((p-1)Ns)]L(pNs)$, and the L_1 error due to the replacement of the simplified wave pattern approaches zero as shown in Theorem 5.2.

Now we are going to estimate $\frac{d}{dt}H(t)$ inside each region $(p-1)Ns < t < pNs$. By the property of the rarefaction wave curves, we immediately have the following lemma.

Lemma 6.3. *Let $\bar{u} \in \Omega$, $\xi,\ \xi' \in \mathbf{R}$, $\xi\xi' > 0$, $k \in \{1, \ldots, n\}$. Define the states and the wave speeds*

$$u = R_k(\xi)(\bar{u}), \quad u' = R_k(\xi')(u),$$
$$\mu = \mu_k(\bar{u}, u), \quad \mu' = \mu_k(u, u'), \quad \mu'' = \mu_k(\bar{u}, u'),$$

where $\mu(u_1, u_2)$ denotes the average speed of the waves in the Riemann problem with the states u_1 and u_2 to the scalar conservation law defined along the rarefaction wave curve $R_k(\bar{u})$, i.e. the scalar conservation law

$$u_t^k + f_x^k(u) = 0, \quad u \in R_k(\bar{u}).$$

Then we have

$$(\xi + \xi')(\lambda'' - \lambda') - \xi(\lambda - \lambda') = 0.$$

Lemma 6.4. *If the values $\xi,\ \xi_j,\ \xi'_j,\ j = 1, 2, \ldots, n$, satisfy*

$$R_n(\xi_n) \circ \cdots \circ R_1(\xi_1)(u) = R_n(\xi'_n) \circ \cdots \circ R_1(\xi'_1) \circ R_i(\xi)(u),$$

then

$$|\xi_i - \xi'_i - \xi| + \sum_{j \neq i} |\xi_j - \xi'_j| = O(1)|\xi| \sum_{j \neq i} |\xi'_j|.$$

And if the values $\xi,\ \xi_j,\ \xi'_j,\ j = 1, 2, \ldots, n$, satisfy

$$R_n(\xi_n) \circ \cdots \circ R_1(\xi_1)(u) = R_n(\xi'_n) \circ \cdots \circ R_1(\xi'_1) \circ H_i(\xi)(u),$$

where $H_i(\xi)(u)$ is an admissible i-shock, then

$$|\xi_i - \xi'_i - \xi| + \sum_{j \neq i} |\xi_j - \xi'_j| = O(1) \left(d + |\xi| \sum_{j \neq i} |\xi'_j| \right),$$

where $d = |H_i(\xi)(u) - R_i(\xi)(u)|$.

By Lemma 6.2, if we let $\alpha^i = (u^-, u^+)$ be an admissible i-shock in u with jump $[\alpha^i] \equiv (u^+ - u^-)^i$, then

$$\begin{aligned}
&\left|q^+(\alpha^i) - q^-(\alpha^i) - [\alpha^i]\right| + \sum_{j \neq i} |q_j^+(\alpha^i) - q_j^-(\alpha^i)| \\
&= O(1)(|\alpha^i| \sum_{j \neq i} \left|q_j^-(\alpha^i)\right| + A(\alpha^i)) \\
&= O(1)(|\alpha^i| \sum_{j \neq i} \left|q_j^+(\alpha^i)\right| + A(\alpha^i)),
\end{aligned} \tag{6.14}$$

where $A(\alpha^i)$ denotes the area corresponding to the shock wave $(u^{-,i}, u^{+,i})$, still denoted by α^i, to the scalar conservation law

$$u_t^i + f_x^i(u) = 0, \quad u \in H_i(u^-).$$

Notice here that even though the distance is measured along rarefaction curves, (6.10), and not along the Hugoniot curves, the above estimates still hold with $A(\alpha^i)$ the area defined by the shock, because the difference is of higher order. If $\alpha^i = (u^-, u^+)$ is an i-rarefaction wave in u with strength $[\alpha^i] \equiv (u^+ - u^-)^i$, then

$$\begin{aligned} \left|q^+(\alpha^i) - q^-(\alpha^i) - [\alpha^i]\right| + \sum_{j\neq i} |q_j^+(\alpha^i) - q_j^-(\alpha^i)| &= O(1)|\alpha^i| \sum_{j\neq i} \left|q_j^-(\alpha^i)\right| \\ &= O(1)|\alpha^i| \sum_{j\neq i} \left|q_j^+(\alpha^i)\right|. \quad (6.15) \end{aligned}$$

Now we are ready to estimate the derivative of the nonlinear functional $H(t)$ and prove the main theorem in this section. According to the construction of the simplified wave patterns of the approximate solutions, the open time interval $I_p \equiv ((p-1)Ns, pNs)$ is the union of two disjoint sets $I_p \equiv I_p^c \cup I_p^d$, where I_p^d are the countable interaction times. $H(t)$ is differentiable for $t \in I_p^c$; and is merely continuous for $t \in I_p^d$. For the change of $H(t)$ when $t \in I_p^c$, $p = 1, \dots, M$, we have the following lemma.

Lemma 6.5. *Suppose that the total variation $T.V.$ of the initial data of the solution is sufficiently small, and that $u_0(x) \in L_1(R)$. Then, for $t \in I_p^c$,*

$$\frac{d}{dt}H(t) \le C(Q(\Lambda_p) + C(\Lambda_p) + T.V.(e+\epsilon)) + \eta(t), \quad p = 1, 2, \dots, N, \quad (6.16)$$

for some choices of the constants K_1 and K_2. Here $\eta(t)$ represents that the error comes from the replacement of the simplified wave pattern, and satisfies

$$\int_0^T \eta(t)dt = 0(1)\left(eT + \epsilon T + C(t \ge 0)\frac{T}{M} + Q(t \ge 0)\left(\frac{T}{M}\right)^{1-\delta} + T.V.T^{\frac{4+\delta}{3}} s^{\frac{2\delta-1}{3}}\right) \to 0,$$

as $s \to 0$ as shown in Theorem 5.2. The function e measures the equidistributedness of the random sequence and ε is the strength of each approximate rarefaction shock, cf. Theorem 5.1.

Proof. Without any ambiguity, we can assume that all the waves propagate at the exact, rather than approximate, speeds in Λ_p. This is true up to the error of the order of

$$e(\Lambda_p) = (Q(\Lambda_p) + C(\Lambda_p) + T.V.(e+\epsilon))Ns + \int_{(p-1)Ns}^{pNs} \eta(t)dt,$$

cf. [23]. We first estimate $\frac{d}{dt}L(t)$. This will be used in the estimation of $Q_d(t)$.

A straightforward calculation gives

$$\begin{aligned} \frac{d}{dt}L(t) &= \sum_{j=1}^{n} \sum_{\alpha^i \in J} \dot{x}(\alpha^i)(|q_j^-(\alpha^i)| - |q_j^+(\alpha^i)|) \\ &= \sum_{j=1}^{n} \sum_{\alpha^i \in J} \left(|q_j^+(\alpha^i)|(\lambda_j^+(\alpha^i) - \dot{x}(\alpha^i)) - |q_j^-(\alpha^i)|(\lambda_j^-(\alpha^i) - \dot{x}(\alpha^i))\right), \quad (6.17) \end{aligned}$$

where we have used $\lambda_j^{\pm}(\alpha^i) = \lambda_j(q_j^{\pm}(\alpha^i))$, cf. (6.10), and the obvious identity

$$\sum_{\alpha^i \in J} |q_j^-(\alpha^i)|\lambda_j^-(\alpha^i) = \sum_{\alpha^i \in J} |q_j^+(\alpha^i)|\lambda_j^+(\alpha^i), \quad j = 1, 2, \ldots, n.$$

Based on Lemmas 6.3 and 6.4, each term in (6.17) can be estimated as in [23] for the cases $i = j$ and $i \neq j$ separately. For brevity, we omit the details. This gives

$$\begin{aligned}\frac{d}{dt}L(t) &\leq 0(1)\sum_{\alpha^i \in J}(A(\alpha^i) + |\alpha^i|\sum_{j\neq i}|q_j^+(\alpha^i)| + e(\Lambda_p))\\ &= 0(1)\sum_{\alpha^i \in J}(A(\alpha^i) + |\alpha^i|\sum_{j\neq i}|q_j^-(\alpha^i)| + e(\Lambda_p)),\end{aligned} \tag{6.18}$$

where $A(\alpha^i) \equiv 0$ if α^i is not an admissible shock.

Now we estimate $\frac{d}{dt}Q_d(\alpha^i)$ of (6.12) for each i-wave α^i,

$$\frac{d}{dt}Q_d(\alpha^i) = \sum_{j>i}|\alpha^i|\frac{d}{dt}\int_{-\infty}^{x(\alpha^i)}|q_j(x,t)|dx + \sum_{j<i}|\alpha^i|\frac{d}{dt}\int_{x(\alpha^i)}^{\infty}|q_j(x,t)|dx.$$

According to the discussion of $\frac{d}{dt}L(t)$, both $\int_{-\infty}^{x(\alpha^i)}|q_j(x,t)|dx$, $j > i$, and $\int_{x(\alpha^i)}^{\infty}|q_j(x,t)|dx$, $j < i$ have the same error terms as $\frac{d}{dt}L(t)$ plus an extra term coming from the difference between the wave speeds $\dot{x}(\alpha^i)$ and $\lambda_j^{\pm}(\alpha^i)$. By the strict hyperbolicity of the system (1.3), we know that there exists a positive constant C_2 such that

$$\lambda_j^-(\alpha^i) - \dot{x}(\alpha^i) > C_2 \quad \text{for } j > i; \qquad \lambda_j^+(\alpha^i) - \dot{x}(\alpha^i) < -C_2, \quad \text{for } j < i.$$

This and the assumption that the total variation of the solutions is sufficiently small yield

$$\begin{aligned}\frac{d}{dt}Q_d(t) &\leq -C_2\sum_{\alpha^i \in J}|\alpha^i|\sum_{j\neq i}|q_j^+(\alpha^i)| + O(1)T.V.\sum_{\alpha^i\in J}A(\alpha^i) + 0(1)e(\Lambda_p)\\ &= -C_2\sum_{\alpha^i \in J}|\alpha^i|\sum_{j\neq i}|q_j^-(\alpha^i)| + O(1)T.V.\sum_{\alpha^i\in J}A(\alpha^i) + 0(1)e(\Lambda_p).\end{aligned} \tag{6.19}$$

Finally we estimate $\frac{d}{dt}E(t)$ as follows:

$$\begin{aligned}\frac{d}{dt}E(t) &= \sum_{j=1}^{n}\sum_{\alpha^i\in J}\dot{x}(\alpha^i)(|q_j^-(\alpha^i)|^2 - |q_j^+(\alpha^i)|^2)\\ &= \sum_{j=1}^{n}\sum_{\alpha^i\in J}\dot{x}(\alpha^i)(|q_j^-(\alpha^i)|^2 - |q_j^+(\alpha^i)|^2 - \phi^i(u(x(\alpha^i)-,t))\\ &\quad + \phi^i(u(x(\alpha^i)+,t)),\end{aligned} \tag{6.20}$$

where $\phi^i(\tilde{u})$ denotes the entropy flux corresponding to the convex entropy $(u^i)^2$ of the scalar conservation law

$$u_t^i + \tilde{f}_x^i(u) = 0,$$

with $\tilde{f}^i(u) = f^i(u)$ and $u \in R_i(\tilde{u})$. The terms in (6.20) with $i \neq j$ can be discussed as in the case for $\frac{d}{dt}L(t)$ by using Lemmas 6.3 and 6.4. We only need to consider the case when $i = j$.

Let $\alpha^i = (u^-, u^+)$ be an admissible i-shock. We also use α^i to denote the admissible shock $(u^{-,i}, u^{+,i})$ to the scalar conservation law

$$u_t^i + \bar{f}_x^i(u) = 0,$$

where $\bar{f}^i(u) = f^i(u)$ and $u \in H_i(u^-)$. By using Lemma 6.3 and, according to the calculation in Lemma 6.1, we have

$$\begin{aligned}
&\dot{x}(\alpha^i)(|q_i^-(\alpha^i)|^2 - |q_i^+(\alpha^i)|^2) - \phi^i(u(x(\alpha^i)-,t)) + \phi^i(u(x(\alpha^i)+,t)) \\
&\quad = (u^{+,i} - u^{-,i})(\tilde{f}^i(u^+) - \tilde{f}^i(u^-)) - 2\int_{u^{-,1}}^{u^{+,1}} \tilde{f}(t)dt \\
&\qquad + 0(1)T.V.\left(|\alpha^i| \sum_{j\neq i} |q_j^{\pm}(\alpha^i)| + A(\alpha^i)\right) \\
&\quad = (u^{+,i} - u^{-,i})(\bar{f}^i(u^+) - \bar{f}^i(u^-)) - 2\int_{u^{-,1}}^{u^{+,1}} \bar{f}(t)dt \\
&\qquad + 0(1)T.V.\left(|\alpha^i| \sum_{j\neq i} |q_j^{\pm}(\alpha^i)| + A(\alpha^i)\right) \\
&\quad = (0(1)T.V. - 2)A(\alpha^i) + 0(1)T.V.|\alpha^i| \sum_{j\neq i} |q_j^{\pm}(\alpha^i)|. \qquad (6.21)
\end{aligned}$$

If α^i is a rarefaction shock with strength not greater than ϵ, then the above discussion yields

$$\begin{aligned}
&\dot{x}(\alpha^i)(|q_i^-(\alpha^i)|^2 - |q_i^+(\alpha^i)|^2) - \phi^i(u(x(\alpha^i)-,t)) + \phi^i(u(x(\alpha^i)+,t)) \\
&\quad = 0(1)T.V.|\alpha^i| \sum_{j\neq i} |q_j^{\pm}(\alpha^i)|. \qquad (6.22)
\end{aligned}$$

Under the assumption that the total variation $T.V.$ is sufficiently small, we have, by putting back the error $e(\Lambda_p)$,

$$\frac{d}{dt}E(t) \leq -C_3 \sum_{\alpha^i \in J} A(\alpha^i) + 0(1)T.V.\left(|\alpha^i| \sum_{j\neq i} |q_j^{\pm}(\alpha^i)| + e(\Lambda_p)\right), \qquad (6.23)$$

where C_3 is a positive constant.

The lemma follows by combining (6.18), (6.19) and (6.23), making use of the smallness of $T.V.$ again. □

For the jump of the functional $H(t)$ crossing the times $t = pNs$, $p = 1, \ldots, M$, we have the following lemma, showing that the difference vanishes as the mesh size s tends to zero.

Lemma 6.6. *Under the hypotheses of Lemma 6.5, we have, for each* $1 \le p \le M$,

$$H(pNs+) - H(pNs-) \le Ce(\Lambda_p). \tag{6.24}$$

Proof. According to the definition of the simplified wave pattern and Theorem 5.2, up to the error of order $T.V.\epsilon$, the difference between the wave patterns at time $t = pNs+$ and $pNs-$ is the appearance of the waves which are either cancelled or newly created in Λ_p. Since the wave propagates at finite speed, the error thus caused is of the order of $D(\Lambda_p)Ns$. This completes the proof of the lemma. □

Now we can state and prove the main theorem in this section.

Theorem 6.1. *Suppose that the total variation $T.V.$ of the initial data of the solution is sufficiently small, and that $u_0(x) \in L_1(R)$. Then, for the exact weak solution $u(x,t)$ of* (1.1) *constructed by Glimm's scheme, there exists a constant G independent of time such that*

$$\|u(x,t))\|_{L_1} \le G\|u(x,s))\|_{L_1},$$

for any s, t, $0 \le s \le t < \infty$.

Proof. Without loss of generality, we will show that $\|u(x,T)\|_{L_1} \le G\|u(x,0)\|_{L_1}$ for any time T. By integrating (6.16) in Lemma 6.5, we have

$$H(pMs-) - H((p-1)Ms+) \le Ce(\Lambda_p), \quad p = 1, 2, \dots, M. \tag{6.25}$$

We sum up (6.24) and (6.25) with respect to p from 1 to M to yield

$$H(T) \le H(0) + C\sum_{p=1}^{M}(Q(\Lambda_p) + C(\Lambda_p))Ns + CT.V.(e+\epsilon)T + C\int_0^T \eta(t)dt,$$

where $\int_0^T \eta(t)dt \to 0$ as $s \to 0$. For any fixed $T = MNs$ and $M = N^2$, we have M, $N \to \infty$ as the mesh size s tends to zero. By the definition of e and ϵ, we have $e \to 0$ and $\epsilon \to 0$. We know that

$$\sum_p (Q(\Lambda_p) + C(\Lambda_p)) \le (Q + C)\{0 \le t \le T\} \equiv A(T) < \infty. \tag{6.26}$$

Thus, for any fixed T,

$$\sum_{p=1}^{M}(Q(\Lambda_p) + C(\Lambda_p) + T.V.(e+\epsilon))Ns \le C\left(A(T)\frac{T}{M} + T.V.(e+\epsilon)T\right) \to 0$$

as $s \to 0$.

Notice that for any fixed M and N, the functional $H(t)$ is equivalent to the L_1-norm of the simplified wave pattern of $u(x,t)$. For the approximate solution $u_r(x,t)$ in the Glimm scheme, we can also define the corresponding functional $\bar{H}(t) = \bar{H}[u_r(\cdot,t)]$ which is equivalent to the $L_1(x)$-norm of $u_r(x,t)$. By Theorem 5.1, we also have the following estimate for $\bar{H}(t)$:

$$\bar{H}(T) \le \bar{H}(0) + C\left(A(T)\frac{T}{M} + T.V.(e+\epsilon)T + \int_0^T \eta(t)dt\right).$$

According to Theorem 4.1, there exist subsequences of the approximate solutions $\{u_r(x,t)\}$ which converge to the exact solution locally in the L_1-norm. Consequently, there exists a constant G independent of T and s such that

$$\|u(x,T)\|_{L_1} \le G\|u(x,0)\|_{L_1}.$$

This completes the proof of the theorem. □

Acknowledgement. The authors wish to thank Fabio Ancona for interesting discussions on the Glimm-type functional. After our paper was written, there is now a different approach to the theory for hyperbolic conservation laws through the zero dissipation limit by Bianchini and Bressan, [3]. While this new approach yields a definitive well-posedness theory, our approach is natural for other studies, such as the regularity and large-time behaviour of the solutions, cf. [19].

References

1. Ancona, F., Marson, A.: Well-posedness for general 2×2 systems of conservation laws. Preprint
2. Bianchini, S.: A note on Riemann problem. Preprint
3. Bianchini, S., Bressan, A.: Vanishing viscosity solutions of nonlinear hyperbolic systems. Preprint
4. Bressan, A.: A locally contractive metric for systems of conservation laws. Ann. Scuola Norm. Sup. Pisa **IV-22**, 109–135 (1995)
5. Bressan, A., Colombo, R.M.: The semigroup generated by 2×2 conservation laws. Arch. Rational Mech. Anal. **133**, 1–75 (1995)
6. Bressan, A., Crasta, G., Piccoli, B.: Well posedness of the Cauchy problem for $n \times n$ systems of conservation laws. Memoir Amer. Math. Soc. **694** (2000)
7. Bressan, A., Liu, T.-P., Yang, T.: L_1 stability estimates for $n \times n$ conservation laws. Arch. Ration. Mech. Anal. **149**(1), 1–22 (1999)
8. Chern, I.-L.: Local and global interaction for nongenuinely nonlinear hyperbolic conservation laws. Preprint
9. Courant, R., Friedrichs, K.O.: *Supersonic Flow and Shock Waves*. Berlin-Heidelberg-New York: Springer-Verlag, 1948
10. Dafermos, C.M.: Polygonal approximations of solutions of the initial value problem for a conservation law. J. Math. Anal. Appl. **38**, 33–41 (1972)
11. DiPerna, R.: Global existence of solutions to nonlinear hyperbolic systems of conservation laws. J. Diff. Eq. **20**, 187–212 (1976)
12. Glimm, J.: Solutions in the large for nonlinear hyperbolic systems of equations. Comm. Pure Appl. Math. **18**, 697–715 (1965)
13. Glimm, J., Lax, P.D.: Decay of solutions of systems of hyperbolic conservation laws. Memoirs Am. Math. Soc. **101**, (1970)
14. Lax, P.D.: Hyperbolic systems of conservation laws II. Comm. Pure Appl. Math. **10**, 537–566 (1957)
15. Lax, P.D.: Shock waves and entropy. In: *Contribution to Nonlinear Functional Analysis*. E. Zarantonello, ed., N.Y.: Academic Press, 1971, pp. 603–634
16. Liu, T.-P.: The Riemann problem for general system of conservation laws. J. Diff. Eq. **18**, 218–234 (1975)
17. Liu, T.-P.: The deterministic version of the Glimm scheme. Commun. Math. Phys. **57**, 135–148 (1977)
18. Liu, T.-P.: Decay to N-waves of solutions of general systems of nonlinear hyperbolic conservation laws. Comm. Pure Appl. Math. **30**, 585–610 (1977)
19. Liu, T.-P.: Admissible solutions of hyperbolic conservation laws. Memoirs Am. Math. Soc. **30**, 240 (1981)
20. Liu, T.-P.: The entropy condition and the admissibility of shocks. J. Math. Anal. Appl. **53**, 78–88 (1976)
21. Liu, T.-P., Yang, T.: Uniform L_1 boundedness of solutions of hyperbolic conservation laws. Meth. Appl. Anal. **4**, 339–355 (1997)
22. Liu, T.-P., Yang, T.: L_1 stability for 2×2 systems of hyperbolic conservation laws. J. Am. Math. Soc. **12**(3), 729–774 (1999)
23. Liu, T.-P., Yang, T.: Well-posedness theory for hyperbolic conservation laws. Comm. Pure Appl. Math. **52**(12), 1553–1586 (1999)

24. Oleinik, O.: Uniqueness and stability of the generalized solution of the Cauchy problem for a quasilinear equation. Usp. Mat. Nauk **14**, 165–170 (1959); Am. Math. Soc. Transl. Ser. 2, **33**, 285–290 (1964)
25. Smoller, J.: *Shock Waves and Reaction–diffusion Equations*. New York: Springer-Verlag, 1983
26. Zumbrun, K.: Decay rates for nonconvex systems of conservation laws. Comm. Pure Appl. Math. **46**(3), 353–386 (1993)

Communicated by P. Constantin