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Singular behavior of vacuum states for compressible fluids

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Dedicated to Roderick S.C. Wong on the occasion of his 60th birthday

Abstract

In this survey paper, we will present the recent work on the study of the compressible fluids with vacuum states by illustrating its interesting and singular behavior through some systems of fluid dynamics, that is, Euler equations, Euler–Poisson equations and Navier–Stokes equations. The main concern is the well-posedness of the problem when vacuum presents and the singular behavior of the solution near the interface separating the vacuum and the gas. Furthermore, the relation of the solutions for the gas dynamics with vacuum to those of the Boltzmann equation will also be discussed. In fact, the results obtained so far for vacuum states are far from being complete and satisfactory. Therefore, this paper can only be served as an introduction to this interesting field which has many open and challenging mathematical problems. Moreover, the problems considered here are limited to the author's interest and knowledge in this area.

Keywords: Vacuum singularity; Euler–Poisson equations; Euler equations; Navier–Stokes equations; Boltzmann equation

1. Introduction

The study on the singularity associated with vacuum state in fluid dynamics can be traced back at least to the collected work of von Neumann [66]. However, there is still no satisfactory answer so far to this important physical problem. One of the reasons could be the existing equations or systems provide either trivial solutions in this region or contain the singularity which prevents the use of present theory.

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In this paper, we will survey some recent work in this direction on some fluid dynamics systems and the Boltzmann equation. One can see that there are more open problems than those which have been solved. Therefore, we expect this paper can provide the readers some updated information on the study of the existence, stability, uniqueness and time evolution of the interface separating the gas and vacuum, called vacuum boundary, to the solutions with vacuum states. Since we are mainly interested in the singular behavior of the solutions with vacuum, the solutions need to have some regularity for this purpose. For this, those work on weak solutions obtained by compensated compactness will not be discussed here.

In the following sections, we will first discuss the behavior of the solutions with vacuum states to some fluid dynamics systems, which include the Euler equations, Euler–Poisson equations and Navier–Stokes equations. Finally, we will also present some special solutions to the Boltzmann equation related to the fluid dynamics in this aspect. To be more precise, we will give some brief discussions on each topic.

It is well known that the system of Euler equations becomes non-strictly hyperbolic when vacuum appears. Therefore, many existence theory for strictly hyperbolic system cannot be applied here and richer nonlinear phenomena are expected. In fact, whether the vacuum state appears in finite time if initial data contain no vacuum states is the main step to obtain the global existence of entropy solutions for large data. This is a long-standing open problem which we will not discuss in this paper. What we will show in Section 2 is that the solution near vacuum boundary has a canonical singular behavior, called the physical boundary condition. This singularity prevents the application of the existing theory on existence for hyperbolic systems even locally in time. The linearized version of this singularity is of the Keldysh type degenerate hyperbolic equation. This is different from the Tricomi-type singularity which is better understood. By looking at the system of Euler equations with the linear frictional damping, it is shown that there is some similarity on the behavior of the evolution of the vacuum boundary to the one for porous media equation. The latter has been well investigated. However, the former is almost open because it comes from a system instead of a single equation.

The system of Euler–Poisson equations consists of a hyperbolic system coupled with a Poisson equation. It is a model system for the time evolution of the self-gravitating gas coming from astrophysics. Starting from 1800s there have been many studies on this system when the initial data have compact support . Since this system can be viewed as a model for the time evolution of gaseous stars, the existence and stability of the stationary solutions are among the important research topics. In the linear theory, there are classical variational principles of Chandrasekhar and Eddington for the stability of the stationary solutions with spherical symmetry. By assuming the gas to be polytropic, according to these two principles, the linear stability depends on the adiabatic constant γ. When $\gamma < \frac{4}{3}$, the stationary solution is instable, while it is stable when $\gamma > \frac{4}{3}$ in three-dimensional space. However, how to justify this for the fully nonlinear system is not complete yet. What we can prove mathematically for the nonlinear system is merely on the stability part. The instability which is more important in physics and interesting in mathematics is now still based on the linearized model.

When the evolution of the gas or fluid becomes viscous and heat conductive, one of the typical systems is the system of Navier–Stokes equations. When the initial density is of compact support, many works have been done on the existence, stability and uniqueness of these solutions. The time evolution of the vacuum boundary is also investigated under some special conditions. Most of these works are in the case when both the viscosity coefficient and the heat conductivity are constants. However, by deriving the Navier–Stokes equations from the Boltzmann equation, the viscosity and heat conductivity are functions of temperature. Since there is no a priori estimate on the temperature for vacuum states, it is not clear how to pose the problem when this physical dependence on temperature is included. Hence, there is almost no result on regular solutions to the full Navier–Stokes equations when the viscosity and heat conductivity are functions of temperature and the density function is of compact support. However, if one considers only the isentropic gas, the dependence on temperature becomes the dependence on the density. In this direction, there are some work on the existence and uniqueness. But even in this case, there is no description of the large-time behavior of the solution because most of the estimates are time dependent coming from Gronwall inequality. To overcome this, new techniques in analysis are needed and new estimates are also required. We should mention that the large-time behavior of the solutions when the viscosity is constant can be given in detail including the expansion rate of the gas region. The non-appearance of vacuum states is proved if initial data contains no vacuum. However, this result cannot be directly generalized to the case when the viscosity depends on the density function.

Finally, the Boltzmann equation is a fundamental equation for the statistical time evolution of a large number of particles in rarefied gas interacting through various physical assumptions on collisions. When the Knudsen number κ (or the mean free path) tends to 0, one can formally obtain the systems of Euler equations and Navier–Stokes equations in different orders of approximations. These limit procedures have been justified in some mathematical settings. In fact, the study on the relations between these fluid dynamics systems and the Boltzmann equation has been one of the main topics in this field since Hilbert raised his sixth problem, Mathematical Treatment of the Axioms of Physics in his famous lecture Mathematical Problems at ICM in 1900.

To illustrate the relation between the Boltzmann equation and the systems for fluid dynamics with vacuum states, we will present some particular solutions to the Boltzmann equation when the initial data connect to vacuum at $|x| = \infty$ with force. We will focus on the hard sphere model of collision. The main objective is to study the solution behavior when κ tends to 0, which should be related to some known results on the solutions to the corresponding fluid dynamics systems, such as Euler equations and Navier–Stokes equations with external force. These problems have physical significance besides their mathematical interest. For example, when we consider the atmosphere around a star, such as our earth, in the outer space the density of the atmosphere is rarefied and the particle experiences the gravitational force from the earth. If we first ignore the radiation from the sun and from the reflection on the surface of the earth, then this situation can be modelled by the above equation. The detailed structure of the solution in this case will be helpful to understand atmosphere distribution in this simplified setting. Notice that the general existence, large-time behavior, stability and the asymptotic phenomena when $\kappa \to 0$ for the Boltzmann equation with a non-trivial time asymptotic nonlinear profile connecting to vacuum are still not known. When the solution tends to a trivial state, i.e., $f(x, t, \xi) \equiv 0$, the existence results can be found in [30,23] using the method introduced in [31].

As mentioned earlier, there are many open questions on vacuum problems for the systems of compressible fluids and the Boltzmann equation. Vacuum states have their unique singularity that have challenging mathematical difficulty and mystery. The above four models only cover part of the works on this subject. Similar problems can also be found in the study of the magnetohydrodynamics where Alfén wave is related to vacuum, and some particle systems.

2. Euler equations

The system of Euler equations consisting of conservation of mass, momentum and energy is one of the typical hyperbolic systems in fluid dynamics. When the solution contains vacuum, the system becomes degenerate in the sense that some of its characteristics coincide. If there is no external force, a state connects to vacuum through rarefaction wave, not shock wave, cf. [66]. And it is also believed that the appearance of vacuum in finite time does not happen if there is no vacuum initially even though a rigorous proof is not known. The intuitive thinking is that the appearance of vacuum state comes only from the interaction of two large rarefaction waves of different families. And numerical computation suggests that in this case the vacuum does not appear in the interaction zone in finite time [40].

In this section, we will try to illustrate the time evolution of the vacuum boundary under the physical boundary condition for some simple physical model. More precisely, we assume that the governed equations for the gas dynamics are Euler equations with linear frictional damping. For physical interpretation of the linear damping term, please refer to [57]. For the previous works on Euler equations related to vacuum, please see [22,46,47,43,51,53,54,70,74,75] and references therein. For the case with frictional damping, the canonical vacuum boundary behavior corresponds to the case when the space derivative of the enthalpy is bounded but not zero which is different from the one for centered rarefaction wave without force, cf. [39]. That is, the pressure has a non-zero finite effect on the evolution of the vacuum boundary. However, for this canonical (physical) case, the system is singular in the sense that it is not symmetrizable with regular coefficients so that the local existence theory for the classical hyperbolic systems cannot be applied. Moreover, the linearized equation at the boundary gives a Keldysh-type equation for which general local existence theory is still not known. Notice that this linearized equation is quite different from the one considered in [62] for weakly hyperbolic equation which is of Tricomi type.

To capture this singularity in the nonlinear setting, a transformation was introduced in [47] and some local existence results in bounded domain were also discussed. The transformed equation is a second-order nonlinear wave equation of an unknown function $\phi(y,t)$ with coefficients as functions of $y^{-1}\phi(y,t)$ and $\phi(0,t)\equiv 0$. Along the vacuum boundary $y=0$ in this setting, the physical boundary condition implies that the coefficients are functions of $\phi_y(0,t)$ which are bounded and away from 0. Hence, the wave equation has no singularity or degeneracy. However, its coefficients have the above special form so that the local existence theory developed for the classical nonlinear wave equation cannot be applied directly [27,34].

Through the transformation capturing the singularity at the vacuum boundary under the physical boundary condition, the coefficients in the reduced wave equation which are power functions of $y^{-1}\phi$ correspond to the fractional differentiation of ϕ. Thus, one can apply the Hardy–Littlewood–Paley theory to obtain some local existence results. Indeed, this was done when the initial data were a small perturbation of a planar wave solution where the enthalpy is linear in the space variable [69].

To appreciate this analysis, consider the one-dimensional compressible Euler for the isentropic flow with frictional damping in Eulerian coordinates

$$\begin{aligned}&\rho_t+(\rho u)_x=0,\\&(\rho u)_t+(\rho u^2+p(\rho))_x=-\rho u,\end{aligned}\tag{2.1}$$

where ρ, u and $p(\rho)$ are density, velocity and pressure, respectively. The linear frictional coefficient is normalized to 1. When the initial density function contains vacuum, the vacuum boundary Γ is defined as

$$\Gamma=cl\{(\boldsymbol{x},t)\mid\rho(\boldsymbol{x},t)>0\}\cap cl\{(\boldsymbol{x},t)|\rho(\boldsymbol{x},t)=0\}.$$

For regular solution, the second equation in (2.1) can be rewritten as

$$u_t+uu_x+i_x=-u,$$

where i is the enthalpy. One can see that the term i_x represents the effect of the pressure on the particle path, especially the vacuum boundary. It is shown in [46,51,54,74,75] that there is no global existence of regular solutions satisfying $i_x \equiv 0$ along the vacuum boundary. That is, in general, i is not C^1 crossing the vacuum boundary. Hence, the canonical behavior of the vacuum boundary should satisfy the condition $i_x \neq 0$ and is bounded. This special feature can be illustrated by the stationary solutions and some self-similar solutions, also for different physical systems, such as Euler–Poisson and Navier–Stokes equations discussed in the next two sections.

When $i_x \equiv 0$ at the vacuum boundary, the proof on the non-global existence of regular solutions is based on the estimation of the support of the density function which should be sub-linear and the time evolution of the second inertia of the gas w.r.t. the origin, i.e., $H(t) \doteq \int \rho|x|^2\,\mathrm{d}x$. For a general system, a non-global existence of regular solution violating the physical boundary condition is given in [75] by considering $n+2$ entropy and entropy flux pairs for a symmetrizable system in n-dimensional space.

Notice that the characteristics of Euler equations is $u \pm c$. For isentropic polytropy gas when $p(\rho)=k\rho^\gamma$ for a positive constant k and the adiabatic constant $\gamma > 1$, we have $i = c^2/(\gamma-1)$. Hence the characteristics are singular with infinite space derivatives at the vacuum boundary under the physical boundary condition. This kind of singularity yields the smooth reflection of the characteristics on the vacuum boundary and thus causes analytical difficulty.

There is an interesting way to view this canonical boundary condition from the study of porous media equation. It is known that the Euler equations with linear damping behave like the porous media equation, $\rho_t = p(\rho)_{xx}$, at least away from vacuum when $t \to \infty$, cf. [28]. Notice also that there are some results with vacuum in the weak sense by using compensated compactness cf. [29]. Since the vacuum boundary is not well-defined in this case, we will not discuss them here.

For porous media equation, there is a well-established theory on the waiting time problem and boundary singularities [2,36]. It can be briefly explained as follows. Consider a parabolic equation

$$V_t(x,t) = (V^m(x,t))_{xx}, \quad m > 1. \tag{2.2}$$

For initial data of compact support, one can show that the canonical boundary behavior is the one with $(V^{m-1})_x$ being bounded and non-zero. That is, if initially $(V^\alpha)_x$ is bounded and non-zero at the edge of the support, then we have the following cases depending on the value of α:

1. When $0 < \alpha \leqslant (m-1)/2$, there exists a solution with the same behavior (i.e., α unchanged) up to a finite time, and then it behaves like $\alpha = m-1$.
2. When $(m-1)/2 < \alpha < m-1$ or $\alpha > m-1$, then the solution has the behavior with $\alpha = m-1$ for $t > 0$.
3. When $\alpha = m-1$, the solution has the same behavior for all time.

If we compare this kind of waiting time behavior to system (2.1), then one can see that they have a lot of similarity. For (2.1), the canonical behavior corresponds to $(\rho^{\gamma-1})_x$ being bounded and non-zero at the vacuum boundary. It turns out that the conjecture on the behavior of $(\rho^\alpha)_x$ being bounded and non-zero at the vacuum boundary for (2.1) can be stated by replacing V by ρ and m by γ, respectively. However, the rigorous proof on this waiting time problem for (2.1) is almost open because it is for a system not just for a scalar equation. The techniques used in proving the theorem for (2.2) may not work here.

We now come back to the local existence of solutions with physical vacuum boundary for Euler equations with linear damping. Since our concern is on the behavior of the solution related to vacuum and any shock wave vanishes at vacuum [44], it is reasonable to consider the problem without shock waves. In fact, any shock wave that appears initially or in finite time will decay to zero in time because of the dissipation from the linear damping. By using the special property of the one-dimensional gas dynamics, we can rewrite system (2.1) by using Lagrangian coordinates to make all the particle paths, in particular the vacuum boundary, as straight lines. In Lagrangian coordinates (2.1) takes the form

$$\begin{aligned} &v_t - u_\xi = 0, \\ &u_t + p(v)_\xi = -u, \end{aligned} \tag{2.3}$$

where $v = 1/\rho$ is the specific volume and $\xi = \int_0^x \rho(y,t)\,dy$. Notice that the physical sigularity, $0 < |i_x| < \infty$, along the vacuum boundary in Eulerian coordinates corresponds to $0 < |p_\xi(v)| < \infty$ in the Lagrangian coordinates.

In order to symmetrize system (2.3) with the physical boundary condition, the following coordinate transformation was introduced in [47]:

$$\xi = y^{2\gamma/(\gamma-1)}.$$

Here for simplicity, we assume that the initial density function satisfies $\rho_0(x) = 0$ for $x < 0$ in the Eulerian coordinates. Then system (2.3) can be rewritten as

$$\begin{aligned} &\phi(v)_t + \bar{\mu} u_y = 0, \\ &u_t + \bar{\mu}\phi(v)_y = -u, \quad y > 0, \quad t > 0, \end{aligned} \tag{2.4}$$

where $\phi(v) = 2\sqrt{\gamma k}/(\gamma-1) v^{-(\gamma-1)/2}$, and

$$\bar{\mu} = \frac{(\gamma-1)\sqrt{k}}{\sqrt{\gamma}} (v y^{2/(\gamma-1)})^{-(\gamma+1)/2} = \beta (y^{-1}\phi)^{(\gamma+1)/(\gamma-1)},$$

for some positive constant β. Notice that near the vacuum boundary, both $\phi(v)_y$ and $\bar{\mu}$ are bounded and non-zero under the physical boundary condition. This allows the application of the Hardy–Littlewood–Paley theory in the proof of local existence, cf. [8,12].

In fact, there are two families of global solutions connecting to vacuum constructed in [43]. One has $p(v)$ as a linear function in ξ with constant velocity, while the velocity in the other solution is linear function in space variable and $p(v)$ is a polynomial of the second order. It was shown that the solutions in the second family converge to the Barenblatt solutions to the porous media equation (2.2) as time tends to infinity. The local existence of solutions for a small perturbation to the solutions of the first family was proved in [69].

3. Euler–Poisson equations

In this section, we will consider the Euler–Poisson equations which can be viewed as a model for the time evolution of self-gravitating gaseous stars:

$$\frac{\partial \rho}{\partial t} + \nabla_x \cdot (\rho u) = 0,$$

$$\frac{\partial(\rho u)}{\partial t} + \nabla_x \cdot (\rho u \otimes u) + \nabla_x p = -\rho \nabla_x \Phi,$$

$$\frac{\partial(\rho s)}{\partial t} + \nabla_x \cdot (\rho s u) = 0, \tag{3.1}$$

$$\Delta_x \Phi = 4\pi g \rho, \tag{3.2}$$

where $t \geqslant 0$, $x \in \mathbf{R}^3$, s is the entropy, and Φ the potential function of the self-gravitational force with g being the gravitational constant. In this section, we consider the equation of state for a polytropic gas:

$$p = k\rho^\gamma \mathrm{e}^s,$$

and the case when the density $\rho(x,t)$ has compact support in the space $x \in \mathbf{R}^3$. Eq. (3.1) is the system of compressible Euler equations with gravitational force governed by the density distribution of the stars through the Poisson equation (3.2). Notice that the adiabatic constant $\gamma = \frac{5}{3}$ for monatomic gas, $\frac{7}{5}$ for a diatomic gas and $\gamma \to 1+0$ for heavier molecules. Other values of γ have significance of their own, cf. [10,38]. For instance, the value $\gamma = \frac{4}{3}$ is important because of the quantum effects [6]. In the following discussion, we will see that γ plays an important role in the existence, stability and uniqueness of the stationary solutions. For instance, the linear theory by the astrophysicists shows that spherical symmetric stationary solutions are unstable when $\gamma \leqslant \frac{4}{3}$, while stable when $\gamma > \frac{4}{3}$.

The above stability criterion was believed long time ago since the work of A. Ritter in the nineteenth century. The justification by linear theory was obtained by Chandrasekhar and Eddington under the assumption of spherical symmetry stated as variational principles. For completeness of the presentation, we briefly include their arguments, cf. [6].

Consider system (3.1)–(3.2), and assume that there is a spherically symmetric stationary profile (ρ_0, u_0) satisfying

$$u_0 = 0, \quad \frac{\mathrm{d}\rho_0(r)}{\mathrm{d}r} < 0, \quad \rho_0(r_m) = 0,$$

where r is the radius and r_m a positive constant. By linearizing system (3.1)–(3.2) by letting

$$\rho = \rho_0 + \varepsilon\rho_1, \quad p = p_0 + \varepsilon p_1, \quad u = \varepsilon u_1, \quad \Phi = \Phi_0 + \varepsilon\Phi_1,$$

the leading term yields

$$\frac{\partial \rho_1}{\partial t} + \nabla_x \cdot (\rho_0 u_1) = 0, \quad \frac{\partial u_1}{\partial t} = -\nabla_x h_1 - \nabla_x \Phi_1,$$

$$\nabla_x \Phi_1 = 4\pi g \rho_1, \tag{3.3}$$

where $h_1 = (\mathrm{d}P/\mathrm{d}\rho)_0\, \rho_1/\rho_0$. Here $(\cdot)_0$ means the value in the bracket is evaluated at the stationary state (u_0, ρ_0). By taking the normal modes as

$$\rho_1(r,t) = \mathrm{Re}\{\rho_a(r)\mathrm{e}^{-\mathrm{i}w_a t}\}, \quad u_1(r,t) = \mathrm{Re}\{u_a(r)\mathrm{e}^{-\mathrm{i}w_a t}\},$$

$$\Phi_1(r,t) = \mathrm{Re}\{\Phi_a(r)\mathrm{e}^{-\mathrm{i}w_a t}\}, \quad h_1(r,t) = \mathrm{Re}\{h_a(r)\mathrm{e}^{-\mathrm{i}w_a t}\},$$

it is straightforward to derive the following identity:

$$\frac{w_a^2}{|w_a|^2}\int \rho_0 |u_a|^2\, \mathrm{d}^3\mathbf{r} = \int \left|\left(\frac{\mathrm{d}\Psi}{\mathrm{d}\rho}\right)_0\right| |\rho_a|^2\, \mathrm{d}^3\mathbf{r} - g\int\int \frac{\mathrm{d}^3\mathbf{r}\,\mathrm{d}^3\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\bar{\rho_a}(\mathbf{r})\rho_a(\mathbf{r}'), \tag{3.4}$$

where $\mathbf{r}$ and $\mathbf{r}'$ are vectors in $\mathbf{R}^3$, and "$\bar{\cdot}$" represents the complex conjugate. Notice that if $w_a^2 \geqslant 0$, then the normal mode is stable and oscillates with frequency w_a. Otherwise, it is unstable. The following is Chandrasekhar's variational principle.

Chandrasekhar's variational principle: A barotropic star with $\mathrm{d}\rho_0/\mathrm{d}r < 0$ and $\rho_0(r_m) = 0$ is stable if the quantity

$$\mathscr{E}[\rho_1] = \int \left| \left(\frac{\mathrm{d}\Psi}{\mathrm{d}\rho} \right)_0 \right| |\rho_a|^2 \, \mathrm{d}^3\mathbf{r} - g \int \int \frac{\mathrm{d}^3\mathbf{r}\, \mathrm{d}^3\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \bar{\rho_a}(\mathbf{r}) \rho_a(\mathbf{r}')$$

is non-negative for all real functions $\rho_a(r)$ that conserve the total mass of star, i.e., $\int \rho_1 \, \mathrm{d}^3\mathbf{r} = 0$.

According to the Antonov–Lebovitz theorem, any spherically symmetric barotropic star with $\mathrm{d}\rho_0/\mathrm{d}r < 0$ and $\rho_0(r_m) = 0$ is stable to all perturbations that are not spherically symmetric. Therefore, one can only look at the spherically symmetric perturbation to study the stability in this setting. Set

$$u_a = v_a \hat{e}_r, \quad \nabla_x h_a = \left(\frac{\mathrm{d}h_a}{\mathrm{d}r} \right) \hat{e}_r, \ldots$$

where $\hat{e}_r$ is the unit vector in radius direction. Then by defining the local adiabatic index

$$\gamma(r) = \left(\frac{\mathrm{d} \ln p}{\mathrm{d} \ln \rho} \right)_0,$$

and the amplitude of the fractional displacement in the radius of a fluid element

$$\zeta_a = \frac{i v_a}{w_a r},$$

(3.3) gives the following identity obtained by Eddington in 1918:

$$\frac{\mathrm{d}}{\mathrm{d}r} \left(p_0 \gamma r^4 \frac{\mathrm{d}\zeta_a}{\mathrm{d}r} \right) + \left\{ w_a^2 \rho_0 r^4 + r^3 \frac{\mathrm{d}}{\mathrm{d}r} [(3\gamma - 4) p_0] \right\} \zeta_a = 0.$$

Then

$$w_a^2 \int_0^{r_m} \rho_0 r^4 |\zeta_a|^2 \, \mathrm{d}r = \int_0^{r_m} \left\{ p_0 \gamma r^4 \left| \frac{\mathrm{d}\zeta_a}{\mathrm{d}r} \right|^2 - r^3 |\zeta_a|^2 \frac{\mathrm{d}}{\mathrm{d}r} [(3\gamma - 4) p_0] \right\} \mathrm{d}r,$$

which yields Eddington's variational principle.

Eddington's Variational Principle: A barotropic star with $\mathrm{d}\rho_0/\mathrm{d}r < 0$ and $\rho_0(r_m) = 0$ is stable to radial perturbations if the quantity

$$\mathscr{F}[\zeta] = \int_0^{r_m} \left\{ p_0 \gamma r^4 \left| \frac{\mathrm{d}\zeta_a}{\mathrm{d}r} \right|^2 - r^3 |\zeta_a|^2 \frac{\mathrm{d}}{\mathrm{d}r} [(3\gamma - 4) p_0] \right\} \mathrm{d}r$$

is non-negative for all function $\zeta_a(r)$.

For isentropic and polytropic star, i.e., $p = k\rho^\gamma$, $\mathscr{F}$ is reduced to

$$\mathscr{F}[\zeta] = \int_0^{r_m} p_0 \gamma r^4 \left| \frac{\mathrm{d}\zeta_a}{\mathrm{d}r} \right|^2 - (3\gamma - 4) \int_0^{r_m} r^3 |\zeta_a|^2 \frac{\mathrm{d}p_0}{\mathrm{d}r} \, \mathrm{d}r.$$

The fact that $\mathrm{d}p_0/\mathrm{d}r < 0$ implies that the star is stable for $\gamma \geqslant \frac{4}{3}$ and unstable otherwise.

Now we turn to the nonlinear theory; consider again first the stationary solutions of non-moving gas $u = 0$ to (3.1) and (3.2):

$$\begin{aligned} &\nabla_x p(\rho) = -\rho \nabla_x \Phi, \qquad \Delta_x \Phi = 4\pi g \rho, \\ &p = k\rho^{\gamma} \mathrm{e}^{s(x)}. \end{aligned} \tag{3.5}$$

Let the gas be confined to a spatially bounded domain $\Omega \subset \mathbf{R}^3$, that is, $\rho|_{\partial\Omega} = 0$, $\rho > 0$ in Ω. We now look for the existence, uniqueness and stability of the stationary solutions to (3.5). Through a nonlinear transformation,

$$Q(\rho, x) = \frac{\gamma}{\gamma - 1} \rho^{\gamma-1} \mathrm{e}^{s(x)},$$

(3.5) can be reduced to a semi-linear elliptic equation with a nonlinear function of $Q(\rho, x)$ as a source,

$$\nabla_x \cdot \left(\mathrm{e}^{-s/\gamma} \nabla_x Q\right) - \frac{1}{\gamma}(\Delta_x S) Q \mathrm{e}^{-s/\gamma} = -4\pi g \left(\frac{\gamma - 1}{\gamma} Q \mathrm{e}^{-s}\right)^{1/(\gamma-1)} \mathrm{e}^{-s/\gamma} \tag{3.6}$$

for which the classical theory of the elliptic equation leads to some existence theorems under some assumptions on γ and $s(x)$. In fact, it was proved in [14] that non-trivial positive solutions of (3.6) exist for $\frac{6}{5} < \gamma < 2$. Moreover, for isentropic gases, the behavior of the solutions, such as uniqueness, multiplicity and radial symmetry, can be discussed more explicitly, cf. [14].

There are extensive studies of weak solutions with shocks for conservation laws and also for Euler–Poisson equations. On the other hand, the regularizing effect of the Poisson equation indicates that the interesting singular behavior of the solutions is not mainly with shocks. Then same as the Euler equations with damping, there is the interesting singular behavior of solutions for the Euler–Poisson equations at the interface separating the gas and the vacuum [14,46], that is, for isentropic fluid,

$$\langle \nabla_x p_\rho, \ \boldsymbol{n} \rangle|_{\partial\Omega} = \langle \nabla_x c^2, \boldsymbol{n} \rangle|_{\partial\Omega} < 0, \quad \text{and bounded},$$

where c is the sound speed. This singular behavior of the solution at the boundary again prevents the symmetrization of the system and the classical local existence theory for symmetrizable hyperbolic systems cannot be applied directly. In the study of the stationary solutions to the Euler–Poisson equations in any bounded region, the physical boundary singularity is guaranteed by Hopf's maximal principle. Moreover, one can show that the integral of the quantity $(\nabla c^2, \boldsymbol{n})$ along the boundary gives exactly the total mass up to a constant factor. The nonlinear stability of the above stationary solutions can be summarized as follows.

Consider a solution with no shocks to the isentropic Euler–Poisson equations which has finite total energy E and total mass M. When $\gamma > \frac{4}{3}$, there are no blowup phenomena where part of the solution collapses to a point with finite mass. This also holds when $\gamma = \frac{4}{3}$ and the total mass is less than the critical mass $M_{\mathrm{c}} = (3K/2\pi)^{3/2} (\mathscr{M}_{4/3})^{-2}$. Here $\mathscr{M}_{4/3}$ is the Marcinkiewicz interpolation constant.

This statement can be proved by the Fourier analysis using the Hardy–Littlewood–Paley inequality, cf. [64], from which the Marcinkiewicz interpolation constant arises naturally. It is consistent with the aforementioned conjecture and linear theory. In that, for $\gamma > \frac{4}{3}$ there is no collapse of the gas to a single point with finite mass, called core collapse. It remains open to construct blowup solutions for the case $\gamma < \frac{4}{3}$. For systems (3.1) and (3.2), the local existence was studied in [5,18,52–54]. The stability of linearized system around a stationary solution with spherical symmetry was also discussed in [38].

For the core collapse at finite time when $\gamma = \frac{4}{3}$, the following example was constructed in [51] for isentropic flow and is later generalized to nonisentropic case in [15].

Assume $\gamma = \frac{4}{3}$ and $p = k(\delta^{\sigma-4/3} y^{(3\sigma-4)/3(\sigma-1)}(r/a(t)))\rho^{4/3}$; there exists a family of isentropic solutions $(\rho, v)(t, r)$ with spherical symmetry in the form of

$$\rho(t,r) = \begin{cases} \dfrac{\delta}{a(t)^3} y^{1/(\sigma-1)}\left(\dfrac{r}{a(t)}\right), & r < a(t)z_\mu, \\ 0, & r \geqslant a(t)z_\mu, \end{cases}$$

$$v(t,r) = \frac{\dot{a}(t)}{a(t)} r\phi\left(\frac{r}{a(t)}\right).$$

Here $r = |x|$, $\delta = [(12(\sigma-1)\pi g)/\sigma K]^{1/(\sigma-2)}$, $\sigma > 1$ and $0 \leqslant \mu \ll \infty$ are constants, z_μ is a positive constant depending only on μ, ϕ is a smooth cut-off function such that $\phi(z) = 1$ for $0 \leqslant z \leqslant z_\mu$ and $\phi(z) = 0$ for $2z_\mu \leqslant z < +\infty$. Here $a = a(t)$ and $y = y(z)$ satisfy the following two ordinary differential equations, respectively:

$$\frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2} = -\frac{\lambda}{a^2}, \quad a(0) = a_0 > 0, \ \ \dot{a}(0) = a_1$$

and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \frac{2}{z}\frac{\mathrm{d}y}{\mathrm{d}z} + y^{1/(\sigma-1)} = \mu, \quad y(0) = 1, \ \ y'(0) = 0,$$

where $\lambda = 4\pi g \delta \mu$.

If $a_1 < \sqrt{(2\lambda/a_0)}$, then the solution collapses at the origin.

Recently, there are some mathematical theories on the rotational gaseous stars governed by (3.1) and (3.2) [49]. In fact, for a stationary solution to (3.1) and (3.2) with given velocity field $v(x)$, the momentum equation can be written as

$$v \cdot \nabla_x v + \frac{1}{\rho}\nabla_x p = -\nabla_x \Phi. \tag{3.7}$$

This combining with the Poisson equation leads to the following elliptic equation:

$$\nabla \cdot (\mathrm{e}^{\alpha s}\nabla_x w) + \Gamma \mathrm{e}^{-\alpha s} w^q - f(x) = 0, \tag{3.8}$$

where

$$q = \frac{1}{(\gamma-1)}, \quad \alpha = \frac{1}{\gamma}, \quad \Gamma = 4\pi g\left(\frac{\gamma-1}{\gamma}\right)^{1/(\gamma-1)}, \quad w = \frac{\gamma}{\gamma-1}(\mathrm{e}^{s/\gamma}\rho)^{\gamma-1}$$

and

$$f(x) = -\nabla \cdot (v \cdot \nabla_x v).$$

However, to satisfy the equation of conservation of mass and energy, the velocity field cannot be arbitrary. In fact, if $u \equiv 0$ or $u(x,t)$ is a rotation around a curve $(x_1, x_2) = (\theta_1(x_3), \theta_2(x_3))$ with angular velocity $\Omega(\eta)$ as a function of $\eta = \sqrt{(x_1-\theta_1)^2 + (x_2-\theta_2)^2}$, then these two equations are trivially satisfied for solutions independent of time. Therefore, the solution to the elliptic equation in these settings leads to the stationary solution to the Euler–Poisson system.

In fact, let $u = \mathrm{e}^{(\alpha/2)s} w$; Eq. (3.8) is reduced to

$$\Delta_x u - a(x)u + K(x)u^q - f(x)\mathrm{e}^{-(\alpha/2)s} = 0, \tag{3.9}$$

where

$$a(x) = \frac{\alpha}{2}\Delta_x s + \frac{\alpha^2}{4}|\nabla_x s|^2, \quad K(x) = k\mathrm{e}^{(3-3\gamma)/2\gamma(\gamma-1)s}.$$

For a given simply connected open region D with smooth boundary, one can consider the positive solutions to Eq. (3.9) in D and with zero Dirichlet boundary condition:

$$u|_{\partial D} = 0. \tag{3.10}$$

When the rotation is around a fixed axis, like x_3, the function $f(x) = 2\Omega(\Omega + \eta\Omega'(\eta)) = g(x_1, x_2)$. In general, the rotation can be around a curve and then the function $f(x)$ is a function of x_i, $i = 1, 2, 3$. For both cases, the existence, uniqueness and the multiplicity of the solutions to (3.9) are investigated in [16,49] based on the theory on the elliptic equation, cf. [1,9,19,20,56] and references therein. For space limitations, we will not present them here.

4. Navier–Stokes equations

In the above two sections, we considered the inviscid fluid with vacuum. Now, we will study the case when the fluid is viscous. Since the system of Navier–Stokes equations is the typical model in this situation, the later discussion is based on this model. Notice that the full Navier–Stokes equations with viscosity and heat conductivity depending on the temperature are very difficult to analyze for the vacuum behavior. The following discussion is mainly on the isentropic fluid, that is, the energy equation and heat conductivity will not be considered here.

The one-dimensional compressible Navier–Stokes equations for isentropic flow in Eulerian coordinates take the form

$$\begin{aligned} &\rho_\tau + (\rho u)_\xi = 0,\\ &(\rho u)_\tau + (\rho u^2 + p(\rho))_\xi = (\mu u_\xi)_\xi, \end{aligned} \tag{4.1}$$

with initial data

$$\rho(\xi, 0) = \rho_0(\xi), \quad u(\xi, 0) = u_0(\xi), \quad a \leqslant \xi \leqslant b, \tag{4.2}$$

where $\xi \in \mathbf{R}^1$ and $\tau > 0$, ρ, u and $p(\rho)$ have the same meaning as those in the previous section, and $\mu \geqslant 0$ is the viscosity coefficient. For simplicity of presentation, we again consider only the polytropic gas where $p(\rho) = k\rho^\gamma$.

We will consider this hyperbolic–parabolic system when the initial data are of compact support, i.e., connecting to vacuum state. Our main concern here is the global existence of solutions and the evolution of the vacuum boundary. Notice that one of the important features of this problem is that the interface separating the gas and the vacuum propagates with finite speed if the initial data are of compact support.

It is interesting to note that the proof of this finite speed propagation is obtained after the lower bound of the density function is given. In other words, this finite speed propagation property is difficult to be justified without the estimate on the density function.

Let us first review some of the works in this direction. When the viscosity coefficient μ is a constant, the study in [25] shows that there is no continuous dependence on the initial data of the solutions to the Navier–Stokes equations (4.1) with vacuum. The main reason for this non-continuous dependence at the vacuum comes from the kinetic viscosity coefficient being independent of the density. It is motivated by the physical consideration that in the derivation of the Navier–Stokes equations from the Boltzmann equation, cf. [21,48], the viscosity is not constant but depends on the temperature. For isentropic flow, this dependence is reduced to the dependence on the density by the laws of Boyle and Gay-Lussac for ideal gas. For example, the viscosity of gas is proportional to the square root of the temperature for hard sphere collision model. Since the temperature is of the order of $\rho^{\gamma-1}$ for the perfect gas where the pressure is proportional to the product of the density and the temperature, for the hard sphere model with $\gamma=\frac{5}{3}$, the viscosity μ is proportional to $\rho^{1/3}$.

The above non-continuous dependence on the initial data for constant viscosity with vacuum is the motivation for the works on the case when the viscosity function is a function of density, such as $\mu=c\rho^{\theta}$, where c and θ are positive constants. Notice that now the viscosity coefficient vanishes at vacuum and this property yields the well-posedness of the Cauchy problem when the initial density is of compact support. In this situation, the local existence of weak solutions to the Navier–Stokes equations with vacuum was first studied in [45], where the initial density was assumed to be connected to vacuum with discontinuities. This property can be maintained for some positive finite time. Moreover, the authors in [61] obtained the global existence of weak solutions when $0<\theta<\frac{1}{3}$ which was later generalized to the cases when $0<\theta<\frac{1}{2}$ and $0<\theta<1$ in [71] and [32], respectively.

It is noticed that the above analysis is based on the uniform positive lower bound of the density in the construction of the approximate solutions, for example, by line method. This estimate is crucial because the other estimates for the convergence of a subsequence of the approximate solutions and the uniqueness of the solution thus obtained follow from this estimation by standard techniques. Notice that this uniform positive lower bound on the density function can only be obtained when the density function connects to vacuum with discontinuities. In this case, the density function is positive for any finite time and thus the viscosity coefficient never vanishes. This good property of the solution is used to prove global existence of solutions to (4.1) when the initial data are of compact support, cf. [32,61,71].

If the density function connects to vacuum continuously, there is no positive lower bound for the density and the viscosity coefficient vanishes at vacuum. This degeneracy in the viscosity coefficient gives rise to a new analysis difficulty because of the weaker regularizing effect on the solutions. For this problem, a local existence result was obtained in [72], and global existence result in [73] for $0<\theta<\frac{2}{9}$ and in [67] for $0<\theta<\frac{1}{3}$. Notice also that the singularity arises at the vacuum boundary when the density function connects to vacuum continuously. This can be seen from the analysis in [68] on the non-global existence of regular solution to Navier–Stokes equations when the density function is of compact support when the viscosity coefficient is constant. The proof there is based on the estimation on the growth rate of the support of the density function in time t. If the growth rate is sub-linear, then the nonlinear functional introduced in [68] gives the non-global existence of regular solutions. The intuitive explanation of this phenomenon comes from the consideration of the pressure in the gas. No matter how smooth the initial data are, the pressure of the gas will build up at the vacuum boundary in finite time and it will push the gas into the vacuum region. This effect cannot be compensated by the dissipation from the viscosity so that the support of the gas stays unchanged. This is different from the system of Euler–Poisson equations for gaseous stars where the pressure and the gravitational force can become balanced to have stationary solutions. In the case of compressible Navier–Stokes equations, the singularity at the vacuum boundary can be overcome by introducing some appropriate weights in the energy estimates as in [67,73].

In the case when the density function is of compact support in both Eulerian and Lagrangian coordinates, the restriction on the solution coming from the boundedness of the support is

$$\int_0^1 \frac{1}{\rho(x,t)}\,\mathrm{d}x < \infty, \tag{4.3}$$

in Lagrangian coordinates (x, t). This is a consequence of the boundedness of velocity in L^∞ norm and is justified after the a priori estimate on the density function is obtained. If the density function of infinite support in Eulerian coordinates, then even though the total mass is assumed to be finite, no restriction like (4.3) is needed. Some new a priori estimates were established in [73] also for this case so that the global existence of weak solution is also obtained when $\frac{1}{3} < \theta < \frac{3}{7}$. Notice that the intervals for θ are disjoint for these two cases because of some technical reasons.

The theorem on non-global existence of regular solutions in [73] generalizes the one for constant viscosity coefficient in [68] to the case when the viscosity coefficient depends on density. This sheds some light on the study of the vacuum problem to the full Navier–Stokes equations for non-isentropic gas when the viscosity and heat conductivity coefficients depend on the temperature. It is noticed that the corresponding vacuum problem for this full Navier–Stokes equations is still open.

The non-continuous dependence result also leads to the study on the initial boundary value problem instead of initial value problem. For this, the free boundary problem of one-dimensional Navier–Stokes equations with one boundary fixed and the other connected to vacuum was investigated in [59], where the global existence of the weak solutions was proved. Similar results were obtained in [60] for the equations of spherically symmetric motion of viscous gases. Moreover, the free boundary problem of the one-dimensional viscous gases which expand into the vacuum has been studied by many people, see [59,60,68] and references therein. A detailed discussion of the regularity and behavior of solutions near the interfaces between the gas and the vacuum is given in [50].

There has been a lot of investigation on the Navier–Stokes equations when the initial density is away from vacuum, both for smooth or discontinuous initial data, and one or multidimensional problems. For these results, please refer to [13,24,35,41,63] and references therein. Recently, the non-appearance of vacuum in the solutions for any finite time if the initial data do not contain vacuum was proved in [26] for constant viscosity.

5. Boltzmann equation

The Boltzmann equation for the rarefied gas is a fundamental equation in statistical physics derived by Boltzmann in 1872. Its close relation to fluid dynamics is known since then, which can be clearly seen by the Hilbert and Chapman–Enskog expansions. In fact, the first order of the Hilbert expansion gives the system of compressible Euler equations, while the second order of the Chapman–Enskog expansion yields the system of Navier–Stokes equations. Based on the thinking that the use of the Boltzmann equation is more appropriate in the region close to the vacuum boundary, the study of the Boltzmann equation with vacuum becomes more important than those studied in the previous sections. However, there are no results on the Boltzmann equation with vacuum and non-trivial large-time behavior or non-trivial fluid dynamics limits so far. Thus, the vacuum problem for the Boltzmann equation with non-trivial large-time behavior and its fluid limit is mathematically very challenging.

In the following discussion, we will first present a decomposition of the Boltzmann equation which yields a system of fluid dynamics for fluid components and an equation for the non-fluid component. Then we will give some particular solutions to the Boltzmann equation with gravitational force and its relation to the systems of fluid dynamics. The stability of these solutions to the Boltzmann equation is interesting and open even though partial results to the fluid dynamics systems have been obtained.

For brevity of presentation, we consider the Boltzmann equation with external force depending only on (x, t), and similar argument works also for the case when it depends also on velocity,

$$f_t + \xi \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_\xi f = \frac{Q(f, f)}{\kappa}, \quad (f, x, t, \xi) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3, \tag{5.1}$$

where the positive constant κ is the Knudsen number [7]. For monatomic gas, the invariance under rotation leads to the definition of the bilinear collision operator $Q(f, g)$ as

$$\begin{aligned} Q(f, g)(\xi) \equiv \frac{1}{2} \int_{\substack{c\mathbb{R}^3 \times S^2 \\ (\xi - \xi_*)\cdot\Omega \geqslant 0}} & (-f(\xi)g(\xi_*) - g(\xi)f(\xi_*) + f(\xi')g(\xi'_*) \\ & + g(\xi')f(\xi'_*))B(|\xi - \xi_*|, \theta)\, \mathrm{d}\xi_*\, \mathrm{d}\Omega, \end{aligned}$$

with $B(\cdot, \cdot)$ being the collision kernel, where θ is the angle between the relative velocity $\xi - \xi_*$ and the unit vector Ω. Moreover, the relation between the velocities before and after collision is

$$\begin{cases} \xi' = \xi - [(\xi - \xi_*) \cdot \Omega]\Omega, \\ \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \Omega]\Omega, \end{cases}$$

coming from conservation of momentum and energy.

We first decompose the solution into the macroscopic, fluid part, the local Maxwellian $\boldsymbol{M} = \boldsymbol{M}(x, t, \xi) = \boldsymbol{M}_{[\rho,u,\theta]}(\xi)$, and the microscopic, non-fluid part $\boldsymbol{G} = \boldsymbol{G}(x, t, \xi)$ of the solution, cf. [48,55]:

$$f = \boldsymbol{M} + \boldsymbol{G}.$$

The local Maxwellian is constructed from the fluid variables, the five conserved quantities, the mass density $\rho(x, t)$, momentum $m(x, t) = \rho u(x, t)$ and energy $\rho(\mathscr{E} + |u|^2/2)$ of the Boltzmann equation:

$$\begin{cases} \rho(x, t) \equiv \int_{\mathbb{R}^3} f(x, t, \xi)\, \mathrm{d}\xi, \\ m^i(x, t) \equiv \int_{\mathbb{R}^3} \psi_i f(x, t, \xi)\, \mathrm{d}\xi \quad \text{for } i = 1, 2, 3, \\ \rho\left(\mathscr{E} + \dfrac{1}{2}|u|^2\right)(x, t) \equiv \int_{\mathbb{R}^3} \psi_4 f(x, t, \xi)\, \mathrm{d}\xi, \end{cases} \tag{5.2}$$

$$\boldsymbol{M} \equiv \boldsymbol{M}_{[\rho,u,\theta]}(\xi) \equiv \frac{\rho}{\sqrt{(2\pi R\theta)^3}} \exp\left(-\frac{|\xi - u|^2}{2R\theta}\right). \tag{5.3}$$

Here $\theta(x, t)$ is the temperature and is related to the internal energy $\mathscr{E}$ through the gas constant R, $\mathscr{E} = \frac{3}{2}R\theta$, and $u(x, t)$ is the fluid velocity. The five fluid variables are conserved quantities because of the following property of the collision invariants ψ_α [7]:

$$\int_{\mathbb{R}^3} \psi_\alpha Q(h, g)\, \mathrm{d}\xi = 0 \quad \text{for any } \alpha = 0, 1, 2, 3, 4, \text{ and for any functions } h, g,$$

where

$$\psi_0 \equiv 1, \quad \psi_i \equiv \xi^i \quad \text{for } i = 1, 2, 3, \quad \psi_4 \equiv \tfrac{1}{2}|\xi|^2. \tag{5.4}$$

With respect to the local Maxwellian, we define an inner product in $\xi \in \mathbb{R}^3$ as

$$\langle h, g\rangle \equiv \int_{\mathbb{R}^3} \frac{1}{\boldsymbol{M}} h(\xi)g(\xi)\,\mathrm{d}\xi,$$

for functions $h,\ g$ of ξ. With respect to this inner product, the sub-manifold of the fluid component is spanned by the following orthogonal basis:

$$\begin{aligned}
&\chi_0(\xi;\rho,u,\theta) \equiv \frac{1}{\sqrt{\rho}}\boldsymbol{M},\\
&\chi_i(\xi;\rho,u,\theta) \equiv \frac{\xi^i - u^i}{\sqrt{R\theta\rho}}\boldsymbol{M} \ \text{ for } \ i = 1, 2, 3,\\
&\chi_4(\xi;\rho,u,\theta) \equiv \frac{1}{\sqrt{6\rho}}\left(\frac{|\xi-u|^2}{R\theta} - 3\right)\boldsymbol{M},\\
&\langle \chi_\alpha, \chi_\beta\rangle = \delta_{\alpha\beta} \quad \text{for } \alpha, \beta = 0, 1, 2, 3, 4.
\end{aligned} \tag{5.5}$$

Hence the macroscopic projection $\boldsymbol{P}_0$ and microscopic projection $\boldsymbol{P}_1$ can be defined as

$$\boldsymbol{P}_0 h \equiv \sum_{\alpha=0}^{4} \langle h, \chi_\alpha\rangle \chi_\alpha, \qquad \boldsymbol{P}_1 h \equiv h - \boldsymbol{P}_0 h. \tag{5.6}$$

The operators $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ are projections, that is,

$$\boldsymbol{P}_0\boldsymbol{P}_0 = \boldsymbol{P}_0, \qquad \boldsymbol{P}_1\boldsymbol{P}_1 = \boldsymbol{P}_1.$$

Note that functions in the range of the microscopic projection $\boldsymbol{P}_1$ are non-fluid. It is clear that for the solution $f(x,t,\xi)$ of the Boltzmann equation,

$$\boldsymbol{P}_0 f = \boldsymbol{M}, \qquad \boldsymbol{P}_1 f = \boldsymbol{G}.$$

With $f = \boldsymbol{M} + \boldsymbol{G}$, the Boltzmann equation becomes

$$(\boldsymbol{M}+\boldsymbol{G})_t + \xi\cdot\nabla_x(\boldsymbol{M}+\boldsymbol{G}) - \nabla_x\phi\cdot\nabla_\xi(\boldsymbol{M}+\boldsymbol{G}) = \frac{1}{\kappa}(2Q(\boldsymbol{G},\boldsymbol{M}) + Q(\boldsymbol{G},\boldsymbol{G})). \tag{5.7}$$

Therefore, it can be rewritten as the conservation laws obtained, as usual, by integrating with respect to ξ of the Boltzmann equation times the collision invariants $\psi_\alpha(\xi)$:

$$\begin{aligned}
&\rho_t + \operatorname{div} m = 0,\\
&m_t^i + \left(\sum\nolimits_{j=1}^{3} u^j m^i\right)_{x^j} + p_{x^i} + \int_{\mathbb{R}^3} \psi_i(\xi\cdot\nabla_x\boldsymbol{G})\,\mathrm{d}\xi = -\rho\nabla_x\phi \quad \text{for } i = 1, 2, 3,\\
&\left[\rho\left(\frac{|u|^2}{2} + \mathscr{E}\right)\right]_t + \sum_{j=1}^{3}\left[u^j\left[\rho\left(\frac{|u|^2}{2}+\mathscr{E}\right) + p\right]\right]_{x^j} + \int_{\mathbb{R}^3}\psi_4(\xi\cdot\nabla_x\boldsymbol{G})\,\mathrm{d}\xi = -\rho u\cdot\nabla_x\phi,
\end{aligned} \tag{5.8}$$

where p is the pressure for the monatomic gases:

$$p = \tfrac{2}{3}\rho\mathscr{E},$$

coupled with

$$\boldsymbol{G}_t + \boldsymbol{P}_1(\xi \cdot \nabla_x \boldsymbol{G} + \xi \cdot \nabla_x \boldsymbol{M}) - \nabla_x \phi \cdot \nabla_\xi \boldsymbol{G} = \frac{1}{\kappa} L\boldsymbol{G} + \frac{1}{\kappa} Q(\boldsymbol{G}, \boldsymbol{G}), \tag{5.9}$$

for the non-fluid component G in the solution. Here

$$L\boldsymbol{G} \equiv Q(\boldsymbol{M} + G, \boldsymbol{M} + G) - Q(G, G). \tag{5.10}$$

In fact, (5.9) gives

$$\boldsymbol{G} = \kappa L^{-1}(\boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{M}) + L^{-1}(\kappa(\partial_t \boldsymbol{G} + \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{G} - \nabla_x \phi \cdot \nabla_\xi \boldsymbol{G}) - Q(\boldsymbol{G}, \boldsymbol{G})), \tag{5.11}$$

so that (5.8) results in, for $i = 1, 2, 3$,

$$\begin{aligned}
&\rho_t + \operatorname{div} m = 0,\\
&m^i_t + \left(\sum_{j=1}^{3} u^j m^i\right)_{x^j} + p_{x^i} + \kappa \int_{\mathbb{R}^3} \psi_i (\xi \cdot \nabla_x L^{-1} \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{M})\, \mathrm{d}\xi\\
&\quad + \int_{\mathbb{R}^3} \psi_i (\xi \cdot \nabla_x L^{-1}(\kappa[\boldsymbol{G}_t + \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{G} - \nabla_x \phi \cdot \nabla_\xi \boldsymbol{G}] - Q(\boldsymbol{G}, \boldsymbol{G})))\, \mathrm{d}\xi = -\rho \nabla_x \phi,\\
&\left[\rho\left(\frac{|u|^2}{2} + \mathscr{E}\right)\right]_t + \sum_{j=1}^{3}\left[u^j\left[\rho\left(\frac{|u|^2}{2} + \mathscr{E}\right) + p\right]\right]_{x^j} + \kappa \int_{\mathbb{R}^3} \psi_4 (\xi \cdot \nabla_x L^{-1} \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{M})\, \mathrm{d}\xi\\
&\quad + \int_{\mathbb{R}^3} \psi_4 (\xi \cdot \nabla_x L^{-1}(\kappa[\boldsymbol{G}_t + \boldsymbol{P}_1 \xi \cdot \nabla_x \boldsymbol{G} - \nabla_x \phi \cdot \nabla_\xi \boldsymbol{G}] - Q(\boldsymbol{G}, \boldsymbol{G})))\, \mathrm{d}\xi = -\rho u \cdot \nabla_x \phi.
\end{aligned} \tag{5.12}$$

The fluid equations, the Euler and Navier–Stokes equations, are in fact part of the above equations. For instance, when the Knudsen number κ is set to 0, system (5.12) becomes the Euler equations as in the Hilbert expansion. When the microscopic part G is set to be 0 in (5.12), it becomes the Navier–Stokes equations as in the Chapman-Enskog expansion. These fluid equations as derived through the Hilbert and Chapman–Enskog expansions are approximations to the Boltzmann equation [11]. Here they are just part of the full Boltzmann equation. Nevertheless, this approach is consistent in spirit with these expansions in that the higher-order terms beyond the first order in the expansions must satisfy a solvability condition, which means that these terms are microscopic.

In the above system, the terms with second space derivative on $\boldsymbol{M}$ are the viscosity and heat conductivity terms in the Navier–Stokes equations which are independent of the density gradient $\nabla_x \rho$.

To have a nontrivial solution containing the vacuum, one can just assume $\phi(x) \to \infty$ as $|x| \to \infty$ so that

$$f = \beta \exp\left(-\alpha\left(\phi(x) + \frac{|\xi|^2}{2}\right)\right)$$

is the stationary solution for any positive constants α and β. Here, we consider the simple case with one-dimensional gravitational force:

$$f_t + \xi \cdot \nabla_x f - g f_{\xi^1} = \frac{1}{\kappa} Q(f, f), \quad x^1 \geqslant 0. \tag{5.13}$$

It is straightforward to check that this equation admits the following two parameter solutions:

$$f = \frac{\rho(x)}{(2\pi R\theta_0)^{3/2}} \exp\left(-\frac{\xi^2}{2\pi R\theta_0}\right), \qquad \rho = \rho_0 \exp\left(\frac{-gx^1}{\pi R\theta_0}\right), \tag{5.14}$$

where the parameters are ρ_0 and θ_0 which can be any positive constants. Here ρ and θ_0 are the density and temperature of the gas, respectively, and R is the gas constant. Notice that the solutions connect to vacuum at $x^1 = \infty$ with exponential decay and are local Maxwellians. Therefore, the collision kernel $Q(f, f)$ is identically zero for these solutions and the behavior of the vanishing Knudsen number cannot be seen in these solutions. However, it is interesting to notice that the corresponding solutions in the fluid variables, i.e.,

$$\rho = \rho_0 \exp\left(\frac{-gx^1}{\pi R\theta_0}\right), \quad u = 0, \quad \theta = \theta_0,$$

are the solutions to both Euler equations

$$\begin{aligned}
&\rho_t t + \sum_{i=1}^{3} (\rho u^i)_{x^i} = 0, \\
&(\rho u^j)_t + \sum_{i=1}^{3} (\rho u^j u^i)_{x^i} + p_{x^j} = -g\rho\delta_j^1, \quad j = 1, 2, 3, \\
&\left(\rho\left(\frac{|u|^2}{2} + E\right)\right)_t + \sum_{i=1}^{3} \left(\left(\rho\left(\frac{|u|^2}{2} + E\right) + p\right) u^i\right)_{x^i} = 0,
\end{aligned} \tag{5.15}$$

and Navier–Stokes equations

$$\begin{aligned}
&\rho_t t + \sum_{i=1}^{3} (\rho u^i)_{x^i} = 0, \\
&(\rho u^j)_t + \sum_{i=1}^{3} (\rho u^j u^i)_{x^i} + p_{x^j} = -g\rho\delta_j^1 + \nabla_x \cdot (\mu_1 \nabla_x u^j + \mu_1 u_{x^j}) + \left(\left(\mu_2 - \frac{2}{3}\mu_1\right) \nabla_x \cdot u\right)_{x^j}, \\
&\left(\rho\left(\frac{|u|^2}{2} + E\right)\right)_t + \sum_{i=1}^{3} \left(\left(\rho\left(\frac{|u|^2}{2} + E\right) + p\right) u^i\right)_{x^i} \\
&\quad = \nabla_x \cdot \left(\mu_1 (u \cdot \nabla_x) u + \frac{1}{2}\mu_1 \nabla_x (u \cdot u)\right. \\
&\qquad \left. + \left(-\frac{2}{3}\mu_1 + \mu_2\right) (\nabla_x \cdot u) u + \kappa \nabla_x \theta\right), \quad j = 1, 2, 3,
\end{aligned} \tag{5.16}$$

with gravitational force, where $\mu_1 = \mu_1(\theta)$ and $\mu_2 = \mu_2(\theta)$ are the coefficients of viscosity and $\kappa = \kappa(\theta)$ is the heat conductivity.

It is interesting to study the other more general solutions to (5.13) which are not local Maxwellians in order to see the asymptotic behavior of the solutions when the Knudsen number κ tends to 0 and the stability of the above special solutions, to understand their generality in the setting of connecting to vacuum at $x^1 = \infty$.

As we can see from the previous sections, a lot of work has been done on Euler and Navier–Stokes equations with vacuum. When we consider the Euler equations (5.15) and the Navier–Stokes equations (5.16) with constant viscosity coefficient and heat conductivity, we can find solutions connecting to vacuum in finite distance with temperature also being 0 at vacuum boundary:

$$\rho = \left(a - \frac{g}{c\mathrm{e}^s}x^1\right)^{1/(\gamma-1)}, \quad u = 0, \quad \theta = R^{-1}\left(a - \frac{g}{c\mathrm{e}^s}x^1\right), \tag{5.17}$$

where $a > 0$, $c > 0$ and s are constants, and the pressure $p = k\rho^\gamma \mathrm{e}^s$. It is easy to check that the local Maxwellian corresponding to these solutions are not solutions to the Boltzmann equation (5.13). But whether they can be viewed as asymptotic states when the Knudsen number κ tends to 0 is not clear. However, the answer to this question may be “no” if we consider the problem as follows. When we derive the Navier–Stokes equations from the Boltzmann equation, the viscosity coefficient and the heat conductivity are not constant but functions of temperature. For the hard sphere model, they are proportional to the square root of the temperature. Under this consideration, it is straightforward to show that the corresponding stationary solutions to the Navier–Stokes are quite different from (5.17). That is, as for the hard sphere model, the density does not have compact support and the temperature tends to infinity as the gas tends to vacuum in the following way:

$$\rho \sim (ax+b)^{-2/3}\mathrm{e}^{-(ax+b)^{1/3}}, \quad \theta \sim (ax+b)^{2/3}, \tag{5.18}$$

where a and b are positive constants. Notice that the local Maxwellian corresponding to all these are not solutions to the Boltzmann equation (5.13). However, it would be interesting to find out whether there is any relation between these solutions and those for Boltzmann equation connecting to vacuum. Notice that the stability of solutions in the form of (5.17) was studied in [59] for isentropic Navier–Stokes equations.

There are many important results on the global existence on small perturbation of a global Maxwellian and the renormalized solutions which are not directly related to our study, and hence we do not present them here, cf. [17,21,65] and references therein. In the constructional proof of [33] on the local existence of solutions, the local existence and uniqueness of solutions were proved by constructing two sequences of approximate solutions, one sequence decreasing and the other increasing. The solutions thus obtained are local because the starting condition for the sequences holds only locally in time in general [33]. For a rarefied gas in an infinite vacuum, this method was used to prove global existence of solutions later in [31]. Recently this method is used in the case with external force, but the solution still tends to 0 as time goes to infinity [23]. The advantage of this approach is that it gives a constructional picture of the solution so that one can understand its behavior more clearly. Furthermore, the approach is non-linear because it does not use any linearization. However, this method cannot be applied to the study of vacuum problem with non-trivial profile directly because the splitting of the collision operator into gain and loss parts for the definition of upper and lower solution sequences does not satisfy the starting condition.

There are also many results on the Boltzmann equation about its relation to compressible or incompressible Euler equations and Navier–Stokes equations. For this, please refer to [3,4,37,42,58] and the references therein.

If we linearize around any of the above background solutions (local Maxwellians), the standard energy-type estimate does not work because the linearized operator with the decay density factor cannot be used to control the non-fluid part in the solutions of the Boltzmann equation. Hence, a full nonlinear approach different from [33] is needed.

So far the external force is assumed to be the gravitational force between the particle and a fixed region of finite mass. It will be more interesting to look at the case when radiation is also taken into account. Then, one needs to include the Maxwell equations where the electro-magnetic fields lead to other complexity and phenomena.

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