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Hypocoercivity of the relativistic Boltzmann and Landau equations in the whole space

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A R T I C L E I N F O

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A B S T R A C T

We study the hypocoercivity property for some kinetic equations in the whole space and obtain the optimal convergence rates of solutions to the equilibrium state in some function spaces. The analysis relies on the basic energy method and the compensating function introduced by Kawashima to the classical Boltzmann equation and developed by Glassey and Strauss in the relativistic setting. It is also motivated by the recent work (Duan et al., 2008 [8]) on the Boltzmann equation by combining the spectrum analysis and energy method. The advantage of the method introduced in this paper is that it can be applied to some complicated system whose detailed spectrum is not known. In fact, only some estimates through the Fourier transform on the conservative transport operator and the dissipation of the linearized operator on the subspace orthogonal to the collision invariants are needed.

Contents

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1. Introduction

In this paper, we consider the relativistic Boltzmann and Landau equations for the dynamics of dilute particles in the whole space. Recall that the relativistic Boltzmann equation takes the form

$$\partial_t F + \hat{v} \cdot \nabla_x F = \mathcal{C}(F, F), \tag{1.1}$$

with initial condition $F(0, x, v) = F_0(x, v)$. Here, $F(t, x, v)$ is the distribution function of the particles at time $t \geqslant 0$, located at $x = (x_1, x_2, x_3) \in \mathbf{R}^3$ with momentum $v = (v_1, v_2, v_3) \in \mathbf{R}^3$. We normalize the speed of light c and the particle mass m to unity. Then the relativistic velocity $\hat{v}$ is defined in the terms of momenta v by

$$\hat{v} = \frac{v}{v_0}, \quad v_0 \equiv \sqrt{1 + |v|^2}.$$

By using the quantities

$$s = 2(u_0 v_0 - u \cdot v + 1), \quad u_0 \equiv \sqrt{1 + |u|^2},$$
$$4g^2 = 2(u_0 v_0 - u \cdot v - 1) = s - 4,$$
$$v_M = \frac{2g\sqrt{1 + g^2}}{u_0 v_0} \equiv \text{Møller velocity},$$

the collision operator $\mathcal{C}(F, F)$ is in the form [3,4,16]

$$\mathcal{C}(F, G)(v) = \int_{\mathbf{R}^3} \int_{\mathbf{S}^2} v_M \sigma(g, \theta) \big[F(u') G(v') - F(u) G(v) \big] \, du \, d\omega, \tag{1.2}$$

where $d\omega$ is a surface measure on the unit sphere $\mathbf{S}^2$, and σ is the scattering kernel satisfying some conditions given later. As usual, we abbreviate $F(t, x, u)$ by $F(u)$, etc., and use prime to represent the moment after collision. For the relativistic model, the conservations of momentum and energy are given by

$$u + v = u' + v', \qquad \sqrt{1 + |u|^2} + \sqrt{1 + |v|^2} = \sqrt{1 + |u'|^2} + \sqrt{1 + |v'|^2}, \tag{1.3}$$

for $u, v \in \mathbf{R}^3$. In (1.2), the scattering angle θ is defined as follows. For given 4 dimension vectors $U = (u_0, u_1, u_2, u_3)$ and $V = (v_0, v_1, v_2, v_3)$, set $U \cdot V = u_0 v_0 - \sum_{k=1}^{3} u_k v_k$ (which is the Lorentz inner product). Then angle θ is given by

$$\cos\theta = \frac{(U - V) \cdot (U' - V')}{(U - V) \cdot (U - V)}.$$

On the other hand, Landau in 1936, introduced the kinetic equation to model a dilute plasma in which particles interact by Coulomb force. When particle velocities are close to the speed of light, the relativistic effect becomes important. The relativistic version of Landau equation was proposed by Belyaev and Budker in 1956 [1], which is a fundamental model to describe the dynamics of a dilute, collisional and fully ionized plasma. It also takes the same form as (1.1) but with a different collision operator

$$\mathcal{C}(g,h)(v) = \nabla_v \cdot \left[\int_{\mathbf{R}^3} \Phi(u,v)\big[h(u)\nabla_v g(v) - g(v)\nabla_u h(u)\big]\, du \right]. \tag{1.4}$$

Here, $\Phi(u,v)$ is a 3×3 matrix given by

$$\Phi(u,v) = \Lambda(u,v) S(u,v),$$

with

$$\Lambda(u,v) = \frac{1}{u_0 v_0}(u_0 v_0 - u\cdot v)^2\big[(u_0 v_0 - u \cdot v)^2 - 1\big]^{\gamma/2},$$
$$S(u,v) = \big[(u_0 v_0 - u\cdot v)^2 - 1\big] I_3 - u \otimes u - v \otimes v + (u_0 v_0 - u \cdot v)(u \otimes v + v \otimes u),$$

where γ is a parameter leading to the standard classification of the hard potential ($\gamma > 0$), Maxwellian molecule ($\gamma = 0$) or soft potential ($\gamma < 0$) [26]. The following study on this model is restricted to the physically interesting case when $\gamma = -3$, that is, the Coulomb interaction in plasma physics.

Since the analysis is about solutions around a global equilibrium state, without loss of generality, we normalize this global relativistic Maxwellian to $\mu(v) \equiv e^{-\sqrt{1+v^2}}$. Then as before, set the perturbation $f(t,x,v)$ around $\mu(v)$ by $F = \mu + \sqrt{\mu} f$. The equation for the perturbation $f(t,x,v)$ becomes

$$\partial_t f + \hat{v} \cdot \nabla_x f + Lf = \Gamma(f,f), \tag{1.5}$$

with $f(0,x,v) = f_0(x,v)$. Here, the linearized collision operator is

$$Lf = \mu^{-1/2}\big\{\mathcal{C}\big(\mu, \mu^{1/2} f\big) + \mathcal{C}\big(\mu^{1/2} f, \mu\big)\big\},$$

and the non-linear collision operator is

$$\Gamma(g_1, g_2) = \mu^{-1/2}\mathcal{C}\big(\mu^{1/2} g_1, \mu^{1/2} g_2\big).$$

It is known that for the relativistic Boltzmann equation, L can be written as $Lf = \nu(v) f - Kf$ with the collision frequency $\nu(v)$ defined by

$$\nu(v) = \int_{\mathbf{R}^3}\int_{\mathbf{S}^2} v_M \sigma(g,\theta)\mu(u)\, d\omega\, du, \tag{1.6}$$

and the operator K by

$$Kf = \int_{\mathbf{R}^3}\int_{\mathbf{S}^2} v_M \sigma(g,\theta)\mu^{1/2}(u)\big[\mu^{1/2}(u')f(v') + \mu^{1/2}(v')f(u') - \mu^{1/2}(v)f(u)\big]\, du\, d\omega.$$

For the scattering kernel in the relativistic Boltzmann equation, as in [13,14], we assume

$$c_1 \frac{g^{\beta+1}}{1+g} \sin^\gamma \theta \leqslant \sigma(g,\theta) \leqslant c_2\big(g^\beta + g^{-\delta}\big) \sin^\gamma \theta, \tag{1.7}$$

where c_1 and c_2 are positive constants, $0 \leqslant \delta < 1/2$, $0 \leqslant \beta < 2-2\delta$, and either $\gamma \geqslant 0$ or

$$|\gamma| < \min\left\{2-\beta, \frac{1}{2}-\delta, \frac{1}{3}(2-2\delta-\beta)\right\}.$$

Under these conditions on $\sigma(g,\theta)$, it was shown in [9] that K is compact on $L^2(\mathbf{R}^3)$. And from [13,14], we know that there is a constant $C > 1$ such that

$$C^{-1} v_0^{\beta/2} \leqslant \nu(v) \leqslant C v_0^{\beta/2} \quad \text{for all } v \in \mathbf{R}^3. \tag{1.8}$$

Note that for the relativistic Landau equation, the collision frequency is defined as

$$\sigma^{ij}(v) = \int_{\mathbf{R}^3} \Phi^{ij}(u,v)\mu(u)\,du. \tag{1.9}$$

By the H-theorem, L is dissipative and the null space of L is spanned by the five collision invariants

$$\mathcal{N} = \operatorname{span}\{\sqrt{\mu}, v_j\sqrt{\mu}, v_0\sqrt{\mu}\}, \quad 1 \leqslant j \leqslant 3.$$

Let $\mathbf{P}$ be the projection of the space $L^2(\mathbf{R}^3)$ to the null space $\mathcal{N}$ in v variable. We can decompose $f(t,x,v)$ as

$$f(t,x,v) = \mathbf{P}f + (\mathbf{I}-\mathbf{P})f. \tag{1.10}$$

Here, $\mathbf{P}f$ represents the macroscopic part and $(\mathbf{I}-\mathbf{P})f$ the microscopic part respectively.

For the later presentation, we need the following notations. Denote $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and $\beta = (\beta_1,\beta_2,\beta_3)$ and

$$\partial^\alpha_\beta \equiv \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}.$$

We use $C^{\bar\beta}_\beta$ to denote the binomial coefficient $\binom{\beta}{\bar\beta}$. To discuss the optimal time decay of solutions to (1.5), the space $Z_q = L^2(\mathbf{R}^3_v; L^q(\mathbf{R}^3_x))$ is used and its norm is given by

$$\|f\|_{Z_q} = \left(\int_{\mathbf{R}^3}\left(\int_{\mathbf{R}^3}\big|f(x,v)\big|^q\,dx\right)^{2/q} dv\right)^{1/2}.$$

In the following, we shall use $\langle\cdot,\cdot\rangle$ to denote the standard L^2 inner product in $\mathbf{R}^3_v$ and $(\cdot,\cdot)$ to denote the standard L^2 inner product in $\mathbf{R}^3_x \times \mathbf{R}^3_v$. And we use $|\cdot|_2$ to denote the L^2 norm in $\mathbf{R}^3_v$ and $\|\cdot\|$ to denote the L^2 norms in $\mathbf{R}^3_x \times \mathbf{R}^3_v$.

For both the relativistic Boltzmann equation and the Landau equation, some weighted norms will be used. Precisely, for the relativistic Boltzmann equation, we use

$$|g|_\nu^2=\int_{\mathbf{R}^3}\nu(v)g^2(v)\,dv,\qquad \|g\|_\nu^2=\int_{\mathbf{R}^3\times\mathbf{R}^3}\nu(v)g^2(v)\,dx\,dv.$$

And for the relativistic Landau equation with (1.9), we use

$$|g|_\sigma^2=\sum_{1\leqslant i,j\leqslant 3}\int_{\mathbf{R}^3}\left\{\sigma^{ij}\partial_i g\partial_j g+\sigma^{ij}\frac{v_iv_j}{v_0^2}g^2\right\}dv,$$

$$\|g\|_\sigma^2=\sum_{1\leqslant i,j\leqslant 3}\int_{\mathbf{R}^3\times\mathbf{R}^3}\left\{\sigma^{ij}\partial_i g\partial_j g+\sigma^{ij}\frac{v_iv_j}{v_0^2}g^2\right\}dv\,dx.$$

Notice that from [26,32], there exists $C>1$ such that

$$C^{-1}\left\{|\nabla_v g|_2^2+|g|_2^2\right\}\leqslant |g|_\sigma^2\leqslant C\left\{|\nabla_v g|_2^2+|g|_2^2\right\}. \tag{1.11}$$

Since in some part of the presentation, the estimates for both systems are in the same form, we will sometimes use a unified notation $|g|_{\mathcal{D}}$ and $\|g\|_{\mathcal{D}}$ to denote either $|g|_\nu$ or $|g|_\sigma$, $\|g\|_\nu$ or $\|g\|_\sigma$ without ambiguity.

In terms of the energy functionals, for the relativistic Boltzmann equation (1.5) and any $l\geqslant 0$, we use

$$\mathcal{E}_l(f)(t)\sim\widetilde{\mathcal{E}}_l(f)(t)=\sum_{|\alpha|\leqslant N}\left\|\nu^l\partial^\alpha f(t)\right\|^2. \tag{1.12}$$

Here and in the following, $\mathcal{A}\sim\mathcal{B}$ means that there exist two generic positive constants C_1 and C_2 such that $C_1\mathcal{A}\leqslant\mathcal{B}\leqslant C_2\mathcal{A}$. Correspondingly, the dissipation functional for (1.5) is given by

$$\mathcal{D}_l(f)(t)\sim\widetilde{\mathcal{D}}_l(f)(t)=\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha\mathbf{P}f(t)\right\|^2+\sum_{|\alpha|\leqslant N}\left\|\nu^l\partial^\alpha(\mathbf{I}-\mathbf{P})f(t)\right\|_\nu^2. \tag{1.13}$$

For the relativistic Landau equation, the energy functional is

$$\mathcal{E}(t)\sim\widetilde{\mathcal{E}}(t)=\sum_{|\alpha|+\beta|\leqslant N}\left\|\partial_\beta^\alpha f(t)\right\|^2, \tag{1.14}$$

with dissipation rate

$$\mathcal{D}(t)\sim\widetilde{\mathcal{D}}(t)=\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha\mathbf{P}f(t)\right\|^2+\sum_{|\alpha|+|\beta|\leqslant N}\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f(t)\right\|_\sigma^2. \tag{1.15}$$

In the following discussion, we will choose the highest order of differentiation with respect to any variable to be N with $N\geqslant 4$. Notice that for the optimal convergence rate analysis, we need some weight in the microscopic momentum for the relativistic Boltzmann equation, but no differentiation with respect to the microscopic momentum. However, for the relativistic Landau equation, it is just opposite.

Throughout this paper, we use C to denote a generic positive constant which may vary from line to line. With the above preparation, the main results can be stated as follows.

Theorem 1.1. *For the relativistic Boltzmann equation, let* $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geqslant 0$ *and* $\sigma(g,\theta)$ *satisfy* (1.7). *There exists a sufficiently small constant* $\varepsilon > 0$ *such that if* $\mathcal{E}_l(f)(0) \leqslant \varepsilon$ *for any* $l \geqslant 0$, *then there exists a unique global solution* $f(t,x,v)$ *to* (1.5) *with* $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geqslant 0$ *satisfying*

$$\mathcal{E}_l(f)(t) + \int_0^t \mathcal{D}_l(f)(s)\,ds \leqslant C\mathcal{E}_l(f)(0). \tag{1.16}$$

If we further assume $l \geqslant 1$ *and* $\|f_0\|_{Z_1} \leqslant \varepsilon$, *then*

$$\|f(t)\|^2 = \|\mathbf{P}f(t)\|^2 + \|(\mathbf{I}-\mathbf{P})f(t)\|^2 \leqslant C(1+t)^{-3/2}, \tag{1.17}$$

$$\sum_{1\leqslant|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P}f(t)\|^2 + \sum_{|\alpha|\leqslant N} \|\nu^l \partial^\alpha(\mathbf{I}-\mathbf{P})f(t)\|^2 \leqslant C(1+t)^{-5/2}. \tag{1.18}$$

Theorem 1.2. *For the relativistic Landau equation, let* $F_0(x,v) = \mu + \sqrt{\mu} f_0(x,v) \geqslant 0$. *There exists a sufficiently small constant* $\varepsilon > 0$ *such that if* $\mathcal{E}(0) \leqslant \varepsilon$, *then there exists a unique global classical solution* $f(t,x,v)$ *to* (1.5). *Moreover,* $F(t,x,v) = \mu + \sqrt{\mu} f(t,x,v) \geqslant 0$ *satisfying*

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(s)\,ds \leqslant C\mathcal{E}(0). \tag{1.19}$$

If we further assume that $\|f_0\|_{Z_1} \leqslant \varepsilon$, *then*

$$\|f(t)\|^2 = \|\mathbf{P}f(t)\|^2 + \|(\mathbf{I}-\mathbf{P})f(t)\|^2 \leqslant C(1+t)^{-3/2}, \tag{1.20}$$

$$\sum_{1\leqslant|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P}f(t)\|^2 + \sum_{|\alpha|+|\beta|\leqslant N} \|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f(t)\|^2 \leqslant C(1+t)^{-5/2}. \tag{1.21}$$

Finally in the introduction, let us review some related works on the Boltzmann equation and the Landau equation. So far, there are several energy methods for the study on the Boltzmann equations near Maxwellian. The earliest one is based on the spectral analysis of the linearized Boltzmann equation and a bootstrapping argument which was initiated by Grad [15] and developed by Ukai [36,37], where the optimal convergence rate to the Maxwellian was given in some weighted L^∞ space in the microscopic velocity. The direct energy method through the macro–micro decomposition which was initiated by Liu and Yu [29] and developed by Liu, Yang and Yu [28], and Guo [17,19] independently in two different ways. In between there is another energy method introduced by Kawashima [25] which is based on the compensating function through the thirteen moments decomposition. For this, the compensating function and the fourteen moments decomposition was given by Glassey and Strauss [14] for the relativistic Boltzmann equation.

Notice that by using only the energy estimate, the optimal convergence rate is difficult to obtain. Recently, an approach based on the combination of the spectrum method and energy method was introduced by Duan, Ukai, Yang and Zhao [8] to obtain the optimal convergence rates in various settings. However, for some complicated equations or systems, if we do not know the detailed structure of the spectrum, it is difficult to apply the method in [8]. For this, the approach introduced in this paper which is based on the compensating function and the energy method is shown to be useful. In this approach, we only need the interaction between conservative operator $\partial_t + v\cdot\nabla_x$ and the dissipation of the linearized operator on the subspace orthogonal to the space spanned by the collision invariants, and some analysis on the non-linear collision operators. And this is exactly the hypocoercivity property studied by Villani and his collaborators recently. In fact, the hypocoercivity theory which is closely related to, but is different from, the hypo-ellipticity theory has become one of the main focuses in the study of problems from mathematical physics. The main feature of this theory is that the coupling of a degenerate diffusion operator and a conservative operator may give the dissipation in all variables, and the convergence to the equilibrium state which lies in a subspace smaller than the kernel of the diffusion operator. The hypocoercivity theory has been investigated extensively for physical models which include the Boltzmann equation, oscillator chains, Fokker–Planck equation, etc., cf. [2,20,30,38–40,31,35] and references therein. And some elegant theorems on the (multiple) commutators in the spirit of Hörmander's celebrated regularity theorem for hypo-ellipticity phenomena have been established, cf. [39].

Related to the hypocoercivity phenomena, there are a lot of works on the convergence rate for the solutions to the kinetic equation. For example, in either a torus or a bounded domain, Desvillettes and Villani [7] developed a framework for the study on the trend to equilibriums for large solutions with an almost exponential decay rate. And Strain and Guo [34] obtained the exponential decay to Maxwellian in the torus for solutions to the Boltzmann and Landau equations with soft potentials near a global Maxwellian. Recently, Duan, Ukai, Yang and Zhao [8] proved the optimal decay estimates on the Cauchy problem of the Boltzmann equation with the external force by combining the energy method and the spectrum analysis. However, so far it seems difficult to apply this method to the relativistic Landau equation because there is no detailed information on the spectrum for this equation with Coulomb interaction. Therefore, the method introduced in this paper has its own advantage and can be applied to the study on more complicated equation or systems.

We now turn back to the problems considered in this paper. Firstly, the general background of the relativistic equations can found in the references [3,4,16,1,21,27]. In fact, a lot of works have been done on the relativistic Boltzmann equation, cf. [3,9,10,12–14,22] and references therein. More precisely, the linearized relativistic Boltzmann equation was solved by Dudyński and Ekiel-Jezewska in [9]. Later Glassey and Strauss obtained the global solution of the relativistic Boltzmann equation near a relativistic Maxwellian in the torus [14], where a more restrictive assumption on the scattering kernel $\sigma(g,\theta)$ is imposed and the solution space is either

$$L^\infty_{\beta_1}\big(\mathbf{R}^3_v; H^{k_1}\big(\mathbf{R}^3_x\big)\big); \quad \text{or} \quad L^\infty_{\beta_1}\big(\mathbf{R}^3_v, C^{k_2}\big(\mathbf{R}^3_x\big)\big), \quad \beta_1 > \frac{3}{2},\ k_1 \geqslant 2,\ k_2 \geqslant 0.$$

Here,

$$L^\infty_{\beta_1}\big(\mathbf{R}^3_v\big) \equiv \big\{ f \;\big|\; \big(1+|v|\big)^{\beta_1} f(v) \in L^\infty\big(\mathbf{R}^3_v\big)\big\}.$$

By adopting Kawashima's compensating function method, in [14], the optimal decay of $(1+t)^{-\frac{3}{4}}$ was obtained for the perturbation in the function space $L^\infty_{\beta_1}(\mathbf{R}^3_v; H^{k_1}(\mathbf{R}^3_x))$, $\beta_1 > \frac{3}{2}$, $k_1 \geqslant 2$. Since $L^\infty_{\beta_1}(\mathbf{R}^3_v) \subset L^2(\mathbf{R}^3_v)$, the function space for solutions in this paper is larger than this space.

On the other hand, the classical (non-relativistic) Landau equation has also been extensively investigated, cf. [5,6,17,41–43] and references therein. Precisely, Desvillettes and Villani [6] proved global existence and uniqueness of classical solutions for spatially homogeneous Landau equation for hard potentials. Degond and Lemou [5] studied the spectral properties and dispersion relation of linearized Landau operator. Guo [17] constructed global classical solutions near a global Maxwellian in a periodic box by energy method. Hsiao and Yu extended Guo's results to the whole space in [24].

Nevertheless, there are less results on the relativistic Landau equation. For this, Lemou [26] extended the result in [5] to the relativistic setting. Recently, global solutions for the relativistic Landau–Maxwell system near a relativistic Maxwellian in a torus were constructed in [32] based on the energy method, and the almost exponential time decay rate to the equilibrium was obtained in [33]. Global solutions to the relativistic Landau equation (1.5) near a relativistic Maxwellian in the whole space were constructed in [23] where the solution space contains the temporal derivatives. Here, the analysis in this paper gives both the global existence and the optimal convergence rate to the equilibrium state, and it does not involve any estimation on the temporal derivatives. Therefore, this method is more direct and robust than the previous energy method on this system.

The rest of the paper will be organized as follows. In the next section, we will review the compensating function developed by Kawashima and Glassey–Strauss and give some basic estimates on the linear equation. The optimal convergence rates for the relativistic Boltzmann equation and the relativistic Landau equation will be given in Sections 3 and 4 respectively.

2. Compensating functions

In this section, we will recall the compensating functions for the relativistic Boltzmann and Landau equations. Most of the estimates can be found in the paper by [14] and we include some derivation here for the convenience of the readers.

As in [14,25], the subspace $\widetilde{W}$ for the 14 moments is defined as the space generated by $\mathcal{N}$ and the images of $\mathcal{N}$ under the mappings $f(v) \mapsto \hat{v}_j f(v)$ $(j = 1, 2, 3)$. That is,

$$\widetilde{W} = \mathrm{span}\{\sqrt{\mu}\varphi_j \mid j = 1, \dots, 14\},$$

where

$$\varphi_1 = 1, \qquad \varphi_{j+1} = v_j, \qquad \varphi_5 = v_0, \qquad \varphi_{j+5} = v_j \hat{v}_j,$$
$$\varphi_9 = v_1 \hat{v}_2, \qquad \varphi_{10} = v_2 \hat{v}_3, \qquad \varphi_{11} = v_3 \hat{v}_1, \qquad \varphi_{j+11} = \hat{v}_j \quad (j = 1, 2, 3).$$

Here, $\mathcal{N} \subset \widetilde{W}$ and the operator of multiplication by $\hat{v}\cdot$ maps $\mathcal{N}$ into $\widetilde{W}$. Denote an orthogonal basis for this 14-dimensional space by e_j, $1 \leqslant j \leqslant 14$, as in [14]. Let P_0 be the orthogonal projection from $L^2(\mathbf{R}^3_v)$ onto $\widetilde{W}$:

$$P_0 f = \sum_{k=1}^{14} \langle f, e_k \rangle e_k.$$

We will now deduce compensating function of the relativistic model equations

$$[\partial_t + \hat{v} \cdot \nabla_x + L]h = g, \tag{2.1}$$

with initial data $h(0, x, v) = h_0(x, v)$ and a source term g. Set $W_k = \langle h, e_k \rangle$. Then we have by using (2.1) that

$$\partial_t W + \sum_j V^j \partial_{x_j} W + \bar{L} W = \bar{g} + R,$$

where V^j $(j = 1, 2, 3)$ and $\bar{L}$ are the symmetric matrices given by

$$\bar{L} = \big\{\big\langle L[e_l], e_k\big\rangle\big\}_{k,l=1}^{14}, \qquad V(\xi) = \sum_{j=1}^{3} V^j \xi_j = \big\{\big\langle (\hat{v} \cdot \xi) e_k, e_l\big\rangle\big\}_{k,l=1}^{14},$$

and $\bar{g}$ is the vector component $\langle g, e_j \rangle$. Here R is the remaining term which has the factor $(\mathbf{I} - P_0)f$. For later use, we will use $\mathcal{R}z$ for the real part of $z \in \mathbf{C}$, and

$$W = [W_I, W_{II}]^T, \qquad W_I = [W_1, \dots, W_5]^T, \qquad W_{II} = [W_6, \dots, W_{14}]^T.$$

The following definition of compensating function was introduced by Kawashima. For later use, we also include the dissipation rate on the microscopic component $|(\mathbf{I} - \mathbf{P})h|^2_{\mathcal{D}}$ in (iii) of the definition.

Definition 2.1. A bounded linear operator $S(\omega)$ with $\omega \in \mathbf{S}^2$ on $L^2(\mathbf{R}^3)$ is called a compensating function for (2.1) if

(i) $S(\cdot)$ is C^∞ on $\mathbf{S}^2$ with values in the space of bounded linear operators on $L^2(\mathbf{R}^3)$, and $S(-\omega) = -S(\omega)$ for all $\omega \in \mathbf{S}^2$,
(ii) $iS(\omega)$ is self-adjoint on $L^2(\mathbf{R}^3)$ for all $\omega \in \mathbf{S}^2$,
(iii) there exists $c_0 > 0$ such that for all $h \in L^2(\mathbf{R}^3)$ and $\omega \in \mathbf{S}^2$,

$$\mathcal{R}\big\langle S(\omega)(\hat{v}\cdot\omega)h, h\big\rangle + \langle Lh, h\rangle \geqslant c_0\big(|\mathbf{P}h|_2^2 + \big|(\mathbf{I}-\mathbf{P})h\big|_{\mathcal{D}}^2\big).$$

To construct the compensating function of the relativistic model equation, the following lemma was proved in [14].

Lemma 2.2. *There exist three* 14×14 *real constant skew-symmetric matrices* R^j $(j = 1, 2, 3)$ *and positive constants* c_1 *and* c_2 *such that*

$$R(\omega) \equiv \sum_{j=1}^{3} R^j \omega_j$$

satisfies

$$\mathcal{R}\big\langle\!\big\langle R(\omega)V(\omega)W, W\big\rangle\!\big\rangle \geqslant c_1|W_I|^2 - c_2|W_{II}|^2$$

for all $W \in \mathbf{C}^{14}$. *Here* $\langle\langle\cdot,\cdot\rangle\rangle$ *represents the inner product on* $\mathbf{C}^{14}$.

Now a compensating function for the relativistic equation (2.1) can be defined as follows. Given $\omega \in \mathbf{S}^2$, set $R(\omega) \equiv \{r_{ij}(\omega)\}_{i,j=1}^{14}$ and let

$$S(\omega)h \equiv \sum_{k,\ell=1}^{14} \lambda r_{k\ell}(\omega)\langle h, e_\ell\rangle e_k \quad \text{for some } \lambda > 0,\ h \in L^2\big(\mathbf{R}^3\big). \tag{2.2}$$

The following lemma is also from [14]. We include the proof here to show that the dissipation rate $|(\mathbf{I}-\mathbf{P})h|_{\mathcal{D}}^2$ can also be included.

Lemma 2.3. *There exists* $\lambda > 0$ *such that* $S(\omega) : L^2(\mathbf{R}^3) \to \widetilde{W}$ *is a compensating function for the relativistic equation* (2.1).

Proof. Recall that $R(\omega) \equiv \sum_{j=1}^{3} R^j\omega_j$, where R^j is a real constant 14×14 and skew-symmetric matrix. Moreover, $S(\cdot)$ is $C^\infty(\mathbf{S}^2)$ and $S(-\omega) = -S(\omega)$.

Let $h_1, h_2 \in L^2(\mathbf{R}_v^3)$. Then

$$\big\langle S(\omega)h_1, h_2\big\rangle = \sum_{k,\ell=1}^{14} r_{k\ell}(\omega)\langle h_1, e_\ell\rangle\overline{\langle h_2, e_k\rangle}. \tag{2.3}$$

Set

$$W = \{W_k\}_{k=1}^{14} = \big\{\langle h_1, e_k\rangle\big\}_{k=1}^{14}, \qquad u = \{u_k\}_{k=1}^{14} = \big\{\langle h_2, e_k\rangle\big\}_{k=1}^{14}.$$

Then

$$\big\langle S(\omega)h_1, h_2\big\rangle = \lambda\big\langle\big\langle R(\omega)W, u\big\rangle\big\rangle,$$

and

$$\big\langle iS(\omega)h_1, h_2\big\rangle = \lambda\big\langle\big\langle iR(\omega)W, u\big\rangle\big\rangle.$$

Since $R(\omega)$ is skew symmetric, $iS(\omega)$ is self-adjoint.

From (2.2), it is straightforward to show that

$$\big\langle S(\omega)(\hat{v}\cdot\omega)h, h\big\rangle \equiv \sum_{k,\ell=1}^{14} \lambda r_{k\ell}(\omega)\big\langle(\hat{v}\cdot\omega)h, e_\ell\big\rangle\overline{\langle h, e_k\rangle}.$$

Put $h = P_0h + (\mathbf{I} - P_0)h$. Then for $W_j = \langle h, e_j\rangle$, we have

$$\begin{aligned}\big\langle(\hat{v}\cdot\omega)h, e_\ell\big\rangle &= \big\langle(\hat{v}\cdot\omega)P_0h, e_\ell\big\rangle + \big\langle(\hat{v}\cdot\omega)(\mathbf{I} - P_0)h, e_\ell\big\rangle \\ &= \sum_{j=1}^{14} W_j \sum_{p=1}^{3}\langle e_j, \omega_p\hat{v}_p e_\ell\rangle + \big\langle(\hat{v}\cdot\omega)(\mathbf{I} - P_0)h, e_\ell\big\rangle.\end{aligned}$$

Notice that

$$\sum_{j=1}^{14} W_j \sum_{p=1}^{3}\langle e_j, \omega_p\hat{v}_p e_\ell\rangle = \sum_{j=1}^{14} W_j \sum_{p=1}^{3}\omega_p\big(V^p\big)_{j\ell} = \sum_{j=1}^{14} W_j\big(V(\omega)\big)_{j\ell} = \sum_{j=1}^{14} W_j\big(V(\omega)\big)_{\ell j}.$$

Thus, we have

$$\begin{aligned}\mathcal{R}\big\langle S(\omega)(\hat{v}\cdot\omega)h, h\big\rangle &= \mathcal{R}\lambda\sum_{k,\ell=1}^{14} r_{k\ell}(\omega)\sum_{j=1}^{14} V_{\ell j}(\omega)W_j\overline{W}_k + \mathcal{R}\lambda\sum_{k,\ell=1}^{14} r_{k\ell}\big\langle(\hat{v}\cdot\omega)(\mathbf{I} - P_0)h, e_\ell\big\rangle\overline{W}_k \\ &= \mathcal{R}\lambda\big\langle\big\langle R(\omega)V(\omega)W, W\big\rangle\big\rangle + \lambda\mathcal{R}\sum_{k,\ell=1}^{14} r_{k\ell}\big\langle(\hat{v}\cdot\omega)(\mathbf{I} - P_0)h, e_\ell\big\rangle\overline{\langle h, e_k\rangle} \\ &\geqslant \lambda\big[c_3|\mathbf{P}h|_2^2 - c_4\big|(\mathbf{I} - \mathbf{P})h\big|_{\mathcal{D}}^2\big] + \lambda\mathcal{R}\sum_{k,\ell=1}^{14} r_{k\ell}\big\langle(\hat{v}\cdot\omega)(\mathbf{I} - P_0)h, e_\ell\big\rangle\overline{\langle h, e_k\rangle},\end{aligned}$$

where we have used Lemma 2.2 and the exponential decay property of e_k in v.

Since

$$\big|\big\langle(\hat{v}\cdot\omega)(\mathbf{I} - P_0)h, e_\ell\big\rangle\big| \leqslant c_5\big|(\mathbf{I} - P_0)h\big|_2 \leqslant c_5\big|(\mathbf{I} - \mathbf{P})h\big|_{\mathcal{D}},$$

we have

$$\begin{aligned}\mathcal{R}\big\langle S(\omega)(\hat{v}\cdot\omega)h, h\big\rangle &\geqslant \lambda\big[c_3|\mathbf{P}h|_2^2 - c_4\big|(\mathbf{I} - \mathbf{P})h\big|_{\mathcal{D}}^2\big] - c_5\lambda|h|_2\big|(\mathbf{I} - \mathbf{P})h\big|_{\mathcal{D}} \\ &\geqslant \lambda(c_3 - \varepsilon)|\mathbf{P}h|_2^2 - \lambda c_\varepsilon\big|(\mathbf{I} - \mathbf{P})h\big|_{\mathcal{D}}^2,\end{aligned}$$

where $\varepsilon > 0$ is a small constant. Since the linearized operator has the following dissipative property, cf. [13,32],

$$\langle Lh, h\rangle \geqslant \delta_0 \big|(\mathbf{I}-\mathbf{P})h\big|_{\mathcal{D}}^2 \quad \text{for some } \delta_0 > 0.$$

A suitable combination of the above estimates yields that there exists $c_0 > 0$ such that

$$\mathcal{R}\big\langle S(\omega)(\hat{v}\cdot\omega)h, h\big\rangle + \langle Lh, h\rangle \geqslant c_0\big(|\mathbf{P}h|_2^2 + \big|(\mathbf{I}-\mathbf{P})h\big|_{\mathcal{D}}^2\big). \tag{2.4}$$

And this shows that $S(\omega)$ is a compensating function and completes the proof of the lemma. □

We now use the compensating function $S(\omega)$ to derive an energy estimate. Set $\omega = \xi/|\xi|$ and take the Fourier transform in x of (2.1). We have

$$\partial_t \hat{h} + i|\xi|(\hat{v}\cdot\omega)\hat{h} + L\hat{h} = \hat{g}. \tag{2.5}$$

By multiplying (2.5) by the conjugate of $\hat{h}$, we have

$$\frac{1}{2}\partial_t|\hat{h}|_2^2 + \langle L\hat{h}, \hat{h}\rangle = \mathcal{R}\langle \hat{h}, \hat{g}\rangle. \tag{2.6}$$

Then applying $-i|\xi|S(\omega)$ to (2.5) gives

$$-i|\xi|S(\omega)\partial_t\hat{h} + |\xi|^2 S(\omega)\big((\hat{v}\cdot\omega)\hat{h}\big) - i|\xi|S(\omega)L\hat{h} = -i|\xi|S(\omega)\hat{g}. \tag{2.7}$$

The inner product of the above equation with $\hat{h}$ yields

$$\mathcal{R}\big\langle -i|\xi|S(\omega)\partial_t\hat{h}, \hat{h}\big\rangle + |\xi|^2\mathcal{R}\big\langle S(\omega)(\hat{v}\cdot\omega)\hat{h}, \hat{h}\big\rangle = |\xi|\mathcal{R}\big\{\big\langle iS(\omega)L\hat{h}, \hat{h}\big\rangle - \big\langle iS(\omega)\hat{g}, \hat{h}\big\rangle\big\}. \tag{2.8}$$

Since $iS(\omega)$ is self-adjoint, the first term is just $-\frac{1}{2}\partial_t[|\xi|\langle iS(\omega)\hat{h}, \hat{h}\rangle]$. By multiplying $(1+|\xi|^2)$ to (2.6), and adding κ times (2.8), we have

$$\begin{aligned}
&\partial_t\left[\frac{(1+|\xi|^2)}{2}|\hat{h}|_2^2 - \frac{\kappa|\xi|}{2}\big\langle iS(\omega)\hat{h}, \hat{h}\big\rangle\right] + \big(1+|\xi|^2-\kappa|\xi|^2\big) \\
&\quad\times \langle L\hat{h}, \hat{h}\rangle + \kappa|\xi|^2\big\{\mathcal{R}\big\langle S(\omega)(\hat{v}\cdot\omega)\hat{h}, \hat{h}\big\rangle + \langle L\hat{h}, \hat{h}\rangle\big\} \\
&\quad= \big(1+|\xi|^2\big)\mathcal{R}\langle \hat{h}, \hat{g}\rangle + \kappa|\xi|\mathcal{R}\big\{\big\langle iS(\omega)L\hat{h}, \hat{h}\big\rangle - \big\langle iS(\omega)\hat{g}, \hat{h}\big\rangle\big\}.
\end{aligned} \tag{2.9}$$

For the second term on the left-hand side of (2.9), when $0 < \kappa < 1$, we have

$$\big(1+|\xi|^2-\kappa|\xi|^2\big)\langle L\hat{h}, \hat{h}\rangle \geqslant (1-\kappa)\big(1+|\xi|^2\big)\cdot\delta_0\big|(\mathbf{I}-\mathbf{P})\hat{h}\big|_{\mathcal{D}}^2.$$

And by Lemma 2.3, the third term on the left-hand side of (2.9) is bounded by

$$\kappa|\xi|^2\big\{\mathcal{R}\big\langle S(\omega)(\hat{v}\cdot\omega)\hat{h}, \hat{h}\big\rangle + \langle L\hat{h}, \hat{h}\rangle\big\} \geqslant \kappa|\xi|^2\cdot c_0\big(|\mathbf{P}\hat{h}|_2^2 + \big|(\mathbf{I}-\mathbf{P})\hat{h}\big|_{\mathcal{D}}^2\big).$$

Notice that

$$S(\omega)Lh = \sum_{k,\ell=1}^{14} \lambda r_{k\ell}(\omega)\langle Lh, e_\ell\rangle e_k = \sum_{k,\ell=1}^{14} \lambda r_{k\ell}(\omega)\big\langle L\big[(\mathbf{I}-\mathbf{P})h\big], e_\ell\big\rangle e_k,$$

and

$$\big|\big\langle L\big[(\mathbf{I}-\mathbf{P})h\big], e_\ell\big\rangle\big| = \big|\big\langle(\mathbf{I}-\mathbf{P})h, Le_\ell\big\rangle\big| \leqslant C\big|(\mathbf{I}-\mathbf{P})h\big|_{\mathcal{D}},$$

where we used the self-adjoint property of L and the exponential decay of $e_\ell(v)$ in v. The second term on the right-hand side of (2.9) is dominated by

$$\begin{aligned} c\kappa|\xi|\big\{\big|\big\langle iS(\omega)\widehat{Lh}, \hat{h}\big\rangle\big| + \big|\big\langle iS(\omega)\hat{g}, \hat{h}\big\rangle\big|\big\} &\leqslant c\kappa|\xi|\left\{\big|(\mathbf{I}-\mathbf{P})\hat{h}\big|_{\mathcal{D}}|\hat{h}|_2 + \sum_{k,\ell=1}^{14}\big|\langle\hat{g}, e_\ell\rangle\big|\cdot\big|\langle\hat{h}, e_k\rangle\big|\right\} \\ &\leqslant c_\varepsilon\kappa\big|(\mathbf{I}-\mathbf{P})\hat{h}\big|_{\mathcal{D}}^2 + \kappa\varepsilon|\xi|^2|\hat{h}|_2^2 + c_\varepsilon\sum_{\ell=1}^{14}\big|\langle\hat{g}, e_\ell\rangle\big|^2. \end{aligned}$$

If we choose $\kappa, \varepsilon > 0$ small enough and combine the above estimates, we know that there exist $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} &\partial_t\big[\big(1+|\xi|^2\big)|\hat{h}|_2^2 - \kappa|\xi|\big\langle iS(\omega)\hat{h}, \hat{h}\big\rangle\big] + \delta_1\big(1+|\xi|^2\big)\big|(\mathbf{I}-\mathbf{P})\hat{h}\big|_{\mathcal{D}}^2 + \delta_2|\xi|^2|\mathbf{P}\hat{h}|_2^2 \\ &\quad\leqslant \big(1+|\xi|^2\big)\mathcal{R}\langle\hat{h}, \hat{g}\rangle + c_\varepsilon\sum_{\ell=1}^{14}\big|\langle\hat{g}, e_\ell\rangle\big|^2. \end{aligned} \tag{2.10}$$

(2.10) will be used to prove the following lemma which was proved in [14] with $q = 1$ and $m = 0$. Set $H^N = H_x^N(L_v^2)$.

Lemma 2.4. *Let $k \geqslant k_1 \geqslant 0$ and $N \geqslant 4$. Assume that*

(i) $h_0 \in H^N \cap Z_q$,
(ii) $g \in C^0([0,\infty); H^N \cap Z_q)$,
(iii) $\mathbf{P}g(t,x,v) = 0$ *for all* $(t,x,v) \in [0,\infty)\times\mathbf{R}^3\times\mathbf{R}^3$.

Then if $h(t,x,v) \in C^0([0,\infty); H^N) \cap C^1([0,\infty); H^{N-1})$ is a solution of (2.1), we have

$$\begin{aligned} \big\|\nabla_x^k h\big\|^2 &\leqslant C(1+t)^{-2\sigma_{q,m}}\big(\big\|\nabla_x^{k_1}h_0\big\|_{Z_q} + \big\|\nabla_x^k h_0\big\|\big)^2 \\ &\quad + \int_0^t (1+t-s)^{-2\sigma_{q,m}}\big(\big\|\nabla_x^{k_1}g\big\|_{Z_q} + \big\|\nabla_x^k g\big\|\big)^2\,ds, \end{aligned} \tag{2.11}$$

for any integer $m = k - k_1 \geqslant 0$, where $q \in [1,2]$ and

$$\sigma_{q,m} = \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{m}{2}.$$

Proof. Firstly, by the assumption (iii), for any small $\varepsilon > 0$, we have

$$\left|\langle \hat{g}, \hat{h}\rangle\right| = \left|\left\langle(\mathbf{I}-\mathbf{P})\hat{g}, \hat{h}\right\rangle\right| \leqslant C_\varepsilon |\hat{g}|_2^2 + \varepsilon\left|(\mathbf{I}-\mathbf{P})\hat{h}\right|_2^2.$$

From (2.10), this gives

$$\begin{aligned}&\partial_t\left[\left(1+|\xi|^2\right)|\hat{h}|_2^2 - \kappa|\xi|\left\langle iS(\omega)\hat{h}, \hat{h}\right\rangle\right]\\&\quad + \delta_1\left(1+|\xi|^2\right)\left|(\mathbf{I}-\mathbf{P})\hat{h}\right|_{\mathcal{D}}^2 + \delta_2|\xi|^2|\mathbf{P}\hat{h}|_2^2 \leqslant C\left(1+|\xi|^2\right)|\hat{g}|_2^2.\end{aligned} \tag{2.12}$$

Thus,

$$\partial_t E[h] + \delta\frac{|\xi|^2}{1+|\xi|^2}E[h] \leqslant C|\hat{g}|_2^2, \tag{2.13}$$

where

$$E[h] = \left|\hat{h}(t,\xi,\cdot)\right|_2^2 - \frac{\kappa|\xi|}{1+|\xi|^2}\left\langle iS(\omega)\hat{h}(t,\xi,\cdot), \hat{h}(t,\xi,\cdot)\right\rangle.$$

Since $\kappa > 0$ can be sufficiently small and $S(\omega)$ is a compensating function, it is clear that

$$\frac{1}{2}\left|\hat{h}(t,\xi,\cdot)\right|_2^2 \leqslant E[h] \leqslant 2\left|\hat{h}(t,\xi,\cdot)\right|_2^2. \tag{2.14}$$

Then (2.13) and (2.14) give

$$\left|\hat{h}(t,\xi,\cdot)\right|_2^2 \leqslant ce^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}\left|\hat{h}_0(\xi)\right|_2^2 + c\int_0^t e^{-\delta(t-s)\frac{|\xi|^2}{1+|\xi|^2}}\left|\hat{g}(s,\xi,\cdot)\right|_2^2\,ds.$$

Multiplying this by $|\xi|^{2k}$ and integrating over ξ yield

$$\begin{aligned}\left\|\nabla_x^k h\right\|^2 &= \int_{\mathbf{R}^3}|\xi|^{2k}\left|\hat{h}(t,\xi,\cdot)\right|_2^2\,d\xi\\&\leqslant c\int_{\mathbf{R}^3}|\xi|^{2k}e^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}\left|\hat{h}_0(\xi)\right|_2^2\,d\xi + c\int_0^t\int_{\mathbf{R}^3}|\xi|^{2k}e^{-\delta(t-s)\frac{|\xi|^2}{1+|\xi|^2}}\left|\hat{g}(s,\xi,\cdot)\right|_2^2\,d\xi\,ds.\end{aligned} \tag{2.15}$$

Set

$$\begin{aligned}I_0 &= \int_{\mathbf{R}^3}|\xi|^{2k}e^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}\left|\hat{h}_0(\xi)\right|_2^2\,d\xi\\&= \int_{|\xi|\leqslant 1}|\xi|^{2k}e^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}\left|\hat{h}_0(\xi)\right|_2^2\,d\xi + \int_{|\xi|\geqslant 1}|\xi|^{2k}e^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}\left|\hat{h}_0(\xi)\right|_2^2\,d\xi.\end{aligned} \tag{2.16}$$

Notice that

$$
\begin{aligned}
\left| \int\limits_{|\xi|\leqslant 1} |\xi|^{2k} e^{-\delta t \frac{|\xi|^2}{1+|\xi|^2}} \left|\hat{h}_0(\xi)\right|_2^2 d\xi \right| &\leqslant \int\limits_{|\xi|\leqslant 1} \left|\xi^{\alpha-\alpha'}\right|^2 e^{-\delta t \frac{|\xi|^2}{2}} \left|\xi^{\alpha'}\hat{h}_0(\xi)\right|_2^2 d\xi \\
&\leqslant \left(\int\limits_{|\xi|\leqslant 1} |\xi|^{2p'm} e^{-\delta p' t \frac{|\xi|^2}{2}} d\xi \right)^{1/p'} \left(\int\limits_{|\xi|\leqslant 1} \left|\xi^{\alpha'}\hat{h}_0(\xi)\right|_2^{2q'} d\xi \right)^{1/q'},
\end{aligned}
$$

where $|\alpha| = k$, $|\alpha'| = k_1$, $m = |\alpha - \alpha'|$ and $p' \in [1, \infty)$ with $\frac{1}{p'} + \frac{1}{q'} = 1$. Since

$$
\int\limits_{|\xi|\leqslant 1} |\xi|^{2p'm} e^{-\delta p' t \frac{|\xi|^2}{2}} d\xi \leqslant C(1+t)^{-3/2-p'm},
$$

and

$$
\left(\int\limits_{|\xi|\leqslant 1} \left|\xi^{\alpha'}\hat{h}_0(\xi)\right|_2^{2q'} d\xi \right)^{1/q'} \leqslant \left\| \partial^{\alpha'} h_0 \right\|_{Z_q}^2, \qquad \frac{1}{q} + \frac{1}{2q'} = 1,
$$

we have

$$
\int\limits_{|\xi|\leqslant 1} |\xi|^{2k} e^{-\delta t \frac{|\xi|^2}{1+|\xi|^2}} \left|\hat{h}_0(\xi)\right|_2^2 d\xi \leqslant c(1+t)^{-3/2-p'm} \left\| \partial^{\alpha'} h_0 \right\|_{Z_q}^2.
$$

On the other hand,

$$
\int\limits_{|\xi|\geqslant 1} |\xi|^{2k} e^{-\delta t \frac{|\xi|^2}{1+|\xi|^2}} \left|\hat{h}_0(\xi)\right|_2^2 d\xi \leqslant c e^{\frac{-\delta t}{2}} \left\| \partial^{\alpha} h_0 \right\|^2.
$$

Thus, we obtain

$$
\begin{aligned}
I_0 &\leqslant C(1+t)^{-3/2p'-m} \left\| \partial^{\alpha'} h_0 \right\|_{Z_q}^2 + c e^{\frac{-\delta t}{2}} \left\| \partial^{\alpha} h_0 \right\|^2 \\
&\leqslant C(1+t)^{-2\sigma_{q,m}} \left(\left\| \nabla_x^{k_1} h_0 \right\|_{Z_q} + \left\| \nabla_x^{k} h_0 \right\| \right)^2.
\end{aligned}
$$

Clearly, the above estimation holds also for the terms involving g in (2.15) so that the final estimate in the lemma follows. □

3. Relativistic Boltzmann equation

The global existence and the optimal convergence rate for the relativistic Boltzmann equation will be considered in this section. Firstly, in the following subsection, we will derive some basic estimates on relativistic Boltzmann collision operator.

3.1. Basic estimates

Lemma 3.1. *For $l \geqslant 0$ and any function g with the following norms well defined, we have*

$$\langle \nu^{2l} Lg, g \rangle \geqslant \frac{1}{2} |\nu^l g|_\nu^2 - C|g|_\nu^2. \tag{3.1}$$

Proof. Notice that

$$\nu^{2l} Lg = \nu^{2l} \nu g - \nu^{2l} Kg. \tag{3.2}$$

Consider $v \in \mathbf{R}^3$ in the regions $|v| \leqslant m$ and $|v| > m$ with $m \geqslant 1$ separately. For the operator K, we know from Lemma 3.7 in [13] that $Kg(v) = \int_{\mathbf{R}^3} K(u,v) g(u)\,du$, and under the assumption (1.7) the kernel $K(u,v)$ satisfies that when $\gamma < 0$,

$$\sup_v \int_{\mathbf{R}^3} |K(u,v)|\,du < \infty, \quad |K(u,v)| \leqslant \frac{c(1+|v|)^{[3|\gamma|+\beta+2\delta]/2} \cdot e^{-c|u-v|}}{[|u \times v|^2 + |u-v|^2]^{1/2} |u-v|^{\delta+|\gamma|}}. \tag{3.3}$$

When $\gamma \geqslant 0$, the estimate (3.3) holds by replacing the $|\gamma|$ by zero and the following estimation also holds. Therefore, we only consider the case when $\gamma < 0$ in the following.

Since K is bounded on $L^2(\mathbf{R}_v^3)$, we have

$$\langle \nu^{2l} Kg \mathbf{1}_{\{|v| \leqslant m\}}, g \rangle \leqslant C_m |g|_\nu^2,$$

and

$$\begin{aligned}
&\langle \nu^{2l} Kg \mathbf{1}_{\{|v|>m\}}, g \rangle \\
&\quad = \int_{|v|>m} \nu^{2l}(v) \left\{ \int_{\mathbf{R}^3} K(u,v) g(u)\,du \right\} g(v)\,dv \\
&\quad \leqslant \int_{|v|>m} \nu^{2l}(v) \left\{ \int_{\mathbf{R}^3} |K(u,v)| g^2(u)\,du \right\}^{1/2} \left\{ \int_{\mathbf{R}^3} |K(u,v)|\,du \right\}^{1/2} |g(v)|\,dv \\
&\quad \leqslant \left\{ \int_{|v|>m} \int_{\mathbf{R}^3} \nu^{2l}(v) |K(u,v)| g^2(u)\,du\,dv \right\}^{1/2} \left\{ \int_{|v|>m} \int_{\mathbf{R}^3} \nu^{2l}(v) |K(u,v)| g^2(v)\,du\,dv \right\}^{1/2}.
\end{aligned} \tag{3.4}$$

For the first factor in the last line of (3.4), we have from (1.8) and (3.3) that

$$\begin{aligned}
&\int_{|v|>m} \int_{\mathbf{R}^3} \nu^{2l}(v) |K(u,v)| g^2(u)\,du\,dv \\
&\quad = \int_{\mathbf{R}^3} g^2(u)\,du \int_{|v|>m} \nu^{2l}(v) |K(u,v)|\,dv \\
&\quad \leqslant C \int_{\mathbf{R}^3} g^2(u)\,du \int_{|v|>m} \frac{\nu^{2l}(v)(1+|v|)^{[3|\gamma|+\beta+2\delta]/2} \cdot e^{-c|u-v|}}{[|u \times v|^2 + |u-v|^2]^{1/2} |u-v|^{\delta+|\gamma|}}\,dv
\end{aligned}$$

$$\leqslant C\int_{\mathbf{R}^3} g^2(u)\,du \int_{|v|>m} \frac{\nu^{2l}(v)(1+|v|)^{[3|\gamma|+\beta+2\delta]/2}\cdot e^{-c|u-v|}}{|(u-v)\times v|\cdot|u-v|^{\delta+|\gamma|}}\,dv$$

$$\leqslant C\int_{\mathbf{R}^3} g^2(u)\,du \int_{|v|>m} \frac{\nu^{2l}(v)(1+|v|)^{[3|\gamma|+\beta+2\delta]/2}\cdot e^{-c|u-v|}}{|(u-v)|\cdot|v|\sin\vartheta\cdot|u-v|^{\delta+|\gamma|}}\,dv$$

$$\leqslant 2C\int_{\mathbf{R}^3} g^2(u)\,du \int_{|v|>m} \frac{\nu^{2l}(v)e^{-c|u-v|}}{|(u-v)|(1+|v|)^{1-[3|\gamma|+\beta+2\delta]/2}\sin\vartheta\cdot|u-v|^{\delta+|\gamma|}}\,dv$$

$$\leqslant \frac{C}{m^{1-[3|\gamma|+\beta+2\delta]/2}}\int_{\mathbf{R}^3} g^2(u)\,du \int_{|v|>m} \frac{(1+|v|^2)^{\beta l/2}\cdot e^{-c|u-v|}}{|(u-v)|\sin\vartheta\cdot|u-v|^{\delta+|\gamma|}}\,dv,$$

where we have used $|(u-v)\times v| = |u-v|\cdot|v|\sin\vartheta$ with $\vartheta\in(0,\pi)$. Clearly, the last integral is bounded by

$$\int_{\mathbf{R}^3} \frac{(1+|v|^2)^{\beta l/2}\cdot e^{-c|u-v|}}{|(u-v)|\sin\vartheta\cdot|u-v|^{\delta+|\gamma|}}\,dv = \int_{\mathbf{R}^3} \frac{(1+|u-v|^2)^{\beta l/2}\cdot e^{-c|v|}}{\sin\vartheta\cdot|v|^{1+\delta+|\gamma|}}\,dv$$

$$\leqslant C\int_{\mathbf{R}^3} \frac{(1+|u|^2+|v|^2)^{\beta l/2}\cdot e^{-c|v|}}{\sin\vartheta\cdot|v|^{1+\delta+|\gamma|}}\,dv$$

$$= C\int_0^\infty\int_0^\pi \frac{(1+|u|^2+\rho^2)^{\beta l/2}\cdot e^{-c\rho}\rho^2\sin\vartheta}{\sin\vartheta\cdot\rho^{1+\delta+|\gamma|}}\,d\rho\,d\vartheta$$

$$\leqslant C\int_0^\infty \big(1+|u|^2+\rho^2\big)^{\beta l/2}\cdot e^{-c\rho/2}\,d\rho \leqslant C\big(1+|u|^2\big)^{\beta l/2}.$$

Therefore,

$$\big\langle \nu^{2l}Kg\mathbf{1}_{\{|v|>m\}}, g\big\rangle = \int_{|v|>m} \nu^{2l}(v)\left\{\int_{\mathbf{R}^3} K(u,v)g(u)\,du\right\}g(v)\,dv$$

$$\leqslant \frac{C}{m^{1/2-[3|\gamma|+\beta+2\delta]/4}}\left\{\int_{\mathbf{R}^3}\big(1+|u|^2\big)^{\beta l/2}g^2(u)\,du\right\}^{1/2}$$

$$\times\left\{\sup_v\int_{\mathbf{R}^3}\big|K(u,v)\big|\,du\cdot\int_{|v|>m}\nu^{2l}(v)g^2(v)\,du\,dv\right\}^{1/2}.$$

Then by (1.8) and (3.3), we have for $\beta\geqslant 0$,

$$\big\langle \nu^{2l}Kg\mathbf{1}_{\{|v|>m\}}, g\big\rangle \leqslant \frac{C}{m^{1/2-[3|\gamma|+\beta+2\delta]/4}}\big|\nu^l g\big|_\nu^2.$$

By noticing that $1/2-[3|\gamma|+\beta+2\delta]/4>0$, we complete the proof of Lemma 3.1 by choosing m large enough. □

Remark 3.1. The estimate in Lemma 3.1 was proved in [18] for the classical Boltzmann equation.

Lemma 3.2. *Let $p \in [1, \infty]$, $l \geqslant 0$ and $\delta \in [0, 1]$. For any $g_1, g_2 \in L^p(\mathbf{R}_v^3)$, it holds that*

$$\left|\nu^{l-\delta}\Gamma(g_1, g_2)\right|_{L^p} \leqslant C\left(\left|\nu^{l+1-\delta}g_1\right|_{L^p}\left|\nu^l g_2\right|_{L^p} + \left|\nu^l g_1\right|_{L^p}\left|\nu^{l+1-\delta}g_2\right|_{L^p}\right). \tag{3.5}$$

Proof. Write

$$\Gamma(g_1, g_2) = \Gamma_1(g_1, g_2) - \Gamma_2(g_1, g_2),$$

with

$$\Gamma_1(g_1, g_2) = \int\limits_{\mathbf{R}^3\times\mathbf{S}^2} v_M\sigma(g, \theta)e^{-u_0/2}g_1(u')g_2(v')\,du\,d\omega,$$

$$\Gamma_2(g_1, g_2) = \int\limits_{\mathbf{R}^3\times\mathbf{S}^2} v_M\sigma(g, \theta)e^{-u_0/2}g_1(u)g_2(v)\,du\,d\omega.$$

From Lemma 3.1 in [13] and the definition of g, we know that

$$\frac{|u-v|}{2u_0^{1/2}v_0^{1/2}} \leqslant g \leqslant \frac{1}{2}u_0^{1/2}v_0^{1/2}.$$

By (1.8), it holds that

$$\int\limits_{\mathbf{R}^3}\left[g^{q\beta} + g^{-q\delta}\right]e^{-qu_0}\,du \leqslant Cv_0^{q\beta/2} + Cv_0^{q\delta/2}\int \frac{u_0^{q\delta/2}e^{-qu_0}}{|u-v|^{q\delta}}\,du$$
$$\leqslant Cv_0^{q\beta/2} + Cv_0^{q\delta}\left[1+|v|\right]^{-q\delta} \leqslant C\nu(v)^q.$$

Firstly, we consider $\Gamma_1(g_1, g_2)$. From Lemma 3.1 in [13], we know that v_M is bounded. Thus, the Hölder inequality gives

$$\begin{aligned}
\left|\Gamma_1(g_1, g_2)\right| &\leqslant \int\limits_{\mathbf{R}^3\times\mathbf{S}^2} \sigma(g, \theta)e^{-u_0/2}\left|g_1(u')\right|\left|g_2(v')\right|\,du\,d\omega \\
&\leqslant \int\limits_{\mathbf{R}^3\times\mathbf{S}^2} \left[g^\beta + g^{-\delta}\right]\sin^\gamma\theta e^{-u_0/2}\left|g_1(u')\right|\left|g_2(v')\right|\,du\,d\omega \\
&\leqslant \left(\int\limits_{\mathbf{R}^3\times\mathbf{S}^2} \left[g^{q\beta} + g^{-q\delta}\right]\sin^\gamma\theta e^{-qu_0/4}\,du\,d\omega\right)^{1/q} \\
&\quad\times\left(\int\limits_{\mathbf{R}^3\times\mathbf{S}^2} \left|g_1(u')\right|^p\left|g_2(v')\right|^p\sin^\gamma\theta e^{-pu_0/4}\,du\,d\omega\right)^{1/p} \\
&\leqslant C\nu(v)\left(\int\limits_{\mathbf{R}^3\times\mathbf{S}^2} \left|g_1(u')\right|^p\left|g_2(v')\right|^p\sin^\gamma\theta e^{-pu_0/4}\,du\,d\omega\right)^{1/p},
\end{aligned}$$

with $p \in [1, \infty)$ and $1/p + 1/q = 1$. Consequently, we have

$$\int_{\mathbf{R}^3} \left|\nu^{l-\delta}(v)\Gamma_1(g_1, g_2)\right|^p dv \leqslant C \int \nu^{(l+1-\delta)p}(v)\left|g_1(u')\right|^p \left|g_2(v')\right|^p \sin^\gamma \theta e^{-pu_0/4}\, dv\, du\, d\omega.$$

Since energy conservation implies

$$\sqrt{1+|v|^2} \leqslant \sqrt{1+\left|u'\right|^2} + \sqrt{1+\left|v'\right|^2},$$

when $\beta \in (0, 2)$, we have

$$\left[\sqrt{1+|v|^2}\right]^{\beta/2} \leqslant \left[\sqrt{1+\left|u'\right|^2}\right]^{\beta/2} + \left[\sqrt{1+\left|v'\right|^2}\right]^{\beta/2},$$

which implies that $\nu(v) \leqslant \nu(u') + \nu(v')$.

Therefore, we obtain

$$\begin{aligned}
&\int_{\mathbf{R}^3} \left|\nu^{l-\delta}(v)\Gamma_1(g_1, g_2)\right|^p dv \\
&\quad\leqslant C \int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \left(\nu^{(l+1-\delta)p}(u') + \nu^{(l+1-\delta)p}(v')\right)\left|g_1(u')\right|^p \left|g_2(v')\right|^p \sin^\gamma \theta e^{-pu_0/4}\, dv\, du\, d\omega \\
&\quad= C \int_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \left(\nu^{(l+1-\delta)p}(u) + \nu^{(l+1-\delta)p}(v)\right)\left|g_1(u)\right|^p \left|g_2(v)\right|^p \\
&\qquad\times \frac{e^{-pu_0'/4}\sqrt{1+|u'|^2}\sqrt{1+|v'|^2}}{\sqrt{1+|u|^2}\sqrt{1+|v|^2}} \sin^\gamma \theta\, dv\, du\, d\omega,
\end{aligned}$$

where we have used the change of the variables $(u, v) \to (u', v')$ and the Jacobian [11]

$$\frac{\partial(u', v')}{\partial(u, v)} = -\frac{\sqrt{1+|u'|^2}\sqrt{1+|v'|^2}}{\sqrt{1+|u|^2}\sqrt{1+|v|^2}}.$$

This proves (3.5) for $\Gamma_1(g_1, g_2)$ for the case when $p \in [1, \infty)$. The case when $p = \infty$ can be proved similarly and the proof for $\Gamma_2(g_1, g_2)$ is simpler so that we skip the details for brevity. This completes the proof of the lemma. □

Remark 3.2. Lemma 3.2 was proved in [13] for the case $p = \infty$, $\delta = 1$ and $l = 0$. The case $p = 2$, $\delta = 1/2$ and $l \geqslant 0$ will be used in the following Lemma 3.3.

Lemma 3.3. *Let $|\alpha| \leqslant N$ and $l \geqslant 0$, we have*

$$\left\|\nu^{l-\frac{1}{2}}\partial^\alpha \Gamma(g_1, g_2)\right\|^2 \leqslant C\left(\widetilde{\mathcal{E}}_l(g_1)(t)\widetilde{\mathcal{D}}_l(g_2)(t) + \widetilde{\mathcal{E}}_l(g_2)(t)\widetilde{\mathcal{D}}_l(g_1)(t)\right). \tag{3.6}$$

Proof. Decompose g_1 and g_2 as in (1.10) and rewrite $\Gamma(g_1, g_2)$ as

$$\Gamma(\mathbf{P}g_1, \mathbf{P}g_2) + \Gamma\big(\mathbf{P}g_1, (\mathbf{I}-\mathbf{P})g_2\big) + \Gamma\big((\mathbf{I}-\mathbf{P})g_1, \mathbf{P}g_2\big) + \Gamma\big((\mathbf{I}-\mathbf{P})g_1, (\mathbf{I}-\mathbf{P})g_2\big). \tag{3.7}$$

Notice that

$$\partial^\alpha \Gamma(g_1, g_2) = \sum_{\alpha_1 \leqslant \alpha} C_\alpha^{\alpha_1} \Gamma\big(\partial^{\alpha_1} g_1, \partial^{\alpha-\alpha_1} g_2\big).$$

By using (3.5) with $\delta = 1/2$ and $p = 2$, we have

$$\begin{aligned}\big\|\nu^{l-\frac{1}{2}}\partial^\alpha \Gamma(\mathbf{P}g_1, \mathbf{P}g_2)\big\|^2 \leqslant C \sum_{\alpha_1 \leqslant \alpha} \int_{\mathbf{R}_x^3} \Big\{ &\big|\nu^{l+\frac{1}{2}}\partial^{\alpha_1}\mathbf{P}g_1\big|_2^2 \big|\nu^l \partial^{\alpha-\alpha_1}\mathbf{P}g_2\big|_2^2 \\ &+ \big|\nu^l \partial^{\alpha_1}\mathbf{P}g_1\big|_2^2 \big|\nu^{l+\frac{1}{2}}\partial^{\alpha-\alpha_1}\mathbf{P}g_2\big|_2^2 \Big\}\, dx. \end{aligned} \tag{3.8}$$

We only estimate the first term of (3.8) because the second term can be estimated similarly. When $|\alpha_1| \leqslant |\alpha|/2$, we have

$$\begin{aligned}\int_{\mathbf{R}_x^3} \big|\nu^{l+\frac{1}{2}}\partial^{\alpha_1}\mathbf{P}g_1\big|_2^2 \big|\nu^l \partial^{\alpha-\alpha_1}\mathbf{P}g_2\big|_2^2\, dx &\leqslant C \sup_{x\in\mathbf{R}^3} \big|\nu^{l+\frac{1}{2}}\partial^{\alpha_1}\mathbf{P}g_1\big|_2^2 \big\|\nu^l \partial^{\alpha-\alpha_1}\mathbf{P}g_2\big\|^2 \\ &\leqslant C \sum_{|\alpha'| \leqslant 1} \big\|\nabla_x \partial^{\alpha_1+\alpha'}\mathbf{P}g_1\big\| \big\|\partial^{\alpha-\alpha_1}\mathbf{P}g_2\big\|^2 \\ &\leqslant C\widetilde{\mathcal{E}}_l(g_2)(t)\widetilde{\mathcal{D}}_l(g_1)(t). \end{aligned}$$

Here we have used the fact that $|\nu^l \partial^{\alpha_1}\mathbf{P}f|_2 \leqslant C|\partial^{\alpha_1}\mathbf{P}f|_2$.

Otherwise, when $|\alpha - \alpha_1| \leqslant |\alpha|/2$, we have

$$\begin{aligned}\int_{\mathbf{R}_x^3} \big|\nu^{l+\frac{1}{2}}\partial^{\alpha_1}\mathbf{P}g_1\big|_2^2 \big|\nu^l \partial^{\alpha-\alpha_1}\mathbf{P}g_2\big|_2^2\, dx &\leqslant C \sup_{x\in\mathbf{R}^3} \big|\nu^l \partial^{\alpha-\alpha_1}\mathbf{P}g_2\big|_2^2 \big\|\nu^{l+\frac{1}{2}}\partial^{\alpha_1}\mathbf{P}g_1\big\|^2 \\ &\leqslant C \sum_{|\alpha'| \leqslant 1} \big\|\nabla_x \partial^{\alpha-\alpha_1+\alpha'}\mathbf{P}g_2\big\| \big\|\partial^{\alpha_1}\mathbf{P}g_1\big\|^2 \\ &\leqslant C\widetilde{\mathcal{E}}_l(g_1)(t)\widetilde{\mathcal{D}}_l(g_2)(t). \end{aligned}$$

For the second term of (3.7), by Lemma 3.2, we obtain

$$\begin{aligned}\big\|\nu^{l-\frac{1}{2}}\partial^\alpha \Gamma\big(\mathbf{P}g_1, (\mathbf{I}-\mathbf{P})g_2\big)\big\|^2 \leqslant C \sum_{\alpha_1 \leqslant \alpha} \int_{\mathbf{R}_x^3} \Big\{ &\big|\nu^{l+\frac{1}{2}}\partial^{\alpha_1}\mathbf{P}g_1\big|_2^2 \big|\nu^l \partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})g_2\big|_2^2 \\ &+ \big|\nu^l \partial^{\alpha_1}\mathbf{P}g_1\big|_2^2 \big|\nu^{l+\frac{1}{2}}\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})g_2\big|_2^2 \Big\}\, dx. \end{aligned} \tag{3.9}$$

Again, we only consider the first term of (3.9). When $|\alpha_1| \leqslant |\alpha|/2$, we have

$$\int_{\mathbf{R}_x^3} \left|\nu^{l+\frac{1}{2}}\partial^{\alpha_1}\mathbf{P}g_1\right|_2^2 \left|\nu^l\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})g_2\right|_2^2 dx \leqslant C \sup_{x\in\mathbf{R}^3}\left|\nu^{l+\frac{1}{2}}\partial^{\alpha_1}\mathbf{P}g_1\right|_2^2 \left\|\nu^l\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})g_2\right\|^2$$

$$\leqslant C\sum_{|\alpha'|\leqslant 1}\left\|\nabla_x\partial^{\alpha_1+\alpha'}\mathbf{P}g_1\right\|\left\|\nu^l\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})g_2\right\|^2$$

$$\leqslant C\widetilde{\mathcal{E}}_l(g_2)(t)\widetilde{\mathcal{D}}_l(g_1)(t).$$

Otherwise, when $|\alpha-\alpha_1|\leqslant|\alpha|/2$, we have

$$\int_{\mathbf{R}_x^3} \left|\nu^{l+\frac{1}{2}}\partial^{\alpha_1}\mathbf{P}g_1\right|_2^2 \left|\nu^l\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})g_2\right|_2^2 dx \leqslant C \sup_{x\in\mathbf{R}^3}\left|\nu^l\partial^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})g_2\right|_2^2 \left\|\nu^{l+\frac{1}{2}}\partial^{\alpha_1}\mathbf{P}g_1\right\|^2$$

$$\leqslant C\sum_{|\alpha'|\leqslant 2}\left\|\nu^l\partial^{\alpha-\alpha_1+\alpha'}(\mathbf{I}-\mathbf{P})g_2\right\|\left\|\partial^{\alpha_1}\mathbf{P}g_1\right\|^2$$

$$\leqslant C\widetilde{\mathcal{E}}_l(g_1)(t)\widetilde{\mathcal{D}}_l(g_2)(t).$$

Since the other terms can be estimated similarly. This completes the proof of (3.6). □

3.2. Energy estimates

Since basically, the energy estimate on the solution without weight in the microscopic momentum is given by using the compensating function and the estimate on the macroscopic component is the same with or without weight, we will only apply the basic energy method to derive the energy estimate on the microscopic component with weight. For this, from the relativistic Boltzmann equation (1.5), the equation on the microscopic component is

$$[\partial_t+\hat{v}\cdot\nabla_x+L](\mathbf{I}-\mathbf{P})f=\Gamma(f,f)-[\partial_t+\hat{v}\cdot\nabla_x]\mathbf{P}f. \tag{3.10}$$

Lemma 3.4. *For the relativistic Boltzmann equation* (1.5), *we have the following weighted estimate*:

$$\frac{d}{dt}\left[\sum_{1\leqslant|\alpha|\leqslant N}\left\|\nu^l\partial^\alpha f\right\|^2+\left\|\nu^l(\mathbf{I}-\mathbf{P})f\right\|^2\right]+\sum_{|\alpha|\leqslant N}\left\|\nu^l(\mathbf{I}-\mathbf{P})\partial^\alpha f\right\|_\nu^2$$
$$\leqslant C\sum_{1\leqslant|\alpha|\leqslant N}\left\|\mathbf{P}\partial^\alpha f\right\|^2+C\sum_{|\alpha|\leqslant N}\left\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\right\|_\nu^2+C\widetilde{\mathcal{E}}_l^{1/2}(f)(t)\widetilde{\mathcal{D}}_l(f)(t). \tag{3.11}$$

Proof. Taking the inner product of (3.10) with $\nu^{2l}(\mathbf{I}-\mathbf{P})f$ gives

$$\frac{1}{2}\frac{d}{dt}\left\|\nu^l(\mathbf{I}-\mathbf{P})f\right\|^2+\left(L(\mathbf{I}-\mathbf{P})f,\nu^{2l}(\mathbf{I}-\mathbf{P})f\right)$$
$$=\left(\Gamma(f,f),\nu^{2l}(\mathbf{I}-\mathbf{P})f\right)-\left([\partial_t+\hat{v}\cdot\nabla_x]\mathbf{P}f,\nu^{2l}(\mathbf{I}-\mathbf{P})f\right). \tag{3.12}$$

We estimate (3.12) term by term. By Lemma 3.1 and Lemma 3.3, we have

$$\left(L(\mathbf{I}-\mathbf{P})f,\nu^{2l}(\mathbf{I}-\mathbf{P})f\right)\geqslant\frac{1}{2}\left\|\nu^l(\mathbf{I}-\mathbf{P})f\right\|_\nu^2-C\left\|(\mathbf{I}-\mathbf{P})f\right\|_\nu^2,$$
$$\left|\left(\Gamma(f,f),\nu^{2l}(\mathbf{I}-\mathbf{P})f\right)\right|\leqslant C\widetilde{\mathcal{E}}_l^{1/2}(f)(t)\widetilde{\mathcal{D}}_l^{1/2}(f)(t)\left\|\nu^l(\mathbf{I}-\mathbf{P})f\right\|_\nu.$$

It is clear that the last term in (3.12) is bounded by

$$\left|\left([\partial_t + \hat{v}\cdot\nabla_x]\mathbf{P}f, \nu^{2l}(\mathbf{I}-\mathbf{P})\right)\right| \leqslant C_\eta\left[\|\nabla_x\mathbf{P}f\|^2 + \|\partial_t\mathbf{P}f\|^2\right] + \eta\left\|\nu^l(\mathbf{I}-\mathbf{P})f\right\|_\nu^2.$$

On the other hand, we can obtain from (3.10) that

$$\frac{1}{C}\|\partial_t\mathbf{P}f\|^2 \leqslant \|\nabla_x\mathbf{P}f\|^2 + \left\|\nabla_x(\mathbf{I}-\mathbf{P})f\right\|^2.$$

By using the above estimates and taking $\eta > 0$ small enough, we have from (3.12) that

$$\begin{aligned}&\frac{d}{dt}\left\|\nu^l(\mathbf{I}-\mathbf{P})f\right\|^2 + \left\|\nu^l(\mathbf{I}-\mathbf{P})f\right\|_\nu^2\\ &\quad\leqslant C\|\nabla_x\mathbf{P}f\|^2 + C\sum_{|\alpha|\leqslant 1}\left\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\right\|_\nu^2 + C\widetilde{\mathcal{E}}_l^{1/2}(f)(t)\widetilde{\mathcal{D}}_l(f)(t).\end{aligned}\tag{3.13}$$

We take ∂^α on Eq. (1.5) with $1 \leqslant |\alpha| \leqslant N$ to obtain

$$[\partial_t + \hat{v}\cdot\nabla_x + L]\partial^\alpha f = \partial^\alpha\Gamma(f,f).$$

Taking the inner product with $\nu^{2l}\partial^\alpha f$ yields

$$\frac{1}{2}\frac{d}{dt}\left\|\nu^l\partial^\alpha f\right\|^2 + \left(L\partial^\alpha f, \nu^{2l}\partial^\alpha f\right) = \left(\partial^\alpha\Gamma(f,f), \nu^{2l}\partial^\alpha f\right).\tag{3.14}$$

From Lemma 3.1, we know that

$$\left(L\partial^\alpha f, \nu^{2l}\partial^\alpha f\right) \geqslant \frac{1}{2}\left\|\nu^l\partial^\alpha f\right\|_\nu^2 - C\left\|\partial^\alpha f\right\|_\nu^2.$$

By Lemma 3.3, the non-linear collision operator have the following estimate

$$\left|\left(\partial^\alpha\Gamma(f,f), \nu^{2l}\partial^\alpha f\right)\right| \leqslant C\widetilde{\mathcal{E}}_l^{1/2}(f)(t)\widetilde{\mathcal{D}}_l^{1/2}(f)(t)\left\|\nu^l\partial^\alpha f\right\|_\nu \leqslant C\widetilde{\mathcal{E}}_l^{1/2}(f)(t)\widetilde{\mathcal{D}}_l(f)(t).$$

By combining the above estimates and taking $\eta > 0$ small enough, we have from (3.14) that

$$\frac{d}{dt}\left\|\nu^l\partial^\alpha f\right\|^2 + \left\|\nu^l\partial^\alpha f\right\|_\nu^2 \leqslant C\left\|\partial^\alpha\mathbf{P}f\right\|^2 + C\left\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\nu^2 + C_\eta\widetilde{\mathcal{E}}_l^{1/2}(f)(t)\widetilde{\mathcal{D}}_l(f)(t).\tag{3.15}$$

A suitable combination of (3.13) and (3.15) implies (3.11) and completes the proof of the lemma. □

In the following, we are going to use the compensating function of the relativistic Boltzmann equation in Section 2 to obtain the estimate on the solution.

Lemma 3.5. *For Eq.* (1.5), *we have the following estimate*

$$\frac{d}{dt}\Big[\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha f\big\|^2-\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big),\widehat{\partial^\alpha f}\big\rangle d\xi\Big]$$
$$+\delta_1\sum_{1\leqslant|\alpha|\leqslant N}\big\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\big\|_\nu^2+\delta_2\sum_{2\leqslant|\alpha|\leqslant N}\big\|\mathbf{P}\partial^\alpha f\big\|^2$$
$$\leqslant C\big(\widetilde{\mathcal{E}}_l^{1/2}(f)(t)+\widetilde{\mathcal{E}}_l(f)(t)\big)\widetilde{\mathcal{D}}_l(f)(t), \tag{3.16}$$

$$\frac{d}{dt}\Big[\sum_{|\alpha|\leqslant N}\big\|\partial^\alpha f\big\|^2-\kappa\int_{\mathbf{R}^3}\big(1+|\xi|^2\big)^{N-1}|\xi|\big\langle iS(\omega)\hat f,\hat f\big\rangle d\xi\Big]$$
$$+\delta_1\sum_{|\alpha|\leqslant N}\big\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\big\|_\nu^2+\delta_2\sum_{1\leqslant|\alpha|\leqslant N}\big\|\mathbf{P}\partial^\alpha f\big\|^2$$
$$\leqslant C\big(\widetilde{\mathcal{E}}_l^{1/2}(f)(t)+\widetilde{\mathcal{E}}_l(f)(t)\big)\widetilde{\mathcal{D}}_l(f)(t), \tag{3.17}$$

where $\kappa>0$ *is small enough.*

Proof. By setting h in (2.10) to be $\partial^\alpha f$, we have

$$\partial_t\big[\big(1+|\xi|^2\big)\big|\widehat{\partial^\alpha f}\big|_2^2-\kappa|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big),\widehat{\partial^\alpha f}\big\rangle\big]+\delta_1\big(1+|\xi|^2\big)\big|(\mathbf{I}-\mathbf{P})\widehat{\partial^\alpha f}\big|_\nu^2+\delta_2|\xi|^2\big|\mathbf{P}\widehat{\partial^\alpha f}\big|_2^2$$
$$\leqslant\big(1+|\xi|^2\big)\mathcal{R}\big\langle\widehat{\partial^\alpha f},\hat g\big\rangle+c_\varepsilon\sum_{\ell=1}^{14}\big|\langle\hat g,e_\ell\rangle\big|^2. \tag{3.18}$$

Let $g=\partial^\alpha\Gamma(f,f)$. Integrating (3.18) over ξ and summing over $1\leqslant|\alpha|\leqslant N-1$ give

$$\partial_t\Big[\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha f\big\|^2-\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big),\widehat{\partial^\alpha f}\big\rangle d\xi\Big]$$
$$+\delta_1\sum_{1\leqslant|\alpha|\leqslant N}\big\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\big\|_\nu^2+\delta_2\sum_{2\leqslant|\alpha|\leqslant N}\big\|\mathbf{P}\partial^\alpha f\big\|^2$$
$$\leqslant\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}\big(1+|\xi|^2\big)\mathcal{R}\big\langle\widehat{\partial^\alpha f},\hat g\big\rangle d\xi+c_\varepsilon\sum_{1\leqslant|\alpha|\leqslant N-1}\sum_{\ell=1}^{14}\int_{\mathbf{R}^3}\big|\langle\hat g,e_\ell\rangle\big|^2 d\xi. \tag{3.19}$$

For the second term on the right-hand side of (3.19), by Lemma 3.2 and the properties of Fourier transform, we obtain

$$\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}\big|\langle\hat g,e_\ell\rangle\big|^2 d\xi=\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}\big|\big\langle\partial^\alpha\Gamma(f,f),e_\ell\big\rangle\big|^2 dx$$
$$\leqslant C\sum_{1\leqslant|\alpha|\leqslant N-1}\sum_{\alpha'\leqslant\alpha}\int_{\mathbf{R}^3}\big[\big|\partial^{\alpha'}f\big|_2^2\big|\partial^{\alpha-\alpha'}f\big|_\nu^2+\big|\partial^{\alpha-\alpha'}f\big|_2^2\big|\partial^{\alpha'}f\big|_\nu^2\big]dx$$
$$\leqslant C\widetilde{\mathcal{E}}_l(f)(t)\widetilde{\mathcal{D}}_l(f)(t).$$

For the first term on the right-hand side of (3.19), we also have

$$\left|\int_{\mathbf{R}^3}(1+|\xi|^2)\mathcal{R}\langle\widehat{\partial^\alpha f},\hat{g}\rangle\,d\xi\right| \leqslant \left|\int_{\mathbf{R}^3}\langle\widehat{\partial^\alpha f},\hat{g}\rangle\,d\xi\right| + \left|\int_{\mathbf{R}^3}|\xi|^2\langle\widehat{\partial^\alpha f},\hat{g}\rangle\,d\xi\right|. \tag{3.20}$$

For the first term in (3.20), by using Lemma 3.3, the expression of g and the properties of Fourier transform, we get

$$\left|\int_{\mathbf{R}^3}\langle\widehat{\partial^\alpha f},\hat{g}\rangle\,d\xi\right| = \left|\left(\partial^\alpha\Gamma(f,f),\partial^\alpha f\right)\right| \leqslant \widetilde{\mathcal{E}}^{1/2}(f)(t)\widetilde{\mathcal{D}}^{1/2}(f)(t)\left\|\partial^\alpha f\right\|_\nu \leqslant C\widetilde{\mathcal{E}}^{1/2}(f)(t)\widetilde{\mathcal{D}}(f)(t),$$

$$\left|\int_{\mathbf{R}^3}|\xi|^2\langle\widehat{\partial^\alpha f},\hat{g}\rangle\,d\xi\right| = \left|\int_{\mathbf{R}^3}\langle\widehat{\partial^{\delta_i}\partial^\alpha f},\widehat{\partial^{\delta_i} g}\rangle\,d\xi\right| = \left|\left(\partial^{\delta_i}\partial^\alpha\Gamma(f,f),\partial^{\delta_i}\partial^\alpha f\right)\right| \leqslant C\widetilde{\mathcal{E}}_l^{1/2}(f)(t)\widetilde{\mathcal{D}}_l(f)(t),$$

where $|\xi|^2 = -(i\xi^{\delta_i})^2$, $|\delta_i| = 1$ and $|\alpha| \leqslant N-1$. Note that (3.19) and the above estimates give the estimate (3.16).

Now let h in (2.10) be f. Multiplying the resulting equation by $(1+|\xi|^2)^{N-1}$ and integrating over ξ yield

$$\begin{aligned}
&\partial_t\left[\int_{\mathbf{R}^3}(1+|\xi|^2)^N|\hat{f}|_2^2\,d\xi - \kappa\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|\langle iS(\omega)\hat{f},\hat{f}\rangle\,d\xi\right] \\
&\quad + \delta_1\int_{\mathbf{R}^3}(1+|\xi|^2)^N\left|(\mathbf{I}-\mathbf{P})\hat{f}\right|_\nu^2\,d\xi + \delta_2\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|^2|\mathbf{P}\hat{f}|_2^2\,d\xi \\
&\leqslant \int_{\mathbf{R}^3}(1+|\xi|^2)^N\mathcal{R}\langle\hat{f},\hat{g}\rangle\,d\xi + c_\varepsilon\sum_{\ell=1}^{14}\int_{\mathbf{R}^3}(1+|\xi|^2)^N\left|\langle\hat{g},e_\ell\rangle\right|^2\,d\xi.
\end{aligned} \tag{3.21}$$

By setting $g = \Gamma(f,f)$ in the above inequality, we have

$$\begin{aligned}
&\frac{d}{dt}\left[\sum_{|\alpha|\leqslant N}\left\|\partial^\alpha f\right\|^2 - \kappa\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|\langle iS(\omega)\hat{f},\hat{f}\rangle\,d\xi\right] \\
&\quad + \delta_1\sum_{|\alpha|\leqslant N}\left\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\right\|_\nu^2 + \delta_2\sum_{1\leqslant|\alpha|\leqslant N}\left\|\mathbf{P}\partial^\alpha f\right\|^2 \\
&\leqslant \sum_{|\alpha|\leqslant N}\int_{\mathbf{R}^3}(1+|\xi|^2)\mathcal{R}\langle\widehat{\partial^\alpha f},\hat{g}\rangle\,d\xi + c_\varepsilon\sum_{|\alpha|\leqslant N}\sum_{\ell=1}^{14}\int_{\mathbf{R}^3}\left|\langle\hat{g},e_\ell\rangle\right|^2\,d\xi.
\end{aligned} \tag{3.22}$$

Finally, by using Lemma 3.2, Lemma 3.3 and the properties of Fourier transform, similar argument as above gives (3.17) and this completes the proof of the lemma. □

3.3. Optimal time decay

The local existence of classical solution in the Sobolev space has been proved, cf. [22]. All we now need is some uniform estimate for both the global existence and the time decay estimate. For the convenience of the readers, we include the local existence result as follows.

Lemma 3.6. *For the relativistic Boltzmann equation, let $l \geqslant 0$. Then for any sufficiently small $\varepsilon > 0$ and $T^* > 0$ with $T^* \leqslant \varepsilon/2$ and $\widetilde{\mathcal{E}}_l(f)(0) \leqslant \varepsilon$, there is a unique classical solution to (1.5) in $[0, T^*) \times \mathbf{R}^3 \times \mathbf{R}^3$ such that*

$$\widetilde{\mathcal{E}}_l(f)(t) + \sum_{|\alpha|\leqslant N} \int_0^t \left\| \nu^l \partial^\alpha f(s) \right\|_\nu^2 ds \leqslant C\widetilde{\mathcal{E}}_l(f)(0),$$

and $\widetilde{\mathcal{E}}_l(f)(t)$ is continuous over $[0, T^*)$. If $F_0(x, v) = \mu + \sqrt{\mu} f_0(x, v) \geqslant 0$, then $F(t, x, v) = \mu + \sqrt{\mu} f(t, x, v) \geqslant 0$.

The proof of Theorem 1.1 can now be given in two parts as follows.

Proof of global existence. By Lemma 3.5, we have

$$\begin{aligned} &\frac{d}{dt}\left[\sum_{|\alpha|\leqslant N} \left\| \partial^\alpha f \right\|^2 - \kappa \int_{\mathbf{R}^3} \left(1 + |\xi|^2\right)^{N-1} |\xi| \left\langle iS(\omega)\hat{f}, \hat{f} \right\rangle d\xi \right] + \delta_2 \sum_{1\leqslant|\alpha|\leqslant N} \left\| \mathbf{P}\partial^\alpha f \right\|^2 \\ &\quad + \delta_1 \sum_{|\alpha|\leqslant N} \left\| (\mathbf{I}-\mathbf{P})\partial^\alpha f \right\|_\nu^2 \leqslant C\left(\widetilde{\mathcal{E}}_l(f)(t) + \widetilde{\mathcal{E}}_l^{1/2}(f)(t)\right)\widetilde{\mathcal{D}}_l(f)(t), \end{aligned} \tag{3.23}$$

where $\kappa > 0$ is small enough. From Lemma 3.4, we have

$$\begin{aligned} &\frac{d}{dt}\left[\sum_{1\leqslant|\alpha|\leqslant N} \left\| \nu^l \partial^\alpha f \right\|^2 + \left\| \nu^l (\mathbf{I}-\mathbf{P}) f \right\|^2 \right] + \sum_{|\alpha|\leqslant N} \left\| \nu^l (\mathbf{I}-\mathbf{P})\partial^\alpha f \right\|_\nu^2 \\ &\quad \leqslant C \sum_{1\leqslant|\alpha|\leqslant N} \left\| \mathbf{P}\partial^\alpha f \right\|^2 + C \sum_{|\alpha|\leqslant N} \left\| (\mathbf{I}-\mathbf{P})\partial^\alpha f \right\|_\nu^2 + C\left(\widetilde{\mathcal{E}}_l(f)(t) + \widetilde{\mathcal{E}}_l^{1/2}(f)(t)\right)\widetilde{\mathcal{D}}_l(f)(t). \end{aligned} \tag{3.24}$$

A suitable linear combination of (3.23) and (3.24) yields

$$\begin{aligned} &\frac{d}{dt}\left\{ \sum_{|\alpha|\leqslant N} \left\| \partial^\alpha f \right\|^2 - \kappa \int_{\mathbf{R}^3} \left(1 + |\xi|^2\right)^{N-1} |\xi| \left\langle iS(\omega)\hat{f}, \hat{f} \right\rangle d\xi + \sum_{1\leqslant|\alpha|\leqslant N} \left\| \nu^l \partial^\alpha f \right\|^2 + \left\| \nu^l (\mathbf{I}-\mathbf{P}) f \right\|^2 \right\} \\ &\quad + \left\{ \sum_{|\alpha|\leqslant N} \left\| \nu^l \partial^\alpha (\mathbf{I}-\mathbf{P}) f \right\|_\nu^2 + \sum_{|\alpha|\leqslant N} \left\| \partial^\alpha (\mathbf{I}-\mathbf{P}) f \right\|_\nu^2 + \sum_{1\leqslant|\alpha|\leqslant N} \left\| \partial^\alpha \mathbf{P} f \right\|^2 \right\} \\ &\leqslant C\left(\widetilde{\mathcal{E}}_l(f)(t) + \widetilde{\mathcal{E}}_l^{1/2}(f)(t)\right)\widetilde{\mathcal{D}}_l(f)(t). \end{aligned} \tag{3.25}$$

On the other hand, we can obtain from (2.2) that

$$\left| \kappa \int_{\mathbf{R}^3} \left(1 + |\xi|^2\right)^{N-1} |\xi| \left\langle iS(\omega)\hat{f}, \hat{f} \right\rangle d\xi \right| \leqslant \kappa \sum_{|\alpha|\leqslant N} \left\| \partial^\alpha f \right\|^2 + \kappa \sum_{|\alpha|\leqslant N-1} \left\| \partial^\alpha f \right\|^2.$$

Therefore, we can define equivalent functionals to (1.12) and (1.13) by

$$\mathcal{E}_l(f)(t) = \sum_{|\alpha|\leqslant N} \|\partial^\alpha f\|^2 - \kappa \int_{\mathbf{R}^3} \left(1+|\xi|^2\right)^{N-1} |\xi| \langle iS(\omega)\hat{f}, \hat{f}\rangle \, d\xi + \sum_{1\leqslant|\alpha|\leqslant N} \|\nu^l \partial^\alpha f\|^2 + \|\nu^l(\mathbf{I}-\mathbf{P})f\|^2,$$

$$\mathcal{D}_l(t) = \sum_{|\alpha|\leqslant N} \|\nu^l \partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\nu^2 + \sum_{|\alpha|\leqslant N} \|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\nu^2 + \sum_{1\leqslant|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P} f\|^2.$$

Hence,

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_l(f)(t) + \mathcal{D}_l(f)(t) &\leqslant C\left(\widetilde{\mathcal{E}}_l(f)(t) + \widetilde{\mathcal{E}}_l^{1/2}(f)(t)\right)\widetilde{\mathcal{D}}_l(f)(t) \\ &\leqslant C\left(\mathcal{E}_l(f)(t) + \mathcal{E}_l^{1/2}(f)(t)\right)\mathcal{D}_l(f)(t). \end{aligned} \tag{3.26}$$

Based on the local existence and the standard continuity argument, if $\widetilde{\mathcal{E}}_l(f)(0) \leqslant \varepsilon$ for $\varepsilon > 0$ small enough, then the global existence follows from the above uniform estimate. □

Proof of the optimal time decay. Here, we will apply the ideas in [8] on the optimal decay rate estimates. We first give the optimal time decay rate of the solution itself.

By using Lemma 2.4, it holds that

$$\begin{aligned}\|f(t)\|^2 &\leqslant C(1+t)^{-3/2}\left(\|f_0\|_{Z_1}^2 + \|f_0\|^2\right) \\ &\quad + C\int_0^t (1+t-s)^{-3/2}\left(\|\Gamma(f,f)\|_{Z_1} + \|\Gamma(f,f)\|\right)^2 ds. \end{aligned} \tag{3.27}$$

By Lemma 3.2 with $l=0$ and $\delta=0$, we know that

$$\|\Gamma(f,f)\|_{Z_1}^2 \leqslant C\|\nu f\|^2\|f\|^2 \leqslant C\mathcal{E}_l(f)(t)\|f\|^2 \leqslant C\varepsilon\|f\|^2,$$

$$\|\Gamma(f,f)\|^2 \leqslant \int_{\mathbf{R}^3} |\nu f|_2^2 |f|_2^2 \, dx \leqslant C\mathcal{E}_l(f)(t)\|f\|^2 \leqslant C\varepsilon\|f\|^2,$$

where we have assumed $l \geqslant 1$ and used $\mathcal{E}_l(f)(t) \leqslant \varepsilon$. Define

$$M_0(t) = \sup_{0\leqslant s\leqslant t}\left\{(1+s)^{3/2}\|f(s)\|^2\right\}. \tag{3.28}$$

Notice that $M_0(t)$ is non-decreasing and for any $0 \leqslant s \leqslant t$,

$$\left(\|\Gamma(f,f)\|_{Z_1} + \|\nabla_x \Gamma(f,f)\|\right)^2 \leqslant C\varepsilon\|f\|^2 \leqslant C\varepsilon(1+t)^{-3/2}M_0(t).$$

With this, we have from (3.27) that

$$M_0(t) \leqslant C\left(\|f_0\|_{Z_1}^2 + \|f_0\|^2\right) + C\varepsilon M_0(t).$$

If we take $\varepsilon > 0$ small enough, then $M_0(t) \leqslant C$ which implies that

$$\|f(t)\|^2 = \|\mathbf{P}f(t)\|^2 + \|(\mathbf{I}-\mathbf{P})f(t)\|^2 \leqslant C(1+t)^{-3/2}. \tag{3.29}$$

We now use Lemma 2.4 to obtain the optimal time decay rate of the derivatives of the solution. For this, we first deduce some energy estimates on the derivatives.

By Lemma 3.5, we have

$$\begin{aligned}&\frac{d}{dt}\Bigg[\sum_{1\leqslant|\alpha|\leqslant N}\|\partial^\alpha f\|^2 - \kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big), \widehat{\partial^\alpha f}\big\rangle d\xi\Bigg] + \delta_2\sum_{2\leqslant|\alpha|\leqslant N}\|\mathbf{P}\partial^\alpha f\|^2\\&\quad+\delta_1\sum_{1\leqslant|\alpha|\leqslant N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\nu^2 \leqslant C\big(\widetilde{\mathcal{E}}_l(f)(t) + \widetilde{\mathcal{E}}_l^{1/2}(f)(t)\big)\widetilde{\mathcal{D}}_l(f)(t),\end{aligned} \tag{3.30}$$

where $\kappa > 0$ is small enough. From Lemma 3.4, we have

$$\begin{aligned}&\frac{d}{dt}\Bigg[\sum_{1\leqslant|\alpha|\leqslant N}\|\nu^l\partial^\alpha f\|^2 + \|\nu^l(\mathbf{I}-\mathbf{P})f\|^2\Bigg] + \sum_{|\alpha|\leqslant N}\|\nu^l(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\nu^2\\&\quad\leqslant C\sum_{1\leqslant|\alpha|\leqslant N}\|\mathbf{P}\partial^\alpha f\|^2 + \sum_{|\alpha|\leqslant N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\nu^2 + C\big(\widetilde{\mathcal{E}}_l(f)(t) + \widetilde{\mathcal{E}}_l^{1/2}(f)(t)\big)\widetilde{\mathcal{D}}_l(f)(t).\end{aligned} \tag{3.31}$$

The standard energy estimate implies

$$\begin{aligned}&\frac{d}{dt}\|(\mathbf{I}-\mathbf{P})f\|^2 + \|(\mathbf{I}-\mathbf{P})f\|_\nu^2\\&\quad\leqslant C\|\nabla_x\mathbf{P}f\|^2 + C\|\nabla_x(\mathbf{I}-\mathbf{P})f\|_\nu^2 + C\big(\widetilde{\mathcal{E}}_l(f)(t) + \widetilde{\mathcal{E}}_l^{1/2}(f)(t)\big)\widetilde{\mathcal{D}}_l(f)(t).\end{aligned} \tag{3.32}$$

A suitable linear combination of (3.30), (3.31) and (3.32) yields

$$\begin{aligned}&\frac{d}{dt}\Bigg\{\sum_{1\leqslant|\alpha|\leqslant N}\|\partial^\alpha f\|^2 - \kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big), \widehat{\partial^\alpha f}\big\rangle d\xi\\&\qquad+\sum_{1\leqslant|\alpha|\leqslant N}\|\nu^l\partial^\alpha f\|^2 + \|\nu^l(\mathbf{I}-\mathbf{P})f\|^2 + \|(\mathbf{I}-\mathbf{P})f\|^2\Bigg\}\\&\qquad+\Bigg\{\sum_{|\alpha|\leqslant N}\|\nu^l\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\nu^2 + \sum_{|\alpha|\leqslant N}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_\nu^2 + \sum_{1\leqslant|\alpha|\leqslant N}\|\partial^\alpha\mathbf{P}f\|^2\Bigg\}\\&\quad\leqslant C\big(\widetilde{\mathcal{E}}_l(f)(t) + \widetilde{\mathcal{E}}_l^{1/2}(f)(t)\big)\widetilde{\mathcal{D}}_l(f)(t) + C\|\nabla_x\mathbf{P}f\|^2.\end{aligned}$$

On the other hand, we obtain from (2.2) that

$$\Bigg|\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big), \widehat{\partial^\alpha f}\big\rangle d\xi\Bigg| \leqslant \kappa\sum_{1\leqslant|\alpha|\leqslant N}\|\partial^\alpha f\|^2 + \kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\|\partial^\alpha f\|^2.$$

Therefore, we can define an energy functional by

$$H_l(f)(t)=\Big\{\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha f\big\|^2-\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big),\widehat{\partial^\alpha f}\big\rangle d\xi$$
$$+\sum_{1\leqslant|\alpha|\leqslant N}\big\|\nu^l\partial^\alpha f\big\|^2+\big\|\nu^l(\mathbf{I}-\mathbf{P})f\big\|^2+\big\|(\mathbf{I}-\mathbf{P})f\big\|^2\Big\},$$

which is bounded by $C\mathcal{D}_l(f)(t)$ for some C and satisfies

$$H_l(f)(t)\sim\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha\mathbf{P}f(t)\big\|^2+\sum_{|\alpha|\leqslant N}\big\|\partial^\alpha(\mathbf{I}-\mathbf{P})f(t)\big\|^2+\sum_{|\alpha|\leqslant N}\big\|\nu^l\partial^\alpha(\mathbf{I}-\mathbf{P})f(t)\big\|^2. \tag{3.33}$$

Hence, we have

$$\frac{d}{dt}H_l(f)(t)+\mathcal{D}_l(f)(t)\leqslant C\big(\widetilde{\mathcal{E}}_l(f)(t)+\widetilde{\mathcal{E}}_l^{1/2}(f)(t)\big)\widetilde{\mathcal{D}}_l(f)(t)+C\|\nabla_x\mathbf{P}f\|^2$$
$$\leqslant C\big(\mathcal{E}_l(f)(t)+\mathcal{E}_l^{1/2}(f)(t)\big)\mathcal{D}_l(f)(t)+C\|\nabla_x\mathbf{P}f\|^2.$$

In the proof of global existence, we know that $\mathcal{E}_l(f)(t)<\varepsilon$ for some $\varepsilon>0$ small enough. Then we have

$$\frac{d}{dt}H_l(f)(t)+\mathcal{D}_l(f)(t)\leqslant C\|\nabla_x\mathbf{P}f\|^2, \tag{3.34}$$

which together with $H_l(f)(t)\leqslant C\mathcal{D}_l(f)(t)$ give

$$\frac{d}{dt}H_l(f)(t)+cH_l(f)(t)\leqslant C\|\nabla_x\mathbf{P}f\|^2. \tag{3.35}$$

By using Lemma 2.4, we can obtain

$$\big\|\nabla_x\mathbf{P}f(t)\big\|^2\leqslant\big\|\nabla_x f(t)\big\|^2$$
$$\leqslant C\lambda_0(1+t)^{-5/2}+C\int_0^t(1+t-s)^{-5/2}\big(\big\|\Gamma(f,f)\big\|_{Z_1}+\big\|\nabla_x\Gamma(f,f)\big\|\big)^2\,ds,$$

where $\lambda_0=\|f_0\|_{Z_1}^2+\|\nabla_x f_0\|^2$.

If $l\geqslant 1$, the Hölder inequality and Lemma 3.2 with $l=0$ and $\delta=0$ give

$$\big\|\Gamma(f,f)\big\|_{Z_1}^2\leqslant C\|\nu f\|^2\|f\|^2\leqslant C\big\|\nu(\mathbf{I}-\mathbf{P})f\big\|^4+C\|\mathbf{P}f\|^4\leqslant C\mathcal{E}_l(f)(t)H_l(f)(t)+C(1+t)^{-3},$$
$$\big\|\nabla_x\Gamma(f,f)\big\|^2\leqslant C\int_{\mathbf{R}^3}\big\{|\nu\nabla_x f|_2^2|f|_2^2+|\nabla_x f|_2^2|\nu f|_2^2\big\}\,dx\leqslant C\mathcal{E}_l(f)(t)H_l(f)(t).$$

Define

$$M(t)=\sup_{0\leqslant s\leqslant t}\big\{(1+s)^{5/2}H_l(f)(s)\big\}. \tag{3.36}$$

Notice that $M(t)$ is non-decreasing and

$$\left(\left\|\Gamma(f,f)\right\|_{Z_1}+\left\|\nabla_x\Gamma(f,f)\right\|\right)^2\leqslant C\varepsilon H_l(f)(t)+C(1+t)^{-3},$$

for any $0\leqslant s\leqslant t$, where we have used the fact that $\mathcal{E}_l(f)(t)\leqslant\varepsilon$.

With this, we have

$$\begin{aligned}\left\|\nabla_x\mathbf{P}f(t)\right\|^2&\leqslant\left\|\nabla_x f(t)\right\|^2\\&\leqslant C\lambda_0(1+t)^{-5/2}+C\big(1+\varepsilon M(t)\big)\int_0^t(1+t-s)^{-5/2}(1+s)^{-5/2}\,ds\\&\leqslant C(1+t)^{-5/2}\big(1+\lambda_0+\varepsilon M(t)\big).\end{aligned}\tag{3.37}$$

On the other hand, by the Gronwall inequality, (3.35) gives

$$H_l(f)(t)\leqslant e^{-ct}H_l(f)(0)+C\int_0^t e^{-c(t-s)}\left\|\nabla_x\mathbf{P}f(s)\right\|^2ds,$$

for some constant $c>0$. Then, (3.37) yields

$$\begin{aligned}H_l(f)(t)&\leqslant e^{-ct}H_l(f)(0)+C\int_0^t e^{-c(t-s)}(1+s)^{-5/2}\,ds\big(1+\lambda_0+\varepsilon M(t)\big)\\&\leqslant C(1+t)^{-5/2}\big(1+H_l(f)(0)+\lambda_0+\varepsilon M(t)\big).\end{aligned}$$

Hence, for any $t\geqslant 0$,

$$M(t)=\sup_{0\leqslant s\leqslant t}\left\{(1+s)^{5/2}H_l(f)(s)\right\}\leqslant C\big(1+H_l(f)(0)+\lambda_0+\varepsilon M(t)\big).$$

Then, if $\varepsilon>0$ is small enough, one has

$$M(t)\leqslant C\big(1+H_l(f)(0)+\lambda_0\big).$$

From (3.36), this gives

$$H_l(f)(t)\leqslant C(1+t)^{-5/2}.\tag{3.38}$$

Notice that

$$H_l(f)(t)\sim\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha\mathbf{P}f(t)\right\|^2+\sum_{|\alpha|\leqslant N}\left\|\nu^l\partial^\alpha(\mathbf{I}-\mathbf{P})f(t)\right\|^2.$$

Thus, (1.18) is proved and this completes the proof of Theorem 1.1. □

4. Relativistic Landau equation

The strategy for obtaining the optimal decay estimate for the relativistic Landau equation is similar to the one given in the last section for the relativistic Boltzmann equation. First, we need the following basic estimates on the relativistic Landau collision operators Lf and $\Gamma(f,f)$.

4.1. Basic estimates

Lemma 4.1. *For two functions $g_1 = g_1(x,v)$ and $g_2 = g_2(x,v)$ with the following norms well defined, it holds that*

$$\big\|\Gamma(g_1,g_2)\big\|_{Z_1} \leqslant C \sum_{|\beta_1|\leqslant 2} \|\partial_{\beta_1} g_1\| \cdot \sum_{|\beta_2|\leqslant 2} \|\partial_{\beta_2} g_2\|. \tag{4.1}$$

Proof. Recall that

$$\begin{aligned}
\Gamma(g_1,g_2) &= \frac{1}{\sqrt{\mu}}\mathcal{C}(\sqrt{\mu}g_1,\sqrt{\mu}g_2)\\
&= \mu^{-1/2}(v)\partial_i \int_{\mathbf{R}^3} \Phi^{ij}(u,v)\mu^{1/2}(u)\mu^{1/2}(v)\big\{g_2(u)\partial_j g_1(v) - \partial_j g_2(u) g_1(v)\big\}\,du\\
&\quad + \mu^{-1/2}(v)\partial_i \int_{\mathbf{R}^3} \Phi^{ij}(u,v)\mu^{1/2}(u)\mu^{1/2}(v)\left\{\frac{u_j}{2u_0} - \frac{v_j}{2v_0}\right\} g_2(u) g_1(v)\,du\\
&= \mu^{-1/2}(v)\partial_i \int_{\mathbf{R}^3} \Phi^{ij}(u,v)\mu^{1/2}(u)\mu^{1/2}(v)\big\{g_2(u)\partial_j g_1(v) - \partial_j g_2(u) g_1(v)\big\}\,du\\
&= \left(\partial_i - \frac{v_i}{2v_0}\right) \int_{\mathbf{R}^3} \Phi^{ij}(u,v)\mu^{1/2}(u)\big\{g_2(u)\partial_j g_1(v) - \partial_j g_2(u) g_1(v)\big\}\,du,
\end{aligned} \tag{4.2}$$

where we have used the fact that

$$\sum_{i=1}^{3} \Phi^{ij}(u,v)\left(\frac{u_i}{u_0} - \frac{v_i}{v_0}\right) = \sum_{j=1}^{3} \Phi^{ij}(u,v)\left(\frac{u_j}{u_0} - \frac{v_j}{v_0}\right) = 0.$$

As [32], we will use the following relativistic differential operator:

$$\Theta_\alpha \equiv \Theta_\alpha(u,v) = \left(\partial_{v_3} + \frac{u_0}{v_0}\partial_{u_3}\right)^{\alpha_3}\left(\partial_{v_2} + \frac{u_0}{v_0}\partial_{u_2}\right)^{\alpha_2}\left(\partial_{v_1} + \frac{u_0}{v_0}\partial_{u_1}\right)^{\alpha_1}.$$

By using Theorem 3 in [32], we have

$$\begin{aligned}
&\partial_i \int_{\mathbf{R}^3} \Phi^{ij}(u,v)\mu^{1/2}(u)\big\{g_2(u)\partial_j g_1(v) - \partial_j g_2(u) g_1(v)\big\}\,du\\
&\quad = \int_{\mathbf{R}^3} \Theta_{e_i}\Phi^{ij}(u,v)\mu^{1/2}(u)\big\{g_2(u)\partial_j g_1(v) - \partial_j g_2(u) g_1(v)\big\}\varphi^{e_i}_{e_i,0,0}(u,v)\,du
\end{aligned}$$

$$+\int_{\mathbf{R}^3} \Phi^{ij}(u,v)\mu^{1/2}(u)\partial_u^{e_i}\big\{g_2(u)\partial_j g_1(v)-\partial_j g_2(u)g_1(v)\big\}\varphi^{e_i}_{0,e_i,0}(u,v)\,du$$

$$+\int_{\mathbf{R}^3} \Phi^{ij}(u,v)\mu^{1/2}(u)\partial_v^{e_i}\big\{g_2(u)\partial_j g_1(v)-\partial_j g_2(u)g_1(v)\big\}\varphi^{e_i}_{0,0,e_i}(u,v)\,du, \tag{4.3}$$

where $\varphi^{e_i}_{\beta_1,\beta_2,\beta_3}(u,v)$ is a smooth function satisfying

$$\big|\varphi^{e_i}_{\beta_1,\beta_2,\beta_3}(u,v)\big| \leqslant Cu_0, \tag{4.4}$$

for any multi-indices $e_i=\beta_1+\beta_2+\beta_3$. By using (4.2), (4.3) and (4.4), we have

$$\begin{aligned}
&\left\|\left(\partial_i-\frac{v_i}{2v_0}\right)\int_{\mathbf{R}^3}\Phi^{ij}(u,v)\mu^{1/2}(u)\big\{g_2(u)\partial_j g_1(v)-\partial_j g_2(u)g_1(v)\big\}\,du\right\|_{L^1_x}\\
&\leqslant C\int_{\mathbf{R}^3} u_0\mu^{1/2}(u)\big[\big|\Phi^{ij}(u,v)\big|+\big|\Theta_{e_i}\Phi^{ij}(u,v)\big|\big]\left\{\left[\sum_{|\beta_1|\leqslant 2}\big\|\partial_{\beta_1}g_1(v)\big\|_{L^2_x}\right]\right.\\
&\quad\left.\times\left[\sum_{|\beta_2|\leqslant 2}\big\|\partial_{\beta_2}g_2(u)\big\|_{L^2_x}\right]+\left[\sum_{|\beta_1|\leqslant 2}\big\|\partial_{\beta_1}g_1(u)\big\|_{L^2_x}\right]\cdot\left[\sum_{|\beta_2|\leqslant 2}\big\|\partial_{\beta_2}g_2(v)\big\|_{L^2_x}\right]\right\}du\\
&\leqslant C\left(\int_{\mathbf{R}^3} u_0^2\mu(u)\big[\big|\Phi^{ij}(u,v)\big|+\big|\Theta_{e_i}\Phi^{ij}(u,v)\big|\big]^2\,du\right)^{1/2}\left\{\left[\sum_{|\beta_1|\leqslant 2}\big\|\partial_{\beta_1}g_1(v)\big\|_{L^2_x}\right]\right.\\
&\quad\left.\times\left[\sum_{|\beta_2|\leqslant 2}\|\partial_{\beta_2}g_2\|\right]+\left[\sum_{|\beta_1|\leqslant 2}\|\partial_{\beta_1}g_1\|\right]\cdot\left[\sum_{|\beta_2|\leqslant 2}\big\|\partial_{\beta_2}g_2(v)\big\|_{L^2_x}\right]\right\}.
\end{aligned}\tag{4.5}$$

By Lemma 2 in [32], we know that on the set $\mathcal{A}=\{|u-v|+|u\times v|\geqslant[|v|+1]/2\}$

$$\big|\Theta_\alpha\Phi^{ij}(u,v)\big|\leqslant Cv_0^{-|\alpha|}u_0^6, \tag{4.6}$$

and on the set $\mathcal{B}=\{|u-v|+|u\times v|\leqslant[|v|+1]/2\}$

$$\big|\Theta_\alpha\Phi^{ij}(u,v)\big|\leqslant Cu_0^7v_0^{-|\alpha|}|u-v|^{-1}. \tag{4.7}$$

Then on the set $\mathcal{A}$, we use (4.6) to obtain

$$\big[\big|\Phi^{ij}(u,v)\big|+\big|\Theta_{e_i}\Phi^{ij}(u,v)\big|\big]\leqslant u_0^6,$$

which implies

$$\left(\int_{\mathbf{R}^3} u_0^2\mu(u)\big[\big|\Phi^{ij}(u,v)\big|+\big|\Theta_{e_i}\Phi^{ij}(u,v)\big|\big]^2\,du\right)^{1/2}\leqslant C_1. \tag{4.8}$$

On the set $\mathcal{B}$, we use (4.7) to obtain

$$\big[\big|\Phi^{ij}(u,v)\big| + \big|\Theta_{e_i}\Phi^{ij}(u,v)\big|\big] \leqslant u_0^7|u-v|^{-1},$$

which implies

$$\left(\int\limits_{\mathbf{R}^3} u_0^2\mu(u)\big[\big|\Phi^{ij}(u,v)\big| + \big|\Theta_{e_i}\Phi^{ij}(u,v)\big|\big]^2\,du\right)^{1/2} \leqslant C_2\big[1+|v|\big]^{-1} \leqslant C_2. \tag{4.9}$$

From (4.5), (4.8) and (4.9), we obtain

$$\big\|\Gamma(g_1,g_2)\big\|_{Z_1} \leqslant C\sum_{|\beta_1|\leqslant 2}\|\partial_{\beta_1}g_1\|\sum_{|\beta_2|\leqslant 2}\|\partial_{\beta_2}g_2\|.$$

And this completes the proof of the lemma. □

Lemma 4.2. *For any $f=f(t,x,v)$ with the following norms well defined, it holds that*

$$\sum_{|\alpha|\leqslant 1}\big\|\partial^\alpha\Gamma(f,f)\big\|^2 \leqslant C\widetilde{\mathcal{E}}(t)\bar{\mathcal{E}}(t) \leqslant C\widetilde{\mathcal{E}}(t)\widetilde{\mathcal{D}}(t), \tag{4.10}$$

where $\bar{\mathcal{E}}(t)$ is defined by

$$\bar{\mathcal{E}}(t) = \sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha\mathbf{P}f(t)\big\|^2 + \sum_{|\alpha|+|\beta|\leqslant N}\big\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f(t)\big\|^2. \tag{4.11}$$

Proof. By (4.2), we have

$$\begin{aligned}\partial^\alpha\Gamma(f,f) &= \frac{1}{\sqrt{\mu}}\mathcal{C}\big(\sqrt{\mu}\partial^{\alpha_1}f,\sqrt{\mu}\partial^{\alpha_2}f\big)\\ &= \sum_{\alpha_1+\alpha_2=\alpha}C_\alpha^{\alpha_1}\left(\partial_i - \frac{v_i}{2v_0}\right)\int\limits_{\mathbf{R}^3}\Phi^{ij}(u,v)\mu^{1/2}(u)\\ &\quad\times\big\{\partial^{\alpha_2}f(u)\partial_j^{\alpha_1}f(v) - \partial_j^{\alpha_2}f(u)\partial^{\alpha_1}f(v)\big\}\,du.\end{aligned} \tag{4.12}$$

By using Theorem 3 in [32], we have

$$\begin{aligned}&\partial_i\int\limits_{\mathbf{R}^3}\Phi^{ij}(u,v)\mu^{1/2}(u)\big\{\partial^{\alpha_2}f(u)\partial_j^{\alpha_1}f(v) - \partial_j^{\alpha_2}f(u)\partial^{\alpha_1}f(v)\big\}\,du\\ &= \int\limits_{\mathbf{R}^3}\Theta_{e_i}\Phi^{ij}(u,v)\mu^{1/2}(u)\big\{\partial^{\alpha_2}f(u)\partial_j^{\alpha_1}f(v) - \partial_j^{\alpha_2}f(u)\partial^{\alpha_1}f(v)\big\}\varphi^{e_i}_{e_i,0,0}(u,v)\,du\\ &\quad+\int\limits_{\mathbf{R}^3}\Phi^{ij}(u,v)\mu^{1/2}(u)\partial_u^{e_i}\big\{\partial^{\alpha_2}f(u)\partial_j^{\alpha_1}f(v) - \partial_j^{\alpha_2}f(u)\partial^{\alpha_1}f(v)\big\}\varphi^{e_i}_{0,e_i,0}(u,v)\,du\\ &\quad+\int\limits_{\mathbf{R}^3}\Phi^{ij}(u,v)\mu^{1/2}(u)\partial_v^{e_i}\big\{\partial^{\alpha_2}f(u)\partial_j^{\alpha_1}f(v) - \partial_j^{\alpha_2}f(u)\partial^{\alpha_1}f(v)\big\}\varphi^{e_i}_{0,0,e_i}(u,v)\,du,\end{aligned} \tag{4.13}$$

where $\varphi^{e_i}_{\beta_1,\beta_2,\beta_3}(u,v)$ is a smooth function satisfying

$$\left|\varphi^{e_i}_{\beta_1,\beta_2,\beta_3}(u,v)\right| \leqslant Cu_0, \tag{4.14}$$

for any multi-indices $e_i = \beta_1+\beta_2+\beta_3$. Without loss of generality, suppose $|\alpha_1| = 0$ and $|\alpha_2| = 1$. By using (4.12), (4.13) and the Hölder inequality, we obtain

$$\begin{aligned}
&\left\| \left(\partial_i - \frac{v_i}{2v_0}\right) \int_{\mathbf{R}^3} \Phi^{ij}(u,v)\mu^{1/2}(u)\left\{\partial^{\alpha_2} f(u)\partial_j f(v) - \partial_j^{\alpha_2} f(u) f(v)\right\} du \right\|_{L^2_x} \\
&\leqslant C \int_{\mathbf{R}^3} u_0\mu^{1/2}(u)\left[\left|\Phi^{ij}(u,v)\right| + \left|\Theta_{e_i}\Phi^{ij}(u,v)\right|\right] \left\{ \left[\sum_{|\beta_1|\leqslant 2} \left\|\partial_{\beta_1} f(v)\right\|_{L^\infty_x}\right] \right. \\
&\quad \times \left[\sum_{|\beta_2|\leqslant 2} \left\|\partial^{\alpha_2}_{\beta_2} f(u)\right\|_{L^2_x}\right] + \left[\sum_{|\beta_1|\leqslant 2} \left\|\partial_{\beta_1} f(u)\right\|_{L^\infty_x}\right] \cdot \left.\left[\sum_{|\beta_2|\leqslant 2} \left\|\partial^{\alpha_2}_{\beta_2} f(v)\right\|_{L^2_x}\right]\right\} du \\
&\leqslant C\left(\int_{\mathbf{R}^3} q_0^2\mu(u)\left[\left|\Phi^{ij}(u,v)\right| + \left|\Theta_{e_i}\Phi^{ij}(u,v)\right|\right]^2 du\right)^{1/2} \left\{ \left[\sum_{|\alpha'|\leqslant 1,\, |\beta_1|\leqslant 2} \left\|\nabla_x \partial^{\alpha'}_{\beta_1} f(v)\right\|_{L^2_x}\right] \right. \\
&\quad \times \left[\sum_{|\beta_2|\leqslant 2} \left\|\partial^{\alpha_2}_{\beta_2} f\right\|\right] + \left[\sum_{|\alpha'|\leqslant 1,\, |\beta_1|\leqslant 2} \left\|\nabla_x \partial^{\alpha'}_{\beta_1} f\right\|\right] \cdot \left.\left[\sum_{|\beta_2|\leqslant 2} \left\|\partial^{\alpha_2}_{\beta_2} f(v)\right\|_{L^2_x}\right]\right\} du.
\end{aligned} \tag{4.15}$$

By (4.8) and (4.9), we get

$$\left(\int_{\mathbf{R}^3} u_0^2\mu(u)\left[\left|\Phi^{ij}(u,v)\right| + \left|\Theta_{e_i}\Phi^{ij}(u,v)\right|\right]^2 du\right)^{1/2} \leqslant C.$$

Taking integration over $\mathbf{R}^3_v$ with respect to the momentum variable gives

$$\sum_{|\alpha|\leqslant 1} \left\|\partial^\alpha \Gamma(f,f)\right\|^2 \leqslant C\left[\sum_{|\alpha|+|\beta|\leqslant N} \left\|\partial^\alpha_\beta f\right\|^2\right] \cdot \left[\sum_{1\leqslant|\alpha|,\, |\alpha|+|\beta|\leqslant N} \left\|\partial^\alpha_\beta f\right\|^2\right] \leqslant C\mathcal{E}(t)\bar{\mathcal{E}}(t).$$

And this completes the proof of the lemma. □

We now recall some basic estimates from [32] which will be used later.

Lemma 4.3. *(See [32].) For any $\eta > 0$, there exist $C_\eta > 0$ and $C > 0$ such that*

$$\langle \partial_\beta Lg, \partial_\beta g\rangle \geqslant |\partial_\beta g|^2_\sigma - \eta \sum_{\beta_1\leqslant\beta} |\partial_{\beta_1} g|^2_\sigma - C_\eta |g|^2_2, \tag{4.16}$$

$$\left|\left\langle \partial^\alpha_\beta \Gamma(f,g), \partial^\alpha_\beta h\right\rangle\right| \leqslant C\sum \left|\partial^{\alpha_1}_{\beta_1} f\right|_2 \left|\partial^{\alpha-\alpha_1}_{\beta_2} g\right|_\sigma + \left|\partial^{\alpha_1}_{\beta_1} f\right|_\sigma \left|\partial^{\alpha-\alpha_1}_{\beta_2} g\right|_2 \left|\partial^\alpha_\beta h\right|_\sigma, \tag{4.17}$$

where $|\alpha| + |\beta| \leqslant N$ and the above summation is over $\alpha_1 \leqslant \alpha$ and $\beta_1 + \beta_2 \leqslant \beta$.

For the non-linear collision operator, we also have the following estimate.

Lemma 4.4. *Let $|\alpha|+|\beta|\leqslant N$. Then there is some constant $C>0$ such that*

$$\big|\big(\partial^\alpha_\beta\Gamma(f,f),\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big)\big|\leqslant C\widetilde{\mathcal{E}}^{1/2}(t)\widetilde{\mathcal{D}}(t). \tag{4.18}$$

Proof. Note that

$$\Gamma(f,f)=\Gamma(\mathbf{P}f,\mathbf{P}f)+\Gamma\big(\mathbf{P}f,(\mathbf{I}-\mathbf{P})f\big)+\Gamma\big((\mathbf{I}-\mathbf{P})f,\mathbf{P}f\big)+\Gamma\big((\mathbf{I}-\mathbf{P})f,(\mathbf{I}-\mathbf{P})f\big).$$

By (4.17) in Lemma 4.3, we have

$$\begin{aligned}&\big|\big(\partial^\alpha_\beta\Gamma(\mathbf{P}f,\mathbf{P}f),\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big)\big|\\&\quad\leqslant C\sum\int_{\mathbf{R}^3}\big[\big|\partial^{\alpha_1}_{\beta_1}\mathbf{P}f\big|_2\big|\partial^{\alpha-\alpha_1}_{\beta_2}\mathbf{P}f\big|_\sigma+\big|\partial^{\alpha_1}_{\beta_1}\mathbf{P}f\big|_\sigma\big|\partial^{\alpha-\alpha_1}_{\beta_2}\mathbf{P}f\big|_2\big]\big|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big|_\sigma\,dx.\end{aligned} \tag{4.19}$$

Here, the summation is over $\alpha_1\leqslant\alpha$ and $\beta_1+\beta_2\leqslant\beta$. Note that there exists $C\geqslant 1$ such that for any β,

$$|\partial_\beta\mathbf{P}f|^2_\sigma\leqslant C|\mathbf{P}f|^2. \tag{4.20}$$

Without loss of generality, we assume $|\alpha_1|+|\beta_1|\leqslant N/2$. Then

$$\begin{aligned}\int_{\mathbf{R}^3}\big|\partial^{\alpha_1}_{\beta_1}\mathbf{P}f\big|_2\big|\partial^{\alpha-\alpha_1}_{\beta_2}\mathbf{P}f\big|_\sigma\big|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big|_\sigma\,dx&\leqslant C\sup_{x\in\mathbf{R}^3}\big|\partial^{\alpha_1}_{\beta_1}\mathbf{P}f\big|_2\big\|\partial^{\alpha-\alpha_1}_{\beta_2}\mathbf{P}f\big\|_\sigma\big\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big\|_\sigma\\&\leqslant C\sum_{|\bar\alpha|\leqslant 1}\big\|\nabla_x\partial^{\bar\alpha}\partial^{\alpha_1}_{\beta_1}\mathbf{P}f\big\|\big\|\partial^{\alpha-\alpha_1}_{\beta_2}\mathbf{P}f\big\|\big\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big\|_\sigma,\end{aligned}$$

which is bounded by the right-hand side of (4.18). The second term in (4.19) can be estimated similarly.

We now turn to estimate $\partial^\alpha_\beta\Gamma(\mathbf{P}f,(\mathbf{I}-\mathbf{P})f)$. By Lemma 4.3 again, we have

$$\begin{aligned}&\big|\big(\partial^\alpha_\beta\Gamma\big(\mathbf{P}f,(\mathbf{I}-\mathbf{P})f\big),\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big)\big|\\&\quad\leqslant C\sum\int_{\mathbf{R}^3}\big[\big|\partial^{\alpha_1}_{\beta_1}\mathbf{P}f\big|_2\big|\partial^{\alpha-\alpha_1}_{\beta_2}(\mathbf{I}-\mathbf{P})f\big|_\sigma+\big|\partial^{\alpha_1}_{\beta_1}(\mathbf{I}-\mathbf{P})f\big|_2\big|\partial^{\alpha-\alpha_1}_{\beta_2}\mathbf{P}f\big|_\sigma\big]\big|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big|_\sigma\,dx.\end{aligned}$$

We only need to consider the first term because the second term can be estimated similarly.

If $|\alpha_1|+|\beta_1|\leqslant N/2$, we have

$$\begin{aligned}&\int_{\mathbf{R}^3}\big|\partial^{\alpha_1}_{\beta_1}\mathbf{P}f\big|_2\big|\partial^{\alpha-\alpha_1}_{\beta_2}(\mathbf{I}-\mathbf{P})f\big|_\sigma\big|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big|_\sigma\,dx\\&\quad\leqslant C\sup_{x\in\mathbf{R}^3}\big|\partial^{\alpha_1}_{\beta_1}\mathbf{P}f\big|_2\big\|\partial^{\alpha-\alpha_1}_{\beta_2}(\mathbf{I}-\mathbf{P})f\big\|_\sigma\big\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big\|_\sigma\\&\quad\leqslant C\sum_{|\bar\alpha|\leqslant 1}\big\|\nabla_x\partial^{\bar\alpha}\partial^{\alpha_1}\mathbf{P}f\big\|\big\|\partial^{\alpha-\alpha_1}_{\beta_2}(\mathbf{I}-\mathbf{P})f\big\|_\sigma\big\|\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\big\|_\sigma.\end{aligned}$$

If $|\alpha-\alpha_1|+|\beta_2|\leqslant N/2$, then

$$\begin{aligned}&\int_{\mathbf{R}^3}\big|\partial_{\beta_1}^{\alpha_1}\mathbf{P}f\big|_2\big|\partial_{\beta_2}^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})f\big|_\sigma\big|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big|_\sigma\,dx\\&\quad\leqslant C\sup_{x\in\mathbf{R}^3}\big|\partial_{\beta_2}^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})f\big|_\sigma\,\big\|\partial^{\alpha_1}\mathbf{P}f\big\|\,\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|_\sigma^2\\&\quad\leqslant C\sum_{|\bar\alpha|\leqslant1}\big\|\nabla_x\partial^{\bar\alpha}\partial_{\beta_2}^{\alpha-\alpha_1}(\mathbf{I}-\mathbf{P})f\big\|_\sigma\,\big\|\partial^{\alpha_1}\mathbf{P}f\big\|\,\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|_\sigma,\end{aligned}$$

which is bounded by the right-hand side of (4.18). The other terms for $\Gamma(f,f)$ can be estimated similarly and this completes the proof of the lemma. □

4.2. Energy estimates

Similar to the relativistic Boltzmann equation, in this section, we will derive the energy estimates on the microscopic component of the solution. Notice that for the relativistic Landau equation, weighted energy estimate is not needed. Firstly, the relativistic Landau equation (1.5) gives

$$[\partial_t+\hat v\cdot\nabla_x+L](\mathbf{I}-\mathbf{P})f=\Gamma(f,f)-[\partial_t+\hat v\cdot\nabla_x]\mathbf{P}f. \tag{4.21}$$

Lemma 4.5. *Let $|\alpha|+|\beta|\leqslant N$ with $|\beta|\geqslant 1$. We have that*

$$\begin{aligned}&\sum_{1\leqslant|\beta|,\,|\alpha|+|\beta|\leqslant N}\left[\frac{d}{dt}\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|^2+\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|_\sigma^2\right]\\&\quad\leqslant C\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha\mathbf{P}f\big\|^2+C\sum_{|\alpha|\leqslant N}\big\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\big\|_\sigma^2+C\widetilde{\mathcal{E}}^{1/2}(t)\widetilde{\mathcal{D}}(t).\end{aligned} \tag{4.22}$$

Proof. Taking ∂_β^α $(|\beta|\geqslant1)$ of Eq. (4.21) gives

$$\begin{aligned}&[\partial_t+\hat v\cdot\nabla_x]\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f+\sum_{\beta_1<\beta}C_\beta^{\beta_1}\partial_{\beta-\beta_1}\hat v\cdot\nabla_x\partial_{\beta_1}^\alpha(\mathbf{I}-\mathbf{P})f+\partial_\beta^\alpha L(\mathbf{I}-\mathbf{P})f\\&\quad=\partial_\beta^\alpha\Gamma(f,f)-[\partial_t+\hat v\cdot\nabla_x]\partial_\beta^\alpha\mathbf{P}f-\sum_{\beta_1<\beta}C_\beta^{\beta_1}\partial_{\beta-\beta_1}\hat v\cdot\nabla_x\partial_{\beta_1}^\alpha\mathbf{P}f.\end{aligned} \tag{4.23}$$

We take the inner product of (4.23) over $\mathbf{R}^3\times\mathbf{R}^3$ with $\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f$. The first term on the left-hand side is equal to $\frac12\frac{d}{dt}\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\|^2$. By using the Hölder inequality, the second term on the left-hand side is bounded by

$$\eta\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|^2+C_\eta\sum_{\beta_1<\beta}\big\|\nabla_x\partial_{\beta_1}^\alpha(\mathbf{I}-\mathbf{P})f\big\|^2. \tag{4.24}$$

From Lemma 4.3, for any $\eta>0$, the last term on the left-hand side is bounded from below by

$$\begin{aligned}&\big(\partial_\beta^\alpha L(\mathbf{I}-\mathbf{P})f,\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big)\\&\quad\geqslant\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|_\sigma^2-\eta\sum_{\beta_1\leqslant\beta}\big\|\partial_{\beta_1}^\alpha(\mathbf{I}-\mathbf{P})f\big\|_\sigma^2-C_\eta\big\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\big\|_\sigma^2.\end{aligned} \tag{4.25}$$

For the term involving the non-linear collision operator, Lemma 4.4 gives

$$\left|\left(\partial_\beta^\alpha \Gamma(f,f), \partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right)\right| \leqslant C\widetilde{\mathcal{E}}^{1/2}(t)\widetilde{\mathcal{D}}(t).$$

Since $|\beta| \geqslant 1$, for any $\eta > 0$, the second term on the right-hand side of (4.23) satisfies

$$\left|\left([\partial_t + \hat{v}\cdot\nabla_x]\partial_\beta^\alpha \mathbf{P}f, \partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right)\right| \leqslant \eta\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2 + C_\eta\left[\left\|\partial_t\partial^\alpha\mathbf{P}f\right\|^2 + \left\|\nabla_x\partial^\alpha\mathbf{P}f\right\|^2\right].$$

From the above inequality, we claim that

$$\begin{aligned}&\left|\left([\partial_t + \hat{v}\cdot\nabla_x]\partial_\beta^\alpha \mathbf{P}f, \partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right)\right| \\ &\quad\leqslant \eta\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2 + C_\eta\left[\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha\mathbf{P}f\right\|^2 + \sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\right\|^2\right].\end{aligned} \tag{4.26}$$

In fact,

$$[\partial_t + \hat{v}\cdot\nabla_x]\partial^\alpha\mathbf{P}f = \partial^\alpha\Gamma(f,f) - [\partial_t + \hat{v}\cdot\nabla_x + L]\partial^\alpha(\mathbf{I}-\mathbf{P})f, \tag{4.27}$$

and

$$\left\langle\partial_t\partial^\alpha\mathbf{P}f, \partial^\alpha\Gamma(f,f) - [\partial_t + L]\partial^\alpha(\mathbf{I}-\mathbf{P})f\right\rangle = 0.$$

By multiplying $\partial_t\partial^\alpha\mathbf{P}f$ to (4.27) and then integrating over $\mathbf{R}^3\times\mathbf{R}^3$, we get

$$\left\|\partial_t\partial^\alpha\mathbf{P}f\right\|^2 \leqslant C\left[\left\|\nabla_x\partial^\alpha\mathbf{P}f\right\|^2 + \left\|\nabla_x\partial^\alpha(\mathbf{I}-\mathbf{P})f\right\|^2\right]. \tag{4.28}$$

For any $\eta > 0$, the last term on the right-hand side of (4.23) is bounded by

$$\left|\left(\partial_{\beta-\beta_1}\hat{v}\cdot\nabla_x\partial_{\beta_1}^\alpha\mathbf{P}f, \partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right)\right| \leqslant \eta\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2 + C_\eta\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha\mathbf{P}f\right\|^2.$$

In combination, we have

$$\begin{aligned}&\frac{d}{dt}\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|^2 + \left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2 \\ &\quad\leqslant C\eta\left(\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2 + \sum_{\beta_1<\beta}\left\|\partial_{\beta_1}^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2\right) + C\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha\mathbf{P}f\right\|^2 \\ &\qquad + \sum_{|\alpha|\leqslant N}\left\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2 + C_\eta\sum_{\beta_1<\beta}\left\|\nabla_x\partial_{\beta_1}^\alpha(\mathbf{I}-\mathbf{P})f\right\|^2 + C\widetilde{\mathcal{E}}^{1/2}(t)\widetilde{\mathcal{D}}(t).\end{aligned} \tag{4.29}$$

For any $1 \leqslant |\beta| \leqslant N$ and any $\eta > 0$ small enough, we take the summation over $|\alpha| + |\beta| \leqslant N$ by a suitable linear combination of (4.29) to give (4.22). And this completes the proof of the lemma. □

Lemma 4.6. *For Eq.* (1.5), *we have the following estimate*

$$\frac{d}{dt}\bigg[\sum_{1\leqslant|\alpha|\leqslant N}\|\partial^\alpha f\|^2-\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big),\widehat{\partial^\alpha f}\big\rangle d\xi\bigg]$$
$$+\delta_1\sum_{1\leqslant|\alpha|\leqslant N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\sigma^2+\delta_2\sum_{2\leqslant|\alpha|\leqslant N}\|\mathbf{P}\partial^\alpha f\|^2\leqslant C\big(\widetilde{\mathcal{E}}^{1/2}(t)+\widetilde{\mathcal{E}}(t)\big)\widetilde{\mathcal{D}}(t), \tag{4.30}$$

$$\frac{d}{dt}\bigg[\sum_{|\alpha|\leqslant N}\|\partial^\alpha f\|^2-\kappa\int_{\mathbf{R}^3}\big(1+|\xi|^2\big)^{N-1}|\xi|\big\langle iS(\omega)\hat f,\hat f\big\rangle d\xi\bigg]$$
$$+\delta_1\sum_{|\alpha|\leqslant N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\sigma^2+\delta_2\sum_{1\leqslant|\alpha|\leqslant N}\|\mathbf{P}\partial^\alpha f\|^2\leqslant C\big(\widetilde{\mathcal{E}}^{1/2}(t)+\widetilde{\mathcal{E}}(t)\big)\widetilde{\mathcal{D}}(t), \tag{4.31}$$

where $\kappa>0$ *is small enough.*

Proof. Set h in (2.10) to be $\partial^\alpha f$. We have

$$\partial_t\big[\big(1+|\xi|^2\big)\big|\widehat{\partial^\alpha f}\big|_2^2-\kappa|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big),\widehat{\partial^\alpha f}\big\rangle\big]+\delta_1\big(1+|\xi|^2\big)\big|(\mathbf{I}-\mathbf{P})\widehat{\partial^\alpha f}\big|_\sigma^2+\delta_2|\xi|^2\big|\mathbf{P}\widehat{\partial^\alpha f}\big|_2^2$$
$$\leqslant\big(1+|\xi|^2\big)\mathcal{R}\big\langle\widehat{\partial^\alpha f},\hat g\big\rangle+c_\varepsilon\sum_{\ell=1}^{14}\big|\langle\hat g,e_\ell\rangle\big|^2. \tag{4.32}$$

Let $g=\partial^\alpha\Gamma(f,f)$. Integrating (4.32) over ξ and summing over $1\leqslant|\alpha|\leqslant N-1$ give

$$\partial_t\bigg[\sum_{1\leqslant|\alpha|\leqslant N}\|\partial^\alpha f\|^2-\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big),\widehat{\partial^\alpha f}\big\rangle d\xi\bigg]$$
$$+\delta_1\sum_{1\leqslant|\alpha|\leqslant N}\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\|_\sigma^2+\delta_2\sum_{2\leqslant|\alpha|\leqslant N}\|\mathbf{P}\partial^\alpha f\|^2$$
$$\leqslant\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}\big(1+|\xi|^2\big)\mathcal{R}\big\langle\widehat{\partial^\alpha f},\hat g\big\rangle d\xi+c_\varepsilon\sum_{1\leqslant|\alpha|\leqslant N-1}\sum_{\ell=1}^{14}\int_{\mathbf{R}^3}\big|\langle\hat g,e_\ell\rangle\big|^2d\xi. \tag{4.33}$$

For the second term on the right-hand side of (4.33), by using Lemma 4.4 and the properties of Fourier transform, we have

$$\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}\big|\langle\hat g,e_\ell\rangle\big|^2d\xi=\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}\big|\big\langle\partial^\alpha\Gamma(f,f),e_\ell\big\rangle\big|^2dx$$
$$\leqslant C\sum_{1\leqslant|\alpha|\leqslant N-1}\sum_{\alpha'\leqslant\alpha}\int_{\mathbf{R}^3}\big[\big|\partial^{\alpha'}f\big|_2^2\big|\partial^{\alpha-\alpha'}f\big|_\sigma^2+\big|\partial^{\alpha-\alpha'}f\big|_2^2\big|\partial^{\alpha'}f\big|_\sigma^2\big]dx$$
$$\leqslant C\widetilde{\mathcal{E}}(t)\widetilde{\mathcal{D}}(t).$$

For the first term on the right-hand side of (4.33), we have

$$\left|\int_{\mathbf{R}^3}(1+|\xi|^2)\mathcal{R}\langle\widehat{\partial^\alpha f},\hat{g}\rangle\,d\xi\right|\leqslant\left|\int_{\mathbf{R}^3}\langle\widehat{\partial^\alpha f},\hat{g}\rangle\,d\xi\right|+\left|\int_{\mathbf{R}^3}|\xi|^2\langle\widehat{\partial^\alpha f},\hat{g}\rangle\,d\xi\right|. \tag{4.34}$$

For the first term of the above inequality, by using Lemma 4.4, the expression of g and the properties of Fourier transform, we get

$$\left|\int_{\mathbf{R}^3}\langle\widehat{\partial^\alpha f},\hat{g}\rangle\,d\xi\right|=\left|\left(\partial^\alpha\Gamma(f,f),\partial^\alpha f\right)\right|\leqslant\widetilde{\mathcal{E}}^{1/2}(t)\widetilde{\mathcal{D}}(t),$$

$$\left|\int_{\mathbf{R}^3}|\xi|^2\langle\widehat{\partial^\alpha f},\hat{g}\rangle\,d\xi\right|=\left|\int_{\mathbf{R}^3}\langle\widehat{\partial^{\delta_i}\partial^\alpha f},\widehat{\partial^{\delta_i}g}\rangle\,d\xi\right|=\left|\left(\partial^{\delta_i}\partial^\alpha\Gamma(f,f),\partial^{\delta_i}\partial^\alpha f\right)\right|\leqslant C\widetilde{\mathcal{E}}^{1/2}(t)\widetilde{\mathcal{D}}(t),$$

where $|\xi|^2=-(i\xi^{\delta_i})^2$, $|\delta_i|=1$ and $|\alpha|\leqslant N-1$. (4.33) and the above estimates give (4.30).

Next, let $h=f$ and $g=\Gamma(f,f)$ in (2.10). Multiplying the equation by $(1+|\xi|^2)^{N-1}$ and integrating over ξ give

$$\begin{aligned}&\partial_t\left[\int_{\mathbf{R}^3}(1+|\xi|^2)^N|\hat{f}|_2^2\,d\xi-\kappa\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|\langle iS(\omega)\hat{f},\hat{f}\rangle\,d\xi\right]\\&\quad+\delta_1\int_{\mathbf{R}^3}(1+|\xi|^2)^N\left|(\mathbf{I}-\mathbf{P})\hat{f}\right|_\sigma^2\,d\xi+\delta_2\int_{\mathbf{R}^3}(1+|\xi|^2)^{N-1}|\xi|^2|\mathbf{P}\hat{f}|_2^2\,d\xi\\&\leqslant\int_{\mathbf{R}^3}(1+|\xi|^2)^N\mathcal{R}\langle\hat{f},\hat{g}\rangle\,d\xi+c_\varepsilon\sum_{\ell=1}^{14}\int_{\mathbf{R}^3}(1+|\xi|^2)^N\left|\langle\hat{g},e_\ell\rangle\right|^2\,d\xi.\end{aligned} \tag{4.35}$$

By using Lemma 4.3, Lemma 4.4 and the properties of Fourier transform, similar argument as above gives (4.31). □

4.3. Optimal time decay

Firstly, notice that the local existence of classical solutions with spatially periodic initial data was proved in [32] and it holds also in the whole space. For the global existence and the optimal decay, again all we need is the uniform estimate. For convenience of the readers, we include the statement of the local existence as follows.

Lemma 4.7. *There exist $\varepsilon>0$ and $T^*>0$ such that if $T^*\leqslant\varepsilon$ and $\widetilde{\mathcal{E}}(0)\leqslant\varepsilon$, there is a unique solution $f(t,x,v)$ to (1.5) in $[0,T^*)\times\mathbf{R}^3\times\mathbf{R}^3$ such that*

$$\widetilde{\mathcal{E}}(t)+\sum_{|\alpha|+|\beta|\leqslant N}\int_0^t\left\|\partial_\beta^\alpha f(s)\right\|_\sigma^2\,ds\leqslant C\widetilde{\mathcal{E}}(0). \tag{4.36}$$

Moreover, $\widetilde{\mathcal{E}}(t)$ is continuous over $[0,T^*)$. If $F_0(x,v)=\mu+\sqrt{\mu}f_0(x,v)\geqslant 0$, then $F(t,x,v)=\mu+\sqrt{\mu}f(t,x,v)\geqslant 0$.

We are now ready to give the proof of Theorem 1.2 in two parts as follows.

Proof of global existence. By Lemma 4.6, we have

$$\frac{d}{dt}\left[\sum_{|\alpha|\leqslant N}\left\|\partial^\alpha f\right\|^2-\kappa\int_{\mathbf{R}^3}\left(1+|\xi|^2\right)^{N-1}|\xi|\langle iS(\omega)\hat{f},\hat{f}\rangle d\xi\right]$$
$$+\delta_1\sum_{|\alpha|\leqslant N}\left\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\right\|_\sigma^2+\delta_2\sum_{1\leqslant|\alpha|\leqslant N}\left\|\mathbf{P}\partial^\alpha f\right\|^2\leqslant C\left(\widetilde{\mathcal{E}}^{1/2}(t)+\widetilde{\mathcal{E}}(t)\right)\widetilde{\mathcal{D}}(t), \tag{4.37}$$

where $\kappa>0$ is small enough. From Lemma 4.5, we have

$$\sum_{1\leqslant|\beta|,\,|\alpha|+|\beta|\leqslant N}\left[\frac{d}{dt}\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|^2+\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2\right]$$
$$\leqslant C\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha\mathbf{P}f\right\|^2+C\sum_{|\alpha|\leqslant N}\left\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2+C\widetilde{\mathcal{E}}^{1/2}(t)\widetilde{\mathcal{D}}(t). \tag{4.38}$$

A suitable linear combination of (4.37) and (4.38) yields

$$\frac{d}{dt}\left\{\sum_{|\alpha|\leqslant N}\left\|\partial^\alpha f\right\|^2-\kappa\int_{\mathbf{R}^3}\left(1+|\xi|^2\right)^{N-1}|\xi|\langle iS(\omega)\hat{f},\hat{f}\rangle d\xi+\sum_{1\leqslant|\beta|,\,|\alpha|+|\beta|\leqslant N}\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|^2\right\}$$
$$+\left\{\sum_{|\alpha|+|\beta|\leqslant N}\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2+\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha\mathbf{P}f\right\|^2\right\}\leqslant C\left(\widetilde{\mathcal{E}}^{1/2}(t)+\widetilde{\mathcal{E}}(t)\right)\widetilde{\mathcal{D}}(t). \tag{4.39}$$

On the other hand, using the boundedness of the operator $S(\omega)$ on $L^2(\mathbf{R}^3)$ gives

$$\left|\kappa\int_{\mathbf{R}^3}\left(1+|\xi|^2\right)^{N-1}|\xi|\langle iS(\omega)\hat{f},\hat{f}\rangle d\xi\right|\leqslant\kappa\sum_{|\alpha|\leqslant N}\left\|\partial^\alpha f\right\|^2+\kappa\sum_{|\alpha|\leqslant N-1}\left\|\partial^\alpha f\right\|^2.$$

Therefore, we can define equivalent functionals to (1.14) and (1.15) by

$$\mathcal{E}(t)=\sum_{|\alpha|\leqslant N}\left\|\partial^\alpha f\right\|^2-\kappa\int_{\mathbf{R}^3}\left(1+|\xi|^2\right)^{N-1}|\xi|\langle iS(\omega)\hat{f},\hat{f}\rangle d\xi+\sum_{1\leqslant|\beta|,\,|\alpha|+|\beta|\leqslant N}\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|^2,$$
$$\mathcal{D}(t)=\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial^\alpha\mathbf{P}f\right\|^2+\sum_{|\alpha|+|\beta|\leqslant N}\left\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\right\|_\sigma^2,$$

to obtain

$$\frac{d}{dt}\mathcal{E}(t)+\mathcal{D}(t)\leqslant C\left(\widetilde{\mathcal{E}}^{1/2}(t)+\widetilde{\mathcal{E}}(t)\right)\widetilde{\mathcal{D}}(t)$$
$$\leqslant C\left(\mathcal{E}^{1/2}(t)+\mathcal{E}(t)\right)\mathcal{D}(t). \tag{4.40}$$

(4.40) together with the smallness on the initial data give the global existence by the standard continuity argument. □

Proof of the optimal time decay. Again the ideas of [8] is used. By Lemma 4.6, we have

$$\frac{d}{dt}\bigg[\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha f\big\|^2-\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big),\widehat{\partial^\alpha f}\big\rangle d\xi\bigg]$$
$$+\delta_1\sum_{1\leqslant|\alpha|\leqslant N}\big\|(\mathbf{I}-\mathbf{P})\partial^\alpha f\big\|_\sigma^2+\delta_2\sum_{2\leqslant|\alpha|\leqslant N}\big\|\mathbf{P}\partial^\alpha f\big\|^2\leqslant C\big(\widetilde{\mathcal{E}}^{1/2}(t)+\widetilde{\mathcal{E}}(t)\big)\widetilde{\mathcal{D}}(t), \tag{4.41}$$

where $\kappa>0$ is small enough. From Lemma 4.5, we have

$$\sum_{1\leqslant|\beta|,\,|\alpha|+|\beta|\leqslant N}\bigg[\frac{d}{dt}\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|^2+\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|_\sigma^2\bigg]$$
$$\leqslant C\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha\mathbf{P}f\big\|^2+C\sum_{|\alpha|\leqslant N}\big\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\big\|_\sigma^2+C\widetilde{\mathcal{E}}^{1/2}(t)\widetilde{\mathcal{D}}(t). \tag{4.42}$$

With (4.28), the standard energy estimate to (4.21) gives

$$\frac{d}{dt}\big\|(\mathbf{I}-\mathbf{P})f\big\|^2+\big\|(\mathbf{I}-\mathbf{P})f\big\|_\sigma^2\leqslant C\|\nabla_x\mathbf{P}f\|^2+C\big\|\nabla_x(\mathbf{I}-\mathbf{P})f\big\|_\sigma^2+C\widetilde{\mathcal{E}}^{1/2}(t)\widetilde{\mathcal{D}}(t). \tag{4.43}$$

A suitable linear combination of (4.41), (4.42) and (4.43) yields

$$\frac{d}{dt}\bigg\{\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha f\big\|^2-\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big),\widehat{\partial^\alpha f}\big\rangle d\xi$$
$$+\sum_{1\leqslant|\beta|,\,|\alpha|+|\beta|\leqslant N}\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|^2+\big\|(\mathbf{I}-\mathbf{P})f\big\|^2\bigg\}$$
$$+\bigg\{\sum_{|\alpha|+|\beta|\leqslant N}\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|_\sigma^2+\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha\mathbf{P}f\big\|^2\bigg\}$$
$$\leqslant C\big(\widetilde{\mathcal{E}}^{1/2}(t)+\widetilde{\mathcal{E}}(t)\big)\widetilde{\mathcal{D}}(t)+C\|\nabla_x\mathbf{P}f\|^2.$$

On the other hand, using the boundedness of the operator $S(\omega)$ on $L^2(\mathbf{R}^3)$ gives

$$\bigg|\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big),\widehat{\partial^\alpha f}\big\rangle d\xi\bigg|\leqslant\kappa\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha f\big\|^2+\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\big\|\partial^\alpha f\big\|^2.$$

Therefore, we can define an equivalent functional to (4.11) by

$$\bar{\mathcal{E}}(t)\sim H(t)=\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial^\alpha f\big\|^2+\sum_{1\leqslant|\beta|,\,|\alpha|+|\beta|\leqslant N}\big\|\partial_\beta^\alpha(\mathbf{I}-\mathbf{P})f\big\|^2+\big\|(\mathbf{I}-\mathbf{P})f\big\|^2$$
$$-\kappa\sum_{1\leqslant|\alpha|\leqslant N-1}\int_{\mathbf{R}^3}|\xi|\big\langle iS(\omega)\big(\widehat{\partial^\alpha f}\big),\widehat{\partial^\alpha f}\big\rangle d\xi,$$

to obtain

$$\frac{d}{dt}H(t)+\mathcal{D}(t)\leqslant C\big(\widetilde{\mathcal{E}}^{1/2}(t)+\widetilde{\mathcal{E}}(t)\big)\widetilde{\mathcal{D}}(t)+C\|\nabla_x\mathbf{P}f\|^2$$
$$\leqslant C\big(\mathcal{E}^{1/2}(t)+\mathcal{E}(t)\big)\mathcal{D}(t)+C\|\nabla_x\mathbf{P}f\|^2.$$

As in the proof of the existence, $\mathcal{E}(t)<\varepsilon$ for some $\varepsilon>0$ small enough. Then, we have

$$\frac{d}{dt}H(t)+\mathcal{D}(t)\leqslant C\|\nabla_x\mathbf{P}f\|^2, \tag{4.44}$$

which together with the fact that $H(t)\leqslant C\mathcal{D}(t)$ give

$$\frac{d}{dt}H(t)+cH(t)\leqslant C\|\nabla_x\mathbf{P}f\|^2. \tag{4.45}$$

By using Lemma 2.4, we obtain

$$\big\|\nabla_x\mathbf{P}f(t)\big\|^2\leqslant\big\|\nabla_x f(t)\big\|^2$$
$$\leqslant C\lambda_0(1+t)^{-5/2}+C\int_0^t(1+t-s)^{-5/2}\big(\big\|\Gamma(f,f)\big\|_{Z_1}+\big\|\nabla_x\Gamma(f,f)\big\|\big)^2\,ds,$$

where $\lambda_0=(\|f_0\|_{Z_1}+\|\nabla_x f_0\|)^2$. By Lemma 4.1 and Lemma 4.2, we have

$$\big\|\nabla_x\Gamma(f,f)\big\|^2\leqslant C\mathcal{E}(t)H(t),$$
$$\big\|\Gamma(f,f)\big\|_{Z_1}^2\leqslant C\sum_{|\beta|\leqslant 2}\big\|\partial_\beta(\mathbf{I}-\mathbf{P})f\big\|^4+C\|\mathbf{P}f\|^4\leqslant C\mathcal{E}(t)H(t)+C\|\mathbf{P}f\|^4.$$

Define

$$M(t)=\sup_{0\leqslant s\leqslant t}\big\{(1+s)^{5/2}H(s)\big\},\qquad M_0(t)=\sup_{0\leqslant s\leqslant t}\big\{(1+s)^{3/2}\big\|f(s)\big\|^2\big\}. \tag{4.46}$$

Notice that $M(t)$ and $M_0(t)$ are non-decreasing and

$$\big(\big\|\Gamma(f,f)\big\|_{Z_1}+\big\|\nabla_x\Gamma(f,f)\big\|\big)^2\leqslant C\mathcal{E}(t)H(t)+C\|\mathbf{P}f\|^4$$
$$\leqslant C\varepsilon(1+t)^{-5/2}M(t)+C(1+t)^{-3}M_0^2(t),$$

for any $0\leqslant s\leqslant t$. With this, we have

$$\big\|\nabla_x\mathbf{P}f(t)\big\|^2\leqslant\big\|\nabla_x f(t)\big\|^2$$
$$\leqslant C\lambda_0(1+t)^{-5/2}+C\big(\varepsilon M(t)+M_0^2(t)\big)\int_0^t(1+t-s)^{-5/2}(1+s)^{-5/2}\,ds$$
$$\leqslant C(1+t)^{-5/2}\big(\lambda_0+\varepsilon M(t)+M_0^2(t)\big), \tag{4.47}$$

where we have used the fact that $\mathcal{E}(t)\leqslant\varepsilon$.

On the other hand, by the Gronwall inequality, (4.45) gives

$$H(t) \leqslant e^{-ct} H(0) + C \sum_{|\alpha|=1} \int_0^t e^{-c(t-s)} \left\| \nabla_x \mathbf{P} f(s) \right\|^2 ds,$$

for some constant $c > 0$. Then, (4.47) yields

$$\begin{aligned} H(t) &\leqslant e^{-ct} H(0) + C \int_0^t e^{-c(t-s)} (1+s)^{-5/2} \, ds \big(\lambda_0 + \varepsilon M(t) + M_0^2(t) \big) \\ &\leqslant C(1+t)^{-5/2} \big(H(0) + \lambda_0 + \varepsilon M(t) + M_0^2(t) \big). \end{aligned}$$

Hence, for any $t \geqslant 0$,

$$M(t) = \sup_{0 \leqslant s \leqslant t} \big\{ (1+s)^{5/2} H(s) \big\} \leqslant C \big(H(0) + \lambda_0 + \varepsilon M(t) + M_0^2(t) \big).$$

Then, if $\varepsilon > 0$ is small enough, one has

$$M(t) \leqslant C \big(H(0) + \lambda_0 + M_0^2(t) \big). \tag{4.48}$$

From (4.46), this gives

$$H(t) \leqslant C(1+t)^{-5/2} \big(H(0) + \lambda_0 + M_0^2(t) \big).$$

By using Lemma 2.4, it holds that

$$\begin{aligned} \left\| f(t) \right\|^2 &\leqslant C(1+t)^{-3/2} \| f_0 \|_{Z_1 \cap L^2}^2 + C \int_0^t (1+t-s)^{-3/2} \big(\left\| \Gamma(f,f) \right\|_{Z_1} + \left\| \Gamma(f,f) \right\| \big)^2 ds \\ &\leqslant C(1+t)^{-3/2} \| f_0 \|_{Z_1 \cap L^2}^2 + C \big(\varepsilon M(t) + M_0^2(t) \big) \int_0^t (1+t-s)^{-3/2} (1+s)^{-5/2} \, ds \\ &\leqslant C(1+t)^{-3/2} \big(\| f_0 \|_{Z_1 \cap L^2}^2 + H(0) + \lambda_0 + M_0^2(t) \big), \end{aligned} \tag{4.49}$$

where we have used the fact that

$$\big(\left\| \Gamma(f,f) \right\|_{Z_1} + \left\| \Gamma(f,f) \right\| \big)^2 \leqslant C\varepsilon (1+t)^{-5/2} M(t) + C(1+t)^{-3} M_0^2(t).$$

From (4.49), we obtain

$$M_0(t) \leqslant C\varepsilon + C M_0^2(t).$$

Then, if $\varepsilon > 0$ is small enough, we have

$$M_0(t) \leqslant C, \tag{4.50}$$

which implies

$$\|f(t)\| \leqslant C(1+t)^{-3/4}. \tag{4.51}$$

From (4.48) and (4.50), we obtain

$$H(t) \leqslant C(1+t)^{-5/2}.$$

On the other hand, we know that

$$H(t) \sim \sum_{1\leqslant|\alpha|\leqslant N} \|\partial^\alpha \mathbf{P} f(t)\|^2 + \sum_{|\alpha|+|\beta|\leqslant N} \|\partial_\beta^\alpha (\mathbf{I}-\mathbf{P}) f(t)\|^2.$$

Thus, (1.21) is proved. This completes the proof of Theorem 1.2. □

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