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TIME ASYMPTOTIC BEHAVIOR OF THE BIPOLAR NAVIER-STOKES-POISSON SYSTEM*

Dedicated to Professor James Glimm on the occasion of his 75th birthday

Hai-Liang Li
Department of Mathematics, Capital Normal University, Beijing 100048, China
E-mail: hailiang.li.math@gmail.com

Tong Yang
Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong
E-mail: matyang@cityu.edu.hk

Chen Zou
Department of Mechanics and Aerospace Engineering, COE, Peking University, Beijing 100871, China
E-mail: zouchen@pku.edu.cn

Abstract The bipolar Navier-Stokes-Poisson system (BNSP) has been used to simulate the transport of charged particles (ions and electrons for instance) under the influence of electrostatic force governed by the self-consistent Poisson equation. The optimal L^2 time convergence rate for the global classical solution is obtained for a small initial perturbation of the constant equilibrium state. It is shown that due to the electric field, the difference of the charge densities tend to the equilibrium states at the optimal rate $(1+t)^{-\frac{3}{4}}$ in L^2-norm, while the individual momentum of the charged particles converges at the optimal rate $(1+t)^{-\frac{1}{4}}$ which is slower than the rate $(1+t)^{-\frac{3}{4}}$ for the compressible Navier-Stokes equations (NS). In addition, a new phenomenon on the charge transport is observed regarding the interplay between the two carriers that almost counteracts the influence of the electric field so that the total density and momentum of the two carriers converges at a faster rate $(1+t)^{-\frac{3}{4}+\varepsilon}$ for any small constant $\varepsilon>0$. The above estimates reveal the essential difference between the unipolar and the bipolar Navier-Stokes-Poisson systems.

Key words bipolar Navier-Stokes-Poisson system; optimal time convergence rate; spectrum analysis; total momentum

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1 Introduction

BNSP has been used to simulate the transport of charged particles under the influence of electrostatic force governed by the self-consistent Poisson equation, cf. [1]. In this paper, we consider the initial value problem (IVP) for BNSP:

$$\begin{cases} \partial_t \rho_i + \nabla \cdot J_i = 0, \\ \partial_t J_i + \nabla \cdot \Big(\dfrac{J_i \otimes J_i}{\rho_i}\Big) + \nabla p(\rho_i) = \mu \Delta \Big(\dfrac{J_i}{\rho_i}\Big) + \lambda \nabla \Big(\nabla \cdot \Big(\dfrac{J_i}{\rho_i}\Big)\Big) + \rho_i \nabla \Phi, \\ \partial_t \rho_e + \nabla \cdot J_e = 0, \\ \partial_t J_e + \nabla \cdot \Big(\dfrac{J_e \otimes J_e}{\rho_e}\Big) + \nabla q(\rho_e) = \mu \Delta \Big(\dfrac{J_e}{\rho_e}\Big) + \lambda \nabla \Big(\nabla \cdot \Big(\dfrac{J_e}{\rho_e}\Big)\Big) - \rho_e \nabla \Phi, \\ \Delta \Phi = \rho_i - \rho_e, \quad \lim\limits_{|x|\to\infty} \Phi(x) \to 0, \\ (\rho_i, \rho_e, J_i, J_e)(x,0) = (\rho_{i0}, \rho_{e0}, J_{i0}, J_{e0})(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where the unknown functions are the charge densities $\rho_i > 0, \rho_e > 0$, the momenta J_i, J_e, and the electrostatic potential Φ. Here, $p = p(\rho_i)$ and $q = q(\rho_e)$ are the pressure functions satisfying $p'(\rho) >$ and $q'(\rho) >$ for $\rho > 0$. And the viscosity coefficients μ, λ satisfy $\mu > 0$ and $\mu + \frac{2}{3}\lambda > 0$.

There have been extensive studies on the large time behavior of the unipolar Navier-Stokes-Poisson system (NSP). The global existence of weak solutions to NSP with general initial data was proved in [3, 17]. The quasi-neutral and some related asymptotic limits were studied in [2, 4, 10, 16]. In the case when the Poisson equation describes the self-gravitonal force for stellar gases, the global existence of weak solutions and asymptotic behavior were also investigated together with the stability analysis, cf. [6, 7, 11] and the references therein. In addition, the global well-posedness of NSP was proved in the Besov type space in [8]. The global existence and the optimal time convergence rates of the classical solution were obtained recently in [12], where it shows that the momentum decays at the rate $(1+t)^{-\frac{1}{4}}$ which is slower than the rate $(1+t)^{-\frac{3}{4}}$ for the compressible Navier-Stokes, cf. [9, 13–15]; while the density tends to its asymptotic state at the same rate $(1+t)^{-\frac{3}{4}}$ as the compressible Navier-Stokes. This implies that the electric field affects the large time behavior of the solution and give rise to different asymptotic behaviors of NS and NSP, cf. [5, 14, 15]. As a continuation of the study in this direction, in this paper, we will study BNSP where the interaction of the two carriers should be taken into account.

In fact, the main concern of this paper is about the existence and dynamic behavior of the global classical solution to the IVP (1.1). In particular, we analyze the effect of the electric filed and the interplay between the two carriers in the transport of charged particles to reveal its influence on the asymptotic behavior of the solutions.

The main result of this paper can be stated as follows.

Theorem 1.1 Let $p'(\rho_i) > 0$ and $q'(\rho_e) > 0$ for $\rho_i > 0$ and $\rho_e > 0$ respectively, and $\bar\rho > 0$ be a constant. If $(\rho_{i0} - \bar\rho, J_{i0}, \rho_{e0} - \bar\rho, J_{e0}) \in H^l(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ with $\delta \triangleq \|(\rho_{i0} - \bar\rho, J_{i0}, \rho_{e0} - \bar\rho, J_{e0})\|_{H^l(\mathbb{R}^3) \bigcap L^1(\mathbb{R}^3)}$ being small enough for $l \geq 4$. Then, there is a unique global classical solution $(\rho_i, J_i, \rho_e, J_e, \Phi)$ to the IVP (1.1) which satisfies

$$(\rho_i - \bar\rho, \rho_e - \bar\rho) \in C(\mathbb{R}^+, H^l(\mathbb{R}^3)) \cap C^1(\mathbb{R}^+, H^{l-1}(\mathbb{R}^3)), \tag{1.2}$$

$$(J_i, J_e) \in C(\mathbb{R}^+, H^l(\mathbb{R}^3)) \cap C^1(\mathbb{R}^+, H^{l-2}(\mathbb{R}^3)), \tag{1.3}$$

$$\Phi \in C(\mathbb{R}^+, L^6(\mathbb{R}^3)), \quad \nabla\Phi \in C(\mathbb{R}^+, H^{l+1}(\mathbb{R}^3)), \tag{1.4}$$

and

$$\|\partial_x^k(\rho_i - \rho_e)(t)\|_{L^2(\mathbb{R}^3)} \le C\delta(1+t)^{-\frac{3}{4}}, \tag{1.5}$$

$$\|\partial_x^k(J_i, J_e)(t)\|_{L^2(\mathbb{R}^3)} + \|\partial_x^k\nabla\Phi(t)\|_{L^2(\mathbb{R}^3)} \le C\delta(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \tag{1.6}$$

for $k = 0, 1$, where $C > 0$ is a constant independent of time. Furthermore, for any small and fixed constant $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ so that

$$\|(\rho_i + \rho_e, J_i + J_e)(t)\|_{L^2(\mathbb{R}^3)} \le C_\varepsilon\delta(1+t)^{-\frac{3}{4}+\varepsilon}, \tag{1.7}$$

$$\|(J_e - J_i)(t)\|_{L^2(\mathbb{R}^3)} \le C\delta(1+t)^{-\frac{1}{4}}. \tag{1.8}$$

Note that $\lim\limits_{\varepsilon\to 0+} C_\varepsilon = \infty$.

Remark 1.1 The faster time convergence rate of $J_i + J_e$ for the total momentum than the individual J_i and J_e is due to the nonlinear interaction between the two carriers that almost counteracts the influence of the electric field so that the dispersion effect leads to the faster decay of the total momentum. We like to point out that the positive constant $\varepsilon > 0$ on the right hand side of (1.7) can not to be taken to be zero in general although it can be chosen arbitrarily small. It is because the electric field still contributes to the large time behavior as one can see later in the analysis. On the other hand, it is interesting to find out the possible condition under which ε can be taken be to zero.

Furthermore, the above time convergence rates (1.5) and (1.6) are optimal for the IVP (1.1) as shown in the following theorem.

Theorem 1.2 Under the same assumption of Theorem 1.1, let $(n_0, \rho_0) = (\rho_{i0} + \rho_{e0} - 2\bar{\rho}, \rho_{e0} - \rho_{i0})$ and assume that their Fourier transforms satisfy $|\widehat{n_0}(\xi)| = \bar{n}_0 > 0$ and $|\widehat{\rho_0}(\xi)| = \bar{\rho}_0 > 0$ for $0 \le |\xi| \ll 1$ with $\bar{n}_0$ and $\bar{\rho}_0$ two constants, and $\widehat{J_{e0}}(\xi) = \widehat{J_{e0}}(\xi) = 0$ for $0 \le |\xi| \ll 1$. Then, the global solution $(\rho_i, J_i, \rho_e, J_e, \Phi)$ given by Theorem 1.1 satisfies

$$c_1(1+t)^{-\frac{3}{4}} \le \|(\rho_i - \rho_e)(t)\|_{L^2(\mathbb{R}^3)} \le c_2(1+t)^{-\frac{3}{4}}, \tag{1.9}$$

$$c_1(1+t)^{-\frac{1}{4}} \le \|(J_i, J_e)(t)\|_{L^2(\mathbb{R}^3)} \le c_2(1+t)^{-\frac{1}{4}}, \tag{1.10}$$

$$c_1(1+t)^{-\frac{1}{4}} \le \|\nabla\Phi(t)\|_{L^2(\mathbb{R}^3)} \le c_2(1+t)^{-\frac{1}{4}}, \tag{1.11}$$

where $c_1, c_2 > 0$ are constants independent of time.

Remark 1.2 Theorems 1.1–1.2 show the influence of electric field and the interaction of two carriers on the time asymptotic dynamics of the transport of the charged particles. In fact, under the influence of the electric field governed by the Poisson equation, the momenta J_i and J_e of the two carriers tend to their equilibrium states at the optimal time convergence rate $(1+t)^{-\frac{1}{4}}$ in L^2-norm which is slower than the rate $(1+t)^{-\frac{3}{4}}$ for the NS, cf. also [12] for the unipolar case. On the other hand, a new phenomenon is observed. That is, the interaction between the two carriers leads to the cancellation of the total momentum. In fact, although the momenta J_i and J_e and their difference $J_i - J_e$ all converge at the slower and optimal time-convergence rate $(1+t)^{-\frac{1}{4}}$, the total momentum $J_i + J_e$ converges at a faster rate $(1+t)^{-\frac{3}{4}+\varepsilon}$ in L^2-norm for any small $\varepsilon > 0$. The convergence rate of the total density and momentum is almost the same as that for the NS, in the sense that the linearized system for total density and momentum $(\rho_i + \rho_e, J_i + J_e)$ converges at exactly the same rate as NS, while the nonlinear system is affected by the transportation and interplay between the two carriers that make $(\rho_i + \rho_e, J_i + J_e)$ converge slightly slower.

The rest of the paper is arranged as follows. In Section 2, we will firstly reformulate the original BNSP and introduce some new variables to rewrite the equations to an equivalent form. Then, we give the spectrum analysis on the two reduced linearized systems and give the L^2 convergence rates. These estimates will be used in Section 3 for the linearized BNSP so that optimal convergence rates for the original nonlinear system will be proved.

Notations Throughout this paper, we will simply use $\|\cdot\|$ for $\|\cdot\|_{L^2(\mathbb{R}^3)}$ without any ambiguity. C and c_i with $i \geq 0$ as an integer denote some general positive constants independent of time.

2 Linearized Systems

We now reformulate the original IVP (1.1) for BNSP and give the optimal time convergence rates of the solution operator to the linearized system in this section.

2.1 Reformulation

Without the loss of generality, we assume $\bar{\rho} = 1$, $p'(1) = q'(1) = 1$, and denote

$$c = \rho_i - 1, \quad d = \rho_e - 1, \tag{2.1}$$

then the IVP (1.1) can be rewritten as

$$\begin{cases} \partial_t c + \nabla \cdot J_i = 0, \\ \partial_t J_i - \mu\Delta J_i - \lambda\nabla(\nabla \cdot J_i) + \nabla c - \nabla\Phi \\ = -\nabla \cdot \Big(\dfrac{J_i \otimes J_i}{c+1}\Big) - [\nabla p(c+1) - \nabla c] - \mu\Delta\Big(\dfrac{cJ_i}{c+1}\Big) - \lambda\nabla\Big(\nabla \cdot \Big(\dfrac{cJ_i}{c+1}\Big)\Big) + c\nabla\Phi, \\ \partial_t d + \nabla \cdot J_e = 0, \\ \partial_t J_e - \mu\Delta J_e - \lambda\nabla(\nabla \cdot J_e) + \nabla d + \nabla\Phi \\ = -\nabla \cdot \Big(\dfrac{J_e \otimes J_e}{d+1}\Big) - [\nabla q(d+1) - \nabla d] - \mu\Delta\Big(\dfrac{dJ_e}{d+1}\Big) - \lambda\nabla\Big(\nabla \cdot \Big(\dfrac{dJ_e}{d+1}\Big)\Big) - d\nabla\Phi, \\ \Phi = (-\Delta)^{-1}(d-c), \\ (c, d, J_i, J_e)(x, 0) = (c_0, d_0, J_{i0}, J_{0e})(x) = (\rho_{i0} - 1, \rho_{e0} - 1, J_{i0}, J_{0e}). \end{cases} \tag{2.2}$$

Denote

$$n = c + d, \quad w = J_e + J_i, \quad \rho = d - c, \quad m = J_e - J_i, \tag{2.3}$$

that is,

$$d = \frac{n+\rho}{2}, \quad c = \frac{n-\rho}{2}, \quad J_i = \frac{w-m}{2}, \quad J_e = \frac{w+m}{2}. \tag{2.4}$$

From (2.2) and (2.3), it follows that

$$\begin{cases} \partial_t n + \nabla \cdot w = 0, \\ \partial_t w - \mu\Delta w - \lambda\nabla(\nabla \cdot w) + \nabla n = f_1, \\ \partial_t \rho + \nabla \cdot m = 0, \\ \partial_t m - \mu\Delta m - \lambda\nabla(\nabla \cdot m) + \nabla\rho + 2\nabla\Phi = f_2, \\ \Phi = (-\Delta)^{-1}\rho, \\ (n, w, \rho, m)(x, 0) = (n_0, w_0, \rho_0, m_0)(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{2.5}$$

where $(n_0, w_0, \rho_0, m_0) =: (c_0 + d_0, J_{e0} + J_{i0}, c_0 - d_0, J_{e0} - J_{i0})$, and

$$
\begin{aligned}
f_1 &=: f_1(\rho, n, w, m, \Phi) \\
&=: -\nabla \left(p(\frac{n-\rho}{2}+1) - \left(\frac{n-\rho}{2}\right) + q(\frac{n+\rho}{2}+1) - \left(\frac{n+\rho}{2}\right)\right) \\
&\quad -\nabla \cdot \left(\frac{(w+m)\otimes(w+m)}{2(n+\rho)+4} + \frac{(w-m)\otimes(w-m)}{2(n-\rho)+4} \right) \\
&\quad -\mu\Delta \left(\frac{(n-\rho)(w-m)}{2(n-\rho)+4} + \frac{(n+\rho)(w+m)}{2(n+\rho)+4} \right) \\
&\quad -\lambda\nabla \left(\nabla \cdot \left(\frac{(n-\rho)(w-m)}{2(n-\rho)+4} + \frac{(n+\rho)(w+m)}{2(n+\rho)+4} \right)\right) - \rho\nabla\Phi, \qquad (2.6) \\
f_2 &=: f_2(\rho, n, w, m, \Phi) \\
&=: -\nabla \left(q(\frac{n+\rho}{2}+1) - \left(\frac{n+\rho}{2}\right) - p(\frac{n-\rho}{2}+1) + \left(\frac{n-\rho}{2}\right)\right) \\
&\quad -\nabla \cdot \left(\frac{(w+m)\otimes(w+m)}{2(n+\rho)+4} - \frac{(w-m)\otimes(w-m)}{2(n-\rho)+4} \right) \\
&\quad -\mu\Delta \left(\frac{(n+\rho)(w+m)}{2(n+\rho)+4} - \frac{(n-\rho)(w-m)}{2(n-\rho)+4} \right) \\
&\quad -\lambda\nabla \left(\nabla \cdot \left(\frac{(n+\rho)(w+m)}{2(n+\rho)+4} - \frac{(n-\rho)(w-m)}{2(n-\rho)+4} \right)\right) - n\nabla\Phi. \qquad (2.7)
\end{aligned}
$$

Obviously, the IVP (2.5) can be formally divided into one for the NS $(2.5)_{1,2}$ and another one for the NSP $(2.5)_{3,4,5}$, which however are coupled with each other through the nonlinear terms. To obtain the global existence and the time asymptotic behavior to the IVP (1.1), it is equivalent to study the corresponding problem for (2.5), that is, to investigate the coupled NS and NSP systems.

In the rest of this section, we consider the IVP for the linearized BNSP, which in fact consists of the linearized NS and the linearized "unipolar" NSP.

2.2 Spectral analysis of linearized NS

Let us firstly consider the IVP for the linearized NS system:

$$
\begin{cases}
\partial_t n + \nabla \cdot w = 0, \\
\partial_t w - \mu\Delta w - \lambda\nabla(\nabla\cdot w) + \nabla n = 0, \\
(n, w)(x, 0) = (n_0, w_0)(x), \quad x \in \mathbb{R}^3.
\end{cases} \tag{2.8}
$$

According to the semigroup theory, the solutions of (2.8) can be formulated as follows. Consider

$$\bar{V}_t = A\bar{V}, \quad \bar{V}(0) = \bar{V}_0, \quad t \ge 0, \tag{2.9}$$

where $\bar{V} = (\bar{n}, \bar{w})^T$ and

$$A = \begin{pmatrix} 0 & -\nabla\cdot \\ -\nabla & \mu\Delta + (\mu+\lambda)\nabla\nabla\cdot \end{pmatrix}, \tag{2.10}$$

then

$$\bar{V}(t) = \mathrm{e}^{tA}\bar{V}_0, \quad t \ge 0. \tag{2.11}$$

To derive the time convergence rates of the solution operator, we need to analyze the differential operator A by Fourier analysis. Similar to [9], we can show that the Fourier transform $\widehat{G}_{NS}$ of the Green function for (2.11) is given by

$$\widehat{G}_{NS}(\xi,t)=\begin{pmatrix}\dfrac{\lambda_+\mathrm{e}^{t\lambda_-}-\lambda_-\mathrm{e}^{t\lambda_+}}{\lambda_+-\lambda_-} & -\dfrac{\mathrm{i}\xi^T(\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-})}{\lambda_+-\lambda_-}\\ -\dfrac{\mathrm{i}\xi(\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-})}{\lambda_+-\lambda_-} & \mathrm{e}^{-\mu|\xi|^2t}\Big(I-\dfrac{\xi\otimes\xi}{|\xi|^2}\Big)+\dfrac{\xi\otimes\xi}{|\xi|^2}\dfrac{\lambda_+\mathrm{e}^{\lambda_+t}-\lambda_-\mathrm{e}^{\lambda_-t}}{\lambda_+-\lambda_-}\end{pmatrix},\tag{2.12}$$

where

$$\lambda_\pm=-\frac{1}{2}(\mu+\lambda)|\xi|^2\pm\frac{1}{2}\sqrt{(\mu+\lambda)^2|\xi|^4-4|\xi|^2}.$$

Moreover, for $|\xi|\ll 1$, we have

$$\frac{\lambda_+\mathrm{e}^{t\lambda_-}-\lambda_-\mathrm{e}^{t\lambda_+}}{\lambda_+-\lambda_-}\sim\mathrm{e}^{-\frac{1}{2}(\mu+\lambda)|\xi|^2t}\Big[\cos(bt)+\frac{1}{2}(\mu+\lambda)\frac{\sin(bt)}{b}|\xi|^2\Big],\tag{2.13}$$

$$\frac{\lambda_+\mathrm{e}^{t\lambda_+}-\lambda_-\mathrm{e}^{t\lambda_-}}{\lambda_+-\lambda_-}\sim\mathrm{e}^{-\frac{1}{2}(\mu+\lambda)|\xi|^2t}\Big[\cos(bt)-\frac{1}{2}(\mu+\lambda)\frac{\sin(bt)}{b}|\xi|^2\Big],\tag{2.14}$$

$$\frac{\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-}}{\lambda_+-\lambda_-}\sim\frac{\sin(bt)}{b}\mathrm{e}^{-\frac{1}{2}(\mu+\lambda)|\xi|^2t},\tag{2.15}$$

where

$$b=\frac{1}{2}\sqrt{4|\xi|^2-(\mu+\lambda)|\xi|^4}\sim|\xi|+\mathcal{O}(|\xi|^2),\quad |\xi|\ll 1.$$

In fact, for (2.8), the large time behavior of the solutions and optimal time-convergence rate have been investigated [9, 13–15], and the result can be summarized as follows.

Proposition 2.1 Denote $\bar{V}(t)\triangleq(\bar{n}(t),\bar{w}(t))^T$, and let $\bar{V}_0\triangleq(n_0,w_0)\in H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3), l\geq 4$. Then, $(\bar{n},\bar{w})$ solves the IVP (2.8) and satisfies for $0\leq k\leq l$ that

$$\|\partial_x^k(\bar{n},\bar{w})(t)\|_{L^2(\mathbb{R}^3)}\leq C(1+t)^{-\frac{3}{4}-\frac{k}{2}}(\|\bar{V}_0\|_{L^1(\mathbb{R}^3)}+\|\partial_x^k\bar{V}_0\|_{L^2(\mathbb{R}^3)}).\tag{2.16}$$

It should be noted that the L^2-convergence rates derived above are optimal. Indeed, we have

Proposition 2.2 Let $\bar{V}_0=(n_0,w_0)\in H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)$, $l\geq 4$. Assume that the Fourier transform $\widehat{n}_0=\mathcal{F}(n_0)$ satisfies $|\widehat{n}_0(\xi)|\geq\bar{n}_0>0$ for $0\leq|\xi|\ll 1$ with $\bar{n}_0$ being a constant, and $\widehat{m}_0(\xi)=0$ for $0\leq|\xi|\ll 1$. Then, the solution $(\bar{n},\bar{w})$ to IVP (2.8) given by Proposition 2.1 satisfies for $t\geq 0$ that

$$c_1(1+t)^{-\frac{3}{4}}\leq\|\bar{n}(t)\|\leq c_2(1+t)^{-\frac{3}{4}},\tag{2.17}$$

$$c_1(1+t)^{-\frac{3}{4}}\leq\|\bar{w}(t)\|\leq c_2(1+t)^{-\frac{3}{4}}.\tag{2.18}$$

Proof We only deal with the lower bound of $\|\bar{w}(t)\|$ for simplicity. From (2.14) and (2.15), we obtain

$$\widehat{\bar{w}}(\xi,t)\sim\begin{cases}-\mathrm{i}\xi\widehat{n}_0\dfrac{\sin(bt)}{b}\mathrm{e}^{-\frac{1}{2}(\mu+\lambda)|\xi|^2t}:=T_1, & |\xi|\ll 1,\\ \mathcal{O}(1)\mathrm{e}^{-R_0t}(|\widehat{n}_0|+|\widehat{w}_0|):=T, & |\xi|\gg 1,\end{cases}$$

with $R_0>0$ being a constant and b was used in (2.13)–(2.15). By Plancherel formula, we have

$$\|\bar{w}\|^2=\|\widehat{\bar{w}}\|^2=\int_{|\xi|\leq\eta}|\widehat{\bar{w}}(\xi,t)|^2\mathrm{d}\xi+\int_{|\xi|\geq\eta}|\widehat{\bar{w}}(\xi,t)|^2\mathrm{d}\xi\geq\int_{|\xi|\leq\eta}|T_1|^2\mathrm{d}\xi-C\mathrm{e}^{-2R_0t}.\tag{2.19}$$

From now on, $\eta > 0$ denotes a small but fixed constant. By direct calculation, we have

$$\int_{|\xi|\le\eta} |T_1|^2 \mathrm{d}\xi \ge C \int_{|\xi|\le\eta} \mathrm{e}^{-(\mu+\lambda)|\xi|^2 t} \sin^2(|\xi|t + O(|\xi|^3 t))\mathrm{d}\xi. \tag{2.20}$$

Since

$$\sin^2(|\xi|t + O(|\xi|^3 t)) \ge \frac{1}{2}\sin^2(|\xi|t) - O([|\xi|^3 t]^2),$$

and

$$\begin{aligned}\int_{|\xi|\le\eta} |T_1|^2 \mathrm{d}\xi \;&\ge\; C_1 \int_{|\xi|\le\eta} \mathrm{e}^{-(\mu+\lambda)|\xi|^2 t} \sin^2(|\xi|t)\mathrm{d}\xi - C \int_{|\xi|\le\eta} \mathrm{e}^{-(\mu+\lambda)|\xi|^2 t}(|\xi|^3 t)^2 \mathrm{d}\xi \\ &:= I_1 - I_2, \end{aligned} \tag{2.21}$$

direct calculation also yields

$$|I_2| \le C(1+t)^{-\frac{5}{2}}. \tag{2.22}$$

On the other hand, for $t \ge t_0$ with $t_0 > 0$ being large enough, we have

$$\begin{aligned} I_1 &= C_1 t^{-\frac{3}{2}} \int_{r\le\eta\sqrt{t}} \mathrm{e}^{-(\mu+\lambda)r^2} \sin^2(r\sqrt{t}) r^2 \mathrm{d}r \\ &\ge C_1(1+t)^{-\frac{3}{2}} \sum_{k=0}^{[\frac{\eta t}{\pi}]} \int_{\frac{k\pi+\frac{\pi}{4}}{\sqrt{t}}}^{\frac{k\pi+\frac{\pi}{2}}{\sqrt{t}}} \mathrm{e}^{-(\mu+\lambda)r^2} \sin^2(r\sqrt{t}) r^2 \mathrm{d}r \\ &\ge \frac{1}{2} C_1(1+t)^{-\frac{3}{2}} \sum_{k=0}^{[\frac{\eta t}{\pi}]} \int_{\frac{k\pi+\frac{\pi}{4}}{\sqrt{t}}}^{\frac{k\pi+\frac{\pi}{2}}{\sqrt{t}}} \mathrm{e}^{-(\mu+\lambda)r^2} r^2 \mathrm{d}r \\ &\ge C_2(1+t)^{-\frac{3}{2}} \int_0^\infty \mathrm{e}^{-(\mu+\lambda)r^2} r^2 \mathrm{d}r \ge C_3(1+t)^{-\frac{3}{2}}, \end{aligned} \tag{2.23}$$

for some positive constants C_i, $i = 1, 2, 3$. From (2.19), (2.21)–(2.22) and (2.23), we obtain the lower bound of the time convergence rate for $\bar{w}$ as

$$\|\bar{w}(t)\| \ge C_3(1+t)^{-\frac{3}{4}}, \quad t \ge t_0.$$

Since $\|\bar{w}(t)\| \ge c(t_0)$ for some positive constant $c(t_0)$ when $0 \le t \le t_0$, the lower bound of (2.18) follows by choosing $c_1 = \min\{C_3, c(t_0)\}$. Similarly, we can obtain the lower bound of the time convergence rate of $\bar{n}$, and this completes the proof of the proposition.

2.3 Spectral analysis of linearized NSP

Consider the IVP for the linearized NSP system:

$$\begin{cases} \partial_t \rho + \nabla\cdot m = 0, \\ \partial_t m - \mu\Delta m - \lambda\nabla(\nabla\cdot m) + \nabla\rho + 2\nabla\Phi = 0, \\ \Phi = (-\Delta)^{-1}\rho, \\ (\rho, m)(x,0) = (\rho_0, m_0)(x), \quad x\in\mathbb{R}^3. \end{cases} \tag{2.24}$$

Again, the global solution $\bar{Y} = (\bar{\rho}, \bar{m})^T$ to linear IVP (2.24) can be formulated as follows. Consider

$$\bar{Y}_t = B\bar{Y}, \quad \bar{Y}(0) = \bar{Y}_0, \quad t \ge 0, \tag{2.25}$$

where

$$B=\begin{pmatrix}0 & -\nabla\cdot\\ -\nabla-\nabla(-\Delta)^{-1} & \mu\Delta+(\mu+\lambda)\nabla\nabla\cdot\end{pmatrix}. \tag{2.26}$$

We have

$$\bar Y(t)=\mathrm{e}^{tB}\bar Y_0,\quad t\geq 0. \tag{2.27}$$

The Fourier transform $\widehat G_{NSP}$ of the Green function for (2.27) is given by

$$\widehat G_{NSP}(\xi,t)=\begin{pmatrix}\dfrac{\lambda_+\mathrm{e}^{t\lambda_-}-\lambda_-\mathrm{e}^{t\lambda_+}}{\lambda_+-\lambda_-} & -\dfrac{\mathrm{i}\xi^T(\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-})}{\lambda_+-\lambda_-}\\ -\dfrac{\mathrm{i}\xi(\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-})}{\lambda_+-\lambda_-} & \mathrm{e}^{-\mu|\xi|^2t}\Big(I-\dfrac{\xi\otimes\xi}{|\xi|^2}\Big)+\dfrac{\xi\otimes\xi}{|\xi|^2}\dfrac{\lambda_+\mathrm{e}^{\lambda_+t}-\lambda_-\mathrm{e}^{\lambda_-t}}{\lambda_+-\lambda_-}\end{pmatrix}$$
$$+\frac{\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-}}{\lambda_+-\lambda_-}\begin{pmatrix}0 & 0\\ \dfrac{-\mathrm{i}\xi}{\frac12|\xi|^2} & 0\end{pmatrix}, \tag{2.28}$$

where

$$\lambda_\pm=-\frac12(\mu+\lambda)|\xi|^2\pm\frac12\sqrt{(\mu+\lambda)^2|\xi|^4-4(|\xi|^2+2)}.$$

Furthermore, for $|\xi|\ll 1$, we have

$$\frac{\lambda_+\mathrm{e}^{t\lambda_-}-\lambda_-\mathrm{e}^{t\lambda_+}}{\lambda_+-\lambda_-}\sim\mathrm{e}^{-\frac12(\mu+\lambda)|\xi|^2t}\Big[\cos(bt)+\frac12(\mu+\lambda)\frac{\sin(bt)}{b}|\xi|^2\Big], \tag{2.29}$$

$$\frac{\lambda_+\mathrm{e}^{t\lambda_+}-\lambda_-\mathrm{e}^{t\lambda_-}}{\lambda_+-\lambda_-}\sim\mathrm{e}^{-\frac12(\mu+\lambda)|\xi|^2t}\Big[\cos(bt)-\frac12(\mu+\lambda)\frac{\sin(bt)}{b}|\xi|^2\Big], \tag{2.30}$$

$$\frac{\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-}}{\lambda_+-\lambda_-}\sim\frac{\sin(bt)}{b}\mathrm{e}^{-\frac12(\mu+\lambda)|\xi|^2t}, \tag{2.31}$$

where

$$b=\frac12\sqrt{4(2+|\xi|^2)-(\mu+\lambda)|\xi|^4}\sim\Big(\sqrt2+\frac{|\xi|^2}{2\sqrt2}\Big)+\mathcal{O}(|\xi|^4),\quad |\xi|\ll 1.$$

The semigroup and spectral analysis of the IVP (2.24) for linearized NSP system was investigated recently in [12], where the L^2 time convergence rates of global strong solution to the IVP (2.24) can be stated as follows.

Proposition 2.3 Denote $\bar Y(t)\triangleq(\bar\rho(t),\bar m(t))^T$, and let $\bar Y_0\triangleq(\rho_0,m_0)\in H^l(\mathbb{R}^3)\cap L^1(\mathbb{R}^3), l\geq 4$. Then, $(\bar\rho,\bar m,\bar E)$ with $\bar E=\nabla(-\Delta)^{-1}\bar\rho$ solves the IVP (2.24) and satisfies for $0\leq k\leq l$ that

$$\|\partial_x^k\bar\rho(t)\|\leq C(1+t)^{-\frac34-\frac k2}(\|\bar Y_0\|_{L^1(\mathbb{R}^3)}+\|\partial_x^k\bar Y_0\|), \tag{2.32}$$

$$\|\partial_x^k(\bar m,\bar E)(t)\|\leq C(1+t)^{-\frac14-\frac k2}(\|\bar Y_0\|_{L^1(\mathbb{R}^3)}+\|\partial_x^k\bar Y_0\|). \tag{2.33}$$

It should be noted that the above estimates are also optimal. Indeed, by using a similar argument as the one in [12], we can show the following lower bound of the time convergence rates of the semigroup. Since the proof is almost the same as the one in [12] for the following proposition, we omit the proof for brevity.

Proposition 2.4 Let $\bar{Y}_0 = (\rho_0, m_0) \in H^l(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ and assume that the Fourier transform $\widehat{\rho}_0$ satisfies $|\widehat{\rho}_0(\xi)| > \bar{\rho}_0 > 0$ for $0 \le |\xi| \ll 1$ with $\bar{\rho}_0$ being a constant. Then, the solution $(\bar{\rho}, \bar{m}, \bar{E})$ to IVP (2.24) given by Proposition 3.1 satisfies for $t \ge 0$ that

$$c_4(1+t)^{-\frac{3}{4}} \le \|\bar{\rho}(t)\| \le c_5(1+t)^{-\frac{3}{4}}, \tag{2.34}$$

$$c_4(1+t)^{-\frac{1}{4}} \le \|\bar{m}(t)\| \le c_5(1+t)^{-\frac{1}{4}}, \tag{2.35}$$

$$c_4(1+t)^{-\frac{1}{4}} \le \|\bar{E}(t)\| \le c_5(1+t)^{-\frac{1}{4}}. \tag{2.36}$$

3 Nonlinear System

3.1 Formulation

With the above preparation, we are ready to work on the IVP (2.5) for (n, w, ρ, m). Let us firstly make use of the Duhamel's principle to express the solution. Denote

$$U = (n, w, \rho, m)^T, \quad U_0 = U(0) = (n_0, w_0, \rho_0, m_0)^T,$$

then we have the vector form of (2.5) denoted by

$$\partial_t U = SU + Q(U), \quad U(0) = U_0, \tag{3.1}$$

where the differential operator S is defined by

$$S = \begin{pmatrix} 0 & -\nabla\cdot & 0 & 0 \\ -\nabla & \mu\Delta + (\mu+\lambda)\nabla\nabla\cdot & 0 & 0 \\ 0 & 0 & 0 & -\nabla\cdot \\ 0 & 0 & -\nabla - \nabla(-\Delta)^{-1} & \mu\Delta + (\mu+\lambda)\nabla\nabla\cdot \end{pmatrix}, \tag{3.2}$$

and the nonlinear term Q is

$$Q(U) = (0, f_1, 0, f_2)^T. \tag{3.3}$$

Thus, we can represent the solution in the mild form

$$U(t) = K(t) * U_0 + \int_0^t K(t-\tau) * Q(U)(\tau) \mathrm{d}\tau, \tag{3.4}$$

with the semigroup $K(t)$ defined by

$$K(t) * U = \mathrm{e}^{tS} U = \mathcal{F}^{-1} \mathrm{e}^{tZ(\xi)} \mathcal{F} U, \quad Z(\xi) = \mathcal{F}(S)(\xi), \quad \xi \in \mathbb{R}^3. \tag{3.5}$$

We need to decompose the Green's function $G \triangleq \mathrm{e}^{tS}$ in terms of its Fourier transform $\hat{G}(\xi)$ to establish the time convergence rate of the original nonlinear problem. Precisely, we can apply the following decomposition to $(\bar{n}, \bar{w}, \bar{\rho}, \bar{m})^T = G * U_0$:

$$\widehat{\bar{n}} = \widehat{R} \cdot \widehat{U}_0 = (\widehat{\mathcal{R}}_1 + \widehat{\mathcal{R}}_2) \cdot \widehat{U}_0, \quad \widehat{\bar{w}} = \widehat{W} \cdot \widehat{U}_0 = \widehat{\mathcal{W}_1} \cdot \widehat{U}_0, \tag{3.6}$$

$$\widehat{\bar{\rho}} = \widehat{N} \cdot \widehat{U}_0 = (\widehat{\mathcal{N}}_1 + \widehat{\mathcal{N}}_2) \cdot \widehat{U}_0, \quad \widehat{\bar{m}} = \widehat{M} \cdot \widehat{U}_0 = (\widehat{\mathcal{M}}_1 + \widehat{\mathcal{M}}_2) \cdot \widehat{U}_0, \tag{3.7}$$

where

$$\widehat{\mathcal{R}}_1=\Big(\frac{\lambda_+\mathrm{e}^{t\lambda_-}-\lambda_-\mathrm{e}^{t\lambda_+}}{\lambda_+-\lambda_-}\quad 0\Big)_{1\times 8},\tag{3.8}$$

$$\widehat{\mathcal{R}}_2=\Big(0\quad -\frac{\mathrm{i}\xi^T(\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-})}{\lambda_+-\lambda_-}\quad 0\quad 0\quad 0\quad 0\Big)_{1\times 8},\tag{3.9}$$

$$\widehat{\mathcal{W}}_1=\Big(-\frac{\mathrm{i}\xi(\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-})}{\lambda_+-\lambda_-}\quad \mathrm{e}^{-\mu|\xi|^2t}\Big(I-\frac{\xi\otimes\xi}{|\xi|^2}\Big)+\frac{\xi\otimes\xi}{|\xi|^2}\frac{\lambda_+\mathrm{e}^{\lambda_+t-\lambda_-\mathrm{e}^{\lambda_-t}}}{\lambda_+-\lambda_-}\quad 0\Big)_{3\times 8},\tag{3.10}$$

$$\widehat{\mathcal{N}}_1=\Big(0\quad 0\quad 0\quad 0\quad \frac{\lambda_+\mathrm{e}^{t\lambda_-}-\lambda_-\mathrm{e}^{t\lambda_+}}{\lambda_+-\lambda_-}\quad 0\quad 0\quad 0\Big)_{1\times 8},\tag{3.11}$$

$$\widehat{\mathcal{N}}_2=\Big(0\quad -\frac{\mathrm{i}\xi^T(\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-})}{\lambda_+-\lambda_-}\Big)_{1\times 8},\tag{3.12}$$

$$\widehat{\mathcal{M}}_1=\Big(0\quad -\frac{\mathrm{i}\xi(\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-})}{\lambda_+-\lambda_-}\quad \mathrm{e}^{-\mu|\xi|^2t}\Big(I-\frac{\xi\otimes\xi}{|\xi|^2}\Big)+\frac{\xi\otimes\xi}{|\xi|^2}\frac{\lambda_+\mathrm{e}^{\lambda_+t-\lambda_-\mathrm{e}^{\lambda_-t}}}{\lambda_+-\lambda_-}\Big)_{3\times 8},\tag{3.13}$$

$$\widehat{\mathcal{M}}_2=\frac{\mathrm{e}^{t\lambda_+}-\mathrm{e}^{t\lambda_-}}{\lambda_+-\lambda_-}\Big(0\quad 0\quad 0\quad 0\quad \frac{-\mathrm{i}\xi}{\frac{1}{2}|\xi|^2}\quad 0\Big)_{3\times 8}.\tag{3.14}$$

And we have the Fourier expression for the electric field $\widehat{\overline{E}}$ as

$$\widehat{\overline{E}}=\frac{-\mathrm{i}\xi}{|\xi|^2}\widehat{\overline{\rho}}=\Big(\frac{-\mathrm{i}\xi}{|\xi|^2}\otimes\widehat{N}\Big)\widehat{U}_0=\Big(\frac{-\mathrm{i}\xi}{|\xi|^2}\otimes\widehat{\mathcal{N}}_1\Big)\widehat{U}_0+\Big(\frac{-\mathrm{i}\xi}{|\xi|^2}\otimes\widehat{\mathcal{N}}_2\Big)\widehat{U}_0,\tag{3.15}$$

from which we can define

$$\widehat{\overline{E}}=\widehat{L}\widehat{U}_0=(\widehat{\mathcal{L}}_1+\widehat{\mathcal{L}}_2)\widehat{U}_0,\tag{3.16}$$

$$\widehat{\mathcal{L}}_1=\Big(\frac{-\mathrm{i}\xi}{|\xi|^2}\otimes\widehat{\mathcal{N}}_1\Big)\widehat{U}_0,\tag{3.17}$$

$$\widehat{\mathcal{L}}_2=\Big(\frac{-\mathrm{i}\xi}{|\xi|^2}\otimes\widehat{\mathcal{N}}_2\Big)\widehat{U}_0.\tag{3.18}$$

It is easy to verify that the global solution (U,E) of the IVP (2.5) can be decomposed into

$$n=R(t)*U_0+\int_0^t\mathcal{R}_2(t-\tau)*Q(U)\mathrm{d}\tau,\tag{3.19}$$

$$w=W(t)*U_0+\int_0^t\mathcal{W}_1(t-\tau)*Q(U)\mathrm{d}\tau,\tag{3.20}$$

$$\rho=N(t)*U_0+\int_0^t\mathcal{N}_2(t-\tau)*Q(U)\mathrm{d}\tau,\tag{3.21}$$

$$m=M(t)*U_0+\int_0^t\mathcal{M}_1(t-\tau)*Q(U)\mathrm{d}\tau,\tag{3.22}$$

$$E=L(t)*U_0+\int_0^t\mathcal{L}_2(t-\tau)*Q(U)\mathrm{d}\tau.\tag{3.23}$$

By Propositions 2.1–3.1, we have the following proposition.

Proposition 3.1 For $|\alpha|\geq 0$, the following time convergence rates for the semigroup hold

$$\|\partial_x^\alpha R(t)*U_0\|\leq C(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}}(\|U_0\|_{L^1}+\|D^\alpha U_0\|),\tag{3.24}$$

$$\|\partial_x^\alpha W(t)*U_0\|\le C(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}}(\|U_0\|_{L^1}+\|D^\alpha U_0\|), \tag{3.25}$$

$$\|\partial_x^\alpha N(t)*U_0\|\le C(1+t)^{-\frac{3}{4}-\frac{|\alpha|}{2}}(\|U_0\|_{L^1}+\|D^\alpha U_0\|), \tag{3.26}$$

$$\|\partial_x^\alpha M(t)*U_0\|\le C(1+t)^{-\frac{1}{4}-\frac{|\alpha|}{2}}(\|U_0\|_{L^1}+\|D^\alpha U_0\|), \tag{3.27}$$

$$\|\partial_x^\alpha L(t)b*U_0\|\le C(1+t)^{-\frac{1}{4}-\frac{|\alpha|}{2}}(\|U_0\|_{L^1}+\|D^\alpha U_0\|). \tag{3.28}$$

Furthermore, by using (2.13)–(2.15), (2.29)–(2.31) and the definitions for $\widehat{\mathcal{R}}_2,\widehat{\mathcal{W}}_1,\widehat{\mathcal{N}}_2,\widehat{\mathcal{M}}_1$, $\widehat{\mathcal{L}}_2$, there exists some constant $c>0$ such that

$$|\widehat{\mathcal{R}}_2(\xi)|\sim\mathcal{O}(1)\mathrm{e}^{-c|\xi|^2t},\quad |\widehat{\mathcal{W}}_1(\xi)|\sim\mathcal{O}(1)\mathrm{e}^{-c|\xi|^2t},\quad |\widehat{\mathcal{N}}_2(\xi)|\sim\mathcal{O}(1)|\xi|\mathrm{e}^{-c|\xi|^2t},$$

$$|\widehat{\mathcal{M}}_1(\xi)|\sim\mathcal{O}(1)\mathrm{e}^{-c|\xi|^2t},\quad |\widehat{\mathcal{L}}_2(\xi)|\sim\mathcal{O}(1)\mathrm{e}^{-c|\xi|^2t},\quad |\xi|\ll 1.$$

Thus, by applying a similar argument as the one for proving Propositions 2.1–3.1, we obtain

Proposition 3.2 For $q=1,2$, the following estimates hold

$$\|\partial_x^\alpha \mathcal{R}_2(t)*U_0\|\le C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{|\alpha|}{2}}(\|U_0\|_{L^q}+\|D^\alpha U_0\|), \tag{3.29}$$

$$\|\partial_x^\alpha \mathcal{W}_1(t)*U_0\|\le C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{|\alpha|}{2}}(\|U_0\|_{L^q}+\|D^\alpha U_0\|), \tag{3.30}$$

$$\|\partial_x^\alpha \mathcal{N}_2(t)*U_0\|\le C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{1}{2}-\frac{|\alpha|}{2}}(\|U_0\|_{L^q}+\|D^\alpha U_0\|), \tag{3.31}$$

$$\|\partial_x^\alpha \mathcal{M}_1(t)*U_0\|\le C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{|\alpha|}{2}}(\|U_0\|_{L^q}+\|D^\alpha U_0\|), \tag{3.32}$$

$$\|\partial_x^\alpha \mathcal{L}_2(t)*U_0\|\le C(1+t)^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})-\frac{|\alpha|}{2}}(\|U_0\|_{L^q}+\|D^\alpha U_0\|), \tag{3.33}$$

and

$$\|\partial_x^\alpha \mathcal{R}_2(t)*U_0\|\le c(1+t)^{-\frac{1}{2}-\frac{|\alpha|}{2}}(\|U_0\|+\|D^{\alpha-1}U_0\|), \tag{3.34}$$

$$\|\partial_x^\alpha \mathcal{N}_2(t)*U_0\|\le c(1+t)^{-\frac{1}{2}-\frac{|\alpha|}{2}}(\|U_0\|+\|D^{\alpha-1}U_0\|). \tag{3.35}$$

3.2 Proof of main results

In this subsection, we will prove Theorems 1.1–1.2. The local existence of classical solutions to IVP (2.5) can be shown by using a standard argument. Here, we omit its proof for brevity. To extend the local strong solution to a global solution, we need to establish the uniform a priori estimate which is given in following lemma.

Lemma 3.1 Under the assumptions of Theorem 1.2 with $l=4$, the solution (n,w,ρ,m,E) with $E=\nabla\Phi$ of the IVP (2.5) satisfies that

$$c(1+t)^{-\frac{3}{4}}\le\|\rho(t)\|\le C(1+t)^{-\frac{3}{4}}, \tag{3.36}$$

$$\|n(t)\|+\|w(t)\|\le C(1+t)^{-\frac{3}{4}+\varepsilon}, \tag{3.37}$$

$$c(1+t)^{-\frac{1}{4}}\le\|m(t)\|+\|E(t)\|\le C(1+t)^{-\frac{1}{4}}, \tag{3.38}$$

$$\|\partial_x^k\rho(t)\|\le C(1+t)^{-\frac{3}{4}-\frac{k}{2}},\quad \|\partial_x^k m(t)\|\le C(1+t)^{-\frac{1}{4}-\frac{k}{2}},\quad k=0,1, \tag{3.39}$$

$$\|\partial_x^j E(t)\|\le C(1+t)^{-\frac{1}{4}-\frac{j}{2}},\quad j=0,1,2, \tag{3.40}$$

$$\|\partial_x(n,w)(t)\|+\|\partial_x^2(n,\rho,w,m)(t)\|+\|\partial_x^3E(t)\|\le C(1+t)^{-\frac{3}{4}}, \tag{3.41}$$

where c and C are positive constants, and $\varepsilon>0$ is given in Theorem 1.1.

Proof First, we denote

$$\begin{aligned}\Lambda(t) \triangleq \sup_{0\le s\le t,k=0,1} &\{(1+s)^{\frac{3}{4}+\frac{k}{2}}\|D_x^k\rho(s)\| + (1+s)^{\frac{3}{4}-\varepsilon}\|(n,w)(s)\| \\ &+(1+s)^{\frac{1}{4}}\|m(s)\| + (1+s)^{\frac{1}{4}}\|E(s)\| + (1+s)^{\frac{3}{4}}\|D_x(\rho,w,m)(s)\| \\ &+(1+s)^{\frac{3}{4}}\|D_x^2(n,w,\rho,m)(s)\| \\ &+\|(D_x^3(n,w,\rho,m),D_x^4(n,w,\rho,m))(s)\|\}. \end{aligned} \tag{3.42}$$

We claim that for any $t \ge 0$,

$$\Lambda(t) \le C\delta, \tag{3.43}$$

where δ is defined in Theorem 1.1. Note that once the claim (3.43) is proved, Lemma 3.1 follows under smallness assumption of δ. The proof of the Claim (3.43) consists of the following three steps.

Step 1 Basic energy estimates From (3.3), the nonlinear term $Q(U)$ can be decomposed into

$$Q(U) = Q_1 + Q_2 + Q_3, \tag{3.44}$$

where

$$Q_1 \sim \mathcal{O}(1)(Dm\cdot m + Dm\cdot w + Dw\cdot w + Dw\cdot m), \tag{3.45}$$

$$Q_2 \sim \mathcal{O}(1)D(D(m\cdot\rho) + D(m\cdot n) + D(w\cdot\rho) + D(w\cdot n)), \tag{3.46}$$

$$Q_3 \sim \mathcal{O}(1)(\rho E + nE). \tag{3.47}$$

Then, by (3.42) and the Nirenberg's inequality $\|f\|_{L^\infty} \le C\|Df\|^{\frac{1}{2}}\|D^2 f\|^{\frac{1}{2}}$ and the fact that

$$\|D^k E\| \le C\|D^{k-1}\rho\|, \qquad k\ge 1, \tag{3.48}$$

we get

$$\|Q(U)\|_{L^1} \le C(1+t)^{-1+\varepsilon}(\Lambda(t))^2, \tag{3.49}$$

$$\|Q(U)\| \le C(1+t)^{-\frac{3}{2}}(\Lambda(t))^2, \tag{3.50}$$

$$\|DQ(U)\| \le C(1+t)^{-\frac{3}{4}}(\Lambda(t))^2, \tag{3.51}$$

$$\|D^2Q(U)\| \le C(1+t)^{-\frac{3}{4}}(\Lambda(t))^2. \tag{3.52}$$

Next, we derive the a priori estimate for the unknown function (n,w,ρ,m,E). Starting from (3.19), by Propositions 3.1–3.2 and by (3.49)–(3.52), we have

$$\begin{aligned}\|(n-R*U_0)(t)\| &\le \int_0^t \|\mathcal{R}_2(t-\tau)*Q(U)(\tau)\|\mathrm{d}\tau \\ &\le C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\|Q(U)\| + \|Q(U)\|_{L^1})(\tau)\mathrm{d}\tau \\ &\le C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\Lambda(t))^2(1+\tau)^{-1+\varepsilon}\mathrm{d}\tau \\ &\le C(\Lambda(t))^2(1+t)^{-\frac{3}{4}+\varepsilon}, \end{aligned} \tag{3.53}$$

which implies

$$C\delta(1+t)^{\frac{3}{4}} - C(\Lambda(t))^2(1+t)^{-\frac{3}{4}+\varepsilon} \le \|n(t)\| \le C\delta(1+t)^{\frac{3}{4}} + C(\Lambda(t))^2(1+t)^{-\frac{3}{4}+\varepsilon}. \tag{3.54}$$

Similarly,

$$\begin{aligned}\|Dn(t)\| &\le \|DR(t)*U_0\| + \int_0^t \|D\mathcal{R}_2(t-\tau)*Q(U)(\tau)\|\mathrm{d}\tau\\ &\le C\delta(1+t)^{-\frac{5}{4}} + \int_0^{\frac{t}{2}} \|D\mathcal{R}_2(t-\tau)*Q(U)(\tau)\|\mathrm{d}\tau\\ &\quad + \int_{\frac{t}{2}}^t \|D\mathcal{R}_2(t-\tau)*Q(U)(\tau)\|\mathrm{d}\tau\\ &\le C\delta(1+t)^{-\frac{5}{4}} + C\int_0^t (1+t-\tau)^{-\frac{5}{4}}(\|Q(U)(\tau)\|_{L^1} + \|DQ(U)(\tau)\|)\mathrm{d}\tau\\ &\le C\delta(1+t)^{-\frac{5}{4}} + C(\Lambda(t))^2\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau\\ &\quad +C(\Lambda(t))^2\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau\\ &\le C\delta(1+t)^{-\frac{5}{4}} + C(\Lambda(t))^2(1+t)^{-\frac{3}{4}+\beta}, \end{aligned} \tag{3.55}$$

where β is a small but fixed constant. And

$$\begin{aligned}\|D^2n(t)\| &\le \|D^2R(t)*U_0\| + \int_0^t \|D^2\mathcal{R}_2(t-\tau)*Q(U)(\tau)\|\mathrm{d}\tau\\ &\le C\delta(1+t)^{-\frac{7}{4}} + \int_0^{\frac{t}{2}} \|D^2\mathcal{R}_2(t-\tau)*Q(U)(\tau)\|\mathrm{d}\tau\\ &\quad + \int_{\frac{t}{2}}^t \|D^2\mathcal{R}_2(t-\tau)*Q(U)(\tau)\|\mathrm{d}\tau\\ &\le C\delta(1+t)^{-\frac{7}{4}} + C\int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{7}{4}}(\|Q(U)(\tau)\|_{L^1} + \|D^2Q(U)(\tau)\|)\mathrm{d}\tau\\ &\quad + \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{7}{4}}(\|Q(U)(\tau)\|_{L^1} + \|D^2Q(U)(\tau))\|\mathrm{d}\tau\\ &\le C\delta(1+t)^{-\frac{7}{4}} + c(\Lambda(t))^2\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{7}{4}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau\\ &\quad +C(\Lambda(t))^2\int_{\frac{t}{2}}^t(1+t-\tau)^{-\frac{7}{4}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau\\ &\le C\delta(1+t)^{-\frac{7}{4}} + C(\Lambda(t))^2(1+t)^{-\frac{3}{2}} + C(\Lambda(t))^2(1+t)^{-\frac{3}{4}}. \end{aligned} \tag{3.56}$$

For w and its derivatives, from (3.20), we have

$$\begin{aligned}\|(w-W*U_0)(t)\| &\le \int_0^t \|\mathcal{W}_1(t-\tau)*Q(U)(\tau)\|\mathrm{d}\tau\\ &\le C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\|Q(U)\| + \|Q(U)\|_{L^1})(\tau)\mathrm{d}\tau\\ &\le C\int_0^t (1+t-\tau)^{-\frac{3}{4}}(\Lambda(t))^2(1+\tau)^{-1+\varepsilon}\mathrm{d}\tau\\ &\le C(\Lambda(t))^2(1+t)^{-\frac{3}{4}+\varepsilon}, \end{aligned} \tag{3.57}$$

which implies

$$C\delta(1+t)^{-\frac{3}{4}} - C(\Lambda(t))^2(1+t)^{-\frac{3}{4}+\varepsilon} \le \|w(t)\| \le C\delta(1+t)^{-\frac{3}{4}} + C(\Lambda(t))^2(1+t)^{-\frac{3}{4}+\varepsilon}, \tag{3.58}$$

with ε given in Theorem 1.1. Moreover,

$$\begin{aligned}\|Dw(t)\| &\leq \|DW(t)*U_0\| + \int_0^t \|D\mathcal{W}_1(t-\tau)*Q(U)(\tau)\|\mathrm{d}\tau \\ &\leq C\delta(1+t)^{-\frac{5}{4}} + \int_0^t (1+t-\tau)^{-\frac{5}{4}}(\|Q(U)(\tau)\|_{L^1} + \|DQ(U)(\tau)\|)\mathrm{d}\tau \\ &\leq C\delta(1+t)^{-\frac{5}{4}} + C(\Lambda(t))^2 \int_0^t (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau \\ &\leq C\delta(1+t)^{-\frac{5}{4}} + C(\Lambda(t))^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau \\ &\quad +C(\Lambda(t))^2 \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{5}{4}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau \\ &\leq C\delta(1+t)^{-\frac{5}{4}} + C(\Lambda(t))^2(1+t)^{-1} + C(\Lambda(t))^2(1+t)^{-\frac{3}{4}},\end{aligned} \tag{3.59}$$

and

$$\begin{aligned}\|D^2w(t)\| &\leq \|D^2W(t)*U_0\| + \int_0^t \|D^2\mathcal{W}_1(t-\tau)*Q(U)(\tau)\|\mathrm{d}\tau \\ &\leq C\delta(1+t)^{-\frac{7}{4}} + \int_0^t (1+t-\tau)^{-\frac{7}{4}}(\|Q(U)(\tau)\|_{L^1} + \|D^2Q(U)(\tau)\|)\mathrm{d}\tau \\ &\leq C\delta(1+t)^{-\frac{7}{4}} + C(\Lambda(t))^2 \int_0^t (1+t-\tau)^{-\frac{7}{4}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau \\ &\leq C\delta(1+t)^{-\frac{7}{4}} + C(\Lambda(t))^2 \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{7}{4}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau \\ &\quad +C(\Lambda(t))^2 \int_{\frac{t}{2}}^t (1+t-\tau)^{-\frac{7}{4}}(1+\tau)^{-\frac{3}{4}}\mathrm{d}\tau \\ &\leq C\delta(1+t)^{-\frac{7}{4}} + C(\Lambda(t))^2(1+t)^{-\frac{3}{2}} + C(\Lambda(t))^2(1+t)^{-\frac{3}{4}}.\end{aligned} \tag{3.60}$$

To yield the same time convergence rate for n and ρ, and the different rates for w and m stated in Proposition 4.1, 4.2 and (3.42), we can use the above method to get the following estimates:

$$\begin{aligned}C\delta(1+t)^{-\frac{3}{4}} - C\Lambda(t))^2(1+t)^{-1} &\leq \|\rho(t)\| \\ &\leq C\delta(1+t)^{-\frac{3}{4}} + C\Lambda(t))^2(1+t)^{-1},\end{aligned} \tag{3.61}$$

$$\begin{aligned}C\delta(1+t)^{-\frac{1}{4}} - C(\Lambda(t))^2(1+t)^{-\frac{3}{4}+\varepsilon} &\leq \|m(t)\| + \|E(t)\| \\ &\leq C\delta(1+t)^{-\frac{1}{4}} + C(\Lambda(t))^2(1+t)^{-\frac{3}{4}+\varepsilon},\end{aligned} \tag{3.62}$$

and

$$\|D\rho\| \leq C\delta(1+t)^{-\frac{5}{4}} + C(\Lambda(t))^2(1+t)^{-\frac{3}{2}+\beta}, \tag{3.63}$$

$$\|D^2\rho\| \leq C\delta(1+t)^{-\frac{7}{4}} + C(\Lambda(t))^2(1+t)^{-2} + C(\Lambda(t))^2(1+t)^{-\frac{3}{4}}, \tag{3.64}$$

$$\|Dm\| \leq C\delta(1+t)^{-\frac{3}{4}} + C(\Lambda(t))^2(1+t)^{-1} + C(\Lambda(t))^2(1+t)^{-\frac{3}{4}}, \tag{3.65}$$

$$\|D^2m\| \leq C\delta(1+t)^{-\frac{5}{4}} + C(\Lambda(t))^2(1+t)^{-\frac{3}{2}} + C(\Lambda(t))^2(1+t)^{-\frac{3}{4}}. \tag{3.66}$$

Step 2 Higher order derivatives estimates The estimates on higher order derivatives can be obtained as follows. By taking $\int (2.5)_1 n\mathrm{d}x + \int (2.5)_2 \cdot w\mathrm{d}x$, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (|n|^2+|w|^2)\mathrm{d}x + C\int |Dw|^2\mathrm{d}x = \int f_1\cdot w\mathrm{d}x. \tag{3.67}$$

Similarly, by taking $\int (2.5)_3 \rho\mathrm{d}x + \int (2.5)_4 \cdot m\mathrm{d}x$, and by using $E=\nabla\Phi$ and the fact that

$$\int \nabla\Phi\cdot m\mathrm{d}x = -\int \Phi\nabla\cdot m\mathrm{d}x = \int \Phi\rho_t\mathrm{d}x = \int \Phi\nabla\cdot E_t\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |E|^2\mathrm{d}x, \tag{3.68}$$

we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (|\rho|^2+|m|^2+|E|^2)\mathrm{d}x + C\int |Dm|^2\mathrm{d}x = \int f_2\cdot m\mathrm{d}x. \tag{3.69}$$

Then the summation of (3.67) and (3.69) gives

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (|n|^2+|w|^2+|\rho|^2+|m|^2+|E|^2)\mathrm{d}x + C\int (|Dw|^2+|Dm|^2)\mathrm{d}x\\ \le& \int (|f_1\cdot w|+|f_2\cdot m|)\mathrm{d}x.\end{aligned} \tag{3.70}$$

By (3.50) and (3.49), it is easy to estimate the right hand side of (3.70) by

$$\int (|f_1\cdot w|+|f_2\cdot m|)\mathrm{d}x \le C(1+t)^{-\frac{5}{4}}\Lambda(t)\|(\rho,m,w,Dm,Dw)\|^2, \tag{3.71}$$

which together with (3.70) give

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (|n|^2+|w|^2+|\rho|^2+|m|^2+|E|^2)\mathrm{d}x + C\int (|Dw|^2+|Dm|^2)\mathrm{d}x\\ \le& C(1+t)^{-\frac{5}{4}}\Lambda(t)\|(\rho,m,w,Dm,Dw)\|^2.\end{aligned} \tag{3.72}$$

Similarly, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int (|D^kn|^2+|D^kw|^2+|D^k\rho|^2+|D^km|^2+|D^kE|^2)\mathrm{d}x + C(\|D^{k+1}w\|^2+\|D^{k+1}m\|^2)\\ \le& C(1+t)^{-\frac{5}{4}}\Lambda(t)(\|(\rho,m,w)\|_{H^k}^2+\|Dw\|_{H^k}^2+\|Dm\|_{H^k}^2),\end{aligned} \tag{3.73}$$

for $k=1,2,3,4$. And this implies that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|n\|_{H^k}^2+\|w\|_{H^k}^2+\|\rho\|_{H^k}^2+\|m\|_{H^k}^2+\|E\|_{H^k}^2) + C(\|Dw\|_{H^k}^2+\|Dm\|_{H^k}^2)\\ \le& C(1+t)^{-\frac{5}{4}}\Lambda(t)(\|(\rho,m,w)\|_{H^k}^2+\|Dw\|_{H^k}^2+\|Dm\|_{H^k}^2).\end{aligned} \tag{3.74}$$

Set

$$H(t) \triangleq \|n(t)\|_{H^k}^2+\|w(t)\|_{H^k}^2+\|\rho(t)\|_{H^k}^2+\|m(t)\|_{H^k}^2+\|E(t)\|_{H^k}^2,\ k=1,2,3,4. \tag{3.75}$$

It follows from (3.74) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}H(t) + C(\|Dw\|_{H^k}^2+\|Dm\|_{H^k}^2) \le C(1+t)^{-\frac{5}{4}}\Lambda(t)H(t). \tag{3.76}$$

Therefore, by the a priori assumption that $\Lambda(t)$ is small, we have by the Gronwall inequality that

$$H(t) \le C\mathrm{e}^{\int_0^t (1+\tau)^{-\frac{5}{4}}\Lambda(\tau)\mathrm{d}\tau} \|(n_0, w_0, \rho_0, m_0, E_0)\|_{H^4} \le C\delta. \tag{3.77}$$

Step 3 Closure of the estimates Finally, by (3.53)–(3.66) and (3.77), we have

$$\Lambda(t) \le C\delta + C(\Lambda(t))^2, \tag{3.78}$$

which leads to (3.43) under the smallness assumption on $\delta > 0$. And this completes the proof of the lemma.

The proof of Theorems 1.1–1.2 By the local existence and the closure of the a priori estimate, the global existence of smooth solution of the IVP (2.5) follows from the standard continuity argument. The estimates (1.5), (1.6), (1.7) and (1.8) in Theorem 1.1 follow from Lemma 3.1. The optimal time convergence rate in Theorem 1.2 is a combination of Proposition 3.2, and the estimates (3.54), (3.58), (3.61), and (3.62).

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