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STABILITY OF THE ONE-SPECIES VLASOV–POISSON–BOLTZMANN SYSTEM*

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Abstract. In this paper, we are concerned with the one-species Vlasov–Poisson–Boltzmann system with a nonconstant background density in full space. There exists a stationary solution when the background density is a small perturbation of a positive constant state. We prove the nonlinear stability of solutions to the Cauchy problem near the stationary state in some Sobolev space without any time derivatives. This result is nontrivial even when the background density is a constant state. In the proof, the macroscopic balance laws are essentially used to deal with the a priori estimates on both the microscopic and macroscopic parts of the solution. Moreover, some interactive energy functionals are introduced to overcome difficulty stemming from the absence of time derivatives in the energy functional.

Key words. Vlasov–Poisson–Boltzmann system, stability, energy estimates

AMS subject classifications. 76P05, 82C40, 82D05

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1. Introduction.

The Vlasov–Poisson–Boltzmann (VPB) system is a physical model describing the time evolution of dilute charged particles (e.g., electrons) in the absence of an external magnetic field [22]. In this paper, we consider the VPB system for one species of particles in the whole space $\mathbb{R}^3$:

$$\partial_t f + \xi \cdot \nabla_x f + \nabla_x \Phi \cdot \nabla_\xi f = Q(f, f), \tag{1.1}$$

$$\Delta_x \Phi = \int_{\mathbb{R}^3} f d\xi - \bar{\rho}(x). \tag{1.2}$$

Here, the unknown $f = f(t, x, \xi)$ is a nonnegative function standing for the number density of gas particles which have position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ at time $t > 0$. The bilinear collision operator Q with hard-sphere interaction [2] is defined by

$$Q(f, g) = \int_{\mathbb{R}^3 \times S^2} (f' g'_* - f g_*) |(\xi - \xi_*) \cdot \omega| d\omega d\xi_*,$$
$$f = f(t, x, \xi), \quad f' = f(t, x, \xi'), \quad g_* = g(t, x, \xi_*), \quad g'_* = g(t, x, \xi'_*),$$
$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega, \quad \omega \in S^2.$$

The self-consistent electric potential $\Phi = \Phi(t, x)$ is coupled with the distribution function $f(t, x, \xi)$ through the Poisson equation. $\bar{\rho}(x)$ denotes the stationary background density satisfying

$$\bar{\rho}(x) \to \rho_\infty \quad \text{as} \quad |x| \to \infty$$

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for a positive constant state $\rho_\infty > 0$. Throughout this paper, we take $\rho_\infty = 1$ for simplicity.

First, in this paper, we prove the existence of the stationary solution to the VPB system (1.1)–(1.2) under some conditions on the background density $\bar\rho(x)$. For this purpose, let us define the weighted norm $\|\cdot\|_{W_k^{m,\infty}}$ by

$$\|g\|_{W_k^{m,\infty}} = \sup_{x\in\mathbb{R}^3}(1+|x|)^k \sum_{|\alpha|\le m} |\partial_x^\alpha g(x)| \tag{1.3}$$

for suitable $g = g(x)$ and integers $m \ge 0$, $k \ge 0$. Actually, one has the following theorem.

THEOREM 1.1. *For integers $m \ge 0$ and $k \ge 0$, suppose that $\|\bar\rho - 1\|_{W_k^{m,\infty}}$ is small enough. Then the following elliptic equation with the exponential nonlinearity*

$$\Delta_x \phi = e^\phi - \bar\rho(x) \tag{1.4}$$

admits a unique solution $\phi = \phi(x)$ satisfying

$$\|\phi\|_{W_k^{m,\infty}} \le C\|\bar\rho - 1\|_{W_k^{m,\infty}} \tag{1.5}$$

for some constant C.

From Theorem 1.1, it is straightforward to check that the VPB system (1.1)–(1.2) has a stationary solution (f_*, Φ_*) given by

$$f_* = e^\phi \mathbf{M}, \quad \Phi_* = \phi,$$

where the global Maxwellian

$$\mathbf{M} = \frac{1}{(2\pi)^{3/2}} \exp\left(-|\xi|^2/2\right)$$

is normalized to have zero bulk velocity and unit density and temperature.

Based on this existence result, we will consider the stability of the stationary state (f_*, Φ_*). For this, set the perturbations $u = u(t, x, \xi)$ and $\Psi = \Psi(t, x)$ by

$$f = e^\phi \mathbf{M} + \sqrt{\mathbf{M}}u, \quad \Phi = \phi + \Psi. \tag{1.6}$$

Then u and Ψ satisfy the perturbed system

$$\begin{aligned} \partial_t u + \xi\cdot\nabla_x u + \nabla_x(\Psi+\phi)\cdot\nabla_\xi u - \frac{1}{2}\xi\cdot\nabla_x(\Psi+\phi)u - \xi\cdot\nabla_x\Psi e^\phi\sqrt{\mathbf{M}} \\ = e^\phi \mathbf{L}u + \Gamma(u,u), \end{aligned} \tag{1.7}$$

$$\Delta_x \Psi = \int_{\mathbb{R}^3} \sqrt{\mathbf{M}} u d\xi, \quad (t,x,\xi) \in (0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3, \tag{1.8}$$

with given initial data

$$u(0,x,\xi) = u_0(x,\xi), \quad (x,\xi) \in \mathbb{R}^3\times\mathbb{R}^3, \tag{1.9}$$

where $\mathbf{L}u$ and $\Gamma(u,u)$ are denoted by

$$\begin{aligned} \mathbf{L}u &= \frac{1}{\sqrt{\mathbf{M}}}\left[Q(\mathbf{M}, \sqrt{\mathbf{M}}u) + Q(\sqrt{\mathbf{M}}u, \mathbf{M})\right], \\ \Gamma(u,u) &= \frac{1}{\sqrt{\mathbf{M}}} Q(\sqrt{\mathbf{M}}u, \sqrt{\mathbf{M}}u). \end{aligned}$$

It is well known that for the linearized collision operator $\mathbf{L}$, one has

$$\begin{aligned}(\mathbf{L}u)(\xi) &= -\nu(\xi)u(\xi) + (Ku)(\xi),\\ \nu(\xi) &= \int_{\mathbb{R}^3\times S^2} |(\xi-\xi_*)\cdot\omega|\mathbf{M}_*\,d\omega d\xi_*,\\ (Ku)(\xi) &= \int_{\mathbb{R}^3\times S^2}\left(-\sqrt{\mathbf{M}}u_* + \sqrt{\mathbf{M}'_*}u' + \sqrt{\mathbf{M}'}u'_*\right)|(\xi-\xi_*)\cdot\omega|\sqrt{\mathbf{M}_*}d\omega d\xi_*\\ &= \int_{\mathbb{R}^n} K(\xi,\xi_*)u(\xi_*)d\xi_*,\end{aligned}$$

where $\nu(\xi)$ is called the collision frequency and K is a self-adjoint compact operator on $L^2(\mathbb{R}^3_\xi)$ with a real symmetric integral kernel $K(\xi,\xi_*)$. The nullspace of the operator $\mathbf{L}$ is the five-dimensional space spanned by the collision invariants

$$\mathcal{N} = \mathrm{Ker}\mathbf{L} = \mathrm{span}\left\{\sqrt{\mathbf{M}};\ \xi_i\sqrt{\mathbf{M}}, i=1,2,3;\ |\xi|^2\sqrt{\mathbf{M}}\right\}. \tag{1.10}$$

From Boltzmann's H-theorem, the linearized collision operator $\mathbf{L}$ is nonpositive and, moreover, $-\mathbf{L}$ is locally coercive in the sense that there is a constant $\lambda>0$ such that

$$-\int_{\mathbb{R}^3} u\mathbf{L}u\,d\xi \geq \lambda\int_{\mathbb{R}^3}\nu(\xi)\left(\{\mathbf{I}-\mathbf{P}\}u\right)^2 d\xi \quad \forall\, u\in D(\mathbf{L}), \tag{1.11}$$

where for fixed (t,x), $\mathbf{P}$ denotes the projection operator from $L^2(\mathbb{R}^3_\xi)$ to $\mathcal{N}$ and $D(\mathbf{L})$ is the domain of $\mathbf{L}$ given by

$$D(\mathbf{L}) = \left\{u\in L^2(\mathbb{R}^3_\xi)\ \middle|\ \nu(\xi)u\in L^2(\mathbb{R}^3_\xi)\right\}.$$

In addition, for the hard-sphere model,

$$\nu(\xi)\sim 1+|\xi|$$

holds true and will be used throughout this paper. For the proof of all those mentioned properties of $\mathbf{L},\nu$, and K, see [2]. For later use, we write $\mathbf{P}u$ in the form of coordinates as follows:

$$\mathbf{P}u = \left\{a(t,x) + \sum_{i=1}^3 b_i(t,x)\xi_i + c(t,x)|\xi|^2\right\}\sqrt{\mathbf{M}},$$

where $a,\ b=(b_1,b_2,b_3)$, and c (used in place of the notation $a^u, b^u=(b_1^u,b_2^u,b_3^u)$, and c^u, for simplicity) are the coefficients of the macroscopic component $\mathbf{P}u$.

Define the energy functional

$$[[u(t)]]^2 \equiv \sum_{|\alpha|+|\beta|\leq N}\|\partial_x^\alpha\partial_\xi^\beta u\|^2 + \sum_{|\alpha|\leq N}\|\partial_x^\alpha\nabla_x\Psi\|^2$$

and the dissipation rate

$$\begin{aligned}[[u(t)]]_\nu^2 \equiv &\sum_{|\alpha|+|\beta|\leq N}\|\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u\|_\nu^2\\ &+\sum_{|\alpha|\leq N-1}\left(\|\partial_x^\alpha\nabla_x(a,b,c)\|^2 + \|\partial_x^\alpha(a+3c)\|^2\right),\end{aligned}$$

where $N \geq 4$ is a fixed integer throughout this paper, and Ψ is given by the formula

$$\Psi(t,x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\left(\int_{\mathbb{R}^3}\sqrt{\mathbf{M}}u(t,y,\xi)d\xi\right)dx.$$

Also, define the solution space

$$X(0,\infty) = \left\{ v = v(t,x,\xi) \;\middle|\; \begin{array}{l} e^{\phi}\mathbf{M} + \sqrt{\mathbf{M}}v \geq 0,\ (t,x,\xi) \in [0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3 \\ [[v(t)]]^2 : [0,\infty) \to [0,\infty) \ \text{ is continuous} \\ \sup_{t\geq 0}[[v(t)]] < \infty \end{array} \right\}.$$

Now, the main result about the stability of the stationary solution is stated as follows.

THEOREM 1.2. *Suppose that $\|\bar{\rho}-1\|_{W_2^{N+1,\infty}}$ is small enough. There are constants $\delta_0 > 0$, $\lambda_0 > 0$, and C_0 such that if $[[u_0]]^2 \leq \delta_0$ and $f_0(x,\xi) \equiv e^{\phi}\mathbf{M}+\sqrt{\mathbf{M}}u_0(x,\xi) \geq 0$, then the Cauchy problem* (1.7)*,* (1.8)*, and* (1.9) *of the VPB system admits a unique global solution $u = u(t,x,\xi) \in X(0,\infty)$ satisfying $f(t,x,\xi) \equiv e^{\phi}\mathbf{M}+\sqrt{\mathbf{M}}u(t,x,\xi) \geq 0$, and*

$$[[u(t)]]^2 + \lambda_0\int_0^t [[u(s)]]_\nu^2 ds \leq C_0[[u_0]]^2 \tag{1.12}$$

for any $t \geq 0$.

Remark 1.1. Notice that the hypothesis of initial data in Theorem 1.2 is the same as that given in [27]. However, the main point of the current paper is that in order to close the a priori estimate, one does not need to include the time derivatives in the a priori estimate, so that there is no need to estimate the time derivatives in the energy estimate. In this sense, both the norms on the initial data and the solution are the same, that is, [[u]]. Moreover, the method taken by this paper can be naturally applied to the time-decay estimates of the linearized system, and the corresponding results are to be reported in the future.

Recently, there have been some works on the study of the Cauchy problem of the VPB system and even the Vlasov–Maxwell–Boltzmann (VMB) system near Maxwellians in the full space $\mathbb{R}^3$. The global existence of classical solutions to the one-species VPB system was first given in [25] under the condition that either the mean free path is sufficiently small or the background density $\bar{\rho}$ is a sufficiently large positive constant. The proof in [25] is based on the elaborated energy method developed in [21, 20] by taking perturbations near the local Maxwellian. Later, those restrictions in [25] were removed in [27], where the convergence rate in time to the global Maxwellian was also obtained through the energy method. The work [27] used some simplified energy estimates found in [26]. On the other hand, the same issue about the stability of the global Maxwellian was considered in [24] for the two-species VMB system, where the similar method can also apply to the case of two-species VPB system. The proof in [24] is based on another kind of energy method independently introduced in a series of works [14, 13, 12] by perturbing the solution near the global Maxwellian.

Here, it should be pointed out that all the energy spaces used in the works mentioned above include the time derivatives. Thus, the system about the time evolution of the distribution function needs to be used in order to define the time derivatives of the initial data, which leads to the fact that the initial data must admit the higher integrability in ξ and higher regularity in x than solutions. The goal of this paper is to remove this restriction by excluding the time derivative in the energy space. Similar work was done in [5, 6] for the Boltzmann equation with or without a stationary potential force, where a refined energy method was introduced and the optimal convergence rate in time for the case with forces was also given in some Sobolev space of position and velocity variables; cf. [5] for more details. In a periodic domain, an energy method with no time derivatives was also adapted in [17] to study the diffusive expansion for solutions around Maxwellians to the two-species VMB system.

We further remark on the result in [24], where the time derivatives need to be put into the energy functional to obtain the stability of global Maxwellians for the two-species VMB system in $\mathbb{R}^3$. Here, recall that perturbation

$$u(t,x,\xi) = [u_+(t,x,\xi), u_-(t,x,\xi)]$$

of the two-species VMB system near the global Maxwellian satisfies

$$\partial_t u + \xi\cdot\nabla_x u + q(E+\xi\times B)\cdot\nabla_\xi u - E\cdot\xi\sqrt{\mathbf{M}}q_1 = \mathbf{L}_{VMB}u + \frac{q}{2}E\cdot\xi u + \Gamma_{VMB}(u,u),$$

where $q = \mathrm{diag}(1,-1)$ and $q_1 = [1,-1]$. In this case, the nullspace of the linearized operator $\mathbf{L}_{VMB}$ is given by

$$\mathrm{span}\{[\sqrt{\mathbf{M}},0],[0,\sqrt{\mathbf{M}}];\ [\xi_i\sqrt{\mathbf{M}},\xi_i\sqrt{\mathbf{M}}], i=1,2,3;\ [|\xi|^2\sqrt{\mathbf{M}},|\xi|^2\sqrt{\mathbf{M}}]\}$$

and the macroscopic projection $\mathbf{P}_{VMB}$ is defined by $\mathbf{P}_{VMB}u = [\mathbf{P}^+_{VMB}u, \mathbf{P}^-_{VMB}u]$ with

$$\mathbf{P}^\pm_{VMB}u = \left\{a_\pm(t,x) + \sum_{i=1}^3 b_i(t,x)\xi_i + c(t,x)|\xi|^2\right\}\sqrt{\mathbf{M}}.$$

In [24], on one hand, the dissipation $\|E(t)\|^2$ of the electric field E by itself was included in the energy dissipation rate in order to control the nonlinear term $\frac{q}{2}E\cdot\xi u$ in the proof of the microscopic a priori estimates. On the other hand, for obtaining the dissipation $\|E(t)\|^2$, the macroscopic equations

$$\partial_t b_i + \partial_i a_\pm \pm E_i = R^\pm \tag{1.13}$$

are used to get

$$\|E_i\| \le \|\partial_t b_i\| + \|\partial_i a_\pm\| + \|R^\pm\|,$$

where the estimates on the spatial derivatives $\partial_i a_\pm$ and the remaining terms $R^\pm$ can be made as in [12]. To close the estimates of time derivatives, the cancelation in (1.13) between $\pm$ is essentially used by taking the summation of two equations to yield

$$\partial_t b_i + \partial_i(a_+ + a_-) = R^+ + R^-,$$

which combined with other macroscopic equations gives estimates on the mixed time-space derivatives of $a_\pm$, b, and c. However, the above argument fails for the one-species VMB because there is no cancelation for the single equation in this case.

As mentioned above, the proof of Theorem 1.2 is based on the refined energy method introduced in [5]. The main idea is to introduce some interactive energy functionals to overcome the difficulty caused by the fact that there is no time derivative in the energy functionals. Let us point out some other ideas in the following proof. First, as observed and essentially used in [27], the macroscopic conservative quantity

$a+3c$ itself is dissipative in L^2-norm, which comes from the macroscopic balance laws (3.9)–(3.10) and the Poisson equation (3.13); cf. Lemma 5.6. This is analogous to some studies of the Navier–Stokes–Poisson equations [19] and a model for semiconductors [16]. Notice that in [16] the damping term is considered instead of the diffusion term so that the exponential time-decay rates were obtained. Next, not all the macroscopic equations given in (3.4)–(3.8) are used in the proof. In fact, (3.4)–(3.5) have been ignored by considering the macroscopic balance laws. Thus the estimates on a in the previous works [5] can be modified in a more simplified way. Finally, the macroscopic balance laws are used to control the evolution of zero-order energy by using the high-order dissipation; cf. Lemma 4.4. For example, it is a little delicate to estimate the term

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \xi\cdot\nabla_x\Psi|u|^2 dxd\xi$$

because of the weak dissipation. However, even though $\nabla_x\Psi$ is not dissipative in L^2-norm, one can use the balance law (3.10) of b to control the above nonlinear term.

Finally, we mention some other related works. The global existence of renormalized solutions with large data was given in [4] and [23], and the time asymptotic behavior of renormalized solutions with extra regularity was proved in [3]. The decay property of solutions to the linearized equation was studied in [9, 10], and the global existence of classical solutions near vacuum was obtained in [15] and [7], respectively.

The rest of this paper is arranged as follows. The existence of the stationary solution will be proved in section 2. In terms of the macro-micro decomposition given in section 3, the a priori energy estimates on the microscopic and macroscopic dissipations are derived in sections 4 and 5, respectively. The proof of the global existence of solutions to the perturbed problem is obtained in the last section.

Notation. Throughout this paper, C denotes a generic positive (generally large) constant and λ denotes a generic positive (generally small) constant, where both C and λ may take different values at different places. When necessary, we write C_0, C_1, etc., to distinguish these constants and use $C(\cdot)$ to show that constants depend on some parameters in the argument. In addition, $A\sim B$ means $\lambda_1 A\leq B\leq \lambda_2 A$ for two generic constants $\lambda_1>0$ and $\lambda_2>0$. We use $\langle\cdot,\cdot\rangle$ to denote the inner product in the Hilbert space $L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$ or $L^2(\mathbb{R}^3_x)$ or $L^2(\mathbb{R}^3_\xi)$, and $\|\cdot\|$ to denote the corresponding L^2-norm. Sometimes we also write $\|\cdot\|_{L^2_{x,\xi}}$, $\|\cdot\|_{L^2_x}$, and $\|\cdot\|_{L^2_\xi}$ when it is needed to be precise. We also define

$$\langle u,v\rangle_\nu\equiv\langle \nu(\xi)u,v\rangle$$

for suitable functions $u=u(x,\xi)$ and $v=v(x,\xi)$ to be the weighted inner product in $L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)$, and use $\|\cdot\|_\nu$ for the corresponding weighted L^2-norm. For the multiple indices $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,\beta_2,\beta_3)$, as usual we denote

$$\partial_x^\alpha\partial_\xi^\beta=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}\partial_{\xi_3}^{\beta_3}.$$

The length of α is $|\alpha|=\alpha_1+\alpha_2+\alpha_3$. For simplicity, we also use ∂_i to denote ∂_{x_i} for each $i=1,2,3$.

2. Existence of stationary solution. In this section, we prove Theorem 1.1 for the existence of stationary solutions to the elliptic equation (1.4) by using the contraction mapping theorem. First, for convenience, recall the equation

$$\Delta_x\phi=e^\phi-\bar{\rho}(x) \tag{2.1}$$

and the norm $\|\cdot\|_{W_k^{m,\infty}}$ defined by (1.3) for integers $m \geq 0$, $k \geq 0$.

Proof of Theorem 1.1. Equation (2.1) can be rewritten as the integral form

$$\phi = \mathbf{T}(\phi) := G * (e^{\phi} - \phi - \bar{\rho}),$$

where $G = G(x)$ given by

$$G(x) = -\frac{1}{4\pi|x|}e^{-|x|}$$

is the fundamental solution to the linear elliptic equation $\Delta_x G - G = 0$. Thus (2.1) admits a solution if and only if the nonlinear mapping $\mathbf{T}$ has a fixed point. Define

$$\mathbb{B}_{m,k}(C) = \{\phi \in W^{m,\infty}(\mathbb{R}^3); \|\phi\|_{W_k^{m,\infty}} \leq C\|\bar{\rho} - 1\|_{W_k^{m,\infty}}\}$$

for some constant C to be determined later. Next, we prove that if $\|\bar{\rho} - 1\|_{W_k^{m,\infty}}$ is small enough, there exists a constant C such that $\mathbf{T} : \mathbb{B}_{m,k}(C) \to \mathbb{B}_{m,k}(C)$ is a contraction mapping. In fact, for simplicity, let us denote

$$g(x) = e^x - x - 1.$$

Then it holds that

$$\mathbf{T}(\phi)(x) = -\int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|}e^{-|x-y|}\left[g(\phi(y)) - (\bar{\rho}(y) - 1)\right] dy. \tag{2.2}$$

Taking derivatives ∂_x^α on both sides of (2.2) and summing them up over $|\alpha| \leq m$, one has

$$\sum_{|\alpha|\leq m} |\partial_x^\alpha \mathbf{T}(\phi)(x)| \leq \int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|}e^{-|x-y|} \sum_{|\alpha|\leq m} [|\partial^\alpha g(\phi(y))| + |\partial^\alpha(\bar{\rho}(y) - 1)|]dy. \tag{2.3}$$

By the definition (1.3) of the norm $\|\cdot\|_{W_k^{m,\infty}}$, one has

$$\sum_{|\alpha|\leq m} |\partial^\alpha(\bar{\rho}(y) - 1)| \leq \frac{\|\bar{\rho} - 1\|_{W_k^{m,\infty}}}{(1+|y|)^k} \tag{2.4}$$

for any $y \in \mathbb{R}^3$. On the other hand, noticing

$$g(\phi) = \int_0^1 \int_0^\theta e^{\tau\phi} d\tau d\theta\, \phi^2,$$

one has

$$\sum_{|\alpha|\leq m} |\partial^\alpha g(\phi)| \leq \frac{C(\|\phi\|_{W_k^{m,\infty}})\|\phi\|_{W_k^{m,\infty}}}{(1+|y|)^{2k}} \tag{2.5}$$

for any $y \in \mathbb{R}^3$, where $C(\cdot)$ is a nondecreasing, nonnegative, and continuous function. Now, we need the following.

Claim. *For any $k \geq 0$, it holds that*

$$\int_{\mathbb{R}^3} \frac{1}{|y|}e^{-|y|}\frac{1}{(1+|x-y|)^k}dy \leq \frac{C_k}{(1+|x|)^k}.$$

Proof. Divide the integral into two parts:

$$I_1 = \int_{|x-y|\geq |x|/2}, \qquad I_2 = \int_{|x-y|<|x|/2}.$$

For I_1, it holds that

$$\begin{aligned} I_1 &\leq \frac{1}{(1+|x|/2)^k} \int_{|x-y|\geq |x|/2} \frac{1}{|y|} e^{-|y|} dy \\ &\leq \frac{2^k}{(1+|x|)^k} \int_{\mathbb{R}^3} \frac{1}{|y|} e^{-|y|} dy \\ &= \frac{4\pi 2^k}{(1+|x|)^k} \int_0^\infty r e^{-r} dr \\ &\leq \frac{C_k}{(1+|x|)^k}. \end{aligned}$$

For I_2, notice that

$$\{|x-y| < |x|/2\} \subseteq \{|x|/2 < |y|\},$$

and then it holds that

$$\begin{aligned} I_2 &= \int_{|x-y|\leq |x|/2} \frac{1}{|y|} e^{-|y|} \frac{1}{(1+|x-y|)^k} dy \\ &\leq e^{-\frac{|x|}{4}} \int_{\mathbb{R}^3} \frac{1}{|y|} e^{-|y|} dy \\ &\leq C e^{-\frac{|x|}{4}}. \end{aligned}$$

Therefore the claim follows from the above estimates on I_1 and I_2. ☐

By putting (2.4)–(2.5) into (2.2)–(2.3), and using the claim, one has

$$\sum_{|\alpha|\leq m} |\partial^\alpha \mathbf{T}\phi(x)| \leq \frac{C(\|\phi\|_{W_k^{m,\infty}})\|\phi\|^2_{W_k^{m,\infty}}}{(1+|x|)^{2k}} + \frac{C\|\bar{\rho}-1\|_{W_k^{m,\infty}}}{(1+|x|)^k}.$$

Thus it follows that

$$\|\mathbf{T}\phi\|_{W_k^{m,\infty}} \leq C\|\bar{\rho}-1\|_{W_k^{m,\infty}} + C(\|\phi\|_{W_k^{m,\infty}})\|\phi\|^2_{W_k^{m,\infty}}. \tag{2.6}$$

Finally, for any $\phi_1 = \phi_1(x)$ and $\phi_2 = \phi_2(x)$, it holds that

$$\mathbf{T}\phi_1 - \mathbf{T}\phi_2 = G * (g(\phi_1) - g(\phi_2))$$

with

$$g(\phi_1) - g(\phi_2) = \int_0^1 g'(\theta\phi_1 + (1-\theta)\phi_2) d\theta\, (\phi_1 - \phi_2).$$

Notice that for any $\phi = \phi(x)$,

$$g'(\phi) = e^\phi - 1 = \int_0^1 e^{\theta\phi} d\theta\, \phi.$$

Then the same computations as for (2.6) yield

$$\|\mathbf{T}\phi_1 - \mathbf{T}\phi_2\|_{W_k^{m,\infty}} \\ \le C(\|\phi_1\|_{W_k^{m,\infty}}, \|\phi_2\|_{W_k^{m,\infty}})(\|\phi_1\|_{W_k^{m,\infty}} + \|\phi_2\|_{W_k^{m,\infty}})\|\phi_1 - \phi_2\|_{W_k^{m,\infty}}, \tag{2.7}$$

where $C(\cdot,\cdot)$ is a nonnegative, nonincreasing, and continuous function in the two arguments. Combining (2.6) and (2.7), the standard argument implies that $\mathbf{T}$ has a unique fixed point ϕ in $\mathbb{B}_{m,k}(C)$ for a proper constant C provided that $\|\bar\rho - 1\|_{W_k^{m,\infty}}$ is small enough. This completes the proof of Theorem 1.1. ∎

Let us conclude this section with a remark. The existence of solutions to the elliptic equation (2.1) has been proved in [11] when the background density $\bar\rho(x) \ge 0$ satisfies

$$|\bar\rho(x) - 1| \le \frac{C}{(1+|x|)^{1/2}},\ x \in \mathbb{R}^3,$$

for a constant C which is not necessarily small. Later, [8] generalized the above condition to

$$|\bar\rho(x) - 1| \le \frac{C}{[\ln(e+|x|)]^{\sigma}},\ x \in \mathbb{R}^3,$$

for constants C and $\sigma > 0$. In Theorem 1.1, both the smallness and space decay at infinity of $\bar\rho(x) - 1$ are assumed in order to deal with the stability of the stationary solution under small perturbations. It is an interesting problem to consider the same issue for the stationary solution obtained in [11] or [8].

3. Macro-micro decomposition. For fixed (t,x), any function $u(t,x,\xi)$ can be uniquely decomposed as

$$\begin{cases} u(t,x,\xi) = u_1 + u_2, \\ u_1 \equiv \mathbf{P}u \in \mathcal{N}, \\ \mathbf{P}u = \left\{a(t,x) + \sum_{i=1}^3 b_i(t,x)\xi_i + c(t,x)|\xi|^2\right\}\sqrt{\mathbf{M}}, \\ u_2 \equiv \{\mathbf{I} - \mathbf{P}\}u \in \mathcal{N}^\perp, \end{cases} \tag{3.1}$$

where u_1 is called the macroscopic part of $u(t,x,\xi)$ with coefficients (a,b,c), and u_2 is the microscopic part of $u(t,x,\xi)$. Plugging this decomposition into the perturbed equation (1.7), one can obtain the macroscopic equation of u_1:

$$\partial_t u_1 + \xi\cdot\nabla_x u_1 + \nabla_x\phi\cdot\nabla_\xi u_1 - \frac{1}{2}\xi\cdot\nabla_x\phi u_1 - \xi\cdot\nabla_x\Psi e^{\phi}\sqrt{\mathbf{M}} \\ = r + \ell + n, \tag{3.2}$$

where

$$r = -\partial_t u_2,$$
$$\ell = -\xi\cdot\nabla_x u_2 - \nabla_x\phi\cdot\nabla_\xi u_2 + \frac{1}{2}\xi\cdot\nabla_x\phi u_2 + e^{\phi}\mathbf{L}u_2,$$
$$n = \Gamma(u,u) - \nabla_x\Psi\cdot\nabla_\xi u + \frac{1}{2}\xi\cdot\nabla_x\Psi u.$$

One can also obtain the evolution equations for each coefficient (a, b, c) of u_1. In fact, by putting the expansion $(3.1)_3$ into (3.2) and collecting the coefficients with respect to the basis $\{e_k\}_{k=1}^{13}$ consisting of

$$\sqrt{\mathbf{M}},\ \left(\xi_i\sqrt{\mathbf{M}}\right)_{1\le i\le 3},\ \left(|\xi_i|^2\sqrt{\mathbf{M}}\right)_{1\le i\le 3},\ \left(\xi_i\xi_j\sqrt{\mathbf{M}}\right)_{1\le i<j\le 3},\ \left(|\xi|^2\xi_i\sqrt{\mathbf{M}}\right)_{1\le i\le 3}, \tag{3.3}$$

then one has the following macroscopic equations on coefficients (a, b, c) of u_1:

$$\partial_t a + b\cdot\nabla_x\phi = \gamma^{(0)}, \tag{3.4}$$

$$\partial_t b_i + \partial_i a - (a\partial_i\phi - 2c\partial_i\phi) - \partial_i\Psi e^{\phi} = \gamma_i^{(1)}, \tag{3.5}$$

$$\partial_t c + \partial_i b_i - b_i\partial_i\phi = \gamma_i^{(2)}, \tag{3.6}$$

$$\partial_i b_j + \partial_j b_i - (b_j\partial_i\phi + b_i\partial_j\phi) = \gamma_{ij}^{(2)}, \quad i\neq j, \tag{3.7}$$

$$\partial_i c - c\partial_i\phi = \gamma_i^{(3)}, \tag{3.8}$$

where all terms on the right-hand side are the coefficients of $r+\ell+n$ with respect to the corresponding elements in the basis (3.3) with the further precise form:

$$\begin{aligned}
\gamma^{(0)} &\equiv -\partial_t\tilde{r}^{(0)} + \ell^{(0)} + n^{(0)},\\
\gamma_i^{(1)} &\equiv -\partial_t\tilde{r}_i^{(1)} + \ell_i^{(1)} + n_i^{(1)},\\
\gamma_i^{(2)} &\equiv -\partial_t\tilde{r}_i^{(2)} + \ell_i^{(2)} + n_i^{(2)},\\
\gamma_{ij}^{(2)} &\equiv -\partial_t\tilde{r}_{ij}^{(2)} + \ell_{ij}^{(2)} + n_{ij}^{(2)}, \quad i\neq j,\\
\gamma_i^{(3)} &\equiv -\partial_t\tilde{r}_i^{(3)} + \ell_i^{(3)} + n_i^{(3)}.
\end{aligned}$$

Here, the coefficients of $r = -\partial_t u_2$ in terms of the basis (3.3) were respectively written as the negative time derivatives of coefficients of $\tilde{r} = u_2$.

On the other hand, a, $b = (b_1, b_2, b_3)$, and c also satisfy the local macroscopic balance laws. In fact, multiplying the unperturbed equation (1.1) by the collision invariants in (1.10) and integrating them over $\mathbb{R}^3_\xi$, we have

$$\begin{aligned}
\partial_t\int_{\mathbb{R}^3} f d\xi + \nabla_x\cdot\int_{\mathbb{R}^3}\xi f d\xi &= 0,\\
\partial_t\int_{\mathbb{R}^3}\xi f d\xi + \nabla_x\cdot\int_{\mathbb{R}^3}\xi\otimes\xi f d\xi - \nabla_x\Phi\int_{\mathbb{R}^3} f d\xi &= 0,\\
\partial_t\int_{\mathbb{R}^3}\frac{1}{2}|\xi|^2 f d\xi + \nabla_x\cdot\int_{\mathbb{R}^3}\frac{1}{2}|\xi|^2\xi f d\xi - \nabla_x\Phi\cdot\int_{\mathbb{R}^3}\xi f d\xi &= 0.
\end{aligned}$$

By using the perturbed forms (1.6) on f and Φ and further using the decomposition (3.1), one can compute all moments appearing in the above system to obtain the macroscopic balance laws on coefficients:

$$\partial_t(a+3c) + \nabla_x\cdot b = 0, \tag{3.9}$$

$$\begin{aligned}\partial_t b + \nabla_x(a+5c) + \nabla_x\cdot\langle\xi\otimes\xi\sqrt{\mathbf{M}}, u_2\rangle = (a+3c)\nabla_x(\phi+\Psi)&\\ + e^{\phi}\nabla_x\Psi&,\end{aligned} \tag{3.10}$$

$$\partial_t(3a+15c) + \nabla_x\cdot(5b) + \nabla_x\cdot\langle|\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle = b\cdot\nabla_x(\phi+\Psi), \tag{3.11}$$

where from (3.11) the evolution of c can be written as

$$\partial_t c + \frac{1}{3}\nabla_x\cdot b + \frac{1}{6}\nabla_x\cdot\langle|\xi|^2\xi\sqrt{\mathbf{M}}, u_2\rangle = \frac{1}{6}b\cdot\nabla_x(\phi+\Psi). \tag{3.12}$$

Finally, from (1.8), Ψ is coupled with $a+3c$ again through the Poisson equation

$$\Delta_x \Psi = a + 3c. \tag{3.13}$$

Similarly to [12] and [5], based on the two macroscopic equations (3.6) and (3.7) only, the macroscopic component $b=(b_1,b_2,b_3)$ satisfies an elliptic-type equation:

$$\begin{aligned} -\Delta_x b_j - \partial_j\partial_j b_j &= \sum_{i\neq j}\partial_j(b_i\partial_i\phi) + \sum_{i\neq j}\partial_j\gamma_i^{(2)} \\ &\quad - \sum_{i\neq j}\partial_i(b_j\partial_i\phi + b_i\partial_j\phi) - \sum_{i\neq j}\partial_i\gamma_{ij}^{(2)} \\ &\quad - 2\partial_j(b_j\partial_j\phi) - 2\partial_j\gamma_j^{(2)}. \end{aligned} \tag{3.14}$$

It should be pointed out that the right-hand side of (3.14) contains the time-space-mixed derivatives of the local velocity-moment functions of the microscopic part u_2, given by

$$\partial_t\left[\sum_{i\neq j}\partial_j\tilde{r}_i^{(2)} - \sum_{i\neq j}\partial_i\tilde{r}_{ij}^{(2)} - 2\partial_j\tilde{r}_j^{(2)}\right].$$

4. Microscopic dissipation. In this section, we devote ourselves to obtaining the microscopic dissipation rate

$$\sum_{|\alpha|+|\beta|\le N}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2 = \|u_2\|_\nu^2 + \sum_{1\le|\alpha|\le N}\|\partial_x^\alpha u_2\|_\nu^2 + \sum_{k=1}^N \sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2$$

in three steps. To the end, let us denote

$$\delta_\phi = \|\phi\|_{W_2^{N+1,\infty}} = \sup_x(1+|x|)^2\sum_{|\alpha|\le N+1}|\partial_x^\alpha\phi(x)|, \tag{4.1}$$

for simplicity of presentation. We first cite some lemmas to be used later.

LEMMA 4.1 (see [1, 18]). *Let $u=u(x)\in H^2(\mathbb{R}^3)$. Then*

(i) $\|u\|_{L^\infty}\le C\|\nabla u\|^{\frac12}\|\nabla^2 u\|^{\frac12}\le C\|\nabla u\|_{H^1}$;

(ii) $\|u\|_{L^6}\le C\|\nabla u\|$;

(iii) $\|u\|_{L^q}\le C\|u\|_{H^1}$, $2\le q\le 6$;

(iv) $\|\frac{u}{|x|}\|\le C\|\nabla u\|$;

(v) $\|\Delta u\|^2=\sum_{|\alpha|=2}\|\partial^\alpha u\|^2$, *and for* $|\alpha|\ge 2$,

$$\|\partial^\alpha u\|\le C\sum_{|\alpha'|\le|\alpha|-2}\|\partial^{\alpha'}\Delta u\|.$$

LEMMA 4.2 (see [14]).

$$\begin{aligned} |\langle\partial_\xi^\beta\Gamma(u,v),w\rangle| \le C\sum_{\beta_1+\beta_2\le\beta}&\left\{\int_{\mathbb{R}^3}\|\nu^{1/2}\partial_\xi^{\beta_1}u\|_{L^2_\xi}\|\partial_\xi^{\beta_2}v\|_{L^2_\xi}\|\nu^{1/2}w\|_{L^2_\xi}dx\right. \\ &\left.+\int_{\mathbb{R}^3}\|\nu^{1/2}\partial_\xi^{\beta_1}v\|_{L^2_\xi}\|\partial_\xi^{\beta_2}u\|_{L^2_\xi}\|\nu^{1/2}w\|_{L^2_\xi}dx\right\}; \end{aligned}$$

$$\|\langle\Gamma(u,v),w\rangle\|_{L^2_x}+\|\langle\Gamma(v,u),w\rangle\|_{L^2_x}\leq C\|\nu^3 w\|_{L^\infty_{x,\xi}}\|u\|_{L^\infty_x(L^2_\xi)}\|v\|.$$

LEMMA 4.3 (see [14]). *Let $|\beta|>0$. Then $\partial_\xi^\beta\nu(\xi)$ is uniformly bounded, and for any small $\epsilon>0$ there exists $C_{\beta,\epsilon}$ such that, for any u,*

$$\|\partial_\xi^\beta[Ku]\|^2\leq\epsilon\sum_{|\beta_1|=|\beta|}\|\partial_\xi^{\beta_1}u\|^2+C_{\beta,\epsilon}\|u\|^2.$$

Here, Lemmas 4.2 and 4.3 were both provided, respectively, in Lemmas 2.3 and 2.2 in [14] due to the modified proof from $\mathbb{T}^3_x$ to $\mathbb{R}^3_x$, and Lemma 4.2 was also used in [12].

Next, as the first step, we consider the energy estimates on u of zero order, whose proof essentially needs the macroscopic balance laws to control the evolution of zero-order energy by using the high-order dissipation.

LEMMA 4.4 (zero order). *There are constants $\lambda>0$, C such that*

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2+\|\nabla_x\Psi\|^2-2\int e^{-\phi}|b|^2c\,dx\right)+\lambda\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)|u_2|^2dxd\xi$$
$$\leq C(\delta_\phi+[[u(t)]])(1+[[u(t)]])[[u(t)]]_\nu^2 \tag{4.2}$$

holds for any $t\geq 0$.

Proof. Multiplying (1.7) by u and taking integration over $\mathbb{R}^3\times\mathbb{R}^3$ gives

$$\frac{1}{2}\frac{d}{dt}\|u\|^2-\langle\xi\cdot\nabla_x\Psi e^\phi\sqrt{\mathbf{M}},u\rangle-\langle e^\phi\mathbf{L}u,u\rangle$$
$$=\frac{1}{2}\langle\xi\cdot\nabla_x(\Psi+\phi)u,u\rangle+\langle\Gamma(u,u),u\rangle.$$

It follows from taking the integration with respect to ξ and integration by parts that

$$-\langle\xi\cdot\nabla_x\Psi e^\phi\sqrt{\mathbf{M}},u\rangle=-\langle\nabla_x\Psi e^\phi,b\rangle=-\langle\nabla_x\Psi,b\rangle-\langle\nabla_x\Psi,(e^\phi-1)b\rangle, \tag{4.3}$$

where as usual one can further use the conservation law (3.9) of mass and the Poisson equation (3.13) to compute

$$-\langle\nabla_x\Psi,b\rangle=\langle\Psi,\nabla_x\cdot b\rangle=-\langle\Psi,\partial_t(a+3c)\rangle=-\langle\Psi,\Delta_x\partial_t\Psi\rangle$$
$$=\langle\nabla_x\Psi,\partial_t\nabla_x\Psi\rangle=\frac{1}{2}\frac{d}{dt}\|\nabla_x\Psi\|^2. \tag{4.4}$$

Thus one arrives at

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2+\|\nabla_x\Psi\|^2\right)-\langle e^\phi\mathbf{L}u,u\rangle$$
$$=\langle\Gamma(u,u),u\rangle+\frac{1}{2}\langle\xi\cdot\nabla_x(\Psi+\phi)u,u\rangle+\langle\nabla_x\Psi,(e^\phi-1)b\rangle. \tag{4.5}$$

For simplicity of later presentation, let us denote

$$I_1=\frac{1}{2}\langle\xi\cdot\nabla_x\phi u,u\rangle,\quad I_2=\frac{1}{2}\langle\xi\cdot\nabla_x\Psi u,u\rangle,\quad I_3=\langle\nabla_x\Psi,(e^\phi-1)b\rangle.$$

Next we estimate each term I_i $(i=1,2,3)$ as follows. First, for I_1 and I_3, it follows directly from Hardy's inequality given in Lemma 4.1 that

$$
\begin{aligned}
I_1 &\le \iint_{\mathbb{R}^3\times\mathbb{R}^3} |\xi\cdot\nabla_x\phi|(|u_1|^2+|u_2|^2)dxd\xi \\
&\le C\|(1+|x|)^2\nabla_x\phi\|_{L^\infty_x}\int_{\mathbb{R}^3}\frac{|(a,b,c)|^2}{(1+|x|)^2}dx + C\|\nabla_x\phi\|_{L^\infty_x}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)|u_2|^2dxd\xi \\
&\le C\delta_\phi\|\nabla_x(a,b,c)\|^2 + C\delta_\phi\|u_2\|_\nu^2 \qquad (4.6)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &\le \left[\sup_x |x|^2(e^\phi-1)\right]\int_{\mathbb{R}^3}\frac{|\nabla_x\Psi\cdot b|}{|x|^2}dx \\
&\le C\left[\sup_x |x|^2|\phi(x)|\right]\left\|\frac{\nabla_x\Psi}{|x|}\right\|_{L^2_x}\left\|\frac{b}{|x|}\right\|_{L^2_x} \\
&\le C\delta_\phi\|\nabla_x^2\Psi\|_{L^2_x}\|\nabla_x b\|_{L^2_x}. \qquad (4.7)
\end{aligned}
$$

For I_2, it is not straightforward to make estimates similarly as in (4.6). Actually, one can write I_2 as the summation of three terms as follows:

$$
I_2 = \frac{1}{2}\langle\xi\cdot\nabla_x\Psi,|u_1|^2\rangle + \langle\xi\cdot\nabla_x\Psi,u_1u_2\rangle + \frac{1}{2}\langle\xi\cdot\nabla_x\Psi,|u_2|^2\rangle, \qquad (4.8)
$$

where by Lemma 4.1 the second and third terms on the right-hand side can be directly estimated:

$$
\begin{aligned}
\frac{1}{2}\langle\xi\cdot\nabla_x\Psi,|u_2|^2\rangle &\le \frac{1}{2}\|\nabla_x\Psi\|_{L^\infty_x}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi||u_2|^2dxd\xi \\
&\le C\|\nabla_x^2\Psi\|_{H^1_x}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)|u_2|^2dxd\xi \qquad (4.9)
\end{aligned}
$$

and

$$
\begin{aligned}
\langle\xi\cdot\nabla_x\Psi,u_1u_2\rangle &\le \|\xi\cdot\nabla_x\Psi u_1\|_{L^2_{x,\xi}}\|u_2\|_{L^2_{x,\xi}} \\
&\le C\|\nabla_x\Psi(a,b,c)\|_{L^2_x}\|u_2\|_{L^2_{x,\xi}} \\
&\le C\|\nabla_x\Psi\|_{L^3_x}\|(a,b,c)\|_{L^6_x}\|u_2\|_{L^2_{x,\xi}} \\
&\le C\|\nabla_x\Psi\|_{H^1_x}\|\nabla_x(a,b,c)\|_{L^2_x}\|u_2\|_{L^2_{x,\xi}}. \qquad (4.10)
\end{aligned}
$$

For the first term on the right-hand side of (4.8), one can use the precise expression of $u_1=\mathbf{P}u$ in (3.1) and take integration with respect to ξ over $\mathbb{R}^3$ to get

$$\begin{aligned}\frac{1}{2}\langle \xi\cdot\nabla_x\Psi,|u_1|^2\rangle &= \frac{1}{2}\left\langle \xi\cdot\nabla_x\Psi,\left[(a+b\cdot\xi+c|\xi|^2)\sqrt{\mathbf{M}}\right]^2\right\rangle\\
&=\langle\nabla_x\Psi,(a+c|\xi|^2)b\cdot\xi\xi\mathbf{M}\rangle\\
&=\sum_{ij}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\partial_i\Psi\mathbf{M}(ab_j\xi_j\xi_i+cb_j\xi_j\xi_i|\xi|^2)dxd\xi\\
&=\sum_{j}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\partial_j\Psi\mathbf{M}(ab_j|\xi_j|^2+cb_j|\xi_j|^2|\xi|^2)dxd\xi\\
&=\int_{\mathbb{R}^3}\nabla_x\Psi\cdot b\left(a\int_{\mathbb{R}^3}\frac{|\xi|^2}{3}\mathbf{M}d\xi+c\int_{\mathbb{R}^3}\frac{|\xi|^4}{3}\mathbf{M}d\xi\right)dx\\
&=\int_{\mathbb{R}^3}\nabla_x\Psi\cdot b(a+5c)dx,\end{aligned}$$

that is,

$$\frac{1}{2}\langle \xi\cdot\nabla_x\Psi,|u_1|^2\rangle=\int_{\mathbb{R}^3}\nabla_x\Psi\cdot b(a+3c)\,dx+2\int_{\mathbb{R}^3}\nabla_x\Psi\cdot bc\,dx, \tag{4.11}$$

where the first term is estimated by

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla_x\Psi\cdot b(a+3c)\,dx&\le\|\nabla_x\Psi\|_{L^3_x}\|b\|_{L^6_x}\|a+3c\|_{L^2_x}\\
&\le C\|\nabla_x\Psi\|_{H^1_x}\|\nabla_x b\|_{L^2_x}\|a+3c\|_{L^2_x},\end{aligned} \tag{4.12}$$

and by using the balance laws (3.10) for b to replace $\nabla_x\Psi$, the second term is rewritten as

$$\begin{aligned}2\int_{\mathbb{R}^3}\nabla_x\Psi\cdot bc\,dx&=2\int_{\mathbb{R}^3}\partial_t be^{-\phi}\cdot bc\,dx+2\int_{\mathbb{R}^3}\nabla_x(a+5c)e^{-\phi}\cdot bc\,dx\\
&\quad+2\int_{\mathbb{R}^3}\nabla_x\langle\xi\otimes\xi\sqrt{\mathbf{M}},u_2\rangle e^{-\phi}\cdot bc\,dx\\
&\quad-2\int_{\mathbb{R}^3}(a+3c)\nabla_x(\phi+\Psi)e^{-\phi}\cdot bc\,dx.\end{aligned} \tag{4.13}$$

Notice that

$$\begin{aligned}2\int_{\mathbb{R}^3}\partial_t be^{-\phi}\cdot bc\,dx&=\int_{\mathbb{R}^3}e^{-\phi}c\frac{d|b|^2}{dt}dx\\
&=\frac{d}{dt}\int_{\mathbb{R}^3}e^{-\phi}|b|^2c\,dx-\langle e^{-\phi}|b|^2,\partial_t c\rangle,\end{aligned} \tag{4.14}$$

where $\partial_t c$ can be further replaced by the balance law (3.12) for c so that one can estimate

$$\begin{aligned}-\langle e^{-\phi}|b|^2,\partial_t c\rangle&=\left\langle e^{-\phi}|b|^2,\frac{1}{3}\nabla_x\cdot b-\frac{1}{6}\nabla_x\cdot\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\rangle+\frac{1}{6}b\cdot\nabla_x(\phi+\Psi)\right\rangle\\
&\le C\|b\|_{L^6_x}\|b\|_{L^3_x}\left(\|\nabla_x\cdot b\|_{L^2_x}+\|\nabla_x\cdot\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\rangle\|_{L^2_x}\right)\\
&\quad+C\|b\|^3_{L^6_x}\left(\|\nabla_x\phi\|_{L^2_x}+\|\nabla_x\Psi\|_{L^2_x}\right)\\
&\le C\|\nabla_x b\|_{L^2_x}\|b\|_{H^1_x}\left(\|\nabla_x b\|_{L^2_x}+\|\nabla_x u_2\|_{L^2_{x,\xi}}\right)\\
&\quad+C\|\nabla_x b\|^3_{L^2_x}\left(\|\nabla_x\phi\|_{L^2_x}+\|\nabla_x\Psi\|_{L^2_x}\right).\end{aligned} \tag{4.15}$$

The remaining three terms on the right-hand side of (4.13) are bounded by

$$\begin{aligned} &2\|\nabla_x(a+5c)\|_{L^2_x}\|b\|_{L^6_x}\|c\|_{L^3_x}+2\|\nabla\cdot\langle\xi\otimes\xi\sqrt{\mathbf{M}},u_2\rangle\|_{L^2_x}\|b\|_{L^6_x}\|c\|_{L^3_x}\\ &\quad+2\|a+3c\|_{L^2_x}\|b\|_{L^6_x}\|c\|_{L^3_x}(\|\nabla_x\phi\|_{L^\infty_x}+\|\nabla_x\Psi\|_{L^\infty_x})\\ &\le C\|\nabla_x(a,c)\|_{L^2_x}\|\nabla_x b\|_{L^2_x}\|c\|_{H^1_x}+C\|u_2\|_{L^2_{x,\xi}}\|\nabla_x b\|_{L^2_x}\|c\|_{H^1_x}\\ &\quad+C\|a+3c\|_{L^2_x}\|\nabla_x b\|_{L^2_x}\|c\|_{H^1_x}(\|\nabla^2_x\phi\|_{H^1_x}+\|\nabla^2_x\Psi\|_{H^1_x}). \end{aligned} \tag{4.16}$$

Putting all the estimates (4.9), (4.10), (4.11), (4.12), (4.13), (4.14), (4.15), and (4.16) into (4.8) yields

$$\begin{aligned} I_2 &\le C\|\nabla^2_x\Psi\|_{H^1_x}\|u_2\|^2_\nu+C\|\nabla_x\Psi\|_{H^1_x}\|\nabla_x(a,b,c)\|\|u_2\|+C\|\nabla_x\Psi\|_{H^1_x}\|\nabla_x b\|\|a+3c\|\\ &\quad+C\|b\|_{H^1_x}\|\nabla_x b\|(\|\nabla_x b\|+\|\nabla_x u_2\|)+C(\|\nabla_x\phi\|+\|\nabla_x\Psi\|)\|\nabla_x b\|^3\\ &\quad+C\|c\|_{H^1_x}\|\nabla_x b\|\left[\|\nabla_x(a,c)\|+\|u_2\|+\left(\|\nabla^2_x\phi\|_{H^1_x}+\|\nabla^2_x\Psi\|_{H^1_x}\right)\|a+3c\|\right]. \end{aligned} \tag{4.17}$$

Collecting all estimates (4.6), (4.7), and (4.17) on I_1, I_2, and I_3, it follows from (4.5) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|u\|^2+\|\nabla_x\Psi\|^2-2\int_{\mathbb{R}^3}e^{-\phi}|b|^2c\,dx\right)-\langle e^\phi\mathbf{L}u,u\rangle\\ &\le\langle\Gamma(u,u),u\rangle+C\delta_\phi\|\nabla_x(a,b,c)\|^2+C\delta_\phi\|u_2\|^2_\nu+C\|\nabla^2_x\Psi\|_{H^1_x}\|u_2\|^2_\nu\\ &\quad+C\|\nabla_x\Psi\|_{H^1_x}\|\nabla_x(a,b,c)\|\|u_2\|+C\|\nabla_x\Psi\|_{H^1_x}\|\nabla_x b\|\|a+3c\|\\ &\quad+C\|b\|_{H^1_x}\|\nabla_x b\|(\|\nabla_x b\|+\|\nabla_x u_2\|)+C(\|\nabla_x\phi\|+\|\nabla_x\Psi\|)\|\nabla_x b\|^3\\ &\quad+C\|c\|_{H^1_x}\|\nabla_x b\|[\|\nabla_x(a,c)\|+\|u_2\|+(\|\nabla^2_x\phi\|_{H^1_x}+\|\nabla^2_x\Psi\|_{H^1_x})\|a+3c\|]\\ &\quad+C\delta_\phi\|\nabla^2_x\Psi\|\|\nabla_x b\|\\ &\le\langle\Gamma(u,u),u\rangle+C\left(\delta_\phi+[[u(t)]]+\delta_\phi[[u(t)]]+[[u(t)]]^2\right)[[u(t)]]^2_\nu, \end{aligned} \tag{4.18}$$

where for the first term on the right-hand side, as in [12], it holds that

$$\langle\Gamma(u,u),u\rangle\le[[u(t)]][[u(t)]]^2_\nu.$$

Therefore, (4.2) follows from (4.18) and the coercivity (1.11) of $-\mathbf{L}$. This completes the proof of Lemma 4.4. □

For the second step, we consider the energy estimates on the pure spatial derivatives of u.

LEMMA 4.5 (pure spatial derivatives). *There are constants $\lambda>0$, C such that*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{1\le|\alpha|\le N}\left(\|\partial^\alpha_x u\|^2+\|\partial^\alpha_x\nabla_x\Psi\|^2\right)+\lambda\sum_{1\le|\alpha|\le N}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)|\partial^\alpha_x u_2|^2dxd\xi\\ &\le C[[u(t)]][[u(t)]]^2_\nu+C\delta_\phi\sum_{|\alpha|\le N-1}\|\partial^\alpha_x\nabla_x(a,b,c)\|^2+C\delta_\phi\sum_{|\alpha|\le N-2}\|\partial^\alpha_x(a+3c)\|^2\\ &\quad+C\delta_\phi\sum_{1\le|\alpha|\le N-1}\|\partial^\alpha_x\nabla_\xi u_2\|^2 \end{aligned} \tag{4.19}$$

holds for any $t\ge0$, provided that $\delta_\phi>0$ is small enough.

Proof. Take any α with $1 \le |\alpha| \le N$. Applying ∂_x^α to the equation (1.7) yields

$$\begin{aligned}\partial_t(\partial_x^\alpha u) + \xi\cdot\nabla_x(\partial_x^\alpha u) + \nabla_x\phi\cdot\nabla_\xi(\partial_x^\alpha u) - \frac{1}{2}\xi\cdot\nabla_x\phi(\partial_x^\alpha u) - \xi\cdot\nabla_x(\partial_x^\alpha\Psi)e^\phi\sqrt{\mathbf{M}}&\\ + \sum_{|\alpha'|\ge 1,\alpha'\le\alpha} C_{\alpha'}^{\alpha}\left[\partial_x^{\alpha'}\nabla_x\phi\cdot\nabla_\xi\partial_x^{\alpha-\alpha'}u - \frac{1}{2}\xi\cdot\partial_x^{\alpha'}\nabla_x\phi\partial_x^{\alpha-\alpha'}u\right]&\\ - \sum_{|\alpha'|\ge 1,\alpha'\le\alpha} C_{\alpha'}^{\alpha}\xi\nabla_x\Psi\partial_x^{\alpha'}e^\phi\sqrt{\mathbf{M}}&\\ = e^\phi\mathbf{L}\partial_x^\alpha u - \partial_x^\alpha\left[\nabla_x\Psi\cdot\nabla_\xi u + \frac{1}{2}\xi\cdot\nabla_x\Psi u\right] + \partial_x^\alpha\Gamma(u,u)&\\ + \sum_{|\alpha'|\ge 1,\alpha'\le\alpha} C_{\alpha'}^{\alpha}\partial_x^{\alpha'}e^\phi\mathbf{L}\partial_x^{\alpha-\alpha'}u.&\end{aligned}$$

Multiplying the above equation by $\partial_x^\alpha u$ and taking integration over $\mathbb{R}^3\times\mathbb{R}^3$ further yields

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha u\|^2 + \langle -\xi\cdot\nabla_x(\partial_x^\alpha\Psi)e^\phi\sqrt{\mathbf{M}}, \partial_x^\alpha u\rangle + \langle -e^\phi\mathbf{L}\partial_x^\alpha u, \partial_x^\alpha u\rangle = \sum_{i=1}^{6} I_i, \tag{4.20}$$

where for simplicity of later presentation we have denoted

$$\begin{aligned}I_1 &= \langle\partial_x^\alpha\Gamma(u,u), \partial_x^\alpha u\rangle,\\ I_2 &= \left\langle \frac{1}{2}\xi\cdot\nabla_x\phi\partial_x^\alpha u, \partial_x^\alpha u\right\rangle,\\ I_3 &= \left\langle -\partial_x^\alpha[\nabla_x\Psi\cdot\nabla_\xi u + \frac{1}{2}\xi\cdot\nabla_x\Psi u], \partial_x^\alpha u\right\rangle\end{aligned}$$

and

$$\begin{aligned}I_4 &= \sum_{|\alpha'|\ge 1,\alpha'\le\alpha} C_{\alpha'}^{\alpha}\left\langle -\partial_x^{\alpha'}\nabla_x\phi\cdot\nabla_\xi\partial_x^{\alpha-\alpha'}u + \frac{1}{2}\xi\cdot\partial_x^{\alpha'}\nabla_x\phi\partial_x^{\alpha-\alpha'}u, \partial_x^\alpha u\right\rangle,\\ I_5 &= \sum_{|\alpha'|\ge 1,\alpha'\le\alpha} C_{\alpha'}^{\alpha}\langle\xi\nabla_x\Psi\partial_x^{\alpha'}e^\phi\sqrt{\mathbf{M}}, \partial_x^\alpha u\rangle,\\ I_6 &= \sum_{|\alpha'|\ge 1,\alpha'\le\alpha} C_{\alpha'}^{\alpha}\langle\partial_x^{\alpha'}e^\phi\mathbf{L}\partial_x^{\alpha-\alpha'}u, \partial_x^\alpha u\rangle.\end{aligned}$$

We estimate each term in (4.20) as follows. First, similarly to the case of zero-order estimates as in (4.3) and (4.4), it follows from integration by parts, conservation law (3.9) of mass, and the Poisson equation (3.13) that

$$\begin{aligned}\langle -\xi\cdot\nabla_x(\partial_x^\alpha\Psi)e^\phi\sqrt{\mathbf{M}}, \partial_x^\alpha u\rangle &= \langle -\nabla_x\partial_x^\alpha\Psi e^\phi, \partial_x^\alpha b\rangle\\ &= -\langle\nabla_x\partial_x^\alpha\Psi, \partial_x^\alpha b\rangle - \langle\nabla_x\partial_x^\alpha\Psi(e^\phi-1), \partial_x^\alpha b\rangle\\ &= \frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha\Psi\|^2 - \langle\nabla_x\partial_x^\alpha\Psi(e^\phi-1), \partial_x^\alpha b\rangle,\end{aligned} \tag{4.21}$$

where by Lemma 4.1 and again the Poisson equation (3.13) the inner product term is estimated by

$$\begin{aligned}|\langle\nabla_x\partial_x^\alpha\Psi(e^\phi-1), \partial_x^\alpha b\rangle| &\le C\|\phi\|_{L^\infty}\|\partial_x^\alpha\nabla_x\Psi\|_{L^2}\|\partial_x^\alpha b\|_{L^2}\\ &\le C\delta_\phi\sum_{|\beta|\le|\alpha|-1}\|\partial_x^\beta(a+3c)\|^2 + C\delta_\phi\|\partial_x^\alpha b\|^2.\end{aligned} \tag{4.22}$$

Here $|\alpha|\geq 1$ was used. Next, let us consider estimates on the terms on the right-hand side of (4.20). For I_1, as in [12], it follows from Lemma 4.2 that

$$I_1 \leq C[[u(t)]][[u(t)]]_\nu^2. \tag{4.23}$$

For I_2, it holds that

$$\begin{aligned} I_2 &\leq \frac{1}{2}\|\nabla_x\phi\|_{L^\infty_x}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\xi||\partial_x^\alpha u|^2 dxd\xi \\ &\leq C\|\nabla_x\phi\|_{L^\infty_x}(\|\partial_x^\alpha u_1\|_\nu^2+\|\partial_x^\alpha u_2\|_\nu^2) \\ &\leq C\|\nabla_x\phi\|_{L^\infty_x}(\|\partial_x^\alpha (a,b,c)\|^2+\|\partial_x^\alpha u_2\|_\nu^2). \end{aligned}\tag{4.24}$$

For I_3, one can write it as

$$\begin{aligned} I_3 &= \left\langle \frac{1}{2}\xi\cdot\nabla_x\Psi\partial_x^\alpha u, \partial_x^\alpha u\right\rangle \\ &\quad+\sum_{|\alpha'|\geq 1,\alpha'\leq\alpha} C_{\alpha'}^\alpha\left\langle -\partial_x^{\alpha'}\nabla_x\Psi\cdot\nabla_\xi\partial^{\alpha-\alpha'}u+\frac{1}{2}\xi\cdot\nabla_x^{\alpha'}\Psi\partial_x^{\alpha-\alpha'}u, \partial_x^\alpha u\right\rangle \\ &= I_{3,1}+\sum_{|\alpha'|\geq 1,\alpha'\leq\alpha} C_{\alpha'}^\alpha I_{3,\alpha'}. \end{aligned}\tag{4.25}$$

Here, $I_{3,1}$ is estimated by

$$\begin{aligned} I_{3,1} &\leq C\|\nabla_x\Psi\|_{L^\infty_x}\|\partial_x^\alpha u\|_\nu^2 \\ &\leq C\|\nabla_x\nabla_x\Psi\|_{H^1_x}(\|\partial_x^\alpha(a,b,c)\|^2+\|\partial_x^\alpha u_2\|_\nu^2) \\ &\leq C[[u(t)]][[u(t)]]_\nu^2. \end{aligned}\tag{4.26}$$

For each α' with $|\alpha'|\geq 1$ and $\alpha'\leq\alpha$, $I_{3,\alpha'}$ is estimated by two cases.

Case 1. $\{|\alpha'|\geq 1,\alpha'\leq\alpha\}\cap\{|\alpha'|\leq|\alpha|-2\}$. In this case, it holds that

$$\begin{aligned} I_{3,\alpha'} &\leq C\int_{\mathbb{R}^3}\|\partial_x^{\alpha'}\nabla_x\Psi\|_{L^\infty_x}\|\nabla_\xi\partial_x^{\alpha-\alpha'}u\|_{L^2_x}\|\partial_x^\alpha u\|_{L^2_x}d\xi \\ &\quad+C\int_{\mathbb{R}^3}|\xi|\|\partial_x^{\alpha'}\nabla_x\Psi\|_{L^\infty_x}\|\partial_x^{\alpha-\alpha'}u\|_{L^2_x}\|\partial_x^\alpha u\|_{L^2_x}d\xi \\ &\leq C\left\{\sum_{2\leq|\alpha|\leq N}\|\partial_x^\alpha\nabla_x\Psi\|\right\}\left\{\sum_{2\leq|\alpha|\leq N-1}\|\nabla_\xi\partial_x^\alpha u\|\right\}\|\partial_x^\alpha u\| \\ &\quad+C\left\{\sum_{2\leq|\alpha|\leq N}\|\partial_x^\alpha\nabla_x\Psi\|\right\}\left\{\sum_{2\leq|\alpha|\leq N-1}\|\partial_x^\alpha u\|_\nu\right\}\|\partial_x^\alpha u\|_\nu \\ &\leq C[[u(t)]][[u(t)]]_\nu^2. \end{aligned}\tag{4.27}$$

Case 2. $\{|\alpha'|\geq 1, \alpha'\leq\alpha\}\cap\{|\alpha'|>|\alpha|-2\}$. In this case, it holds that

$$
\begin{aligned}
I_{3,\alpha'} &\leq C\int_{\mathbb{R}^3}\|\nabla_\xi\partial_x^{\alpha-\alpha'}u\|_{L^\infty_x}\|\partial_x^{\alpha'}\nabla_x\Psi\|_{L^2_x}\|\partial_x^\alpha u\|_{L^2_x}d\xi\\
&\quad + C\int_{\mathbb{R}^3}|\xi|\|\partial_x^{\alpha-\alpha'}u\|_{L^\infty_x}\|\partial_x^{\alpha'}\nabla_x\Psi\|_{L^2_x}\|\partial_x^\alpha u\|_{L^2_x}d\xi\\
&\leq C\left\{\sum_{1\leq|\alpha|\leq N}\|\partial_x^\alpha\nabla_x\Psi\|\right\}\int_{\mathbb{R}^3}\left\{\sum_{1\leq|\alpha|\leq 3}\|\nabla_\xi\partial_x^\alpha u\|_{L^2_x}\right\}\|\partial_x^\alpha u\|_{L^2_x}d\xi\\
&\quad + C\left\{\sum_{1\leq|\alpha|\leq N}\|\partial_x^\alpha\nabla_x\Psi\|\right\}\int_{\mathbb{R}^3}|\xi|\left\{\sum_{1\leq|\alpha|\leq 3}\|\partial_x^\alpha u\|_{L^2_x}\right\}\|\partial_x^\alpha u\|_{L^2_x}d\xi\\
&\leq C\left\{\sum_{1\leq|\alpha|\leq N}\|\partial_x^\alpha\nabla_x\Psi\|\right\}\left\{\|\partial_x^\alpha u\|\sum_{1\leq|\alpha|\leq 3}\|\nabla_\xi\partial_x^\alpha u\|+\|\partial_x^\alpha u\|_\nu\sum_{1\leq|\alpha|\leq 3}\|\partial_x^\alpha u\|\right\}\\
&\leq C[[u(t)]][[u(t)]]_\nu^2. \qquad (4.28)
\end{aligned}
$$

Therefore, by putting (4.26), (4.27), and (4.28) into (4.25), one has

$$
I_3\leq C[[u(t)]][[u(t)]]_\nu^2. \qquad (4.29)
$$

For I_4, it holds that

$$
\begin{aligned}
I_4 &\leq \delta_\phi\|\partial_x^\alpha u\|_\nu^2+\frac{C}{\delta_\phi}\sum_{1\leq|\alpha'|\leq|\alpha|}\left(\|\partial_x^{\alpha'}\nabla_x\phi\cdot\nabla_\xi\partial_x^{\alpha-\alpha'}u\|^2+\|\partial_x^{\alpha'}\nabla_x\phi\partial_x^{\alpha-\alpha'}u\|_\nu^2\right)\\
&\leq C\delta_\phi\|\partial_x^\alpha u_2\|_\nu^2+C\delta_\phi\|\partial_x^\alpha(a,b,c)\|^2+C\delta_\phi\sum_{1\leq|\alpha|\leq N-1}\left(\|\partial_x^\alpha\nabla_\xi u\|^2+\|\partial_x^\alpha u\|^2\right)\\
&\leq C\delta_\phi\sum_{1\leq|\alpha|\leq N}\|\partial_x^\alpha u_2\|_\nu^2+C\delta_\phi\sum_{1\leq|\alpha|\leq N}\|\partial_x^\alpha(a,b,c)\|^2\\
&\quad + C\delta_\phi\sum_{1\leq|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_\xi u_2\|^2. \qquad (4.30)
\end{aligned}
$$

For I_5, it holds that

$$
\begin{aligned}
I_5 &\leq \delta_\phi\|\partial_x^\alpha u\|^2+\frac{C}{\delta_\phi}\sum_{1\leq|\alpha'|\leq|\alpha|}\|\partial_x^{\alpha-\alpha'}\nabla_x\Psi\partial_x^{\alpha'}e^\phi\|^2\\
&\leq C\delta_\phi\|\partial_x^\alpha u_2\|^2+C\delta_\phi\|\partial_x^\alpha(a,b,c)\|^2+C\delta_\phi\sum_{1\leq|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_x\Psi\|^2\\
&\leq C\delta_\phi\|\partial_x^\alpha u_2\|^2+C\delta_\phi\|\partial_x^\alpha(a,b,c)\|^2+C\delta_\phi\sum_{|\alpha|\leq N-2}\|\partial_x^\alpha(a+3c)\|^2. \qquad (4.31)
\end{aligned}
$$

Finally, for I_6, as in [12], it holds that

$$
\begin{aligned}
I_6 &= \sum_{1\leq|\alpha'|\leq|\alpha|}C^\alpha_{\alpha'}\langle\partial_x^{\alpha'}e^\phi\mathbf{L}\partial_x^{\alpha-\alpha'}u_2,\partial_x^\alpha u_2\rangle\\
&\leq \delta_\phi\|\partial_x^\alpha u_2\|_\nu^2+\frac{C}{\delta_\phi}\sum_{1\leq|\alpha'|\leq|\alpha|}\|\nu^{-1/2}\partial_x^{\alpha'}e^\phi(-\nu+K)\partial_x^{\lambda-\lambda_1}u_2\|^2\\
&\leq \delta_\phi\|\partial_x^\alpha u_2\|_\nu^2+C\delta_\phi\sum_{1\leq|\alpha|\leq N-1}\|\partial_x^\alpha u_2\|_\nu^2. \qquad (4.32)
\end{aligned}
$$

Thus, by collecting all estimates (4.21), (4.22), (4.23), (4.24), (4.29), (4.30), (4.31), and (4.32) and using the coercivity (1.11) of $-\mathbf{L}$, it follows from (4.20) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\partial_x^\alpha u\|^2+\|\partial_x^\alpha\nabla_x\Psi\|^2)+\lambda\iint_{\mathbb{R}^3\times\mathbb{R}^3}\nu(\xi)|\partial_x^\alpha u_2|^2dxd\xi\\&\leq C[[u(t)]][[u(t)]]_\nu^2+C\delta_\phi\sum_{1\leq|\alpha|\leq N}\|\partial_x^\alpha(a,b,c)\|^2+C\delta_\phi\sum_{|\alpha|\leq N-2}\|\partial_x^\alpha(a+3c)\|^2\\&\quad+C\delta_\phi\sum_{1\leq|\alpha|\leq N}\|\partial_x^\alpha u_2\|_\nu^2+C\delta_\phi\sum_{1\leq|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_\xi u_2\|^2,\end{aligned}$$

which leads to the desired estimate (4.19) after taking summation over $1\leq|\alpha|\leq N$, provided that $\delta_\phi>0$ is small enough. This completes the proof of Lemma 4.5. ☐

As the final step, we consider the energy estimates on u with mixed space and velocity derivatives in the following

LEMMA 4.6 (space-velocity-mixed derivatives). *Let $1\leq k\leq N$. There are constants $\lambda>0$, C such that*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\leq N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|^2+\lambda\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\leq N}}\iint e^\phi\nu(\xi)|\partial_x^\alpha\partial_\xi^\beta u_2|^2dxd\xi\\&\leq C(\delta_\phi+[[u(t)]]+[[u(t)]]^2)[[u(t)]]_\nu^2+C\sum_{|\alpha|\leq N-k}\|\partial_x^\alpha\nabla_x(a,b,c)\|^2\\&\quad+C\sum_{|\alpha|\leq N-k+1}\|\partial_x^\alpha u_2\|_\nu^2+C\chi_{\{2\leq k\leq N\}}\sum_{\substack{1\leq|\beta|\leq k-1\\|\alpha|+|\beta|\leq N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2\end{aligned}\tag{4.33}$$

holds for any $t\geq0$, provided that $\delta_\phi>0$ is small enough. Here χ_A denotes the characteristic function of a set A.

Proof. Applying the microscopic projection $\{\mathbf{I}-\mathbf{P}\}$ to (1.7), one has the microscopic evolution equation

$$\begin{aligned}&\partial_t u_2+\{\mathbf{I}-\mathbf{P}\}\xi\cdot\nabla_x u+\{\mathbf{I}-\mathbf{P}\}\left(\nabla_x(\phi+\Psi)\cdot\nabla_\xi u-\frac{1}{2}\xi\cdot\nabla_x(\phi+\Psi)u\right)\\&\qquad=e^\phi\mathbf{L}u_2+\Gamma(u,u).\end{aligned}\tag{4.34}$$

Notice that the macroscopic quantities $\partial_t u_1$ and $\xi\cdot\nabla_x\Psi e^\phi\sqrt{\mathbf{M}}$ disappear after taking the microscopic projection. One can further rewrite (4.34) as

$$\begin{aligned}&\partial_t u_2+\xi\cdot\nabla_x u_2+\nabla_x(\phi+\Psi)\cdot\nabla_\xi u_2+e^\phi\nu(\xi)u_2\\&=e^\phi Ku_2+\Gamma(u,u)+\frac{1}{2}\xi\cdot\nabla_x(\phi+\Psi)u_2\\&\quad-\{\mathbf{I}-\mathbf{P}\}\left(\nabla_x\phi\cdot\nabla_\xi u_1-\frac{1}{2}\xi\cdot\nabla_x\phi u_1+\xi\cdot\nabla_x u_1\right)\\&\quad-\{\mathbf{I}-\mathbf{P}\}(\nabla_x\Psi\cdot\nabla_\xi u_1-\frac{1}{2}\xi\cdot\nabla_x\Psi u_1)\\&\quad+\mathbf{P}\left(\nabla_x\phi\cdot\nabla_\xi u_2-\frac{1}{2}\xi\cdot\nabla_x\phi u_2+\xi\cdot\nabla_x u_2\right)\\&\quad+\mathbf{P}\left(\nabla_x\Psi\cdot\nabla_\xi u_2-\frac{1}{2}\xi\cdot\nabla_x\Psi u_2\right),\end{aligned}\tag{4.35}$$

where the right-hand side is the summation of seven terms, and for simplicity we denote them by S_i ($1 \le i \le 7$), respectively. Let $1 \le k \le N$, and fix α, β with $|\beta| = k$ and $|\alpha| + |\beta| \le N$. By taking derivatives $\partial_x^\alpha \partial_\xi^\beta$ for (4.35), multiplying it by $\partial_x^\alpha \partial_\xi^\beta u_2$, and then taking integration over $\mathbb{R}^3 \times \mathbb{R}^3$, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + \iint_{\mathbb{R}^3\times\mathbb{R}^3} e^\phi \nu(\xi)|\partial_x^\alpha \partial_\xi^\beta u_2|^2 dx d\xi \\
&= \sum_{i=1}^{7}\langle \partial_x^\alpha \partial_\xi^\beta S_i, \partial_x^\alpha \partial_\xi^\beta u_2\rangle - \sum_{|\alpha'|\ge 1, \alpha'\le\alpha} C_{\alpha'}^{\alpha}\langle I_{\alpha'}^1 + I_{\alpha'}^2, \partial_x^\alpha \partial_\xi^\beta u_2\rangle \\
&\quad - \sum_{|\beta'|\ge 1, \beta'\le\beta} C_{\beta'}^{\beta}\langle I_{\beta'}^1 + I_{\beta'}^2, \partial_x^\alpha \partial_\xi^\beta u_2\rangle,
\end{aligned}
\tag{4.36}
$$

where $I_{\alpha'}^1, I_{\alpha'}^2$ are given by

$$
I_{\alpha'}^1 = \nabla_x \partial_x^{\alpha'}(\phi + \Psi)\cdot\nabla_\xi \partial_x^{\alpha-\alpha'}\partial_\xi^\beta u_2, \quad I_{\alpha'}^2 = \partial_x^{\alpha'} e^\phi \partial_\xi^\beta[\nu(\xi)\partial_x^{\alpha-\alpha'}u_2],
$$

and $I_{\beta'}^1, I_{\beta'}^2$ are given by

$$
I_{\beta'}^1 = \partial_\xi^{\beta'}\xi\cdot\nabla_x\partial_x^\alpha\partial_\xi^{\beta-\beta'}u_2, \quad I_{\beta'}^2 = e^\phi\partial_\xi^{\beta'}\nu(\xi)\partial_x^\alpha\partial_\xi^{\beta-\beta'}u_2.
$$

Now let us estimate each inner product term in (4.36), and in what follows we take $0 < \epsilon \le 1$ to be determined later and suppose $0 < \delta_\phi \le 1$ for simplicity. First, it holds that

$$
\begin{aligned}
&\langle \partial_x^\alpha \partial_\xi^\beta S_1, \partial_x^\alpha \partial_\xi^\beta u_2\rangle \\
&= \sum_{\alpha'\le\alpha} C_{\alpha'}^{\alpha}\langle \partial_x^{\alpha'} e^\phi \partial_\xi^\beta K \partial_x^{\alpha-\alpha'}u_2, \partial_x^\alpha \partial_\xi^\beta u_2\rangle \\
&\le \epsilon\|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + \frac{C}{\epsilon}\sum_{\alpha'\le\alpha}\|\partial_\xi^\beta K\partial_x^{\alpha-\alpha'}u_2\|^2 \\
&\le \epsilon\|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + C\epsilon\sum_{\alpha'\le\alpha}\sum_{|\beta'|=|\beta|}\|\partial_\xi^{\beta'}\partial_x^{\alpha-\alpha'}u_2\|^2 + \frac{C_{\beta,\epsilon^2}}{\epsilon}\sum_{\alpha'\le\alpha}\|\partial_x^{\alpha-\alpha'}u_2\|^2 \\
&\le \epsilon\|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + C\epsilon\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + C_\epsilon\sum_{|\alpha|\le N-k}\|\partial_x^\alpha u_2\|^2,
\end{aligned}
$$

where Lemma 4.3 was used. As in [12], again by Lemma 4.3, one has

$$
\langle \partial_x^\alpha \partial_\xi^\beta S_2, \partial_x^\alpha \partial_\xi^\beta u_2\rangle = \langle \partial_x^\alpha \partial_\xi^\beta \Gamma(u,u), \partial_x^\alpha \partial_\xi^\beta u_2\rangle \le C[[u(t)]][[u(t)]]^2.
$$

For S_3, it holds that

$$
\begin{aligned}
&\langle \partial_x^\alpha \partial_\xi^\beta S_3, \partial_x^\alpha \partial_\xi^\beta u_2\rangle \\
&= \left\langle \frac{1}{2}\partial_x^\alpha \partial_\xi^\beta[\xi\cdot\nabla_x(\phi+\Psi)u_2], \partial_x^\alpha \partial_\xi^\beta u_2\right\rangle \\
&\le C\left(\delta_\phi + \sum_{|\alpha'|\le N}\|\partial_x^{\alpha'}\nabla_x\Psi\|\right)\sum_{\alpha'\le\alpha,\beta'\le\beta}\|\partial_x^{\alpha'}\partial_\xi^{\beta'}u_2\|_\nu^2 \\
&\le C\delta_\phi\sum_{|\alpha|\le N-k}\|\partial_x^\alpha u_2\|_\nu^2 + C\delta_\phi\sum_{\substack{1\le|\beta|\le k\\|\alpha|+|\beta|\le N}}\|\partial_x^\alpha \partial_\xi^\beta u_2\|_\nu^2 + C[[u(t)]][[u(t)]]_\nu^2.
\end{aligned}
$$

S_4 and S_5 are estimated by

$$\begin{aligned}
&\langle \partial_x^\alpha \partial_\xi^\beta S_4, \partial_x^\alpha \partial_\xi^\beta u_2 \rangle \\
&= -\left\langle \partial_x^\alpha \partial_\xi^\beta \{\mathbf{I}-\mathbf{P}\} \left(\nabla_x \phi \cdot \nabla_\xi u_1 - \frac{1}{2} \xi \cdot \nabla_x \phi u_1 + \xi \cdot \nabla_x u_1 \right), \partial_x^\alpha \partial_\xi^\beta u_2 \right\rangle \\
&\leq \epsilon \|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + C_\epsilon \delta_\phi^2 \sum_{|\alpha| \leq N-k-1} \|\partial_x^\alpha \nabla_x (a,b,c)\|^2 + C_\epsilon \sum_{|\alpha| \leq N-k} \|\partial_x^\alpha \nabla_x (a,b,c)\|^2 \\
&\leq \epsilon \|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + C_\epsilon \sum_{|\alpha| \leq N-k} \|\partial_x^\alpha \nabla_x (a,b,c)\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\langle \partial_x^\alpha \partial_\xi^\beta S_5, \partial_x^\alpha \partial_\xi^\beta u_2 \rangle \\
&= -\left\langle \partial_x^\alpha \partial_\xi^\beta \{\mathbf{I}-\mathbf{P}\} \left(\nabla_x \Psi \cdot \nabla_\xi u_1 - \frac{1}{2} \xi \cdot \nabla_x \Psi u_1 \right), \partial_x^\alpha \partial_\xi^\beta u_2 \right\rangle \\
&\leq \epsilon \|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + C_\epsilon \sum_{|\alpha| \leq N-1} \|\partial_x^\alpha \nabla_x \Psi\|^2 \sum_{|\lambda| \leq N-1} \|\partial_x^\alpha \nabla_x (a,b,c)\|^2 \\
&\leq \epsilon \|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + C_\epsilon [[u(t)]]^2 [[u(t)]]_\nu^2.
\end{aligned}$$

Similarly, S_6 and S_7 are estimated by

$$\begin{aligned}
&\langle \partial_x^\alpha \partial_\xi^\beta S_6, \partial_x^\alpha \partial_\xi^\beta u_2 \rangle \\
&= \left\langle \partial_x^\alpha \partial_\xi^\beta \mathbf{P} \left(\nabla_x \phi \cdot \nabla_\xi u_2 - \frac{1}{2} \xi \cdot \nabla_x \phi u_2 + \xi \cdot \nabla_x u_2 \right), \partial_x^\alpha \partial_\xi^\beta u_2 \right\rangle \\
&\leq \epsilon \|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + C_\epsilon \delta_\phi^2 \sum_{|\alpha| \leq N-k} \|\partial_x^\alpha u_2\|^2 + C_\epsilon \sum_{|\alpha| \leq N-k} \|\partial_x^\alpha \nabla_x u_2\|^2 \\
&\leq \epsilon \|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + C_\epsilon \sum_{|\alpha| \leq N-k+1} \|\partial_x^\alpha u_2\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\langle \partial_x^\alpha \partial_\xi^\beta S_7, \partial_x^\alpha \partial_\xi^\beta u_2 \rangle \\
&= \left\langle \partial_x^\alpha \partial_\xi^\beta \mathbf{P} \left(\nabla_x \Psi \cdot \nabla_\xi u_2 - \frac{1}{2} \xi \cdot \nabla_x \Psi u_2 \right), \partial_x^\alpha \partial_\xi^\beta u_2 \right\rangle \\
&\leq \epsilon \|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + C_\epsilon \sum_{|\alpha| \leq N-1} \|\partial_x^\alpha \nabla_x \Psi\|^2 \sum_{|\alpha| \leq 1} \|\partial_x^\alpha \nabla_x u_2\|^2 \\
&\leq \epsilon \|\partial_x^\alpha \partial_\xi^\beta u_2\|^2 + C_\epsilon [[u(t)]]^2 [[u(t)]]_\nu^2.
\end{aligned}$$

For the remaining terms about $I_{\alpha'}^1, I_{\alpha'}^2$ and $I_{\beta'}^1, I_{\beta'}^2$, one can estimate them as follows.

First, for $I^1_{\alpha'}$, it holds that

$$
\begin{aligned}
&-\sum_{|\alpha'|\geq 1,\alpha'\leq\alpha} C^{\alpha}_{\alpha'}\langle I^1_{\alpha'},\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\rangle\\
&=-\sum_{|\alpha'|\geq 1,\alpha'\leq\alpha} C^{\alpha}_{\alpha'}\langle \nabla_x\partial^{\alpha'}_x(\phi+\Psi)\cdot\nabla_{\xi}\partial^{\alpha-\alpha'}_x\partial^{\beta}_{\xi}u_2,\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\rangle\\
&\leq C\delta_{\phi}\sum_{|\alpha'|\geq 1,\alpha'\leq\alpha}\|\nabla_{\xi}\partial^{\alpha-\alpha'}_x\partial^{\beta}_{\xi}u_2\|\|\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|\\
&\quad+C\sum_{|\alpha'|\geq 1,\alpha'\leq\alpha}\int_{\mathbb{R}^3}|\nabla_x\partial^{\alpha'}_x\Psi|\cdot\|\nabla_{\xi}\partial^{\alpha-\alpha'}_x\partial^{\beta}_{\xi}u_2\|_{L^2_{\xi}}\|\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|_{L^2_{\xi}}dx,
\end{aligned}
$$

where the first term on the right-hand side is bounded by

$$
C\delta_{\phi}\|\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|^2+C\delta_{\phi}\sum_{\substack{|\beta|=k+1\\|\alpha|+|\beta|\leq N}}\|\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|^2,
$$

and the second term on the right-hand side is bounded by

$$
\begin{aligned}
&C\left\{\sum_{|\alpha|=N-1}\|\nabla_x\partial^{\alpha}_x\Psi\|\right\}\left\{\sum_{|\beta|=1}\sup_{x\in\mathbb{R}^3}\|\nabla_{\xi}\partial^{\beta}_{\xi}u_2\|_{L^2_{\xi}}\right\}\|\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|_{L^2_{x,\xi}}\\
&\quad+C\left\{\sum_{1\leq|\alpha'|\leq N-2}\|\nabla_x\partial^{\alpha'}_x\Psi\|_{L^{\infty}_x}\right\}\left\{\sum_{|\alpha'|\geq 1,\alpha'\leq\alpha}\|\nabla_{\xi}\partial^{\alpha-\alpha'}_x\partial^{\beta}_{\xi}u_2\|_{L^2_{x,\xi}}\right\}\|\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|_{L^2_{x,\xi}}\\
&\leq C\left\{\sum_{|\alpha|=N-1}\|\nabla_x\partial^{\alpha}_x\Psi\|\right\}\left\{\sum_{|\alpha|\leq 1,|\beta|=1}\|\nabla_x\nabla_{\xi}\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|_{L^2_{x,\xi}}\right\}\|\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|_{L^2_{x,\xi}}\\
&\quad+C\left\{\sum_{2\leq|\alpha|\leq N}\|\nabla_x\partial^{\alpha}_x\Psi\|_{L^2_x}\right\}\left\{\sum_{|\alpha|+|\beta|\leq N}\|\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|_{L^2_{x,\xi}}\right\}\|\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|_{L^2_{x,\xi}}\\
&\leq C[[u(t)]][[u(t)]]^2_{\nu}.
\end{aligned}
$$

For $I^2_{\alpha'}$, it holds that

$$
\begin{aligned}
&-\sum_{|\alpha'|\geq 1,\alpha'\leq\alpha} C^{\alpha}_{\alpha'}\langle I^2_{\alpha'},\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\rangle\\
&=-\sum_{|\alpha'|\geq 1,\alpha'\leq\alpha} C^{\alpha}_{\alpha'}\langle \partial^{\alpha'}_x e^{\phi}\partial^{\beta}_{\xi}[\nu(\xi)\partial^{\alpha-\alpha'}_x u_2],\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\rangle\\
&\leq C\delta_{\phi}\sum_{|\alpha'|\geq 1,\alpha'\leq\alpha}\|\nu^{-1/2}\partial^{\beta}_{\xi}[\nu(\xi)\partial^{\alpha-\alpha'}_x u_2]\|\|\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|_{\nu}\\
&\leq C\delta_{\phi}\sum_{|\alpha|\leq N-k-1}\|\partial^{\alpha}_x u_2\|^2+C\delta_{\phi}\sum_{\substack{1\leq|\beta|\leq k\\|\alpha|+|\beta|\leq N}}\|\partial^{\alpha}_x\partial^{\beta}_{\xi}u_2\|^2_{\nu}.
\end{aligned}
$$

Similarly, for $I^1_{\beta'}$ and $I^2_{\beta'}$, one has

$$\begin{aligned}
&\sum_{|\beta'|\geq 1,\beta'\leq\beta} C^\beta_{\beta'}\langle I^1_{\beta'}, \partial^\alpha_x\partial^\beta_\xi u_2\rangle\\
&=\sum_{|\beta'|\geq 1,\beta'\leq\beta} C^\beta_{\beta'}\langle \partial^{\beta'}_\xi \xi\cdot\nabla_x\partial^\alpha_x\partial^{\beta-\beta'}_\xi u_2, \partial^\alpha_x\partial^\beta_\xi u_2\rangle\\
&\leq \epsilon\|\partial^\alpha_x\partial^\beta_\xi u_2\|^2 + C_\epsilon\sum_{|\alpha|\leq N-k}\|\partial^\alpha_x\nabla_x u_2\|^2 + C_\epsilon \sum_{\substack{1\leq|\beta|\leq k-1\\ |\alpha|+|\beta|\leq N}}\|\partial^\alpha_x\partial^\beta_\xi u_2\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{|\beta'|\geq 1,\beta'\leq\beta} C^\beta_{\beta'}\langle I^2_{\beta'}, \partial^\alpha_x\partial^\beta_\xi u_2\rangle\\
&=\sum_{|\beta'|\geq 1,\beta'\leq\beta} C^\beta_{\beta'}\langle e^\phi\partial^{\beta'}_\xi \nu(\xi)\partial^\alpha_x\partial^{\beta-\beta'}_\xi u_2, \partial^\alpha_x\partial^\beta_\xi u_2\rangle\\
&\leq \epsilon\|\partial^\alpha_x\partial^\beta_\xi u_2\|^2 + C_\epsilon\sum_{|\alpha|\leq N-k}\|\partial^\alpha_x u_2\|^2 + C_\epsilon \sum_{\substack{1\leq|\beta|\leq k-1\\ |\alpha|+|\beta|\leq N}}\|\partial^\alpha_x\partial^\beta_\xi u_2\|^2.
\end{aligned}$$

Therefore, the desired inequality (4.33) follows by putting all the above estimates into (4.36), taking summation over $\{|\beta|=k, |\alpha|+|\beta|\leq N\}$, and then choosing a properly small $0<\epsilon\leq 1$. This completes the proof of Lemma 4.6. ∎

5. Macroscopic dissipation. In this section, we shall devote ourselves to obtaining the macroscopic dissipation. Similarly to the case of the Boltzmann equation in [21, 12], the high-order derivatives of (a,b,c) are dissipative, and moreover, as observed in [27], the conservative quantity $a+3c$ is also dissipative. In fact, the dissipation of $a+3c$ can also be seen from studies of the Navier–Stokes–Poisson equations such as in [19] and a model for semiconductors in [16].

Let us fist introduce a definition about the interactive energy functional.

DEFINITION 5.1. *$\mathcal{E}_{\rm int}(\cdot)$ is said to be an interactive energy functional of $u(t,x,\xi)$ corresponding to some Sobolev space X with norm $\|\cdot\|_X$ if the following two conditions hold:*

(i) *$\mathcal{E}_{\rm int}(u(t))$ is the linear combination of some inner products over L^2_x or $L^2_{x,\xi}$ between the macroscopic component $\mathbf{P}u$ and the microscopic component $\{\mathbf{I}-\mathbf{P}\}u$ or between the coefficients (a,b,c) of the macroscopic component $\mathbf{P}u$.*

(ii) *There exists a positive constant C such that*

$$|\mathcal{E}_{\rm int}(u(t))|\leq C\|u(t)\|^2_X$$

holds true for any $t\geq 0$.

In what follows, we shall prove that there actually exists an interactive energy functional such that its dissipation rate contains L^2-norms of $a+3c$ and also all the high-order derivatives of (a,b,c). In fact, one has the following theorem.

THEOREM 5.2. *Suppose that δ_ϕ given in (4.1) is small enough. There is an interactive energy functional $\mathcal{E}_{\rm int}(\cdot)$ corresponding to $L^2_\xi(H^N_x)$ such that*

$$\frac{d}{dt}\mathcal{E}_{\rm int}(u(t))+\lambda\mathcal{D}_{\rm mac}(u(t))\leq C\sum_{|\alpha|\leq N}\|\partial^\alpha_x u_2\|^2 + C[[u(t)]]^2[[u(t)]]^2_\nu, \tag{5.1}$$

where $\mathcal{D}_{\rm mac}(u(t))$ *is the macroscopic dissipation rate given by*

$$\mathcal{D}_{\rm mac}(u(t))=\sum_{|\alpha|\leq N-1}(\|\partial_x^\alpha\nabla_x(a,b,c)\|^2+\|\partial_x^\alpha(a+3c)\|^2),$$

and $\mathcal{E}_{\rm int}(u(t))$ *is the linear combination of the following terms over* $|\alpha|\leq N-1$ *and* $1\leq j\leq 3$*:*

$$\mathcal{I}_\alpha^a(u(t))=\langle-\partial_x^\alpha\nabla_x\cdot b,\partial_x^\alpha(a+3c)\rangle\,, \tag{5.2}$$

$$\mathcal{I}_{\alpha,j}^b(u(t))=\left\langle\sum_{i\neq j}\partial_j\partial_x^\alpha\tilde{r}_i^{(2)}-\sum_{i\neq j}\partial_i\partial_x^\alpha\tilde{r}_{ij}^{(2)}-2\partial_j\partial_x^\alpha\tilde{r}_j^{(2)},\partial_x^\alpha b_j\right\rangle, \tag{5.3}$$

$$\mathcal{I}_{\alpha,j}^c(u(t))=\langle\partial_x^\alpha\tilde{r}_i^{(3)},\partial_i\partial_x^\alpha c\rangle. \tag{5.4}$$

Theorem 5.2 follows from the individual estimates on a,b, and c, which are based on the analysis of the macroscopic equations (3.6), (3.7), and (3.8) as well as (3.14), the macroscopic balance laws (3.9), (3.10), and (3.12), and the Poisson equation (3.13). It should be pointed out that, in contrast to the previous work in [12, 24, 5], we shall not use (3.4) and (3.5) in the whole proof, which actually have been replaced by (3.9) and (3.10).

First, let us give a lemma without proofs, which shows that among those terms on the right-hand side of the macroscopic equations (3.4)–(3.8), the coefficients of the separated part $\tilde{r}$, the linear part ℓ, and the nonlinear part n can be bounded by the microscopic dissipation rate. Roughly speaking, the idea of proofs is just based on the fact that the velocity-coordinate projector is bounded uniformly in t and x, and the velocity polynomials and velocity derivatives can be absorbed by the global Maxwellian $\mathbf{M}$ which exponentially decays in ξ.

LEMMA 5.3. *Suppose that* δ_ϕ *given in* (4.1) *is finite. Then for any* $|\alpha|\leq N$ *and* $1\leq i,j\leq N$*, it holds that*

$$\left\|\partial_x^\alpha\left(\tilde{r}^{(0)},\tilde{r}_i^{(1)},\tilde{r}_i^{(2)},\tilde{r}_{ij}^{(2)},\tilde{r}_i^{(3)}\right)\right\|\leq C\|\partial_x^\alpha u_2\|.$$

Moreover, for any $|\alpha|\leq N-1$ *and* $1\leq i,j\leq N$*, it holds that*

$$\left\|\partial_x^\alpha\left(\ell^{(0)},\ell_i^{(1)},\ell_i^{(2)},\ell_{ij}^{(2)},\ell_i^{(3)}\right)\right\|\leq C\sum_{|\beta|\leq|\alpha|+1}\|\partial_x^\beta u_2\|$$

and

$$\left\|\partial_x^\alpha\left(n^{(0)},n_i^{(1)},n_i^{(2)},n_{ij}^{(2)},n_i^{(3)}\right)\right\|\leq C[[u(t)]][[u(t)]]_\nu.$$

Next, the estimates about the dissipation of a,b, and c are given in the following three lemmas.

LEMMA 5.4. *Suppose that* δ_ϕ *given in* (4.1) *is small enough. Then there are constants* $\lambda>0$, C *such that for any* $0<\epsilon\leq 1$*, one has*

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\leq N-1}\sum_{i=1}^3\mathcal{I}_{\alpha,i}^c(u(t))+\lambda\sum_{|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_x c\|^2\\&\leq C\epsilon\sum_{|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_x b\|^2+\frac{C}{\epsilon}\left\{\sum_{|\alpha|\leq N}\|\partial_x^\alpha u_2\|^2+[[u(t)]]^2[[u(t)]]_\nu^2\right\}\end{aligned} \tag{5.5}$$

for any $t \geq 0$, where

$$\mathcal{I}^c_{\alpha,j}(u(t)) = \langle \partial^\alpha_x \tilde{r}^{(3)}_i, \partial_i \partial^\alpha_x c\rangle.$$

Proof. Let $|\alpha| \leq N-1$. It follows from (3.8) that

$$\begin{aligned}
\|\partial_i \partial^\alpha_x c\|^2 &= -\langle \partial_t \partial^\alpha_x \tilde{r}^{(3)}_i, \partial_i \partial^\alpha_x c\rangle + \langle \partial^\alpha_x [\ell^{(3)}_i + n^{(3)}_i], \partial_i \partial^\alpha_x c\rangle + \langle \partial^\alpha_x [c\partial_i \phi], \partial_i \partial^\alpha_x c\rangle \\
&= -\frac{d}{dt}\langle \partial^\alpha_x \tilde{r}^{(3)}_i, \partial_i \partial^\alpha_x c\rangle + \langle \partial^\alpha_x \tilde{r}^{(3)}_i, \partial_i \partial^\alpha_x \partial_t c\rangle \\
&\quad + \langle \partial^\alpha_x [\ell^{(3)}_i + n^{(3)}_i], \partial_i \partial^\alpha_x c\rangle + \langle \partial^\alpha_x [c\partial_i \phi], \partial_i \partial^\alpha_x c\rangle \\
&= -\frac{d}{dt}\mathcal{I}^c_{\alpha,i} + I_1 + I_2 + I_3. \qquad (5.6)
\end{aligned}$$

For I_1, it holds that

$$I_1 = -\langle \partial_i \partial^\alpha_x \tilde{r}^{(3)}_i, \partial^\alpha_x \partial_t c\rangle \leq \epsilon \|\partial^\alpha_x \partial_t c\|^2 + \frac{1}{4\epsilon}\|\partial_i \partial^\alpha_x \tilde{r}^{(3)}_i\|^2,$$

where one can further use the balance law (3.12) for c,

$$\partial_t c = -\frac{1}{3}\nabla_x b - \frac{1}{6}\nabla_x \cdot \langle |\xi|^2 \xi \sqrt{\mathbf{M}}, u_2\rangle + \frac{1}{6} b \cdot \nabla_x(\phi + \Psi),$$

to replace the time derivative $\partial_t c$ to obtain

$$\begin{aligned}
\|\partial^\alpha_x \partial_t c\|^2 &\leq C\|\partial^\alpha_x \nabla_x \cdot b\|^2 + \|\partial^\alpha_x \nabla_x \cdot \langle |\xi|^2 \xi \sqrt{\mathbf{M}}, u_2\rangle\|^2 + \|\partial^\alpha_x [b \cdot \nabla_x(\phi + \Psi)]\|^2 \\
&\leq C\|\partial^\alpha_x \nabla_x \cdot b\|^2 + C\|\partial^\alpha_x \nabla_x u_2\|^2 + C\delta_\phi \sum_{|\alpha| \leq N-1} \|\partial^\alpha_x \nabla_x b\|^2 \\
&\quad + C\left\{\sum_{|\alpha| \leq N-1} \|\partial^\alpha_x \nabla_x b\|^2\right\}\left\{\sum_{|\alpha| \leq N-1} \|\partial^\alpha_x \nabla_x \Psi\|^2\right\} \\
&\leq C \sum_{|\alpha| \leq N-1} (\|\partial^\alpha_x \nabla_x b\|^2 + \|\partial^\alpha_x \nabla_x u_2\|^2) + C[[u(t)]]^2[[u(t)]]^2_\nu.
\end{aligned}$$

For I_3, it holds that

$$\begin{aligned}
I_3 = \langle \partial^\alpha_x [c\partial_i \phi], \partial_i \partial^\alpha_x c\rangle &\leq \delta_\phi \|\partial_i \partial^\alpha_x c\|^2 + \frac{1}{4\delta_\phi}\|\partial^\alpha_x (c\partial_i \phi)\|^2 \\
&\leq \delta_\phi \|\partial_i \partial^\alpha_x c\|^2 + C\delta_\phi \sum_{|\alpha| \leq N-1} \|\partial^\alpha_x \nabla_x c\|^2 \\
&\leq C\delta_\phi \sum_{|\alpha| \leq N-1} \|\partial^\alpha_x \nabla_x c\|^2.
\end{aligned}$$

Putting all estimates into (5.6) and taking summation over $|\alpha| \leq N-1$, one has

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha| \leq N-1}\sum_{i=1}^3 \mathcal{I}^c_{\alpha,i}(u(t)) + \frac{1}{2}\sum_{|\alpha| \leq N-1} \|\partial^\alpha_x \nabla_x c\|^2 \\
&\leq C\epsilon \sum_{|\alpha| \leq N-1} (\|\partial^\alpha_x \nabla_x b\|^2 + \|\partial^\alpha_x \nabla_x u_2\|^2) + C\delta_\phi \sum_{|\alpha| \leq N-1} \|\partial^\alpha_x \nabla_x c\|^2 \\
&\quad + C\epsilon[[u(t)]]^2[[u(t)]]^2_\nu + \frac{C}{\epsilon}\sum_{|\alpha| \leq N-1}\sum_{i=1}^3 \|\partial^\alpha_x (\partial_i \tilde{r}^{(3)}_i, \ell^{(3)}_i, n^{(3)}_i)\|.
\end{aligned}$$

Further using Lemma 5.3 and the smallness of δ_ϕ, (5.5) follows. This completes the proof of Lemma 5.4. □

LEMMA 5.5. *Suppose that δ_ϕ given in* (4.1) *is small enough. Then there are constants $\lambda > 0$, C such that for any $0 < \epsilon \le 1$, one has*

$$\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\le N-1}\sum_{j=1}^{3}\mathcal{I}^b_{\alpha,j}(u(t)) + \lambda\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x b\|^2\\
&\le C\epsilon\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x c\|^2 + C\epsilon\sum_{|\alpha|\le N}\|\partial_x^\alpha(a+3c)\|^2\\
&\quad + \frac{C}{\epsilon}\left\{\sum_{|\alpha|\le N}\|\partial_x^\alpha u_2\|^2 + [[u(t)]]^2[[u(t)]]_\nu^2\right\}
\end{aligned} \tag{5.7}$$

for any $t \ge 0$, where

$$\mathcal{I}^b_{\alpha,j}(u(t)) = \left\langle \sum_{i\ne j}\partial_j\partial_x^\alpha\tilde{r}_i^{(2)} - \sum_{i\ne j}\partial_i\partial_x^\alpha\tilde{r}_{ij}^{(2)} - 2\partial_j\partial_x^\alpha\tilde{r}_j^{(2)}, \partial_x^\alpha b_j\right\rangle.$$

Proof. Let $|\alpha| \le N-1$. In terms of the elliptic-type equation (3.14) for each $j \in \{1,2,3\}$, the elementary energy estimate gives

$$\begin{aligned}
&\|\nabla_x\partial_x^\alpha b_j\|^2 + \|\partial_j\partial_x^\alpha b_j\|^2\\
&= -\left\langle \partial_t\left[\sum_{i\ne j}\partial_j\partial_x^\alpha\tilde{r}_i^{(2)} - \sum_{i\ne j}\partial_i\partial_x^\alpha\tilde{r}_{ij}^{(2)} - 2\partial_j\partial_x^\alpha\tilde{r}_j^{(2)}\right], \partial_x^\alpha b_j\right\rangle\\
&\quad + \left\langle \sum_{i\ne j}\partial_j\partial_x^\alpha\ell_i^{(2)} - \sum_{i\ne j}\partial_i\partial_x^\alpha\ell_{ij}^{(2)} - 2\partial_j\partial_x^\alpha\ell_j^{(2)}, \partial_x^\alpha b_j\right\rangle\\
&\quad + \left\langle \sum_{i\ne j}\partial_j\partial_x^\alpha n_i^{(2)} - \sum_{i\ne j}\partial_i\partial_x^\alpha n_{ij}^{(2)} - 2\partial_j\partial_x^\alpha n_j^{(2)}, \partial_x^\alpha b_j\right\rangle\\
&\quad + \left\langle \sum_{i\ne j}\partial_j\partial_x^\alpha(b_i\partial_i\phi) - \sum_{i\ne j}\partial_i\partial_x^\alpha(b_j\partial_i\phi + b_i\partial_j\phi) - \partial_j\partial_x^\alpha(b_j\partial_j\phi), \partial_x^\alpha b_j\right\rangle\\
&= \sum_{i=1}^{4} I_i.
\end{aligned} \tag{5.8}$$

We shall estimate each term I_i $(1 \le i \le 4)$ in (5.8). For I_1, in what follows, let us denote

$$\mathfrak{R}_j = \sum_{i\ne j}\partial_j\partial_x^\alpha\tilde{r}_i^{(2)} - \sum_{i\ne j}\partial_i\partial_x^\alpha\tilde{r}_{ij}^{(2)} - 2\partial_j\partial_x^\alpha\tilde{r}_j^{(2)}$$

for simplicity of later presentation; then it holds that

$$I_1 = -\frac{d}{dt}\langle\mathfrak{R}_j, \partial_x^\alpha b_j\rangle + \langle\mathfrak{R}_j, \partial_x^\alpha\partial_t b_j\rangle = -\frac{d}{dt}\langle\mathfrak{R}_j, \partial_x^\alpha b_j\rangle + I_{1,1} + I_{1,2}, \tag{5.9}$$

where we used the balance law (3.10)

$$\partial_t b_j = -\partial_j(a+5c) - \nabla_x\cdot\langle\xi\xi_j\sqrt{\mathbf{M}}, u_2\rangle + (a+3c)\partial_j(\phi+\Psi) + e^\phi\partial_j\Psi$$

to replace $\partial_t b_j$, and $I_{1,1}$ and $I_{1,2}$ are given by

$$I_{1,1} = \langle \mathfrak{R}_j, \partial_x^\alpha[-\partial_j(a+5c) - \nabla_x \cdot \langle \xi\xi_j\sqrt{\mathbf{M}}, u_2\rangle + (a+3c)\partial_j(\phi+\Psi)]\rangle,$$
$$I_{1,2} = \langle \mathfrak{R}_j, \partial_x^\alpha(e^\phi \partial_j\Psi)\rangle.$$

Here, $I_{1,1}$ is bounded by

$$I_{1,1} \le \frac{1}{4\epsilon}\|\mathfrak{R}_j\|^2 + \epsilon\|\partial_x^\alpha[-\partial_j(a+5c) - \nabla_x \cdot \langle \xi\xi_j\sqrt{\mathbf{M}}, u_2\rangle + (a+3c)\partial_j(\phi+\Psi)]\|^2, \tag{5.10}$$

where further one has

$$\begin{aligned}
&\left\|\partial_x^\alpha[-\partial_j(a+5c) - \nabla_x \cdot \langle \xi\xi_j\sqrt{\mathbf{M}}, u_2\rangle + (a+3c)\partial_j(\phi+\Psi)]\right\|^2 \\
&\le C\|\partial_j\partial_x^\alpha(a,c)\|^2 + C\|\partial_x^\alpha\nabla_x u_2\|^2 + C\|\partial_x^\alpha[(a+3c)\partial_j\phi]\|^2 + C\|\partial_x^\alpha[(a+3c)\partial_j\Psi]\|^2 \\
&\le C\|\partial_j\partial_x^\alpha(a,c)\|^2 + C\|\partial_x^\alpha\nabla_x u_2\|^2 \\
&\quad + C\sum_{|\beta|\le|\alpha|}\left(\|\partial_x^\beta\partial_j\phi\|_{L^\infty}^2 + \|\partial_x^\beta\partial_j\Psi\|_{L^\infty}^2\right)\sum_{|\beta|\le|\alpha|}\|\partial_x^\beta(a+3c)\|^3 \\
&\le C\|\partial_j\partial_x^\alpha(a,c)\|^2 + C\|\partial_x^\alpha\nabla_x u_2\|^2 \\
&\quad + C\left(\delta_\phi^2 + \sum_{2\le|\alpha|\le N}\|\partial_x^\alpha\Psi\|^2\right)\sum_{|\alpha|\le N-1}\|\partial_x^\alpha(a+3c)\|^2.
\end{aligned} \tag{5.11}$$

For $I_{1,2}$, it follows from integration by parts that

$$\begin{aligned}
I_{1,2} &= -\sum_{i\ne j}\langle\partial_x^\alpha\tilde{r}_i^{(2)}, \partial_j\partial_x^\alpha[e^\phi\partial_j\Psi]\rangle + \sum_{i\ne j}\langle\partial_x^\alpha\tilde{r}_{ij}^{(2)}, \partial_i\partial_x^\alpha[e^\phi\partial_j\Psi]\rangle \\
&\quad + 2\langle\partial_x^\alpha\tilde{r}_j^{(2)}, \partial_j\partial_x^\alpha[e^\phi\partial_j\Psi]\rangle \\
&\le \epsilon\sum_{ij}\|\partial_i\partial_x^\alpha[e^\phi\partial_j\Psi]\|^2 + \frac{1}{4\epsilon}\sum_{ij}\|\partial_x^\alpha(\tilde{r}_i^{(2)}, \tilde{r}_{ij}^{(2)})\|^2.
\end{aligned} \tag{5.12}$$

Notice that

$$\begin{aligned}
\partial_i\partial_x^\alpha[e^\phi\partial_j\Psi] &= \partial_x^\alpha\partial_i\partial_j\Psi + \partial_i\partial_x^\alpha[(e^\phi-1)\partial_j\Psi] \\
&= \partial_x^\alpha\partial_i\partial_j\Psi + \sum_{\beta\le\alpha'}C_\beta^{\alpha'}\partial_x^{\alpha'-\beta}(e^\phi-1)\partial_x^\beta\partial_j\Psi,
\end{aligned}$$

where $|\alpha'| = |\alpha|+1$ with $\alpha_i' = \alpha_i+1$, and hence one has

$$\begin{aligned}
\|\partial_i\partial_x^\alpha[e^\phi\partial_j\Psi]\|^2 &\le 2\|\partial_x^\alpha\partial_i\partial_j\Psi\|^2 + C\sum_{|\beta|\le|\alpha|+1}\|\partial_x^{\alpha'-\beta}(e^\phi-1)\partial_x^\beta\partial_j\Psi\|^2 \\
&\le 2\|\partial_x^\alpha\partial_i\partial_j\Psi\|^2 + C\||x|\partial_x^{\alpha'}(e^\phi-1)\|_{L^\infty}^2\left\|\frac{\partial_j\Psi}{|x|}\right\|^2 \\
&\quad + C\sum_{1\le|\beta|\le|\alpha|+1}\|\partial_x^{\alpha'-\beta}(e^\phi-1)\|_{L^\infty}^2\|\partial_x^\beta\partial_j\Psi\|^2 \\
&\le 2\|\partial_x^\alpha\partial_i\partial_j\Psi\|^2 + C\delta_\phi^2\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x^2\Psi\|^2.
\end{aligned} \tag{5.13}$$

Therefore it follows from (5.12) and (5.13) that

$$I_{1,2} \le C\epsilon(1+\delta_\phi^2)\sum_{|\alpha|\le N-1}\|\partial_x^\alpha\nabla_x^2\Psi\|^2 + \frac{1}{4\epsilon}\sum_{ij}\|\partial_x^\alpha(\tilde r_i^{(2)},\tilde r_{ij}^{(2)})\|^2$$

$$\le C\epsilon(1+\delta_\phi^2)\sum_{|\alpha|\le N-1}\|\partial_x^\alpha(a+3c)\|^2 + \frac{1}{4\epsilon}\sum_{ij}\|\partial_x^\alpha(\tilde r_i^{(2)},\tilde r_{ij}^{(2)})\|^2. \tag{5.14}$$

Thus from (5.9) as well as (5.10), (5.11), and (5.14), I_1 is bounded as

$$\begin{aligned} I_1 \le& -\frac{d}{dt}\mathcal{I}_{\alpha,j}^b(u(t)) + C\epsilon\left(\|\partial_x^\alpha\nabla_x(a,c)\|^2 + \|\partial_x^\alpha\nabla_x u_2\|^2\right)\\ &+ C\epsilon\left(1+\delta_\phi^2+\sum_{2\le|\alpha|\le N}\|\partial_x^\alpha\Psi\|^2\right)\sum_{|\alpha|\le N-1}\|\partial_x^\alpha(a+3c)\|^2\\ &+\frac{C}{\epsilon}\sum_{ij}\left(\left\|\nabla_x\partial_x^\alpha(\tilde r_i^{(2)},\tilde r_{ij}^{(2)})\right\| + \left\|\partial_x^\alpha(\tilde r_i^{(2)},\tilde r_{ij}^{(2)})\right\|\right). \end{aligned}\tag{5.15}$$

For I_2 and I_3, it holds that

$$I_2 + I_3 \le \frac{1}{2}\|\nabla_x\partial_x^\alpha b_j\|^2 + C\sum_{ij}\left\|\partial_x^\alpha(\ell_i^{(2)},\ell_{ij}^{(2)},n_i^{(2)},n_{ij}^{(2)})\right\|^2. \tag{5.16}$$

Finally for I_4, it holds that

$$\begin{aligned} I_4 \le& \,\delta_\phi\|\nabla_x\partial_x^\alpha b_j\|^2\\ &+\frac{C}{\delta_\phi}\left\|\sum_{i\ne j}\partial_j\partial_x^\alpha(b_i\partial_i\phi) - \sum_{i\ne j}\partial_i\partial_x^\alpha(b_j\partial_i\phi + b_i\partial_j\phi) - \partial_j\partial_x^\alpha(b_j\partial_j\phi)\right\|^2\\ \le& \,C\delta_\phi\sum_{j=1}^3\sum_{|\beta|\le|\alpha|}\|\nabla_x\partial_x^\beta b_j\|^2. \end{aligned}\tag{5.17}$$

By putting estimates (5.15), (5.16), and (5.17) into (5.8) and then taking summation over $|\alpha|\le N-1$, one has

$$\begin{aligned} &\frac{d}{dt}\sum_{|\alpha|\le N-1}\sum_{j=1}^3\mathcal{I}_{\alpha,j}^b(u(t)) + \frac{1}{2}\sum_{|\alpha|\le N-1}\|\nabla_x\partial_x^\alpha b\|^2\\ &\le C\delta_\phi\sum_{|\alpha|\le N-1}\|\nabla_x\partial_x^\alpha b\|^2 + C\epsilon\sum_{|\alpha|\le N-1}\left(\|\partial_x^\alpha\nabla_x(a,c)\|^2 + \|\partial_x^\alpha\nabla_x u_2\|^2\right)\\ &\quad + C\epsilon\left(1+\delta_\phi^2+\sum_{2\le|\alpha|\le N}\|\partial_x^\alpha\Psi\|^2\right)\sum_{|\alpha|\le N-1}\|\partial_x^\alpha(a+3c)\|^2\\ &\quad + \frac{C}{\epsilon}\sum_{|\alpha|\le N-1}\sum_{ij}\left\|\nabla_x\partial_x^\alpha(\tilde r_i^{(2)},\tilde r_{ij}^{(2)})\right\|^2\\ &\quad + \frac{C}{\epsilon}\sum_{|\alpha|\le N-1}\sum_{ij}\left\|\partial_x^\alpha(\tilde r_i^{(2)},\tilde r_{ij}^{(2)},\ell_i^{(2)},\ell_{ij}^{(2)},n_i^{(2)},n_{ij}^{(2)})\right\|^2, \end{aligned}$$

which gives the desired estimate (5.7) by using Lemma 5.3 and the smallness of δ_ϕ. This completes the proof of Lemma 5.5. □

LEMMA 5.6. *Suppose that $\delta_\phi > 0$ given in* (4.1) *is small enough. Then there are constants $\lambda > 0$, C such that*

$$\frac{d}{dt}\sum_{|\alpha|\le N-1}\mathcal{I}^a_\alpha(u(t)) + \lambda\sum_{|\alpha|\le N}\|\partial^\alpha_x(a+3c)\|^2$$
$$\le C\sum_{|\alpha|\le N-1}\|\partial^\alpha_x\nabla_x(b,c)\|^2 + C\left\{\sum_{|\alpha|\le N-1}\|\partial^\alpha_x\nabla_x u_2\|^2 + [[u(t)]]^2[[u(t)]]^2_\nu\right\} \tag{5.18}$$

holds for any $t \ge 0$, where

$$\mathcal{I}^a_\alpha(u(t)) = \langle -\partial^\alpha_x\nabla_x\cdot b, \partial^\alpha_x(a+3c)\rangle\,.$$

Proof. Recall the balance law (3.10) for b,

$$\begin{aligned}&\partial_t b + \nabla_x(a+3c) + 2\nabla_x c + \nabla_x\cdot\langle\xi\otimes\xi\sqrt{\mathbf{M}}, u_2\rangle\\ &= (a+3c)\nabla_x\phi + (a+3c)\nabla_x\Psi + \nabla_x\Psi + (e^\phi-1)\nabla_x\Psi,\end{aligned}$$

and take the divergence $\nabla_x\cdot$ to get

$$\begin{aligned}&\partial_t(\nabla_x\cdot b) + \Delta_x(a+3c) + 2\Delta_x c + \nabla_x\cdot\nabla_x\cdot\langle\xi\otimes\xi\sqrt{\mathbf{M}}, u_2\rangle\\ &= \nabla_x\cdot[(a+3c)\nabla_x\phi] + \nabla_x\cdot[(a+3c)\nabla_x\Psi] + (a+3c) + \nabla_x\cdot[(e^\phi-1)\nabla_x\Psi],\end{aligned} \tag{5.19}$$

where we used the Poisson equation (3.13). Fix α with $|\alpha| \le N-1$, and by the standard energy estimate, it follows from (5.19) that

$$\begin{aligned}&\frac{d}{dt}\langle -\partial^\alpha_x\nabla_x\cdot b, \partial^\alpha_x(a+3c)\rangle + \|\partial^\alpha_x(a+3c)\|^2 + \|\nabla_x\partial^\alpha_x(a+3c)\|^2\\ &= \|\partial^\alpha_x\nabla_x\cdot b\|^2 + 2\langle\Delta_x\partial^\alpha_x c, \partial^\alpha_x(a+3c)\rangle\\ &\quad + \langle\nabla_x\cdot\nabla_x\cdot\langle\xi\otimes\xi\sqrt{\mathbf{M}}, \partial^\alpha_x u_2\rangle, \partial^\alpha_x(a+3c)\rangle\\ &\quad - \langle\nabla_x\cdot\partial^\alpha_x[(a+3c)\nabla_x\phi], \partial^\alpha_x(a+3c)\rangle\\ &\quad - \langle\nabla_x\cdot\partial^\alpha_x[(a+3c)\nabla_x\Psi], \partial^\alpha_x(a+3c)\rangle\\ &\quad - \langle\nabla_x\cdot\partial^\alpha_x[(e^\phi-1)\nabla_x\Psi], \partial^\alpha_x(a+3c)\rangle,\end{aligned} \tag{5.20}$$

where we used the conservation law (3.9) of mass to obtain

$$\begin{aligned}&\langle\partial_t\partial^\alpha_x\nabla_x\cdot b, \partial^\alpha_x(a+3c)\rangle\\ &= \frac{d}{dt}\langle\partial^\alpha_x\nabla_x\cdot b, \partial^\alpha_x(a+3c)\rangle - \langle\partial^\alpha_x\nabla_x\cdot b, \partial^\alpha_x\partial_t(a+3c)\rangle\\ &= \frac{d}{dt}\langle\partial^\alpha_x\nabla_x\cdot b, \partial^\alpha_x(a+3c)\rangle + \|\partial^\alpha_x\nabla_x\cdot b\|^2.\end{aligned}$$

Thus by applying integration by parts and the Cauchy–Schwarz inequality to (5.20), one has

$$\begin{aligned}&\frac{d}{dt}\mathcal{I}^a_\alpha(u(t)) + \|\partial^\alpha_x(a+3c)\|^2 + \frac{1}{2}\|\nabla_x\partial^\alpha_x(a+3c)\|^2\\ &\le \|\partial^\alpha_x\nabla_x\cdot b\|^2 + C\|\nabla_x\partial^\alpha_x c\|^2 + C\|\nabla_x\partial^\alpha_x u_2\|^2 + C\|\partial^\alpha_x[(a+3c)\nabla_x\phi]\|^2\\ &\quad + C\|\partial^\alpha_x[(a+3c)\nabla_x\Psi]\|^2 + C\|\partial^\alpha_x[(e^\phi-1)\nabla_x\Psi]\|^2.\end{aligned} \tag{5.21}$$

Furthermore, one can estimate the L^2-norms for those product terms in (5.21) as follows. First, it holds that

$$
\begin{aligned}
\|\partial_x^\alpha[(a+3c)\nabla_x\phi]\|^2 &\le C\sum_{|\beta|\le|\alpha|}\|\partial_x^\beta\nabla_x\phi\|_{L^\infty}^2\sum_{|\beta|\le|\alpha|}\|\partial_x^\beta(a+3c)\|^2\\
&\le C\delta_\phi^2\sum_{|\beta|\le|\alpha|}\|\partial_x^\beta(a+3c)\|^2
\end{aligned}
\tag{5.22}
$$

and

$$
\begin{aligned}
&\|\partial_x^\alpha[(a+3c)\nabla_x\Psi]\|^2\\
&\le C\|(a+3c)\partial_x^\alpha\nabla_x\Psi\|^2+C\sum_{|\beta|<|\alpha|}\|\partial_x^{\alpha-\beta}(a+3c)\partial_x^\beta\nabla_x\Psi\|^2\\
&\le C\|a+3c\|_{L^\infty}^2\|\partial_x^\alpha\nabla_x\Psi\|^2\\
&\quad+C\sum_{|\beta|<|\alpha|}\|\partial_x^\beta\nabla_x\Psi\|_{L^\infty}^2\sum_{|\beta|<|\alpha|}\|\partial_x^{\alpha-\beta}(a+3c)\|^2\\
&\le C\|\nabla_x(a+3c)\|_{H_x^1}^2\|\partial_x^\alpha\nabla_x\Psi\|^2\\
&\quad+C\sum_{1\le|\beta|\le N}\|\partial_x^\beta\nabla_x\Psi\|^2\sum_{1\le|\beta|\le N-1}\|\partial_x^\beta(a+3c)\|^2,
\end{aligned}
$$

which by $N\ge 4$ implies

$$
\begin{aligned}
\|\partial_x^\alpha[(a+3c)\nabla_x\Psi]\|^2 &\le C\sum_{1\le|\beta|\le N}\|\partial_x^\beta\nabla_x\Psi\|^2\sum_{1\le|\beta|\le N-1}\|\partial_x^\beta(a+3c)\|^2\\
&\le C[[u(t)]]^2[[u(t)]]_\nu^2.
\end{aligned}
\tag{5.23}
$$

Similarly as before, it holds that

$$
\|\partial_x^\alpha[(e^\phi-1)\nabla_x\Psi]\|^2\le C\delta_\phi^2\sum_{1\le|\beta|\le N-1}\|\partial_x^\beta\nabla_x\Psi\|^2\le C\delta_\phi^2\sum_{|\beta|\le N-2}\|\partial_x^\beta(a+3c)\|^2.
\tag{5.24}
$$

Thus, by putting (5.22), (5.23), and (5.24) into (5.21) and then taking summation over $|\alpha|\le N-1$, one has

$$
\begin{aligned}
&\frac{d}{dt}\sum_{|\alpha|\le N-1}\mathcal{I}_\alpha^a(u(t))+\sum_{|\alpha|\le N-1}\|\partial_x^\alpha(a+3c)\|^2+\frac{1}{2}\sum_{|\alpha|\le N-1}\|\nabla_x\partial_x^\alpha(a+3c)\|^2\\
&\le C\sum_{|\alpha|\le N-1}(\|\partial_x^\alpha\nabla_x(b,c)\|^2+\|\partial_x^\alpha\nabla_x u_2\|^2)+C[[u(t)]]^2[[u(t)]]_\nu^2\\
&\quad+C\delta_\phi^2\sum_{|\alpha|\le N-1}\|\partial_x^\alpha(a+3c)\|^2,
\end{aligned}
$$

which yields the desired estimate (5.18) by using the smallness of δ_ϕ. This completes the proof of Lemma 5.6. □

Proof of Theorem 5.2. Let us multiply (5.5) and (5.7) by a constant $M>0$ and then take summation of both of them as well as (5.18). One can first choose $M>0$ sufficiently large such that the first term on the right-hand side of (5.18) is absorbed by the dissipation of b and c. By fixing $M>0$, one can further choose $0<\epsilon\le 1$

sufficiently small such that the first term on the right-hand side of (5.5) and the first two terms on the right-hand side of (5.7) are absorbed by the full dissipation of b, c, and $a+3c$. Therefore, one has

$$\begin{aligned}\frac{d}{dt}&\left[M\sum_{|\alpha|\le N-1}\sum_{i=1}^{3}\left(\mathcal{I}^b_{\alpha,i}(u(t))+\mathcal{I}^c_{\alpha,i}(u(t))\right)+\sum_{|\alpha|\le N-1}\mathcal{I}^a_{\alpha}(u(t))\right]\\ &+\lambda\sum_{|\alpha|\le N-1}\left(\|\partial^\alpha_x\nabla_x(a+3c,b,c)\|^2+\|\partial^\alpha_x(a+3c)\|^2\right)\\ &\qquad\le C\sum_{|\alpha|\le N}\|\partial^\alpha_x u_2\|^2+C[[u(t)]]^2[[u(t)]]^2_\nu.\end{aligned}$$

The further linear combination of the above inequality and (5.5), by taking $0<\epsilon\le 1$ small enough, leads to the desired estimate (5.1). The rest is to verify that $\mathcal{E}_{\rm int}(\cdot)$ is indeed an interactive energy functional corresponding to $L^2_\xi(H^N_x)$. Actually, by the definitions (5.2), (5.3), and (5.4), one has

$$\begin{aligned}\mathcal{E}_{\rm int}(u(t))&\le C\sum_{|\alpha|\le N-1}\sum_{j=1}^{3}\left(|\mathcal{I}^a_{\alpha}(u(t))|^2+|\mathcal{I}^b_{\alpha,j}(u(t))|^2+|\mathcal{I}^c_{\alpha,j}(u(t))|^2\right)\\ &\le C\sum_{|\alpha|\le N-1}(\|\partial^\alpha_x\nabla_x\cdot b\|^2+\|\partial^\alpha_x(a+3c)\|^2\\ &\qquad\qquad+\|\partial^\alpha_x\nabla_x u_2\|^2+\|\partial^\alpha_x b\|^2+\|\partial^\alpha_x u_2\|^2+\|\partial^\alpha_x\nabla_x c\|^2)\\ &\le C\sum_{|\alpha|\le N}(\|\partial^\alpha_x(a,b,c)\|^2+\|\partial^\alpha_x u_2\|^2),\end{aligned}$$

which implies that

$$\begin{aligned}\mathcal{E}_{\rm int}(u(t))&\le C\sum_{|\alpha|\le N}(\|\mathbf{P}\partial^\alpha_x u(t)\|^2+\|\{\mathbf{I}-\mathbf{P}\}\partial^\alpha_x u(t)\|^2)\\ &\le C\sum_{|\alpha|\le N}\|\partial^\alpha_x u(t)\|^2=C\|u(t)\|^2_{L^2_\xi(H^N_x)}.\end{aligned}$$

This completes the proof of Theorem 5.2.

6. Stability of stationary solution. In this section, we prove that the stationary solution $(f_*,\Phi_*)=(e^{\phi}\mathbf{M},\phi)$ obtained in Theorem 1.1 for the VPB system (1.1)–(1.2) is stable under the small initial perturbation in the sense of Theorem 1.2. Theorem 1.2 about the global existence of solutions to the Cauchy problem (1.7)–(1.9) will be proved by the following local existence together with uniform a priori estimates as well as the standard continuum argument.

PROPOSITION 6.1 (local existence). *There exist constants $\delta_1>0$ and $T^*>0$ such that if $[[u_0]]^2\le\delta_1$ and $\|\bar\rho-1\|_{W^{N+1,\infty}_k}\le\delta_1$, then there is a unique solution $u(t,x,\xi)$ in $[0,T^*]\times\mathbb{R}^3\times\mathbb{R}^3$ to the Cauchy problem* (1.7)–(1.9) *of the VPB system such that*

$$\frac{1}{2}[[u(t)]]^2+\sum_{|\alpha|+|\beta|\le N}\int_0^t\|\partial^\alpha_x\partial^\beta_\xi u(s)\|^2_\nu ds\le\delta_1$$

for any $t \in [0, T^*]$. *Moreover,* $[[u(t)]]^2 : [0, T^*] \to \mathbb{R}$ *is continuous. If* $f_0(x,\xi) \equiv e^{\phi}\mathbf{M} + \sqrt{\mathbf{M}}u_0(x,\xi) \geq 0$*, then* $f(t,x,\xi) \equiv e^{\phi}\mathbf{M} + \sqrt{\mathbf{M}}u(t,x,\xi) \geq 0$.

The proof of Proposition 6.1 can be given the same way as in [12] without considering estimates on the time derivatives, and thus is omitted for simplicity. Next, we devote ourselves to obtaining the uniform a priori estimates on the basis of the estimates on the microscopic and macroscopic dissipations given in sections 4 and 5. For this purpose, let us suppose that the Cauchy problem (1.7)–(1.9) admits a solution $u(t,x,\xi)$ in $[0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$ for some $T > 0$, such that

$$\sup_{0 \leq t \leq T} [[u(t)]]^2 \leq \delta$$

for $\delta > 0$ small enough. Recall that δ_ϕ given in (4.1) is also small enough by (1.5) and the smallness of $\|\bar{\rho} - 1\|_{W_2^{N+1,\infty}}$.

The aim is to obtain the full dissipation

$$[[u(t)]]_\nu^2 = \mathcal{D}_{\rm mic}(u(t)) + \mathcal{D}_{\rm mac}(u(t)),$$

with

$$\mathcal{D}_{\rm mic}(u(t)) = \sum_{|\alpha|+|\beta| \leq N} \|\partial_x^\alpha \partial_\xi^\beta \{\mathbf{I} - \mathbf{P}\}u\|_\nu^2,$$

$$\mathcal{D}_{\rm mac}(u(t)) = \sum_{|\alpha| \leq N-1} \left(\|\partial_x^\alpha \nabla_x (a,b,c)\|^2 + \|\partial_x^\alpha (a + 3c)\|^2 \right).$$

Step 1. The microscopic dissipation rate $\mathcal{D}_{\rm mic}(u(t))$ can be obtained by taking the proper linear combinations of (4.2), (4.19), and (4.33). In fact, the sum of (4.2) and (4.19) gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\mathcal{E}_x(u(t)) + \lambda \sum_{|\alpha| \leq N} \|\partial_x^\alpha u_2\|^2 \\ &\leq C(\delta_\phi + \sqrt{\delta})(1 + \sqrt{\delta})[[u(t)]]_\nu^2 \\ &\quad + C\delta_\phi \sum_{|\alpha| \leq N-1} \|\partial_x^\alpha \nabla_x (a,b,c)\|^2 + C\delta_\phi \sum_{|\alpha| \leq N-2} \|\partial_x^\alpha (a + 3c)\|^2 \\ &\quad + C\delta_\phi \sum_{1 \leq |\alpha| \leq N-1} \|\partial_x^\alpha \nabla_\xi u_2\|^2, \end{aligned} \tag{6.1}$$

where

$$\mathcal{E}_x(u(t)) = \sum_{|\alpha| \leq N} \left(\|\partial_x^\alpha u\|^2 + \|\partial_x^\alpha \nabla_x \Psi\|^2 \right) - 2\int_{\mathbb{R}^3} e^{-\phi}|b|^2 c dx.$$

Notice that

$$2\left| \int_{\mathbb{R}^3} e^{-\phi}|b|^2 c\, dx \right| \leq C\|c\|_{L_x^\infty} \|b\|_{L_x^2}^2 \leq C\|\nabla_x c\|_{H_x^1} \|\mathbf{P}u\|_{L_{x,\xi}^2}^2 \leq C\sqrt{\delta}\|u\|_{L_{x,\xi}^2}^2,$$

and since $\delta > 0$ is small enough, it holds that

$$\mathcal{E}_x(u(t)) \sim \sum_{|\alpha| \leq N} \left(\|\partial_x^\alpha u\|^2 + \|\partial_x^\alpha \nabla_x \Psi\|^2 \right).$$

The linear combination of (4.33) with k ranging from 1 to N gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\mathcal{E}_{x,\xi}(u(t))+\lambda\sum_{\substack{|\beta|\geq 1\\|\alpha|+|\beta|\leq N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2\\&\leq C(\delta_\phi+\sqrt{\delta}+\delta)[[u(t)]]_\nu^2\\&\quad+C\sum_{|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_x(a,b,c)\|^2+C\sum_{|\alpha|\leq N}\|\partial_x^\alpha u_2\|_\nu^2,\end{aligned}\tag{6.2}$$

where

$$\mathcal{E}_{x,\xi}(u(t))=\sum_{k=1}^N C_{N,k}\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\leq N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|^2$$

for some proper positive constants $C_{N,k}$. Therefore, the linear combination of (6.1) and (6.2) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\mathcal{E}_{M_1}(u(t))+\lambda\mathcal{D}_{\mathrm{mic}}(u(t))\\&\leq C(\delta_\phi+\sqrt{\delta})(1+\sqrt{\delta})[[u(t)]]_\nu^2\\&\quad+C\sum_{|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_x(a,b,c)\|^2+C\delta_\phi\sum_{|\alpha|\leq N-2}\|\partial_x^\alpha(a+3c)\|^2,\end{aligned}\tag{6.3}$$

where

$$\mathcal{E}_{M_1}(u(t))=M_1\mathcal{E}_x(u(t))+\mathcal{E}_{x,\xi}(u(t))$$

with the constant $M_1>0$ large enough. It can be seen that

$$\mathcal{E}_{M_1}(u(t))\sim[[u(t)]]^2.$$

Step 2. The linear combination of (6.1) and (5.1) gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[M_2\mathcal{E}_x(u(t))+2\mathcal{E}_{\mathrm{int}}(u(t))\right]+\lambda\sum_{|\alpha|\leq N}\|\partial_x^\alpha u_2\|^2+\lambda\mathcal{D}_{\mathrm{mac}}(u(t))\\&\leq C(\delta_\phi+\sqrt{\delta})(1+\sqrt{\delta})[[u(t)]]_\nu^2+C\delta_\phi\sum_{1\leq|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_\xi u_2\|^2\end{aligned}\tag{6.4}$$

for the constant $M_2>0$ large enough. By Theorem 5.2 and also condition (ii) in Definition 5.1, one has

$$M_2\mathcal{E}_x(u(t))+2\mathcal{E}_{\mathrm{int}}(u(t))\sim\mathcal{E}_x(u(t)).$$

Step 3. Finally, the linear combination of (6.3) and (6.4) gives

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_{M_1,M_2,M_3}(u(t))+\lambda[\mathcal{D}_{\mathrm{mic}}(u(t))+\mathcal{D}_{\mathrm{mac}}(u(t))]\leq C(\delta_\phi+\sqrt{\delta})(1+\sqrt{\delta})[[u(t)]]_\nu^2,$$

where

$$\mathcal{E}_{M_1,M_2,M_3}(u(t))=M_3\left[M_2\mathcal{E}_x(u(t))+2\mathcal{E}_{\mathrm{int}}(u(t))\right]+\mathcal{E}_{M_1}(u(t)),$$

with the constant $M_3 > 0$ large enough. Notice that

$$\mathcal{E}_{M_1,M_2,M_3}(u(t)) \sim M_3\mathcal{E}_x(u(t)) + [[u(t)]]^2 \sim [[u(t)]]^2,$$
$$\mathcal{D}_{\rm mic}(u(t)) + \mathcal{D}_{\rm mac}(u(t)) \sim [[u(t)]]_\nu^2.$$

Since δ and δ_ϕ are small enough, one has

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}_{M_1,M_2,M_3}(u(t)) + \lambda[[u(t)]]_\nu^2 \leq 0.$$

By further taking the time integration, it follows that there is $\delta_2 > 0, \lambda_1 > 0$, and $C_1 > 0$ such that as long as

$$\sup_{0\leq t\leq T}[[u(t)]]^2 \leq \delta_2 \quad \text{and} \quad \|\bar{\rho}-1\|_{W_2^{N+1,\infty}} \leq \delta_2$$

hold with $T > 0$, one has

$$[[u(t)]]^2 + \lambda_1\int_0^t [[u(s)]]_\nu^2 ds \leq C_1[[u_0]]^2 \tag{6.5}$$

for any $0 \leq t \leq T$.

Proof of Theorem 1.2. Define

$$M = \min\{\delta_1, \delta_2\} > 0$$

and choose initial data u_0 and the background density $\bar{\rho}$ such that $f_0 \equiv e^\phi\mathbf{M} + \sqrt{\mathbf{M}}u_0 \geq 0$ and

$$[[u_0]]^2 \leq \frac{M}{2(C_1+1)}, \quad \|\bar{\rho}-1\|_{W_2^{N+1,\infty}} \leq M.$$

Define

$$T_\infty = \sup_t\left\{t \geq 0 \,\middle|\, \sup_{0\leq s\leq t}[[u(s)]]^2 \leq M\right\}.$$

Since

$$[[u_0]]^2 \leq \frac{M}{2} < M \leq \delta_1 \quad \text{and} \quad \|\bar{\rho}-1\|_{W_2^{N+1,\infty}} \leq M \leq \delta_1,$$

it follows from Proposition 6.1 that $T_\infty > 0$ holds true. If T_∞ is finite, the definition of T_∞ implies

$$\sup_{0\leq t\leq T}[[u(t)]]^2 = M \leq \delta_2. \tag{6.6}$$

By further using

$$\|\bar{\rho}-1\|_{W_2^{N+1,\infty}} \leq M \leq \delta_2,$$

it follows from (6.5) that

$$\sup_{0\leq t\leq T}[[u(t)]]^2 \leq C_1[[u_0]]^2 \leq \frac{C_1}{2(C_1+1)}M \leq \frac{M}{2} < M,$$

which is in contradiction to (6.6). Then $T_\infty = \infty$ holds. Furthermore, $u \in X(0,\infty)$ and (1.12) follow by Proposition 6.1 and (6.5). Hence, Theorem 1.2 is proved. ∎

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