L^1 and BV-type Stability of the Boltzmann Equation with External Forces

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Abstract

Based on the existence theory on the Boltzmann equation with external forces in infinite vacuum, in this paper, we will study the L^1 and BV-type stability of the classical solutions for small initial data. The stability results generalize those for the Boltzmann equation without force to the case with external force. In particular, we show that the stability can be established for the soft potentials directly, while the stability for the hard potentials and hard sphere model is obtained through the contruction of some nonlinear functionals. The functionals thus contructed generalize those constructed in [19] for the case without force to capture the effect of the force term on the time evolution of the solutions.

1 Introduction

For the rarefied gas in the whole space $\mathbf{R}_x^3$, let $f(t,x,v)$ be the distribution function for particles at time $t \geq 0$ with location $x=(x_1,x_2,x_3) \in \mathbf{R}_x^3$ and velocity $v=(v_1,v_2,v_3) \in \mathbf{R}_v^3$. In the presence of the external force, the time evolution of f is governed by the Boltzmann equation as a fundamental equation in statistical physics,

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = J(f,f). \tag{1.1}$$

In the following discussion, we denote the initial data by

$$f(0,x,v) = f_0(x,v). \tag{1.2}$$

Here $E = E(t,x,v)$ is the external force. And the collision operator $J(f,f)$ describing the binary elastic collision takes the form:

$$J(f,f) = Q(f,f) - fR(f), \tag{1.3}$$

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with

$$Q(f,f)(t,x,v)=\int_{[0,2\pi]\times[0,\pi/2]\times\mathbf{R}^3_{v_*}} B(\theta,|v-v_*|)f(t,x,v')f(t,x,v'_*)\,d\varepsilon d\theta dv_*, \tag{1.4}$$

and

$$fR(f)(t,x,v)=f(t,x,v)\int_{[0,2\pi]\times[0,\pi/2]\times\mathbf{R}^3_{v_*}} B(\theta,|v-v_*|)f(t,x,v_*)\,d\varepsilon d\theta dv_*. \tag{1.5}$$

Here (v,v_*) and (v',v'_*) are the pre-collision and the post-collision velocities respectively, satisfying

$$v'=v-\langle v-v_*,\omega\rangle\omega,\ \ v'_*=v_*+\langle v-v_*,\omega\rangle\omega,\quad \omega\in S^2_+,$$

by the conservation of momentum and energy. And ε and θ are the polar and azimuthal angles of v'. $B(\theta,|v-v_*|)$ is the collision kernel characterizing the collision of the gas particles coming from different physical settings with various interaction potentials.

Throughout this paper, we assume that the collision kernel B is nonnegative and continuous in its arguments and satisfies the following condition:

$$\frac{B(\theta,|v-v_*|)}{|\sin\theta\cos\theta|}\le Cb_{\delta_1,\delta_2}(|v-v_*|), \tag{1.6}$$

where

$$b_{\delta_1,\delta_2}(|v-v_*|)=1+|v-v_*|^{\delta_1}+|v-v_*|^{\delta_2},\quad -2<\delta_1\le 0\le\delta_2\le 1. \tag{1.7}$$

Notice that for the hard-sphere model,

$$B(\theta,|v-v_*|)=C|v-v_*|\sin\theta\cos\theta, \tag{1.8}$$

which satisfies (1.6) with $\delta_1=0$ and $\delta_2=1$. Notice also that both the hard and soft potentials with angular cut-off satisfy the condition (1.6). And later for simpler presentation, we call the cases with $\delta_2=0$ and $\delta_1=0$ as soft and hard potentials respectively.

As usual, we will later rewrite the equation (1.1) by integration along the bi-characteristics. For any fixed $(x,v)\in\mathbf{R}^3_x\times\mathbf{R}^3_v$, the forward bi-characteristics $[X^t(x,v),V^t(x,v)]$ is defined by

$$\begin{cases}\dfrac{dX^t(x,v)}{dt}=V^t(x,v), \quad \dfrac{dV^t(x,v)}{dt}=E(t,X^t(x,v),V^t(x,v)),\\ (X^t,V^t)_{t=0}=(x,v).\end{cases} \tag{1.9}$$

Then the mild form of the Boltzmann equation becomes

$$f^\#(t,x,v)=f_0(x,v)+\int_0^t J^\#(f,f)(s,x,v)\,ds, \tag{1.10}$$

where

$$\begin{cases}f^\#(t,x,v)=f(t,X^t(x,v),V^t(x,v)),\\ J^\#(f,f)(s,x,v)=J(f,f)(s,X^s(x,v),V^s(x,v)).\end{cases}$$

We now introduce some norms for the solutions under consideration as in [4]. For any $f=f(t,x,v)$ and $f_0=f_0(x,v)$, define

$$|||f|||^E_{\alpha,\beta}=\sup_{t,x,v}\frac{|f^\#(t,x,v)|}{h_\alpha(|x|)m_\beta(|v|)},\quad \|f_0\|_{\alpha,\beta}=\sup_{x,v}\frac{|f_0(x,v)|}{h_\alpha(|x|)m_\beta(|v|)}, \tag{1.11}$$

where the weight functions h_α and m_β have algebraic decay rates and are in the form of

$$h_\alpha(|x|) = (1+|x|^2)^{-\alpha},\ \alpha > 0 \ \text{ and } \ m_\beta(|v|) = (1+|v|^2)^{-\beta},\ \beta > 0. \tag{1.12}$$

For simplicity, throughout this paper, $\mathcal{O}(1)$ denotes the generic positive constant which may vary for different equations. And for any function $f(t,x,v)$, we use notations:

$$\|f(t)\|_1 = \|f(t,\cdot,\cdot)\|_{L^1(\mathbf{R}^3_x\times\mathbf{R}^3_v)},\quad \|f(t)\|_\infty = \sup_{x,v}|f(t,x,v)|,$$

and

$$\|\nabla_x f(t)\|_p = \sum_{i=1}^3 \|\partial_{x_i} f(t)\|_p,\quad \|\nabla_v f(t)\|_p = \sum_{i=1}^3 \|\partial_{v_i} f(t)\|_p,$$

where $p = 1$ or ∞.

In this paper, the external force E is assumed to satisfy the following conditions:

(E1) $E(\cdot,\cdot,\cdot) \in C_b^0(\mathbf{R}^+_t\times\mathbf{R}^3_x\times\mathbf{R}^3_v)$ and $\nabla_x E(t,\cdot,\cdot)$, $\nabla_v E(t,\cdot,\cdot) \in C_b^0(\mathbf{R}^3_x\times\mathbf{R}^3_v)$ for any fixed $t > 0$;

(E2) There exist constants $\varepsilon_0 > 0$ and $0 < \varepsilon_1 < 1$ such that

$$\int_0^\infty \|E(t)\|_\infty dt \le \varepsilon_0, \tag{1.13}$$

and

$$\int_0^\infty \left(\|\nabla_v E(t)\|_\infty + (1+t)\|\nabla_x E(t)\|_\infty\right) dt \le \varepsilon_1. \tag{1.14}$$

Before stating the main result in this paper, we first give the following existence theorem on the classical solution in infinite vacuum to the Boltzmann equation with external forces for small initial data. Notice that even though Theorem 1.1 is slightly different from the existence theorem in [12], where the assumption for the collision kernel B is

$$\frac{B(\theta,|v-v_*|)}{|\sin\theta\cos\theta|} \le \frac{1+|v-v_*|}{|v-v_*|^\mu},\quad 0\le\mu<1,$$

the same proof leads to Theorem 1.1 for more general collision kernel B satisfying (1.6). Therefore, we omit the proof of Theorem 1.1 for brevity.

Theorem 1.1. *Let $\alpha > 1/2$ and $\beta > 2$. Suppose that the collision kernel B satisfies (1.6) with $-2 < \delta_1 \le 0 \le \delta_2 \le 1$ and the external force E satisfies (E1) and (E2) with $\varepsilon_0 > 0$ and $0 < \varepsilon_1 < 1$. Fix any $C_1 > 1/(1-\varepsilon_1)$. If $0 \le f_0(x,v) \in C^1(\mathbf{R}^3_x\times\mathbf{R}^3_v)$ such that*

$$\|f_0\|_{\alpha,\beta} + \|\nabla_x f_0\|_{\alpha,\beta} + \|\nabla_v f_0\|_{\alpha,\beta} \le \delta \tag{1.15}$$

with $\delta > 0$ sufficiently small, then there exists a unique classical solution $0 \le f(t,x,v) \in C^1(\mathbf{R}^+_t\times\mathbf{R}^3_x\times\mathbf{R}^3_v)$ to the initial value problem (1.1) and (1.2) of the Boltzmann equation satisfying

$$|||f|||^E_{\alpha,\beta} + |||\nabla_x f|||^E_{\alpha,\beta} + |||(1+t)^{-1}\nabla_v f|||^E_{\alpha,\beta} \le C_1\delta. \tag{1.16}$$

Our main result about the stability is as follows.

Theorem 1.2. *Assume that all the conditions in Theorem 1.1 hold. Moreover, assume*

$$\mathrm{div}_v E(t,x,v) \equiv 0,$$

for any $(t,x,v) \in \mathbf{R}_t^+ \times \mathbf{R}_x^3 \times \mathbf{R}_v^3$. *Suppose that* f *and* g *are the classical solutions to the Boltzmann equation (1.1) corresponding to the initial data* f_0 *and* g_0 *satisfying (1.15). If* $\delta > 0$ *is sufficiently small, then the following* L^1 *stability and* BV*-type estimate hold:*

$$\|f(t) - g(t)\|_1 \le \mathcal{O}(1)\|f_0 - g_0\|_1, \quad \forall\, t \ge 0, \tag{1.17}$$

and

$$\|\nabla_x f(t)\|_1 + \|(1+t)^{-1}\nabla_v f(t)\|_1 \le \mathcal{O}(1)(\|\nabla_x f_0\|_1 + \|\nabla_v f_0\|_1), \quad \forall\, t \ge 0, \tag{1.18}$$

when one of the following two cases is satisfied:

- *Case 1. When* $-2 < \delta_1 \le 0$ *and* $\delta_2 = 0$.
- *Case 2. when* $-2 < \delta_1 \le 0$, $0 < \delta_2 \le 1$, $\alpha > 3$ *and* $\beta > 4$. *Moreover, the external force* $E(t,x,v)$ *also satisfies*

$$\int_0^\infty (1+s)\|E(s)\|_\infty ds < \infty. \tag{1.19}$$

To better understand the statement in Theorem 1.2, we give the following remarks on the consequences of the assumption.

Remark 1.1. In Theorem 1.2, the assumption that $\mathrm{div}_v E \equiv 0$ is to insure that the mapping $(x,v) \to (X^t(x,v), V^t(x,v))$ for any $t \ge 0$ preserves the measure. Under this condition, for any $1 \le p \le \infty$ and $h(t,x,v)$, we have that

$$\|h^{\#}(t,\cdot,\cdot)\|_{L^p(\mathbf{R}_x^3 \times \mathbf{R}_v^3)} = \|h(t,\cdot,\cdot)\|_{L^p(\mathbf{R}_x^3 \times \mathbf{R}_v^3)}, \quad t \ge 0, \tag{1.20}$$

where $h^{\#}$ takes values along the bi-characteristics (1.9) generated by the force field E. Please refer to [5] for the proof.

Remark 1.2. In the BV-type estimate (1.18), it is natural to add the weight $(1+t)^{-1}$ in the L^1 norm for $\nabla_v f$. In fact, one can just look at the free transport equation

$$\partial_t f + v \cdot \nabla_x f = 0.$$

The solution is $f(t,x,v) = f_0(x - vt, v)$. When the initial f_0 is smooth, we have

$$\nabla_x f = \nabla_x f_0 \quad \text{and} \quad \nabla_v f = -t\nabla_x f_0 + \nabla_v f_0.$$

Remark 1.3. Let's point out the difference between Case 1 and Case 2 in Theorem 1.2. We see from Case 2 that for the hard potential, the finiteness of the total mass of gas in the phase space is required. Furthermore the external force E decays with higher rates than the one for the soft potential. In the proof, the estimate for the soft potential can be obtained directly, while the estimate for the hard potential is based on the construction of some nonlinear functionals.

Now we review some previous works on the related topics and then give the main ideas in this paper. Some general knowledge on these topics can be found in the literatures on the Boltzmann equation, such as [3, 6, 7, 28]. In the absence of the external force, the Cauchy problem and the initial boundary value problem for the Boltzmann equation have been extensively studied, see [10, 20, 22, 23, 25, 26] and references therein. We only mention some works related to the problems considered in this paper. For the space homogeneous case, Arkeryd [1] proved the Lyapunov-type weighted L^1 stability. For the space nonhomogeneous case, Ha [18, 19] first obtained the uniform L^1 stability and BV-type estimates by using a Lyapunov functional. Recently, a new Lyapunov functional and L^1 stability for the Vlasov-Poisson system were presented by Chae-Ha [8]. For the other interesting issues such as convergence to the Maxwellian, interested readers please refer to [9, 21].

For the Botlzmann equation with the external force in infinite vacuum, the local existence theorem was given by Glikson [13, 14] and Asano [2]. Then Bellomo-Lachowicz-Palzewski-Toscani [4] gave a general framework on the global existence of mild solutions, see also the recent result [11]. For classical solutions, the global existence was obtained by Guo [17] for the soft potential and Duan-Yang-Zhu [12] for the general potential. For solutions near a global Maxwellian, Ukai-Yang-Zhao [27] proved the stability of stationary Maxwellian solutions to the Boltzmann equation with external forces through the energy method. For this, please refer to some related results in [15, 16].

Part of the ideas in the proof of this paper comes from [18, 19] by Ha. There are two main observations for the L^1 stability and BV-type estimates: one is the decay in time of $f^\#$ in the space $L^1(\mathbf{R}^3_v)$, the other is the decay in time of $J^\#(f,g)$ in the space $L^1(\mathbf{R}^3_x \times \mathbf{R}^3_v)$. To obtain our result, we directly use the Gronwall's inequality to deal with the case of the soft potential. For the case of the hard potential, some new nonlinear functionals, which reduce to the same functionals in [19] when the external force vanishes, are constructed to control the the factor $|v-v_*|^{\delta_2}$ in the collision kernel $B(\theta, |v-v_*|)$.

The rest of this paper is arranged as follows. In Section 2, some preliminary lemmas are given for later use. In Section 3, the L^1 stability estimate is obtained by considering the following two cases: the soft potential and the hard potential. Finally, in Section 4, the BV-type estimate is obtained by the similar arguments for the L^1 stability.

2 Preliminary

For any fixed $(t,x,v) \in \mathbf{R}^+_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v$, we define the backward bi-characteristics $[X(s;t,x,v), V(s;t,x,v)]$ by solutions to the ODE system

$$\begin{cases} \dfrac{dX(s;t,x,v)}{ds} = V(s;t,x,v), \quad \dfrac{dV(s;t,x,v)}{ds} = E(s, X(s;t,x,v), V(s;t,x,v)), \\ (X(s;t,x,v), V(s;t,x,v))_{s=t} = (x,v). \end{cases} \tag{2.1}$$

Notice that for any $s>0$, if the mapping $(X^s, V^s) : \mathbf{R}^3_x \times \mathbf{R}^3_v \to \mathbf{R}^3_x \times \mathbf{R}^3_v$ defined by

$$(X^s, V^s)(x,v) = (X^s(x,v), V^s(x,v))$$

is one-to one and onto, then

$$(X^s, V^s)(X(0;t,x,v), V(0;t,x,v)) = (X(s;t,x,v), V(s;t,x,v))$$

for any $(t,x,v)\in \mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$ and $s\in[0,t]$. Thus for any $f(t,x,v)$ with $|||f|||_{\alpha,\beta}^E<\infty$, we have that

$$|f(t,x,v)|\leq |||f|||_{\alpha,\beta}^E h_\alpha(|X(0;t,x,v)|)m_\beta(|V(0;t,x,v)|). \tag{2.2}$$

Next we list some basic lemmas for later use. Interested readers may refer to [3, 4, 12, 19] for the details of the proofs.

Lemma 2.1. *Suppose that the external force E satisfies (1.13) with $\varepsilon_0>0$. Then we have that for any $t\geq 0$, $s\geq 0$ and $(x,v)\in\mathbf{R}_x^3\times\mathbf{R}_v^3$,*

$$\begin{aligned}
&(i) && |V^t(x,v)-V^s(x,v)|\leq\varepsilon_0,\\
&(ii) && \left|\frac{1}{t}\int_0^t V^s(x,v)ds-v\right|\leq\varepsilon_0,\\
&(iii) && \left|\frac{1}{t}\int_0^t V^s(x,v)ds-V^t(x,v)\right|\leq 2\varepsilon_0.
\end{aligned}$$

In terms of the backward bi-characteristics, we have that for any $(t,x,v)\in\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$ and $0\leq s_1,s_2\leq t$,

$$\begin{aligned}
&(i)' && |V(s_1;t,x,v)-V(s_2;t,x,v)|\leq\varepsilon_0,\\
&(ii)' && \left|\frac{1}{t}\int_0^t V(s;t,x,v)ds-V(0;t,x,v)\right|\leq\varepsilon_0,\\
&(iii)' && \left|\frac{1}{t}\int_0^t V(s;t,x,v)ds-v\right|\leq 2\varepsilon_0.
\end{aligned}$$

Remark 2.1. For any $(t,x,v)\in\mathbf{R}_t^+\times\mathbf{R}_x^3\times\mathbf{R}_v^3$, we can rewrite $X^t(x,v)$ and $X(0;t,x,v)$ in the following forms:

$$\begin{cases} X^t(x,v)=x+(a(t,x,v)+v)t,\\ X(0;t,x,v)=x-(b(t,x,v)+v)t,\end{cases} \tag{2.3}$$

where

$$a(t,x,v)=\frac{1}{t}\int_0^t V^s(x,v)ds-v,\quad b(t,x,v)=\frac{1}{t}\int_0^t V(s;t,x,v)ds-v. \tag{2.4}$$

By Lemma 2.1, we have that

$$\sup_{t,x,v}|a(t,x,v)|\leq\varepsilon_0\ \text{ and }\ \sup_{t,x,v}|b(t,x,v)|\leq 2\varepsilon_0. \tag{2.5}$$

Lemma 2.2. *For any $\alpha>0$ and $(x,y)\in\mathbf{R}^3\times\mathbf{R}^3$, we have*

$$(1+|y|+|y|^2)^{-\alpha}\leq\frac{h_\alpha(|x|)}{h_\alpha(|x+y|)}\leq(1+|y|+|y|^2)^\alpha.$$

Lemma 2.3. *For any $\alpha>1/2$ and $u\in\mathbf{R}^3$ with $u\neq 0$, we have*

$$\sup_x\int_0^\infty h_\alpha(|x+su|)\,ds\leq\frac{\mathcal{O}(1)}{|u|}.$$

Lemma 2.4. *For any $0\leq\gamma<3$ and $\beta>3/2$, we have that*

$$\sup_v\int_{\mathbf{R}^3}\frac{1}{|v-u|^\gamma}m_\beta(|u|)du\leq\mathcal{O}(1).$$

Lemma 2.5. *Let $\alpha > 3$ and $\beta > 4$. Suppose that the collision kernel B satisfies (1.6) with $-2 < \delta_1 \le 0 \le \delta_2 \le 1$ and the external force E satisfies (1.13). Then there exists a positive constant η with $0 < \eta < \beta - 2$ such that for any $(t, x, v) \in \mathbf{R}_t^+ \times \mathbf{R}_x^3 \times \mathbf{R}_v^3$, we have*

$$\left|Q^{\#}(f, g)(t, x, v)\right| + \left|f^{\#} R^{\#}(g)(t, x, v)\right| \le \frac{\mathcal{O}(1)|||f|||_{\alpha,\beta}^E |||g|||_{\alpha,\beta}^E}{(1+t)^{\min\{3+\delta_1, 2\}}} h_{\alpha-1/2}(|x|) m_{\beta-\eta}(|v|). \quad (2.6)$$

3 L^1 stability

3.1 Soft potentials

In this subsection, we shall prove the L^1 stability estimate (1.17) for the Case 1 in Theorem 1.2. For this purpose, let f and g be two classical solutions to the Boltzmann equation (1.1) corresponding to the initial data f_0 and g_0 satisfying (1.15) in Theorem 1.1. Define the nonnegative symmetric bilinear operator $\mathcal{S}$ by

$$\mathcal{S}(f, g)(t, x, v) = [Q(f, g) + fR(g) + Q(g, f) + gR(f)](t, x, v),$$

and the nonlinear functionals $\mathcal{L}(t)$ and $\Lambda_{\delta_1,\delta_2}(t)$ by

$$\mathcal{L}(t) = \int_{\mathbf{R}_x^3 \times \mathbf{R}_v^3} |f - g|^{\#}(t, x, v) dx dv,$$

$$\Lambda_{\delta_1,\delta_2}(t) = \int_{\mathbf{R}_x^3 \times \mathbf{R}_v^3 \times \mathbf{R}_{v_*}^3} b_{\delta_1,\delta_2}(|v - v_*|)|f - g|(t, x, v)(f + g)(t, x, v_*) dx dv dv_*.$$

First we have the following basic estimates for the case of the general potential, which is used in the proof.

Lemma 3.1. *For any $-2 < \delta_1 \le 0 \le \delta_2 \le 1$, we have*

$$\left\|\mathcal{S}^{\#}(f, g)(t)\right\|_1 \le \mathcal{O}(1) \int_{\mathbf{R}_x^3 \times \mathbf{R}_v^3 \times \mathbf{R}_{v_*}^3} b_{\delta_1,\delta_2}(|v - v_*|) f(t, x, v) g(t, x, v_*) dx dv dv_*, \quad (3.1)$$

and

$$\frac{d\mathcal{L}(t)}{dt} \le \mathcal{O}(1) \left\|\mathcal{S}^{\#}(|f - g|, f + g)(t)\right\|_1 \le \mathcal{O}(1)\Lambda_{\delta_1,\delta_2}(t). \quad (3.2)$$

Proof. We only prove (3.2) because (3.1) follows directly from the definition. Since f and g are solutions to the Boltzmann equation (1.1), then

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + E(t, x, v) \cdot \nabla_v f &= J(f, f), \\ \partial_t g + v \cdot \nabla_x g + E(t, x, v) \cdot \nabla_v g &= J(g, g). \end{aligned}$$

Taking difference of the above two equations and multiplying it by $\mathrm{sign}(f - g)$ give

$$\partial_t |f - g| + v \cdot \nabla_x |f - g| + E(t, x, v) \cdot \nabla_v |f - g| \le \mathcal{S}(|f - g|, f). \quad (3.3)$$

Similarly, interchanging f and g yields

$$\partial_t |g - f| + v \cdot \nabla_x |g - f| + E(t, x, v) \cdot \nabla_v |g - f| \le \mathcal{S}(|g - f|, g). \quad (3.4)$$

It follows from (3.3) and (3.4) that

$$\partial_t|f-g| + v\cdot\nabla_x|f-g| + E(t,x,v)\cdot\nabla_v|f-g| \leq \frac{1}{2}\mathcal{S}(|f-g|, f+g). \tag{3.5}$$

Along the forward bi-characteristics, the above inequality can be rewritten as

$$\partial_t|f-g|^{\#} \leq \mathcal{O}(1)\mathcal{S}^{\#}(|f-g|, f+g). \tag{3.6}$$

By integrating (3.6) over $\mathbf{R}_x^3\times\mathbf{R}_v^3$ with respect to (x,v) and using (3.1), we have (3.2). And this completes the proof of Lemma 3.1.

Furthermore, for the case of the soft potential, i.e. $\delta_2 = 0$, the following lemma shows that $\Lambda_{\delta_1,0}(t)$ can be controlled by $\mathcal{L}(t)$ so that the L^1 stability can be obtained directly.

Lemma 3.2. *Suppose that the conditions of Theorem 1.2 hold. For the case when $-2<\delta_1\leq 0$ and $\delta_2=0$, we have that*

$$\Lambda_{\delta_1}(t) \leq \frac{\mathcal{O}(1)}{(1+t)^{3+\delta_1}}\mathcal{L}(t), \tag{3.7}$$

where $\Lambda_{\delta_1}(t)=\Lambda_{\delta_1,0}(t)$.

Proof. Write $\Lambda_{\delta_1}(t)=\Lambda_{\delta_1,0}(t)$. It follows from the representation of $\Lambda_{\delta_1,0}(t)$ that

$$\begin{aligned}
\Lambda_{\delta_1}(t) &= \int_{\mathbf{R}_x^3\times\mathbf{R}_v^3}|f-g|(t,x,v)dxdv\int_{\mathbf{R}_{v_*}^3} b_{\delta_1,0}(|v-v_*|)(f+g)(t,x,v_*)dv_*\\
&\leq \mathcal{O}(1)\delta\int_{\mathbf{R}_x^3\times\mathbf{R}_v^3}|f-g|(t,x,v)dxdv\\
&\qquad\qquad \int_{\mathbf{R}_{v_*}^3} b_{\delta_1,0}(|v-v_*|)h_\alpha(|X(0;t,x,v_*)|)m_\beta(|V(0;t,x,v_*)|)dv_*\\
&\leq \mathcal{O}(1)\delta\int_{\mathbf{R}_x^3\times\mathbf{R}_v^3}|f-g|(t,x,v)dxdv\\
&\qquad\qquad \int_{\mathbf{R}_{v_*}^3} b_{\delta_1,0}(|v-v_*|)h_\alpha(|x-(b(t,x,v_*)+v_*)t|)m_\beta(|v_*|)dv_*\\
&\leq \mathcal{O}(1)\delta\int_{\mathbf{R}_x^3\times\mathbf{R}_v^3}|f-g|(t,x,v)dxdv\\
&\qquad\qquad \sup_{|b|\leq 2\varepsilon_0}\int_{\mathbf{R}_{v_*}^3} b_{\delta_1,0}(|v-v_*|)h_\alpha(|x-(b+v_*)t|)m_\beta(|v_*|)dv_*.
\end{aligned} \tag{3.8}$$

For any constant vector $b\in\mathbf{R}^3$ with $|b|\leq 2\varepsilon_0$, let's define

$$I_{\delta_1}(t,x,v) = \int_{\mathbf{R}_{v_*}^3} b_{\delta_1,0}(|v-v_*|)h_\alpha(|x-(b+v_*)t|)m_\beta(|v_*|)dv_*. \tag{3.9}$$

We claim that

$$\sup_{x,v} I_{\delta_1}(t,x,v) \leq \frac{\mathcal{O}(1)}{(1+t)^{3+\delta_1}}. \tag{3.10}$$

In fact, for any $t\geq 0$, we have from Lemma 2.4 that

$$I_{\delta_1}(t,x,v) \leq \mathcal{O}(1)\int_{\mathbf{R}_{v_*}^3}(1+|v-v_*|^{\delta_1})m_\beta(|v_*|)dv_* \leq \mathcal{O}(1). \tag{3.11}$$

Furthermore, for any $t \geq 1$, we let $v_* t = \bar{v}$ to obtain

$$\begin{aligned} I_{\delta_1}(t,x,v) &\leq \frac{\mathcal{O}(1)}{t^3}\int_{\mathbf{R}^3_{\bar{v}}}\left(1+\left|v-\frac{\bar{v}}{t}\right|^{\delta_1}\right)h_\alpha(|x-bt-\bar{v}|)d\bar{v} \\ &\leq \frac{\mathcal{O}(1)}{t^{3+\delta_1}}\int_{\mathbf{R}^3_{\bar{v}}}(1+|vt-\bar{v}|^{\delta_1})h_\alpha(|x-bt-\bar{v}|)d\bar{v} \\ &\leq \frac{\mathcal{O}(1)}{t^{3+\delta_1}}\int_{\mathbf{R}^3_u}(1+|x-(b+v)t-u|^{\delta_1})h_\alpha(|u|)du \\ &\leq \frac{\mathcal{O}(1)}{t^{3+\delta_1}}, \end{aligned} \tag{3.12}$$

where Lemma 2.4 is used again. Combining (3.11) and (3.12) yields (3.10), which together with (3.8) implies (3.7). The proof of Lemma 3.2 is completed.

Remark 3.1. Both Lemmas 3.1 and 3.2 imply that

$$\frac{d\mathcal{L}(t)}{dt} \leq \frac{\mathcal{O}(1)}{(1+t)^{3+\delta_1}}\mathcal{L}(t). \tag{3.13}$$

Since $\delta_1 > -2$, i.e. $3+\delta_1 > 1$, the Gronwall's inequality immediately leads to

$$\mathcal{L}(t) \leq \mathcal{O}(1)\mathcal{L}(0). \tag{3.14}$$

This shows that if the collision kernel B satisfies (1.6) with $-2 < \delta_1 \leq 0$ and $\delta_2 = 0$, then L^1 stability estimate (1.17) holds.

3.2 General potentials

In this subsection, we consider the L^1 stability of solutions to the Boltzmann equation for the Case 2 in Theorem 1.2. For brevity of the presentation, we divide the Case 2 into the following subcases:

- *Subcase 2.1.* $\delta_1 = 0$, $0 < \delta_2 \leq 1$, $\alpha > 3$, $\beta > 4$ *and the inequality (1.19) holds;*
- *Subcase 2.2.* $-2 < \delta_1 < 0$, $0 < \delta_2 \leq 1$, $\alpha > 3$, $\beta > 4$ *and the inequality (1.19) holds.*

In fact, the proof of the Subcase 2.2 follows from the Case 1 and the Subcase 2.1.

Notice that the estimate similar to Lemma 3.2 fails in Case 2 because of the factor $|v-v_*|^{\delta_2}$ in the collision kernel $B(\theta, |v-v_*|)$. To overcome this difficulty, we will construct a new Lyapunov functional motivated by the works [18, 19] on the L^1 stability of solutions to the Boltzmann equation without the external force. For this purpose, let's define

$$v_\infty(x,v) = \int_0^\infty E\left(\theta, X^\theta(x,v), V^\theta(x,v)\right)d\theta. \tag{3.15}$$

Lemma 3.3. *Suppose that*

$$\int_0^\infty (1+s)\|E(s)\|_\infty ds < \infty.$$

Then for any $(x,v) \in \mathbf{R}^3_x \times \mathbf{R}^3_v$, $v_\infty(x,v)$ *is well-defined with*

$$|v_\infty(x,v)| \leq \varepsilon_0, \tag{3.16}$$

and

$$a(t,x,v) \to v_\infty(x,v) \quad as \quad t \to \infty, \tag{3.17}$$

where $a(t,x,v)$ *is defined by (2.4).*

Proof. It is obvious that $v_\infty(x,v)$ is well-defined if (3.17) holds. We only need to consider the limit (3.17). Notice that

$$\begin{aligned}
|a(t,x,v)-v_\infty(x,v)| &= \left|\frac{1}{t}\int_0^t V^s(x,v)ds - v - v_\infty(x,v)\right| \\
&= \left|\frac{1}{t}\int_0^t\int_0^s E(\theta,X^\theta(x,v),V^\theta(x,v))d\theta ds - v_\infty(x,v)\right| \\
&= \left|\frac{1}{t}\int_0^t\int_s^\infty E(\theta,X^\theta(x,v),V^\theta(x,v))d\theta ds\right| \\
&\leq \frac{1}{t}\int_0^t\int_s^\infty \|E(\theta)\|_\infty d\theta ds \\
&\leq \int_t^\infty \|E(\theta)\|_\infty d\theta + \frac{1}{t}\int_0^\infty (1+s)\|E(s)\|_\infty ds. \qquad (3.18)
\end{aligned}$$

Hence (3.17) holds. The proof of Lemma 3.3 is completed.

In order to control the integral $\Lambda_{\delta_1,\delta_2}(t)$ for the case when $0<\delta_2\leq 1$, as in [18, 19], let's define functionals as follows:

$$\begin{aligned}
\Lambda_{\delta_2}(t) = \int_{\mathbf{R}^3_x\times\mathbf{R}^3_v} & |f-g|^{\#}(t,x,v)dxdv \\
&\int_{\mathbf{R}^3_{v_*}} |v_\infty(x,v)+v-v_*|^{\delta_2}(f+g)(t,X^t(x,v),v_*)dv_* \qquad (3.19)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}_{\delta_2}(t) = \int_{\mathbf{R}^3_x\times\mathbf{R}^3_v} & |f-g|^{\#}(t,x,v)dxdv \\
&\int_{\mathbf{R}^3_{v_*}\times\mathbf{R}^+_\tau} |v_\infty(x,v)+v-v_*|^{\delta_2-1} \\
&\qquad (f+g)(t,X^t(x,v)+\tau n(v_\infty(x,v)+v-v_*),v_*)dv_*d\tau, \qquad (3.20)
\end{aligned}$$

where $n(z)=z/|z|$ denotes the unit vector along z-direction for any nonzero vector $z\in\mathbf{R}^3$.

Remark 3.2. If the external force E vanishes, i.e. $E\equiv 0$, then we have

$$v_\infty(x,v)\equiv 0,$$

and

$$X^t(x,v)=x+vt, \quad V^t(x,v)=v.$$

Thus the functionals $\Lambda_{\delta_2}(t)$ and $\mathcal{D}_{\delta_2}(t)$ reduce to

$$\begin{aligned}
\Lambda_{\delta_2}(t) = \int_{\mathbf{R}^3_x\times\mathbf{R}^3_v} & |f-g|^{\#}(t,x,v)dxdv \\
&\int_{\mathbf{R}^3_{v_*}} |v-v_*|^{\delta_2}(f+g)^{\#}(t,x+t(v-v_*),v_*)dv_*
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{D}_{\delta_2}(t) = \int_{\mathbf{R}^3_x\times\mathbf{R}^3_v} & |f-g|^{\#}(t,x,v)dxdv \\
&\int_{\mathbf{R}^3_{v_*}\times\mathbf{R}^+_\tau} |v-v_*|^{\delta_2-1}(f+g)^{\#}(t,x+t(v-v_*)+\tau n(v-v_*),v_*)dv_*d\tau,
\end{aligned}$$

which are exactly the same as those in [19].

Furthermore, we define the integral $I_{\delta_2}(t,x,v)$ by

$$I_{\delta_2}(t,x,v) = \int_{\mathbf{R}^3_{v_*}\times\mathbf{R}^+_\tau} |v_\infty(x,v)+v-v_*|^{\delta_2-1} \\ h_\alpha(|X(0;t,X^t(x,v)+\tau n(v_\infty(x,v)+v-v_*),v_*)|) \\ m_\beta(|V(0;t,X^t(x,v)+\tau n(v_\infty(x,v)+v-v_*),v_*)|)dv_*d\tau. \tag{3.21}$$

We first show that $\mathcal{D}_{\delta_2}(t)$ can be bounded by $\mathcal{L}(t)$, which comes from the following lemma.

Lemma 3.4. *If $0\le\delta_2\le 1$ and the external force E satisfies (1.13), then we have*

$$\sup_{t,x,v} I_{\delta_2}(t,x,v) \le \mathcal{O}(1). \tag{3.22}$$

Proof. Fix any $(t,x,v)\in\mathbf{R}^+_t\times\mathbf{R}^3_x\times\mathbf{R}^3_v$. It follows from Lemmas 2.1-2.4 that

$$\begin{aligned} I_{\delta_2}(t,x,v) &\le \sup_{|b|\le 2\varepsilon_0}\int_{\mathbf{R}^3_{v_*}\times\mathbf{R}^+_\tau} |v_\infty(x,v)+v-v_*|^{\delta_2-1} m_\beta(|v_*|) \\ &\qquad h_\alpha\left(\left|X^t(x,v)-(b+v_*)t+\tau n(v_\infty(x,v)+v-v_*)\right|\right) dv_*d\tau \\ &\le \sup_{|b|\le 2\varepsilon_0}\int_{\mathbf{R}^3_{v_*}} |v_\infty(x,v)+v-v_*|^{\delta_2-1} m_\beta(|v_*|)dv_* \\ &\qquad \int_{\mathbf{R}^+_\tau} h_\alpha\left(\left|X^t(x,v)-(b+v_*)t+\tau n(v_\infty(x,v)+v-v_*)\right|\right) d\tau \\ &\le \mathcal{O}(1)\sup_{|b|\le 2\varepsilon_0}\int_{\mathbf{R}^3_{v_*}} |v_\infty(x,v)+v-v_*|^{\delta_2-1} m_\beta(|v_*|)dv_* \\ &\le \mathcal{O}(1). \end{aligned} \tag{3.23}$$

Thus the proof of Lemma 3.4 is completed.

Remark 3.3. From Lemma 3.4, it is easy to see that for any $t\ge 0$, we have

$$\mathcal{D}_{\delta_2}(t) \le \mathcal{O}(1)\mathcal{L}(t). \tag{3.24}$$

Next we consider the L^1 stability for the Subcase 2.1.

Lemma 3.5. *Suppose that the conditions of Theorem 1.2 hold. For the case when $\delta_1 = 0$, $0<\delta_2\le 1$, $\alpha>3$ and $\beta>4$, we have that*

$$\frac{d\mathcal{D}_{\delta_2}(t)}{dt} \le -(1-\mathcal{O}(1)\delta)\Lambda_{\delta_2}(t) + \mathcal{O}(1)\lambda_1(t)\mathcal{L}(t), \tag{3.25}$$

where

$$\lambda_1(t) = \frac{\delta}{(1+t)^3} + \frac{\delta^2}{(1+t)^{\min\{3+\delta_1,2\}}} + \delta(1+t)\|E(t)\|_\infty + \delta\int_t^\infty \|E(\theta)\|_\infty d\theta. \tag{3.26}$$

Proof. First, notice that

$$
\begin{aligned}
\partial_t\, & [f(t, \quad X^t(x,v)+\tau n(v_\infty(x,v)+v-v_*), v_*)] \\
&= (\partial_t f)(t, X^t(x,v)+\tau n, v_*) + V^t(x,v)\cdot\nabla_x f(t, X^t(x,v)+\tau n, v_*) \\
&= J(f,f)(t, X^t(x,v)+\tau n, v_*) - v_*\cdot\nabla_x f(t, X^t(x,v)+\tau n, v_*) \\
&\quad -E(t, X^t(x,v)+\tau n, v_*)\cdot\nabla_v f(t, X^t(x,v)+\tau n, v_*) \\
&\quad +V^t(x,v)\cdot\nabla_x f(t, X^t(x,v)+\tau n, v_*) \\
&= J(f,f)(t, X^t(x,v)+\tau n, v_*) \\
&\quad -E(t, X^t(x,v)+\tau n, v_*)\cdot\nabla_v f(t, X^t(x,v)+\tau n, v_*) \\
&\quad +(V^t(x,v)-v-v_\infty(x,v))\cdot\nabla_x f(t, X^t(x,v)+\tau n, v_*) \\
&\quad +\partial_\tau(|v_\infty(x,v)+v-v_*| f(t, X^t(x,v)+\tau n, v_*)),
\end{aligned}
\tag{3.27}
$$

where for simplicity we have used n to denote the unit vector $n(v_\infty(x,v)+v-v_*)$. Hence (3.27) together with (3.6) yield

$$
\begin{aligned}
&\partial_t \left[|f-g|^\#(t,x,v)(f+g)(t, X^t(x,v)+\tau n, v_*)\right] \\
&= \partial_t |f-g|^\#(t,x,v)(f+g)(t, X^t(x,v)+\tau n, v_*) \\
&\quad +|f-g|^\#(t,x,v)\partial_t\left[(f+g)(t, X^t(x,v)+\tau n, v_*)\right] \\
&\le \mathcal{S}^\#(|f-g|, f+g)(t,x,v)(f+g)(t, X^t(x,v)+\tau n, v_*) \\
&\quad +|f-g|^\#(t,x,v)(\mathcal{S}(f,f)+\mathcal{S}(g,g))(t, X^t(x,v)+\tau n, v_*) \\
&\quad +|f-g|^\#(t,x,v)\|E(t)\|_\infty(|\nabla_v f|+|\nabla_v g|)(t, X^t(x,v)+\tau n, v_*) \\
&\quad +|f-g|^\#(t,x,v)|V^t(x,v)-v-v_\infty(x,v)|(|\nabla_x f|+|\nabla_x g|)(t, X^t(x,v)+\tau n, v_*) \\
&\quad +\partial_\tau\left[|v_\infty(x,v)+v-v_*||f-g|^\#(t,x,v)(f+g)(t, X^t(x,v)+\tau n, v_*)\right].
\end{aligned}
\tag{3.28}
$$

Multiplying the above inequality by $|v_\infty(x,v)+v-v_*|^{\delta_2-1}$ and integrating it over the domain $D = \mathbf{R}^3_x\times\mathbf{R}^3_v\times\mathbf{R}^3_{v_*}\times\mathbf{R}^+_\tau$ leads to

$$
\frac{d\mathcal{D}_{\delta_2}(t)}{dt} \le -\Lambda_{\delta_2}(t) + \sum_{i=1}^{4} J_i(t), \tag{3.29}
$$

where $J_i(t)$, $i = 1, 2, 3, 4$ are defined as follows:

$$J_1(t) = \int_D |v_\infty(x,v) + v - v_*|^{\delta_2 - 1}\mathcal{S}^\#(|f-g|, f+g)(t,x,v)$$
$$(f+g)(t, X^t(x,v) + \tau n, v_*)dxdvdv_*d\tau,$$

$$J_2(t) = \int_D |v_\infty(x,v) + v - v_*|^{\delta_2 - 1}|f-g|^\#(t,x,v)$$
$$(\mathcal{S}(f,f) + \mathcal{S}(g,g))(t, X^t(x,v) + \tau n, v_*)dxdvdv_*d\tau,$$

$$J_3(t) = \int_D |v_\infty(x,v) + v - v_*|^{\delta_2 - 1}\|E(t)\|_\infty|f-g|^\#(t,x,v)$$
$$(|\nabla_v f| + |\nabla_v g|)(t, X^t(x,v) + \tau n, v_*)dxdvdv_*d\tau,$$

$$J_4(t) = \int_D |v_\infty(x,v) + v - v_*|^{\delta_2 - 1}|V^t(x,v) - v - v_\infty(x,v)||f-g|^\#(t,x,v)$$
$$(|\nabla_x f| + |\nabla_x g|)(t, X^t(x,v) + \tau n, v_*)dxdvdv_*d\tau.$$

For $J_1(t)$, it follows from Lemma 3.4 that

$$\begin{aligned}
J_1(t) = &\int_{\mathbf{R}^3_x \times \mathbf{R}^3_v} \mathcal{S}^\#(|f-g|, f+g)(t,x,v)dxdv \\
&\quad \int_{\mathbf{R}^3_{v_*} \times \mathbf{R}^+_\tau} |v_\infty(x,v) + v - v_*|^{\delta_2 - 1}(f+g)(t, X^t(x,v) + \tau n, v_*)dv_*d\tau \\
\le\ & \mathcal{O}(1)\delta \int_{\mathbf{R}^3_x \times \mathbf{R}^3_v} \mathcal{S}^\#(|f-g|, f+g)(t,x,v)dxdv \\
&\quad \int_{\mathbf{R}^3_{v_*} \times \mathbf{R}^+_\tau} |v_\infty(x,v) + v - v_*|^{\delta_2 - 1}h_\alpha(|X(0;t,X^t(x,v) + \tau n, v_*)|) \\
&\qquad m_\beta(|V(0;t,X^t(x,v) + \tau n, v_*)|)dv_*d\tau \\
\le\ & \mathcal{O}(1)\delta \int_{\mathbf{R}^3_x \times \mathbf{R}^3_v} \mathcal{S}^\#(|f-g|, f+g)(t,x,v)dxdv \\
\le\ & \mathcal{O}(1)\delta\Lambda_{0,\delta_2}(t).
\end{aligned} \tag{3.30}$$

Moreover, we have from Lemmas 2.1, 3.2 and 3.3 that

$$\begin{aligned}
\Lambda_{0,\delta_2}(t) = &\int_{\mathbf{R}^3_x \times \mathbf{R}^3_v} |f-g|(t,x,v)dxdv \int_{\mathbf{R}^3_{v_*}} (2 + |v - v_*|^{\delta_2})(f+g)(t,x,v_*)dv_* \\
= &\int_{\mathbf{R}^3_x \times \mathbf{R}^3_v} |f-g|^\#(t,x,v)dxdv \\
&\quad \int_{\mathbf{R}^3_{v_*}} (2 + |V^t(x,v) - v_*|^{\delta_2})(f+g)(t, X^t(x,v), v_*)dv_* \\
\le\ & \mathcal{O}(1) \int_{\mathbf{R}^3_x \times \mathbf{R}^3_v} |f-g|^\#(t,x,v)dxdv \int_{\mathbf{R}^3_{v_*}} (f+g)(t, X^t(x,v), v_*) \\
&\quad \left(1 + |v_\infty(x,v)|^{\delta_2} + |V^t(x,v) - v|^{\delta_2} + |v_\infty(x,v) + v - v_*|^{\delta_2}\right) dv_* \\
\le\ & \mathcal{O}(1)\Lambda_{\delta_2}(t) + \mathcal{O}(1)\int_{\mathbf{R}^3_x \times \mathbf{R}^3_v \times \mathbf{R}^3_{v_*}} |f-g|(t,x,v)(f+g)(t,x,v_*)dxdvdv_* \\
\le\ & \mathcal{O}(1)\Lambda_{\delta_2}(t) + \frac{\mathcal{O}(1)\delta}{(1+t)^3}\mathcal{L}(t).
\end{aligned} \tag{3.31}$$

By putting (3.31) into (3.30), we have

$$J_1(t) \le \mathcal{O}(1)\delta\Lambda_{\delta_2}(t) + \frac{\mathcal{O}(1)\delta}{(1+t)^3}\mathcal{L}(t). \tag{3.32}$$

From Lemmas 2.5 and 3.4, $J_2(t)$ is estimated as follows:

$$\begin{aligned}
J_2(t) = &\int_{\mathbf{R}_x^3\times\mathbf{R}_v^3} |f-g|(t,x,v)dxdv \int_{\mathbf{R}_{v_*}^3\times\mathbf{R}_\tau^+} |v_\infty(x,v)+v-v_*|^{\delta_2-1}\\
&\qquad (\mathcal{S}(f,f)+\mathcal{S}(g,g))(t,X^t(x,v)+\tau n,v_*)dv_*d\tau\\
\le &\frac{\mathcal{O}(1)\delta^2}{(1+t)^{\min\{3+\delta_1,2\}}} \int_{\mathbf{R}_x^3\times\mathbf{R}_v^3} |f-g|(t,x,v)dxdv \int_{\mathbf{R}_{v_*}^3\times\mathbf{R}_\tau^+} |v_\infty(x,v)+v-v_*|^{\delta_2-1}\\
&\qquad h_{\alpha-1/2}\left(\left|X(0;t,X^t(x,v)+\tau n,v_*)\right|\right)\\
&\qquad m_{\beta-\eta}\left(\left|V(0;t,X^t(x,v)+\tau n,v_*)\right|\right)dv_*d\tau\\
\le &\frac{\mathcal{O}(1)\delta^2}{(1+t)^{\min\{3+\delta_1,2\}}}\mathcal{L}(t).
\end{aligned} \tag{3.33}$$

Similarly, for $J_3(t)$, we have

$$\begin{aligned}
J_3(t) = &\|E(t)\|_\infty \int_{\mathbf{R}_x^3\times\mathbf{R}_v^3} |f-g|^{\#}(t,x,v)dxdv\\
&\int_{\mathbf{R}_{v_*}^3\times\mathbf{R}_\tau^+} |v_\infty(x,v)+v-v_*|^{\delta_2-1}(|\nabla_v f|+|\nabla_v g|)(t,X^t(x,v)+\tau n,v_*)dv_*d\tau\\
\le &\mathcal{O}(1)\delta(1+t)\|E(t)\|_\infty \int_{\mathbf{R}_x^3\times\mathbf{R}_v^3} |f-g|^{\#}(t,x,v)dxdv\\
&\int_{\mathbf{R}_{v_*}^3\times\mathbf{R}_\tau^+} |v_\infty(x,v)+v-v_*|^{\delta_2-1}\\
&\qquad h_\alpha\left(\left|X(0;t,X^t(x,v)+\tau n,v_*)\right|\right) m_\beta\left(\left|V(0;t,X^t(x,v)+\tau n,v_*)\right|\right)dv_*d\tau\\
\le &\mathcal{O}(1)\delta(1+t)\|E(t)\|_\infty\mathcal{L}(t).
\end{aligned} \tag{3.34}$$

Finally, to estimate $J_4(t)$, by noticing that

$$\begin{aligned}
|V^t(x,v) - &v - v_\infty(x,v)|\\
&= \left|\int_0^t E(\theta,X^\theta(x,v),V^\theta(x,v))d\theta - \int_0^\infty E(\theta,X^\theta(x,v),V^\theta(x,v))d\theta\right|\\
&= \left|\int_t^\infty E(\theta,X^\theta(x,v),V^\theta(x,v))d\theta\right|\\
&\le \int_t^\infty \|E(\theta)\|_\infty d\theta,
\end{aligned} \tag{3.35}$$

we have

$$J_4(t) \le \mathcal{O}(1)\delta \int_t^\infty \|E(\theta)\|_\infty d\theta\mathcal{L}(t). \tag{3.36}$$

Combining (3.29) and (3.32)-(3.36) gives (3.25). Thus the proof of Lemma 3.5 is completed.

Corollary 3.1. *Suppose that the conditions of Theorem 1.2 hold. For the case when $\delta_1 = 0$, $0 < \delta_2 \le 1$, $\alpha > 3$ and $\beta > 4$, if $\delta > 0$ is sufficiently small, then we have the L^1 stability estimate (1.17).*

Proof. Under the conditions in Corollary 3.1, it follows from Lemma 3.2 and the inequality (3.31) that

$$\frac{d\mathcal{L}(t)}{dt} \le \mathcal{O}(1)\Lambda_{0,\delta_2}(t) \le \mathcal{O}(1)\Lambda_{\delta_2}(t) + \frac{\mathcal{O}(1)\delta}{(1+t)^3}\mathcal{L}(t). \tag{3.37}$$

As in [19], we construct the nonlinear functional

$$\mathcal{H}(t) = \mathcal{L}(t) + K\mathcal{D}_{\delta_2}(t), \tag{3.38}$$

where K is a positive constant to be determined later. By Lemma 3.5 and (3.37), we have

$$\frac{d\mathcal{H}(t)}{dt} \le \left(\mathcal{O}(1) - K(1-\mathcal{O}(1)\delta)\right)\Lambda_{\delta_2}(t) + K\lambda_1(t)\mathcal{H}(t). \tag{3.39}$$

Since $0 < \delta \ll 1$, we can choose K sufficiently large independent of δ such that

$$\frac{d\mathcal{H}(t)}{dt} + \Lambda_{\delta_2}(t) \le \mathcal{O}(1)\lambda_1(t)\mathcal{H}(t). \tag{3.40}$$

Recall the definition of $\lambda_1(t)$ in (3.26). By using the inequality (1.19), we have

$$\begin{aligned}
\int_0^t \lambda_1(s)ds &\le \mathcal{O}(1)\delta + \delta\int_0^t\int_s^\infty \|E(\theta)\|_\infty d\theta ds \\
&= \mathcal{O}(1)\delta + \delta t\int_t^\infty \|E(\theta)\|_\infty d\theta + \delta\int_0^t s\|E(s)\|_\infty ds \\
&\le \mathcal{O}(1)\delta + 2\delta\int_0^\infty \theta\|E(\theta)\|_\infty d\theta \\
&\le \mathcal{O}(1)\delta.
\end{aligned} \tag{3.41}$$

Hence (3.40), (3.41) together with the Gronwall's inequality yield

$$\mathcal{H}(t) + \int_0^t \Lambda_{\delta_2}(s)ds \le \mathcal{O}(1)\mathcal{H}(0). \tag{3.42}$$

From Remark 3.3 and the definition of $\mathcal{H}(t)$ in (3.38), $\mathcal{H}(t)$ is equivalent with the L^1 distance of two solutions, that is,

$$\|f(t)-g(t)\|_1 \le \mathcal{H}(t) \le \mathcal{O}(1)\|f(t)-g(t)\|_1, \quad \forall\ t \ge 0. \tag{3.43}$$

Thus,

$$\|f(t)-g(t)\|_1 \le \mathcal{H}(t) \le \mathcal{O}(1)\mathcal{H}(0) \le \mathcal{O}(1)\|f_0-g_0\|_1. \tag{3.44}$$

And the proof of Corollary 3.1 is completed.

Now we study the L^1 stability of solutions to the Boltzmann equation in the Subcase 2.2 which follows from Lemmas 3.2 and 3.5.

Corollary 3.2. *Suppose that the conditions of Theorem 1.2 hold. For the case when $-2 < \delta_1 < 0$, $0 < \delta_2 \le 1$, $\alpha > 3$ and $\beta > 4$, if $\delta > 0$ is sufficiently small, then we have the L^1 stability estimate (1.17).*

Proof. First, Lemma 3.2 and (3.31) imply

$$
\begin{aligned}
\Lambda_{\delta_1,\delta_2}(t) &= \int_{\mathbf{R}^3_x\times\mathbf{R}^3_v\times\mathbf{R}^3_{v_*}} b_{\delta_1,\delta_2}(|v-v_*|)|f-g|(t,x,v)(f+g)(t,x,v_*)dxdvdv_*\\
&\leq \mathcal{O}(1)\int_{\mathbf{R}^3_x\times\mathbf{R}^3_v\times\mathbf{R}^3_{v_*}} [b_{\delta_1,0}(|v-v_*|)+b_{0,\delta_2}(|v-v_*|)]\\
&\qquad\qquad |f-g|(t,x,v)(f+g)(t,x,v_*)dxdvdv_*\\
&\leq \left\{\frac{\mathcal{O}(1)\delta}{(1+t)^{3+\delta_1}}+\frac{\mathcal{O}(1)\delta}{(1+t)^3}\right\}\mathcal{L}(t)+\mathcal{O}(1)\Lambda_{\delta_2}(t).
\end{aligned}\tag{3.45}
$$

Hence, by Lemma 3.1, we have

$$
\frac{d\mathcal{L}(t)}{dt}\leq \mathcal{O}(1)\Lambda_{\delta_1,\delta_2}(t)\leq \mathcal{O}(1)\Lambda_{\delta_2}(t)+\left\{\frac{\mathcal{O}(1)\delta}{(1+t)^{3+\delta_1}}+\frac{\mathcal{O}(1)\delta}{(1+t)^3}\right\}\mathcal{L}(t).\tag{3.46}
$$

To control the term $\Lambda_{\delta_2}(t)$ on the right hand of (3.46) for the case when $-2<\delta_1<0$ and $0<\delta_2\leq 1$, similar to (3.29), we have again from (3.45) that

$$
\begin{aligned}
\frac{d\mathcal{D}_{\delta_2}(t)}{dt} &\leq -\Lambda_{\delta_2}(t)+\sum_{i=1}^4 J_i(t)\\
&\leq -\Lambda_{\delta_2}(t)+\mathcal{O}(1)\delta\Lambda_{\delta_1,\delta_2}(t)+\sum_{i=2}^4 J_i(t)\\
&\leq -(1-\mathcal{O}(1)\delta)\Lambda_{\delta_2}(t)+\mathcal{O}(1)\delta\lambda_2(t)\mathcal{L}(t),
\end{aligned}\tag{3.47}
$$

where

$$
\lambda_2(t)=\frac{\delta}{(1+t)^{3+\delta_1}}+\lambda_1(t).\tag{3.48}
$$

Since $3+\delta_1>1$, we have

$$
\int_0^t \lambda_2(s)ds\leq \mathcal{O}(1)\delta.\tag{3.49}
$$

Therefore, the L^1 stability estimate (1.17) can be obtained by the same argument as Corollary 3.1. And this completes the proof of the corollary.

4 BV-type stability

In this section, we will consider the BV-type estimate (1.18) by the following series of lemmas. Throughout this section, we assume all the conditions in Theorem 1.2 hold, and f is the classical solution to the Boltzmann equation in Theorem 1.1.

First, the proof for the case of the soft potentials is direct as for the L^1 stability.

Lemma 4.1. *Suppose that the conditions of Theorem 1.2 hold. Consider the case when $-2<\delta_1\leq 0$ and $\delta_2=0$. If $\delta>0$ is sufficiently small, then BV-type estimate (1.18) holds.*

Proof. Since f is the classical solution to the Boltzmann equation

$$
\partial_t f+v\cdot\nabla_x f+E(t,x,v)\cdot\nabla_v f=J(f,f),
$$

we have by taking x_i and v_i derivatives on both sides that

$$\begin{aligned}\partial_t(\partial_{x_i} f) &+v\cdot\nabla_x(\partial_{x_i} f)+E(t,x,v)\cdot\nabla_v(\partial_{x_i} f)\\ &= J(\partial_{x_i} f,f)+J(f,\partial_{x_i} f)-\partial_{x_i}E(t,x,v)\cdot\nabla_v f,\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\partial_t(\partial_{v_i} f) &+v\cdot\nabla_x(\partial_{v_i} f)+E(t,x,v)\cdot\nabla_v(\partial_{v_i} f)\\ &= J(\partial_{v_i} f,f)+J(f,\partial_{v_i} f)-\partial_{v_i}E(t,x,v)\cdot\nabla_v f-\partial_{x_i} f.\end{aligned} \tag{4.2}$$

After integration along the forward bi-characteristics, we have

$$\begin{aligned}|\partial_{x_i} f|^{\#}(t) \le |\partial_{x_i} f_0| &+ \int_0^t \mathcal{S}^{\#}(|\partial_{x_i} f|,f)(s)ds\\ &+\int_0^t \|\partial_{x_i}E(s)\|_\infty |\nabla_v f|^{\#}(s)ds,\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}|\partial_{v_i} f|^{\#}(t) \le |\partial_{v_i} f_0| &+ \int_0^t \mathcal{S}^{\#}(|\partial_{v_i} f|,f)(s)ds\\ &+\int_0^t \|\partial_{v_i}E(s)\|_\infty |\nabla_v f|^{\#}(s)ds+\int_0^t |\partial_{x_i} f|^{\#}(s)ds.\end{aligned} \tag{4.4}$$

Since $\delta_2=0$, similar to Lemma 3.2, we have

$$\int_{\mathbf{R}^3_x\times\mathbf{R}^3_v} \mathcal{S}^{\#}(|\partial f|,f)(t)dxdv \le \frac{\mathcal{O}(1)\delta}{(1+t)^{3+\delta_1}}\|\partial f(t)\|_1, \quad \partial=\partial_{x_i} \text{ or } \partial_{v_i}. \tag{4.5}$$

Hence, integrating (4.3) and (4.4) over $\mathbf{R}^3_x\times\mathbf{R}^3_v$ with respect to (x,v) yields

$$\begin{aligned}\|\partial_{x_i} f(t)\|_1 \le \|\partial_{x_i} f_0\|_1 &+ \int_0^t \frac{\mathcal{O}(1)\delta}{(1+s)^{3+\delta_1}}\|\partial_{x_i} f(s)\|_1 ds\\ &+\int_0^\infty \|(1+s)\partial_{x_i}E(s)\|_\infty ds \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1,\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\|\partial_{v_i} f(t)\|_1 \le \|\partial_{v_i} f_0\|_1 &+ (1+t)\int_0^t \frac{\mathcal{O}(1)\delta}{(1+s)^{3+\delta_1}}\|(1+s)^{-1}\partial_{v_i} f(s)\|_1 ds\\ &+(1+t)\int_0^\infty \|\partial_{v_i}E(s)\|_\infty ds \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1\\ &+(1+t)\sup_t \|\partial_{x_i} f(t)\|_1.\end{aligned} \tag{4.7}$$

By the Gronwall's inequality, it follows from (4.6) that

$$\begin{aligned}\|\partial_{x_i} f(t)\|_1 &\le \mathcal{O}(1)\|\partial_{x_i} f_0\|_1\\ &+\exp\{\mathcal{O}(1)\delta\}\int_0^\infty \|(1+s)\partial_{x_i}E(s)\|_\infty ds \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1.\end{aligned} \tag{4.8}$$

By putting the above inequality into (4.7) and using (1.14), we obtain

$$\begin{aligned}\|(1+t)^{-1}\nabla_v f\|_1 \le \mathcal{O}(1)(\|\nabla_x f_0\|_1 &+ \|\nabla_v f_0\|_1)\\ &+\varepsilon_1 \exp\{\mathcal{O}(1)\delta\}\sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1\\ &+\int_0^t \frac{\mathcal{O}(1)\delta}{(1+s)^{3+\delta_1}}\|(1+s)^{-1}\nabla_v f(s)\|_1 ds.\end{aligned} \tag{4.9}$$

Again the Gronwall's inequality yields

$$
\begin{aligned}
\|(1+t)^{-1}\nabla_v f\|_1 \le\ & \mathcal{O}(1)(\|\nabla_x f_0\|_1 + \|\nabla_v f_0\|_1)\\
&+\varepsilon_1 \exp\{\mathcal{O}(1)\delta\} \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1. \qquad (4.10)
\end{aligned}
$$

Since $0 < \varepsilon_1 < 1$, we can choose $\delta > 0$ sufficiently small such that

$$
\varepsilon_1 \exp\{\mathcal{O}(1)\delta\} < 1. \qquad (4.11)
$$

Then it follows from (4.10) that

$$
\sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1 \le \mathcal{O}(1)(\|\nabla_x f_0\|_1 + \|\nabla_v f_0\|_1). \qquad (4.12)
$$

By the above inequality and (4.8), we have

$$
\sup_t \|\nabla_x f(t)\|_1 \le \mathcal{O}(1)(\|\nabla_x f_0\|_1 + \|\nabla_v f_0\|_1). \qquad (4.13)
$$

Thus BV-type estimate (1.18) holds if $-2 < \delta_1 \le 0$ and $\delta_2 = 0$, and this completes the proof of Lemma 4.1.

For the case of the hard potentials, we need to use the following nonlinear functionals:

$$
\begin{aligned}
\Lambda_{\delta_2}^{\mathrm{BV}(x_i)}(t) = & \int_{\mathbf{R}_x^3\times\mathbf{R}_v^3} |\partial_{x_i} f|^{\#}(t,x,v)dxdv\\
&\int_{\mathbf{R}_{v_*}^3} |v_\infty(x,v)+v-v_*|^{\delta_2} f(t,X^t(x,v),v_*)dv_*,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_{\delta_2}^{\mathrm{BV}(x_i)}(t) = & \int_{\mathbf{R}_x^3\times\mathbf{R}_v^3} |\partial_{x_i} f|^{\#}(t,x,v)dxdv\\
&\int_{\mathbf{R}_{v_*}^3\times\mathbf{R}_\tau^+} |v_\infty(x,v)+v-v_*|^{\delta_2-1}\\
&\qquad f(t,X^t(x,v)+\tau n(v_\infty(x,v)+v-v_*),v_*)dv_*d\tau,
\end{aligned}
$$

and

$$
\begin{aligned}
\Lambda_{\delta_2}^{\mathrm{BV}(v_i)}(t) = & \int_{\mathbf{R}_x^3\times\mathbf{R}_v^3} |\partial_{v_i} f|^{\#}(t,x,v)dxdv\\
&\int_{\mathbf{R}_{v_*}^3} |v_\infty(x,v)+v-v_*|^{\delta_2} f(t,X^t(x,v),v_*)dv_*,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{D}_{\delta_2}^{\mathrm{BV}(v_i)}(t) = & \int_{\mathbf{R}_x^3\times\mathbf{R}_v^3} |\partial_{v_i} f|^{\#}(t,x,v)dxdv\\
&\int_{\mathbf{R}_{v_*}^3\times\mathbf{R}_\tau^+} |v_\infty(x,v)+v-v_*|^{\delta_2-1}\\
&\qquad f(t,X^t(x,v)+\tau n(v_\infty(x,v)+v-v_*),v_*)dv_*d\tau.
\end{aligned}
$$

Lemma 4.2. *Suppose that the conditions of Theorem 1.2 hold. For the case when $\delta_1 = 0$, $0 < \delta_2 \le 1$, $\alpha > 3$ and $\beta > 4$, if $\delta > 0$ is sufficiently small, then there exist constants K_x, $K_v > 0$ independent of δ such that*

$$
\begin{aligned}
&\|\partial_{x_i} f(t)\|_1 + K_x \mathcal{D}_{\delta_2}^{\mathrm{BV}(x_i)}(t) + \int_0^t \Lambda_{\delta_2}^{\mathrm{BV}(x_i)}(s)ds\\
&\le \mathcal{O}(1)\|\partial_{x_i} f_0\|_1\\
&\quad +(1+\mathcal{O}(1)\delta)\exp\{\mathcal{O}(1)\delta\}\int_0^\infty \|(1+s)\partial_{x_i} E(s)\|_\infty ds \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1, \qquad (4.14)
\end{aligned}
$$

and

$$
\begin{aligned}
\|(1+t)^{-1}\partial_{v_i}f(t)\|_1 + &\frac{K_v}{1+t}\mathcal{D}_{\delta_2}^{\mathrm{BV}(v_i)}(t) + \frac{1}{1+t}\int_0^t \Lambda_{\delta_2}^{\mathrm{BV}(v_i)}(s)ds \\
&\le \mathcal{O}(1)\|\partial_{v_i}f_0\|_1 + (1+\mathcal{O}(1)\delta)\sup_t \|\partial_{x_i}f(t)\|_1 \\
&\quad + \mathcal{O}(1)\delta\int_0^\infty \|\partial_{v_i}E(s)\|_\infty ds \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1 \\
&\quad + \mathcal{O}(1)\int_0^t \lambda_1(s)\|(1+s)^{-1}\partial_{v_i}f(s)\|_1 ds, \qquad (4.15)
\end{aligned}
$$

where $\lambda_1(t)$ *is defined by (3.26).*

Proof. By integrating (4.3) over $\mathbf{R}_x^3 \times \mathbf{R}_v^3$ with respect to (x,v), similar to the proof of Lemma 3.5, we have

$$
\begin{aligned}
\|\partial_{x_i}f(t)\|_1 \le \|\partial_{x_i}f_0\|_1 &+ \mathcal{O}(1)\int_0^t \Lambda_{\delta_2}^{\mathrm{BV}(x_i)}(s)ds \\
&+ \int_0^t \frac{\mathcal{O}(1)\delta}{(1+s)^3}\|\partial_{x_i}f(s)\|_1 ds \\
&+ \int_0^\infty \|(1+s)\partial_{x_i}E(s)\|_\infty ds \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1, \qquad (4.16)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}_{\delta_2}^{\mathrm{BV}(x_i)}(t) \le \mathcal{D}_{\delta_2}^{\mathrm{BV}(x_i)}(0) &- (1-\mathcal{O}(1)\delta)\int_0^t \Lambda_{\delta_2}^{\mathrm{BV}(x_i)}(s)ds \\
&+ \mathcal{O}(1)\int_0^t \lambda_1(s)\|\partial_{x_i}f(s)\|_1 ds \\
&+ \mathcal{O}(1)\delta\int_0^\infty \|(1+s)\partial_{x_i}E(s)\|_\infty ds \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1. \qquad (4.17)
\end{aligned}
$$

If $\delta > 0$ is sufficiently small, then there exists $K_x > 0$ such that

$$
\begin{aligned}
\|\partial_{x_i}f(t)\|_1 + &K_x\mathcal{D}_{\delta_2}^{\mathrm{BV}(x_i)}(t) + \int_0^t \Lambda_{\delta_2}^{\mathrm{BV}(x_i)}(s)ds \\
&\le \|\partial_{x_i}f_0\|_1 + K_x\mathcal{D}_{\delta_2}^{\mathrm{BV}(x_i)}(0) \\
&\quad + (1+\mathcal{O}(1)\delta)\int_0^\infty \|(1+s)\partial_{x_i}E(s)\|_\infty ds \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1 \\
&\quad + \mathcal{O}(1)\int_0^t \lambda_1(s)\|\partial_{x_i}f(s)\|_1 ds. \qquad (4.18)
\end{aligned}
$$

Notice that

$$
\int_0^\infty \lambda_1(s)ds \le \mathcal{O}(1)\delta \quad \text{and} \quad \mathcal{D}_{\delta_2}^{\mathrm{BV}(x_i)}(0) \le \mathcal{O}(1)\|\partial_{x_i}f_0\|_1. \qquad (4.19)
$$

(4.18) together with the Gronwall's inequality give (4.14).

Similarly, we have that

$$
\begin{aligned}
\|(1+t)^{-1}\partial_{v_i}f(t)\|_1 \leq \|\partial_{v_i}f_0\|_1 &+ \frac{\mathcal{O}(1)}{1+t}\int_0^t \Lambda_{\delta_2}^{\mathrm{BV}(v_i)}(s)ds \\
&+\int_0^t \frac{\mathcal{O}(1)\delta}{(1+s)^3}\|(1+s)^{-1}\partial_{v_i}f(s)\|_1 ds \\
&+\mathcal{O}(1)\delta\int_0^\infty \|\partial_{v_i}E(s)\|_\infty ds \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1 \\
&+\sup_t \|\partial_{x_i}f(t)\|_1, \qquad (4.20)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{1+t}\mathcal{D}_{\delta_2}^{\mathrm{BV}(v_i)}(t) \leq \mathcal{D}_{\delta_2}^{\mathrm{BV}(v_i)}(0) &- \frac{1-\mathcal{O}(1)\delta}{1+t}\int_0^t \Lambda_{\delta_2}^{\mathrm{BV}(v_i)}(s)ds \\
&+\mathcal{O}(1)\int_0^t \lambda_1(s)\|(1+s)^{-1}\partial_{v_i}f(s)\|_1 ds \\
&+\mathcal{O}(1)\delta\int_0^\infty \|\partial_{v_i}E(s)\|_\infty ds \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1 \\
&+\mathcal{O}(1)\delta \sup_t \|\partial_{x_i}f(t)\|_1. \qquad (4.21)
\end{aligned}
$$

Therefore, when $\delta>0$ is sufficiently small, there exists $K_v>0$ such that (4.15) holds. The proof of Lemma 4.2 is completed.

Corollary 4.1. *Under the assumptions of Lemma 4.2, if $\delta>0$ is sufficiently small, then BV-type estimate (1.18) holds.*

Proof. By putting (4.14) into (4.15) and using (1.14), we have

$$
\begin{aligned}
&\|(1+t)^{-1}\nabla_v f(t)\|_1 + \frac{K_v}{1+t}\sum_{i=1}^3 \mathcal{D}_{\delta_2}^{\mathrm{BV}(v_i)}(t) + \frac{1}{1+t}\int_0^t \sum_{i=1}^3 \Lambda_{\delta_2}^{\mathrm{BV}(v_i)}(s)ds \\
&\leq \mathcal{O}(1)(\|\nabla_x f_0\|_1 + \|\nabla_v f_0\|_1) \\
&\quad +(1+\mathcal{O}(1)\delta)^2\exp\{\mathcal{O}(1)\delta\} \\
&\qquad\qquad \int_0^\infty (\|\nabla_v E(s)\|_\infty + (1+s)\|\nabla_x E(s)\|_\infty)\, ds \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1 \\
&\quad +\int_0^t \lambda_1(s)\|(1+s)^{-1}\partial_{v_i}f(s)\|_1 ds \\
&\leq \mathcal{O}(1)(\|\nabla_x f_0\|_1 + \|\nabla_v f_0\|_1) + (1+\mathcal{O}(1)\delta)^2\exp\{\mathcal{O}(1)\delta\}\varepsilon_1 \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1 \\
&\quad +\int_0^t \lambda_1(s)\|(1+s)^{-1}\nabla_v f(s)\|_1 ds. \qquad (4.22)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\|(1+t)^{-1}\nabla_v f(t)\|_1 &\leq \mathcal{O}(1)(\|\nabla_x f_0\|_1 + \|\nabla_v f_0\|_1) \\
&\quad +(1+\mathcal{O}(1)\delta)^2\exp\{\mathcal{O}(1)\delta\}\varepsilon_1 \sup_t \|(1+t)^{-1}\nabla_v f(t)\|_1. \qquad (4.23)
\end{aligned}
$$

Since $0<\varepsilon_1<1$, we can choose $\delta>0$ sufficiently small such that

$$
(1+\mathcal{O}(1)\delta)^2\exp\{\mathcal{O}(1)\delta\}\varepsilon_1 < 1. \qquad (4.24)
$$

Thus, this gives

$$\|(1+t)^{-1}\nabla_v f(t)\|_1 \leq \mathcal{O}(1)(\|\nabla_x f_0\|_1 + \|\nabla_v f_0\|_1). \tag{4.25}$$

Combining (4.14) and (4.25) yields (1.18). The proof of Corollary 4.1 is completed.

Finally, based on Lemma 4.2 and Corollary 4.1, we have the BV-type estimate (1.18) for the general potential by the argument used in Corollary 4.1.

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