

ON THE ILL-POSEDNESS OF THE PRANDTL EQUATIONS IN THREE SPACE DIMENSIONS

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ABSTRACT. In this paper, we give an instability criterion for the Prandtl equations in three space variables, which shows that the monotonicity condition of tangential velocity fields is not sufficient for the well-posedness of the three dimensional Prandtl equations, in contrast to the classical well-posedness theory of the Prandtl equations in two space variables under the Oleinik monotonicity assumption of the tangential velocity. Both of linear stability and nonlinear stability are considered. This criterion shows that the monotonic shear flow is linear stable for the three dimensional Prandtl equations if and only if the tangential velocity field direction is invariant with respect to the normal variable, and this result is an exact complement to our recent work [8] on the well-posedness theory for the three dimensional Prandtl equations with special structure.

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1. INTRODUCTION

The inviscid limit of the viscous flow has been known as a challenging mathematical problem that contains many unsolved problems. For the incompressible Navier-Stokes equations confined in a domain with boundary, in particular with the non-slip boundary condition, the justification of the inviscid limit remains basically open, c.f. [2] and references therein. The main obstruction comes from the formation of boundary layers near the physical boundary, in which the tangential velocity component changes dramatically.

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The foundation of the boundary layer theories was established by Prandtl [13] in 1904 when he introduced the classical Prandtl equations by considering the incompressible Navier-Stokes equations with non-slip boundary condition. His observation reveals that outside the layer of thickness of $\sqrt{\nu}$ with ν being the viscosity coefficient, the convection dominates so that the flow can be described approximately by the incompressible Euler equations, however, within the layer of thickness of $\sqrt{\nu}$ in the vicinity of the boundary, the convection and viscosity balance so that the flow is governed by the Prandtl equations that is degenerate and mixed type. Since then, there have been a lot of mathematical studies on the Prandtl equations, however, the existing theories are basically limited to the two space dimensional case except the one in analytic framework by Sammartino and Caffisch [14] and others [17]. On the other hand, in two dimensional space, the classical work by Oleinik and her collaborators [12] gives the local in time well-posedness when the tangential velocity component is monotone in the normal direction, by using the Crocco transformation. Recently, this well-posedness result of the two dimensional Prandtl equations is re-studied in [1, 10] by direct energy method. In addition to the monotone condition on the velocity, if a favorable pressure condition is imposed, then global in time weak solution was also obtained in two dimensional space, see [16].

The stability mechanism of the three dimensional Prandtl equations is very challenging and delicate mainly due to the possible appearance of secondary flows in the three dimensional boundary layer flow as explained in Moore [11], and it is an open question proposed by Oleinik and Samokhin in the monograph [12]. Recently, in [8] the authors construct a local solution to the three dimensional Prandtl equations when the tangential velocity field direction is invariant with respect to the normal variable under certain monotonicity condition. In addition, this special boundary layer flow is linearly stable with respect to any perturbation, and the global in time weak solution is also obtained under an additional favorable pressure condition [9].

The purpose of this paper is to investigate the instability of boundary layer flows in three space dimensions without the special structure proposed in [8], even when the two tangential velocity components of the background state are monotonic. This reveals the essential difference of the Prandtl equations between two and three space dimensions. For this, let us first review the recent extensive studies on the instability of the two space dimensional flow around a background state of shear flow with non-monotonicity.

In fact, without the monotonicity assumption on the tangential component of the velocity, boundary separation will occur. For this, there are many physical observations and mathematical studies. For example, Van Dommelen and Shen in [15] illustrated the “Van Dommelen singularity” by considering an impulsively started circular cylinder to show the blowup of the normal velocity, and E and Enquist in [3] precisely constructed some finite time blowup solutions to the two-dimensional Prandtl equations. Started by Grenier’s work in 2000, there are some extensive investigation on the instability of the two-dimensional Prandtl equations when the background shear flow has some degeneracy. Precisely, corresponding to the well known Rayleigh criterion for the Euler flow, Grenier [6] showed that the unstable Euler shear flow yields instability of the Prandtl equations. It was shown in [4] that a non-degenerate critical point in the shear flow of the Prandtl equations leads to a strong linear ill-posedness of the Prandtl equations in the Sobolev space

framework. Moreover, [5] strengthens the result of [4] for any unstable shear flow. Along this direction, the ill-posedness in the nonlinear setting was proved in [7] to show that the Prandtl equations are ill-posed near non-stationary and non-monotonic shear flows so that the asymptotic boundary layer expansion is not valid for non-monotonic shear layer flows in Sobolev spaces.

To describe the problem to be studied in this paper, consider the following incompressible Navier-Stokes equations

$$(1.1) \quad \begin{cases} \partial_t \mathbf{u}^\nu + (\mathbf{u}^\nu \cdot \nabla) \mathbf{u}^\nu + \nabla p^\nu - \nu \Delta \mathbf{u}^\nu = 0, \\ \nabla \cdot \mathbf{u}^\nu = 0, \\ \mathbf{u}^\nu|_{z=0} = 0, \end{cases}$$

in $\{t > 0, (x, y) \in \mathbb{T}^2, z \in \mathbb{R}^+\}$ with boundary at $\{z = 0\}$, here $\mathbf{u}^\nu = (u^\nu, v^\nu, w^\nu)^T$. According to Prandtl's observation, set the ansatz for \mathbf{u}^ν near $\{z = 0\}$ as

$$(1.2) \quad \begin{cases} u^\nu(t, x, y, z) = u(t, x, y, \frac{z}{\sqrt{\nu}}) + o(1), \\ v^\nu(t, x, y, z) = v(t, x, y, \frac{z}{\sqrt{\nu}}) + o(1), \\ w^\nu(t, x, y, z) = \sqrt{\nu} w(t, x, y, \frac{z}{\sqrt{\nu}}) + o(\sqrt{\nu}), \end{cases}$$

and plug it in the Navier-Stokes equations (1.1), one finds that the boundary layer profile $(u, v, w)(t, x, y, z)$ (here we replace $\frac{z}{\sqrt{\nu}}$ by z for simplicity of notations) satisfies

$$(1.3) \quad \begin{cases} \partial_t u + (u \partial_x + v \partial_y + w \partial_z) u + \partial_x p^E(t, x, y, 0) = \partial_z^2 u, \\ \partial_t v + (u \partial_x + v \partial_y + w \partial_z) v + \partial_y p^E(t, x, y, 0) = \partial_z^2 v, \\ \partial_x u + \partial_y v + \partial_z w = 0, \\ (u, v, w)|_{z=0} = 0, \quad \lim_{z \rightarrow +\infty} (u, v) = (u^E, v^E)(t, x, y, 0), \end{cases}$$

which is the famous Prandtl layer equations. Here, the pressure p^E is related to the outer Euler flow $\mathbf{u}^E = (u^E, v^E, 0)(t, x, y, 0)$ through

$$\partial_t \mathbf{u}^E + (\mathbf{u}^E \cdot \nabla) \mathbf{u}^E + \nabla p^E = 0.$$

The main results of this paper show that when the background state is a shear flow $(u^s(t, z), v^s(t, z), 0)$ of (1.3) with initial data $(U_s(z), V_s(z))$, even under the monotonicity condition that $U'_s(z), V'_s(z) > 0$, the Prandtl equations (1.3) are both linearly and nonlinearly unstable under a very general assumption that

$$(1.4) \quad \exists z_0 > 0, \text{ s.t. } \frac{d}{dz} \left(\frac{V'_s}{U'_s} \right) (z_0) \neq 0 \quad \text{or} \quad \frac{d}{dz} \left(\frac{U'_s}{V'_s} \right) (z_0) \neq 0.$$

Note that in [8], existence of solutions to the three space dimensional Prandtl equations was proved with special structure and in that case, the tangential components of the solution $(u, v, w)(t, x, y, z)$ satisfies $\frac{\partial}{\partial z} \left(\frac{v}{u} \right) \equiv 0$. In fact, in this case, the appearance of the secondary flow that is the key factor in instability is avoided. Thus, by combining with the results obtained in [8], we know that the condition (1.4) is not only sufficient but also necessary for the linear instability of the three dimensional Prandtl equation (1.3) linearized around the monotonic shear flow $(u^s(t, z), v^s(t, z), 0)$.

The rest of the paper will be organized as follows. In Section 2, we first state the main results on the linear and nonlinear instability of the three-dimensional Prandtl equations with background state as monotonic shear flow. Then, we prove the linear

instability result of the shear flow in Section 3, and the nonlinear instability will be studied in Section 4. In the Appendix, we give a well-posedness result for the linearized three-dimensional Prandtl equations in the analytic setting with respect to only one horizontal variable, under the assumption that one component of the tangential velocity field of background shear flow is monotonic.

2. MAIN RESULTS

By assuming that the outer Euler flow is uniform in (1.3), consider the following boundary value problem of three dimensional Prandtl equations in $\Omega \triangleq \{(t, x, y, z) : t > 0, (x, y) \in \mathbb{T}^2, z \in \mathbb{R}^+\}$,

$$(2.1) \quad \begin{cases} \partial_t u + (u\partial_x + v\partial_y + w\partial_z)u - \partial_z^2 u = 0, \\ \partial_t v + (u\partial_x + v\partial_y + w\partial_z)v - \partial_z^2 v = 0, \\ \partial_x u + \partial_y v + \partial_z w = 0, \\ (u, v, w)|_{z=0} = 0, \quad \lim_{z \rightarrow +\infty} (u, v) = (U_0, V_0) \end{cases}$$

for positive constants U_0 and V_0 . To understand this problem, we start with the simple situation of shear flow. Let $u^s(t, z)$ and $v^s(t, z)$ be smooth solutions of the heat equations:

$$(2.2) \quad \begin{cases} \partial_t u^s - \partial_z^2 u^s = 0, & \partial_t v^s - \partial_z^2 v^s = 0, \\ (u^s, v^s)|_{z=0} = 0, & \lim_{z \rightarrow +\infty} (u^s, v^s) = (U_0, V_0), \\ (u^s, v^s)|_{t=0} = (U_s, V_s)(z), \end{cases}$$

with $(u^s - U_0, v^s - V_0)$ rapidly tending to 0 when $z \rightarrow +\infty$. It is straightforward to check that the shear velocity profile $(u^s, v^s, 0)(t, z)$ satisfies the problem (2.1).

The question we answer in this paper is whether such trivial profile is stable even when $u^s(t, z)$ and $v^s(t, z)$ are strictly monotonic in $z > 0$. For this, we first focus on the linear stability problem, and consider the linearization of the problem (2.1) around $(u^s, v^s, 0)$:

$$(2.3) \quad \begin{cases} \partial_t u + (u^s \partial_x + v^s \partial_y)u + w u_z^s - \partial_z^2 u = 0, & \text{in } \Omega, \\ \partial_t v + (u^s \partial_x + v^s \partial_y)v + w v_z^s - \partial_z^2 v = 0, & \text{in } \Omega, \\ \partial_x u + \partial_y v + \partial_z w = 0, & \text{in } \Omega, \\ (u, v, w)|_{z=0} = 0, & \lim_{z \rightarrow +\infty} (u, v) = 0. \end{cases}$$

To present the linear instability result that is motivated by the work [4] for two dimensional problem, we first introduce some notations. As in [4], for any $\alpha, m \geq 0$, denote by

$$\begin{aligned} L_\alpha^2(\mathbb{R}^+) &:= \{f = f(z), z \in \mathbb{R}^+; \|f\|_{L_\alpha^2} \triangleq \|e^{\alpha z} f\|_{L^2} < \infty\}, \\ H_\alpha^m(\mathbb{R}^+) &:= \{f = f(z), z \in \mathbb{R}^+; \|f\|_{H_\alpha^m} \triangleq \|e^{\alpha z} f\|_{H^m} < \infty\}, \\ W_\alpha^{m, \infty}(\mathbb{R}^+) &:= \{f = f(z), z \in \mathbb{R}^+; \|f\|_{W_\alpha^{m, \infty}} \triangleq \|e^{\alpha z} f\|_{W^{m, \infty}} < \infty\}, \end{aligned}$$

and the functional space for $\forall \beta > 0$:

$$E_{\alpha, \beta} := \left\{ f = f(x, y, z) = \sum_{k_1, k_2 \in \mathbb{Z}} e^{i(k_1 x + k_2 y)} f_{k_1, k_2}(z), \|f_{k_1, k_2}\|_{L_\alpha^2} \leq C_{\alpha, \beta} e^{-\beta \sqrt{k_1^2 + k_2^2}} \right\},$$

with

$$\|f\|_{E_{\alpha,\beta}} \triangleq \sup_{k_1, k_2 \in \mathbb{Z}} e^{\beta \sqrt{k_1^2 + k_2^2}} \|f_{k_1, k_2}\|_{L_{\alpha}^2}.$$

The same notations are also used for the vector functions without confusion.

As in [4], we first have the following existence result for the problem (2.3) when the data are analytic in the tangential variables (x, y) .

Proposition 1. *Let $(u^s - U_0, v^s - V_0) \in C(\mathbb{R}^+; W_{\alpha}^{1,\infty}(\mathbb{R}^+))$. Then, there exists a $\rho > 0$ such that for all T with $\beta - \rho T > 0$, and $(u_0, v_0) \in E_{\alpha,\beta}$, the linear problem (2.3) with the initial data $(u, v)|_{t=0} = (u_0, v_0)$ has a unique solution*

$$(u, v) \in C([0, T]; E_{\alpha, \beta - \rho T}), \quad (u, v)(t, \cdot) \in E_{\alpha, \beta - \rho t}.$$

The proof of this Proposition is the same as that given in [4, Proposition 1], so we omit it for brevity.

If we impose monotonic condition on the tangential velocity components of the shear flow $(u^s, v^s, 0)$, then another well-posedness result can be obtained. For this, similar to the previous notations, we introduce the following function spaces: for any $\alpha, \beta > 0$, set

$$(2.4) \quad K_{\alpha}^m := \left\{ f = f(x, z); \|f\|_{K_{\alpha}^m} \triangleq \|e^{\alpha z} f\|_{H^m(\mathbb{T}_x; L^2(\mathbb{R}_z^+))} < \infty \right\},$$

and

$$F_{\alpha, \beta}^m := \left\{ f = f(x, y, z) = \sum_{k \in \mathbb{Z}} e^{iky} f_k(x, z); \|f_k\|_{K_{\alpha}^m} \leq C_{\alpha, \beta} e^{-\beta|k|}, \forall k \right\}$$

with

$$\|f\|_{F_{\alpha, \beta}^m} \triangleq \sup_{k \in \mathbb{Z}} e^{\beta|k|} \|f_k\|_{K_{\alpha}^m}.$$

The following result shows that the linear problem (2.3) is still well-posed when the analyticity with respect to one horizontal variable given in Proposition 1 is replaced by some monotonicity assumption.

Proposition 2. *Let $(u^s - U_0, v^s - V_0) \in C(\mathbb{R}^+; W_{\alpha}^{3,\infty}(\mathbb{R}^+))$, $\alpha > 0$ satisfying $u_z^s > 0$ and*

$$\frac{u_{zz}^s}{u_z^s}, \frac{v_z^s}{u_z^s} \in C(\mathbb{R}^+; W^{1,\infty}(\mathbb{R}^+)),$$

and the initial data $(u, v)|_{t=0} = (u_0, v_0)$ of the problem (2.3) satisfying

$$(2.5) \quad (u_0, v_0)(x, y, z) \in F_{\alpha, \beta}^m, \quad \partial_z \left(\frac{u_0}{u_z^s(0, \cdot)} \right) \in F_{\alpha, \beta}^m.$$

Then, there exists a $\rho > 0$ such that for all T with $\beta - \rho T > 0$, the linear problem (2.3) has a unique solution $(u, v)(t, x, y, z)$ satisfying

$$(u, v) \in L^{\infty}(0, T; F_{\alpha, \beta - \rho T}^m), \quad \partial_z(u, v) \in L^2(0, T; F_{\alpha, \beta - \rho T}^m)$$

with $(u, v)(t, \cdot) \in F_{\alpha, \beta - \rho t}^m$.

The proof of this proposition will be given in the Appendix.

From the above Proposition 1 and Proposition 2, we know that when the monotonic condition is imposed to one tangential component of background velocity field, such as u^s , the analyticity requirement for the velocity field with respect to the corresponding horizontal variable x can be replaced by the Sobolev regularity, while the velocity field is still analytic in the other horizontal variable y , then one still

has the local in time well-posedness for the linearized system in three dimensional space.

Following this argument, it is natural and interesting to study whether the well-posedness of the linearized Prandtl equations (2.3) still holds in the Sobolev framework if one imposes monotonicity conditions on both tangential velocity components of background state but without analyticity assumption anymore. The study of this paper gives a negative answer to the question. In fact, the following theorem shows a strong linear instability of three dimensional Prandtl equations around basically shear flow in the Sobolev framework except those with special structure studied in [8].

To state the result, we need some more notations. Denote by the operator $T \in \mathcal{L}(E_{\alpha,\beta}, E_{\alpha,\beta'})$:

$$(2.6) \quad T(t, s)((u_0, v_0)) := (u, v)(t, \cdot),$$

where (u, v) is the solution of (2.3) with $(u, v)|_{t=s} = (u_0, v_0)$. Introduce the function spaces for $m, \alpha \geq 0$,

$$(2.7) \quad \mathcal{H}_\alpha^m := H^m(\mathbb{T}_{x,y}^2; L_\alpha^2(\mathbb{R}_z^+)).$$

Since the space $E_{\alpha,\beta}$ is dense in the space \mathcal{H}_α^m , we can extend the operator T from the space $E_{\alpha,\beta}$ to \mathcal{H}_α^m , and define

$$\|T(t, s)\|_{\mathcal{L}(\mathcal{H}_\alpha^{m_1}, \mathcal{H}_\alpha^{m_2})} := \sup_{(u_0, v_0) \in E_{\alpha,\beta}} \frac{\|T(t, s)(u_0, v_0)\|_{\mathcal{H}_\alpha^{m_2}}}{\|(u_0, v_0)\|_{\mathcal{H}_\alpha^{m_1}}} \in \mathbb{R}^+ \cup \{\infty\},$$

where the infinity means that T can not be extended to $\mathcal{L}(\mathcal{H}_\alpha^{m_1}, \mathcal{H}_\alpha^{m_2})$.

The main result on linear instability of shear flow is stated as follows.

Theorem 1. *Let $(u^s, v^s)(t, z)$ be the solution of the problems (2.2) satisfying $(u^s - U_0, v^s - V_0) \in C^0(\mathbb{R}^+; W_\alpha^{4,\infty}(\mathbb{R}^+) \cap H_\alpha^4(\mathbb{R}^+)) \cap C^1(\mathbb{R}^+; W_\alpha^{2,\infty}(\mathbb{R}^+) \cap H_\alpha^2(\mathbb{R}^+))$.*

Assume that the initial data of (2.2) satisfies that

$$(2.8) \quad \exists z_0 > 0, \text{ s.t. } (U'_s(z_0))^2 + (V'_s(z_0))^2 \neq 0, \quad V'_s(z_0)U''_s(z_0) \neq U'_s(z_0)V''_s(z_0).$$

Then we have the following two instability statements.

i). There exists $\sigma > 0$ such that for all $\delta > 0$,

$$(2.9) \quad \sup_{0 \leq s \leq t \leq \delta} \|e^{-\sigma(t-s)\sqrt{|\partial_\tau|}} T(t, s)\|_{\mathcal{L}(\mathcal{H}_\alpha^m, \mathcal{H}_\alpha^{m-\mu})} = +\infty, \quad \forall m > 0, \mu \in [0, \frac{1}{4}),$$

where the operator ∂_τ represents the tangential derivative ∂_x or ∂_y ;

ii). There exists an initial shear layer (U_s, V_s) to (2.2) and $\sigma > 0$, such that for all $\delta > 0$,

$$(2.10) \quad \sup_{0 \leq s \leq t \leq \delta} \|e^{-\sigma(t-s)\sqrt{|\partial_\tau|}} T(t, s)\|_{\mathcal{L}(\mathcal{H}_\alpha^{m_1}, \mathcal{H}_\alpha^{m_2})} = +\infty, \quad \forall m_1, m_2 > 0.$$

Remark 2.1. *From Theorem 1, we know that the three dimensional Prandtl equations can be linearly unstable around the shear flow $(u^s, v^s, 0)(t, z)$ even under the monotonic conditions $u_z^s > 0$ and $v_z^s > 0$. On the other hand, if we impose the monotonic condition $U'_s > 0$, then (2.8) is equivalent to*

$$(2.11) \quad \frac{d}{dz} \left(\frac{V'_s}{U'_s} \right) \neq 0.$$

And then, by virtue of the boundary condition $U_s(0) = V_s(0) = 0$, (2.11) is equivalent to

$$(2.12) \quad \frac{d}{dz} \left(\frac{V_s}{U_s} \right) \neq 0.$$

Thus, the result of Theorem 1 is exactly a complement to the well-posedness result of the three dimensional Prandtl equations obtained by the authors in [8] for flow with special structure, that is $\frac{d}{dz} \left(\frac{V_s}{U_s} \right) \equiv 0$. For simplicity, we will assume that $U'_s(z_0) \neq 0$ in the following argument.

Finally, under the above assumption (2.8) we shall have nonlinear instability for the original problem (2.1) of the three dimensional nonlinear Prandtl equations. To state the result, let us first recall the definition of local well-posedness from [7].

Definition 2.1. *The problem (2.1) with the initial data $(u, v)|_{t=0} = (u_0, v_0)(x, y, z)$ is locally well-posed, if there exist positive continuous functions $T(\cdot, \cdot)$, $C(\cdot, \cdot)$, some $\alpha > 0$ and some integer $m \geq 1$ such that for any initial data (u_0^1, v_0^1) and (u_0^2, v_0^2) with*

$$(u_0^1 - U_0, v_0^1 - V_0) \in \mathcal{H}_\alpha^m, \quad (u_0^2 - U_0, v_0^2 - V_0) \in \mathcal{H}_\alpha^m,$$

there are unique distributional solutions (u^1, v^1) and (u^2, v^2) satisfying that for $i = 1, 2$, $(u^i, v^i)|_{t=0} = (u_0^i, v_0^i)$ and

$$(u^i - U_0, v^i - V_0) \in L^\infty(0, T; L^2(\mathbb{T}^2 \times \mathbb{R}^+)) \cap L^2(0, T; H^1(\mathbb{T}^2 \times \mathbb{R}^+)), \quad i = 1, 2,$$

and the following estimate holds

$$(2.13) \quad \begin{aligned} & \| (u^1, v^1) - (u^2, v^2) \|_{L^\infty(0, T; L^2(\mathbb{T}^2 \times \mathbb{R}^+))} + \| (u^1, v^1) - (u^2, v^2) \|_{L^2(0, T; H^1(\mathbb{T}^2 \times \mathbb{R}^+))} \\ & \leq C \left(\| (u_0^1 - U_0, v_0^1 - V_0) \|_{\mathcal{H}_\alpha^m}, \| (u_0^1 - U_0, v_0^1 - V_0) \|_{\mathcal{H}_\alpha^m} \right) \| (u_0^1 - u_0^2, v_0^1 - v_0^2) \|_{\mathcal{H}_\alpha^m}, \end{aligned}$$

where $T = T \left(\| (u_0^1 - U_0, v_0^1 - V_0) \|_{\mathcal{H}_\alpha^m}, \| (u_0^1 - U_0, v_0^1 - V_0) \|_{\mathcal{H}_\alpha^m} \right)$.

The second main result of this paper is the following ill-posedness of the nonlinear problem (2.1).

Theorem 2. *Under the same assumption as given in Theorem 1, the problem (2.1) of the three dimensional nonlinear Prandtl equations is not locally well-posed in the sense of Definition 2.1.*

3. LINEAR INSTABILITY

In this section, we will prove Theorem 1 to show the linear instability of three dimensional Prandtl equations. The proof is divided into the following four subsections.

3.1. The linear instability mechanism. In this subsection, we develop the method introduced in [4] to analyse the linear instability mechanism of three dimensional Prandtl equations. More precisely, we will find some high frequency mode in the tangential variables that grow exponentially in time. To illustrate this kind of instability mechanism, as in [4], we first replace the background shear flow

in (2.3) by its initial data so that the background profile is independent of time. Corresponding to (2.3), let us consider the following problem:

$$(3.1) \quad \begin{cases} \partial_t u + (U_s \partial_x + V_s \partial_y)u + wU'_s - \partial_z^2 u = 0, & \text{in } \Omega, \\ \partial_t v + (U_s \partial_x + V_s \partial_y)v + wV'_s - \partial_z^2 v = 0, & \text{in } \Omega, \\ \partial_x u + \partial_y v + \partial_z w = 0, & \text{in } \Omega, \\ (u, v, w)|_{z=0} = 0, \quad \lim_{z \rightarrow +\infty} (u, v) = 0. \end{cases}$$

Noting that the coefficients in (3.1) are also independent of the tangential variables x and y , it is convenient to work on the Fourier variables with respect to x and y .

Recalling in Remark 2.1 we assume that $U'_s(z_0) \neq 0$, so we can set

$$(3.2) \quad a \triangleq \frac{V'_s(z_0)}{U'_s(z_0)},$$

and then, the condition (2.8) yields that the initial tangential velocity $V_s - aU_s$ has a non-degenerate critical point at $z = z_0$. Thus, we may look for solutions of (3.1) in the form of

$$(3.3) \quad (u, v, w)(t, x, y, z) = e^{ik(y-ax+\lambda(k)t)}(\hat{u}^k, \hat{v}^k, \hat{w}^k)(z),$$

with some large integer k . To insure that the right hand side of (3.3) is 2π -periodic both in x and y , that is, both of k and ak being integers, we need that the constant a given in (3.2) is a rational number, this condition can be easily satisfied. Indeed, the assumption $U'_s(z_0) \neq 0$ and condition (2.8) imply that

$$(3.4) \quad U'_s(z) \neq 0, \quad V'_s(z)U''_s(z) \neq U'_s(z)V''_s(z)$$

holds in a neighborhood of z_0 . So, by using the continuity of $(V'_s, U'_s, V''_s, U''_s)$ and the denseness of the rational numbers in \mathbb{R} , there is a point z_1 in the neighborhood of z_0 , such that the condition (2.8) holds for z_1 and $\frac{V'_s(z_1)}{U'_s(z_1)}$ is a rational number. Therefore, in the following discussion we can always assume that

$$(3.5) \quad a = \frac{l}{q},$$

for some co-prime integers l and q .

Letting $\epsilon \triangleq \frac{1}{k} \ll 1$, combining (3.3) with the divergence free condition in (3.1), we rewrite (3.3) in the form:

$$(3.6) \quad \begin{cases} (u, v)(t, x, y, z) = e^{i\epsilon^{-1}(y-ax+\lambda_\epsilon t)}(u_\epsilon, v_\epsilon)(z), \\ w(t, x, y, z) = -i\epsilon^{-1}e^{i\epsilon^{-1}(y-ax+\lambda_\epsilon t)}w_\epsilon(z). \end{cases}$$

Then, the divergence free condition in (3.1) yields that

$$(3.7) \quad -au_\epsilon + v_\epsilon = w'_\epsilon.$$

Substituting (3.6) into (3.1), we obtain

$$(3.8) \quad \begin{cases} (\lambda_\epsilon + V_s - aU_s)u_\epsilon - U'_s w_\epsilon + i\epsilon u_\epsilon^{(2)} = 0, \\ (\lambda_\epsilon + V_s - aU_s)v_\epsilon - V'_s w_\epsilon + i\epsilon v_\epsilon^{(2)} = 0, \\ (u_\epsilon, v_\epsilon, w_\epsilon)(0) = 0, \quad \lim_{z \rightarrow +\infty} (u_\epsilon, v_\epsilon) = 0. \end{cases}$$

Set

$$(3.9) \quad W_s(z) \triangleq (V_s - aU_s)(z),$$

combining (3.7) with (3.8) implies that

$$(3.10) \quad \begin{cases} (\lambda_\epsilon + W_s)w'_\epsilon - W'_s w_\epsilon + i\epsilon w_\epsilon^{(3)} = 0, \\ w_\epsilon(0) = w'_\epsilon(0) = 0. \end{cases}$$

Note that the equation for $w_\epsilon(z)$ in (3.10) is the same as (2.3) studied in [4] for two dimensional Prandtl equations. Therefore, according to [4], we have the following result:

Lemma 3.1. *For the equation (3.10), if z_0 is a non-degenerate critical point of $W_s(z)$, then $w_\epsilon(z)$ has the following formal approximate expansion in ϵ :*

$$(3.11) \quad \begin{cases} \lambda_\epsilon & \sim -W_s(z_0) + \epsilon^{\frac{1}{2}}\tau, \\ w_\epsilon(z) & \sim H(z - z_0)[W_s(z) - W_s(z_0) + \epsilon^{\frac{1}{2}}\tau] + \epsilon^{\frac{1}{2}}W\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right), \end{cases}$$

where H is the Heaviside function, τ is a complex constant with $\Im\tau < 0$, and the function $W(Z)$ solves the following ODE:

$$(3.12) \quad \begin{cases} \left(\tau + W_s''(z_0)\frac{Z^2}{2}\right)W' - W_s''(z_0)ZW + iW^{(3)} = 0, & Z \neq 0, \\ [W]|_{Z=0} = -\tau, [W']|_{Z=0} = 0, [W'']|_{Z=0} = -W_s''(z_0), \\ \lim_{Z \rightarrow \pm\infty} W = 0, & \text{exponentially,} \end{cases}$$

where the notation $[u]|_{Z=0} = \lim_{\delta_1 \rightarrow 0^+} u(\delta_1) - \lim_{\delta_2 \rightarrow 0^-} u(\delta_2)$ denotes the jump of a related function $u(Z)$ across $Z = 0$.

As in [4], the asymptotic expansion of the solution $w_\epsilon(z)$ given in (3.11) shows that the approximate solution of $w_\epsilon(z)$ can be divided into the "regular" part and the "shear layer" part as:

$$w_\epsilon^a(z) := w_\epsilon^{reg}(z) + w_\epsilon^{sl}(z)$$

with

$$w_\epsilon^{reg} \triangleq H(z - z_0)[W_s(z) - W_s(z_0) + \epsilon^{\frac{1}{2}}\tau],$$

and

$$w_\epsilon^{sl} \triangleq \epsilon^{\frac{1}{2}}W\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right).$$

Note that $w_\epsilon^{reg}(0) = (w_\epsilon^{reg})'(0) = 0$. The "shear layer" part w_ϵ^{sl} is to cancel the discontinuities of the "regular" part w_ϵ^{reg} at $z = z_0$, such that the approximation $w_\epsilon^a \in C^2(\mathbb{R}^+)$.

The formal asymptotic expansion (3.11)₁ for the eigenvalue indicates strong instability of (3.1), that is, back to the Fourier representation (3.6), the tangential velocity (u, v) grows like $e^{\frac{t}{\sqrt{\epsilon}}}$. To complete this process, we will construct the formal approximation of $(u_\epsilon, v_\epsilon)(z)$. The construction of $(u_\epsilon, v_\epsilon)(z)$ is based on the relation (3.7) and the approximation (3.11), which implies that

$$(3.13) \quad (v_\epsilon - au_\epsilon)(z) \sim H(z - z_0)W'_s(z) + \epsilon^{\frac{1}{4}}W'\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right).$$

From (3.13), we assume that the formal approximation for $(u_\epsilon, v_\epsilon)(z)$ can be chosen as follows:

$$(3.14) \quad \begin{cases} u_\epsilon(z) \sim H(z - z_0)U'_s(z) + h\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right) + \epsilon^{\frac{1}{4}}U\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right), \\ v_\epsilon(z) \sim H(z - z_0)V'_s(z) + ah\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right) + \epsilon^{\frac{1}{4}}(aU + W')\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right), \end{cases}$$

where the functions $h(Z)$ and $U(Z)$ is rapidly decay as $Z \rightarrow \pm\infty$ as explained in the following. One can easily verifies that

$$H(z - z_0)(U'_s(z), V'_s(z))|_{z=0} = 0, \quad \lim_{z \rightarrow +\infty} (u_\epsilon, v_\epsilon) = 0.$$

Similar to the ‘‘shear layer’’ part w_ϵ^{sl} defined in (3.11), the functions $h(Z)$ and $U(Z)$ are used to cancel the discontinuities in $H(z - z_0)U'_s(z)$ and $H(z - z_0)V'_s(z)$ at $z = z_0$, so that the approximations in (3.14) belongs to $C^1(\mathbb{R}^+)$. Moreover, $h(Z)$ and $U(Z)$ are used to balance the approximation in the orders of $O(\sqrt{\epsilon})$ and $O(\epsilon^{\frac{3}{4}})$ respectively. For this, from (3.8), (3.11) and (3.14), $h(Z)$ and $U(Z)$ satisfy the following problems respectively,

$$(3.15) \quad \begin{cases} \left(\tau + W_s''(z_0) \frac{Z^2}{2} \right) h - U'_s(z_0)W + ih'' = 0, & Z \neq 0, \\ [h]|_{Z=0} = -U'_s(z_0), \quad [h']|_{Z=0} = 0, \\ \lim_{Z \rightarrow +\infty} h = 0, \end{cases}$$

and

$$(3.16) \quad \begin{cases} \left(\tau + W_s''(z_0) \frac{Z^2}{2} \right) U - U_s''(z_0)ZW + iU'' = 0, & Z \neq 0, \\ [U]|_{Z=0} = 0, \quad [U']|_{Z=0} = -U_s''(z_0), \\ \lim_{Z \rightarrow +\infty} U = 0. \end{cases}$$

Comparing the problem (3.15) with (3.16), respectively (3.12), it follows that $\frac{U'_s(z_0)}{W_s''(z_0)}W''(Z)$, respectively $\frac{U_s''(z_0)}{W_s''(z_0)}W'(Z)$, solves the problem (3.15), respectively (3.16). Consequently, we can choose the formal expansion of $(u_\epsilon, v_\epsilon)(z)$ as

$$(3.17) \quad \begin{cases} u_\epsilon(z) \sim H(z - z_0)U'_s(z) + \frac{U'_s(z_0)}{W_s''(z_0)}W''\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right) + \epsilon^{\frac{1}{4}}\frac{U_s''(z_0)}{W_s''(z_0)}W'\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right), \\ v_\epsilon(z) \sim H(z - z_0)V'_s(z) + \frac{V'_s(z_0)}{W_s''(z_0)}W''\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right) + \epsilon^{\frac{1}{4}}\frac{V_s''(z_0)}{W_s''(z_0)}W'\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right). \end{cases}$$

Therefore, we have concluded the following results for the reduced boundary value problem (3.1).

Proposition 3. *For the large frequency $k = \frac{1}{\epsilon}$, the approximate solutions of the problem (3.1) can be expressed as (3.6) with*

$$(3.18) \quad \begin{cases} \lambda_\epsilon & \sim -W_s(z_0) + \epsilon^{\frac{1}{2}}\tau, \\ w_\epsilon(z) & \sim H(z - z_0) \left[W_s(z) - W_s(z_0) + \epsilon^{\frac{1}{2}}\tau \right] + \epsilon^{\frac{1}{2}}W\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right), \\ (u_\epsilon(z), v_\epsilon(z)) & \sim H(z - z_0)(U'_s(z), V'_s(z)) + W''\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right) \frac{1}{W_s''(z_0)}(U'_s(z_0), V'_s(z_0)), \\ & + \epsilon^{\frac{1}{4}}W'\left(\frac{z - z_0}{\epsilon^{\frac{1}{4}}}\right) \frac{1}{W_s''(z_0)}(U_s''(z_0), V_s''(z_0)), \end{cases}$$

where the complex constant τ and function $W(Z)$ are given in Lemma 3.1.

Remark 3.1. *Recalling from [4], we know that the pair $(\tau, W(Z))$ given in Lemma 3.1 has the following form*

$$(3.19) \quad \begin{cases} \tau & = \left| \frac{W_s''(z_0)}{2} \right|^{\frac{1}{2}} \tilde{\tau}, \\ W(Z) & = \left| \frac{W_s''(z_0)}{2} \right|^{\frac{1}{2}} \left[(\tilde{\tau} + \left| \frac{W_s''(z_0)}{2} \right|^{\frac{1}{2}} Z^2) \tilde{W} \left(\left| \frac{W_s''(z_0)}{2} \right|^{\frac{1}{4}} Z \right) - 1_{\mathbb{R}^+} (\tilde{\tau} + \left| \frac{W_s''(z_0)}{2} \right|^{\frac{1}{2}} Z^2) \right]. \end{cases}$$

where the complex constant $\tilde{\tau}$ has a negative imaginary part, i.e., $\Im\tilde{\tau} < 0$, and the function $\tilde{W}(\tilde{Z})$ is a smooth solution of the following third order ordinary differential equation:

$$(3.20) \quad \begin{cases} (\tilde{\tau} + \text{sign}(W_s''(z_0))\tilde{Z}^2) \frac{d}{d\tilde{Z}} \tilde{W} + i \frac{d^3}{d\tilde{Z}^3} \left((\tau + \text{sign}(W_s''(z_0))\tilde{Z}^2) \tilde{W} \right) = 0, \\ \lim_{\tilde{Z} \rightarrow -\infty} \tilde{W} = 0, \quad \lim_{\tilde{Z} \rightarrow +\infty} \tilde{W} = 1. \end{cases}$$

3.2. Construction of approximate solutions. Inspiring by the construction of approximate solutions to the simplified problem (3.1) given in the above subsection, and also by the argument given in [4], we are going to construct the approximate solution of the original linearized problem (2.3).

Let $(u^s(t, z), v^s(t, z))$ satisfy the assumptions of Theorem 1, and denote by

$$w_a^s(t, z) \triangleq v^s(t, z) - au^s(t, z),$$

where the constant a is given in (3.2). Then, we know that z_0 is a non-degenerate critical point of $w_a^s(0, z)$. Without loss of generality, we assume that $\partial_z^2 w_a^s(0, z_0) < 0$, then the differential equation

$$(3.21) \quad \begin{cases} \partial_t \partial_z w_a^s(t, f(t)) + \partial_z^2 w_a^s(t, f(t)) f'(t) = 0, \\ f(0) = z_0, \end{cases}$$

defines a non-degenerate critical point $f(t)$ of $w_a^s(t, \cdot)$ when $0 < t < t_0$ for some small $t_0 > 0$. Note that such $f(t)$ can also be determined by the following equation:

$$\frac{d}{dt} \left(\frac{v_z^s(t, f(t))}{u_z^s(t, f(t))} \right) = 0, \quad f(0) = z_0.$$

Since the approximation solution of (3.1) given in Proposition 3 is obtained with the background state being frozen at the initial data $(u^s, v^s)|_{t=0} = (U_s, V_s)(z)$, to construct approximate solutions of the original problem (2.3) with background state being the shear flow in the time interval $0 < t < t_0$, we need to do some modification as in [4]. Recall Proposition 3 and Remark 3.1 in the above subsection, let $\tau, W(Z)$ be given in (3.19), and set

$$(3.22) \quad W_{sl}(Z) := (\tau - Z^2)W(Z) - 1_{\mathbb{R}^+}(\tau - Z^2).$$

Then, for $\epsilon > 0$ we introduce

$$(3.23) \quad \lambda_\epsilon(t) := -w_a^s(t, f(t)) + \epsilon^{\frac{1}{2}} \left| \frac{\partial_z^2 w_a^s(t, f(t))}{2} \right|^{\frac{1}{2}} \tau,$$

and the ‘‘regular’’ part of velocities

$$(3.24) \quad \begin{cases} U_\epsilon^{reg}(t, z) = H(z - f(t))u_z^s(t, z), & V_\epsilon^{reg}(t, z) = H(z - f(t))v_z^s(t, z), \\ W_\epsilon^{reg}(t, z) = H(z - f(t)) \left[w_a^s(t, z) - w_a^s(t, f(t)) + \epsilon^{\frac{1}{2}} \left| \frac{\partial_z^2 w_a^s(t, f(t))}{2} \right|^{\frac{1}{2}} \tau \right], \end{cases}$$

as well as the ‘‘shear layer’’ part of w_ϵ

$$(3.25) \quad W_\epsilon^{sl}(t, z) := \epsilon^{\frac{1}{2}} \varphi(z - f(t)) \left| \frac{\partial_z^2 w_a^s(t, f(t))}{2} \right|^{\frac{1}{2}} W_{sl} \left(\left| \frac{\partial_z^2 w_a^s(t, f(t))}{2} \right|^{\frac{1}{4}} \cdot \frac{z - f(t)}{\epsilon^{\frac{1}{4}}} \right).$$

Here, φ is a smooth truncation function near 0, and W_{sl} is given in (3.22). Therefore, from Proposition 3 and by (3.24) and (3.25), the approximate solution of the problem (2.3) can be defined as:

$$(3.26) \quad (u_\epsilon, v_\epsilon, w_\epsilon)(t, x, y, z) = e^{i\epsilon^{-1}(y-ax)}(U_\epsilon, V_\epsilon, W_\epsilon)(t, z)$$

with

$$(3.27) \quad \begin{aligned} (U_\epsilon, V_\epsilon)(t, z) &= ie^{i\epsilon^{-1} \int_0^t \lambda_\epsilon(s) ds} \left\{ (U_\epsilon^{reg}, V_\epsilon^{reg})(t, z) + \frac{\partial_z^2 W_\epsilon^{sl}(t, z)}{\partial_z^2 w_a^s(t, f(t))} (u_z^s, v_z^s)(t, f(t)) \right. \\ &\quad \left. + \frac{\partial_z W_\epsilon^{sl}(t, z)}{\partial_z^2 w_a^s(t, f(t))} (\partial_z^2 u^s, \partial_z^2 v^s)(t, f(t)) \right\}, \\ W_\epsilon(t, z) &= \epsilon^{-1} e^{i\epsilon^{-1} \int_0^t \lambda_\epsilon(s) ds} (W_\epsilon^{reg}(t, z) + W_\epsilon^{sl}(t, z)). \end{aligned}$$

Moreover, in order that the function (3.26) is 2π -periodic in x and y , we take $\epsilon = \frac{1}{qk}$ with the integers q given in (3.5) and $k \in \mathbb{N}$.

It is straightforward to check that for $(u_\epsilon, v_\epsilon, w_\epsilon)$ defined in (3.26),

$$(u_\epsilon, v_\epsilon, w_\epsilon)|_{z=0} = 0, \quad \lim_{z \rightarrow +\infty} (u_\epsilon, v_\epsilon) = 0,$$

and the divergence free condition holds. Also,

$$(u_\epsilon, v_\epsilon)(t, x, y, z) = e^{i\epsilon^{-1}(y-ax)}(U_\epsilon, V_\epsilon)(t, z)$$

is analytic in the tangential variables x, y and H_α^2 in z . Moreover, there are positive constants C_0 and σ_0 , independent of ϵ , such that

$$(3.28) \quad C_0^{-1} e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}} \leq \|(U_\epsilon, V_\epsilon)(t, \cdot)\|_{L_\alpha^2} \leq C_0 e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}}.$$

Plugging the relation (3.26) into the original linearized Prandtl equations (2.3), it follows that

$$(3.29) \quad \begin{cases} \partial_t u_\epsilon + (u^s \partial_x + v^s \partial_y) u_\epsilon + w_\epsilon u_z^s - \partial_z^2 u_\epsilon = r_\epsilon^1, \\ \partial_t v_\epsilon + (u^s \partial_x + v^s \partial_y) v_\epsilon + w_\epsilon v_z^s - \partial_z^2 v_\epsilon = r_\epsilon^2, \\ \partial_x u_\epsilon + \partial_y v_\epsilon + \partial_z w_\epsilon = 0, \quad \text{in } \Omega, \\ (u_\epsilon, v_\epsilon, w_\epsilon)|_{z=0} = 0, \quad \lim_{z \rightarrow +\infty} (u_\epsilon, v_\epsilon) = 0, \end{cases}$$

where the remainder term is represented by

$$(3.30) \quad (r_\epsilon^1, r_\epsilon^2)(t, x, y, z) := e^{i\epsilon^{-1}(y-ax)}(R_\epsilon^1, R_\epsilon^2)(t, z)$$

with

$$(3.31) \quad \begin{aligned} R_\epsilon^1 &= \partial_t U_\epsilon + i\epsilon^{-1} w_a^s(t, z) U_\epsilon - \partial_z^2 U_\epsilon + u_z^s(t, z) W_\epsilon, \\ R_\epsilon^2 &= \partial_t V_\epsilon + i\epsilon^{-1} w_a^s(t, z) V_\epsilon - \partial_z^2 V_\epsilon + v_z^s(t, z) W_\epsilon. \end{aligned}$$

Note that the representation (3.24) implies

$$(3.32) \quad (\partial_t - \partial_z^2) U_\epsilon^{reg} = (\partial_t - \partial_z^2) V_\epsilon^{reg} = 0, \quad z \neq f(t).$$

Then, from the representation of (U_ϵ, V_ϵ) and W_ϵ given in (3.27) and by using the equations (3.20) and (3.32), we conclude that for $z \neq f(t)$,

$$(3.33) \quad \begin{aligned} R_\epsilon^1(t, z) = & e^{i\epsilon^{-1} \int_0^t \lambda_\epsilon(s) ds} \left\{ -\epsilon^{-1} \left[w_a^s(t, z) - w_a^s(t, f(t)) - \partial_z^2 w_a^s(t, f(t)) \frac{(z - f(t))^2}{2} \right] \right. \\ & \cdot \left[\frac{u_z^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z^2 W_\epsilon^{sl} + \frac{\partial_z^2 u^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z W_\epsilon^{sl} \right] \\ & + \epsilon^{-1} \left[u_z^s(t, z) - u_z^s(t, f(t)) - u_{zz}^s(t, f(t))(z - f(t)) \right] W_\epsilon^{sl} \\ & \left. + i\partial_t \left(\frac{u_z^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z^2 W_\epsilon^{sl} + \frac{\partial_z^2 u^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z W_\epsilon^{sl} \right) + O(\epsilon^\infty) \right\}, \end{aligned}$$

and

$$(3.34) \quad \begin{aligned} R_\epsilon^2(t, z) = & e^{i\epsilon^{-1} \int_0^t \lambda_\epsilon(s) ds} \left\{ -\epsilon^{-1} \left[w_a^s(t, z) - w_a^s(t, f(t)) - \partial_z^2 w_a^s(t, f(t)) \frac{(z - f(t))^2}{2} \right] \right. \\ & \cdot \left[\frac{v_z^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z^2 W_\epsilon^{sl} + \frac{\partial_z^2 v^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z W_\epsilon^{sl} \right] \\ & + \epsilon^{-1} \left[v_z^s(t, z) - v_z^s(t, f(t)) - v_{zz}^s(t, f(t))(z - f(t)) \right] W_\epsilon^{sl} \\ & \left. + i\partial_t \left(\frac{v_z^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z^2 W_\epsilon^{sl} + \frac{\partial_z^2 v^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z W_\epsilon^{sl} \right) + O(\epsilon^\infty) \right\}. \end{aligned}$$

The terms $O(\epsilon^\infty)$ in (3.33) and (3.34) represent the remainders with exponential decay in z that follows from the fact that W_ϵ^{sl} decays exponentially and the derivatives of $\varphi(\cdot - f(t))$ vanish near $f(t)$. Then, with the same σ_0 given in (3.28), we have that $(R_\epsilon^1, R_\epsilon^2)(t, z)$ satisfy

$$(3.35) \quad \|(R_\epsilon^1, R_\epsilon^2)(t, \cdot)\|_{L_\alpha^2} \leq C_1 \epsilon^{-\frac{1}{4}} e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}},$$

where the constant C_1 is independent of ϵ .

Therefore, we conclude

Proposition 4. *For the linear problem (2.3), the approximate solution $(u_\epsilon, v_\epsilon, w_\epsilon)$ given in (3.26) and (3.27), satisfies the problem (3.29), where the source term $(r_\epsilon^1, r_\epsilon^2)$ in the form (3.30) has the bound (3.35).*

Remark 3.2. *The estimate (3.35) follows from the fact that the "shear layer" part $\partial_z^2 W_\epsilon^{sl}$ cancels the terms $\epsilon^{-1} u_z^s(t, f(t)) W_\epsilon^{sl}$ and $\epsilon^{-1} v_z^s(t, f(t)) W_\epsilon^{sl}$ in (3.33) and (3.34) respectively, by using the equation (3.15) for instance. And this error bound leads to the choice of $\mu < \frac{1}{4}$ in (2.9). It is slight different from the two dimensional case studied in [4] where $\mu < \frac{1}{2}$, because here we require that the initial data of u^s and v^s do not degenerate simultaneously at a point, $z = z_0$, while in the two dimensional problem, it is assumed that there is a degeneracy at the critical point.*

3.3. Proof of Theorem 1(i). At this stage, based on the approximate solution given in Proposition 4, we can use the method from [4] to prove Theorem 1. We now sketch the proof as follows.

Verification of (2.9) for the tangential differential operator by contradiction.

Suppose that (2.9) does not hold for ∂_x , that is, for all $\sigma > 0$, there exists

$\delta > 0, m \geq 0$ and $\mu \in [0, \frac{1}{4})$, that is,

$$(3.36) \quad \sup_{0 \leq s \leq t \leq \delta} \|e^{-\sigma(t-s)} \sqrt{|\partial_x|} T(t, x)\|_{\mathcal{L}(\mathcal{H}_\alpha^m, \mathcal{H}_\alpha^{m-\mu})} < +\infty.$$

Introduce the operator

$$T_\epsilon(t, s) : L_\alpha^2(\mathbb{R}^+) \mapsto L_\alpha^2(\mathbb{R}^+)$$

as

$$(3.37) \quad T_\epsilon(t, s)((U_0, V_0)) := e^{-i\epsilon^{-1}(y-ax)} T(t, s) \left(e^{i\epsilon^{-1}(y-ax)} (U_0, V_0) \right)$$

with $T(t, s)$ being defined in (2.6). From (3.36), we have

$$(3.38) \quad \|T_\epsilon(t, s)\|_{\mathcal{L}(L_\alpha^2)} \leq C_2 \epsilon^{-\mu} e^{\frac{\sqrt{a}\sigma(t-s)}{\sqrt{\epsilon}}}, \quad \forall 0 \leq s \leq t \leq \delta$$

for a constant C_2 independent of ϵ .

Denote by

$$L_\epsilon := e^{-i\epsilon^{-1}(y-ax)} L e^{i\epsilon^{-1}(y-ax)},$$

where L is the linearized Prandtl operator around the shear flow $(u^s, v^s, 0)$. Let $(U, V)(t, z)$ be a solution to the problem

$$\begin{cases} \partial_t(U, V) + L_\epsilon(U, V) = 0, \\ (U, V)|_{t=0} = (U_\epsilon, V_\epsilon)(0, z). \end{cases}$$

From the definition (3.37), we have that

$$(U, V)(t, z) = T_\epsilon(t, 0)((U_\epsilon, V_\epsilon)(0, z)).$$

Then, from (3.38) it follows that for all $t \leq \delta$,

$$(3.39) \quad \|(U, V)(t, \cdot)\|_{L_\alpha^2} \leq C_2 \epsilon^{-\mu} e^{\frac{\sqrt{a}\sigma t}{\sqrt{\epsilon}}} \|(U_\epsilon, V_\epsilon)(0, \cdot)\|_{L_\alpha^2} \leq C_3 \epsilon^{-\mu} e^{\frac{\sqrt{a}\sigma t}{\sqrt{\epsilon}}}$$

holds for a constant C_3 independent of ϵ . From (3.33) and (3.34), we know that

$$\partial_t(U_\epsilon, V_\epsilon) + L_\epsilon(U_\epsilon, V_\epsilon) = (R_\epsilon^1, R_\epsilon^2).$$

Thus, the difference $(\tilde{U}, \tilde{V}) := (U, V) - (U_\epsilon, V_\epsilon)$ can be obtained by the Duhamel representation:

$$(3.40) \quad (\tilde{U}, \tilde{V})(t, \cdot) = \int_0^t T_\epsilon(t, s)((R_\epsilon^1, R_\epsilon^2)(s, \cdot)) ds, \quad \forall t \leq \delta.$$

Combining (3.35), (3.38) and (3.40), and choosing $\sqrt{a}\sigma < \sigma_0$ yields that

$$(3.41) \quad \|(\tilde{U}, \tilde{V})(t, \cdot)\|_{L_\alpha^2} \leq C_1 C_2 \epsilon^{-\mu - \frac{1}{4}} \int_0^t e^{\frac{\sqrt{a}\sigma(t-s)}{\sqrt{\epsilon}}} e^{\frac{\sigma_0 s}{\sqrt{\epsilon}}} ds \leq C_4 \epsilon^{\frac{1}{4} - \mu} e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}},$$

where the constant $C_4 > 0$ is independent of ϵ . Then, by using (3.28), we obtain that for $t < \delta$ and sufficiently small ϵ ,

$$(3.42) \quad \begin{aligned} \|(U, V)(t, \cdot)\|_{L_\alpha^2} &\geq \|(U_\epsilon, V_\epsilon)(t, \cdot)\|_{L_\alpha^2} - \|(\tilde{U}, \tilde{V})(t, \cdot)\|_{L_\alpha^2} \\ &\geq C_0^{-1} e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}} - C_4 \epsilon^{\frac{1}{4} - \mu} e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}} \geq \frac{1}{2} C_0^{-1} e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}}. \end{aligned}$$

As $\sqrt{a}\sigma < \sigma_0$, comparing (3.39) with (3.42), the contradiction arises when $t > \frac{\sqrt{\epsilon}}{\sigma_0 - \sqrt{a}\sigma} (\ln(2C_0 C_3) - \ln \epsilon)$ with sufficiently small ϵ . Thus, the proof of Theorem 1(i) is completed.

3.4. Proof of Theorem 1(ii). The aim of this subsection is to prove Theorem 1(ii). By comparing (2.9) with (2.10), we only need to show that there exists an initial data (U_s, V_s) for the shear flow to (2.2) such that (2.9) still holds for arbitrary $\mu > 0$. Recall the proof of part i) in the above subsection, the task can be attributed to find some (U_s, V_s) such that the remainder $(R_\epsilon^1, R_\epsilon^2)(t, z)$, generated in (3.31) by the approximation (3.27), has the following estimate:

$$(3.43) \quad \|(R_\epsilon^1, R_\epsilon^2)(t, \cdot)\|_{L_\alpha^2} \leq C(\epsilon^N + t^{2N})e^{\frac{\sigma_0 t}{\sqrt{\epsilon}}}$$

for some $N \in \mathbb{R}^+$, $N + \frac{1}{2} > \mu$. Once this is achieved, as in [4], the desired conclusion holds.

Similar to [4], the special initial shear layer $(U_s, V_s)(z)$ can be chosen such that

$$(U_s, V_s)|_{z=0} = 0, \quad (U_s, V_s)(z) \rightarrow (U_0, V_0), \text{ exponentially as } z \rightarrow +\infty,$$

and in a small neighborhood of z_0 ,

$$(3.44) \quad \begin{cases} U_s(z) = U_s''(z_0)\frac{(z-z_0)^2}{2} + q(z-z_0) + U_s(z_0), \\ V_s(z) = V_s''(z_0)\frac{(z-z_0)^2}{2} + l(z-z_0) + V_s(z_0), \end{cases}$$

where q, l are integers given in (3.5), and the constants $U_s''(z_0), U_s(z_0), V_s''(z_0), V_s(z_0)$ satisfy

$$V_s''(z_0) - aU_s''(z_0) \neq 0, \quad U_s(z_0) \neq 0$$

with the constant a being given in (3.2). Then, for such $(U_s, V_s)(z)$, we will show that (3.43) holds. We only estimate the term R_ϵ^1 , as the same argument works for R_ϵ^2 . From (3.33), decompose R_ϵ^1 as follows for $z \neq f(t)$:

$$(3.45) \quad R_\epsilon^1 := e^{i\epsilon \int_0^t \lambda(s) ds} \left(R_{\epsilon,1}^1 + R_{\epsilon,2}^1 + R_{\epsilon,3}^1 + O(\epsilon^\infty) \right),$$

where

$$\begin{aligned} R_{\epsilon,1}^1(t, z) &= -\epsilon^{-1} \left[w_a^s(t, z) - w_a^s(t, f(t)) - \partial_z^2 w_a^s(t, f(t)) \frac{(z-f(t))^2}{2} \right] \\ &\quad \cdot \left[\frac{u_z^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z^2 W_\epsilon^{sl} + \frac{\partial_z^2 u^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z W_\epsilon^{sl} \right], \\ R_{\epsilon,2}^1(t, z) &= \epsilon^{-1} \left[u_z^s(t, z) - u_z^s(t, f(t)) - u_{zz}^s(t, f(t))(z-f(t)) \right] W_\epsilon^{sl}, \\ R_{\epsilon,3}^1(t, z) &= i\partial_t \left(\frac{u_z^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z^2 W_\epsilon^{sl} + \frac{\partial_z^2 u^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z W_\epsilon^{sl} \right). \end{aligned}$$

Therefore, it remains to show that

$$(3.46) \quad \|R_{\epsilon,i}^1(t, \cdot)\|_{L_\alpha^2} \leq C(\epsilon^N + t^{2N}), \quad i = 1, 2, 3.$$

Firstly, for $R_{\epsilon,1}^1$, note that the function $w_a^s(t, z) = v^s(t, z) - au^s(t, z)$ satisfies

$$(3.47) \quad \partial_t w_a^s - \partial_z^2 w_a^s = 0,$$

and in a small neighborhood of z_0 ,

$$(3.48) \quad w_a^s(0, z) = (V_s''(z_0) - aU_s''(z_0)) \frac{(z-z_0)^2}{2} + V(z_0) - aU_s(z_0).$$

Thus, by using the Taylor expansion and (3.47), it follows that for any $N \in \mathbb{N}$,

$$\begin{aligned}
(3.49) \quad |R_{\epsilon,1}^1(t, z)| &= \epsilon^{-1} \left| \int_{f(t)}^z \frac{(\tilde{z} - f(t))^2}{2} \partial_z^3 w_a^s(t, \tilde{z}) d\tilde{z} \right| \cdot \left| \frac{u_z^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z^2 W_\epsilon^{sl} + \frac{\partial_z^2 u^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} \partial_z W_\epsilon^{sl} \right| \\
&\leq C \epsilon^{-1} \int_{f(t)}^z \frac{(\tilde{z} - f(t))^2}{2} \sum_{k=0}^{2N-1} \frac{t^k}{k!} |\partial_t^k \partial_z^3 w_a^s(0, \tilde{z})| d\tilde{z} \cdot \left(|\partial_z^2 W_\epsilon^{sl}| + |\partial_z W_\epsilon^{sl}| \right) + O(t^{2N}) \\
&\leq C \epsilon^{-1} \int_{f(t)}^z \frac{(\tilde{z} - f(t))^2}{2} \sum_{k=0}^{2N-1} \frac{t^k}{k!} |\partial_z^{3+2k} w_a^s(0, \tilde{z})| d\tilde{z} \cdot \left(|\partial_z^2 W_\epsilon^{sl}| + |\partial_z W_\epsilon^{sl}| \right) + O(t^{2N}).
\end{aligned}$$

From (3.48), we know that

$$\partial_z^{3+2k} w_a^s(0, z) \equiv 0, \quad \forall k \in \mathbb{N},$$

when z is in a small neighborhood of $f(t)$ as $t > 0$ is small. This implies that the integral in the last line of (3.49) is supported away from $z = f(t)$. Then, combining with the exponential decrease of the ‘‘shear layer’’ W_ϵ^{sl} , (3.49) yields

$$(3.50) \quad \|R_{\epsilon,1}^1(t, \cdot)\|_{L_x^2} \leq C(\epsilon^N + t^{2N}).$$

For the term $R_{\epsilon,2}^1$, we can use similar arguments as above to obtain

$$(3.51) \quad \|R_{\epsilon,2}^1(t, \cdot)\|_{L_x^2} \leq C(\epsilon^N + t^{2N}).$$

Next, from the expression (3.25) of $W_\epsilon^{sl}(t, z)$ and the relation

$$\partial_t u^s = \partial_z^2 u^s, \quad \partial_t w_a^s = \partial_z^2 w_a^s,$$

a straightforward calculation implies that

$$\begin{aligned}
(3.52) \quad R_{\epsilon,3}^1(t, z) &= \partial_z^3 u^s(t, f(t)) g_1(t, z) + \partial_z^4 u^s(t, f(t)) g_2(t, z) + \partial_z^4 w_a^s(t, f(t)) g_3(t, z) \\
&\quad + f'(t) g_4(t, z) + O(\epsilon^\infty),
\end{aligned}$$

where each function $g_i(t, z)$, $1 \leq i \leq 4$ can be expressed as a linear combination of the terms

$$O(\epsilon^{-\frac{1}{4}}) \varphi(z - f(t)) W_{sl}^{(j)} \left(\left| \frac{\partial_z^2 w_a^s(t, f(t))}{2} \right|^{\frac{1}{4}} \cdot \frac{z - f(t)}{\epsilon^{\frac{1}{4}}} \right), \quad 0 \leq j \leq 3$$

with W_{sl} being defined in (3.22). Then, as for $R_{\epsilon,1}^1$, the first three terms on the right hand side of (3.52) have the same bounds as in (3.50). For the fourth term given in (3.52), by noticing that from (3.21),

$$f'(t) = -\frac{\partial_t \partial_z w_a^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))} = \frac{\partial_z^3 w_a^s(t, f(t))}{\partial_z^2 w_a^s(t, f(t))},$$

we can verify that $f'(t) g_4(t, z)$ also satisfies the same estimate as (3.50). In conclusion, we have

$$(3.53) \quad \|R_{\epsilon,3}^1(t, \cdot)\|_{L_x^2} \leq C(\epsilon^N + t^{2N}).$$

Thus, combining (3.50), (3.51) and (3.53), we have the estimate (3.46), and then obtain the proof of Theorem 1(ii) by taking N large enough.

Finally, we state the following result to finish this section, which can be obtained by similar arguments as above,

Proposition 5. *There exists a shear layer (u^s, v^s) to (2.2) with $u_z^s, v_z^s > 0$, such that for all $\delta > 0$,*

$$(3.54) \quad \sup_{0 \leq s \leq t \leq \delta} \|T(t, s)\|_{\mathcal{L}(\mathcal{H}_\alpha^m, L^2)} = +\infty, \quad \forall m > 0.$$

4. NONLINEAR INSTABILITY

In this section, we will prove the nonlinear ill-posedness result of the three dimensional Prandtl equations stated in Theorem 2, it will mainly follow the argument of [7]. First, let us give a preliminary result on the uniqueness of solutions to the linear problem (2.3) as follows.

Lemma 4.1. *Suppose that $(u^s, v^s)(t, z)$ is a solution to the problem (2.2) satisfying that*

$$\sup_{t \geq 0} \left(\sup_{z \geq 0} |(u^s, v^s)| + \int_0^{+\infty} z |(\partial_z u^s, \partial_z v^s)|^2 dz \right) < +\infty.$$

Let $(u, v) \in L^\infty(0, T; L^2(\mathbb{T}^2 \times \mathbb{R}^+))$ with $(\partial_z u, \partial_z v) \in L^2((0, T) \times \mathbb{T}^2 \times \mathbb{R}^+)$ be a solution to the problem (2.3) with the vanish initial data $(u, v)|_{t=0} = 0$. Then, for all $t > 0$, $(u, v) \equiv 0$.

The proof of this lemma is similar to the one given in [5, Proposition 2.1] or [7, Proposition 2.2], we omit it here for simplicity.

Proof of Theorem 2. (1) First, by using (3.54), we know that for the shear flow (u^s, v^s) given in Proposition 5, for fixed $\delta_0 > 0$ and any $m, n \in \mathbb{N}$, there exist s_n, t_n with $0 \leq s_n \leq t_n \leq \delta_0$, functions $(u_0^n, v_0^n)(x, y, z)$, and solutions (u_L^n, v_L^n) to the linearized problem (2.3), such that $(u_L^n, v_L^n)|_{t=s_n} = (u_0^n, v_0^n)$ and

$$(4.1) \quad \|(u_0^n, v_0^n)\|_{\mathcal{H}_\alpha^m} = 1, \quad \|(u_L^n, v_L^n)(t_n)\|_{L^2} \geq n.$$

(2) Now, we prove this theorem by contradiction. Assume that the problem (2.1) is locally well-posedness for some integer $m \geq 0$ in the sense of Definition 2.1. Denote by

$$(4.2) \quad (u_{s_n}^s, v_{s_n}^s)(t, z) := (u^s, v^s)(t + s_n, z),$$

and

$$(4.3) \quad (u_{0,\delta}^n, v_{0,\delta}^n)(x, y, z) := (u^s, v^s)(s_n, z) + \delta(u_0^n, v_0^n)(x, y, z)$$

with a small positive constant δ . Let (u_δ^n, v_δ^n) be the solution to the problem (2.1) with the initial data $(u_\delta^n, v_\delta^n)|_{t=0} = (u_{0,\delta}^n, v_{0,\delta}^n)$. Thus, applying Definition 2.1 to two solutions (u_δ^n, v_δ^n) and $(u_{s_n}^s, v_{s_n}^s)$, it yields that there exist positive continuous functions $T(\cdot, \cdot)$ and $C(\cdot, \cdot)$, such that

$$(4.4) \quad \begin{aligned} & \| (u_\delta^n - u_{s_n}^s, v_\delta^n - v_{s_n}^s) \|_{L^\infty(0, T; L^2(\mathbb{T}^2 \times \mathbb{R}^+))} + \| (u_\delta^n - u_{s_n}^s, v_\delta^n - v_{s_n}^s) \|_{L^2(0, T; H^1(\mathbb{T}^2 \times \mathbb{R}^+))} \\ & \leq C\delta \| (u_0^n, v_0^n) \|_{\mathcal{H}_\alpha^m} = C\delta, \end{aligned}$$

where $T = T\left(\|(u_{0,\delta}^n - U_0, v_{0,\delta}^n - V_0)\|_{\mathcal{H}_\alpha^m}, \|(u_{s_n}^s - U_0, v_{s_n}^s - V_0)\|_{\mathcal{H}_\alpha^m}\right)$ and $C = C\left(\|(u_{0,\delta}^n - U_0, v_{0,\delta}^n - V_0)\|_{\mathcal{H}_\alpha^m}, \|(u_{s_n}^s - U_0, v_{s_n}^s - V_0)\|_{\mathcal{H}_\alpha^m}\right)$.

Combining (4.1), (4.2) and (4.3), we know that $\|(u_{0,\delta}^n - U_0, v_{0,\delta}^n - V_0)\|_{\mathcal{H}_\alpha^m}$ and $\|(u_{s_n}^s - U_0, v_{s_n}^s - V_0)\|_{\mathcal{H}_\alpha^m}$ are uniformly bounded in δ and n . Thus, we can take

the functions $T(\cdot, \cdot)$ and $C(\cdot, \cdot)$ independent of δ and n , and in the following we use T, C to replace $T(\cdot, \cdot), C(\cdot, \cdot)$ for simplicity.

(3) From the estimate (4.4), we know that the sequence

$$(\tilde{u}_\delta^n, \tilde{v}_\delta^n) := \frac{1}{\delta}(u_\delta^n - u_{s_n}^s, v_\delta^n - v_{s_n}^s)$$

is bounded in $L^\infty(0, T; L^2(\mathbb{T}^2 \times \mathbb{R}^+)) \cap L^2(0, T; H^1(\mathbb{T}^2 \times \mathbb{R}^+))$ uniformly in δ and n , which yields that there is (u^n, v^n) such that, up to a subsequence, as $\delta \rightarrow 0$,

$$(4.5) \quad (\tilde{u}_\delta^n, \tilde{v}_\delta^n) \rightharpoonup (u^n, v^n), \quad \text{weakly} - * \text{ in } L^\infty(0, T; L^2(\mathbb{T}^2 \times \mathbb{R}^+)) \cap L^2(0, T; H^1(\mathbb{T}^2 \times \mathbb{R}^+)),$$

and

$$(4.6) \quad \|(u^n, v^n)\|_{L^\infty(0, T; L^2(\mathbb{T}^2 \times \mathbb{R}^+))} + \|(u^n, v^n)\|_{L^2(0, T; H^1(\mathbb{T}^2 \times \mathbb{R}^+))} \leq C, \quad \forall n \geq 1.$$

Next, since both (u_δ^n, v_δ^n) and $(u_{s_n}^s, v_{s_n}^s)$ solve the problem (2.1), we have

$$(4.7) \quad \begin{cases} \partial_t(\tilde{u}_\delta^n, \tilde{v}_\delta^n) + \mathcal{P}_n(\tilde{u}_\delta^n, \tilde{v}_\delta^n) = \delta N(\tilde{u}_\delta^n, \tilde{v}_\delta^n), \\ (\tilde{u}_\delta^n, \tilde{v}_\delta^n)|_{t=0} = (u_0^n, v_0^n), \end{cases}$$

where \mathcal{P}_n is the linearized Prandtl operator at the shear profile $(u_{s_n}^s, v_{s_n}^s)$, and $N(\cdot, \cdot)$ is the nonlinear term,

$$(4.8) \quad \begin{aligned} N(\tilde{u}_\delta^n, \tilde{v}_\delta^n) := & \left(-\tilde{u}_\delta^n \partial_x \tilde{u}_\delta^n - \tilde{v}_\delta^n \partial_y \tilde{u}_\delta^n + \int_0^z (\partial_x \tilde{u}_\delta^n + \partial_y \tilde{v}_\delta^n) d\tilde{z} \cdot \partial_z \tilde{u}_\delta^n, \right. \\ & \left. -\tilde{u}_\delta^n \partial_x \tilde{v}_\delta^n - \tilde{v}_\delta^n \partial_y \tilde{v}_\delta^n + \int_0^z (\partial_x \tilde{u}_\delta^n + \partial_y \tilde{v}_\delta^n) d\tilde{z} \cdot \partial_z \tilde{v}_\delta^n \right). \end{aligned}$$

Therefore, we want to show that the limit function (u^n, v^n) satisfies the linearized Prandtl equations in the sense of distribution. For this, we only need to prove that the right hand side of the equation in (4.7) goes to zero as $\delta \rightarrow 0$ in the sense of distribution.

Indeed, note that from (4.8) the nonlinear term can be rewritten as

$$N(\tilde{u}_\delta^n, \tilde{v}_\delta^n) = \left(N_1(\tilde{u}_\delta^n, \tilde{v}_\delta^n), N_2(\tilde{u}_\delta^n, \tilde{v}_\delta^n) \right)$$

with

$$\begin{aligned} N_1(\tilde{u}_\delta^n, \tilde{v}_\delta^n) &= -\partial_x(\tilde{u}_\delta^n)^2 - \partial_y(\tilde{u}_\delta^n \tilde{v}_\delta^n) + \partial_z \left(\int_0^z (\partial_x \tilde{u}_\delta^n + \partial_y \tilde{v}_\delta^n) d\tilde{z} \cdot \tilde{u}_\delta^n \right), \\ N_2(\tilde{u}_\delta^n, \tilde{v}_\delta^n) &= -\partial_x(\tilde{u}_\delta^n \tilde{v}_\delta^n) - \partial_y(\tilde{v}_\delta^n)^2 + \partial_z \left(\int_0^z (\partial_x \tilde{u}_\delta^n + \partial_y \tilde{v}_\delta^n) d\tilde{z} \cdot \tilde{v}_\delta^n \right). \end{aligned}$$

For any compact set K of $[0, T] \times \mathbb{T}^2 \times \mathbb{R}^+$ and smooth function φ supported in K , we have

$$\begin{aligned} & \left| \int_{[0, T] \times \mathbb{T}^2 \times \mathbb{R}^+} N_1(\tilde{u}_\delta^n, \tilde{v}_\delta^n) \varphi dx dy dz dt \right| \\ & \leq C_{K, \varphi} \int_K \left(|\tilde{u}_\delta^n|^2 + |\tilde{u}_\delta^n \tilde{v}_\delta^n| + \left| \int_0^z (\partial_x \tilde{u}_\delta^n + \partial_y \tilde{v}_\delta^n) d\tilde{z} \cdot \tilde{u}_\delta^n \right| \right) dx dy dz dt \\ & \leq C_{K, \varphi} \left(\|\tilde{u}_\delta^n\|_{L^2(K)}^2 + \|\tilde{v}_\delta^n\|_{L^2(K)}^2 + \left\| \int_0^z (\partial_x \tilde{u}_\delta^n + \partial_y \tilde{v}_\delta^n) d\tilde{z} \right\|_{L^2(K)}^2 \right), \end{aligned}$$

where $C_{K,\varphi}$ is a positive constant depending on K and $W^{1,\infty}$ norm of φ . From the obvious inequality,

$$\left| \int_0^z (\partial_x \tilde{u}_\delta^n + \partial_y \tilde{v}_\delta^n) d\tilde{z} \right| \leq z^{\frac{1}{2}} \left(\int_{\mathbb{R}_z^+} |\partial_x \tilde{u}_\delta^n + \partial_y \tilde{v}_\delta^n|^2 dz \right)^{\frac{1}{2}},$$

we get

$$\left\| \int_0^z (\partial_x \tilde{u}_\delta^n + \partial_y \tilde{v}_\delta^n) d\tilde{z} \right\|_{L^2(K)}^2 \leq C_K \|\partial_x \tilde{u}_\delta^n + \partial_y \tilde{v}_\delta^n\|_{L^2}^2 \leq C_K (\|\tilde{u}_\delta^n\|_{L^2(H^1)}^2 + \|\tilde{v}_\delta^n\|_{L^2(H^1)}^2)$$

for some positive constant C_K depending on K . Thus, it follows that

$$\left| \int_{[0,T] \times \mathbb{T}^2 \times \mathbb{R}^+} N_1(\tilde{u}_\delta^n, \tilde{v}_\delta^n) \varphi dx dy dz \right| \leq C_{K,\varphi} (\|\tilde{u}_\delta^n\|_{L^2(H^1)}^2 + \|\tilde{v}_\delta^n\|_{L^2(H^1)}^2).$$

Similarly, one can deduce

$$\left| \int_{[0,T] \times \mathbb{T}^2 \times \mathbb{R}^+} N_2(\tilde{u}_\delta^n, \tilde{v}_\delta^n) \varphi dx dy dz \right| \leq C_{K,\varphi} (\|\tilde{u}_\delta^n\|_{L^2(H^1)}^2 + \|\tilde{v}_\delta^n\|_{L^2(H^1)}^2).$$

Then, by using the uniform boundedness of $(\tilde{u}_\delta^n, \tilde{v}_\delta^n)$ in $L^\infty(0, T; L^2(\mathbb{T}^2 \times \mathbb{R}^+)) \cap L^2(0, T; H^1(\mathbb{T}^2 \times \mathbb{R}^+))$ with respect to δ , it implies that the nonlinear term $\delta N(\tilde{u}_\delta^n, \tilde{v}_\delta^n)$ converges to zero in the sense of distribution. Thus, letting $\delta \rightarrow 0$ in (4.7), we obtain that (u^n, v^n) solves the following linear problem in the sense of distribution,

$$(4.9) \quad \begin{cases} \partial_t(u^n, v^n) + \mathcal{P}_n(u^n, v^n) = 0, \\ (u^n, v^n)|_{t=0} = (u_0^n, v_0^n). \end{cases}$$

(4) Shift the time variable t to $t - s_n$ in (4.9), and denote by

$$(\tilde{u}^n, \tilde{v}^n)(t, \cdot) := (u^n, v^n)(t - s_n, \cdot).$$

Then, (4.9) becomes

$$(4.10) \quad \begin{cases} \partial_t(\tilde{u}^n, \tilde{v}^n) + \mathcal{P}(\tilde{u}^n, \tilde{v}^n) = 0, \\ (\tilde{u}^n, \tilde{v}^n)|_{t=s_n} = (u_0^n, v_0^n), \end{cases}$$

which means that $(\tilde{u}^n, \tilde{v}^n)$ solves the linearized problem (2.3) with $(\tilde{u}^n, \tilde{v}^n)|_{t=s_n} = (u_0^n, v_0^n)$. By virtue of the uniqueness given in Lemma 4.1, it follows that

$$(\tilde{u}^n, \tilde{v}^n) = (u_L^n, v_L^n), \quad \text{on } [s_n, T].$$

Therefore, from (4.1) and (4.6) we get a contradiction:

$$n \leq \|(u_L^n, v_L^n)(t_n)\|_{L^2} = \|(\tilde{u}^n, \tilde{v}^n)(t_n)\|_{L^2} = \|(u^n, v^n)(t_n - s_n)\|_{L^2} \leq C, \quad \forall n \geq 1,$$

as the positive constant C , given in (4.4), is independent of n . So we obtain the proof of Theorem 2. \square

5. APPENDIX

In this Appendix, we present the main steps of the proof of Proposition 2 given in Section 2, which shows that the three dimensional linearized Prandtl equations is well-posed locally in time when one component of the background tangential velocity, such as u^s , is monotonic in the normal variable, and we study the problem in the analytic setting only in the horizontal variable y .

Proof of Proposition 2. Let the solution of the linear problem (2.3) have the form

$$(u, v)(t, x, y, z) = \sum_{k \in \mathbb{Z}} e^{iky} (u_k, v_k)(t, x, z).$$

Plugging this relation into (2.3), it follows that

$$(5.1) \quad \begin{cases} \partial_t u_k + u^s \partial_x u_k - \partial_z^2 u_k + ikv^s u_k - u_z^s \int_0^z (\partial_x u_k + ikv_k) d\tilde{z} = 0, \\ \partial_t v_k + u^s \partial_x v_k - \partial_z^2 v_k + ikv^s v_k - v_z^s \int_0^z (\partial_x u_k + ikv_k) d\tilde{z} = 0, \\ (u_k, v_k)|_{z=0} = 0. \end{cases}$$

By assuming that $u^s(t, z)$ is monotonic in z , i.e. $\partial_z u^s > 0$, we employ the transformation given in [1] for the first component of the tangential velocity in the above problem,

$$h_k(t, x, z) \triangleq \partial_z \left(\frac{u_k(t, x, z)}{u_z^s(t, z)} \right), \text{ or } u_k(t, x, z) = u_z^s(t, z) \int_0^z h_k(t, x, \tilde{z}) d\tilde{z},$$

and set

$$\tilde{v}_k(t, x, z) \triangleq \left(v_k - \frac{v_z^s}{u_z^s} u_k \right)(t, x, z).$$

Then, from the problem (5.1) we know that (h_k, \tilde{v}_k) satisfies the following problem,

$$(5.2) \quad \begin{cases} \partial_t h_k + u^s \partial_x h_k - \partial_z^2 h_k - 2\partial_z \left(\frac{u_{zz}^s}{u_z^s} h_k \right) + ik(v^s h_k - \tilde{v}_k) = 0, \\ \partial_t \tilde{v}_k + u^s \partial_x \tilde{v}_k - \partial_z^2 \tilde{v}_k + ikv^s \tilde{v}_k - 2u_z^s \partial_z \left(\frac{v_z^s}{u_z^s} \right) h_k = 0, \\ \partial_z h_k|_{z=0} = 0, \quad \tilde{v}_k|_{z=0} = 0, \end{cases}$$

where we use $\partial_z^2 u_k|_{z=0}$, $u_{zz}^s|_{z=0} = 0$ to derive the boundary condition of h_k .

For the problem (5.2), by the energy method one can have

$$(5.3) \quad \begin{aligned} & \| (h_k, \tilde{v}_k)(t, \cdot) \|_{K_\alpha^m}^2 + \int_0^t \| (\partial_z h_k, \partial_z \tilde{v}_k)(s, \cdot) \|_{K_\alpha^m}^2 ds \\ & \leq C \left(\| (h_k, \tilde{v}_k)(0, \cdot) \|_{K_\alpha^m}^2 + \max\{1, |k|\} \int_0^t \| (h_k, \tilde{v}_k)(s, \cdot) \|_{K_\alpha^m}^2 ds \right), \end{aligned}$$

where the positive constant C depends on α and (u^s, v^s) . Applying the Gronwall inequality to (5.3), it implies that there exists a $\rho > 0$, depending on α and (u^s, v^s) , such that

$$(5.4) \quad \| (h_k, \tilde{v}_k)(t, \cdot) \|_{K_\alpha^m}^2 + \int_0^t \| (\partial_z h_k, \partial_z \tilde{v}_k)(s, \cdot) \|_{K_{\alpha,0}^m}^2 ds \leq C e^{\rho|k|t} \| (h_k, \tilde{v}_k)(0, \cdot) \|_{K_\alpha^m}^2.$$

From the assumption (2.5), we have

$$(5.5) \quad \| (h_k, \tilde{v}_k)(0, \cdot) \|_{K_\alpha^m} \leq C_0 e^{-\beta|k|},$$

for some positive constant C_0 . As $u^s \in C(\mathbb{R}^+; W_{\alpha}^{3,\infty}(\mathbb{R}^+))$, one has

$$(5.6) \quad \| u_k(t, \cdot) \|_{K_\alpha^m} = \| (u_z^s \int_0^z h_k d\tilde{z})(t, \cdot) \|_{K_\alpha^m} \leq C_1 \| h_k(t, \cdot) \|_{K_\alpha^m},$$

with the constant $C_1 > 0$ depending on α and u^s . Therefore, from the estimates (5.4)-(5.6) and the relation $v_k = \tilde{v}_k + \frac{v_z^s}{u_z^s} u_k$ it follows that

$$(5.7) \quad \| (u_k, v_k)(t, \cdot) \|_{K_\alpha^m} \leq C_2 e^{-(\beta-\rho t)|k|},$$

where the positive constant C_2 is independent of k . From the estimate (5.7) we complete the proof of this proposition. \square

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