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Pointwise estimates and L_p convergence rates to diffusion waves for p-system with damping

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Abstract

By introducing a new approximate Green function, we obtain the pointwise estimates on the solutions of Euler equations with linear frictional damping, from which we can deduce the optimal L_p $(1 \leqslant p \leqslant +\infty)$ convergence rates to the nonlinear diffusion waves. The pointwise estimates and L_p $(1 \leqslant p < 2)$ convergence rates given in this paper are new.

MSC: 35K; 76N; 35L

Keywords: p-System with damping; Nonlinear diffusion waves; Approximate Green function; Pointwise estimates

1. Introduction

In this paper, we study the time-asymptotic behavior of solutions to the p-system with frictional damping, which in Lagrangian coordinates can be written as

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u, \quad \alpha > 0, \ p' < 0, \end{cases} \tag{1.1}$$

with the initial data

$$(v, u)(x, 0) = (v_0(x), u_0(x)).$$

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Here $v(x,t) > 0$ and $u(x,t)$ represent the specific volume and velocity, respectively, and $\alpha > 0$ is the frictional coefficient. The pressure $p(v)$ is assumed to be a smooth function of v with $p(v) > 0,\ p'(v) < -C_0 < 0$ for v under consideration.

It was proved in [3] that the solutions to (1.1) time asymptotically behave like those governed by the Darcy's law in L_2 and L_∞ norms if the solution is away from vacuum. That is, as t tends to infinity, the solution $(v(x,t), u(x,t))$ of (1.1) approaches to the solution $(\bar{v}(x,t), \bar{u}(x,t))$ governed by the following system with the same end states as $v(x,0)$ at infinity:

$$\begin{cases} \bar{v}_t = -\frac{1}{\alpha} p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\alpha \bar{u}. \end{cases} \tag{1.2}$$

The convergence rates in L_p $(2 \leqslant p \leqslant \infty)$ norms were studied in [9,10] by using the energy method and an approximate Green function.

However, the pointwise estimates and L_p $(1 \leqslant p < 2)$ convergence rate cannot be obtained by the method used in [10]. The reason can be explained as follows. Since the large time behavior of the solution to the Euler equations with linear damping is governed by a parabolic equation derived by using Darcy's law, the main idea in [10] is to consider a linear parabolic equation by putting the second-order derivative with respect to time variable to the right-hand side of the equation and treating it as a source. Notice that this second-order derivative term is linear. The advantage of this method is that the obtained linear equation becomes a heat equation with variable coefficient which depends on the diffusion wave. Therefore, the approximate Green function can be defined easily which is a variation of the heat kernel. However, the second-order derivative with respect to time in the source requires higher-order derivatives on the solutions in the analysis. Consequently, to close the decay estimates of the solution needs a crude decay estimates on the highest order derivatives of the solution with respect to time obtained by energy method. Since the energy method is crucial there, to our knowledge, this will not give any detailed pointwise estimates. But the estimates for L_p with $p \geqslant 2$ can be obtained, cf. [10].

How to overcome the above difficulty is the main purpose of this paper. Here, we find a new way to construct an approximate Green function for the purpose of obtaining the pointwise estimates on the solutions so that a sharper result on the convergence rates in L_p $(1 \leqslant p \leqslant \infty)$ can be established. In particular, the L_1 estimate is obtained. Our idea is to keep the second-order derivative with respect to time variable on the left-hand side of the equation. Therefore, the linear equation now is of the second order, and of hyperbolic type with linear damping and variable coefficient. Notice that here the source term is nonlinear where the highest order derivative has a factor of lower-order derivatives. In the following analysis, we first give a detailed pointwise estimates of the Green function to the above linear hyperbolic equation by treating the variable coefficient as a parameter through Fourier analysis. Then we introduce a new method to define an approximate Green function to the original linearized equation. Based on these analysis, we can estimate the error due to the approximate Green function and the integral of the product of the approximate Green function and the nonlinear source to obtain the desired estimates. We should mention that some of the techniques used here in the Fourier analysis on the Green function come from the study on Navier–Stokes equations and hyperbolic systems with relaxation in [6,12].

Furthermore, as a preparation for the study of planar diffusion wave in multi-dimensional space, we give estimates on higher-order derivatives on the solutions in this one-dimensional setting. For this, we make some interesting improvement on the analysis in closing the a priori estimates in the last section.

There are other interesting problems concerning (1.1) with partial results obtained so far. One of them is the behavior of the solutions when vacuum appears. For this problem, the equivalence between (1.1) and (1.2) is known for some special families of solutions by construction and the L_∞ solutions by compensated compactness. Another problem is the existence of global weak solutions with finite total variation. This was proved for the case when the end states of the initial data at infinities coincide. But the problem without this restriction is still open. Interested readers please refer to [1,2,4,5,13] and reference therein. Some interesting works on the convergence to the Darcy's law from the compressible Euler flow and the study of limiting behavior of nonhomogeneous hyperbolic systems when the relaxed equilibrium is described by parabolic equations are done in [7,8]. There are also some works on the case with boundary and the case for nonisentropic gas which we do not refer here because it is irrelevant to this paper.

The rest of the paper is arranged as follows: The main theorems are stated in Section 2. In Section 3, we will first study the Green function with a parameter in details by Fourier analysis. In Section 4, the new approximate Green function is introduced and the proof of the main theorems is given in the last section.

Throughout this paper, C and b will be used to denote a generic positive large and small constants, respectively.

2. The main result

As in [3,9,10], we are interested in the large time behavior of the solution of (1.1) with initial data satisfying

$$(v,u)(x,0)\rightarrow(v_\pm,u_\pm)\quad \text{as } x\rightarrow\pm\infty \tag{2.1}$$

with v_+ not necessarily being equal to v_-. Denote the self-similar solution, i.e. the diffusion wave, of (1.2) in the form of $\varphi(\frac{x}{\sqrt{t+1}})$ by $\bar{v}(x,t)$ with the same end states at infinities as $v(x,0)$, i.e.,

$$\bar{v}(x,t)=\phi\left(\frac{x}{\sqrt{t+1}}\right),\quad \bar{v}(\pm\infty,t)=v_\pm. \tag{2.2}$$

Also set

$$\bar{u}(x,t) \equiv -\frac{1}{\alpha} p(\bar{v})_x. \tag{2.3}$$

From [3,10], we know that $\bar{v}(x,t)$ has the following exponential decay property in $\frac{x^2}{t+1}$:

$$\begin{aligned} &|\bar{v}(x,t) - v_+|_{x^2<1+t} + |v(x,t) - v_-|_{x^2>1+t} \\ &\quad \leqslant C|v_+ - v_-| e^{-C\alpha \frac{x^2}{1+t}}, \\ &|\partial_x^k \bar{v}(x,t)| \leqslant C|v_+ - v_-|(1+t)^{-k/2} e^{-C\alpha \frac{x^2}{1+t}}. \end{aligned} \tag{2.4}$$

Since the u component of the solution is expected to decay exponentially in t at $x = \pm\infty$, an auxiliary functions $(\tilde{u}, \tilde{v})$ was introduced in [3] as follows:

$$\tilde{u}(x,t) = e^{-\alpha t}\left(u_- + (u_+ - u_-)\int_{-\infty}^{x} m_0(y)\,dy\right) \tag{2.5}$$

and

$$\tilde{v}(x,t) = \frac{u_+ - u_-}{-\alpha} e^{-\alpha t} m_0(x), \tag{2.6}$$

where $m_0(x)$ is a smooth function with compact support satisfying

$$\int_{-\infty}^{\infty} m_0(x)\,dx = 1.$$

It is easy to check that $(\tilde{u}, \tilde{v})$ satisfies

$$\begin{cases} \tilde{v}_t - \tilde{u}_x = 0, \\ \tilde{u}_t = -\alpha \tilde{u}. \end{cases}$$

Let the initial data $v_0(x)$ be a small perturbation of a diffusion wave $\bar{v}(x,0)$. We are going to study how the solution behaves pointwise as t tends to infinity. As in [3], there exists a shift x_0 such that the initial data satisfies:

$$\int_{-\infty}^{\infty} (v_0(y) - \bar{v}(y + x_0, 0) - \tilde{v}(y,0))\,dy = 0.$$

By the first equation of (1.1), this implies that the function $V(x,t)$ defined by

$$V(x,t) = \int_{-\infty}^{x} (v(y,t) - \bar{v}(y + x_0, t) - \tilde{v}(y,t))\,dy, \tag{2.7}$$

satisfies $V(\pm\infty, t) = 0$. Here x_0 is a constant uniquely determined by

$$\int_{-\infty}^{\infty} (v(x,0) - \bar{v}(x + x_0, 0))\, dx = \frac{u_+ - u_-}{-\alpha}. \tag{2.8}$$

For later use, denote

$$U(x,t) = u(x,t) - \bar{u}(x + x_0, t) - \tilde{u}(x,t), \tag{2.9}$$

$V_0(x) = V(x,0)$ and $U_0(x) = U(x,0) = V_t(x,0)$. From (1.1), (1.2), (2.5)–(2.9), we have

$$\begin{cases} V_t - U = 0, \\ U_t + (p(V_x + \bar{v} + \tilde{v}) - p(\bar{v}))_x + \alpha U = \frac{1}{\alpha} p(\bar{v})_{xt}, \\ (V, U)|_{t=0} \equiv (V_0, U_0)(x) \to 0 \quad \text{as } x \to \pm\infty. \end{cases} \tag{2.10}$$

By linearizing the second equation of (2.10) about $\bar{v}$, we have the following system:

$$\begin{cases} V_t - U = 0, \\ U_t + (p'(\bar{v}) V_x)_x + \alpha U = F_1 + F_2, \\ (V, U)|_{t=0} \equiv (V_0, U_0)(x) \to 0 \quad \text{as } x \to \pm\infty, \end{cases} \tag{2.11}$$

where $F_j(x,t) = (\tilde{F}_j(x,t))_x$ $(j = 1, 2)$, and

$$\begin{aligned} \tilde{F}_1(x,t) &= \frac{1}{\alpha} p(\bar{v})_t, \\ \tilde{F}_2(x,t) &= -(p(V_x + \bar{v} + \tilde{v}) - p(\bar{v}) - p'(\bar{v}) V_x). \end{aligned} \tag{2.12}$$

From now on, we will study system (2.11). First, the following theorem is a direct consequence of the a priori estimates obtained in [3,9], and we omit its proof for brevity.

Theorem 2.1. *For sufficiently small* ε_0, *if*

$$|u_+ - u_-| + |v_+ - v_-| + ||V_0||_{H^{n+3}} + ||U_0||_{H^{n+2}} \leqslant \varepsilon_0$$

then there exists a global in time solution $(V(x,t), U(x,t))$ *of* (2.10) *satisfying*

$$||V(t)||_{H^{n+3}} + ||U(t)||_{H^{n+2}} \leqslant C\varepsilon_0, \tag{2.13}$$

where n is a positive integer.

Our main results of this paper are stated in the following.

Theorem 2.2. *For $m \geqslant 3$, if $V_0(x)$ and $U_0(x)$ satisfy the condition of Theorem* 2.1 *for $n = 2m$, and for some sufficiently small ε_0 and any positive number $N > \frac{1}{2}$,*

$$
\begin{aligned}
&|u_+ - u_-| + |v_+ - v_-| \leqslant \varepsilon_0, \\
&\sum_{h \leqslant m} |\partial_x^h V_0(x)| + \sum_{h \leqslant m-2} |\partial_x^h U_0(x)| \leqslant \varepsilon_0 (1 + x^2)^{-N},
\end{aligned}
\tag{2.14}
$$

then there exists a global in time solution $(V(x,t), U(x,t))$ of (2.10), *which satisfies*

$$
\begin{aligned}
&|\partial_x^k V(x,t)| \leqslant C\varepsilon_0 (1+t)^{-(k+1)/2} B_N(x,t), \\
&|\partial_x^l U(x,t)| \equiv |\partial_t \partial_x^l V(x,t)| \leqslant C\varepsilon_0 (1+t)^{-(l+3)/2} B_N(x,t),
\end{aligned}
\tag{2.15}
$$

where $k \leqslant m-1, l \leqslant m-3$ and

$$
B_N(x,t) = \left(1 + \frac{x^2}{1+t}\right)^{-N}.
$$

That is, in the original function (u, v), we have

$$
\begin{aligned}
&|\partial_x^{k-1}(v(x,t) - \bar{v}(x+x_0,t) - \tilde{v}(x,t))| \leqslant C\varepsilon_0 (1+t)^{-(k+1)/2} B_N(x,t), \\
&|\partial_x^l (u(x,t) - \bar{u}(x+x_0,t) - \tilde{u}(x,t))| \leqslant C\varepsilon_0 (1+t)^{-(l+3)/2} B_N(x,t).
\end{aligned}
$$

Remark 2.1. If the initial perturbation decays exponentially in x, then the perturbation decays exponentially in x for any time. In fact, if

$$
\begin{aligned}
&|u_+ - u_-| + |v_+ - v_-| \leqslant \varepsilon_0, \\
&\sum_{h \leqslant m} |\partial_x^h V_0(x)| + \sum_{h \leqslant m-2} |\partial_x^h U_0(x)| \leqslant \varepsilon_0 e^{-Cx^2},
\end{aligned}
\tag{2.16}
$$

then, for $k \leqslant m-1,\ l \leqslant m-3$, $(V(x,t), U(x,t))$ satisfies:

$$
\begin{aligned}
&|\partial_x^k V(x,t)| \leqslant C\varepsilon_0 (1+t)^{-(k+1)/2} e^{-\frac{bx^2}{1+t}}, \\
&|\partial_x^l U(x,t)| \leqslant C\varepsilon_0 (1+t)^{-(l+3)/2} e^{-\frac{bx^2}{1+t}}.
\end{aligned}
\tag{2.17}
$$

As a corollary of Theorem 2.2, we have the following theorem.

Theorem 2.3. *For $p \in [1, \infty]$ and $m \geqslant 2$, if $V_0(x)$ and $U_0(x)$ satisfy the conditions of Theorem* 2.1 *for $n = 2m$ and* (2.14), *then there exists a global in time solution*

$(V(x,t), U(x,t))$ *of* (2.10), *which satisfies*:

$$\|\partial_x^k V(t)\|_{L_p} \leqslant C\varepsilon_0(1+t)^{-\frac{1}{2}(1+k-\frac{1}{p})},$$
$$\|\partial_x^l U(t)\|_{L_p} \leqslant C\varepsilon_0(1+t)^{-\frac{1}{2}(3+l-\frac{1}{p})}, \tag{2.18}$$

where $k \leqslant m-1$, $l \leqslant m-3$. *That is, for the functions* (u,v), *we have*

$$\|\partial_x^{k-1}(v(x,t)-\bar{v}(x+x_0,t)-\tilde{v}(x,t))\|_{L_p} \leqslant C\varepsilon_0(1+t)^{-\frac{1}{2}(1+k-\frac{1}{p})},$$
$$\|\partial_x^l(u(x,t)-\bar{u}(x+x_0,t)-\tilde{u}(x,t))\|_{L_p} \leqslant C\varepsilon_0(1+t)^{-\frac{1}{2}(3+l-\frac{1}{p})}.$$

3. Green function with a parameter

In this section, we first consider a linear partial differential equation with constant coefficient and parameter λ as follows:

$$V_{tt} - \lambda V_{xx} + \alpha V_t = 0. \tag{3.1}$$

If $G^*(\lambda;x,t)$ is the Green function of (3.1), then it satisfies:

$$G^*_{tt}(\lambda;x,t) - \lambda(G^*(\lambda;x,t))_{xx} + \alpha G^*_t(\lambda;x,t) = 0,$$
$$G^*(\lambda;x,0) = 0, \quad G^*_t(\lambda;x,0) = \delta(x), \tag{3.2}$$

where λ is a parameter satisfying $C_1 > \lambda > C_0 > 0$ with constants C_1 and C_0, and δ is the Dirac function.

We now establish estimates of Green function of (3.2) which will be used to obtain the pointwise estimates for the approximate Green function. This can be done by Fourier analysis as in [6] for the Navier–Stokes equations and [12] for hyperbolic system with relaxation.

The Fourier transform of $f(x)$ is

$$\hat{f}(\xi,t) = \int_{\mathbf{R}} f(x,t)e^{-ix\xi}\,dx,$$

and the inverse Fourier transform is

$$f(x,t) \equiv (\mathscr{F}^{-1}\hat{f})(x,t) = (2\pi)^{-1}\int_{\mathbf{R}} \hat{f}(\xi,t)e^{ix\xi}\,d\xi.$$

The symbol of the operator for Eq. (3.1) is

$$\tau^2 + \lambda\xi^2 + \alpha\tau. \tag{3.3}$$

Here, τ and ξ correspond to $\frac{\partial}{\partial t}$ and D_x respectively, and $D_x = \frac{1}{i}\frac{\partial}{\partial_x}$. It is easy to see that the eigenvalues of (3.3) for τ are

$$\tau = \mu_{\pm}(\xi) \equiv \tfrac{1}{2}(-\alpha \pm \sqrt{\alpha^2 - 4\lambda\xi^2}). \tag{3.4}$$

From (3.2), we have by direct calculation that

$$\hat{G}^*(\lambda; \xi, t) = (\alpha^2 - 4\lambda\xi^2)^{-1/2}(e^{\mu_+(\xi)t} - e^{\mu_-(\xi)t}). \tag{3.5}$$

For convenience, we sometimes decompose $\hat{G}^*(\lambda; \xi, t) = \hat{G}^+(\lambda; \xi, t) + \hat{G}^-(\lambda; \xi, t)$, where

$$\hat{G}^{\pm}(\lambda; \xi, t) = \pm\mu_0^{-1} e^{\mu_{\pm}(\xi)t}, \quad \mu_0 = (\alpha^2 - 4\lambda\xi^2)^{1/2}.$$

Notice that

$$\partial_\lambda(\hat{G}^{\pm})(\lambda; \xi, t) = (2\xi^2\mu_0^{-2} \mp \xi^2 t\mu_0^{-1})\hat{G}^{\pm}(\lambda; \xi, t). \tag{3.6}$$

For $\hat{G}^*(\lambda; \xi, t)$ and $\hat{G}^{\pm}(\lambda; \xi, t)$, we first have the following lemma from standard consideration.

Lemma 3.1. *$\hat{G}^*(\lambda; \xi, t)$ is a holomorphic function of ξ. $\hat{G}^{\pm}(\lambda; \xi, t)$ also is a holomorphic function of ξ except isolated singularities $\xi = \pm\frac{\alpha}{2\sqrt{\lambda}}$.*

In the following, we are going to obtain some detailed properties of the Green function $G^*(\lambda; x, t)$. Denote

$$\begin{aligned} G^*(\lambda; x, t) &= \int_{-\varepsilon}^{\varepsilon} \hat{G}^*(\lambda; \xi, t) e^{ix\xi}\, d\xi + \int_{\varepsilon \leqslant |\xi| \leqslant R} \hat{G}^*(\lambda; \xi, t) e^{ix\xi}\, d\xi \\ &\quad + \int_{R \leqslant |\xi| \leqslant \infty} \hat{G}^*(\lambda; \xi, t) e^{ix\xi}\, d\xi \\ &=: G_1^*(\lambda; x, t) + G_2^*(\lambda; x, t) + G_3^*(\lambda; x, t). \end{aligned}$$

For $G^{\pm}$, the corresponding $G_j^{\pm}$ $(j = 1, 2, 3)$ can be defined in the same way.

Lemma 3.2. *For sufficiently small ε, there exist positive constants C and b, such that*

$$|\partial_t^l \partial_\lambda^h D_x^k G_1^+(\lambda; x, t)| \leqslant C(1+t)^{-(1+2l+k)/2} e^{-bx^2/(1+t)}, \tag{3.7}$$

$$|\partial_t^l \partial_\lambda^h D_x^k G_1^-(\lambda; x, t)| \leqslant C e^{-\alpha t/2} e^{-bx^2/(1+t)} \tag{3.8}$$

for any non-negative integers l, h and k.

Proof. Since $\hat{G}_\pm$ is a holomorphic function of ξ at the origin. Thus, we can move the path of integration from $[-\varepsilon, \varepsilon]$ to $\Sigma(-\varepsilon, \varepsilon, c)$, where

$$\Sigma(a,b,c) = \{\xi | \mathrm{Re}\,\xi = a; \mathrm{Im}\,\xi : 0 \to c\} \bigcup \{\xi | \mathrm{Im}\,\xi = c; \mathrm{Re}\,\xi : a \to b\} \bigcup \{\xi | \mathrm{Re}\,\xi = b; \mathrm{Im}\,\xi : c \to 0\}.$$

Let $\xi = \zeta + i\eta$, ζ and η are real numbers. Since $|\xi| \leqslant \varepsilon$ and $\varepsilon > 0$ is sufficiently small, we have

$$\mathrm{Re}(\mu_+ t + ix\xi) = (\alpha^{-1}\lambda\eta^2 t - x\eta) - \alpha^{-1}\lambda\zeta^2 t + O(1)(\eta^4 + \zeta^4)t,$$

$$\mathrm{Re}(\mu_- t + ix\xi) = -\alpha t + (-\alpha^{-1}\lambda\eta^2 t - x\eta) + \alpha^{-1}\lambda\zeta^2 t + O(1)(\eta^4 + \zeta^4)t.$$

We only prove (3.7) in the following since the corresponding estimate on G_1^- can be obtained in the same way. In fact, from the above identities, we have

$$\begin{aligned}
&\partial_t^l \partial_\lambda^h D_x^k \int_{-\varepsilon}^{\varepsilon} \hat{G}^+(\lambda; \xi, t) e^{ix\xi}\, d\xi \\
&= O(1) \int_{-\varepsilon}^{\varepsilon} e^{(\alpha^{-1}\lambda\eta^2 t - x\eta)} e^{-\alpha^{-1}\lambda\zeta^2 t} e^{O(1)(\eta^4+\zeta^4)t} (\eta^2 + \zeta^2)^{k/2+l} \\
&\quad \times ((\eta^2+\zeta^2)(1+t))^h\, d\zeta \\
&\quad + O(1) \int_0^c e^{(\alpha^{-1}\lambda\eta^2 t - x\eta)} e^{-\alpha^{-1}\lambda\zeta^2 t} e^{O(1)(\eta^4+\zeta^4)t} (\eta^2 + \zeta^2)^{k/2+l} \\
&\quad \times ((\eta^2+\zeta^2)(1+t))^h\, d\eta.
\end{aligned}$$

For $\frac{\alpha|x|}{2\lambda t} < \frac{\varepsilon}{2}$, we let $c = \frac{\alpha x}{2\lambda t}$, then $(\alpha^{-1}\lambda\eta^2 t - x\eta)|_{\eta=c} = \frac{-\alpha x^2}{2\lambda t}$. Denote the left-hand of (3.7) by $R_{l,h,k}^+$, then

$$\begin{aligned}
R_{l,h,k}^+ &\leqslant C(1+t)^{-k/2-l} e^{\frac{-\alpha x^2}{16\lambda t}} \int_{-\varepsilon}^{\varepsilon} e^{-4\alpha^{-1}\lambda\zeta^2 t} e^{O(1)(\eta^4+\zeta^4)t} \\
&\quad \times ((\eta^2+\zeta^2)(1+t))^{k/2+h+l}\, d\zeta \\
&\quad + C(1+t)^{-k/2-l} e^{-4\alpha^{-1}\lambda\varepsilon^2 t} \int_0^{\frac{\alpha x}{2\lambda t}} e^{(4\alpha^{-1}\lambda\eta^2 t - |x|\eta)} e^{O(1)(\eta^4+\zeta^4)t} \\
&\quad \times ((\eta^2+\zeta^2)(1+t))^{k/2+l+h}\, d\eta \\
&\leqslant C(1+t)^{-(k+1)/2-l} (e^{-bx^2/(1+t)} + e^{-bt}).
\end{aligned}$$

For $\frac{\alpha|x|}{2\lambda t} \geqslant \frac{\varepsilon}{2}$, we let $c = \frac{\varepsilon}{2}\operatorname{sign} x$, then

$$\begin{aligned} R^+_{l,h,k} &\leqslant C(1+t)^{-k/2-l} e^{-\varepsilon|x|/4} \int_{-\varepsilon}^{\varepsilon} e^{-4\alpha^{-1}\lambda\zeta^2 t} e^{O(1)(\eta^4+\zeta^4)t} \\ &\quad \times ((\eta^2+\zeta^2)(1+t))^{k/2+h+l}\, d\zeta \\ &\quad + C(1+t)^{-k/2-l} e^{-3\alpha^{-1}\lambda\varepsilon^2 t} \int_0^{\varepsilon/2} e^{-|x|\eta} e^{O(1)(\eta^4+\zeta^4)t} \\ &\quad \times ((\eta^2+\zeta^2)(1+t))^{k/2+l+h}\, d\eta \\ &\leqslant C(1+t)^{-(k+1)/2-l}(e^{-bt} + e^{-b_1|x|}) \\ &\leqslant C(1+t)^{-(k+1)/2-l} e^{-bt}. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.3. *For fixed ε and R, there exist positive constants b and C, such that*

$$|\partial_t^l \partial_\lambda^h D_x^k G_2^*(\lambda; x, t)| \leqslant Ce^{-bt}. \tag{3.9}$$

Proof. For any fixed ε and R, we can choose $b_1 > 0$ sufficiently small such that if $\varepsilon \leqslant |\xi| \leqslant R$

$$|e^{\mu_+(\xi)t}| \leqslant Ce^{-b_1 t}.$$

As for $e^{\mu_-(\xi)t}$, since $\operatorname{Re}\mu_-(\xi) \leqslant -\frac{\alpha}{2}$, we have

$$|e^{\mu_-(\xi)t}| \leqslant Ce^{-\alpha t/2}.$$

Thus if $|\mu_0(\xi)| \geqslant \varepsilon_1 > 0$, one has

$$|\hat{G}_2^*(\lambda; \xi, t)| \leqslant Ce^{-b_2 t}.$$

Notice that $e^{\mu_+(\xi)t} - e^{\mu_-(\xi)t} = O(1)e^{-\alpha t/2}\mu_0(\xi)t$ if $|\mu_0(\xi)| < \varepsilon_1$, and ε_1 is sufficiently small, we also get above inequality. Thus, for any fixed ε and R, we deduce that

$$|G_2^*(\lambda; x, t)| \leqslant Ce^{-b_2 t} \int_{\varepsilon \leqslant |\xi| \leqslant R} d\xi \leqslant Ce^{-b_2 t}.$$

By applying $\partial_t^l \partial_\lambda^h D_x^k$ on both sides of (3.1), we can prove similarly that

$$|\partial_t^l \partial_\lambda^h D_x^k G_2^*(\lambda; x, t)| \leqslant Ce^{-bt} \int_{\varepsilon \leqslant |\xi| \leqslant R} (1+|\xi|)^{k+l+h}\, d\xi \leqslant Ce^{-bt}$$

for some $b > 0$, which implies (3.9). □

Now we consider $G_3^*(\lambda; x, t)$. By letting

$$\mu_\eta = \sqrt{\eta - 4\lambda/\alpha^2},$$

and then taking the Taylor expansion of μ_η in η, we have

$$\mu_\eta = i\frac{2\sqrt{\lambda}}{\alpha} + i\eta\left(\frac{\alpha}{\sqrt{\lambda}}\right) + O(1)(\eta^2). \tag{3.10}$$

Since

$$\mu_\pm(\xi) = \tfrac{1}{2}(-\alpha \pm \xi\alpha\sqrt{\xi^{-2} - 4\lambda/\alpha^2}),$$

when $|\xi|$ is sufficiently large, we arrive at

$$\mu_\pm(\xi) = \frac{1}{2}\left(-\alpha \pm 2i\sqrt{\lambda}\xi \pm i\frac{\alpha^2}{\sqrt{\lambda}\xi} + O(|\xi|^{-3})\right).$$

This implies that

$$e^{\mu_\pm(\xi)t} = e^{-\alpha t/2}e^{\pm i\sqrt{\lambda}\xi t}e^{\pm i\frac{\alpha^2 t}{2\sqrt{\lambda}\xi}}(1 + O(1)(|\xi|^{-3})t). \tag{3.11}$$

Eq. (3.11) will be used in the proofs of the following lemmas.

Lemma 3.4. *For R sufficiently large, there exists a constant $C > 0$, such that*

$$|G_3^*(\lambda; x, t)| \leqslant Ce^{-\alpha t/4}. \tag{3.12}$$

Moreover, there exists a distribution

$$K_{l,k,h}^*(\lambda; x, t) = e^{-\alpha t/2}\sum_{j=0}^{l+k+h-1}(q_{j+h}^+(t)\delta^{(l+k+h-j)}(x + \sqrt{\lambda}t) + q_{j+h}^-(t)\delta^{(l+k+h-j)}(x - \sqrt{\lambda}t)),$$

such that when $l + k \geqslant 1$, we have

$$|\partial_t^l\partial_\lambda^h D_x^k G_3^*(\lambda; x, t) - K_{l,k,h}^*(\lambda; x, t)| \leqslant Ce^{-\alpha t/4}, \tag{3.13}$$

where $q_{j+h}^\pm(t)$, $j = 0, \ldots, l + k$ are polynomials of t with degrees not greater than $j + h$ correspondingly.

Proof. Since for $|\xi|$ large enough,

$$\mu_0^{-1}(\xi) = \frac{-i}{2\sqrt{\lambda}\xi} + O(|\xi|^{-2}),$$

Eq. (3.11) gives

$$\hat{G}^*(\lambda;\xi,t) = \frac{-i}{2\sqrt{\lambda}\xi} e^{-\alpha t/2}(e^{i\sqrt{\lambda}\xi t} - e^{-i\sqrt{\lambda}\xi t}) + O(|\xi|^{-2})(1 + t + O(|\xi|^{-1})t)e^{-\alpha t/2}.$$

Thus

$$\begin{aligned} |G_3^*(\lambda;x,t)| &\leqslant Ce^{-\alpha t/2}\left|\int_{R\leqslant|\xi|\leqslant\infty} \frac{-i}{2\sqrt{\lambda}\xi}(e^{i\sqrt{\lambda}\xi t} - e^{-i\sqrt{\lambda}\xi t})e^{ix\xi}\,d\xi\right| \\ &\quad + Ce^{-\alpha t/4} \\ &\leqslant Ce^{-\alpha t/2}\left|\int_R^\infty \frac{\sin(x-\sqrt{\lambda}t)\xi - \sin(x+\sqrt{\lambda}t)\xi}{\xi}\,d\xi\right| \\ &\quad + Ce^{-\alpha t/4} \\ &\leqslant Ce^{-\alpha t/4}, \end{aligned}$$

which implies (3.12). For the general case, since

$$\begin{aligned} &\partial_t^l \partial_\lambda^h \xi^k \hat{G}^*(\lambda;\xi,t) \\ &= O(1)e^{-\alpha t/2}\left(e^{i\sqrt{\lambda}\xi}\left(\sum_{j=0}^{l+k+h} q_{j+h}^+(t)|\xi|^{-j} + q_{l+k+2h+1}^+(t)|\xi|^{-l-k-h-1}\right)\right. \\ &\quad \left. + e^{-i\sqrt{\lambda}\xi}\left(\sum_{j=0}^{l+k+h} q_{j+h}^-(t)|\xi|^{-j} + q_{l+k+2h+1}^-(t)|\xi|^{-l-k-h-1}\right)\right) \\ &\quad \times |\xi|^{-1+k+l+h} \end{aligned}$$

if we set

$$\begin{aligned} \hat{K}_{l,k,h}^*(\lambda;\xi,t) = O(1)e^{-\alpha t/2}\int_{-\infty}^\infty &\left(e^{i\sqrt{\lambda}\xi}\left(\sum_{j=0}^{l+k+h-1} q_{j+h}^+(t)|\xi|^{-j}\right)\right. \\ &\left. + e^{-i\sqrt{\lambda}\xi}\left(\sum_{j=0}^{l+k+h-1} q_{j+h}^-(t)|\xi|^{-j}\right)\right)|\xi|^{-1+k+h+l}e^{ix\xi}\,d\xi, \end{aligned}$$

where $q_m^{\pm}(t)$ is polynomial of t with degree not greater than m, and $|q_{j+h}^{+}(t)| = |q_{j+h}^{-}(t)|$ then we can deduce that

$$
\begin{aligned}
&|\partial_t^l D_x^k \partial_\lambda^h G_3^*(\lambda; x, t) - K_{l,k,h}^*(\lambda; x, t)| \\
&\leqslant \left| \int_{|\xi| \geqslant R} (\partial_t^l \xi^k \partial_\lambda^h \hat{G}^*(\lambda; x, t) - \hat{K}_{l,k,h}^*(\lambda; \xi, t)) e^{ix\xi}\, d\xi \right| \\
&\quad + \left| \int_{|\xi| \leqslant R} \hat{K}_{l,k,h}^*(\lambda; \xi, t)\, d\xi \right| \\
&\leqslant Ce^{-\alpha t/3} \left| \int_{|\xi| \geqslant R} (e^{i\sqrt{\lambda}\xi} q_{l+k+2h}^{+}(t)\xi^{-1} + e^{-i\sqrt{\lambda}\xi} q_{l+k+2h}^{-}(t)\xi^{-1}) e^{ix\xi}\, d\xi \right| \\
&\quad + e^{-\alpha t/3} \left| \int_{|\xi| \geqslant R} e^{O(|\xi|^{-1})t} O(|\xi|^{-2}) e^{ix\xi}\, d\xi \right| + Ce^{-\alpha t/4}.
\end{aligned}
$$

Similar to the proof of (3.12), we get (3.13). □

Lemma 3.5. *For $M > 0$ sufficiently large, there exists a constant $C > 0$, such that when $|x|/(1+t) \geqslant M$, we have*

$$
|\partial_t^l D_x^k \partial_\lambda^h G^*(\lambda; x, t) - K_{l,k,h}^*(\lambda; x, t)| \leqslant Ce^{-\alpha t/4} e^{-|x|^2/(1+t)}. \tag{3.14}
$$

Proof. By Lemma 3.1, we can move the path of integration to $\Sigma(-\infty, \infty, 2x/(1+t))$. Then

$$
G^*(\lambda; x, t) = C \int_{-\infty}^{\infty} \hat{G}^*(\lambda; \zeta + 2ix/(1+t), t) e^{ix\zeta} e^{-2x^2/(1+t)}\, d\zeta.
$$

Since $|x|/(1+t) > M$ and M is sufficiently large, we know that $|\xi| = |\zeta + 2ix/(1+t)|$ is large enough and then (3.11) can be applied to variable $\zeta + 2ix/(1+t)$. Combining this observation with the above identity imply that

$$
\begin{aligned}
&G^*(\lambda; x, t) \\
&= O(1) e^{-\alpha t/2} \Bigg(e^{-(2x^2 + \sqrt{\lambda} x t)/(1+t)} \int_{-\infty}^{\infty} (\zeta + 2ix/(1+t))^{-1} e^{i(x+\sqrt{\lambda} t)\zeta}\, d\zeta \\
&\quad + e^{-(2x^2 - \sqrt{\lambda} x t)/(1+t)} \int_{-\infty}^{\infty} (\zeta + 2ix/(1+t))^{-1} e^{i(x-\sqrt{\lambda} t)\zeta}\, d\zeta \Bigg) \\
&\quad + O(1) e^{-\alpha t/2} e^{-2x^2/(1+t)} \\
&\qquad \times \int_{-\infty}^{\infty} O((\zeta^2 + (2x/(1+t))^2)^{-1} (1 + t e^{O(\zeta^2 + (2x/(1+t))^{-2}) t})\, d\zeta.
\end{aligned}
$$

Since

$$\frac{1}{\zeta+2ix/(1+t)}=\frac{1}{\zeta}-\frac{2ix/(1+t)}{\zeta^2+(2x/(1+t))^2}-\frac{(2x/(1+t))^2}{\zeta^2+(2x/(1+t))^2}\frac{1}{\zeta},$$

we have

$$\int_{-\infty}^{\infty}(\zeta\pm 2ix/(1+t))^{-1}e^{i(x+\sqrt{\lambda}t)\zeta}\,d\zeta$$

$$\leqslant C\left(|x\pm\sqrt{\lambda}t|+\frac{|x|}{1+t}+\frac{|x|^2}{(1+t)|x\pm\sqrt{\lambda}t|}\right).$$

In summary, by noticing that $|x|/(1+t)>M$, we have

$$|G^{*}(\lambda;x,t)|\leqslant Ce^{-\alpha t/2}e^{-x^2/(1+t)}$$

$$\times\left(|x\pm\sqrt{\lambda}t|+\frac{|x|}{1+t}+\frac{|x|^2}{(1+t)|x\pm\sqrt{\lambda}t|}+1+e^{t/M}\right)$$

$$\leqslant Ce^{-\alpha t/2}e^{-bx^2/(1+t)}.$$

Thus, we have (3.14) for $l=k=h=0$. The general case can be proved similarly as in Lemma 3.5. □

Combining Lemmas 3.2 with 3.5, we have

Theorem 3.1. *For non-negative integers* l, k *and* h, *there exists a distribution* $K^{*}_{l,k,h}(\lambda;x,t)$, *such that*

$$|\partial_t^l D_x^k \partial_\lambda^h G^{*}(\lambda;x,t)-K^{*}_{l,k,h}(\lambda;x,t)|\leqslant C(1+t)^{-(1+k+2l)/2}e^{-bx^2/t}. \qquad (3.15)$$

Proof. If $|x|/(1+t)\leqslant M$, we have

$$-t\leqslant-\frac{M^2x^2}{1+t},\quad |x\pm\sqrt{\lambda}t|\leqslant(M+\sqrt{\lambda})(1+t).$$

Thus, (3.15) holds for $|x|/(1+t)\leqslant M$ by Lemmas 3.2–3.4. For $|x|/(1+t)>M$, Lemma 3.5 implies (3.15), and this completes the proof. □

4. Approximate Green function

In this section, we construction the approximate Green function for the unknown function $V(x,t)$.

$$V_{tt} - (a(x,t)V_x)_x + \alpha V_t = F, \tag{4.1}$$

where $a(x,t) = -p'(\bar{v}(x,t)) > C_0 > 0$, $F = F_1 + F_2$. The constructed approximate Green function $G(x,t;y,s)$ for (4.1) satisfies the basic requirement

$$G(x,t;y,t) = 0, \quad G_s(x,t;y,t) = \delta(y-x). \tag{4.2}$$

By multiplying (4.1) whose variables are now changed to (y,s) by G and integrating over the region $(y,s) \in \mathbf{R} \times (0,t)$, we have

$$\begin{aligned} V(x,t) &= \int_{-\infty}^{\infty} G_s(x,t;y,0)V_0(y)\,dy + \int_{-\infty}^{\infty} G(x,t;y,0)(\alpha V_0(y) + V_t(y,0))\,dy \\ &\quad + \int_0^t \int_{-\infty}^{\infty} G(x,t;y,s)F(y,s)\,dy\,ds \\ &\quad + \int_0^t \int_{-\infty}^{\infty} ((G_{ss} - \alpha G_s)(x,t;y,s) \\ &\quad - (a(y,s)G_y(x,t;y,s))_y)V(y,s)\,dy\,ds. \end{aligned} \tag{4.3}$$

If $a(y,s)$ is a constant and G is a Green function of (4.1), then we know that the last integral of (4.3) is equal to zero. However, it now depends on the profile of the diffusion wave. Therefore, we can only try to minimize the term $G_{ss} - \alpha G_s - (aG_y)_y$. For this purpose, we compare it with the linear partial differential equation with constant coefficient and a parameter λ, i.e. (3.1). From the discussion of the last section, the Green function $G^*(\lambda;x,t)$ satisfies (3.2) and Theorem 3.1. Define an approximate Green function for (4.1) as follows:

$$G(x,t;y,s) = G^*(a(y,\sigma(t,s));x-y,t-s), \tag{4.4}$$

where $\sigma(t,s) \in C^3(\mathbf{R}^2)$ and

$$\sigma(t,s) = \begin{cases} s, & s > t/2 + 1, \\ t/2, & s \leqslant t/2. \end{cases}$$

Moreover, we can choose $\sigma(t,s)$ to be smooth when $s \in (t/2, t/2+1)$, such that

$$\sum_{1 \leqslant l_1 + l_2 \leqslant 3} |\partial_t^{l_1} \partial_s^{l_2} \sigma(t,s)| \leqslant C. \tag{4.5}$$

When $t > 1$, we also obtain

$$\sigma(t,s)^{-1} \leqslant \frac{C}{1+t}. \tag{4.6}$$

It is clear that the approximate Green function defined in (4.4) satisfies the condition (4.2), and

$$G^*_{33}(a(y,\sigma); x-y, t-s) - a(y,\sigma)G^*_{22}(a(y,\sigma); x-y, t-s) + \alpha G^*_3(a(y,\sigma); x-y, t-s) = 0, \tag{4.7}$$

where $G^*_j = G^*_j(a(y,\sigma), x-y, t-s)$ denotes the partial derivative with respect to the jth $(j = 1,2,3,4)$ variable of G^*.

The function $G(x,t;y,s)$ is not symmetric with respect to the variables x,t and y,s. Instead, we have the following relations:

$$\begin{aligned} \partial_x G &= -\partial_y G + \partial_a(G^*)a_y, \\ \partial_t G &= -\partial_s G + \partial_a(G^*)(a_s + a_t), \end{aligned} \tag{4.8}$$

where a_y, a_t and a_s represent the derivatives of a with respect to y,t,s, respectively. Since $a(y,\sigma(t,s)) = -p'(\bar{v}(y,\sigma(t,s)))$, it follows from [10] that

$$|\partial_t^{l_1}\partial_s^{l_2}\partial_y^k a(y,\sigma(t,s))| \leqslant C\varepsilon_0(1+t)^{-(k/2+l_1+l_2)} \tag{4.9}$$

for $k + l_1 + l_2 \geqslant 1$. Now we give a pointwise estimate to the approximate Green function $G(x,t;y,s)$. In fact, from Theorem 3.1 and the above discussion, it is straightforward to have the following theorem.

Theorem 4.1. *For $l = 0,1$ and a positive integer k, there exists a distribution*

$$\begin{aligned} &K_{l,k}(x,y;t,s) \\ &= e^{-\alpha(t-s)/2}\sum_{j=0}^{l+k-1}(\tilde{q}_j^+(t-s)\delta^{(l+k-j)}(x-y+\sqrt{a(y,\sigma(t,s))}) \\ &\quad + \tilde{q}_j^-(t-s)\delta^{(l+k-j)}(x-y-\sqrt{a(y,\sigma(t,s))}), \end{aligned} \tag{4.10}$$

such that

$$\begin{aligned} &\left|\sum_{l_1+l_2=l,k_1+k_2=k}(\partial_t^{l_1}\partial_s^{l_2}\partial_x^{k_1}\partial_y^{k_2}G(x,t;y,s) - K_{l,k}(x,y;t,s))\right| \\ &\leqslant C((1+t)^{-(1+k+2l)/2} + (1+t-s)^{-(1+k+2l)/2})e^{-b(x-y)^2/(t-s)}, \end{aligned} \tag{4.11}$$

where $l = 0, 1$ and $\tilde{q}_j^{\pm}$ $(j = 0, \ldots, l + k - 1)$ are polynomials of t with degrees not greater than j correspondingly.

Given a function $g(y, s)$, and $k \geqslant 1$, from (4.8) and (4.9), we have

$$\begin{aligned}
&\int_a^b \int_{-\infty}^{\infty} \partial_x^{h+k} \partial_t^l G(x, t; y, s) g(y, s)\, dy\, ds \\
&\quad = \int_a^b \int_{-\infty}^{\infty} \partial_x^h \partial_t^l G(x, t; y, s) \partial_y^k g(y, s)\, dy\, ds \\
&\qquad + O(1)\varepsilon_0 \sum_{k' < k} \int_a^b \int_{-\infty}^{\infty} ((1 + t - s)^{-(1+2l+h)/2} \\
&\qquad\quad \times (1 + t)^{-(k-k')}) e^{-b(x-y)^2/(t-s)} \\
&\qquad + K_{l,h}(x, y; t, s)) \partial_y^{k'} g(y, s)\, dy\, ds.
\end{aligned} \tag{4.12}$$

Set the error due to the approximate Green function by R_G,

$$R_G \equiv G_{ss}(x, t; y, s) - \alpha G_s(x, t; y, s) - (a(y, s) G_y(x, t; y, s))_y.$$

Then (4.7) implies that

$$\begin{aligned}
R_G = &(a(y, \sigma) - a(y, s)) G_{22}^* + 2G_{13}^* a_s(y, \sigma) + G_{11}^* (a_s(y, \sigma))^2 \\
&+ G_1^* a_{ss}(y, \sigma) - a(y, s)(2G_{12}^* a_y(y, \sigma) + G_{11}^* (a_y(y, \sigma)^2) \\
&+ G_1^* a_{yy}(y, \sigma) - \alpha G_1^* a_s(y, \sigma).
\end{aligned} \tag{4.13}$$

Notice that

$$\begin{aligned}
&|a(y, s) - a(y, \sigma)| \\
&\quad = \begin{cases} O(1)\varepsilon_0 (1 + t - s)^{1/2} (1 + s)^{-1/2} e^{-\frac{by^2}{1+t}}, & s < t/2 + 1; \\ 0, & s \geqslant t/2 + 1. \end{cases}
\end{aligned} \tag{4.14}$$

It follows from Theorem 4.1, (4.9) and (4.13) that there exists a distribution

$$\begin{aligned}
&K_G(x, y; t, s) \\
&\quad = (a(y, \sigma) - a(y, s)) K_{2,0,0}^* + 2K_{0,1,1}^* a_s(y, \sigma) + K_{0,0,2}^* (a_s(y, \sigma))^2 \\
&\qquad + K_{0,0,1}^* a_{ss}(y, \sigma) - a(y, s)(2K_{1,0,1}^* a_y(y, \sigma) + K_{0,0,2}^* (a_y(y, \sigma)^2) \\
&\qquad + K_{0,0,1}^* a_{yy}(y, \sigma) - \alpha K_{0,0,1}^* a_s(y, \sigma),
\end{aligned}$$

such that

$$R_G = O(1)\varepsilon_0 \Theta(y,t,s)((1+t-s)^{-1/2}e^{-b(x-y)^2/(t-s)} + K_G(x,y;t,s)), \quad (4.15)$$

where

$$\Theta(y,t,s) = ((1+t)^{-1} + (1+t-s)^{-1/2}(1+s)^{-1/2})e^{-\frac{by^2}{1+t}}.$$

Furthermore, it can be shown by straightforward calculation that there exists a distribution

$$\begin{aligned}
&K_{G,l,k}(x,y;t,s)\\
&\quad = e^{-\alpha(t-s)/2}\sum_{j=0}^{l+k+1}(Q_j^+(t-s)\delta^{(l+k+2-j)}(x-y+\sqrt{a(y,\sigma(t,s))})\\
&\qquad + Q_j^-(t-s)\delta^{(l+k+2-j)}(x-y-\sqrt{a(y,\sigma(t,s))}),
\end{aligned}$$

such that

$$\begin{aligned}
&\sum_{l_1+l_2=l,k_1+k_2=k}\partial_t^{l_1}\partial_s^{l_2}\partial_x^{k_1}\partial_y^{k_2}R_G(x,t;y,s)\\
&\quad = O(1)\varepsilon_0\Theta(y,t,s)((1+t-s)^{-\frac{1+k+2l}{2}}e^{-b(x-y)^2/(t-s)}\\
&\qquad + K_{G,l,k}(x,y;t,s)). \qquad (4.16)
\end{aligned}$$

Here $l = 0,1$ and $Q_j^{\pm}$ $(j = 0,\dots,l+k+1)$ are polynomials of t with degrees not greater than j correspondingly.

In deducing the pointwise estimates, we need to perform integrations by parts for R_G and a given function $g(y,s)$. For later use, we list the following two identities:

$$\begin{aligned}
&\int_a^b\int_{-\infty}^{\infty}\partial_x^k R_G(x,t;y,s)g(y,s)\,dy\,ds\\
&\quad = \int_a^b\int_{-\infty}^{\infty}R_G(x,t;y,s)\partial_y^k g(y,s)\,dy\,ds\\
&\qquad + O(1)\varepsilon_0\sum_{l<k}\int_a^b\int_{-\infty}^{\infty}\Theta(y,t,s)\\
&\qquad\quad \times((1+t)^{-(k-l)/2)}(1+t-s)^{-1/2}e^{-b(x-y)^2/(t-s)}\\
&\qquad + K_G(x,y;t,s))\partial_y^l g(y,s)\,dy\,ds \qquad (4.17)
\end{aligned}$$

and

$$\int_a^b \int_{-\infty}^{\infty} \partial_t \partial_x^k R_G(x,t;y,s) g(y,s)\, dy\, ds$$

$$= \int_a^b \int_{-\infty}^{\infty} R_G(x,t;y,s) \partial_s \partial_y^k g(y,s)\, dy\, ds$$

$$- \int_{-\infty}^{\infty} \partial_x^k R_G(x,t;y,s) g(y,s)\, dy|_{s=a}^{s=b}$$

$$+ O(1)\varepsilon_0 \sum_{l<k} \int_a^b \int_{-\infty}^{\infty} \Theta(y,t,s)((1+t)^{-(2+k-l)/2)}(1+t-s)^{-1/2}$$

$$\times e^{-b(x-y)^2/(t-s)} + K_G(x,y;t,s))\partial_y^l g(y,s)\, dy\, ds. \tag{4.18}$$

All the above pointwise estimates and identities will be used in the next section to obtain pointwise estimates and the L_p convergence rates on the solutions to (4.1).

5. The Proof of Theorem 2.2

From (4.3), we have

$$\partial_t^l \partial_x^k V(x,t) = I_1^{l,k} + I_2^{l,k} + I_3^{l,k}, \tag{5.1}$$

where

$$I_1^{l,k} = \int_{-\infty}^{\infty} \partial_t^l \partial_x^k G_s(x,t;y,0) V(y,0)\, dy$$

$$- \int_{-\infty}^{\infty} \partial_t^l \partial_x^k G(x,t;y,0)(\alpha V(y,0) + V_t(y,0))\, dy,$$

$$I_2^{l,k} = \partial_t^l \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x,t;y,s) F(y,s)\, dy\, ds,$$

$$I_3^{l,k} = \partial_t^l \int_0^t \int_{-\infty}^{\infty} \partial_x^k R_G(x,t;y,s) V(y,s)\, dy\, ds.$$

We will derive the estimates on $I_j^{l,k}$ $(j=1,2,3)$ in the following. First, we give a lemma which will be used in the analysis.

Lemma 5.1. *If* $0<C_0\leqslant\lambda\leqslant C_1$, *then for a positive number* $N>\frac{1}{2}$, *we have*

$$B_N(x\pm\sqrt{\lambda}t,t)\leqslant C(1+t)^{2N}B_N(x,t), \tag{5.2}$$

$$\int_{-\infty}^{\infty} e^{-b(x-y)^2/t}(1+y^2)^{-N}\,dy\leqslant CB_N(x,t) \tag{5.3}$$

and

$$\int_{-\infty}^{\infty} e^{-b(x-y)^2/(t-s)}B_N(y,s)\,dy\leqslant C\sqrt{\frac{(1+t-s)(1+s)}{1+t}}B_N(x,t). \tag{5.4}$$

Proof. Since $e^{-x^2/t}\leqslant CB_N(x,t)$, (5.2) and(5.3) are direct consequences of Lemmas 3.4 and 5.1 of [11]. For (5.4), we only need to prove that

$$J_N=:\int_{-\infty}^{\infty} B_N(x-y,t-s)B_N(y,s)\,dy\leqslant C\sqrt{\frac{(1+t-s)(1+s)}{1+t}}B_N(x,t).$$

If $x^2\leqslant t$, then $1\leqslant 2^N B_N(x,t)$. Thus,

$$J_N\leqslant\int_{-\infty}^{\infty} B_N(x-y,t-s)\,dy\leqslant C\varepsilon_0(1+t-s)^{1/2}.$$

Also we can obtain $I_N\leqslant C\varepsilon_0(1+s)^{1/2}$ similarly. Thus, (5.4) hold when $x^2\leqslant t$. For case when $x^2>t$, since

$$\left(1+\frac{(x-y)^2}{1+t-s}\right)^{-N}\left(1+\frac{y^2}{1+s}\right)^{-N}$$
$$\leqslant\begin{cases} C(1+\frac{x^2}{1+t-s})^{-N}(1+\frac{y^2}{1+s})^{-N}, & |x|\geqslant|y|/2,\\ C(1+\frac{(x-y)^2}{1+t-s})^{-N}(1+\frac{x^2}{1+s})^{-N}, & |x|\leqslant|y|/2,\end{cases}$$

we have

$$J_N \leqslant C\int_{-\infty}^{\infty}\left(1+\frac{x^2}{1+t-s}\right)^{-N}\left(1+\frac{y^2}{1+s}\right)^{-N} dy$$

$$+ C\int_{-\infty}^{\infty}\left(1+\frac{(x-y)^2}{1+t-s}\right)^{-N}\left(1+\frac{x^2}{1+s}\right)^{-N} dy$$

$$\leqslant C\left(\left(\frac{1+t-s}{1+t}\right)^{N}(1+s)^{1/2}+\left(\frac{1+s}{1+t}\right)^{N}(1+t-s)^{1/2}\right)B_N(x,t)$$

$$\leqslant C\sqrt{\frac{(1+t-s)(1+s)}{1+t}}B_N(x,t).$$

Thus the lemma is proved. □

For $I_1^{l,k}$, we know from Theorem 4.1 and the conditions in Theorem 2.2 that

$$I_1^{l,k} = \int_{-\infty}^{\infty}(\partial_t^l\partial_x^k G_s(x,t;y,0)-K_{l+1,k}(x-y,t))V(y,0)\,dy$$

$$+\int_{-\infty}^{\infty}(\partial_t^l\partial_x^k G(x,t;y,0)-K_{l,k}(x-y,t))(\alpha V+V_t)(y,0)\,dy$$

$$+\int_{-\infty}^{\infty}(K_{l+1,k}(x-y,t)V(y,0)+K_{l,k}(x-y,t)(\alpha V+V_t)(y,0))\,dy$$

$$\leqslant C\varepsilon_0(1+t)^{-\frac{1+k+2l}{2}}$$

$$\times\begin{cases}\int_{-\infty}^{\infty} e^{-(x-y)^2/t}(1+y^2)^{-N}\,dy & \\ \quad +e^{-\alpha t/3}(1+x^2)^{-N}, & 2l+k\leqslant m-1,\\ \quad 1, & 2l+k>m-1.\end{cases}$$

By Lemma 5.1, we know that

$$|I_1^{l,k}|\leqslant C\varepsilon_0(1+t)^{-(1+k+2l)/2}\begin{cases}B_N(x,t), & 2l+k\leqslant m-1,\\ 1, & 2l+k>m-1.\end{cases} \tag{5.5}$$

For $I_2^{l,k}$, we first set

$$\Phi^{k,l}(x,t) = (1+t)^{v(1+k+2l)} \begin{cases} (1+\frac{x^2}{1+t})^N, & 2l+k \leqslant m-1, \\ 1, & 2l+k > m-1, \end{cases}$$

and for $l \leqslant 1$ and a even integer m, set

$$M(t) = \sup_{0 \leqslant s \leqslant t, x \in \mathbf{R}, 2l+k \leqslant 2m} \Phi^{k,l}(x,s) |\partial_t^l \partial_x^k V(x,s)|, \tag{5.6}$$

where

$$v(h) = \frac{1}{2} \begin{cases} h, & h \leqslant m, \\ m-2, & h = m+1, m+2, \\ m-4, & h = m+3, m+4, \\ \vdots & \vdots \\ 0, & h = 2m-1, 2m. \end{cases}$$

It follows easily from [3,10] that

$$\partial_t^l \partial_x^k \bar{v}(x,t) \leqslant C\varepsilon_0 (1+t)^{-(k+2l)/2} e^{-\frac{bx^2}{1+t}}. \tag{5.7}$$

Thus,

$$|\partial_t^l \partial_x^k \tilde{F}_1(x,t)| \leqslant C\varepsilon_0 (1+t)^{-(k+2l+2)/2} e^{-\frac{bx^2}{1+t}}. \tag{5.8}$$

Since

$$\tilde{F}_2 = -(p'(\bar{v})\tilde{v} + p''(\bar{v} + \theta(V_x + \tilde{v}))((V_x + \tilde{v})^2/2),$$

with $0 < \theta < 1$, we have

$$|\partial_t^l \partial_x^k \tilde{v}(x,t)| \leqslant C|u_+ - u_-| e^{-\alpha t} \partial_x^k m_0(x).$$

Noticing that $m_0(x)$ is a smooth function with compact support, we have

$$|\partial_t^l \partial_x^k \tilde{v}(x,t)| \leqslant C\varepsilon_0 e^{-\alpha t} e^{-x^2}. \tag{5.9}$$

By (5.6) and Theorem 2.1, we have for $l \leqslant 1, 1+2l+k \leqslant 2m$

$$|\partial_t^l \partial_x^k \partial_x (V_x^2)| \leqslant \begin{cases} CM^2(t)(1+t)^{-\frac{3+k+2l}{2}} B_{2N}(x,t), & 1+2l+k \leqslant m-2, \\ CM^2(t)(1+t)^{-\frac{m}{2}} B_N(x,t), & 1+2l+k = m-1, m, \\ CM^2(t)(1+t)^{-\frac{m-2}{2}} B_N(x,t), & 1+2l+k = m+1, m+2, \\ \vdots & \vdots \\ CM^2(t)(1+t)^{-1} B_N(x,t), & 1+2l+k = 2m-3, 2m-2, \\ C\varepsilon_0 M(t)(1+t)^{-1} B_N(x,t), & 1+2l+k = 2m-1, 2m. \end{cases} \tag{5.10}$$

Also, (5.7), (5.9) and (5.10) yield

$$|\partial_t^l \partial_x^k F_2| \leqslant C(\varepsilon_0 + M^2(t)) \begin{cases} (1+t)^{-\frac{3+k+2l}{2}} B_{2N}(x,t), & 1+2l+k \leqslant m-2, \\ (1+t)^{-\frac{m}{2}} B_N(x,t), & 1+2l+k = m-1, m, \\ (1+t)^{-\frac{m-2}{2}} B_N(x,t), & 1+2l+k = m+1, m+2, \\ \vdots & \vdots \\ (1+t)^{-1} B_N(x,t), & 1+2l+k = 2m-1, 2m. \end{cases} \tag{5.11}$$

By (4.12), for each k, we can choose a $\bar{k}$ according to (5.10) and (5.11) so that

$$\begin{aligned} I_2^{0,k} &= \int_{t/2}^{t} \int_{-\infty}^{\infty} (\partial_x^{k-\tilde{k}} G - K_{0,k-\tilde{k}})(x,y,t,s) \partial_y^{\tilde{k}} F(y,s)\, dy\, ds \\ &\quad + \int_{t/2}^{t} \int_{-\infty}^{\infty} K_{0,k-\tilde{k}}(x,y,t,s) \partial_y^{\tilde{k}} F(y,s)\, dy\, ds \\ &\quad + O(1)\varepsilon_0 \sum_{h<\tilde{k}} \int_{t/2}^{t} \int_{-\infty}^{\infty} ((1+t-s)^{-(1+k-\tilde{k})/2} (1+t)^{-(\tilde{k}-h)/2} \end{aligned}$$

$$\times e^{-b(x-y)^2/(t-s)} + K_{0,k-\tilde{k}}(x,y,t,s)(1+t)^{-(\tilde{k}-h)/2})\partial_y^h F(y,s)\,dy\,ds$$

$$+ \int_0^{t/2}\int_{-\infty}^{\infty} (\partial_x^k G(x,y,t,s) - K_{0,k}(x,y;t,s))F(y,s)\,dy\,ds$$

$$+ \int_0^{t/2}\int_{-\infty}^{\infty} K_{0,k}(x,y;t,s)F(y,s)\,dy\,ds.$$

Thus, if $k \leqslant 2m-1$, then Theorem 4.1, Lemma 5.1 and (5.11) give

$$|I_2^{0,k}| \leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-v(1+k)}B_N(x,t). \tag{5.12}$$

By using (4.12) again, we have

$$I_2^{1,k} = \int_{-\infty}^{\infty} \partial_x^k G(x,y,t,s)F(y,s)|_{s=t}$$

$$+ \int_{t/2}^{t}\int_{-\infty}^{\infty} (\partial_x^{k-\tilde{k}}\partial_t G - K_{1,k-\tilde{k}})(x,y,t,s)\partial_y^{\tilde{k}} F(y,s)\,dy\,ds$$

$$+ \int_{t/2}^{t}\int_{-\infty}^{\infty} K_{1,k-\tilde{k}}(x,y,t,s)\partial_y^{\tilde{k}} F(y,s)\,dy\,ds$$

$$+ O(1)\varepsilon_0 \sum_{h<\tilde{k}} \int_{t/2}^{t}\int_{-\infty}^{\infty} ((1+t-s)^{-(3+k-\tilde{k})/2}(1+t)^{-(\tilde{k}-h)/2}$$

$$\times e^{-b(x-y)^2/(t-s)} + K_{1,k-\tilde{k}}(x,y,t,s)(1+t)^{-(\tilde{k}-h)/2})\partial_y^h F(y,s)\,dy\,ds$$

$$+ \int_0^{t/2}\int_{-\infty}^{\infty} (\partial_t\partial_x^k G(x,y,t,s) - K_{1,k}(x,y;t,s))F(y,s)\,dy\,ds$$

$$+ \int_0^{t/2}\int_{-\infty}^{\infty} K_{1,k}(x,y;t,s)F(y,s)\,dy\,ds.$$

Then, if $k \leqslant 2m-3$, Theorem 4.1, Lemma 5.1 and (5.11) give that

$$|I_2^{1,k}| \leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-v(1+2+k)}B_N(x,t). \tag{5.13}$$

As for $I_3^{l,k}$, (4.17) yields

$$
\begin{aligned}
I_3^{0,k} = & \int_{t/2}^{t}\int_{-\infty}^{\infty}(R_G-K_G)(x,t;y,s)\partial_y^k V(y,s)\,dy\,ds \\
& + \int_{t/2}^{t}\int_{-\infty}^{\infty}K_G(x,t;y,s)\partial_y^k V(y,s)\,dy\,ds \\
& + O(1)\varepsilon_0\sum_{l<k}\int_{t/2}^{t}\int_{-\infty}^{\infty}\Theta(y,t,s)(1+t)^{-(k-l)/2}((1+t-s)^{-1/2} \\
& \times e^{-b(x-y)^2/(t-s)} + K_G(x,y;t,s))\partial_y^l V(y,s))\,dy\,ds \\
& + \int_{0}^{t/2}\int_{-\infty}^{\infty}(\partial_x^k R_G(x,t;y,s) - K_{G,0,k}(x,y;t,s))V(y,s)\,dy\,ds \\
& + \int_{0}^{t/2}\int_{-\infty}^{\infty}K_{G,0,k}(x,y;t,s)V(y,s)\,dy\,ds.
\end{aligned}
$$

By Lemma 5.1, Theorem 2.1 and (4.16), we can obtain

$$
|I_3^{0,k}| \leqslant CM^2(t)(1+t)^{-v(1+k)}B_N(x,t), \tag{5.14}
$$

when $k\leqslant 2m-1$.

Similarly, for $I_3^{1,k}$, (4.18) gives

$$
\begin{aligned}
I_3^{1,k} = & \int_{-\infty}^{\infty}\partial_x^k R_G(x,t;y,s)V(y,s)\,dy|_{s=t} \\
& + \int_{t/2}^{t}\int_{-\infty}^{\infty}(R_G-K_G)(x,t;y,s)\partial_s\partial_y^k V(y,s)\,dy\,ds \\
& + \int_{t/2}^{t}\int_{-\infty}^{\infty}K_G(x,t;y,s)\partial_s\partial_y^k V(y,s)\,dy\,ds \\
& + O(1)\varepsilon_0\sum_{l<k}\int_{t/2}^{t}\int_{-\infty}^{\infty}\Theta(y,t,s)(1+t)^{-(k-l)/2}((1+t-s)^{-1/2}
\end{aligned}
$$

$$\times e^{-b(x-y)^2/(t-s)} + K_G(x,y;t,s))\partial_y^l V(y,s)\,dy\,ds$$

$$+\int_0^{t/2}\int_{-\infty}^{\infty}(\partial_t\partial_x^k R_G(x,t;y,s) - K_{G,1,k}(x,y;t,s))V(y,s)\,dy\,ds$$

$$+\int_0^{t/2}\int_{-\infty}^{\infty} K_{G,1,k}(x,y;t,s))V(y,s)\,dy\,ds.$$

By using Lemma 5.1 and (4.16) again, we can also obtain for $k\leqslant 2m-3$ that,

$$|I_3^{1,k}|\leqslant CM^2(t)(1+t)^{-v(3+k)}B_N(x,t). \tag{5.15}$$

Combining (5.6), (5.12)–(5.14) with (5.15), we have

$$|\partial_t^l\partial_x^k V_x(x,y,t)|\leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-v(1+k+2l)}B_N(x,t). \tag{5.16}$$

Noticing that $U(x,t) = V_t(x,t)$, (5.16) gives

$$M(t)\leqslant C(\varepsilon_0 + M^2(t)).$$

Since ε_0 is sufficiently small, by continuity argument we have $M(t)\leqslant C\varepsilon_0$. That is, when $k\leqslant m-1$ and $l\leqslant m-3$,

$$|\partial_x^k V(x,t)|\leqslant C\varepsilon_0(1+t)^{-(1+k)/2}B_N(x,t),$$

$$|\partial_x^l U(x,t)|\leqslant C\varepsilon_0(1+t)^{-(3+k)/2}B_N(x,t), \tag{5.17}$$

and the proof of Theorem 2.2 is complete. □

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