Kinetic Equations:
Fluid Dynamical Limits and Viscous Heating

Claude Bardos
University Denis Diderot
Paris, France and
Laboratory Jacques Louis Lions, University Pierre et Marie Curie
75251 Paris Cédex 05, France

C. David Levermore
Department of Mathematics, University of Maryland
College Park, MD 20742-4015, US

Seiji Ukai
Liu Bie Ju Centre for Mathematical Sciences, City University of Hong Kong
Kowloon, Hong Kong, China

Tong Yang
Department of Mathematics, City University of Hong Kong
Kowloon, Hong Kong, china

Dedicated to Professor Yoshio Sone on the occasion of his 70th birthday

Abstract

In the long-time scale, we consider the fluid dynamical limits for the kinetic equations when the fluctuation is decomposed into even and odd parts with respect to the microscopic velocity with different scalings. It is shown that when the background state is an absolute Maxwellian, the limit fluid dynamical equations are the incompressible Navier-Stokes equations with viscous heating. This is different from the case when the even and odd parts of the fluctuation have the same scaling where the standard incompressible Navier-Stokes equations without viscous heating are obtained. On the other hand, when the background is a local Maxwellian, it is shown that the above even-odd decomposition leads to a non-classical fluid dynamical system without viscous heating which has been used to describe the ghost effect in the kinetic theory. In addition, the above even-odd decomposition is justified rigorously for the Boltzmann equation for the former case when the background is an absolute Maxwellian.

1 Introduction

The first derivations of fluid dynamics from kinetic equations go back to Maxwell [Mx] and Boltzmann [Bl]. These early derivations rested on arguments as how the various terms in a kinetic equation balance each other. These balance arguments seemed arbitrary to some extent. Hence, Hilbert [Hl] proposed that such derivations should be based on a systematic expansion in a small non-dimensional parameter $\epsilon > 0$, sometimes called the Knudsen number, which is the ratio of microscopic to macroscopic time scales. A bit later Enskog [En] proposed a somewhat different systematic expansion, now often called the Chapman-Enskog expansion, in the same small parameter ϵ. With this parameter introduced, classical kinetic equations take the form

$$\partial_t F + v\cdot\nabla_x F = \frac{1}{\epsilon}\mathcal{C}(F)\,. \tag{1.1}$$

Here $F = F(t, x, v)$ is a nonnegative mass density of particles with position x in a smooth domain $\Omega \subset R^d$ and microscopic velocity v in R^d, while $\mathcal{C}(F)$ is a collision operator that acts only on the v variable. Either the Hilbert or Chapman-Enskog expansion yields the compressible Euler equations at leading order, and the compressible Navier-Stokes equations, Burnett equations, and so-called super Burnett equations at subsequent orders. Justifying these formal approximations has been proven difficult, in part because many basic well-posedness and regularity questions are still mostly open for these fluid equations. Here we mention only [C], [N], [UA] on the justification of the compressible Euler approximation.

Several papers [BGL, DMEL, BU, So] have therefore studied direct derivations of incompressible Navier-Stokes equations, about which more is known. Denote the Maxwellian in space dimension d with density ρ, bulk velocity u and temperature θ by

$$\mathcal{M}(\rho, u, \theta) = \frac{\rho}{(2\pi\theta)^{\frac{d}{2}}}\exp\big(-\frac{|v-u|^2}{2\theta}\big).$$

More specifically, they consider F to be a perturbation about an absolute Maxwellian $M = M(v)$. By an appropriate choice of a Galilean frame and choice of mass and velocity units, it can be assumed that M has the form

$$M \equiv \mathcal{M}(1, 0, 1) = \frac{1}{(2\pi)^{d/2}}\exp(-\tfrac{1}{2}|v|^2)\,. \tag{1.2}$$

It is assumed that any boundary conditions are consistent with M being either an exact stationary solution, or a sufficiently good approximation of a stationary solution of (1.1). Because F is near M, it will therefore evolve on a longer time scale than that appearing in (1.1). Upon introducing this longer time scale, these papers considered a scaled kinetic equation of the form

$$\epsilon\,\partial_t F^\epsilon + v\cdot\nabla_x F^\epsilon = \frac{1}{\epsilon}\mathcal{C}(F^\epsilon)\,, \tag{1.3}$$

where the phase space density $F^\epsilon = F^\epsilon(t, x, v)$ is of the form

$$F^\epsilon = M + \epsilon\,\widetilde{F}^\epsilon\,. \tag{1.4}$$

Here $\widetilde{F}^\epsilon = \widetilde{F}^\epsilon(t, x, v)$ denotes the fluctuation of the phase space density about M. The incompressible Navier-Stokes equations are then derived from (1.3) and (1.4) in the limit of vanishing Knudsen number ϵ. In this set-up, the Mach number and the Knudsen number are of the same order so as to obtain a nonzero viscosity [BGL].

Here, we should mention that the recent major breakthrough to the Navier-Stokes limit of the Boltzmann equation was made by Golse and Saint-Raymond, [GS]. Roughly speaking, this important result shows that the limits of suitably rescaled sequences of DiPerna-Lions renormalized solutions to the Boltzmann equation are the Leray solutions to the incompressible Navier-Stokes equations. However, since we are concerned with the formal derivations and the limit of "regular" solutions, we will not go into details in this direction. Notice also that the fluid dynamical limits with viscous heating involve higher order corrections so that there is no room for weak convergence, such as from renormalized solutions of the Boltzmann equation to weak solutions of the Navier-Stokes equations, [BGL],[GS].

The associated fluid dynamical variables, mass density ρ^ϵ, bulk velocity u^ϵ, and temperature θ^ϵ, are defined in terms of F^ϵ by

$$\rho^\epsilon \equiv \int_{\mathbb{R}^d} F^\epsilon \, dv \,, \qquad \rho^\epsilon u^\epsilon \equiv \int_{\mathbb{R}^d} v \, F^\epsilon \, dv \,, \qquad \rho^\epsilon \theta^\epsilon \equiv \tfrac{1}{d} \int_{\mathbb{R}^d} |v - u^\epsilon|^2 F^\epsilon \, dv \,. \tag{1.5}$$

By the form (1.4) of F^ϵ, these variables may be expressed in terms of their fluctuations as

$$\rho^\epsilon = 1 + \epsilon \, \widetilde{\rho}^\epsilon \,, \qquad u^\epsilon = \epsilon \, \widetilde{u}^\epsilon \,, \qquad \theta^\epsilon = 1 + \epsilon \, \widetilde{\theta}^\epsilon \,. \tag{1.6}$$

Then under certain hypotheses these fluctuations will converge as $\epsilon \to 0+$ in a convenient topology:

$$\widetilde{\rho}^\epsilon \to \widetilde{\rho} \,, \qquad \widetilde{u}^\epsilon \to \widetilde{u} \,, \qquad \widetilde{\theta}^\epsilon \to \widetilde{\theta} \,, \tag{1.7}$$

where the limiting velocity fluctuation is governed by

$$\nabla_x \cdot \widetilde{u} = 0 \,, \tag{1.8a}$$

$$\partial_t \widetilde{u} + \widetilde{u} \cdot \nabla_x \widetilde{u} + \nabla_x \widetilde{p} = \mu^* \Delta \widetilde{u} \,, \tag{1.8b}$$

while the limiting density and temperature fluctuations are governed by

$$\widetilde{\rho} + \widetilde{\theta} = 0 \,, \tag{1.9a}$$

$$\tfrac{d+2}{2} (\partial_t \widetilde{\theta} + \widetilde{u} \cdot \nabla_x \widetilde{\theta}) = \kappa^* \Delta \widetilde{\theta} \,. \tag{1.9b}$$

Here (1.8) are the standard incompressible Navier-Stokes motion equations with viscosity μ^*, (1.9a) is the Boussinesq relation between the density and temperature fluctuations, and (1.9b) is the temperature equation with thermal conductivity κ^*. Both μ^* and κ^* are positive and have formulas in terms of the linearization of the collision operator $\mathcal{C}$ about the absolute Maxwellian M.

These equations can also be derived from the compressible Navier-Stokes equations for an ideal gas. Expressed in terms of ρ^ϵ, u^ϵ, and θ^ϵ and in terms of the same long-time scaling used in (1.3), these equations are

$$\epsilon \, \partial_t \rho^\epsilon + \nabla_x \cdot (\rho^\epsilon u^\epsilon) = 0 \tag{1.10a}$$

$$\rho^\epsilon \big(\epsilon \, \partial_t + u^\epsilon \cdot \nabla_x \big) u^\epsilon + \nabla_x p^\epsilon = \epsilon \, \nabla_x \cdot [\mu^\epsilon \sigma^\epsilon] \tag{1.10b}$$

$$\tfrac{d+2}{2} \rho^\epsilon \big(\epsilon \, \partial_t + u^\epsilon \cdot \nabla_x \big) \theta^\epsilon - \big(\epsilon \, \partial_t + u^\epsilon \cdot \nabla_x \big) p^\epsilon = \epsilon \, \tfrac{1}{2} \mu^\epsilon |\sigma^\epsilon|^2 + \epsilon \, \nabla_x \cdot [\kappa^\epsilon \nabla_x \theta^\epsilon] \,. \tag{1.10c}$$

Here the pressure p^ϵ is given by the ideal gas law as $p^\epsilon = \rho^\epsilon \theta^\epsilon$, the viscosity μ^ϵ and thermal conductivity κ^ϵ are given as positive functions of ρ^ϵ and θ^ϵ denoted by $\mu^\epsilon = \mu(\rho^\epsilon, \theta^\epsilon)$ and $\kappa^\epsilon = \kappa(\rho^\epsilon, \theta^\epsilon)$, while the strain-rate tensor σ^ϵ is given by

$$\sigma^\epsilon \equiv \nabla_x u^\epsilon + (\nabla_x u^\epsilon)^T - \tfrac{2}{d} \nabla_x \cdot u^\epsilon \, I \,, \tag{1.11}$$

and $|\sigma^\epsilon|^2 \equiv \mathrm{tr}(\sigma^{\epsilon \, 2}) = \sigma^\epsilon : \sigma^\epsilon$. If one assumes that ρ^ϵ, u^ϵ, and θ^ϵ have the form given by (1.6) and that their fluctuations converge as $\epsilon \to 0+$ in a convenient topology as in (1.7), then their limiting fluctuations satisfy (1.8) and (1.9).

There is however more than one incompressible limit for the compressible Navier-Stokes equations [BLP]. If one assumes that, rather than (1.6), the fluid variables may be expressed in terms of their fluctuations as

$$\rho^\epsilon = 1 + \epsilon^2 \widetilde{\rho}^\epsilon \,, \qquad u^\epsilon = \epsilon \, \widetilde{u}^\epsilon \,, \qquad \theta^\epsilon = 1 + \epsilon^2 \widetilde{\theta}^\epsilon \,, \tag{1.12}$$

and that these fluctuations converge as $\epsilon \to 0+$ in a convenient topology as in (1.7), then the limiting velocity fluctuation is again governed by the Navier-Stokes motion equations (1.8), but the limiting density and temperature fluctuations are now governed by

$$\widetilde{p} = \widetilde{\rho} + \widetilde{\theta} \,, \tag{1.13a}$$

$$\tfrac{d+2}{2} (\partial_t \widetilde{\theta} + \widetilde{u} \cdot \nabla_x \widetilde{\theta}) - (\partial_t \widetilde{p} + \widetilde{u} \cdot \nabla_x \widetilde{p}) = \tfrac{1}{2} \mu^* |\widetilde{\sigma}|^2 + \kappa^* \Delta \widetilde{\theta} \,. \tag{1.13b}$$

Here $\widetilde{p}$ is the limit of the pressure fluctuation $\widetilde{p}^\epsilon$ defined by

$$\widetilde{p}^\epsilon \equiv \widetilde{\rho}^\epsilon + \widetilde{\theta}^\epsilon + \epsilon^2 \widetilde{\rho}^\epsilon \widetilde{\theta}^\epsilon \,; \tag{1.14}$$

the limiting viscosity and thermal conductivity are given by $\mu^* = \mu(1,1)$ and $\kappa^* = \kappa(1,1)$. However, now by virtue of (1.8a), $\widetilde{\sigma}$ is a traceless tensor and one has:

$$\widetilde{\sigma} \equiv \nabla_x \widetilde{u} + (\nabla_x \widetilde{u})^T \ \text{ and } \ \tfrac{1}{2} |\widetilde{\sigma}|^2 = \sum_{ij} \Big(\frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} \Big) + \sum_i \Big(\frac{\partial u_i}{\partial x_i} \Big)^2 \,. \tag{1.15}$$

Observe that, with the Navier-Stokes equation for $\widetilde{u}$ and the "Boussinesq relation" (1.13a), (1.13b) is also equivalent to

$$\partial_t \Big(\frac{|\widetilde{u}|^2}{2} + \frac{d}{2} \widetilde{\theta} - \widetilde{\rho} \Big) + \nabla_x \cdot \Big(\widetilde{u} \Big(\frac{|\widetilde{u}|^2}{2} + \frac{d+2}{2} \widetilde{\theta} \Big) \Big) = \mu^* \nabla_x \big(u \big(\nabla_x \widetilde{u} + (\nabla_x \widetilde{u})^T \big) + \kappa^* \Delta \widetilde{\theta} \tag{1.13b'}$$

which describes the balance of internal energy with a left hand side written in conservation form.

Equations (1.13) differ from equations (1.9) in that they include the viscous heating term $\frac{1}{2} \mu^* |\widetilde{\sigma}|^2$ and driving terms involving the limiting pressure fluctuation $\widetilde{p}$. These terms have clear antecedents in the compressible Navier Stokes equations (1.10). They appeared in (1.13) because the scaling (1.12) gives a balance that includes more physics at the leading order [cf. BLP].

One of the main purposes of the present paper is to show that the same type of scaling made at the level of a kinetic equation will lead to the same limit equations. In particular, we show that the structure of the limit equations depends on the collision operator (assumed to be in a "reasonable physical class") only through the viscosity and heat conduction defined by this operator. We again derive this incompressible limit from the scaled kinetic equation (1.3), but we now decompose the fluctuation $\widetilde{F}^\epsilon$ into its odd and even parts as functions of v, denoted by $\widetilde{F}_o^\epsilon$ and $\widetilde{F}_e^\epsilon$, and assume that

$$F^\epsilon = M + \epsilon \, \widetilde{F}_o^\epsilon + \epsilon^2 \widetilde{F}_e^\epsilon \,. \tag{1.16}$$

Our formal derivation is universal in that it is valid not only for the genuine Boltzmann operator but also for any collision operator $\mathcal{C}$ that retains certain basic abstract "physical" properties of the Boltzmann operator. It is also moment-based, in the style of the derivation given in [BGL] of (1.8) and (1.9) from (1.3) and (1.4). This approach captures the physical spirit of the balance arguments used by Maxwell and Boltzmann while retaining the systematic mechanism provided by an asymptotic development in the small parameter ϵ. This is manifest through its emphasis of the role of the leading order terms in weak formulations. It thereby requires less control to establish the convergence of these quantities than to establish the validity of expansion-based derivations.

The incompressible limits thus obtained, however, are not the only limits of the scaled Boltzmann equation (1.3). A second purpose of the present paper is to show that another asymptotic development for small parameter ϵ leads to a non-classical system of fluid dynamical equations.

Even though the Boltzmann equation has close relation to the classical systems in fluid dynamics, it provides more information in the mesoscopic level so that it describes some phenomena which can not be modeled by using the classical systems of Euler and Navier-Stokes equations. This kind of interesting phenomena, such as the thermal creep flow in a rarefied gas, was known already at the time of Maxwell. The mathematical formulation and numerical computation on the basis of kinetic equations have been studied since 1960's, cf. [So2]. The explicit forms of the fluid dynamical equations derived there are reproduced in (4.6) in Section 4. One of the main features is that they exhibit the "ghost effect".

We will show that these non-classical equations can be derived again from the scaled Boltzmann equation (1.3) but under the new scaling

$$F^\epsilon = \mathcal{M}(\rho^\epsilon, \epsilon\widetilde{u}^\epsilon, \theta^\epsilon) + \epsilon\mathbf{G}^\epsilon, \tag{1.17}$$

where $\mathcal{M}(\rho^\epsilon, \epsilon\widetilde{u}^\epsilon, \theta^\epsilon)$ is the local Maxwellian determined by the solution F^ϵ itself while $\mathbf{G}^\epsilon$ is the microscopic component of the solution. When ϵ tends to zero, even though the bulk velocity approaches to zero, the scaled flow velocity $\widetilde{u}$ as the limit of $\widetilde{u}^\epsilon$ appears in the equations governing the motion of the limit density ρ and the limit temperature θ. The infinitesimal quantity $\widetilde{u}$ is not a "real-world" quantity and hence a "ghost". Thus, this fluid dynamical system describes a non-classical phenomenon such that the "real-world" quantities ρ and θ are governed by the "ghost" $\widetilde{u}$. Notice that the appearance of the "ghost effect" comes through the boundary or from infinity in physics, see Remark 4.1. Moreover, this system does not contain the viscous heating, see Remark 4.2.

Straightforward calculation shows that the above decomposition with scaling is equivalent to the following even-odd decomposition

$$F^\epsilon = \mathcal{M}(\rho, 0, \theta) + \epsilon\,\widetilde{F}^\epsilon_o + \epsilon^2\widetilde{F}^\epsilon_e, \tag{1.18}$$

where ρ and θ are not constant, but functions of (x, t). In other words, $\mathcal{M}(\rho, 0, \theta)$ is a local Maxwellian which is different from the case in (1.16). Moreover, it is stressed that in contrast to (1.16), the odd-even decomposition in (1.18) is a consequence of the setting of the scaling but not the assumption, see Theorem 4.1, where the formal derivation based on the decomposition (1.17) is given. The purpose there is to give a systematic derivation of the non-classical fluid dynamical system. The well-posedness of the system and the justification of the limit will not be discussed in this paper.

In the above, we saw that the same equation (1.3) has different limits if different scalings are introduced to the solutions. This fact may be in analogy with the Weierstrass-Picard theorem on the behavior of holomolphic functions near essential singularities: $\epsilon = 0$ is something like an essential singularity of the scaled Boltzmann equation (1.3) and different scalings result in different limits.

After the above formal derivations, we go on to show that the convergence can be established in the setting of the Boltzmann equation for the hard sphere model and cutoff hard potentials in $\mathbb{R}^3$ when the background is an absolute Maxwellian. How to justify the case for the non-classical fluid dynamic limit is not in the scope of this paper and is left for future investigation. Our result parallels the convergence proof of Bardos and Ukai [BU] for the formal moment-based derivation found in [BGL] of (1.8) and (1.9) from (1.3) and (1.4). It works with classical solutions of the Boltzmann equation defined in certain Grad spaces.

Our paper is laid out as follows. In Section 2, we will identify the general class of collision operators $\mathcal{C}$ for which our formal derivation holds. In this setting, we will develop identities that will play a central role in the subsequent formal derivations. In Section 3, we will present the formal moment-based derivation with even-odd decomposition of the fluctuation when the background is an absolute Maxwellian. When the background is a local Maxwellian, the formal derivation of the non-classical fluid dynamic limit will be given in Section 4. Section 5 establishes the limit for certain classical solutions of the Boltzmann equation for the hard sphere model and the cutoff hard potentials in $\mathbb{R}^3$ to justify the incompressible limit with viscous heating derived in Section 3.

2 Properties of the Collision Operator

The collision operator $\mathcal{C}$ is assumed to be defined over a domain $\mathrm{D}(\mathcal{C})$ that is contained within the cone of nonnegative functions of v with convenient decay at infinity. It is assumed that $\mathcal{C}$ enjoys the following properties which relate to local conservation, local dissipation, Galilean invariance, and its expansion about an equilibrium. These properties are shared by a wide range of classical collision operators. They are expressed with the following notations. The integral of any scalar or vector-valued integrable function $f = f(v)$ over R^d will be denoted by $\langle f \rangle$, so that

$$\langle f \rangle = \int f(v)\, dv\,. \tag{2.1}$$

All functions in this paper are understood to be measurable in all variables.

2.1: Basic Assumptions about the Collision Operator.

First, the operator $\mathcal{C}$ is assumed to have 1, v, and $|v|^2$ as locally conserved quantities; this means

$$\langle \mathcal{C}(f) \rangle = 0\,, \qquad \langle v\, \mathcal{C}(f) \rangle = 0\,, \qquad \langle |v|^2 \mathcal{C}(f) \rangle = 0\,, \qquad \text{for every } f \in \mathrm{D}(\mathcal{C})\,. \tag{2.2}$$

Moreover, it is assumed that every locally conserved quantity is a linear combination of these three, so that for any $g = g(v)$ with convenient decay at infinity the following statements are equivalent:

$$\begin{aligned} &\text{i)} \quad \langle g\, \mathcal{C}(f) \rangle = 0\,, \qquad \text{for every } f \in \mathrm{D}(\mathcal{C})\,;\\ &\text{ii)} \quad g \in \mathbb{E} \equiv \operatorname{span}\{1, v_1, v_2, \cdots, v_d, |v|^2\}. \end{aligned} \tag{2.3}$$

The relations (2.2) represent the physical laws of mass, momentum, and energy conservation during collisions and (2.3) states that there are no other local conservation laws.

Second, the operator $\mathcal{C}$ is assumed to satisfy the local dissipation relation

$$\langle \log f\, \mathcal{C}(f) \rangle \leq 0\,, \qquad \text{for every } f \in \mathrm{D}(\mathcal{C})\,. \tag{2.4}$$

The quantity on the left of (2.4) is the so-called local entropy dissipation rate. The local equilibrium of $\mathcal{C}$ are assumed to be characterized by the vanishing of the local entropy dissipation rate and to be given by the class of Maxwellian densities, i.e., those of the form

$$f = \mathcal{M}(\rho, u, \theta) \equiv \frac{\rho}{(2\pi\theta)^{\frac{d}{2}}} \exp\left(- \frac{|v-u|^2}{2\theta} \right), \tag{2.5}$$

for some $(\rho, u, \theta) \in \mathbb{R}_+ \times R^d \times \mathbb{R}_+$. More precisely, for every $f \in \mathrm{D}(\mathcal{C})$ the following statements are assumed to be equivalent:

$$\begin{aligned} &\text{i)} \quad \langle \log(f)\, \mathcal{C}(f) \rangle = 0\,,\\ &\text{ii)} \quad \mathcal{C}(f) = 0\,,\\ &\text{iii)} \quad f \text{ is a Maxwellian density given by (2.5).} \end{aligned} \tag{2.6}$$

These assumptions about $\mathcal{C}$ merely abstract some of the consequences of Boltzmann's celebrated H-theorem.

Third, the operator $\mathcal{C}$ is assumed to commute with the actions of translational and orthogonal transformations on v. Specifically, given any $f = f(v)$, then for every vector $u \in R^d$ and for every orthogonal matrix $O \in \mathbb{R}^{d\times d}$ define functions $\mathcal{A}_u f$ and $\mathcal{A}_o f$ by

$$\mathcal{A}_u f = \mathcal{A}_u f(v) \equiv f(v-u)\,, \qquad \mathcal{A}_o f = \mathcal{A}_o f(v) \equiv f(O^T v)\,. \tag{2.7}$$

It is assumed that if f is in $\mathrm{D}(\mathcal{C})$, then so are $\mathcal{A}_u f$ and $\mathcal{A}_o f$ with

$$\mathcal{A}_u \mathcal{C}(f) = \mathcal{C}(\mathcal{A}_u f)\,, \qquad \mathcal{A}_o \mathcal{C}(f) = \mathcal{C}(\mathcal{A}_o f)\,. \tag{2.8}$$

These relations reflect the Galilean invariance of the microscopic collisional dynamics and implies that when $\Omega = R^d$, the kinetic equation (1.3) formally retains Galilean invariance.

Fourth, it is assumed that when acting on smooth functions with convenient decay at infinity, the collision operator $\mathcal{C}$ is four times Fréchet differentiable, the first, second, and third derivatives being denoted by $\mathcal{C}'$, $\mathcal{C}''$, and $\mathcal{C}'''$. In particular for any given Maxwellian $\mathcal{M}$ of the form (2.5), the operator $\mathcal{C}$ has the formal Taylor expansion

$$\frac{1}{\mathcal{M}}\mathcal{C}\big(\mathcal{M}(1+\epsilon\,\widetilde{g})\big) = -\epsilon\,\mathcal{L}_{\mathcal{M}}\widetilde{g} + \epsilon^2 \mathcal{Q}_{\mathcal{M}}(\widetilde{g},\widetilde{g}) + \epsilon^3 \mathcal{T}_{\mathcal{M}}(\widetilde{g},\widetilde{g},\widetilde{g}) + O(\epsilon^4)\,. \tag{2.9}$$

with the identifications:

$$\begin{aligned} \mathcal{L}_{\mathcal{M}}\widetilde{g} &= -\frac{1}{\mathcal{M}}\mathcal{C}'(\mathcal{M})(\mathcal{M}\widetilde{g}),\\ \mathcal{Q}_{\mathcal{M}}(\widetilde{g},\widetilde{g}) &= \frac{1}{2\mathcal{M}}\mathcal{C}''(\mathcal{M})(\mathcal{M}\widetilde{g},\mathcal{M}\widetilde{g}),\\ \mathcal{T}_{\mathcal{M}}(\widetilde{g},\widetilde{g},\widetilde{g}) &= \frac{1}{3!\mathcal{M}}\mathcal{C}'''(\mathcal{M})(\mathcal{M}\widetilde{g},\mathcal{M}\widetilde{g},\mathcal{M}\widetilde{g}). \end{aligned} \tag{2.9'}$$

The linear and symmetric multi-linear operators $\mathcal{L}_{\mathcal{M}}$, $\mathcal{Q}_{\mathcal{M}}$, and $\mathcal{T}_{\mathcal{M}}$ are defined by the relation (2.9') in $\mathbb{H}_{\mathcal{M}}$, the Hilbert space with the weighted inner product

$$\big(\widetilde{h}\,|\,\widetilde{g}\,\big)_{\mathcal{M}} \equiv \langle \widetilde{h}\,\mathcal{M}\widetilde{g}\rangle\,. \tag{2.10}$$

They are assumed to be closed operators from $\mathbb{H}_{\mathcal{M}}$ into $\mathbb{H}_{\mathcal{M}}$ with domains $\mathrm{D}(\mathcal{L}_{\mathcal{M}})$, $\mathrm{D}(\mathcal{Q}_{\mathcal{M}})$, and $\mathrm{D}(\mathcal{T}_{\mathcal{M}})$ which contain the set of smooth functions $\widetilde{g}$ with at most polynomial growth as $|v| \to \infty$, and therefore, which are dense in $\mathbb{H}_{\mathcal{M}}$.

2.2: Some Consequences of the Basic Assumptions and Further Hypotheses.

The assumptions made in the previous section contain much more information on the structure of the collision operator. Given any Maxwellian $\mathcal{M} = \mathcal{M}(\rho,u,\theta)$ of the form (2.5), many properties of $\mathcal{L}_{\mathcal{M}}$, $\mathcal{Q}_{\mathcal{M}}$, and $\mathcal{T}_{\mathcal{M}}$ now follow directly from those of $\mathcal{C}$.

First, combining the local conservation relations (2.2) with the definition (2.9) of $\mathcal{L}_{\mathcal{M}}$, $\mathcal{Q}_{\mathcal{M}}$, and $\mathcal{T}_{\mathcal{M}}$ gives:

Proposition 2.1: Let $\xi \in \mathbb{E}$, where $\mathbb{E}$ is the space of locally conserved quantities defined in (2.3). Then

$$\langle \xi\,\mathcal{M}\mathcal{L}_{\mathcal{M}}\widetilde{g}\rangle = 0\,, \qquad \text{for every } \widetilde{g} \in \mathrm{D}(\mathcal{L}_{\mathcal{M}})\,; \tag{2.11a}$$

$$\langle \xi\,\mathcal{M}\mathcal{Q}_{\mathcal{M}}(\widetilde{g},\widetilde{g})\rangle = 0\,, \qquad \text{for every } \widetilde{g} \in \mathrm{D}(\mathcal{Q}_{\mathcal{M}})\,; \tag{2.11b}$$

$$\langle \xi\,\mathcal{M}\mathcal{T}_{\mathcal{M}}(\widetilde{g},\widetilde{g},\widetilde{g})\rangle = 0\,, \qquad \text{for every } \widetilde{g} \in \mathrm{D}(\mathcal{T}_{\mathcal{M}})\,. \tag{2.11c}$$

Second, combining the local dissipation relation (2.4) with the definition (2.9) of $\mathcal{L}_{\mathcal{M}}$, $\mathcal{Q}_{\mathcal{M}}$, and $\mathcal{T}_{\mathcal{M}}$ and the expansion

$$\log\big(\mathcal{M}(1+\epsilon\,\widetilde{g})\big) = \log(\mathcal{M}) + \epsilon\,\widetilde{g} - \tfrac{1}{2}\epsilon^2\widetilde{g}^2 + \tfrac{1}{3}\epsilon^3\widetilde{g}^3 - \cdots\,, \tag{2.12}$$

while using (2.9) with the fact $\log(\mathcal{M}) \in \mathbb{E}$, gives:

Proposition 2.2: For every $\widetilde{g} \in C^\infty(R^d)$ with at most polynomial growth as $|v| \to \infty$, one has

$$\begin{aligned} 0 &\le -\left\langle \log\big(\mathcal{M}(1+\epsilon\,\widetilde{g})\big)\,\mathcal{C}\big(\mathcal{M}(1+\epsilon\,\widetilde{g})\big)\right\rangle \\ &= \epsilon^2 \langle \widetilde{g}\,\mathcal{M}\mathcal{L}_{\mathcal{M}}\widetilde{g}\rangle - \epsilon^3\left(\tfrac{1}{2}\langle \widetilde{g}^2\mathcal{M}\mathcal{L}_{\mathcal{M}}\widetilde{g}\rangle + \langle \widetilde{g}\,\mathcal{M}\mathcal{Q}_{\mathcal{M}}(\widetilde{g},\widetilde{g})\rangle\right) \\ &\quad + \epsilon^4\left(\tfrac{1}{3}\langle \widetilde{g}^3\mathcal{M}\mathcal{L}_{\mathcal{M}}\widetilde{g}\rangle + \tfrac{1}{2}\langle \widetilde{g}^2\mathcal{M}\mathcal{Q}_{\mathcal{M}}(\widetilde{g},\widetilde{g})\rangle - \langle \widetilde{g}\,\mathcal{M}\mathcal{T}_{\mathcal{M}}(\widetilde{g},\widetilde{g},\widetilde{g})\rangle\right) + O(\epsilon^5)\,. \end{aligned} \tag{2.13}$$

From the above expansion one deduces the relation

$$0 \le \langle \widetilde{g}\,\mathcal{M}\mathcal{L}_{\mathcal{M}}\widetilde{g}\rangle, \tag{2.14}$$

or the fact that the "unbounded" operator $\mathcal{L}_{\mathcal{M}}$ is non-negative.

Let $\mathcal{L}_{\mathcal{M}}^\dagger$ denote the adjoint of $\mathcal{L}_{\mathcal{M}}$ over $\mathbb{H}_{\mathcal{M}}$ with its domain denoted by $\mathrm{D}(\mathcal{L}_{\mathcal{M}}^\dagger)$. It will be assumed that the domains of these two operators coincide

$$\mathrm{D}(\mathcal{L}_{\mathcal{M}}^\dagger) = \mathrm{D}(\mathcal{L}_{\mathcal{M}})\,. \tag{2.15}$$

With (2.14) and the classical Hilbertian theory (cf. Brezis [Br]), the operators $\mathcal{L}_{\mathcal{M}}$ and $\mathcal{L}_{\mathcal{M}}^\dagger$ are maximal positive, their spectra is contained in the half plane

$$Re\,\lambda \ge 0,$$

and the present analysis relies, as usual, on the fact that 0 belongs to this spectra. Let $\mathrm{N}(\mathcal{L}_{\mathcal{M}})$ and $\mathrm{N}(\mathcal{L}_{\mathcal{M}}^\dagger)$ denote the null spaces of $\mathcal{L}_{\mathcal{M}}$ and $\mathcal{L}_{\mathcal{M}}^\dagger$, and let $\mathrm{R}(\mathcal{L}_{\mathcal{M}})$ and $\mathrm{R}(\mathcal{L}_{\mathcal{M}}^\dagger)$ denote their ranges. With the positivity, one has $\widetilde{g} \in \mathrm{N}(\mathcal{L}_{\mathcal{M}} + \mathcal{L}_{\mathcal{M}}^\dagger)$ if and only if $\langle \widetilde{g}\,\mathcal{M}\mathcal{L}_{\mathcal{M}}\widetilde{g}\rangle_{\mathcal{M}} = 0$, which then implies

$$\mathrm{N}(\mathcal{L}_{\mathcal{M}}) = \mathrm{N}(\mathcal{L}_{\mathcal{M}}^\dagger) \subset \mathrm{N}(\mathcal{L}_{\mathcal{M}} + \mathcal{L}_{\mathcal{M}}^\dagger)\,. \tag{2.16}$$

Third, let $\mathcal{M}^\epsilon = \mathcal{M}(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$, where $(\rho^\epsilon, u^\epsilon, \theta^\epsilon)$ is an arbitrary analytic parametrization that has an expansion for small ϵ of the form

$$\mathcal{M}^\epsilon = \mathcal{M}\Big(1 + \epsilon\, m^{(1)} + \epsilon^2 m^{(2)} + \epsilon^3 m^{(3)} + O(\epsilon^4)\Big)\,, \tag{2.17}$$

then one can use (2.9) to expand the relation $\mathcal{C}(\mathcal{M}^\epsilon) = 0$ in ϵ, and thereby obtain:

Proposition 2.3: Given $m^{(j)}$ as defined in (2.17), one has

$$\mathcal{L}_{\mathcal{M}} m^{(1)} = 0\,, \tag{2.18a}$$

$$\mathcal{L}_{\mathcal{M}} m^{(2)} = \mathcal{Q}_{\mathcal{M}}(m^{(1)}, m^{(1)})\,, \tag{2.18b}$$

$$\mathcal{L}_{\mathcal{M}} m^{(3)} = 2\mathcal{Q}_{\mathcal{M}}(m^{(1)}, m^{(2)}) + \mathcal{T}_{\mathcal{M}}(m^{(1)}, m^{(1)}, m^{(1)})\,. \tag{2.18c}$$

The family of Maxwellians

$$\mathcal{M}^\epsilon = \mathcal{M}(1+\epsilon^2\widetilde{\rho}, \epsilon\,\widetilde{u}, 1+\epsilon^2\widetilde{\theta}) = \frac{1+\epsilon^2\widetilde{\rho}}{(2\pi(1+\epsilon^2\widetilde{\theta}))^{\frac{d}{2}}}\exp\left(-\frac{|v-\epsilon\,\widetilde{u}|^2}{2(1+\epsilon^2\widetilde{\theta})}\right) \tag{2.19}$$

is motivated for our application by the scaling in (1.12), in which the density and temperature fluctuations are of order two while the velocity fluctuation of order one. It has the following Taylor expansion:

$$\mathcal{M}^\epsilon = M\ \Big(1 + \epsilon\, m^{(1)} + \epsilon^2 m^{(2)} + \epsilon^3 m^{(3)} + O(\epsilon^4)\Big) \tag{2.20}$$

with $m^{(1)}$, $m^{(2)}$, and $m^{(3)}$ given by:

$$\begin{aligned}
m^{(1)} &= \widetilde{u}\cdot v\,,\\
m^{(2)} &= \widetilde{\rho} + (\tfrac{1}{2}|\widetilde{u}|^2 + \tfrac{d}{2}\widetilde{\theta})\tfrac{2}{d}(\tfrac{1}{2}|v|^2 - \tfrac{d}{2}) + \tfrac{1}{2}A(v)\cdot(\widetilde{u}\vee\widetilde{u})\,,\\
m^{(3)} &= \widetilde{\rho}\widetilde{u}\cdot v + \widetilde{\theta}\widetilde{u}\cdot B(v) + \tfrac{1}{3}C(v)\cdot(\widetilde{u}\vee\widetilde{u}\vee\widetilde{u})\,.
\end{aligned} \tag{2.21}$$

In (2.21) the traceless symmetric matrix $A(v)$, the vector $B(v)$, and the symmetric three-tensor $C(v)$ are defined by

$$\begin{aligned}
A &= A(v) = v\vee v - \frac{1}{d}|v|^2 I\,,\\
B &= B(v) = \left(\tfrac{1}{2}|v|^2 v - \tfrac{d+2}{2}v\right),\\
C &= C(v) = \tfrac{1}{2}(v\vee v\vee v - 3I\vee v)\,,
\end{aligned} \tag{2.22}$$

where $\vee$ denotes the symmetric tensor product.

In particular, for $\mathcal{M} = M$ given by (1.2) one has the identities

$$\mathcal{L}_M(m^{(1)}) = \mathcal{L}_M(\widetilde{u}\cdot v) = 0\,, \tag{2.23a}$$

$$\begin{aligned}
\mathcal{L}_M(m^{(2)}) =&\mathcal{L}_M(\widetilde{\rho} + (\tfrac{1}{2}|\widetilde{u}|^2 + \tfrac{d}{2}\widetilde{\theta})\tfrac{2}{d}(\tfrac{1}{2}|v|^2 - \tfrac{d}{2}) + \tfrac{1}{2}A(v)\cdot(\widetilde{u}\vee\widetilde{u}))\\
&= \mathcal{Q}_M(\widetilde{u}\cdot v, \widetilde{u}\cdot v)\,.
\end{aligned} \tag{2.23b}$$

Since the parameters $\widetilde{\rho}, \widetilde{\theta}$ and $\widetilde{u}$ are independent, the relation (2.23b) is equivalent to the relations

$$\mathcal{L}_M(\widetilde{\rho} + \widetilde{\theta}(\tfrac{1}{2}|v|^2 - \tfrac{d}{2})) = 0 \tag{2.23c}$$

and

$$\mathcal{L}_M(\tfrac{1}{2}A(v)\cdot(\widetilde{u}\vee\widetilde{u})) = \mathcal{Q}_M(\widetilde{u}\cdot v, \widetilde{u}\cdot v)\,. \tag{2.23d}$$

Eventually, the last equation of (2.22) can also be written as:

$$\mathcal{L}_M(m^{(3)}) = 2\mathcal{Q}_M(\widetilde{u}\cdot v, m^{(2)}) + \mathcal{T}_M(\widetilde{u}\cdot v, \widetilde{u}\cdot v, \widetilde{u}\cdot v)\,. \tag{2.24}$$

Finally, fix a local equilibrium $\mathcal{M} = \mathcal{M}(\rho, u, \theta)$. Every orthogonal matrix $O \in \mathbb{R}^{d\times d}$ defines the transformation $\mathcal{O}_o \equiv \mathcal{A}_u\mathcal{A}_o\mathcal{A}_u^{-1}$ where $\mathcal{A}_u$ and $\mathcal{A}_o$ are defined by (2.7). It is easily checked that $\mathcal{O}_o\mathcal{M} = \mathcal{M}$ and that $\mathcal{O}_o$ is an orthogonal transformation over $\mathbb{H}_{\mathcal{M}}$. Upon using (2.9) to expand the commutation relations (2.8) about $\mathcal{M}$, one obtains that the operators $\mathcal{L}_{\mathcal{M}}$, $\mathcal{Q}_{\mathcal{M}}$, and $\mathcal{T}_{\mathcal{M}}$ commute with the orthogonal transformations $\mathcal{O}_o$:

Proposition 2.4: Let $O \in \mathbb{R}^{d\times d}$ be an orthogonal matrix. If $\widetilde{g} \in \mathrm{D}(\mathcal{L}_{\mathcal{M}})$, then $\mathcal{O}_o\widetilde{g} \in \mathrm{D}(\mathcal{L}_{\mathcal{M}})$, and

$$\mathcal{O}_o\mathcal{L}_{\mathcal{M}}\widetilde{g} = \mathcal{L}_{\mathcal{M}}\mathcal{O}_o\widetilde{g} \text{ and } \mathcal{O}_o\mathcal{L}_{\mathcal{M}}^{\dagger}\widetilde{g} = \mathcal{L}_{\mathcal{M}}^{\dagger}\mathcal{O}_o\widetilde{g}\,. \tag{2.25a}$$

If $\widetilde{g} \in \mathrm{D}(\mathcal{Q}_{\mathcal{M}})$, then $\mathcal{O}_o\widetilde{g} \in \mathrm{D}(\mathcal{Q}_{\mathcal{M}})$, and

$$\mathcal{O}_o\mathcal{Q}_{\mathcal{M}}(\widetilde{g}, \widetilde{g}) = \mathcal{Q}_{\mathcal{M}}(\mathcal{O}_o\widetilde{g}, \mathcal{O}_o\widetilde{g})\,. \tag{2.25b}$$

If $\widetilde{g} \in \mathrm{D}(\mathcal{T}_{\mathcal{M}})$, then $\mathcal{O}_o\widetilde{g} \in \mathrm{D}(\mathcal{T}_{\mathcal{M}})$, and

$$\mathcal{O}_o\mathcal{T}_{\mathcal{M}}(\widetilde{g}, \widetilde{g}, \widetilde{g}) = \mathcal{T}_{\mathcal{M}}(\mathcal{O}_o\widetilde{g}, \mathcal{O}_o\widetilde{g}, \mathcal{O}_o\widetilde{g})\,. \tag{2.25c}$$

By specializing Proposition 2.4 to the case $O = -I$, the operators $\mathcal{L}_{\mathcal{M}}$, $\mathcal{Q}_{\mathcal{M}}$, and $\mathcal{T}_{\mathcal{M}}$ are seen to respect even and odd symmetries. For example, if $\widetilde{g}_e$ and $\widetilde{g}_o$ denote any functions that are even and odd in $v-u$ respectively, then these symmetries imply that $\mathcal{L}_{\mathcal{M}}\widetilde{g}_e$, $\mathcal{Q}_{\mathcal{M}}(\widetilde{g}_e,\widetilde{g}_e)$, and $\mathcal{Q}_{\mathcal{M}}(\widetilde{g}_o,\widetilde{g}_o)$ are even functions of $v-u$, while $\mathcal{L}_{\mathcal{M}}\widetilde{g}_o$, $\mathcal{Q}_{\mathcal{M}}(\widetilde{g}_o,\widetilde{g}_e)$, and $\mathcal{T}_{\mathcal{M}}(\widetilde{g}_o,\widetilde{g}_o,\widetilde{g}_o)$ are odd.

2.3: The Pseudo Inverse of the Linearized Collision Operator.

In order to carry out our formal calculations, we will need to make additional assumptions regarding the linearization $\mathcal{L}_{\mathcal{M}}$ of the collision operator $\mathcal{C}$ about a Maxwellian $\mathcal{M}$.

With the relation (2.11a) of Proposition 2.1, the space $\mathbb{E}$ of locally conserved quantities is a subset of $\mathrm{N}(\mathcal{L}_{\mathcal{M}}^{\dagger})$. With the relations (2.23) it is also a subspace of $\mathrm{N}(\mathcal{L}_{\mathcal{M}})$. Eventually one has

$$\mathbb{E} \subset \mathrm{N}(\mathcal{L}_{\mathcal{M}}) = \mathrm{N}(\mathcal{L}_{\mathcal{M}}^{\dagger}) \subset \mathrm{N}(\mathcal{L}_{\mathcal{M}} + \mathcal{L}_{\mathcal{M}}^{\dagger}) \,. \tag{2.26}$$

These properties of $\mathcal{L}_{\mathcal{M}}$ are not generally sufficient to carry out formal derivations of the Navier-Stokes equations, so we now add two assumptions regarding $\mathcal{L}_{\mathcal{M}}$ to the basic assumptions in Section 2.1.

First, we shall assume that inclusions (2.26) are equalities. More precisely, for every $\widetilde{g} \in \mathrm{D}(\mathcal{L}_{\mathcal{M}})$ the following statements are assumed to be equivalent:

$$\begin{aligned} \text{i)} \quad & \langle \widetilde{g}\, \mathcal{M}\mathcal{L}_{\mathcal{M}}\widetilde{g}\rangle = 0 \,, \\ \text{ii)} \quad & \mathcal{L}_{\mathcal{M}}\widetilde{g} = 0 \,, \\ \text{iii)} \quad & \mathcal{L}_{\mathcal{M}}^{\dagger}\widetilde{g} = 0 \,, \\ \text{iv)} \quad & \widetilde{g} \in \mathbb{E} \,. \end{aligned} \tag{2.27}$$

An important special case which includes all classical collision operators is when $\mathcal{L}_{\mathcal{M}}$ is self-adjoint ($\mathcal{L}_{\mathcal{M}}^{\dagger} = \mathcal{L}_{\mathcal{M}}$). In this case (2.26) implies that (i), (ii), and (iii) in (2.26) are always equivalent, so the only assertion in (2.27) is the equivalence of (iv) to the others.

Second, we assume that $\mathcal{L}_{\mathcal{M}}$ satisfies the Fredholm alternative $\mathrm{R}(\mathcal{L}_{\mathcal{M}}) = \mathbb{E}^{\perp}$. Specifically, this means we are assuming that $\mathrm{R}(\mathcal{L}_{\mathcal{M}})$ is closed, a property that does not hold for all classical collision operators but does hold for that of Boltzmann in the case of Maxwell molecules or hard potentials with an angular cutoff [Gr]. One could include the case of the soft potentials if the Fredholm assumption is replaced with the weaker one that $\mathcal{L}_{\mathcal{M}}$ satisfies the Fredholm alternative in a space that is appropriately related to $\mathbb{H}_{\mathcal{M}}$, but for simplicity we will not do so here.

The Fredholm alternative implies that the equations

$$\mathcal{L}_{\mathcal{M}}\widetilde{g} = \widetilde{h} \text{ and } \mathcal{L}_{\mathcal{M}}^{\dagger}\widetilde{g}^{*} = \widetilde{h} \tag{2.28}$$

have a solution if and only if $\widetilde{h} \in \mathbb{E}^{\perp}$, in which case these solutions ($\widetilde{g}$ and $\widetilde{g}^{*}$) are unique in $\mathbb{E}^{\perp}$. Therefore, the pseudo-inverse operators $\mathcal{L}_{\mathcal{M}}^{-1}$ and $(\mathcal{L}_{\mathcal{M}}^{\dagger})^{-1}$ are uniquely defined by the relations:

$$\begin{aligned} \mathcal{L}_{\mathcal{M}}\mathcal{L}_{\mathcal{M}}^{-1} = \mathcal{I} - \mathcal{P}_{\mathcal{M}} \,, \quad & \mathcal{L}_{\mathcal{M}}^{-1}\mathcal{L}_{\mathcal{M}} \subset \mathcal{I} - \mathcal{P}_{\mathcal{M}} \,, \\ \mathcal{L}_{\mathcal{M}}^{\dagger}(\mathcal{L}_{\mathcal{M}}^{\dagger})^{-1} = \mathcal{I} - \mathcal{P}_{\mathcal{M}} \,, \quad & (\mathcal{L}_{\mathcal{M}}^{\dagger})^{-1}\mathcal{L}_{\mathcal{M}}^{\dagger} \subset \mathcal{I} - \mathcal{P}_{\mathcal{M}} \,, \end{aligned} \tag{2.29}$$

where $\mathcal{P}_{\mathcal{M}}$ denotes the orthogonal projection of $\mathbb{H}_{\mathcal{M}}$ onto $\mathbb{E}$ explicitly given by the formula:

$$\begin{aligned} \mathcal{P}_{\mathcal{M}}\widetilde{g} =& \frac{1}{\rho}\bigg[\langle \mathcal{M}\widetilde{g}\rangle + \frac{(v-u)\cdot\langle (v-u)\mathcal{M}\widetilde{g}\rangle}{\theta} \\ & + \left(\frac{|v-u|^2}{2\theta} - \frac{d}{2}\right)\frac{2}{d}\left\langle \left(\frac{|v-u|^2}{2\theta} - \frac{d}{2}\right)\mathcal{M}\widetilde{g}\right\rangle\bigg] \,. \end{aligned} \tag{2.30}$$

The dissipation relation (2.14) ensures that the quadratic forms associated with $\mathcal{L}_{\mathcal{M}}$ and $\mathcal{L}_{\mathcal{M}}^{-1}$ enjoy the positivity properties:

$$
\begin{aligned}
\langle \widetilde{g}\, \mathcal{M}\mathcal{L}_{\mathcal{M}}\widetilde{g}\rangle &= \langle \widetilde{g}\, \mathcal{M}\mathcal{L}_{\mathcal{M}}^{\dagger}\widetilde{g}\rangle > 0\,, \qquad \text{for every nonzero } \widetilde{g} \in \mathbb{E}^{\perp} \cap \mathrm{D}(\mathcal{L}_{\mathcal{M}})\,,\\
\langle \widetilde{g}\, \mathcal{M}\mathcal{L}_{\mathcal{M}}^{-1}\widetilde{g}\rangle &= \langle \widetilde{g}\, \mathcal{M}(\mathcal{L}_{\mathcal{M}}^{\dagger})^{-1}\widetilde{g}\rangle > 0\,, \qquad \text{for every nonzero } \widetilde{g} \in \mathbb{E}^{\perp}\,.
\end{aligned} \tag{2.31}
$$

In the case that $\mathcal{L}_{\mathcal{M}}$ is self-adjoint this means that $\mathcal{L}_{\mathcal{M}}$ and $\mathcal{L}_{\mathcal{M}}^{-1}$ are positive definite over $\mathbb{E}^{\perp}$.

Since the measure dv is invariant under action by any $\mathcal{O}_o$, it follows from Proposition 2.4 that

$$
\begin{aligned}
\langle \widetilde{g}\, \mathcal{M}\mathcal{L}_{\mathcal{M}}\widetilde{g}\rangle \;&=\; \langle \mathcal{O}_o(\widetilde{g}\, \mathcal{M}\mathcal{L}_{\mathcal{M}}\widetilde{g})\rangle \;=\; \langle (\mathcal{O}_o\widetilde{g})\, \mathcal{M}\mathcal{L}_{\mathcal{M}}\mathcal{O}_o\widetilde{g}\rangle\,,\\
\langle \widetilde{g}\, \mathcal{M}\mathcal{L}_{\mathcal{M}}^{-1}\widetilde{g}\rangle &= \langle \mathcal{O}_o(\widetilde{g}\, \mathcal{M}\mathcal{L}_{\mathcal{M}}^{-1}\widetilde{g})\rangle = \langle (\mathcal{O}_o\widetilde{g})\, \mathcal{M}\mathcal{L}_{\mathcal{M}}^{-1}\mathcal{O}_o\widetilde{g}\rangle\,,
\end{aligned} \tag{2.32}
$$

for every orthogonal matrix $O \in \mathbb{R}^{d\times d}$. This so-called orthogonal symmetry simplifies the evaluation of many tensors that appear subsequently.

The tensor and vector-valued functions $A(v)$ and $B(v)$ belong to $\mathbb{E}^{\perp} = \mathrm{N}(\mathcal{L}_{\mathcal{M}})^{\perp}$ (this can be proved by direct computation or deduced from (2.18) and (2.19)), and therefore the functions

$$
\mathcal{L}_{M}^{-1}A\,, \qquad \mathcal{L}_{M}^{-1}B\,, \qquad (\mathcal{L}_{M}^{\dagger})^{-1}A\,, \qquad (\mathcal{L}_{M}^{\dagger})^{-1}B\,,
$$

are well defined.

Due to its frequent appearance in the computations, the vector valued function $(\mathcal{L}_{M}^{\dagger})^{-1}B$ is denoted by $\widetilde{B}$. The following formulas are obtained

$$
\begin{aligned}
&\langle A_{ij}\, M\, \mathcal{L}_{M}^{-1}A_{kl}\rangle = \langle A_{ij}\, M\, (\mathcal{L}_{M}^{\dagger})^{-1}A_{kl}\rangle\\
&= \tfrac{1}{(d-1)(d+2)}\, \langle A\!:\!M\, \mathcal{L}_{M}^{-1}A\rangle\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\big),
\end{aligned} \tag{2.33a}
$$

$$
\langle B_i\, M\, (\mathcal{L}_{M}^{\dagger})^{-1}B_j\rangle = \langle B_i\, M\, \widetilde{B}_j\rangle = \tfrac{1}{d}\, \langle B\!\cdot\! M\, \widetilde{B}\rangle\delta_{ij}\,, \tag{2.33b}
$$

leading to the definition of two strictly positive numbers which represent the viscosity and the thermal diffusivity:

$$
\mu^* = \tfrac{1}{(d-1)(d+2)}\, \langle A\!:\!M\, \mathcal{L}_{M}^{-1}A\rangle\,, \tag{2.34a}
$$

$$
\kappa^* = \tfrac{1}{d}\, \langle B\!\cdot\! M\, \widetilde{B}\rangle\,. \tag{2.34b}
$$

In the analysis of the energy balance equation, two other tensor value functions appear:

$$
\langle A_{ij}\, M v_k \widetilde{B}_l\rangle \text{ and } \langle (\mathcal{L}_{M}^{-1}(A))_{ij}\, M v_k \widetilde{B}_l\rangle\,.
$$

They have the same symmetry properties as the tensor given by (2.33a). In particular, the following formulas will be used:

$$
\langle A_{ij}\, M\, v_k\widetilde{B}_l\rangle - \langle A_{ik}\, M\, v_j\widetilde{B}_l\rangle = \frac{2}{d}\kappa^*\big(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl}\big) \tag{2.35}
$$

and

$$
\begin{aligned}
&\langle \mathcal{L}_{M}^{-1}(A)_{ij}\, M v_k \widetilde{B}_l\rangle =\\
&\qquad \tfrac{1}{(d-1)(d+2)}\, \langle \mathcal{L}_{M}^{-1}(A)\!:\! M v \otimes \widetilde{B}_l\rangle\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\big)\,.
\end{aligned} \tag{2.36}
$$

Proofs and comments:

To obtain the formulas (2.33a) and (2.33b), one uses the symmetries of the tensors $A(v)$ and $B(v)$, the relation (2.30) for the operators and the fact that $\mathrm{tr}(A) = 0$. The fact that the constants μ^* and κ^* are strictly positive is a consequence of the strict positivity of the operator $\mathcal{L}_{_M}^{-1}$ defined on $\mathbb{E}^\perp$ (use (2.14)).

To prove (2.35), we use the expression $A_{ij}(v) = v_i v_j - \frac{|v|^2}{d}\delta_{ij}$ and observe that

$$\langle v_k(\tfrac{1}{d}|v|^2)M\widetilde{B}_l\rangle = \tfrac{2}{d}\langle v_k\big(\frac{|v|^2-(d+2)}{2}\big)M\widetilde{B}_l\rangle = \tfrac{2}{d}\kappa^*\delta_{kl}\,.$$

To obtain the relation (2.36), we use the symmetry of $\widetilde{B}$ deduced from the symmetry of B, the symmetry of $\mathcal{L}^{-1}(A)$ and the fact that this tensor has zero trace. With the form of the right hand side of (2.36) and the incompressibility condition $\nabla_x\cdot\widetilde{u} = 0$, the explicit value of the constant

$$\langle\mathcal{L}_{_M}^{-1}(A)_{ij}\, M v_k\widetilde{B}_l\rangle$$

does not appear in our derivation.

2.4: A New Identity.

The derivation of the internal energy balance equation involves higher order terms and therefore it requires one extra identity which is the object of the

Proposition 2.6. For any vector $u \in R^d$, one has that

$$2\langle\mathcal{Q}_{_M}(v\cdot u, \mathcal{L}_{_M}^{-1}A)M\widetilde{B}\rangle + \langle u\cdot vAM\widetilde{B}\rangle = \langle(\mathcal{L}_{_M}^{-1}A)MAu\rangle\,, \tag{2.37}$$

or equivalently "component-wise"

$$2\langle(\mathcal{Q}_{_M}(v_iu_i, (\mathcal{L}_{_M}^{-1}(A))_{kl})M\widetilde{B}_j\rangle + \langle u_iv_iA_{kl}M\widetilde{B}_j\rangle = \langle(\mathcal{L}_{_M}^{-1}(A))_{kl}Mu_iA_{ij}\rangle \tag{2.37'}$$

Proof: Rotational invariance has been used for the previous relation. In the present step the translational invariance is also used.

For $u \in \mathbb{R}^d$ the functions:

$$\begin{aligned}\mathcal{M}_u &= \mathcal{A}_u M = \frac{1}{(2\pi)^{\frac{d}{2}}}\exp\Big(-\frac{|v-u|^2}{2}\Big)\,,\\ A_u &= \mathcal{A}_u A = A(v-u)\\ B_u &= \mathcal{A}_u B = B(v-u)\end{aligned} \tag{2.38}$$

are introduced. The commutation relation (2.8) implies that one has:

$$\mathcal{L}_{\mathcal{M}_u}^{-1}A_u = \mathcal{A}_u\mathcal{L}_M^{-1}A$$

and therefore, by parity reasons, for any $u \in \mathbb{R}^d$ and any $s \in \mathbb{R}$ one has

$$s \mapsto h(s) = \langle\mathcal{L}_{\mathcal{M}_{su}}^{-1}(A_{su}), \mathcal{M}_{su}B_{su}\rangle = 0\,. \tag{2.39}$$

The proof of the identity (2.37) is done with the computation of the derivative of $h(s)$ at $s = 0$.

First, one has the following trivial identities:

$$\begin{aligned}&\frac{d}{ds}\big(\mathcal{M}_{(su)}\big)|_{s=0} = u\cdot v\mathcal{M}, \quad \frac{d}{ds}\big((\mathcal{M}_{(su)})^{-1}\big)|_{s=0} = -u\cdot v\mathcal{M}^{-1}\,,\\ &\frac{d}{ds}\big(A_{su}\big)|_{s=0} = -(v\otimes u + u\otimes v - \frac{2}{d}v\cdot uI)\,,\\ &\frac{d}{ds}\big(B_{su}\big)|_{s=0} = -(Au + \frac{d+2}{d}(\frac{|v|^2-d}{2}u))\,.\end{aligned} \tag{2.40}$$

Second, using the differentiability up to the order two of the collision operator, the derivative of $\mathcal{L}_{\mathcal{M}_{su}}$ at $s=0$ is given by

$$
\begin{aligned}
\frac{d}{ds}\mathcal{L}_{\mathcal{M}_{su}}(\widetilde{g}) &= -\frac{d}{ds}\frac{d}{d\epsilon}\big(\frac{1}{\mathcal{M}_{su}}\mathcal{C}\big(\mathcal{M}_{su}(1+\epsilon\,\widetilde{g})\big)\big)\;|_{\epsilon=0} \\
&= -\frac{d}{d\epsilon}\Big[\Big(\frac{d}{ds}\big(\frac{1}{\mathcal{M}_{su}}\big)\Big)\mathcal{C}\big(\mathcal{M}_{su}(1+\epsilon\,\widetilde{g})\big) \\
&+ \big(\frac{1}{\mathcal{M}_{su}}\big)\mathcal{C}'\big(\mathcal{M}_{su}(1+\epsilon\,\widetilde{g})\big)\big(\frac{d}{ds}\mathcal{M}_{su}(1+\epsilon\,\widetilde{g})\big)\Big]\;|_{\epsilon=0} \\
&= -\Big(\frac{d}{ds}\big(\frac{1}{\mathcal{M}_{su}}\big)\Big)\mathcal{C}'\big(\mathcal{M}_{su}\big)(\mathcal{M}_{su}\widetilde{g}) \\
&- \frac{1}{\mathcal{M}_{su}}\mathcal{C}'\big(\mathcal{M}_{su}\big)\big(\frac{d}{ds}\mathcal{M}_{su}\widetilde{g}\big) \\
&- \frac{1}{\mathcal{M}_{su}}\mathcal{C}''\big(\mathcal{M}_{su}\big)\big((\mathcal{M}_{su}\widetilde{g}),\big(\frac{d}{ds}\mathcal{M}_{su}\big)\big)\,.
\end{aligned}
\tag{2.41}
$$

With $s=0$ and $\epsilon=0$ in (2.41), the relations (2.40) and the Taylor expansion (2.9), one obtains the formula:

$$
\frac{d}{ds}(\mathcal{L}_{\mathcal{M}_{su}})_{s=0}(\widetilde{g}) = -(u\cdot v)\mathcal{L}_{_M}\widetilde{g} + \mathcal{L}_{_M}(u\cdot v\widetilde{g}) - 2\mathcal{Q}_{_M}(u\cdot v,\widetilde{g})\,. \tag{2.42}
$$

Since the operator $\mathcal{L}_{\mathcal{M}_{su}}$ is defined on the space $\mathbb{E}^{\perp}$ which is independent of s, one has

$$
\frac{d}{ds}\big(\mathcal{L}^{-1}_{\mathcal{M}_{su}}\big) = -\mathcal{L}^{-1}_{\mathcal{M}_{su}}\frac{d}{ds}\big(\mathcal{L}_{\mathcal{M}_{su}}\big)\mathcal{L}^{-1}_{\mathcal{M}_{su}}\,. \tag{2.43}
$$

Since

$$
\begin{aligned}
&\langle B_{su}\mathcal{M}_{su}\mathcal{L}^{-1}_{\mathcal{M}_{su}}\big(\frac{d}{ds}A_{(su)}\big)\rangle \\
&= -\langle(\mathcal{L}^{\dagger}_{\mathcal{M}_{su}})^{-1}B_{su}\mathcal{M}_{su}\big((v-su)\otimes u + u\otimes(v-su) - \frac{2}{d}(v-su)\cdot uI\big)\rangle = 0\,,
\end{aligned}
\tag{2.44}
$$

the differentiation at $s=0$ of the relation (2.39) gives:

$$
0 = -\langle(\mathcal{L}^{-1}_{_M}A)MAu\rangle + \langle Bu\cdot v\,M\mathcal{L}^{-1}_{_M}A\rangle - \langle(\mathcal{L}^{\dagger})^{-1}BM\big(\frac{d}{ds}\mathcal{L}_{\mathcal{M}_{su}}\big)_{s=0}(\mathcal{L}^{-1}A)\rangle\,. \tag{2.45}
$$

For the last term on the right hand side of (2.45) the relation (2.42) is used so that the identity (2.37) is obtained.

3 Formal Derivation of the Incompressible System with Viscous Heating

In this section, the formal convergence theorem is proven. The word "formal" refers to the fact that the solutions of the rescaled equation are assumed to be bounded in a space of functions where all the convergence results needed will be true. It is assumed that for any ϵ the solution of the scaled kinetic equation

$$\epsilon\, \partial_t F^\epsilon + v\cdot\nabla_x F^\epsilon = \frac{1}{\epsilon}\mathcal{C}(F^\epsilon)\,, \tag{3.1}$$

fluctuates about an absolute Maxwellian according to the formula:

$$F^\epsilon = M(1 + \epsilon\, \widetilde{G}^\epsilon_o + \epsilon^2 \widetilde{G}^\epsilon_e) \tag{3.2}$$

where $\widetilde{G}^\epsilon_o$ is v-odd and $\widetilde{G}^\epsilon_e$ is v-even.

Theorem 3.1: Let $F^\epsilon(t,x,v)$ be a sequence of nonnegative solutions to the scaled kinetic equation (3.1) such that, when written according to formula (3.2), the sequences $\widetilde{G}^\epsilon_o$ and $\widetilde{G}^\epsilon_e$ are bounded in a convenient Banach subspace of $C(\mathbb{R}^+_t(\mathcal{D}(\mathbb{R}^d_v \times \mathbb{R}^d_x))$ where the following Taylor expansion

$$\begin{aligned}\frac{1}{M}\mathcal{C}\big(M(1+\epsilon\,\widetilde{G}^\epsilon_o + \epsilon^2\widetilde{G}^\epsilon_e)\big) = -\epsilon\,\mathcal{L}\widetilde{G}^\epsilon_o + \epsilon^2\big(-\mathcal{L}\widetilde{G}^\epsilon_e + \mathcal{Q}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_o)\big)\\ + \epsilon^3\big(\mathcal{T}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_o) + 2\mathcal{Q}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_e)\big) + R_\epsilon\,,\end{aligned} \tag{3.3}$$

with $R_\epsilon = O(\epsilon^4)$, is uniformly valid.

Assume also that in the sense of distribution, i.e. in $\mathcal{D}(R^d_v \times R^d_x)$, the functions $\widetilde{G}^\epsilon_o$, $\widetilde{G}^\epsilon_e$ and $\mathcal{Q}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_o)$ converge to the three corresponding functions $\widetilde{G}_o$, $\widetilde{G}_e$ and $\mathcal{Q}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_o)$ as ϵ goes to zero. Finally, assume for the moment that the following convergence in $\mathcal{D}(\mathbb{R}^d_x)$ holds:

$$(v|\widetilde{G}^\epsilon_o)_M \to (v|\widetilde{G}_o)_M\,,\quad (v\widetilde{G}^\epsilon_o|\widetilde{B})_M \to (v\,\widetilde{G}_o|\widetilde{B})_M\,,$$

$$(v\vee v|\widetilde{G}^\epsilon_e)_M \to (v\vee v|\widetilde{G}_e)_M\,,\quad (v\widetilde{G}^\epsilon_e|\widetilde{B})_M \to (v\widetilde{G}_e|\widetilde{B})_M\,,$$

and

$$\{2(\mathcal{Q}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_e)|\widetilde{B})_M + (\mathcal{T}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_o),\widetilde{B})_M\}$$

to

$$\{2(\mathcal{Q}(\widetilde{G}_o,\widetilde{G}_e)|\widetilde{B})_M + (\mathcal{T}(\widetilde{G}_o,\widetilde{G}_o,\widetilde{G}_o),\widetilde{B})_M\}\,.$$

Assume also that the reminder term R_ϵ in the formula (3.3) satisfies the estimate:

$$\lim_{\epsilon\to 0}\epsilon^{-2}(R_\epsilon|\widetilde{B})_M = 0\,.$$

Then the limits $\widetilde{G}_o$ and $\widetilde{G}_e$ have the form

$$\widetilde{G}_o = v\cdot\widetilde{u}\,, \tag{3.4}$$

$$\widetilde{G}_e = \big(\widetilde{\rho} + (\tfrac{1}{2}|\widetilde{u}|^2 + \tfrac{d}{2}\widetilde{\theta})\tfrac{2}{d}(\frac{|v|^2-d}{2}) + \tfrac{1}{2}A(v):(\widetilde{u}\vee\widetilde{u})\big) - \mathcal{L}^{-1}A(v):\nabla_x\widetilde{u}\,. \tag{3.5}$$

The velocity $\widetilde{u}$ is divergence free while the density and temperature fluctuations, $\widetilde{\rho}$ and $\widetilde{\theta}$, satisfy the Boussinesq relation:

$$\nabla_x\cdot\widetilde{u} = 0\,, \tag{3.6a}$$

$$\widetilde{p} = \widetilde{\rho} + \widetilde{\theta}\,. \tag{3.6b}$$

Moreover, the functions $\widetilde{\rho}$, $\widetilde{u}$, and $\widetilde{\theta}$ are weak solutions of the equations

$$\partial_t\widetilde{u} + \widetilde{u}\cdot\nabla_x\widetilde{u} - \mu^*\Delta_x\widetilde{u} = -\nabla_x\widetilde{p}\,, \quad \nabla_x\cdot\widetilde{u} = 0\,, \tag{3.7}$$

and

$$\partial_t(\tfrac{d}{2}\widetilde{\theta} + \tfrac{1}{2}|\widetilde{u}|^2 - \widetilde{\rho}) + \nabla_x\cdot\big(\widetilde{u}(\tfrac{d+2}{2}\widetilde{\theta} + \tfrac{1}{2}|\widetilde{u}|^2)\big) = \kappa^*\Delta_x\widetilde{\theta} + \mu^*\nabla_x\cdot\big(\widetilde{u}(\nabla_x\widetilde{u} + (\nabla_x\widetilde{u})^T)\big)\,. \tag{3.8}$$

Proof: With the expansion (3.2), the following "moment equations" are deduced from the local conservation properties:

$$\epsilon^2\partial_t(1|\widetilde{G}^\epsilon_e)_M + \nabla_x\cdot(v|\widetilde{G}^\epsilon_o)_M = 0\,, \tag{3.9a}$$

$$\partial_t(v|\widetilde{G}^\epsilon_o)_M + \nabla_x\cdot(v\vee v\,\widetilde{G}^\epsilon_e)_M = 0\,, \tag{3.9b}$$

$$\epsilon^2\partial_t((\tfrac{1}{2}|v|^2 - \tfrac{d+2}{2})|\widetilde{G}^\epsilon_e))_M + \nabla_x\cdot(v\,(\tfrac{1}{2}|v|^2 - \tfrac{d+2}{2})|\widetilde{G}^\epsilon_o)_M = 0\,. \tag{3.9c}$$

Proposition 2.4 implies that $\mathcal{L}\widetilde{G}_e$, $\mathcal{Q}(\widetilde{G}_o,\widetilde{G}_o)$, and $\mathcal{Q}(\widetilde{G}_e,\widetilde{G}_e)$ are v-even, while $\mathcal{L}\widetilde{G}_o$, $\mathcal{Q}(\widetilde{G}_o,\widetilde{G}_e)$, and $\mathcal{T}(\widetilde{G}_o,\widetilde{G}_o,\widetilde{G}_o)$ are v-odd. It is therefore possible to deduce from (3.2) and (3.3) the equations

$$\epsilon^2\partial_t\widetilde{G}^\epsilon_o + \epsilon^2 v\cdot\nabla_x\widetilde{G}^\epsilon_e + \mathcal{L}\widetilde{G}^\epsilon_o = 2\epsilon^2\mathcal{Q}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_e) + \epsilon^2\mathcal{T}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_o) + O(\epsilon^3)\,, \tag{3.10}$$

and

$$\epsilon^2\partial_t\widetilde{G}^\epsilon_e + v\cdot\nabla_x\widetilde{G}^\epsilon_o + \mathcal{L}\widetilde{G}^\epsilon_e = \mathcal{Q}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_o) + O(\epsilon^2)\,. \tag{3.11}$$

From (3.10), one deduces that

$$\widetilde{G}_o = \lim_{\epsilon\to0}\widetilde{G}^\epsilon_o$$

is an odd solution of the equation $\mathcal{L}\widetilde{G}_o = 0$. Hence, one has

$$\widetilde{G}_o = \widetilde{u}(x,t)\cdot v\,, \tag{3.12}$$

which with (3.9a) also implies the relation

$$0 = \nabla_x\cdot(v|\widetilde{G}_o)_M = \nabla_x\cdot\widetilde{u}\,, \tag{3.13}$$

and proves the incompressibility. Equation (3.11) leads, for $\epsilon\to0$, to the relation:

$$\widetilde{G}_e = \lim_{\epsilon\to0}\widetilde{G}^\epsilon_e$$

$$\mathcal{L}\widetilde{G}_e = -v\cdot\nabla_x(v\cdot\widetilde{u}) + \mathcal{Q}(\widetilde{u}\cdot v,\widetilde{u}\cdot v)\,. \tag{3.14}$$

With the (2.21) and (2.23b) one has

$$\mathcal{Q}(\widetilde{u}\cdot v,\widetilde{u}\cdot v) = \tfrac{1}{2}\mathcal{L}(A(v):(\widetilde{u}\vee\widetilde{u}))\,.$$

Moreover with the incompressibility condition, the first term of the right side of (3.14) can also be written as

$$v\cdot\nabla_x(v\cdot\widetilde{u}) = v\vee v:\nabla_x\widetilde{u} = (v\vee v - \tfrac{1}{d}|v|^2 I):\nabla_x\widetilde{u} = A(v):\nabla_x\widetilde{u}\,, \tag{3.15}$$

which proves that $\widetilde{G}_e$ is a solution of the equation:

$$\mathcal{L}\widetilde{G}_e = -\nabla_x(A(v) : \widetilde{u}) + \tfrac{1}{2}\mathcal{L}(A(v) : (\widetilde{u} \vee \widetilde{u}))\,. \tag{3.16}$$

The Fredholm property is used to uniquely determine, from the above equation, $\widetilde{G}_e$ up to an even term $z_e \in \mathbb{E} = \mathrm{N}(\mathcal{L}_{M})$, which, in accordance with the expansion (2.20)−(2.21) of the Maxwellian

$$\frac{1+\epsilon^2\widetilde{\rho}}{(2\pi(1+\epsilon^2\widetilde{\theta}))^{\frac{d}{2}}}\exp\Big(-\frac{|v-\epsilon\,\widetilde{u}|^2}{2(1+\epsilon^2\widetilde{\theta})}\Big)\,,$$

is written as

$$z_e = \widetilde{\rho} + (\tfrac{1}{2}|\widetilde{u}|^2 + \tfrac{d}{2}\widetilde{\theta})\tfrac{2}{d}(\frac{|v|^2-d}{2})\,. \tag{3.17}$$

Therefore, the macroscopic variables $\widetilde{\rho}$ and $\widetilde{\theta}$ are introduced, and also m^i denoting the (x,v,t) dependent scalar functions given by (cf (2.21)):

$$\begin{aligned}
m^{(1)} &= \widetilde{u}\cdot v\,,\\
m^{(2)} &= \widetilde{\rho} + (\tfrac{1}{2}|\widetilde{u}|^2 + \tfrac{d}{2}\widetilde{\theta})\tfrac{2}{d}(\tfrac{1}{2}|v|^2 - \tfrac{d}{2}) + \tfrac{1}{2}A(v)\cdot(\widetilde{u}\vee\widetilde{u}) \;= z_e + \tfrac{1}{2}A(v)\cdot(\widetilde{u}\vee\widetilde{u})\\
m^{(3)} &= \widetilde{\rho}\widetilde{u}\cdot v + \widetilde{\theta}\widetilde{u}\cdot B(v) + \tfrac{1}{3}C(v)\cdot(\widetilde{u}\vee\widetilde{u}\vee\widetilde{u})\,.
\end{aligned}$$

Eventually, one has:

$$\begin{aligned}
\widetilde{G}_e =& z_e + \tfrac{1}{2}(A(v) : (\widetilde{u}\vee\widetilde{u})) - \mathcal{L}^{-1}(A(v)) : \nabla_x\widetilde{u}\\
=& m^{(2)} - \mathcal{L}^{-1}(A(v)) : \nabla_x\widetilde{u}\,.
\end{aligned} \tag{3.18}$$

With $\epsilon \to 0+$ the equation (3.9b) (conservation of macroscopic momentum) gives the Navier Stokes equation. More precisely:

$$\partial_t(v|\widetilde{G}_o)_{M} = -\nabla_x\cdot(v\vee v|\widetilde{G}_e)_{M} = -\nabla_x\cdot(A(v)|\widetilde{G}_e)_{M} - \nabla_x(\frac{|v|^2}{d}|\widetilde{G}_e)_{M}\,, \tag{3.19}$$

inserting in (3.19) the expression of $\widetilde{G}_o$ given by (3.12) and the expression of $\widetilde{G}_e$ given by (3.18) leads to the equation:

$$\begin{aligned}
\partial_t\widetilde{u} =& -\nabla_x(A(v)|\big(z_e + \tfrac{1}{2}(A(v) : (\widetilde{u}\vee\widetilde{u})) - \mathcal{L}^{-1}(A(v)) : \nabla_x\widetilde{u}\big))_{M}\\
& - \nabla_x(\frac{|v|^2}{d}|\big(z_e + \tfrac{1}{2}(A(v) : (\widetilde{u}\vee\widetilde{u})) - \mathcal{L}^{-1}(A(v)) : \nabla_x\widetilde{u}\big))_{M}\\
=& \nabla_x\big((A(v) : (\mathcal{L}^{-1}(A(v))))_{M} : \nabla_x\widetilde{u}\big)\\
& -\nabla_x\big(\tfrac{1}{2}(A(v) : A(v))\big)_{M} : (\widetilde{u}\vee\widetilde{u}\big) + (\frac{|v|^2}{d}|\widetilde{G}_e)_{M}\,,
\end{aligned} \tag{3.20}$$

where the notation $(g(v))_{M} = (g(v)|1)_{M}$ is used. Explicit computations of moments give:

$$(\tfrac{1}{d}|v|^2|\widetilde{G}_e))_{M} = (\frac{|v|^2}{d}\big(\widetilde{\rho} + (\tfrac{1}{2}|\widetilde{u}|^2 + \frac{d}{2}\widetilde{\theta})\frac{2}{d}(\frac{|v|^2-d}{2})\big))_{M} = \frac{|\widetilde{u}|^2}{d} + (\widetilde{\theta}+\widetilde{\rho}) \tag{3.21}$$

and

$$\tfrac{1}{2}(A(v) : (\widetilde{u}\vee\widetilde{u})))_{M} = \widetilde{u}\vee\widetilde{u} - \tfrac{1}{d}|\widetilde{u}|^2 I\,. \tag{3.22}$$

Therefore, the equation (3.20) becomes:

$$\partial_t\widetilde{u} - \nabla_x\big((A(v) : \mathcal{L}^{-1}A(v))_{M} : \nabla_x\widetilde{u}\big) + \nabla_x(\widetilde{u}\otimes\widetilde{u}) + \nabla_x(\widetilde{\theta}+\widetilde{\rho}) = 0\,. \tag{3.23}$$

On the other hand, with (2.33a) one has:

$$\nabla_x((A(v):\mathcal{L}^{-1}A(v))_{_M}:\nabla_x\widetilde{u}) = \mu^*\nabla_x\big((\nabla_x\widetilde{u})+(\nabla_x\widetilde{u})^T-\frac{2}{d}\nabla_x\cdot\widetilde{u}\big)\,;$$
$$(\text{ with the incompressibility})$$
$$=\mu^*\Delta_x\widetilde{u}\,. \tag{3.24}$$

Therefore the Boussinesq relation (3.6b) and the Navier-Stokes equation (3.7) are proven.

To obtain the energy balance equation (3.8), we use (3.9c) in the following form:

$$\partial_t((\frac{|v|^2-(d+2)}{2})|\widetilde{G}_e^\epsilon)_{_M}+\nabla_x\cdot(\widetilde{G}_o^\epsilon|\frac{B(v)}{\epsilon^2})_{_M}=0\,. \tag{3.25}$$

The first term on the left side of (3.25) converges to

$$(\frac{|v|^2-(d+2)}{2}|\widetilde{G}_e)_{_M}=(\frac{|v|^2-(d+2)}{2}|z_e)_{_M}\,, \tag{3.26}$$

which is easily computed to give:

$$\lim_{\epsilon\to 0}(\frac{|v|^2-(d+2)}{2}|\widetilde{G}_e^\epsilon)_{_M}=\tfrac{1}{2}|\widetilde{u}|^2+\tfrac{d}{2}\widetilde{\theta}-\widetilde{\rho} \tag{3.27}$$

and corresponding to the time derivative in (3.8). For the derivation of the limit of the second term, write:

$$\lim_{\epsilon\to 0}(\widetilde{G}_o^\epsilon|\frac{B}{\epsilon^2})_{_M}=\lim_{\epsilon\to 0}(\frac{1}{\epsilon^2}\mathcal{L}\widetilde{G}_o^\epsilon|\widetilde{B})_{_M} \tag{3.28}$$

and use the equation (3.10) to obtain:

$$\begin{aligned}\lim_{\epsilon\to 0}(\widetilde{G}_o^\epsilon|\frac{B(v)}{\epsilon^2})_{_M}&=-(\partial_t\widetilde{G}_o|\widetilde{B})_{_M}-(v\cdot\nabla_x\widetilde{G}_e|\widetilde{B})_{_M}\\&\quad+2(\mathcal{Q}(\widetilde{G}_o,\widetilde{G}_e)|\widetilde{B})_{_M}+(\mathcal{T}(\widetilde{G}_o,\widetilde{G}_o,\widetilde{G}_o)|\widetilde{B})_{_M}\,.\end{aligned} \tag{3.29}$$

With $\widetilde{B}\in\mathbb{E}^\perp$ one has

$$(\partial_t\widetilde{G}_o,\widetilde{B})_{_M}=0\,.$$

Therefore,

$$\lim_{\epsilon\to 0}(\widetilde{G}_o^\epsilon|\frac{B}{\epsilon^2})_{_M}=2(\mathcal{Q}(\widetilde{G}_o,\widetilde{G}_e)|\widetilde{B})_{_M}+(\mathcal{T}(\widetilde{G}_o,\widetilde{G}_o,\widetilde{G}_o)|\widetilde{B})_{_M}-\nabla_x\cdot(v\,\widetilde{G}_e|\widetilde{B})_{_M}\,. \tag{3.30}$$

For the computation of the first two terms on the right side of (3.30), write, using (3.12) and (3.18):

$$\begin{aligned}&2(\mathcal{Q}(\widetilde{G}_o,\widetilde{G}_e)|\widetilde{B})_{_M}+(\mathcal{T}(\widetilde{G}_o,\widetilde{G}_o,\widetilde{G}_o)|\widetilde{B})_{_M}\\&=2(\mathcal{Q}(m^{(1)},m^{(2)})|\widetilde{B})_{_M}+(\mathcal{T}(m^{(1)},m^{(1)},m^{(1)})|\widetilde{B})_{_M}\\&-2(\mathcal{Q}(u\cdot v,\mathcal{L}^{-1}(A(v)):\nabla_x u)|\widetilde{B})_{_M}\,.\end{aligned} \tag{3.31}$$

For the first two terms of (3.31) the relation (2.18c) can be used to give:

$$\begin{aligned}&2(\mathcal{Q}(m^{(1)},m^{(2)})|\widetilde{B})_{_M}+(\mathcal{T}(m^{(1)},m^{(1)},m^{(1)})|\widetilde{B})_{_M}\\&=(\mathcal{L}(m^{(3)})|(\mathcal{L}^\dagger)^{-1}(B))_{_M}\\&=\big(\big(\widetilde{\rho}\widetilde{u}\cdot v+\widetilde{\theta}B(v)\cdot\widetilde{u}+\tfrac{1}{3}C(v)\cdot(\widetilde{u}\vee\widetilde{u}\vee\widetilde{u})\big)|B(v)\big)_{_M}\\&=\widetilde{u}(\tfrac{d+2}{2}\widetilde{\theta}+\tfrac{1}{2}|\widetilde{u}|^2)\,.\end{aligned} \tag{3.32}$$

Inserted in (3.28) and then in (3.25), the right hand side of (3.32) gives the advection term in (3.8).

Eventually, to conclude the proof of the formula (3.8) one has to show that:

$$\begin{aligned} &\nabla_x\cdot\nabla_x\cdot(v\,\widetilde{G}_e|\widetilde{B})_M + \nabla_x\cdot\big(2(\mathcal{Q}(u\cdot v,\mathcal{L}^{-1}(A(v)):\nabla_x u)|\widetilde{B})_M \\ =&\kappa^*\Delta_x\widetilde{\theta} + \mu^*\nabla_x\cdot\big(\widetilde{u}((\nabla_x\widetilde{u}) + (\nabla_x\widetilde{u})^T)\big)\,. \end{aligned} \tag{3.33}$$

Inserting the expression of $\widetilde{G}_e$ given by (3.14) in the first term of the right hand side of (3.33) gives:

$$\begin{aligned} &\nabla_x\cdot(v\widetilde{G}_e|\widetilde{B})_M = \\ &\nabla_x\cdot(v\big[\widetilde{\rho} + (\tfrac{1}{2}|\widetilde{u}|^2 + \frac{d}{2}\widetilde{\theta})\frac{2}{d}(\frac{|v|^2-d}{2}) + \tfrac{1}{2}(A(v):(\widetilde{u}\vee\widetilde{u})) - \mathcal{L}^{-1}(A(v)):\nabla_x\widetilde{u}\big]|\widetilde{B})_M \\ &= \nabla_x\cdot(\widetilde{\theta}B|\widetilde{B})_M + \nabla_x\cdot(\frac{|\widetilde{u}|^2}{d}B|\widetilde{B})_M \\ &+ \nabla_x\cdot(\tfrac{1}{2}(vA(v):(\widetilde{u}\vee\widetilde{u}))|\widetilde{B})_M - \nabla_x\cdot(v\mathcal{L}^{-1}(A(v)):\nabla_x\widetilde{u}|\widetilde{B})_M\,. \end{aligned} \tag{3.34}$$

Therefore, the proof of (3.33) is reduced to the proof of the two following relations:

$$\nabla_x\cdot\nabla_x\cdot(v\mathcal{L}^{-1}(A(v)):\nabla_x\widetilde{u})|\widetilde{B})_M = 0 \tag{3.35}$$

and

$$\begin{aligned} 2(\mathcal{Q}(\widetilde{u}\cdot v,\mathcal{L}^{-1}(A(v)):\nabla_x\widetilde{u})|\widetilde{B})_M + \nabla_x\cdot(\tfrac{1}{2}(vA(v):(\widetilde{u}\vee\widetilde{u}))|\widetilde{B})_M + \nabla_x(\kappa^*\frac{|\widetilde{u}|^2}{d}) = \\ \mu^*\widetilde{u}\big((\nabla_x\widetilde{u}) + (\nabla_x\widetilde{u})^T\big)\,. \end{aligned} \tag{3.36}$$

For the relation (3.35), we can use

$$(\mathcal{L}^{-1}(v_iA(v))_{kl}|\widetilde{B}_j)_M = c\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\big),$$

(cf (2.36)) and the incompressibility ($\nabla_x\cdot\widetilde{u} = 0$).

The proof of (3.36) is done component-wise: The left hand side being a summation in i,k,l of terms in the form:

$$2(\mathcal{Q}(\widetilde{u}_iv_i,\mathcal{L}^{-1}(A(v))_{kl}|\widetilde{B}_j)_M\frac{\partial\widetilde{u}_k}{\partial x_l} + (v_lA_{ki}|\widetilde{B}_j)_M\widetilde{u}_i\frac{\partial\widetilde{u}_k}{\partial x_l} + \tfrac{2\kappa^*}{d}\widetilde{u}_i\frac{\partial\widetilde{u}_i}{\partial x_j}$$

With the formula (2.35) and the incompressibility, one has:

$$\begin{aligned} &\sum_{i,k,l}\big((v_lA_{ki}|\widetilde{B}_j)_M - (v_iA_{kl}|\widetilde{B}_j)_M\big)\widetilde{u}_i\frac{\partial\widetilde{u}_k}{\partial x_l} = \tfrac{2\kappa^*}{d}\big(\sum_{i,k,l}(\delta_{ij}\delta_{kl} - \delta_{ki}\delta_{lj})\widetilde{u}_i\frac{\partial\widetilde{u}_k}{\partial x_l}\big) \\ &= \tfrac{2\kappa^*}{d}\widetilde{u}_j\sum_k\frac{\partial\widetilde{u}_k}{\partial x_k} - \tfrac{2\kappa^*}{d}\sum_i\widetilde{u}_i\frac{\partial\widetilde{u}_i}{\partial x_j} = -\tfrac{2\kappa^*}{d}\sum_i\widetilde{u}_i\frac{\partial\widetilde{u}_i}{\partial x_j}\,. \end{aligned} \tag{3.37}$$

With (3.37) the right hand side of (3.36) coincides with:

$$2(\mathcal{Q}(\widetilde{u}_iv_i,\mathcal{L}^{-1}(A(v))_{kl}|\widetilde{B}_j)_M\frac{\partial\widetilde{u}_k}{\partial x_l} + (v_iA_{kl}|\widetilde{B}_j)_M\widetilde{u}_i\frac{\partial\widetilde{u}_k}{\partial x_l}.$$

Eventually, the "new identity" (2.37) can be used to give:

$$
\begin{aligned}
&2(\mathcal{Q}(\widetilde{u}_i v_i, \mathcal{L}^{-1}(A(v))_{kl}|\widetilde{B}_j)_{\scriptscriptstyle M}\frac{\partial \widetilde{u}_k}{\partial x_l} + (v_l A_{ki}|\widetilde{B}_j)_{\scriptscriptstyle M}\,\widetilde{u}_i\frac{\partial \widetilde{u}_k}{\partial x_l} + \tfrac{2}{d}\kappa^*\widetilde{u}_i\frac{\partial \widetilde{u}_i}{\partial x_i} \\
&= 2(\mathcal{Q}(\widetilde{u}_i v_i, \mathcal{L}^{-1}(A(v))_{kl}|\widetilde{B}_j)_{\scriptscriptstyle M}\frac{\partial \widetilde{u}_k}{\partial x_l} + (v_i\widetilde{u}_i A_{kl}|\widetilde{B}_j)_{\scriptscriptstyle M}\frac{\partial \widetilde{u}_k}{\partial x_l} \\
&= \widetilde{u}_i(A_{ij}|\mathcal{L}^{-1}(A(v))_{kl})_{\scriptscriptstyle M}\frac{\partial \widetilde{u}_k}{\partial x_l},
\end{aligned}
$$

which, with (2.33a), coincides with

$$
\mu^*\widetilde{u}\big((\nabla_x\widetilde{u}) + (\nabla_x\widetilde{u})^T\big),
$$

and the proof of (3.36) is completed.

4 Formal Derivation of the Non-classical System

As stated in Introduction, the limits of the scaled Boltzmann equation (1.3) are not exhausted by the incompressible limits (1.8), (1.9), and (1.13). The aim of this section is to show that a mere change from the uniform Maxwellian to a local Maxwellian in the decomposition (1.4) of F^ϵ gives rise to a different system of fluid dynamical equations in the limit $\epsilon \to 0+$. More precisely, we will take the decomposition (1.17) and derive a system of fluid dynamical equations which is different not only from (1.8),(1.9), and (1.13), but also from any other classical systems including the Euler and Navier-Stokes equations, compressible or incompressible.

This system of fluid dynamical equations exhibits the "ghost effect" mentioned in Section 1. Unlike the incompressible limits, however, it does not contain the viscous heating. Furthermore, the even-odd decomposition in velocity variables v like in (1.16) is a consequence of the setting but not the assumption, and the counterparts of the hypothesis (1.6) and (1.12) on the asymptotic behavior of ρ^ϵ and θ^ϵ do not result in any essential distinction. This means that the viscous heating appears only in a higher correction.

In contrast to Theorem 3.1, the formal convergence theorem will be established for the "genuine" Boltzmann equation for three space dimension and the convergence hypothesis will be stated in terms of the asymptotic expansions of both macroscopic and microscopic quantities. The main result of this section is as follows.

Theorem 4.1. *Write the solution $F^\epsilon(t,x,v)$ of the scaled Boltzmann equation*

$$\epsilon\partial_t F^\epsilon + v\cdot\nabla_x F^\epsilon = \frac{1}{\epsilon}\mathcal{C}(F^\epsilon, F^\epsilon), \tag{4.1}$$

in the form of the micro-macro decomposition:

$$F^\epsilon = \mathcal{M}^\epsilon + \epsilon\mathbf{G}^\epsilon, \tag{4.2}$$

where

$$\mathcal{M}^\epsilon = \mathcal{M}(\rho^\epsilon, u^\epsilon, \theta^\epsilon) \tag{4.3}$$

is a local Maxwellian determined by the solutions F^ϵ themselves while

$$\mathbf{G}^\epsilon \perp \mathbb{E}^\epsilon \tag{4.4}$$

is the microscopic component where $\mathbb{E}^\epsilon$ stands for the collision invariant subspace associated with the local Maxwellian $\mathcal{M}^\epsilon$. See [LY], [LYY] *for details. And, $\mathcal{C}$ is the usual collision operator, which is given by (5.4) in Section 5.*

We assume that the following asymptotic expansions for small $\epsilon > 0$ hold in sufficiently strong Banach spaces which vary depending on the relevant quantities.

$$\begin{aligned}
&\rho^\epsilon = \rho + \epsilon^2\widetilde{\rho} + O(\epsilon^3),\\
&u^\epsilon = \epsilon\widetilde{u}^\epsilon, \qquad \widetilde{u}^\epsilon = \widetilde{u} + \epsilon\check{u} + O(\epsilon^2),\\
&\theta^\epsilon = \theta + \epsilon^2\widetilde{\theta} + O(\epsilon^3),\\
&\mathbf{G}^\epsilon = \mathbf{G}_o + \epsilon\mathbf{G}_e + O(\epsilon^2).
\end{aligned} \tag{4.5}$$

Then, the leading parameters $\rho, \widetilde{u}, \theta$ of the local Maxwellian $\mathcal{M}^\epsilon$ solve the "ghost effect" equations (4.6) *below while the microscopic first and second order components $\mathbf{G}_o$ and $\mathbf{G}_e$ are odd and even functions of v*

respectively. The relevant equations are,

$$\begin{aligned}
&\nabla_x(\rho\theta) = 0,\\
&\partial_t\rho + \nabla_x\cdot(\rho\widetilde{u}) = 0,\\
&\partial_t(\rho\widetilde{u}) + \nabla_x\cdot(\rho\widetilde{u}\otimes\widetilde{u}) + \nabla_x P^* = \nabla_x\cdot D(\widetilde{u},\theta) - \nabla_x\cdot\Sigma(\theta,\rho),\\
&\tfrac{3}{2}\partial_t(\rho\theta) + \tfrac{5}{2}\nabla_x\cdot(\rho\widetilde{u}) = \tfrac{5}{2}\nabla_x\cdot(\kappa(\theta)\nabla_x\theta),
\end{aligned} \tag{4.6}$$

where P^* is a unknown scalar pressure while

$$\begin{aligned}
\kappa(\theta) &= \gamma_2(\theta)\sqrt{\theta},\\
D(u,\theta) &= \gamma_1(\theta)\sqrt{\theta}\left(\nabla_x u + (\nabla_x u)^{\mathrm{T}} - \frac{2}{3}\nabla_x\cdot u\,\mathrm{I}\right),\\
\Sigma(\theta,\rho) &= \frac{\gamma_3(\theta)}{\rho}\Sigma_1(\theta) + \frac{\gamma_7(\theta)}{\rho\theta}\Sigma_2(\theta),\\
\Sigma_1(\theta) &= \nabla_x^2\theta - \frac{1}{3}\Delta_x\theta\,\mathrm{I},\\
\Sigma_2(\theta) &= \nabla_x\theta\otimes\nabla_x\theta - \frac{1}{3}|\nabla_x\theta|^2\,\mathrm{I},
\end{aligned}$$

where $\gamma_j(\theta)$ are positive functions of $\theta > 0$ determined exclusively by the linearized collision operator $\mathcal{L}_M$ and the Burnett functions $A(v)$ and $B(v)$. Their explicit formulas can be found in [So1].

Remark 4.1. *The system (4.6) implies that the infinitesimal (ghost) bulk velocity $\widetilde{u}$ governs the evolution of the "real-world" mass density ρ and temperature θ. This is the "ghost effect" introduced in [S]. Notice that $(4.6)_a$ means that*

the pressure $P_0 = \frac{3}{2}\rho\theta$ may be a function of t but not of x.

Clearly, P_0 is to be determined by the condition on the boundary or at infinity in the x-space but not by the initial condition. Thus, the ghost effect is in action only through the boundary effect or the effect from the infinity.

Remark 4.2. *Suppose that ρ and θ are constant (both in t and x). Then, $(4.6)_a$ is a trivial equation while $(4.6)_{b,c}$ reduce to the classical incompressible Navier-Stokes equations $(1.8)_{a,b}$. Further, $(4.6)_d$ becomes trivial but not the heat conductive equation $(1.9)_b$. This means the latter is the first or higher order correction to $(4.6)_d$.*

Also, the viscous heating does not appear in (4.6). Actually, it can be shown to be the third order correction, although we do not go into it here.

Remark 4.3. *The asymptotic expansions (4.5) do not contain the first order terms for ρ^ϵ and θ^ϵ. However, the inclusion of them do not induce any essential change to (4.6). We will come back to this point in Theorem 4.2, after finishing the proof of the above theorem.*

Proof of Theorem 4.1: Plugging the decomposition (4.2) into (4.1) yields

$$\epsilon\partial_t\mathcal{M}^\epsilon + v\cdot\nabla_x\mathcal{M}^\epsilon + \epsilon^2\partial_t\mathbf{G}^\epsilon + \epsilon v\cdot\nabla_x\mathbf{G}^\epsilon = \mathcal{L}_{\mathcal{M}^\epsilon}\mathbf{G}^\epsilon + \epsilon\mathcal{C}(\mathbf{G}^\epsilon,\mathbf{G}^\epsilon), \tag{4.7}$$

where

$$\mathcal{L}_{\mathcal{M}^\epsilon} g = 2\mathcal{C}(\mathcal{M}^\epsilon, g)$$

is the linearized operator of $\mathcal{C}$ around $\mathcal{M}^\epsilon$. Under the hypothesis (4.5), as $\epsilon \to 0+$, (4.7) has the limit

$$v \cdot \nabla_x \mathcal{M} = \mathcal{L}\mathbf{G}_o, \tag{4.8}$$

with

$$\mathcal{M} = \mathcal{M}(\rho, 0, \theta), \qquad \mathcal{L} = \mathcal{L}_{\mathcal{M}}. \tag{4.9}$$

If (4.8) is taken as an equation for $\mathbf{G}_o$, it requires the solvability condition $v \cdot \nabla_x \mathcal{M} \perp \mathbb{E}$ or

$$\nabla_x \cdot < v\phi_j(v)\mathcal{M} >= 0, \quad j = 0, \cdots, 4. \tag{4.10}$$

The inner products for $j = 0, 4$ are zero because $\mathcal{M}$ is even in v but $v\phi_j(v)$ are odd for such j, and for $i, j = 1, 2, 3$, we have $< v_i v_j, \mathcal{M} >= \delta_{ij}\rho\theta$ whence follows $(4.6)_a$:

$$\nabla_x(\rho\theta) = 0. \tag{4.11}$$

Once the solvability condition (4.10) is satisfied, the solution of (4.8) is easily computed to give

$$\mathbf{G}_o = \mathcal{L}^{-1}\Big(v \cdot \nabla_x \mathcal{M}\Big) = \frac{1}{\sqrt{\theta}}\widetilde{B}\Big(\frac{v}{\sqrt{\theta}}\Big) \cdot \nabla_x \theta, \qquad \widetilde{B}(v) = \mathcal{L}^{-1}B(v), \tag{4.12}$$

which is odd in v because so is the Burnett function $B(v)$, and $\mathcal{L}$ together with its inverse preserves the parity.

Now, subtraction (4.7) from (4.8) yields

$$\begin{aligned} \epsilon\partial_t \mathcal{M}^\epsilon + &\epsilon v \cdot \nabla_x(\mathcal{M}^\epsilon - \mathcal{M}) + \epsilon^2 \partial_t \mathbf{G}^\epsilon + \epsilon v \cdot \nabla_x \mathbf{G}^\epsilon \\ &= \mathcal{L}^\epsilon \mathbf{G}^\epsilon - \mathcal{L}\mathbf{G}_o + \epsilon \mathcal{C}(\mathbf{G}^\epsilon, \mathbf{G}^\epsilon). \end{aligned} \tag{4.13}$$

Since (4.5) implies

$$\mathcal{M}^\epsilon - \mathcal{M} = \epsilon m_o + O(\epsilon^2), \qquad m_o = \frac{1}{\theta} v \cdot \widetilde{u}\mathcal{M}, \tag{4.14}$$

we have

$$\begin{aligned} \mathcal{L}^\epsilon \mathbf{G}^\epsilon - \mathcal{L}\mathbf{G}_o &= 2\mathcal{C}(\mathcal{M}^\epsilon - \mathcal{M}, \mathbf{G}^\epsilon) + 2\mathcal{C}(\mathcal{M}, \mathbf{G}^\epsilon - \mathbf{G}_o) \\ &= \epsilon\Big\{2\mathcal{C}(m_o, \mathbf{G}^\epsilon) + \mathcal{L}\mathbf{G}_e\Big\} + O(\epsilon^2). \end{aligned}$$

Thus (4.13), when divided by ϵ, is

$$\begin{aligned} \partial_t \mathcal{M}^\epsilon + &v \cdot \nabla_x m_o + \epsilon \partial_t \mathbf{G}^\epsilon + v \cdot \nabla_x \mathbf{G}^\epsilon \\ &= 2\mathcal{C}(m_o, G^\epsilon) + \mathcal{L}\mathbf{G}_e + \mathcal{C}(\mathbf{G}^\epsilon, \mathbf{G}^\epsilon) + O(\epsilon), \end{aligned}$$

which yields in the limit

$$\partial_t \mathcal{M} + v \cdot \nabla_x m_o + v \cdot \nabla_x \mathbf{G}_o = 2\mathcal{C}(m_o, \mathbf{G}_o) + \mathcal{L}\mathbf{G}_e + \mathcal{C}(\mathbf{G}_o, \mathbf{G}_o). \tag{4.15}$$

Put

$$H = \partial_t \mathcal{M} + v \cdot \nabla_x m_o + v \cdot \nabla_x \mathbf{G}_o,$$

and take (4.15) as an equation for $\mathbf{G}_e$. Then, the solvability condition is $H \perp \mathbb{E}$, that is,

$$< \phi_j, H >= 0, \qquad j = 0, \cdots, 4. \tag{4.16}$$

Consider the case $j = 0$. Since $\mathbf{G}_o \perp \mathbb{E}$ as well as $\mathbf{G}$ by definition (4.2), we have

$$< \phi_0, H >= \partial_t < \mathcal{M} > +\nabla_x \cdot < v m_o >= 0,$$

which gives, in particular by (4.14),

$$\partial_t \rho + \nabla_x \cdot (\rho \widetilde{u}) = 0, \tag{4.17}$$

which is $(4.6)_b$.

For $j = 1, 2, 3$, (4.16) reduces to

$$\nabla_x \cdot < (v \otimes v)(m_o + \mathbf{G}_o) >= 0, \tag{4.18}$$

but this is a trivial equation because $v \otimes v$ is even in v while m_o and $\mathbf{G}_o$ are odd so that $< (v \otimes v)(m_o + \mathbf{G}_o) >= 0$.

Finally, for $j = 4$, we have

$$\frac{1}{2} < |v|^2 \mathcal{M} > = 3\rho\theta, \quad \frac{1}{5} < v|v|^2 m_o >= 5\rho\widetilde{u}\theta$$
$$\frac{1}{2} < v|v|^2 \mathbf{G}_o > = \frac{1}{\sqrt{2\theta}} < v|v|^2 \otimes \widetilde{B}\Big(\frac{v}{\sqrt{\theta}}\Big) > \nabla_x \theta.$$

Then, (4.16) gives

$$\frac{3}{2}\partial_t(\rho\theta) + \frac{5}{2}\nabla_x \cdot (\rho u) = \frac{5}{2}\nabla_x \cdot (\kappa(\theta)\nabla_x \theta), \tag{4.19}$$

which is just $(4.6)_d$.

It remains to derive $(4.6)_c$. Unlike the equations $(4.6)_{b,d}$, it comes from the second order terms of (4.13). More precisely, we shall compute

$$\epsilon^{-2} < v \ (4.13) >, \tag{4.20}$$

and go to the limit $\epsilon \to 0+$ to deduce $(4.6)_c$. Firstly, since $< v\mathcal{M} >= 0$, we get

$$\begin{aligned} \epsilon^{-2}\partial_t < v\epsilon\mathcal{M}^\epsilon >= \epsilon^{-1}\partial_t < v(\mathcal{M}^\epsilon - \mathcal{M})\} > \\ = \partial_t < v m_o > +O(\epsilon) \to \partial_t(\rho\widetilde{u}) \quad (\epsilon \to 0+). \end{aligned} \tag{4.21}$$

Secondly, the expansion (4.14) can be strengthened as

$$\mathcal{M}^\epsilon - \mathcal{M} = \epsilon m_o + \epsilon^2 m_e + O(\epsilon^3), \tag{4.22}$$

for an even function m_e. Then, since $< v \otimes v m_o >= 0$ due to even-odd combination, we have

$$\begin{aligned} &\epsilon^{-2}\nabla_x \cdot < v \otimes v(\mathcal{M}^\epsilon - \mathcal{M}) > \\ &\qquad = \epsilon^{-1}\nabla_x \cdot < v \otimes v(m_o + \epsilon m_e) > +O(\epsilon) \\ &\qquad \to \nabla_x \cdot < v \otimes v m_e > \qquad (\epsilon \to 0+). \end{aligned} \tag{4.23}$$

Since m_e is seen to be even in each v_i and symmetric with respect to v_1, v_2, v_3, we have

$$< v \otimes v m_e >= p_1^* I, \tag{4.24}$$

with $p_1^* =< v_i^2 m_e >$ which is independent of i.

Thirdly, notice that

$$< v \otimes v \mathbf{G}_o >= 0$$

due to even-oddness. Consequently, we get

$$\epsilon^{-1} < v \otimes v(\mathbf{G}^\epsilon - \mathbf{G}) >=< v \otimes v\mathbf{G}_e > +O(\epsilon) \tag{4.25}$$
$$\to< A(v)\mathbf{G}_e > -\frac{1}{3} < |v|^2\mathbf{G}_e > I =< \widetilde{A}, \mathcal{L}\mathbf{G}_e > -p_2^* I \qquad (\epsilon \to 0+),$$

where $A(v)$ is the Burnett function defined in (2.22), $\widetilde{A}(v) = \mathcal{L}^{-1}A(v)$, and $p_2^* =< |v|^2\mathbf{G}_e > /3$.

Plugging (4.21), (4.23), and (4.25) into (4.20) yields

$$\partial_t(\rho\widetilde{u}) + \nabla_x P^* = -\nabla_x\cdot < \widetilde{A}, \mathcal{L}\mathbf{G}_e > . \tag{4.26}$$

To compute the last term, we now use (4.15) in the form,

$$\mathcal{L}\mathbf{G}_e = (I - \mathbf{P})H - 2\mathcal{C}(m_o, \mathbf{G}_o) - \mathcal{C}(\mathbf{G}_o, \mathbf{G}_o). \tag{4.27}$$

Since $(I - \mathcal{P})\mathcal{M} = 0$, the left hand side is determined exclusively by m_o and $\mathbf{G}_o$, which includes, in turn, quantities depending only on $\rho, \widetilde{u}, \theta$ as seen by (4.14) and (4.12) respectively. After some manipulation, we then have $(4.6)_c$. This completes the proof of the theorem.

We conclude this section with the point raised in Remark 4.3.

Theorem 4.2. *The reinforcement of* (4.5) *with the asymptotic expansions*

$$\rho^\epsilon = \rho + \epsilon\rho_1 + \epsilon^2\widetilde{\rho} + O(\epsilon^3), \quad \theta^\epsilon = \theta + \epsilon\theta_1 + \epsilon^2\widetilde{\theta} + O(\epsilon^3),$$

does not change (4.6) *at all formally but supplements it with the extra relation*

$$\nabla_x(\rho\theta_1 + \rho_1\theta) = 0. \tag{4.28}$$

Proof. First, (4.11) or $(4.6)_a$ does not change. The main difference is that m_o and m_e in (4.14) and (4.22) now have even and odd parts respectively. Elementary calculation yields

$$m_o = m_{oo} + m_{oe} \equiv \frac{1}{\theta} v \cdot \widetilde{u}\mathcal{M} + \Big(\frac{\rho_1}{\rho} - (\tfrac{3}{2} - \tfrac{1}{2}\frac{|v|^2}{\theta})\frac{\theta_1}{\theta}\Big)\mathcal{M}.$$

Notice that m_{oo} is the same as the old m_o. On the other hand, $< vm_{oe} >= 0$ and hence $< vm_o >=< vm_{oo} >$, which recovers (4.17) and hence $(4.6)_b$. Further, since

$$< v_iv_jm_{oo} >= 0, \qquad < v_iv_jm_{oe} >= (\rho\theta_1 + \rho_1\theta)\delta_{ij}$$

hold, (4.18) is reduced to the extra equation (4.28). And, since

$$\frac{1}{5} < v|v|^2m_o >= \frac{1}{5} < v|v|^2m_{oo} >= 5\rho\widetilde{u}\theta,$$

(4.19), i.e., $(4.6)_d$, survives.

To deduce $(4.6)_c$, we shall compute (4.20) with the new m_o and m_e. We already know that (4.21) is true with m_{oo} in place of m_o. Write m_e of (4.22) as a sum of the even and odd parts:

$$m_e = m_{ee} + m_{eo}.$$

m_{ee} is not the same as the old m_e. Although $< v \otimes vm_o >\neq 0$, (4.23) is still valid with the new m_e by virtue of the extra relation (4.28), and so is (4.24) with m_{ee}, giving a different p_1^*.

The equations (4.25), (4.26), and (4.27) are the same as before but with the new m_o. As a consequence, the only additional term we should compute for the inner product $< \widetilde{A}, \mathcal{L}\mathbf{G}_e >$ is the inner product

$$< \widetilde{A}, \mathcal{C}(m_{oe}, G_o) > .$$

And, this is zero because $\widetilde{A}$ is even and $\mathcal{C}(m_{oe}, G_o)$ odd in v. Thus $(4.6)_c$ follows in the exactly same way as before. This completes the proof of the theorem.

5 Regularity Results and Convergence Proofs

In this section a complete proof of the validity of the formal Theorem 3.1 is proposed for the "genuine " Boltzmann equation. The analysis is done for the three-dimensional space. For the macroscopic quantities, the Sobolev spaces $H^l(\mathbb{R}^3_x)$ are used with norms denoted by $\|\cdot\|_l$. Furthermore, any solution of the equation

$$-\Delta_x f = \nabla_x \cdot g\,;\ \int_{\mathbb{R}^3} f(x)dx = 0\,, g \in H^l(\mathbb{R}^3_x) \tag{5.1}$$

can be represented as

$$f(x) = \mathcal{G}(g)(x) = ((-\Delta_x)^{-1}\nabla_x\cdot g)(x) = \tfrac{1}{4\pi}\int_{\mathbb{R}} \frac{x-y}{|x-y|^3} g(y)dy, \tag{5.2}$$

with $\mathcal{G}$ defining a linear continuous operator from the space $H^l(\mathbb{R}^3_x)$ with value in the space H^0_{l+1}:

$$\begin{aligned} H^0_l &= \{f \in C(\mathbb{R}^3_x);\ \nabla_x f \in H^l(\mathbb{R}^3_x)\}, \\ |f|_l = \|f\|_{H^0_l} &= \{\sup_{x\in\mathbb{R}^3_x} |f(x)|\} + \|\nabla_x f\|_l\}\,. \end{aligned} \tag{5.3}$$

The collision operator is given by the formula

$$\mathcal{C}(F) = \iint (F_1'F' - F_1F)q(v_1 - v, \omega)\, dv_1 d\omega\,. \tag{5.4}$$

The variable ω lies on the unit sphere $S^2 = \{\omega \in \mathbb{R}^3 : |\omega| = 1\}$ endowed with its rotationally invariant unit measure. The F, F_1, F' and F_1' appearing in the integrand are understood to mean $F(t, x, \cdot)$ evaluated at the velocities v, v_1, v' and v_1' respectively, where the primed velocities are defined by

$$v' = v + \omega\,\omega\cdot(v_1 - v)\,, \qquad v_1' = v_1 - \omega\,\omega\cdot(v_1 - v)\,, \tag{5.5}$$

for any given $(v, v_1, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times S^2$.

The positive function $q(v_1 - v, \omega)$ is the cross section, and following [U1] one observes that the proof done below applies for the hard sphere model

$$q(v_1 - v, \omega) = \sigma|(v_1 - v)\cdot\omega|, \tag{5.6}$$

and more generally for the inverse power law with angular cuttoff

$$q(v_1 - v, \omega) = |(v_1 - v)|^\gamma \Big|\frac{(v_1 - v)\cdot\omega}{|(v_1 - v)|}\Big|^{-\gamma'}\Xi\Big(\Big(\frac{(v_1 - v)\cdot\omega}{|(v_1 - v)|}\Big), \tag{5.7}$$

with $s > 4$, $\gamma = 1 - \frac{4}{s}$, $\gamma' = 1 + \frac{2}{s}$ and Ξ a smooth positive function which vanishes in a small neighborhood of 0.

The definitions and notations introduced in the previous sections are systematically used. In particular one has

$$\begin{aligned} \mathcal{C}(F, G) &= \frac{1}{2}\iint (F_1'G' + G_1'F' - F_1G - G_1F)q(v_1 - v, \omega)\, dv_1, \\ \mathcal{L}_M G &= -2M^{-1}\mathcal{C}(M, MG), \\ \mathcal{Q}(\widetilde{G}, U) &= M^{-1}\mathcal{C}(M\widetilde{G}, MU). \end{aligned} \tag{5.8}$$

The linearized operator, $\mathcal{L}_M$, can be decomposed as follows

$$\mathcal{L}_M u = \nu(v)u + Ku,\ \ Ku(v) = \int_{R^d} M^{-1}(v)K(v,v')u(v')dv' \tag{5.9}$$

with the function ν and the operator K satisfying the following relations:

$$\begin{aligned} 0 < \nu_0 \leq \nu(v) \leq \nu_1 |v|^{\gamma}, \\ K(v,v') = K(v',v), \\ \int_{R^d} M^{-1}(v)|K(v,v')||v'|^{-\beta} dv' \leq k_0 |v|^{-(\beta+1)}, \\ \int_{R^d} M^{-1}(v)|K(v,v')|^2 dv' \leq k_1, \end{aligned} \tag{5.10}$$

with suitable constants γ, β, k etc.. As a consequence, it satisfies the basic assumptions in Section 2. In particular, its kernel is the five dimensional space

$$\mathbb{E} = \text{span } \{1, v, \frac{|v|^2 - 3}{2}\} \tag{5.11}$$

The orthonormal projection on $\mathbb{E}$ is denoted by $\mathcal{P}$ and any function may be decomposed into its hydrodynamic part and its non hydrodynamic part according to the formula and notations :

$$f = \mathcal{P}f + (I - \mathcal{P})f = \mathcal{P}f +^{\perp} \mathcal{P}f\,. \tag{5.12}$$

It is well known that the incompressible Navier-Stokes equation (3.7) for initial data u_0 in the Sobolev space $H^l(\mathbb{R}^3_x)$ (with $l > 3/2$) do have a unique classical solution for $0 \leq t \leq T$ with $T = \infty$ when $\|u_0\|$ is small enough (in $H^l(\mathbb{R}^3_x)$) with respect to the viscosity. Similar type of results have been established both for the Boltzmann equation and for some macroscopic limits. Such macroscopic limits include both the Euler limit (cf. [N], [U2], [UA]) and the incompressible limit (cf. [BU].) In this setting, the natural counterpart of the Sobolev spaces are the Grad spaces

$$\begin{aligned} H_{l,\beta} \equiv \{f = f(\cdot,v)\,|\,(1+|v|^{\beta})M^{\frac{1}{2}}(v)f \in L^{\infty}(\mathbb{R}^3_v; H^l), \\ \sup_{|v|>R} (1+|v|^{\beta})\|M^{\frac{1}{2}}(v)f(\cdot,v)\|_l \to 0 \quad (R \to \infty)\}, \end{aligned} \tag{5.13}$$

where the corresponding norms are denoted by $\|f\|_{l,\beta}$.

The fact that a large variety of formal macroscopic limits of the Boltzmann equation lead both to well and ill posed problems (classical and non-classical equations including Burnett and Prandtl equations, and "ghost effect" equations) justifies that complete proofs should be given.

On the other hand, the contribution of the present section involves higher order corrections. Therefore there is no room for weak convergence, say from renormalized solutions of the Boltzmann equation to weak solutions of the Navier-Stokes equation as in [BGL] and [GS], and the initial data will be assumed in the space $H_{l,\beta}$ with l and β large enough.

The relevance of the Grad spaces is mainly due to the following facts (cf. Theorem 2.1.3 of [U1]).

i) For $l > 3/2$ and $\beta > 5/2$, $\mathcal{Q}$ is a bilinear continuous operator in $H_{l,\beta}$:

$$\|\nu^{-1}\mathcal{Q}(f,g)\|_{l,\beta} \leq C\|f\|_{l,\beta}\|g\|_{l,\beta}\,. \tag{5.14}$$

ii) The pseudo inverse $\mathcal{L}_M^{-1}$ defined on $\mathbb{E}^\perp$ is continuous in any Grad space:

$$\mathcal{P}f = 0,\ \mathcal{L}_M f = g \Rightarrow \|f\|_{l,\beta} \le C\|g\|_{l,\beta}\,. \tag{5.15}$$

In this context, a slightly weaker version of the following theorem was proven in [BU].

Theorem 5.1. *Assume that the sequence $\widetilde{G}_0^\epsilon$ converges to a hydrodynamical fluctuation in the following sense:*

$$\widetilde{G}_0^\epsilon = \{\rho_0 + \widetilde{u}_0 \cdot v + \frac{1}{2}(|v|^2 - d)\theta_0\} + \epsilon w_0^\epsilon, \tag{5.16}$$

with $\rho_0, \widetilde{u}_0$ and $\theta_0 \in H_l(R_x^d)$, and w^ϵ uniformly bounded in $H_{l,\beta}$ ($l > \frac{3}{2}$, $\beta > \frac{5}{2}$). Then, for the corresponding solution of the rescaled Boltzmann equation

$$\epsilon\partial_t \widetilde{G}^\epsilon + v\cdot\nabla_x \widetilde{G}^\epsilon + \frac{1}{\epsilon}\mathcal{L}_M \widetilde{G}^\epsilon = \mathcal{Q}(\widetilde{G}^\epsilon, \widetilde{G}^\epsilon), \quad \widetilde{G}^\epsilon(x,v,0) = \widetilde{G}_0^\epsilon\,, \tag{5.17}$$

there exists an ϵ-independent time T such that $\widetilde{G}^\epsilon$ is ϵ-uniformly bounded in

$$L^\infty(0,T;H_{l,\beta})\,.$$

Furthermore:

i) For any $\delta > 0$ the sequence $\widetilde{G}^\epsilon$ converges in $C([\delta,T];H_{l,\beta})$ to an hydrodynamic fluctuation which is solution of the incompressible Navier-Stokes, Boussinesq and the θ-linear heat diffusion equations:

$$\begin{aligned}
&\lim_{\epsilon\to 0} \widetilde{G}^\epsilon = \{\rho(x,t) + \widetilde{u}(x,t)\cdot v + \frac{1}{2}(|v|^2 - d)\theta(x,t)\} \\
&\partial_t \widetilde{u} + \widetilde{u}\cdot\nabla_x \widetilde{u} + \nabla_x \widetilde{p} = \mu^* \Delta \widetilde{u}\,, \\
&\rho(x,t) + \theta(x,t) = 0\,, \quad \nabla_x \cdot \widetilde{u} = 0, \\
&\tfrac{d+2}{2}(\partial_t \widetilde{\theta} + \widetilde{u}\cdot\nabla_x \widetilde{\theta}) = \kappa^* \Delta \widetilde{\theta}\,.
\end{aligned} \tag{5.18}$$

with initial data given by:

$$\begin{aligned}
&\widetilde{u}(x,0) = \widetilde{u}_0(x) + (\nabla_x p_0)(x),\ -\Delta_x p_0 = \nabla_x\cdot(\widetilde{u}_0 \cdot \nabla_x \widetilde{u}_0)\,; \\
&\theta(x,0) = \tfrac{1}{2}(\theta_0(x) - \rho_0(x))\,.
\end{aligned} \tag{5.19}$$

ii) The convergence of $\widetilde{G}^\epsilon$ holds in $C([0,T];H_{l,\beta})$ if the initial data satisfy the "compatibility conditions":

$$\nabla_x\cdot\widetilde{u}_0 = 0,\ \textit{and}\ \rho_0 + \theta_0 = 0. \tag{5.20}$$

iii) For ρ_0, $\widetilde{u}_0$ and θ_0 small enough in $H_l(\mathbb{R}_x^d)$, T can be taken to be $+\infty$.

In view of the adaptation to the present situation, the main steps of the proof are recalled below.

The operators

$$B_\epsilon = -\epsilon v \cdot \nabla_x + \mathcal{L}_M \tag{5.21}$$

are introduced and shown to be generators of contraction semigroups in any Grad spaces $H_{l,\beta}$. Then the equation (5.17) is written according to the Duhamel principle

$$\widetilde{G}^\epsilon = \exp\frac{tB_\epsilon}{\epsilon^2}\widetilde{G}_0^\epsilon + \frac{1}{\epsilon}\int_0^t \exp\frac{(t-s)B_\epsilon}{\epsilon^2}\mathcal{Q}(\widetilde{G}^\epsilon, \widetilde{G}^\epsilon)(s)ds. \tag{5.22}$$

The spaces

$$X_{l,\beta} = L^\infty(0,T;H_{l,\beta}), \tag{5.23}$$

are introduced and their norm is denoted by $\|\!|.\|\!|_{l,\beta}$.

For $l > 3/2$ and $\beta > 3/2 + 1$

$$\Psi^\epsilon(\widetilde{G},\widetilde{H}) = \frac{1}{\epsilon}\int_0^t \exp\frac{(t-s)B_\epsilon}{\epsilon^2}\mathcal{Q}(\widetilde{G},\widetilde{H})(s)ds, \tag{5.24}$$

defines a bilinear continuous map from $X_{l,\beta}\times X_{l,\beta}$ to $X_{l,\beta}$. With a detailed analysis of the spectral properties of the semi-group $\exp tB_\epsilon$, one shows the validity of the following ϵ-uniform estimates:

$$\begin{aligned}&\|\!|\Psi^\epsilon(\widetilde{G},\widetilde{H})\|\!|_{l,\beta} \le CT\|\!|\widetilde{G}\|\!|_{l,\beta}\|\!|\widetilde{H}\|\!|_{l,\beta},\ \text{for } T<\infty,\\ &\|\!|\Psi^\epsilon(\widetilde{G},\widetilde{H})\|\!|_{l,\beta} \le C\|\!|\widetilde{G}\|\!|_{l,\beta}\|\!|\widetilde{H}\|\!|_{l,\beta},\ \text{for } T=\infty.\end{aligned} \tag{5.25}$$

From (5.22) and (5.25), one deduces that:

(a) For T small enough with respect to $\|\!|\widetilde{G}_0^0\|\!|_{l,\beta}$, the sequence $\widetilde{G}^\epsilon$ is uniformly bounded in $L^\infty(0,T;H_{l,\beta})$.

(b) For $\|\!|\widetilde{G}_0^0\|\!|_{l,\beta}$ small enough the sequence $\widetilde{G}^\epsilon$ is uniformly bounded in $L^\infty(0,\infty;H_{l,\beta})$.

The Statement ii) of Theorem 5.1 concerns the initial layer which may appear in the term

$$\exp\frac{tB_\epsilon}{\epsilon^2}\widetilde{G}_0^\epsilon$$

of the Duhamel formula (5.22). Following classical spectral analysis of the linearized Boltzmann operator, one writes (with $\mathcal{F}_x$ denoting the x Fourier transform):

$$\begin{aligned}\exp\frac{tB_\epsilon}{\epsilon^2}f &= \mathcal{F}_x^{-1}e^{-t\lambda_o^\epsilon(\xi)}\mathcal{F}_x\mathcal{P}_of + \mathcal{F}_x^{-1}e^{-t\lambda_e^\epsilon(\xi)}\mathcal{F}_x\mathcal{P}_ef\\ &+\exp\frac{tB_\epsilon}{\epsilon^2}(\mathcal{P}f-(\mathcal{P}_o+\mathcal{P}_e)f) + \exp\frac{tB_\epsilon}{\epsilon^2}(f-\mathcal{P}f)\end{aligned} \tag{5.26}$$

In (5.26) $\mathcal{P}_o$ is the projection on the space of odd hydrodynamical fluctuations $\widetilde{u}(x)\cdot v$ which satisfy the divergence free condition, while $\mathcal{P}_e$ is the projection on the even hydrodynamical fluctuations $\rho+\frac{1}{2}(|v|^2-d)\theta$ which satisfy the Boussinesq equation $\rho+\theta=0$. Further, $\lambda_o^\epsilon(\xi)$ and $\lambda_e^\epsilon(\xi)$ are the eigenvalues of the Fourier transform of the linearized Boltzmann operator associated with the projections $\mathcal{P}_o$ and $\mathcal{P}_e$ respectively. Eventually, they satisfy

$$\mathrm{Re}(\lambda_o^\epsilon(\xi)) \le -\beta_o|\xi|^2, \qquad \mathrm{Re}(\lambda_e^\epsilon(\xi)) \le -\beta_e|\xi|^2,$$

for some positive constants β_o and β_e respectively.

For any f in $H_{l,\beta}$ the function $\exp\frac{tB_\epsilon}{\epsilon^2}(f-\mathcal{P}f)$ converges to zero in $C([0,T];H_{l,\beta})$. On the other hand using the stationary phase method in Fourier space, one shows that the function $\exp\frac{tB_\epsilon}{\epsilon^2}(\mathcal{P}f-(\mathcal{P}_o+\mathcal{P}_e)f)$ converges to zero in $C([\delta,T];H_{l,\beta})$ for any $\delta>0$. Therefore, $\widetilde{G}^\epsilon$ converges in $C([\delta,T],H_{l,\beta})$ for any $\delta>0$ to a hydrodynamical fluctuation with initial value given by:

$$\begin{aligned}\widetilde{G}(x,0) =&(\mathcal{P}_0+\mathcal{P}_e)\{\rho_0(x)+\widetilde{u}_0(0,x)\cdot v+\frac{1}{2}(|v|^2-3)\theta_0(0,x)\}\\ =&\widetilde{u}(x,0)\cdot v+\theta(x,0)\big(\frac{1}{2}(|v|^2-3)-1\big)\end{aligned} \tag{5.27}$$

with $\widetilde{u}(x,0)$ and $\theta(x,0)$ given by (5.19).

Eventually, one observes that

$$\exp \frac{tB_\epsilon}{\epsilon^2}(\mathcal{P}f - (\mathcal{P}_o + \mathcal{P}_e)f)$$

converges to zero in $C([0,T]; H_{l,\beta})$ if and only if

$$\mathcal{P}f - (\mathcal{P}_o + \mathcal{P}_e)f = 0 \tag{5.28}$$

which is equivalent to

$$\nabla_x \cdot \widetilde{u}_0 = 0 \quad \text{and} \quad \rho_0 + \theta_0 = 0\,. \tag{5.29}$$

For the present derivation which involves higher order correction, the following class of "well prepared" initial data is considered:

$$\begin{aligned}\widetilde{G}_0^\epsilon = \widetilde{u}_0 \cdot v + \epsilon\{\tfrac{1}{2}(A(v) : (\widetilde{u}_0 \vee \widetilde{u}_0)) - \mathcal{L}_M^{-1}(A(v)) : \nabla_x \widetilde{u}_0\}\\ +\epsilon\{\widetilde{\rho}_0 + (\tfrac{1}{2}|\widetilde{u}_0|^2 + \tfrac{d}{2}\widetilde{\theta}_0)\tfrac{2}{d}(\frac{|v|^2-d}{2})\} + \epsilon^2 w_0^\epsilon\end{aligned} \tag{5.30}$$

In (5.30) the hydrodynamic functions $\widetilde{u}_0$, $\widetilde{\rho}_0$ and $\widetilde{\theta}_0$ are assumed to satisfy the following regularity hypothesis:

$$\begin{aligned}\widetilde{u}_0 \in H_{l+2},\ \widetilde{\rho}_0 \in H_{l+1} \text{ and } \widetilde{\theta}_0 \in H_{l+1},\\ \|w_0^\epsilon\|_{l+1,\beta+1} \le C,\ \epsilon \text{ independently},\end{aligned} \tag{5.31}$$

in the Sobolev space H_{l+1} and $w^\epsilon \in \mathbb{E}^\perp$ is assumed ϵ uniformly bounded in $H_{l+1,\beta+1}$. Extension of [BU] leads to the following

Proposition 5.1. *i) With well prepared initial data as given by* (5.30)*, the solution* $\widetilde{G}_\epsilon$ *converges in* $C([\delta,T]; H_{l+1,\beta+1})$ *with* $\delta = 0$ *if and only if the initial data satisfies the relation*

$$\nabla_x \cdot \widetilde{u}_0 = 0. \tag{5.32}$$

Furthermore, for $\widetilde{u}_0$ *and* $0 < \epsilon \le \epsilon_0$ *small enough,* T *can be taken equal to* $+\infty$.

ii) With the relation (5.32)*, the sequence* $\partial_t \widetilde{G}_\epsilon$ *is uniformly bounded in*

$$L^\infty(0,T; H_{l,\beta})\,,$$

and converges in

$$C([\delta,T]; H_{l,\beta}),$$

with $\delta = 0$ *if and only if the following supplementary compatibility condition is satisfied.*

$$-\Delta_x(\widetilde{\theta}_0 + \widetilde{\rho}_0) = \sum_{ij} \frac{\partial \widetilde{u}_{0j}}{\partial x_i}\frac{\partial \widetilde{u}_{0i}}{\partial x_j}\,. \tag{5.33}$$

Proof: The point i) is a direct application of the results of [BU]. Furthermore, it is already clear that $\partial_t \widetilde{G}_\epsilon$ converges in a weak sense to $v \cdot \partial_t u$ which is a continuous function with initial data at $t = 0$ given by

$$v \cdot \partial_t u_{|t=0} = v \cdot \{\mu^* \Delta_x \widetilde{u}_0 - \nabla_x(\widetilde{u}_0 \otimes \widetilde{u}_0) - \nabla_x p_0\}, \tag{5.34}$$

with p_0 given by

$$-\Delta_x p_0 = \sum_{ij} \partial_{x_i}\widetilde{u}_0 \partial_{x_j}\widetilde{u}_0, \tag{5.35}$$

or

$$p_0 = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x-y|}\sum_{ij}\partial_{x_i}\widetilde{u}_0\partial_{x_j}\widetilde{u}_0(y)dy\,. \tag{5.36}$$

Therefore, the present issue is the uniform bound in $L^\infty(0,T;H_{l,\beta})$ and the strong convergence in $C([\delta,T];H_{l,\beta})$ of $\partial_t\widetilde{G}_\epsilon$.

Differentiation of (5.17) gives for $\dot{\widetilde{G}}_\epsilon = \partial_t\widetilde{G}_\epsilon$ the following equation:

$$\dot{\widetilde{G}}^\epsilon = \exp\frac{tB_\epsilon}{\epsilon^2}\dot{\widetilde{G}}^\epsilon_0 + 2\Psi^\epsilon(\widetilde{G}^\epsilon,\dot{\widetilde{G}}^\epsilon)(t), \tag{5.37}$$

and, in the same way as in [BU], the estimate:

$$|||\dot{\widetilde{G}}^\epsilon|||_{l,\beta} \le |||\exp\frac{tB_\epsilon}{\epsilon^2}\dot{\widetilde{G}}^\epsilon_0|||_{l,\beta} + C\int_0^t (t-s)^{-1/2}|||\widetilde{G}^\epsilon(s)|||_{l,\beta}|||\dot{\widetilde{G}}^\epsilon(s)|||_{l,\beta}ds, \tag{5.38}$$

which is a linear Gronwall-type estimate with respect to the function $\dot{\widetilde{G}}^\epsilon(s)$. Therefore, $|||\dot{\widetilde{G}}^\epsilon|||_{l,\beta}$ is uniformly bounded in $L^\infty(0,T;H_{l,\beta})$ if the first term of the right hand side of (5.38) remains bounded. The initial value of $\dot{\widetilde{G}}^\epsilon_0$ is deduced from the equation (5.17) and the relation (5.30) (well prepared initial data) :

$$\begin{aligned}\dot{\widetilde{G}}^\epsilon_0 &= -\tfrac{1}{\epsilon}v\cdot\nabla_x\widetilde{G}^\epsilon_0 - \tfrac{1}{\epsilon^2}\mathcal{L}_M\widetilde{G}^\epsilon_0 + \tfrac{1}{\epsilon}\mathcal{Q}(\widetilde{G}^\epsilon_0,\widetilde{G}^\epsilon_0)\\ &= W^\epsilon_1 + W^\epsilon_2 + W^\epsilon_3,\end{aligned} \tag{5.39}$$

where

$$\begin{aligned}W^\epsilon_1 &= -\frac{1}{\epsilon}\{v\cdot\nabla_x\widetilde{u}_0\cdot v + \mathcal{L}_M(\tfrac12(A(v):(\widetilde{u}_0\vee\widetilde{u}_0))\\ &\quad - \mathcal{L}_M^{-1}\big(A(v)\big):\nabla_x\widetilde{u}_0) + \mathcal{Q}(\widetilde{u}_0\cdot v,\widetilde{u}_0\cdot v)\big)\},\\ W^\epsilon_2 &= -v\cdot\nabla_x\{(\widetilde{\rho}_0 + (\tfrac12|\widetilde{u}_0|^2 + \tfrac32\widetilde{\theta}_0)(\frac{|v|^2-3}{2}))\\ &\quad + \tfrac12(A(v):(\widetilde{u}_0\vee\widetilde{u}_0)) - \mathcal{L}_M^{-1}\big((A(v)):\nabla_x\widetilde{u}_0\big)\},\\ W^\epsilon_3 &= 2\mathcal{Q}(\widetilde{u}_0\cdot v,\widetilde{\rho}_0 + (\tfrac12|\widetilde{u}_0|^2 + \tfrac32\widetilde{\theta}_0)(\frac{|v|^2-3}{2})) - \mathcal{L}_M w^\epsilon_0 + \epsilon h_\epsilon,\end{aligned}$$

with ϵh_ϵ uniformly bounded in $H_{l,\beta}$. Since the initial data $\widetilde{u}_0$ is divergence free, one has

$$v\cdot\nabla_x\widetilde{u}_0\cdot v = A(v):\nabla_x\widetilde{u}_0, \tag{5.40}$$

which implies that there is no term of order -1 in (5.39). The assumption made on the regularity of the initial data implies that the rest of the expansion is bounded in $H_{l,\beta}$. This proves the first point of ii).

A direct computation shows that :

$$\mathcal{P}(\dot{\widetilde{G}}^\epsilon_0) = v\cdot\{\mu^*\Delta_x\widetilde{u}_0 - \nabla_x(\widetilde{u}_0\otimes\widetilde{u}_0) - \nabla_x(\widetilde{\theta}_0+\widetilde{\rho}_0)\} + \epsilon h_\epsilon\,. \tag{5.41}$$

From (5.41) one deduces that

$$(\mathcal{P}_0+\mathcal{P}_e)\dot{\widetilde{G}}^\epsilon_0 = v\cdot\{\mu^*\Delta_x\widetilde{u}_0 - \nabla_x(\widetilde{u}_0\otimes\widetilde{u}_0) - \nabla_x(\widetilde{\theta}_0+\widetilde{\rho}_0) - \nabla_x p_0\}, \tag{5.42}$$

with p_0 given by the relation

$$-\Delta_x p_0 = \sum_{ij} \partial_{x_i} \widetilde{u}_0 \partial_{x_j} \widetilde{u}_0 + \Delta_x(\widetilde{\theta}_0 + \widetilde{\rho}_0)\,. \tag{5.43}$$

Eventually, one obtains

$$\begin{aligned}\exp \frac{tB_\epsilon}{\epsilon^2} \dot{\widetilde{G}}_0^\epsilon &= \exp \frac{tB_\epsilon}{\epsilon^2} \Big(v \cdot \{ \mu^* \Delta_x \widetilde{u}_0 - \nabla_x(\widetilde{u}_0 \otimes \widetilde{u}_0) - \nabla_x(\widetilde{\theta}_0 + \widetilde{\rho}_0) - \nabla_x p_0 \} \Big) \\ &+ \exp \frac{tB_\epsilon}{\epsilon^2} (\mathcal{P} - (\mathcal{P}_o + \mathcal{P}_e)) \dot{\widetilde{G}}_0^\epsilon + \epsilon h_\epsilon,\end{aligned} \tag{5.44}$$

with

$$\exp \frac{tB_\epsilon}{\epsilon^2} (\mathcal{P} - (\mathcal{P}_o + \mathcal{P}_e)) \dot{\widetilde{G}}_0^\epsilon = \exp \frac{tB_\epsilon}{\epsilon^2} (v \cdot \nabla_x p_0). \tag{5.45}$$

The convergence in $C([\delta, T]; H_{l,\beta})$ of $\dot{\widetilde{G}}$ follows from (5.44), and with (5.44) one shows that δ can be taken to be zero if and only if $\nabla_x p_0 = 0$, i.e., if and only if the relation (5.32) holds, which concludes the proof of the Proposition 5.1.

At present, as in the formal derivations, the solution of the rescaled Boltzmann equation is decomposed into odd and even parts according to scaling and projection on $\mathbb{E}$ and $\mathbb{E}^\perp$:

$$\begin{aligned}\widetilde{G}^\epsilon &= \widetilde{G}_o^\epsilon + \epsilon \widetilde{G}_e^\epsilon, \\ \widetilde{G}_o^\epsilon &= \widetilde{u}^\epsilon \cdot v +^\perp \widetilde{G}_o^\epsilon, \\ \widetilde{G}_e^\epsilon &= \widetilde{\rho}^\epsilon + \frac{1}{6}(|v|^2 - 3)(|\widetilde{u}|^2 + 3\widetilde{\theta}^\epsilon) +^\perp \widetilde{G}_e^\epsilon,\end{aligned} \tag{5.46}$$

and one has the following:

Theorem 5.2. *Let $\widetilde{G}^\epsilon$ be the solution of the rescaled Boltzmann equation* (5.17) *with well prepared initial data given by* (5.30)*. Then:*

i) $\widetilde{G}_0^\epsilon$ *converges to the odd function $\widetilde{u} \cdot v$ in $C([\delta, T]; H_{l,\beta})$ with $\delta = 0$ if and only if the initial data satisfy the compatibility conditions* (5.20).

ii) $\widetilde{G}_e^\epsilon$ *converges in $C([\delta, T]; H_{l-1,\beta})$ with $\delta = 0$ if and only if* (5.20) *and the following supplementary compatibility condition is satisfied.*

$$-\Delta_x(\widetilde{\theta}_0 + \widetilde{\rho}_0) = \sum_{ij} \frac{\partial \widetilde{u}_{0j}}{\partial x_i} \frac{\partial \widetilde{u}_{0i}}{\partial x_j}\,. \tag{5.47}$$

Furthermore, for $\widetilde{u}_0$ and $0 < \epsilon \leq \epsilon_0$ small enough, T can be taken to be $+\infty$.

This theorem substantiates the convergence hypotheses in Theorem 3.1. Notice the "derivative loss" by order 1 in the convergence of the even part $\widetilde{G}^\epsilon$ compared with the odd part $\widetilde{G}_o^\epsilon$.

Proof of Theorem 5.2: As stated in Theorem 5.1 and Proposition 5.1, it is already known that $\widetilde{G}^\epsilon$ converges in $C([\delta, T]; H_{l+1,\beta+1})$ to the odd fluctuation $\widetilde{u} \cdot v$ and that $\dot{\widetilde{G}}^\epsilon$ is uniformly bounded in $L^\infty(0, T; H_{l,\beta})$ with strong convergence in $C(\delta, T; H_{l,\beta})$, with $\delta = 0$ allowed under the assumption specified there. This implies the same bound and convergence for $\widetilde{G}_o^\epsilon$, $\epsilon \widetilde{G}_e^\epsilon$, $\partial_t \widetilde{G}_o^\epsilon$ and $\epsilon \partial_t \widetilde{G}_e^\epsilon$. Therefore, the points i) and ii) of Theorem 5.2 follow once the convergence of $\widetilde{G}_e^\epsilon$ is established.

Firstly, we will check ${}^\perp \widetilde{G}_e^\epsilon$. With the equations,

$$\epsilon^2 \partial_t \widetilde{G}_o^\epsilon + \epsilon^2 v \cdot \nabla_x \widetilde{G}_e^\epsilon + \mathcal{L} \widetilde{G}_o^\epsilon = 2\epsilon^2 \mathcal{Q}(\widetilde{G}_o, \widetilde{G}_e), \tag{5.48}$$

and

$$\epsilon^2\partial_t\widetilde{G}_e^\epsilon + v\cdot\nabla_x\widetilde{G}_o^\epsilon + \mathcal{L}\widetilde{G}_e^\epsilon = \mathcal{Q}(\widetilde{G}_o^\epsilon,\widetilde{G}_o^\epsilon) + \epsilon^2\mathcal{Q}(\widetilde{G}_e,\widetilde{G}_e), \tag{5.49}$$

one has

$$\begin{aligned}\mathcal{L}_M({}^\perp\widetilde{G}_o^\epsilon) &= 2\epsilon\mathcal{Q}(\widetilde{G}_o^\epsilon,\epsilon\widetilde{G}_e^\epsilon) - \epsilon^2\partial_t\widetilde{G}_o^\epsilon - \epsilon v\cdot\nabla_x\epsilon\widetilde{G}_e^\epsilon,\\ \mathcal{L}_M({}^\perp\widetilde{G}_e^\epsilon) &= \mathcal{Q}(\widetilde{G}_o^\epsilon,\widetilde{G}_o^\epsilon) + \epsilon^2\mathcal{Q}(\widetilde{G}_e,\widetilde{G}_e) - \epsilon^2\partial_t\widetilde{G}_e^\epsilon - v\cdot\nabla_x\widetilde{G}_o^\epsilon,\end{aligned} \tag{5.50}$$

and the continuity of the pseudo inverse implies that

$$\|{}^\perp\widetilde{G}_o^\epsilon\|_{L^\infty(0,T,H_{l,\beta})} \leq C\epsilon. \tag{5.51}$$

Therefore ${}^\perp\widetilde{G}_o^\epsilon$ converges to zero in $C([0,T];H_{l,\beta})$ while ${}^\perp\widetilde{G}_e^\epsilon$ converges in $C([\delta,T];H_{l,\beta})$ to the solution of the equation

$$\mathcal{L}({}^\perp\widetilde{G}_e) = \mathcal{Q}(\widetilde{u}\cdot v,\widetilde{u}\cdot v) - v\cdot\nabla_x\widetilde{u}\cdot v, \tag{5.52}$$

which is given (with the divergence free condition) by

$${}^\perp\widetilde{G}_e = \tfrac{1}{2}(A(v):(\widetilde{u}\vee\widetilde{u})) - \mathcal{L}_M^{-1}(A(v)):\nabla_x\widetilde{u}\,. \tag{5.53}$$

To complete the proof, it remains to analyze the convergence of the hydrodynamic part of $\widetilde{G}_e^\epsilon$:

$$\mathcal{P}(\widetilde{G}_e^\epsilon) = \widetilde{\rho}^\epsilon + \tfrac{1}{6}(|\widetilde{u}^\epsilon|^2 + 3\widetilde{\theta}^\epsilon)(|v|^2-3). \tag{5.54}$$

The behavior of $\widetilde{\rho}^\epsilon$ and $\widetilde{\theta}^\epsilon$ is given by the behavior of the functions:

$$\begin{aligned}\sigma^\epsilon &\equiv \frac{1}{3}(|v|^2\,|\mathcal{P}\widetilde{G}_e^\epsilon)_M = \widetilde{\rho}^\epsilon + \widetilde{\theta}^\epsilon + \tfrac{1}{3}|\widetilde{u}^\epsilon|^2\\ \omega^\epsilon &\equiv \frac{1}{5}(|v|^2-5\,|\mathcal{P}_e\widetilde{G}_e^\epsilon)_M = \tfrac{3}{2}\widetilde{\theta}^\epsilon - \widetilde{\rho}^\epsilon + \tfrac{1}{2}|\widetilde{u}^\epsilon|^2\,,\end{aligned} \tag{5.55}$$

keeping in mind the strong convergence of $\widetilde{u}^\epsilon$ in $C([\delta,T];H_l)$ and the obvious formulas:

$$\begin{aligned}\widetilde{\rho}^\epsilon &= \tfrac{1}{5}(3\sigma^\epsilon - 2\omega^\epsilon),\\ \widetilde{\theta}^\epsilon &= \tfrac{2}{5}\big(\sigma^\epsilon+\omega^\epsilon\big) - \tfrac{1}{3}|\widetilde{u}^\epsilon|^2.\end{aligned} \tag{5.56}$$

Multiplication of the equation (5.48) by v gives (cf. (3.9))

$$\partial_t(v|\widetilde{G}_o^\epsilon)_M + \nabla_x\cdot\big((v\vee v - \frac{|v|^2}{3})\,\widetilde{G}_e^\epsilon)\big)_M + \nabla_x(\frac{|v|^2}{3}\,\widetilde{G}_e^\epsilon)_M = 0\,, \tag{5.57}$$

or

$$\begin{aligned}\nabla_x\sigma^\epsilon &= \nabla_x(\frac{|v|^2}{3}\,\mathcal{P}\widetilde{G}_e^\epsilon)_M = \nabla_x(\frac{|v|^2}{3}\,\widetilde{G}_e^\epsilon)_M\\ &= -\nabla_x\cdot(A(v){}^\perp\widetilde{G}_e^\epsilon))_M - \partial_t(v\widetilde{G}_o^\epsilon)_M\,,\end{aligned} \tag{5.58}$$

and eventually,

$$-\Delta_x\sigma^\epsilon = \nabla_x\cdot(\nabla_x\cdot(A(v)|{}^\perp\widetilde{G}_e^\epsilon))_M) - \nabla_x\cdot(\partial_t(v|\widetilde{G}_o^\epsilon)_M). \tag{5.59}$$

With (5.53) one observes that

$$\nabla_x\cdot(\nabla_x\cdot(A(v)|{}^\perp\widetilde{G}_e^\epsilon))_M)$$

converges in $C([\delta,T];H_l)$ to

$$\nabla_x\cdot(\nabla_x\cdot(A(v):(\tfrac{1}{2}(A(v):(\widetilde{u}\vee\widetilde{u}))-\mathcal{L}^{-1}(A(v)):\nabla_x\widetilde{u})) \\ =\sum_{ij}\partial_{x_i}\widetilde{u}\partial_{x_j}\widetilde{u}-\Delta_x\frac{|\widetilde{u}|^2}{3}. \tag{5.60}$$

On the other hand, as a direct consequence of Proposition 5.1, $\nabla_x(\partial_t(v|\widetilde{G}^\epsilon_o)_M)$ converges to 0 in $C([\delta,T];H_{l-1})$ with $\delta=0$ if and only if

$$-\Delta_x(\widetilde{\rho}_0+\widetilde{\theta}_0)=\sum_{ij}\partial_{x_i}\widetilde{u}_0\partial_{x_j}\widetilde{u}_0 \tag{5.61}$$

Same convergence properties hold for $-\Delta_x(\sigma^\epsilon)$, and with the continuity property of the operator $\mathcal{G}$ given by (4.2), one has

$$\lim_{\epsilon\to0,\ \text{in } C([\delta,T];H^0_l)}\sigma^\epsilon=-\mathcal{G}\Big(\sum_{ij}\partial_{x_i}\widetilde{u}\partial_{x_j}\widetilde{u}\Big)+\frac{|\widetilde{u}|^2}{3}, \tag{5.62}$$

which proves the convergence of σ^ϵ.

In order to prove the convergence of ω^ϵ, starting from the moment equation (3.9),

$$\partial_t((\frac{|v|^2-5)}{2})|\widetilde{G}^\epsilon_e)_M+\nabla_x\cdot(\widetilde{G}^\epsilon_o|\frac{B(v)}{\epsilon^2})_M=0, \tag{5.63}$$

one deduces that

$$\partial_t\omega^\epsilon=\partial_t((\frac{|v|^2-5)}{2})|\mathcal{P}_e\widetilde{G}^\epsilon_e)_M \\ =\partial_t((\frac{|v|^2-5}{2})|\widetilde{G}^\epsilon_e)_M=-\nabla_x\cdot(\widetilde{G}^\epsilon_o|\frac{B(v)}{\epsilon^2})_M, \tag{5.64}$$

As in the formal proof, the equation

$$\epsilon^2\partial_t\widetilde{G}^\epsilon_o+\epsilon^2 v\cdot\nabla_x\widetilde{G}^\epsilon_e+\mathcal{L}\widetilde{G}^\epsilon_o=2\epsilon^2\mathcal{Q}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_e), \tag{5.65}$$

is used for the term

$$\nabla_x\cdot(\widetilde{G}^\epsilon_o|\frac{B(v)}{\epsilon^2})_M,$$

giving

$$(\widetilde{G}^\epsilon_o|\frac{B(v)}{\epsilon^2})_M=(\mathcal{L}_M\widetilde{G}^\epsilon_o|\frac{\widetilde{B}(v)}{\epsilon^2})_M \\ =(\{2\mathcal{Q}(\widetilde{G}^\epsilon_o,\widetilde{G}^\epsilon_e)-v\cdot\nabla_x\widetilde{G}^\epsilon_e-\partial_t\widetilde{G}^\epsilon_o\}|\widetilde{B}(v))_M. \tag{5.66}$$

With the already proven convergence results (in particular the limit given by (5.53)), one can represent the right hand side of (5.66) in the following form:

$$(\mathcal{L}_M\widetilde{G}^\epsilon_o|\frac{\widetilde{B}(v)}{\epsilon^2})_M=(2\mathcal{Q}(\widetilde{u}^\epsilon\cdot v,(\tfrac{1}{2}|\widetilde{u}^\epsilon|^2+\frac{3}{2}\widetilde{\theta}^\epsilon)\frac{1}{3}(|v|^2-3)+\tfrac{1}{2}(A(v):(\widetilde{u}^\epsilon\vee\widetilde{u}^\epsilon)))|\widetilde{B})_M \\ -v\cdot\nabla_x(\widetilde{\theta}^\epsilon)(\frac{|v|^2-3}{2})|\widetilde{B})_M+Z^\epsilon_1(x,t), \tag{5.67}$$

where $Z_1^\epsilon(x,t)$ denotes a function uniformly bounded in $L^\infty(0,T,H_l)$ strongly converging in $C([\delta,T];H_l)$ and produces a linear parabolic equation for ω^ϵ:

$$\partial_t\omega^\epsilon + \nabla_x\cdot\big(\widetilde{u}^\epsilon(\tfrac{5}{2}\widetilde{\theta}^\epsilon + \tfrac{1}{2}|\widetilde{u}^\epsilon|^2)\big) - \kappa^*\Delta_x\widetilde{\theta}^\epsilon = Z_2^\epsilon \tag{5.68}$$

with some Z_2^ϵ having the same property as Z_1^ϵ. In (5.68), $\widetilde{\theta}^\epsilon$ is expressed in term of ω^ϵ and σ^ϵ, giving

$$\partial_t\omega^\epsilon + \nabla_x\cdot\big(\widetilde{u}^\epsilon(\omega^\epsilon+\sigma^\epsilon)\big) - \tfrac{2}{5}\kappa^*\Delta_x(\omega^\epsilon+\sigma^\epsilon) = Z_3^\epsilon, \tag{5.69}$$

or

$$\partial_t\omega^\epsilon + \nabla_x\cdot(\widetilde{u}^\epsilon\omega^\epsilon) - \tfrac{2}{5}\kappa^*\Delta_x\omega^\epsilon = \tfrac{2}{5}\kappa^*\Delta_x\sigma^\epsilon - \nabla_x\cdot\sigma^\epsilon + Z_3^\epsilon, \tag{5.70}$$

Z_3^ϵ being similar to Z_1^ϵ. Since σ^ϵ is bounded in $L^\infty(0,T;H_l^0)$ and converging on $C([\delta,T];H_l^0)$ (for any $\delta>0$), eventually one has

$$\partial_t\omega^\epsilon + \nabla_x\cdot(\widetilde{u}^\epsilon\omega^\epsilon) - \tfrac{2}{5}\kappa^*\Delta_x\omega^\epsilon = Z_4^\epsilon, \tag{5.71}$$

where Z_4^ϵ bounded in $L^\infty(0,T;H_{l-2}^0)$ and converging in $C([\delta,T];H_{l-2}^0)$. With the initial data given by

$$\omega^\epsilon(x,0) = \tfrac{3}{2}\widetilde{\theta}_0 - \widetilde{\rho}_0 + \tfrac{1}{2}|\widetilde{u}_0|^2,$$

and by virtue of the well-known parabolic estimates applied to (5.71), we conclude the convergence of ω^ϵ in $C([0,T];H_{l-1})$.

Now, the convergence of $\widetilde{\rho}^\epsilon$ and θ^ϵ follows from the formula (5.56), and as a consequence, the convergence of $\mathcal{P}\widetilde{G}_e^\epsilon$. This completes the proof of Theorem 5.2.

Remark 5.1. *Theorem 5.1 gives an improvement of [BU] on the convergence spaces. In [BU], the convergence is proved in the space $C([\delta,T]\times K, L_\beta^\infty)$ for any compact $K\subset\mathbb{R}^3$, using some compactness property of the relevant semigroup, whereas, in the above theorem, it is in $C([\delta,T];H_{l,\beta})$. This improvement became possible by the newly derived parabolic equation* (5.71) *for ω^ϵ.*

Remark 5.2. *The parabolic equation* (5.71) *may be contrasted with the elliptic equation which is established in [G] based on the micro-macro decomposition of the Boltzmann equation near the uniform Maxwellian. Recently, it was shown in [D] that the Boltzmann equation can be written as a perturbation of the linear compressible Navier-Stokes equations with the source term depending only on the space derivatives of the microscopic component of the solution. The equation* (5.71) *can be taken to be a higher order correction of it.*

Acknowledgements :

First we, the authors, have to thank the editors of this issue for giving us the opportunity to contribute by our paper to the issue of the Institute of Mathematics of the Academia Sinica at Taipei dedicated to the 70th birthday of Professor Yoshio Sone. Over the years we have enjoyed his warm friendship. We learned a lot from him both from direct discussions and from his books and papers. In our community he was the first to understand how diverse are the macroscopic limits of the Boltzmann equation; he introduced us to the "ghost effect " and to the importance of the Knudsen layer. Our contributions to these fields show the importance of our scientific debt to Professor Sone.

Second we acknowledge for support: These recent years the first author has been supported by the Wolfgang Pauli Institute in Vienna. The third author's research is supported by Liu Bie Ju Centre for Mathematical Sciences and Department of Mathematics, City University of Hong Kong. The research of the fourth author was supported the RGC Competitive Earmarked Research Grant of Hong Kong, CityU#102606.

REFERENCES

[BGL] C. Bardos, F. Golse, D. Levermore, *Fluid Dynamic Limits of Kinetic Equations I: Formal Derivations*, J. Stat. Phys. **63** (1991), 323–344.

[BGL2] C. Bardos, F. Golse, D. Levermore, *Fluid Dynamic Limits of Kinetic Equations II: Convergence Proofs for the Boltzmann Equation*, Comm. Pure Appl. Math. 46, 667-753 (1993).

[BLP] B.J. Bayly, C.D. Levermore, and T. Passot, *Density Variations in Weakly Compressible Fluid Flow*, Phys. Fluids A **4** (1992), 945–954.

[BU] C. Bardos and S. Ukai, *The Classical Incompressible Navier-Stokes Limit of the Boltzmann Equation*, Math. Models & Meth Appl. Sci. **1** (1991), 235–257.

[Bl] L. Boltzmann, *Weitere studien uber das Warme gleichgenicht unfer Gasmolekular*, Sitzungsberichte der Akademie der Wissenschaften, Wien, 66(1872), 275-370.

[C] R. Caflish, *The fluid dynamic limit of the nonlinear Boltzmann equation*, Commun. Pure Appl. Math., **33**(1980), 651-666.

[D] R. Duan, On the Cauchy problem for the Boltzmann equation in the whole space: Global existence and uniform stability in $L^2_\xi(H^N_x)$, in preparation.

[DG] L. Desvillettes and F. Golse, *A Remark Concerning the Champman-Enskog Asymptotic*, in *Advances in Kinetic Theory and Computing*, Series on Advances in Mathematics for Applied Sciences **22**, B. Perthame, ed., World Scientific (1994), 191–203.

[En] D. Enskog, *Kinetische Theorie der Vorgänge, in Mässig Verdünnten Gasen I. Allgemeiner Teil*, Almqvist & Wiksell, Uppsala, 1917.

[Hl] D. Hilbert, *Mathematical Problems*, International Congress of Mathematicians, Paris 1900. Translated and reprinted in Bull. Am. Math. Soc. 37, 407-436 (2000).

[GS] F. Golse and Saint-Raymond, *The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels*, Invent. Math. 155(2004), no. 1, 81-161.

[Gr] H. Grad, *Asymptotic Equivalence of the Navier-Stokes and Nonlinear Boltzmann Equations*, Proc. Amer. Math. Symp. on Applications of Partial Differential Equations. (1964), 154–183.

[L] J. Leray, *Sur le mouvement d'un fluide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.

[Mx] J.C. Maxwell, *On the dynamical theory of gases*, Phil. Trans. Roy. Soc. London, 157(1866), 49-88.

[N] T.Nishida,*Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation*, Comm. Math. Phys., **61**(1978), 119-148.

[S] Y. Sone, *Asymptotic Theory of a Steady Flow of Rarefied Gas Past Bodies for Small Knudsen Numbers*, in *Advances in Kinetic Theory and Continuum Mechanics*, Proceedings of a symposium held in honor of Professor Henri Cabannes (Paris, July 6, 1990), R. Gatignol and Soubbaramayer eds., Springer-Verlag, Berlin, 1991, 19–31.

[So2] Y. Sone *Kinetic Theory and Fluid Dynamics*, Birkhäuser, Boston, 2002.

[So3] Y. Sone, *Molecular Gas Dynamics, Theory, Techniques, and Applications*, Birkhäuser, Boston, 2006.

[U1] S. Ukai, *Solutions of the Boltzmann equation*, Pattern and Waves – Qualitative Analysis of Nonlinear Differential Equations (eds. M.Mimura and T.Nishida), Studies of Mathematics and Its Applications **18**, pp37-96, Kinokuniya-North-Holland, Tokyo, 1986.

[U2] S. Ukai, *The incompressible limit and the initial layer of the compressible Euler Equation*, J. Math. Kyoto Univ. **26** (1986) 323–331.

[UA] S.Ukai and K.Asano, The Euler limit and initial layer of the Boltzmann equation, Hokkaido Math.J., **12**(1983), 311-332.