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L^p convergence rates of planar waves for multi-dimensional Euler equations with damping

Jie Liao [a,1], Weike Wang [a,2], Tong Yang [b,a,*,3]

[a] Department of Mathematics, Shanghai Jiao Tong University, Shanghai, PR China
[b] Department of Mathematics, City University of Hong Kong, Hong Kong, PR China

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ABSTRACT

In this paper, the L^p convergence rates of planar diffusion waves for multi-dimensional Euler equations with damping are considered. The analysis relies on a newly introduced frequency decomposition and Green function based energy method. It is a combination of the L^p estimate on the low frequency component by using an approximate Green function and L^2 estimate on the high frequency component through the energy method. By noticing that the low frequency component in the approximate Green function has the algebraic decay which governs the large time behavior, while the high frequency component has the exponential decay but with singularity, their combination leads to a global algebraic decay estimate. To use the decay property only of the low frequency component in the approximate Green function avoids the singularity in the high frequency component so that it simplifies and improves the previous works on this system. This new approach of the combination of the Green function and energy method through the frequency decomposition can also be applied to the hyperbolic–parabolic systems satisfying the Kawashima condition, and also the systems whose derivatives of the coefficients have suitable time decay properties.

* Corresponding author at: Department of Mathematics, City University of Hong Kong, Hong Kong, PR China.
E-mail address: matyang@cityu.edu.hk (T. Yang).

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1. Introduction

In this paper, we consider the stability and L^p convergence rates of planar diffusion waves for multi-dimensional Euler equations with damping. Even though there are extensive studies on the nonlinear profiles for the system of Euler equations with frictional damping in one-dimensional space, there are much less results on the multi-dimensional problems. Consider

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u_i)_t + \operatorname{div}(\rho u_i u) + P(\rho)_{x_i} = -\kappa \rho u_i, \quad 1 \leqslant i \leqslant n, \end{cases} \tag{1.1}$$

with initial data

$$(\rho, u)(x, 0) = \big(\rho_0(x), u_0(x)\big), \tag{1.2}$$

where $\rho(x,t)$, $u(x,t) = (u_1, \ldots, u_n)(x,t)$ and $P = P(\rho(x,t))$ represent the density, velocity and pressure respectively, and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is the space variable. Sometimes, we use the notation (x_1, x') with $x' \in \mathbb{R}^{n-1}$ for x to distinguish the first component in the space variable. The constant $\kappa > 0$ is the frictional damping coefficient. Without loss of generality, we only consider the case when $n = 2, 3$ in this paper.

As usual, we assume the pressure $P(\rho)$ is a smooth function in a neighborhood of a constant state ρ^* with $P'(\rho) > 0$. In the following discussion, we also assume that the initial data is a small perturbation of the diffusion profile constructed later with small wave strength. Let the initial data $\rho_0(x)$ satisfy

$$\lim_{x_1 \to \pm\infty} \rho_0(x) = \rho_\pm, \tag{1.3}$$

where $\rho_0(x) > 0$ and $\rho_\pm > 0$ are two constants with $\rho_- \neq \rho_+$. As in [20], to define the multi-dimensional planar diffusion wave, we first consider the one-dimensional diffusion equation

$$\partial_t \phi = \kappa^{-1} P(\phi)_{x_1 x_1}, \tag{1.4}$$

which can be derived from the Euler equations with frictional damping in one-dimensional case by imposing the Darcy's law, cf. [4]. Then a multi-dimensional diffusion wave $\phi(x,t)$ is a one-dimensional profile in multi-dimensional space. That is, $\phi(x,t) = \varphi(x_1/\sqrt{1+t})$ is a self-similar solution of Eq. (1.4) connecting two end states $\rho_\pm$ at $x_1 = \pm\infty$. Denote $\zeta = \frac{x_1}{\sqrt{t+1}}$, then $\varphi(\zeta)$ satisfies

$$-\frac{1}{2}\zeta \partial_\zeta \varphi = \frac{1}{\kappa} \partial_\zeta \big(P'\big(\varphi(\zeta)\big)\partial_\zeta \varphi\big).$$

For simplicity, let the initial velocity $u_0(x)$ vanish as $x_1 \to \pm\infty$, that is,

$$\lim_{x_1 \to \pm\infty} u_0(x) = 0, \tag{1.5}$$

which implies that there is no mass flux coming in from $x_1 = \pm\infty$. This assumption could be removed in a technical way similar to the argument for one-dimensional problem because the momentum at $x_1 = \pm\infty$ decays exponentially induced by the linear frictional damping.

We now recall some notations for the one-dimensional problem used in [20]. Consider (1.1) and (1.2) in one space dimension:

$$\begin{cases} \rho_t + (\rho u_1)_{x_1} = 0, \\ (\rho u_1)_t + \big(\rho u_1^2\big)_{x_1} + P(\rho)_{x_1} = -\kappa \rho u_1, \\ (\rho, u_1)(x_1, 0) = (\tilde{\rho}, \tilde{u}_1)(x_1, 0). \end{cases} \tag{1.6}$$

Denote the solution of (1.6) by $(\tilde{\rho}, \tilde{u}_1)(x_1, t)$. When

$$\lim_{x_1 \to \pm\infty} \tilde{\rho}(x_1, 0) = \rho_\pm, \qquad \lim_{x_1 \to \pm\infty} \tilde{u}_1(x_1, 0) = 0,$$

the time-asymptotic behavior of $(\tilde{\rho}, \tilde{u}_1)(x_1, t)$ has been well studied which is shown to be a nonlinear profile governed by Darcy's law, cf. [1,4,13–15,19] and references therein. Roughly speaking, the solution $\tilde{\rho}(x_1, t)$ converges to the diffusion wave $\varphi(x_1/\sqrt{1+t})$ up to a constant shift in x_1. This time-asymptotic behavior towards the diffusion wave has been generalized to two-dimensional case in [20] with L^2 decay rates obtained. In this paper, we will establish the L^p $(2 \leqslant p \leqslant \infty)$ convergence rates for the case when $n = 2, 3$ by a new method based on frequency decomposition.

As in [20], we do not compare the solution to the problem (1.1)–(1.2) with the diffusion wave $\phi(x, t)$ directly. Instead, we will compare it with the solution to one-dimensional problem (1.6). For this, without loss of generality, let us first assume the initial density $\tilde{\rho}(x_1, 0)$ in (1.6) satisfies

$$\int_{-\infty}^{+\infty} \big(\tilde{\rho}(x_1, 0) - \varphi(x_1)\big)\, dx_1 = 0.$$

For the multi-dimensional problem, the shift function $\delta_0(x')$ can be chosen as in [20] such that the initial density function satisfies

$$\int_{-\infty}^{+\infty} \big(\rho(x, 0) - \varphi\big(x_1 + \delta_0(x')\big)\big)\, dx_1 = 0. \tag{1.7}$$

Note that $\delta_0(x')$ is then uniquely determined by

$$\delta_0(x') = \frac{1}{\rho_+ - \rho_-} \int_{-\infty}^{\infty} \big(\rho(x, 0) - \varphi(x_1)\big)\, dx_1,$$

for $\rho_- \neq \rho_+$. Moreover, we assume that basically the shift is uniform in directions other than x_1 at infinity, that is,

$$\lim_{|x'| \to +\infty} \frac{1}{\rho_+ - \rho_-} \int_{-\infty}^{+\infty} \big(\rho(x, 0) - \varphi(x_1)\big)\, dx_1 = \delta_*, \tag{1.8}$$

where δ_* is a constant. Note that this assumption simplifies the problem and it remains unsolved for general perturbation when this assumption fails. An immediate consequence of this assumption is that

$$\lim_{|x'| \to +\infty} \delta_0(x') = \delta_*.$$

With these notations, the main purpose here is to show that the solutions (ρ, u) of (1.1) with (1.2) converge to $(\bar{\rho}, \bar{u})$ with certain time decay rates, where

$$\begin{cases} \bar{\rho}(x, t) = \tilde{\rho}\big(x_1 + \delta(x', t), t\big), \\ \bar{u}_1(x, t) = \tilde{u}_1\big(x_1 + \delta(x', t), t\big), \\ \bar{u}_i(x, t) = 0, \quad i \geqslant 2, \\ \delta(x', t) = \delta_* + e^{-\kappa t}\big(\delta_0(x') - \delta_*\big), \end{cases} \tag{1.9}$$

where $\tilde{\rho}$ and $\tilde{u}_1$ are given in (1.6). In the following discussion, we will also assume that the shift generated by the initial data satisfies

$$\left|\partial_{x'}^{\beta}\big(\delta_0(x')-\delta_*\big)\right| \leqslant C\big(1+|x'|^2\big)^{-N}, \tag{1.10}$$

for any multi-index β and any positive integers N. Here, C is a constant depending only on β. This assumption implies that the shift $\delta_0(x')$ decays to δ_* almost exponentially. Again, this assumption can be reduced to the constraint on the initial perturbation.

Before stating the main result, we need a few more notations. Throughout this paper, we denote any generic constant by C. The usual Sobolev space is denoted by $W^{s,p}(\mathbb{R}^n)$, $s\in\mathbf{Z}_+$, $p\in[1,\infty]$, with the norm

$$\|f\|_{W^{s,p}} := \sum_{|\alpha|=0}^{s}\left\|\partial^{\alpha} f\right\|_{L^p},$$

where ∂^α used for ∂_x^α without confusion. In particular, $W^{s,2}=H^s$. Set

$$\begin{aligned} V(x,t) &= \rho(x,t)-\bar{\rho}(x,t)=\rho(x,t)-\tilde{\rho}\big(x_1+\delta(x',t),t\big),\\ U_1(x,t) &= u_1(x,t)-\bar{u}_1(x,t)=u_1(x,t)-\tilde{u}_1\big(x_1+\delta(x',t),t\big),\\ U_i(x,t) &= u_i(x,t),\quad i\geqslant 2, \end{aligned} \tag{1.11}$$

and denote

$$\nu(x,0)=\int_{-\infty}^{x_1} V(x_1,x',0)\,dx_1,\qquad \nu_t(x,0)=\int_{-\infty}^{x_1} V_t(x_1,x',0)\,dx_1. \tag{1.12}$$

Notice that the time derivative on the initial data is not well defined in (1.12). However, $V_t(x_1,x',0)$ can be defined by the compatibility of the initial data through Eq. (2.3).

We can now state the main result in this paper.

Theorem 1.1. *Let $(\bar{\rho},\bar{u})(x,t)$ be defined in* (1.9) *as a planar diffusion wave with a shift $\delta(x',t)$ satisfying the above conditions. For $k\geqslant 4$ and $n=2,3$, assume that the initial data $(\rho,u)(x,0)$ satisfy the smallness assumption*

$$|\rho_+-\rho_-|+\left\|(\nu,\nu_t)(\cdot,0)\right\|_{L^2\cap L^1}+\|\rho-\bar{\rho}\|_{L^1}+\left\|(\rho-\bar{\rho},u-\bar{u})(\cdot,0)\right\|_{H^k}\leqslant\epsilon_0, \tag{1.13}$$

where $\epsilon_0>0$ is a small constant. Then there exists a unique classical and global solution $(\rho,u)\in C([0,\infty),H^{k-2})\cap C^1((0,\infty),H^{k-3})$ to the system (1.1) *and* (1.2)*. Moreover, for $|\gamma|\leqslant k-2$, $p\in[2,\infty]$, we have*

$$\begin{aligned} \left\|\partial_x^{\gamma}(\rho-\bar{\rho})\right\|_{L^p} &\leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}},\\ \left\|\partial_x^{\gamma}(u-\bar{u})\right\|_{L^p} &\leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+2}{2}}. \end{aligned} \tag{1.14}$$

Remark 1.1. In general, if the shift of the profile is not exactly captured, the decay rate should be $\frac{1}{2}$ lower than the one given in the above theorem even in one space dimensional case. Here, the reason that the above decay estimate holds is that the shift due to the initial perturbation is introduced in (1.7) so that when we apply the Green function, the term corresponding to the initial data yields an extra $(1+t)^{-\frac{1}{2}}$ decay after taking the anti-derivative of the initial perturbation. Moreover, under the condition (1.10) on the initial shift, even though the anti-derivative of the perturbation cannot be defined for all time as the shift function is not precisely defined, we know that δ_* is exactly the final shift when t tends to infinity of the profile because the initial perturbation will spread out eventually. By knowing the final shift, it is then exactly the advantage of the introduced method in this paper to obtain the above decay rate by using the frequency decomposition energy method and the approximate Green function.

Remark 1.2. The above theorem shows that the L^p decay rates of $\partial_x^\gamma(u-\bar{u})$ is $\frac{1}{2}$ order faster than that of $\partial_x^\gamma(\rho-\bar{\rho})$. Also note that we only obtain the L^p $(2 \leqslant p \leqslant \infty)$ convergence rates of the perturbation with its derivatives of order up to $k-2$ for the initial perturbation in $H^k(\mathbb{R}^n)$.

As mentioned before, to prove the above theorem, we introduce a new frequency decomposition method to combine the Green function and energy method. There are two advantages of this method. Firstly, to capture the low frequency component in the approximate Green function avoids the singularity in the high frequency component and this gives the precise algebraic decay estimate of the perturbation. Secondly, to apply the energy method only on the high frequency component, one can use the Poincaré-type inequalities (5.10) and (5.28). Notice that the lack of Poincaré inequality in the whole space is usually the essential difficulty in the energy estimate which is contrast to the problem in a torus. This in some sense illustrates the essence of the Green function on the decay rate related to the frequency. In fact, this new approach by combining of the Green function and energy method through frequency decomposition can be also applied to the hyperbolic–parabolic systems satisfying the Kawashima condition, cf. [6], and also the systems whose derivatives of the coefficients have suitable time decay properties.

Notice that the result in this paper gives the L^p $(2 \leqslant p \leqslant \infty)$ convergence rates of the perturbation with derivatives of order only up to $k-2$ even though the initial perturbation has derivatives defined up to order k. The reason is that the L^p estimates obtained by using the approximate Green function are only for the low frequency component, and the estimates on the high frequency component and its derivatives are obtained by energy method only in L^2 framework. Thus, the Sobolev embedding gives the L^∞-norm on lower order derivatives and the others follow from the interpolation.

Finally, we mention some related works on the conservation laws in multi-dimensional space such as those on the shock profiles or the planar diffusion waves, cf. [2,3,5,7,11,12,16,17] and references therein; and some works on the pointwise estimates for hyperbolic–parabolic systems, cf. [8–10] and references therein.

The rest of the paper is arranged as follows. In Section 2, we will reformulate the system around a planar diffusion wave defined in (1.9) and then state some known properties of this background diffusion wave. In Section 3, we will study the approximate Green function for a linear system with a parameter by using Fourier analysis, and then give the estimates on its part on the low frequency component. The main L^p estimates on the low frequency component by using the approximate Green function and L^2 estimates on the high frequency component by using the energy method will be carried out in Sections 4 and 5 respectively. Finally, we will complete the proof of Theorem 1.1 in Section 6.

2. Preliminaries

In this section, we will first derive the equations for the perturbation functions V and U_i, $i = 1, \ldots, n$, defined in (1.11). Then we will recall some results on the background diffusion waves, and enclose some lemmas which will be used later in this paper.

2.1. Reduced system

We first derive the system for the perturbation of a nonlinear planar diffusion wave. Recall that the solutions to (1.1) and (1.6) satisfy the following systems respectively,

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (u_i)_t + u \cdot \nabla u_i + \rho^{-1} P(\rho)_{x_i} = -\kappa u_i, \quad i = 1, \dots, n, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \tilde{\rho}_t + (\tilde{\rho}\tilde{u}_1)_{x_1} = 0, \\ (\tilde{u}_1)_t + \tilde{u}_1(\tilde{u}_1)_{x_1} + \tilde{\rho}^{-1} P(\tilde{\rho})_{x_1} = -\kappa \tilde{u}_1. \end{cases} \tag{2.2}$$

For simplicity, we denote $U = (U_1, \dots, U_n)$, $u = (u_1, \dots, u_n)$ and $\bar{u} = (\bar{u}_1, \dots, \bar{u}_n)$. Then, from (2.1), (2.2) and (1.11), we have

$$\begin{aligned} V_t + (\bar{\rho} + V) \operatorname{div} U &= R_\rho - (U \cdot \nabla)(\bar{\rho} + V) - V \operatorname{div} \bar{u} - (\bar{u} \cdot \nabla) V \\ &= R_\rho - (U \cdot \nabla)(\bar{\rho} + V) - V(\bar{u}_1)_{x_1} - \bar{u}_1 V_{x_1}, \end{aligned} \tag{2.3}$$

where

$$R_\rho = -\tilde{\rho}_1'\big(x_1 + \delta(x', t), t\big)\delta_t(x', t) = \big(-\bar{\rho}(x, t)\delta_t(x', t)\big)_{x_1}.$$

Similarly, the equations for U_i are

$$(U_1)_t + (\bar{\rho} + V)^{-1}\big(P(\bar{\rho} + V) - P(\bar{\rho})\big)_{x_1} + \kappa U_1 = \frac{P(\bar{\rho})_{x_1} V}{\bar{\rho}(\bar{\rho} + V)} + R_{u_1} - R_1, \tag{2.4}$$

and for $i \geqslant 2$,

$$(U_i)_t + (\bar{\rho} + V)^{-1}\big(P(\bar{\rho} + V) - P(\bar{\rho})\big)_{x_i} + \kappa U_i = \frac{P(\bar{\rho})_{x_i} V}{\bar{\rho}(\bar{\rho} + V)} - \bar{\rho}^{-1} P(\bar{\rho})_{x_i} - R_i, \tag{2.5}$$

where

$$\begin{aligned} R_{u_1} &= -\big(\tilde{u}_1\big(x_1 + \delta(x', t), t\big)\delta_t(x', t)\big)_{x_1}, \\ R_1 &= U \cdot \nabla(\bar{u}_1 + U_1) + \bar{u}_1 (U_1)_{x_1}, \\ R_i &= U \cdot \nabla U_i + \bar{u}_1 (U_i)_{x_1}, \quad 2 \leqslant i \leqslant n. \end{aligned}$$

Thus, the system for the perturbation (V, U) can be summarized as

$$\begin{cases} V_t + (\bar{\rho} + V) \operatorname{div} U = Q, \\ (U_i)_t + (\bar{\rho} + V)^{-1}\big(\mathcal{P}(V, \bar{\rho}) V\big)_{x_i} + \kappa U_i = H_i, \quad 1 \leqslant i \leqslant n, \end{cases} \tag{2.6}$$

where $\mathcal{P}(V, \bar{\rho}) = \int_0^1 P'(\bar{\rho} + \theta V)\, d\theta$, and

$$\begin{aligned} Q &= R_\rho - (U \cdot \nabla)(\bar{\rho} + V) - V(\bar{u}_1)_{x_1} - (\bar{u}_1) V_{x_1}, \\ H_1 &= R_{u_1} + \frac{P(\bar{\rho})_{x_1} V}{\bar{\rho}(\bar{\rho} + V)} - R_1, \\ H_i &= -\frac{P(\bar{\rho})_{x_i}}{\bar{\rho}} + \frac{P(\bar{\rho})_{x_i} V}{\bar{\rho}(\bar{\rho} + V)} - R_i, \quad 2 \leqslant i \leqslant n. \end{aligned} \tag{2.7}$$

Moreover, we can deduce the equation for $V(x,t)$ from (2.6) as

$$V_{tt} - \Delta\big(\mathcal{P}(V,\bar\rho)V\big) + \kappa V_t = \tilde{Q}(V,U,\bar\rho,\bar u_1), \tag{2.8}$$

where

$$\tilde{Q}(V,U,\bar\rho,\bar u) = \big[(R_\rho)_t + \kappa(R_\rho)\big] - (\kappa + \partial_t)(V\bar u_1)_{x_1} - \operatorname{div}\big((\bar\rho + V)_t U\big) - \operatorname{div}\big((\bar\rho + V)H\big), \tag{2.9}$$

with $H = (H_1, \ldots, H_n)$. By linearizing (2.8) around $\bar\rho$, we have

$$\begin{aligned} V_{tt} - \Delta\big(a(x,t)V\big) + \kappa V_t &= \tilde{Q}(V,U,\bar\rho,\bar u_1) + \Delta\big(\mathcal{P}_1(\bar\rho,V)V^2\big) \\ &=: F(V,U,\bar\rho,\bar u), \end{aligned} \tag{2.10}$$

where $a(x,t) = P'(\bar\rho)$ and

$$\mathcal{P}_1(\bar\rho,V) = \int_0^1 \left(\int_0^{\theta_1} P''(\bar\rho + \theta_2 V)\, d\theta_2 \right) d\theta_1.$$

Since

$$(R_\rho)_t + \kappa(R_\rho) = \big(-\bar\rho_t(x,t)\delta_t(x',t)\big)_{x_1},$$

direct calculation shows that $F(V,U,\bar\rho,\bar u)$ is in divergence form, that is,

$$F = \sum \big(F^i\big)_{x_i} + \sum \big(F^{ij}\big)_{x_i x_j}, \tag{2.11}$$

where

$$\begin{aligned} F^1 &= -\bar\rho_t\delta_t - (\bar\rho\bar u_1\delta_t)_{x_1}, \qquad F^i = -P(\bar\rho)_{x_i}, \quad 2 \leqslant i \leqslant n, \\ F^{11} &= \big(\bar\rho\big(2\bar u_1 U_1 + U_1^2\big) + V(\bar u_1 + U_1)^2\big) + \big(\mathcal{P}_1(\bar\rho,V)V^2\big), \\ F^{1i} &= F^{i1} = 2\big((\bar\rho + V)(\bar u_1 + U_1)U_i\big), \quad 2 \leqslant i \leqslant n, \\ F^{ij} &= (\bar\rho + V)U_iU_j + \delta_{ij}\big(\mathcal{P}_1(\bar\rho,V)V^2\big), \quad 2 \leqslant i,j \leqslant n. \end{aligned}$$

Here δ_{ij} is the Kronecker symbol. On the other hand, by linearizing $(2.6)_2$ around $\bar u$, we have

$$U_t + \bar\rho^{-1}\nabla\big(a(x,t)V\big) + \kappa U = \bar H, \tag{2.12}$$

where

$$\begin{aligned} \bar H_1 &= R_u - \bar\rho^{-1}\big(\mathcal{P}_1(\bar\rho,V)V^2\big)_x - \frac{P(\bar\rho + V)_x V}{\bar\rho(\bar\rho + V)} - R_1, \\ \bar H_i &= -\frac{P(\bar\rho)_{x_i}}{\bar\rho} - \frac{P(\bar\rho + V)_{x_i} V}{\bar\rho(\bar\rho + V)} - \bar\rho^{-1}\big(\mathcal{P}_1(\bar\rho,V)V^2\big)_{x_i} - R_i, \quad 2 \leqslant i \leqslant n. \end{aligned} \tag{2.13}$$

2.2. Background profile

For later use, we include the following known estimates on the background planar wave, cf. [20]. By the definition of $\varphi(x_1)$, we know for any integer N,

$$\sup_{x_1>0}\left|\varphi(x_1)-\rho_+\right|+\sup_{x_1<0}\left|\varphi(x_1)-\rho_-\right|\leqslant C|\rho_+-\rho_-|\left(1+x_1^2\right)^{-N},$$
$$\left|\partial_{x_1}^h\varphi(x_1)\right|\leqslant C|\rho_+-\rho_-|\left(1+x_1^2\right)^{-N}\quad(h>0).$$

Recall that we have assumed in (1.10) for any multi-index β,

$$\left|\partial_{x'}^\beta\left(\delta_0(x')-\delta_*\right)\right|\leqslant C\left(1+|x'|^2\right)^{-N}.$$

Thus, we have

Lemma 2.1. *If there exists a small positive constant E_ρ such that the initial data $(\tilde\rho,\tilde u_1)(x_1,0)$ satisfies for some $m\geqslant 2$,*

$$|\rho_+-\rho_-|+\left\|\int_{-\infty}^{x_1}\left(\tilde\rho(z,0)-\varphi(z)\right)dz\right\|_{L^2}+\left\|\tilde\rho(\cdot,0)-\varphi(\cdot)\right\|_{H^m}+\left\|\tilde u_1(\cdot,0)-\psi(\cdot,0)\right\|_{H^m}\leqslant E_\rho,\quad(2.14)$$

then for any multi-index $|\alpha|\leqslant m$, we have

$$\sup_{x'}\left\|\partial^\alpha(\bar\rho_{x_1},\bar u_1)(\cdot,x',t)\right\|_{L^2(\mathbb{R}^1_{x_1})}\leqslant CE_\rho(1+t)^{-\frac{1+|\alpha|}{2}-\frac14},$$
$$\left\|\partial^\alpha(\bar\rho_{x_1},\bar u_1)(\cdot,t)\right\|_{L^\infty(\mathbb{R}^n)}\leqslant CE_\rho(1+t)^{-\frac{1+|\alpha|}{2}}.\quad(2.15)$$

In addition, for $2\leqslant i\leqslant n$,

$$\left\|\partial^\alpha\left(\bar\rho_{x_i},(\bar u_1)_{x_i}\right)(t)\right\|_{L^2(\mathbb{R}^n)}\leqslant CE_\rho e^{-\kappa t},$$
$$\left\|\partial^\alpha\left(\bar\rho_{x_i},(\bar u_1)_{x_i}\right)(t)\right\|_{L^\infty(\mathbb{R}^n)}\leqslant CE_\rho e^{-\kappa t}.\quad(2.16)$$

2.3. Some basic estimates

For the convenience of the readers, some estimates on the Fourier transformation and some function spaces with their interpolations are included here for later use. As usual, denote the Fourier transformation by

$$\hat f(\xi,t)\equiv(\mathcal{F}f)(\xi,t)=\int_{\mathbb{R}^n}f(z,t)e^{-\sqrt{-1}z\xi}\,dz,$$

and the inverse Fourier transform by

$$f(z,t)\equiv\left(\mathcal{F}^{-1}\hat f\right)(z,t)=(2\pi)^{-n}\int_{\mathbb{R}^n}\hat f(\xi,t)e^{\sqrt{-1}z\xi}\,d\xi.$$

The following lemma is from [18].

Lemma 2.2. *If $\hat{f}(\xi, t)$ has compact support in the variable ξ and there exists a constant $b > 0$ such that*

$$\left|\partial_\xi^\beta\left(\xi^\alpha \hat{f}(\xi, t)\right)\right| \leqslant C\left(|\xi|^{(|\alpha|+k-|\beta|)_+} + |\xi|^{|\alpha|+k} t^{|\beta|/2}\right)\left(1 + t|\xi|^2\right)^m e^{-b|\xi|^2 t}$$

for any multi-indices α and β with $|\beta| \leqslant 2N$, then

$$\left|\partial_z^\alpha f(z, t)\right| \leqslant C_N t^{-\frac{n+|\alpha|+k}{2}} B_N(z, t),$$

where k and m can be any given integers, $(a)_+ = \max(0, a)$ and $B_N(z, t) = (1 + \frac{|z|^2}{1+t})^{-N}$.

The following three formulas, basically the triangle inequality, the Young's inequality and the interpolation estimate will be used. For the convenience of the readers, we include them here for later reference.

Lemma 2.3. *Let (X, μ), (Y, ν) be two measure spaces and $(X \times Y, \mu \times \nu)$ be their product. And let $f(x, y)$ be a measurable function on $(X \times Y, \mu \times \nu)$. If $f(\cdot, y) \in L^p(X, \mu)$ for a.e. $y \in Y$, and $1 \leqslant p \leqslant \infty$, with*

$$\int_Y \left\|f(\cdot, y)\right\|_{L^p(X,\mu)} d\nu(y) = A \leqslant \infty,$$

then

$$\left\|\int_Y f(\cdot, y)\, d\nu(y)\right\|_{L^p(X,\mu)} \leqslant \int_Y \left\|f(\cdot, y)\right\|_{L^p(X,\mu)} d\nu(y).$$

Lemma 2.4. *For $z, z' \in \mathbb{R}^n$, if $K(z, z')$ is a measurable function on $\mathbb{R}^n \times \mathbb{R}^n$ and*

$$\left\|K(\cdot, z')\right\|_{L^q} \leqslant A, \qquad \left\|K(z, \cdot)\right\|_{L^q} \leqslant B,$$

then for $f(z) \in L^p(\mathbb{R}^n)$, the integral operator

$$Tf(z) = \int_{\mathbb{R}^n} K(z, z') f(z')\, dz'$$

satisfies

$$\|Tf\|_{L^r} \leqslant A^{\frac{1}{r}} B^{1-\frac{1}{p}} \|f\|_{L^p},$$

where $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$ and $1 \leqslant p, q, r \leqslant \infty$.

Lemma 2.5. *Assume $1 \leqslant s \leqslant r \leqslant t \leqslant \infty$ and*

$$\frac{1}{r} = \frac{\theta}{s} + \frac{1-\theta}{t}, \quad 0 \leqslant \theta \leqslant 1.$$

Suppose also $u \in L^s(\mathbb{R}^n) \cap L^t(\mathbb{R}^n)$. Then $u \in L^r(\mathbb{R}^n)$ and

$$\|u\|_{L^r(\mathbb{R}^n)} \leqslant \|u\|_{L^s(\mathbb{R}^n)}^\theta \|u\|_{L^t(\mathbb{R}^n)}^{1-\theta}.$$

3. Approximate Green function

Since the linearized system around a planar wave has variable coefficients, the exact Green function to this system is difficult to analyze. For this, in [20], an approximate Green function with a parameter was introduced and the L^2 estimate was given. In this section, we will introduce the frequency decomposition to this approximate Green function. Based on this decomposition, we give the L^p estimate on the low frequency component.

We now give the approximate Green function for the linearized equation (2.10) which takes the form

$$V_{tt} - \Delta\big(a(x,t)V\big) + \kappa V_t = F(V, U, \bar{\rho}, \bar{m}). \tag{3.1}$$

Since $a(x,t) = P'(\bar{\rho}(x,t))$ and $\bar{\rho}(x,t)$ is a bounded function, we have $0 < C_0 < a = a(x,t) = P'(\bar{\rho}(x,t)) < C_1$ with constants C_1 and C_0. As in [20], we construct an approximate Green function $G(x,t;y,s)$ for the homogeneous part of (3.1) so that $G(x,t;y,s)$ meets the basic requirement

$$G(x,t;y,t) = 0, \qquad G_s(x,t;y,t) = \delta(x-y), \tag{3.2}$$

where δ is the Dirac function. Multiplying (3.1) whose variables are now changed to (y,s) by G and integrating with respect to y and s over the region $\mathbb{R}^n \times [0,t]$, (3.2) gives

$$\begin{aligned} V(x,t) = {} & \int_{\mathbb{R}^n} G_s(x,t;y,0)V(y,0)\,dy - \int_{\mathbb{R}^n} G(x,t;y,0)(\kappa V + V_s)(y,0)\,dy \\ & - \int_0^t \int_{\mathbb{R}^n} G(x,t;y,s)F(y,s)\,dy\,ds \\ & + \int_0^t \int_{\mathbb{R}^n} (G_{ss} - a\Delta_y G - \kappa G_s)(x,t;y,s)V(y,s)\,dy\,ds. \end{aligned} \tag{3.3}$$

If $a(y,s)$ is a constant and G is the Green function of the homogeneous part of (3.1), then the last integral in (3.3) is zero. However, when $a(y,s)$ is not a constant, it is difficult to give an explicit expression of the Green function. Therefore, we will try to minimize the term $G_{ss} - a\Delta_y G - \kappa G_s$ by choosing a suitable function G which is called the approximate Green function. Note that the following choice of approximate Green function may not be optimal and the choice is not unique. For this purpose, we first consider the following linear partial differential equation

$$\partial_{tt} V - \mu\Delta V + \kappa V_t = 0, \tag{3.4}$$

where μ is a bounded parameter with $C_0 < \mu < C_1$. Denote the Green function of (3.4) by $G^\sharp(\mu;x,t)$ satisfying

$$\begin{cases} \big(G^\sharp_{tt} - \mu\Delta G^\sharp + \kappa G^\sharp_t\big)(\mu;x,t) = 0, \\ G^\sharp(\mu;x,0) = 0, \quad G^\sharp_t(\mu;x,0) = \delta(x). \end{cases} \tag{3.5}$$

The Fourier transformation of (3.5) gives the following equation for $\hat{G}^\sharp$

$$\begin{cases} \big(\hat{G}^\sharp_{tt} + \mu|\xi|^2\hat{G}^\sharp + \kappa \hat{G}^\sharp_t\big)(\mu;\xi,t) = 0, \\ \hat{G}^\sharp(\mu;\xi,0) = 0, \quad \hat{G}^\sharp_t(\mu;\xi,0) = 1. \end{cases} \tag{3.6}$$

Direct calculation gives

$$\hat{G}^\sharp(\mu;\xi,t)=\frac{e^{\lambda_+t}-e^{\lambda_-t}}{\lambda_+-\lambda_-},\tag{3.7}$$

where

$$\lambda_\pm(\xi)\equiv\frac{1}{2}\left(-\kappa\pm\sqrt{\kappa^2-4\mu|\xi|^2}\right).\tag{3.8}$$

Sometimes, we write $\hat{G}^\sharp=\hat{E}^+ + \hat{E}^-$ with

$$\hat{E}^+=\eta_0 e^{\lambda_+t},\qquad \hat{E}^-=\eta_0 e^{\lambda_-t},\qquad \eta_0=(\lambda_+-\lambda_-)^{-1}.$$

The approximate Green function is defined by

$$G(x,t;y,s)=G^\sharp\big(a\big(y,\sigma(t,s)\big);x-y,t-s\big),\tag{3.9}$$

with $a(y,\sigma(t,s))=P'(\bar{\rho}(y,\sigma(t,s)))$. It is clear that (3.9) satisfies the basic requirement (3.2). Here, the function $\sigma(t,s)$ is chosen such that $\sigma(t,s)\in C^3([2,\infty]\times[0,\infty])$,

$$\sigma(t,s)=\begin{cases} s, & s>t/2,\\ t/2, & s\leqslant t/2-1,\end{cases}$$

and

$$\sum_{1\leqslant l_1+l_2\leqslant 3}\left|\partial_t^{l_1}\partial_s^{l_2}\sigma(t,s)\right|\leqslant C,\quad s\in(t/2-1,t/2).$$

Notice that $\sigma^{-1}(t,s)\leqslant C(1+t)^{-1}$ for $t>2$ so that we have by Lemma 2.1

$$(1+t)\left|\partial_s a\big(y,\sigma(t,s)\big)\right|+(1+t)^2\left|\partial_s^2 a\big(y,\sigma(t,s)\big)\right|\leqslant CE_\rho,\tag{3.10}$$

where E_ρ is defined in (2.14).

Notice that the decay of the derivatives of the function $a(y,\sigma(t,s))$ with respect to time will be used in the following analysis. Denote the low frequency component in the approximate Green function $G(x,t;y,s)$ by

$$G_L(x,t;y,s)=\chi(D_x)G(x,t;y,s),\tag{3.11}$$

where $\chi(D_x)$, $D_x=\frac{1}{\sqrt{-1}}\partial_x=\frac{1}{\sqrt{-1}}(\partial_{x_1},\dots,\partial_{x_n})$, is the pseudo-differential operator with symbol $\chi(\xi)$ as a smooth cut-off function satisfying

$$\chi(\xi)=\begin{cases}1, & |\xi|<\varepsilon,\\ 0, & |\xi|>2\varepsilon,\end{cases}\tag{3.12}$$

for some chosen constant ε in $(0,\varepsilon_0)$ with $\varepsilon_0=\frac{1}{2}\min\{1,\sqrt{\frac{\kappa^2}{4C_1}}\}$, in which C_1 is the upper bound of μ given after (3.4). Set

$$\hat{G}_L^\sharp(\mu;\xi,t)=\chi(\xi)\cdot\hat{G}^\sharp(\mu;\xi,t)=\chi\cdot(\hat{E}^+ + \hat{E}^-),\qquad G_L^\sharp(\mu;x,t)=E_L^+ + E_L^-.$$

Instead of studying the approximate Green function for all frequencies, we now only estimate its low frequency component G_L defined in (3.11). Notice that the Green function $G(x,t;y,s)$ is closely related to $G^\sharp(a;x-y,t-s)$. Thus, we first state some properties of $G_L^\sharp(\mu;x,t)=E_L^+ + E_L^-$ as follows which can be obtained by straightforward calculation.

Proposition 3.1. *For $|\xi|\leqslant\varepsilon_0$, ε_0 as given above, then there exists a constant $b>0$ such that for any indices h,l,α and β*

$$\left|\partial_\mu^h\partial_t^l\partial_\xi^\beta\big(\xi^\alpha\hat{E}^+(\mu;\xi,t)\big)\right|\leqslant C\big(|\xi|^{(2l+|\alpha|-|\beta|)_+}+|\xi|^{|\alpha|+2l}t^{|\beta|/2}\big)\big(1+t|\xi|^2\big)^{|\beta|+1+2h}e^{-b|\xi|^2t},$$

and

$$\left|\partial_\mu^h\partial_\xi^\beta\big(\xi^\alpha\hat{E}^-(\mu;\xi,t)\big)\right|\leqslant Ce^{-\kappa t}\big(|\xi|^{(|\alpha|-|\beta|)_+}+|\xi|^{|\alpha|}t^{|\beta|/2}\big)\big(1+t|\xi|^2\big)^{|\beta|+1+2h}e^{-b|\xi|^2t},$$

where C is uniform w.r.t. μ.

As in [19], Lemma 2.2 and Proposition 3.1 yield the following proposition.

Proposition 3.2. *For any indices h,l and α,*

$$\left|\partial_\mu^h\partial_t^l\partial_x^\alpha E_L^+(\mu;x,t)\right|\leqslant C(1+t)^{-\frac{n+2l+|\alpha|}{2}}B_N(x,t),$$
$$\left|\partial_\mu^h\partial_t^l\partial_x^\alpha E_L^-(\mu;x,t)\right|\leqslant Ce^{-\kappa t/2}(1+t)^{-\frac{n+|\alpha|}{2}}B_N(x,t),$$

where C is uniform w.r.t. μ.

Recall that the approximate Green function defined in (3.9) is given by

$$G(x,t;y,s)=G^\sharp\big(a\big(y,\sigma(t,s)\big);x-y,t-s\big),$$

which is not symmetric with respect to the variables (x,t) and (y,s). However, straightforward calculation gives their relations as

$$\partial_{x_i}G=-\partial_{y_i}G+\partial_a\big(G^\sharp\big)\,a_{x_i},\qquad\partial_tG=-\partial_sG+\partial_a\big(G^\sharp\big)\,(a_s+a_t).\tag{3.13}$$

Moreover, we have

$$\begin{aligned}G_L(x,t;y,s)&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\chi(\xi)e^{\sqrt{-1}(x-y)\xi}\hat{G}^\sharp\big(a\big(y,\sigma(t,s)\big),\xi,t-s\big)\,d\xi\\&=G_L^\sharp\big(a\big(y,\sigma(t,s)\big);x-y,t-s\big).\end{aligned}$$

Since

$$\left|\partial_t^{l_1}\partial_s^{l_2}\partial_y^\alpha a\big(y,\sigma(t,s)\big)\right|\leqslant CE_\rho(1+\sigma)^{-(\min(l_1+l_2,1)+\frac{|\alpha|}{2})},\tag{3.14}$$

by using Proposition 3.2, (3.13) and (3.14), we have

$$\left|\partial_x^\alpha\partial_y^\beta\partial_s^l\partial_t^hG_L(x,t;y,s)\right|\leqslant C(1+t-s)^{-\frac{n}{2}-\frac{2\min(l+h,1)+|\alpha|+|\beta|}{2}}B_N(x-y,t-s).$$

Therefore, we have the following proposition.

Proposition 3.3. *For $q \in [1,\infty]$ and any indices h, l, α and β, we have*

$$\sup_y \left\| \partial_x^\alpha \partial_y^\beta \partial_s^l \partial_t^h G_L(\cdot, t; y, s)\right\|_{L^q(\mathbb{R}_x^n)} \leqslant C(1+t-s)^{-\frac{n}{2}(1-\frac{1}{q})-\frac{2\min(l+h,1)+|\alpha|+|\beta|}{2}},$$
$$\sup_x \left\| \partial_x^\alpha \partial_y^\beta \partial_s^l \partial_t^h G_L(x, t; \cdot, s)\right\|_{L^q(\mathbb{R}_y^n)} \leqslant C(1+t-s)^{-\frac{n}{2}(1-\frac{1}{q})-\frac{2\min(l+h,1)+|\alpha|+|\beta|}{2}}. \tag{3.15}$$

4. L^p estimates on the low frequency component

In this section, we will establish the L^p estimates on the low frequency component by using the approximate Green function. Assume that $|\alpha| \leqslant k$ in this section. To derive the L^p estimates for the low frequency part, recall (3.3),

$$\begin{aligned} V(x,t) = {} & \int_{\mathbb{R}^n} G_s(x,t;y,0)V(y,0)\,dy - \int_{\mathbb{R}^n} G(x,t;y,0)(\kappa V + V_s)(y,0)\,dy \\ & - \int_0^t \int_{\mathbb{R}^n} G(x,t;y,s)F(y,s)\,dy\,ds \\ & + \int_0^t \int_{\mathbb{R}^n} (G_{ss} - a\Delta_y G - \kappa G_s)(x,t;y,s)V(y,s)\,dy\,ds. \end{aligned}$$

Set

$$\begin{aligned} I_1^\alpha &= \chi(D_x)\int_{\mathbb{R}^n} \partial_x^\alpha G_s(x,t;y,0)V(y,0)\,dy = \int_{\mathbb{R}^n} \partial_x^\alpha (G_L)_s(x,t;y,0)V(y,0)\,dy, \\ I_2^\alpha &= -\chi(D_x)\int_{\mathbb{R}^n} \partial_x^\alpha G(x,t;y,0)(\kappa V + V_s)(y,0)\,dy = -\int_{\mathbb{R}^n} \partial_x^\alpha G_L(x,t;y,0)(\kappa V + V_s)(y,0)\,dy, \\ -I_3^\alpha &= \chi(D_x)\int_0^t \int_{\mathbb{R}^n} \partial_x^\alpha G(x,t;y,s)F(y,s)\,dy\,ds = \int_0^t \int_{\mathbb{R}^n} \partial_x^\alpha G_L(x,t;y,s)F(y,s)\,dy\,ds, \\ I_4^\alpha &= \chi(D_x)\int_0^t \int_{\mathbb{R}^n} \partial_x^\alpha R_G(x,t;y,s)V(y,s)\,dy\,ds = \int_0^t \int_{\mathbb{R}^n} \partial_x^\alpha R_{G_L}(x,t;y,s)V(y,s)\,dy\,ds, \end{aligned} \tag{4.1}$$

where

$$R_G \equiv G_{ss}(x,t;y,s) - a(y,s)\Delta_y G(x,t;y,s) - \kappa G_s(x,t;y,s),$$

and

$$\chi(D_x)R_G = R_{G_L}.$$

Since

$$\left(G_{tt}^\sharp - a\Delta G^\sharp + \kappa G_t^\sharp\right)\left(a(y,s); x-y, t-s\right) = 0,$$

we have

$$\begin{aligned} R_{G_L}(x,t;y,s) = \bigg[& G^\sharp_{L;0,0}\big(a(y,s);x-y,t-s\big)a_s(y,\sigma)^2 - 2G^\sharp_{L;0,n+1}\big(a(y,s);x-y,t-s\big)a_s(y,\sigma) \\ & + G^\sharp_{L;0}\big(a(y,s);x-y,t-s\big)a_{ss}(y,\sigma) - \kappa G^\sharp_{L;0}\big(a(y,s);x-y,t-s\big)a_s(y,\sigma) \\ & + a(y,s)\Bigg(\sum_{i=1}^n \big[G^\sharp_{L;0,i}\big(a(y,s);x-y,t-s\big)a_{y_i}(y,\sigma) \\ & - G^\sharp_{L;0,0}\big(a(y,s);x-y,t-s\big)\big(a^2_{y_i}\big)(y,\sigma)\big] + G^\sharp_{L;0}\big(a(y,s);x-y,t-s\big)\Delta_y a(y,\sigma)\Bigg)\bigg] \\ & + \big[\big(a(y,\sigma)-a(y,s)\big)\Delta G^\sharp_L\big(a(y,s);x-y,t-s\big)\big] \\ =: {} & R^1_{G_L} + R^2_{G_L}. \end{aligned} \tag{4.2}$$

Here $R^i_{G_L}$, $i=1,2$, is the corresponding term in the above summation in the above equation. To denote the derivatives, we use the notations $G^\sharp_{L;0}(a;x,t) = \partial_a G^\sharp_L(a;x,t)$, $G^\sharp_{L;i}(a;x,t) = \partial_{x_i} G^\sharp_L(a;x,t)$, $G^\sharp_{L;n+1}(a;x,t) = \partial_t G^\sharp_L(a;x,t)$, $G^\sharp_{L;0,i}(a;x,t) = \partial_a\partial_{x_i} G^\sharp_L(a;x,t)$, and $G^\sharp_{L;0,n+1}(a;x,t) = \partial_a\partial_t G^\sharp_L(a;x,t)$, etc.

Then, set $V_L(x,t) = \chi(D_x)V(x,t)$, from (3.3) and (4.1), we have

$$\partial^\alpha_x V_L(x,t) = I^\alpha_1 + I^\alpha_2 + I^\alpha_3 + I^\alpha_4. \tag{4.3}$$

We will estimate the right-hand side of (4.3) term by term. By Proposition 3.3 and Lemma 2.3, it is straightforward to obtain

$$\big\| I^\alpha_1 \big\|_{L^p(\mathbb{R}^n_x)} \leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+2}{2}} \|V_0\|_{L_1}. \tag{4.4}$$

For I^α_2, set

$$\tilde{\nu}_0(y) = \nu_t(y,0) + \kappa\nu(y,0),$$

where ν is defined in (1.10). Then

$$\big|I^\alpha_2\big| = \bigg|\int_{\mathbb{R}^n} \partial_{y_1}\partial^\alpha_x G_L(x,t;y,0)\tilde{\nu}_0(y)\,dy\bigg|.$$

Also by using Proposition 3.3 and Lemma 2.3, we have

$$\big\| I^\alpha_2 \big\|_{L^p(\mathbb{R}^n_x)} \leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}} \|\tilde{\nu}_0\|_{L_1}. \tag{4.5}$$

Before estimating I^α_3 and I^α_4, we first set

$$\begin{cases} \mathcal{M}_V = \sup\limits_{0\leqslant s\leqslant t,\,|\alpha|\leqslant k-2} (1+t)^{\frac{n}{2}(1-\frac{1}{p})+\frac{|\alpha|+1}{2}} \big\|\partial^\alpha_x V\big\|_{L^p} + \sup\limits_{0\leqslant s\leqslant t,\,|\alpha|=k,\,k-1} (1+t)^{\frac{n}{4}+\frac{|\alpha|+1}{2}} \big\|\partial^\alpha_x V\big\|_{L^2}, \\ \mathcal{M}_U = \sup\limits_{0\leqslant s\leqslant t,\,|\alpha|\leqslant k-2} (1+t)^{\frac{n}{2}(1-\frac{1}{p})+\frac{|\alpha|+2}{2}} \big\|\partial^\alpha_x U\big\|_{L^p} + \sup\limits_{0\leqslant s\leqslant t,\,|\alpha|=k,k-1} (1+t)^{\frac{n}{4}+\frac{k+1}{2}} \big\|\partial^\alpha_x U\big\|_{L^2}, \end{cases} \tag{4.6}$$

and

$$\mathcal{M} = \mathcal{M}_V + \mathcal{M}_U. \tag{4.7}$$

Next we estimate each term in F defined in (2.11). Recall

$$F(y,s) = \sum_i \left(F^i\right)_{y_i} + \sum_{i,j} \left(F^{ij}\right)_{y_i y_j},$$

where

$$\begin{gathered} F^1 = -(\bar{\rho}_s \delta_s) - (\bar{\rho}\bar{u}_1\delta_s)_{y_1}, \qquad F^i = -P(\bar{\rho})_{y_i}, \quad 2 \leqslant i \leqslant n, \\ F^{11} = \left(\bar{\rho}\left(2\bar{u}_1 U_1 + U_1^2\right) + V(\bar{u}_1 + U_1)^2\right) + \left(\mathcal{P}_1(\bar{\rho}, V)V^2\right), \\ F^{1i} = F^{i1} = 2\left((\bar{\rho} + V)(\bar{u}_1 + U_1)U_i\right), \quad 2 \leqslant i \leqslant n, \\ F^{ij} = (\bar{\rho} + V)U_i U_j + \delta_{ij}\left(\mathcal{P}_1(\bar{\rho}, V)V^2\right), \quad 2 \leqslant i, j \leqslant n. \end{gathered}$$

It is easy to check that for any multi-indices α and γ,

$$\begin{cases} \left\| \partial_y^\alpha F^i \right\|_{L^p(\mathbb{R}_y^n)} \leqslant C E_\rho e^{-\kappa s}, \quad |\alpha| \leqslant k, \\ \left\| \partial_y^\gamma F^{ij} \right\|_{L^p(\mathbb{R}_y^n)} \leqslant C \mathcal{M}^2 (1+s)^{-(n+1+\frac{|\gamma|}{2}) + \frac{n}{2p}}, \quad |\gamma| \leqslant k-2. \end{cases} \tag{4.8}$$

We are now ready to estimate I_3^α. For $|\gamma| \leqslant k-2$,

$$\begin{aligned} \left\| I_3^\gamma \right\|_{L^p(\mathbb{R}_x^n)} &\leqslant \int_0^t \left\| \int_{\mathbb{R}^n} \partial_x^\gamma G_L F \, dy \right\|_{L^p(\mathbb{R}_x^n)} ds \\ &= \int_0^{\frac{t}{2}} \left\| \int_{\mathbb{R}^n} \left(-\sum \partial_x^\gamma \partial_{y_i} G_L F^i + \sum \partial_x^\gamma \partial_{y_i y_j} G_L F^{ij} \right) dy \right\|_{L^p(\mathbb{R}_x^n)} ds + \int_{\frac{t}{2}}^t \left\| \int_{\mathbb{R}^n} \partial_x^\gamma G_L F \, dy \right\|_{L^p(\mathbb{R}_x^n)} ds \\ &\leqslant \int_0^{\frac{t}{2}} (1+t-s)^{-\frac{n}{2}(1-\frac{1}{p}) - \frac{|\gamma|+1}{2}} E_\rho e^{-\kappa s} \, ds + \int_{\frac{t}{2}}^t (1+t-s)^{-\frac{n}{2}(1-\frac{1}{p}) - \frac{|\gamma|+1}{2}} E_\rho e^{-\kappa s} \, ds \\ &\quad + \int_0^{\frac{t}{2}} (1+t-s)^{-\frac{n}{2}(1-\frac{1}{p}) - \frac{|\gamma|+2}{2}} \mathcal{M}^2 (1+s)^{-(n+1) + \frac{n}{2}} \, ds \\ &\quad + \int_{\frac{t}{2}}^t (1+t-s)^{-1} \mathcal{M}^2 (1+s)^{-(n+1+\frac{|\gamma|}{2}) + \frac{n}{2p}} \, ds \\ &\leqslant C\left(E_\rho + \mathcal{M}^2\right)(1+t)^{-\frac{n}{2}(1-\frac{1}{p}) - \frac{|\gamma|+1}{2}}, \end{aligned}$$

where E_ρ is defined in (2.14). Note that the first inequality above follows from Lemma 2.3.

The cases when $|\gamma| = k-1$ and k can be estimated similarly, and the only difference is the estimation on the terms like

$$\int_{\frac{t}{2}}^{t} \left\| \int_{\mathbb{R}^n} \partial_x^\gamma \partial_{y_i y_j} G_L F^{ij}\, dy \right\|_{L^p(\mathbb{R}^n_x)} ds, \quad |\gamma| \leqslant k-2.$$

On the other hand, these terms can be estimated by replacing the derivatives of G_L w.r.t. x to the derivatives of G_L w.r.t. y using (3.13). Then by using integration by parts $k-2$ times to transfer the derivatives on G_L to F^{ij}, we have by (4.8) and Proposition 3.3 that

$$\begin{aligned}
\int_{\frac{t}{2}}^{t} \left\| \int_{\mathbb{R}^n} \partial_x^\gamma G_L \partial_{y_i y_j} F^{ij}\, dy \right\|_{L^p(\mathbb{R}^n_x)} ds &\leqslant C\mathcal{M}^2 \int_{\frac{t}{2}}^{t} (1+t-s)^{-\frac{|\gamma|+2-(k-2)}{2}} (1+s)^{-(n+1+\frac{k-2}{2})+\frac{n}{2p}}\, ds \\
&\leqslant C\mathcal{M}^2 (1+t)^{-(n+1+\frac{k-2}{2})+\frac{n}{2p}} \\
&\leqslant C\mathcal{M}^2 (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}}.
\end{aligned}$$

Therefore, we have the L^p estimate on I_3^α as

$$\left\| I_3^\alpha \right\|_{L^p(\mathbb{R}^n_x)} \leqslant C\mathcal{M}^2 (1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}. \tag{4.9}$$

We now turn to estimate the term I_4^α which is the error coming from the approximate Green function. For illustration, we only consider

$$J_1^\alpha = \int_0^t \int_{\mathbb{R}^n} \partial_x^\alpha G^\sharp_{L;0}\big(a(y,\sigma), x-y, t-s\big) a_s(y,\sigma) V(y,s)\, dy\, ds,$$

and

$$J_2^\alpha = \int_0^t \int_{\mathbb{R}^n} \partial_x^\alpha R^2_{G_L}(x,t;y,s) V(y,s)\, dy\, ds,$$

because the other terms in I_4^α can be estimated similarly. Notice that (3.10) gives

$$\left| a_s(y,\sigma) \right| \leqslant C E_\rho (1+t)^{-1}.$$

Similar to the estimation on I_3^α, we have, for $|\gamma| \leqslant k-2$,

$$\begin{aligned}
\left\| J_1^\gamma \right\|_{L^p(\mathbb{R}^n_x)} &\leqslant \int_0^t \left\| \int_{\mathbb{R}^n} \partial_x^\gamma G^\sharp_{L;0}\big(a(y,\sigma), x-y, t-s\big) a_s(y,\sigma) V(y,s)\, dy \right\|_{L^p(\mathbb{R}^n_x)} ds \\
&\leqslant \int_0^{t/2} (1+t-s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|}{2}} E_\rho \mathcal{M} (1+t)^{-1} (1+s)^{-\frac{n}{2}(1-\frac{1}{1})-1/2}\, ds
\end{aligned}$$

$$+\int_{t/2}^{t}(1+t-s)^{-\frac{n}{2}(1-\frac{1}{1})}E_\rho\mathcal{M}(1+t)^{-1}(1+s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}}\,ds$$

$$\leqslant CE_\rho\mathcal{M}(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}}, \tag{4.10}$$

and for $|\gamma|=k-1$ and k,

$$\|J_1^\gamma\|_{L^p(\mathbb{R}_x^n)}\leqslant E_\rho\mathcal{M}\int_0^{t/2}(1+t-s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|}{2}}(1+t)^{-1}(1+s)^{-\frac{n}{2}(1-\frac{1}{1})-1/2}\,ds$$

$$+E_\rho\mathcal{M}\int_{t/2}^{t}(1+t-s)^{-\frac{|\gamma|+2-k}{2}}E(1+t)^{-1}(1+s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{k-2+1}{2}}\,ds$$

$$\leqslant CE_\rho\mathcal{M}(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}}. \tag{4.11}$$

For J_2^α, since

$$|a(y,s)-a(y,\sigma)|\leqslant\int_s^\sigma|a_\tau(y,\tau)|\,d\tau\leqslant\begin{cases}CE_\rho\Theta(t,s), & s<t/2,\\ 0, & s\geqslant t/2,\end{cases}$$

where

$$\Theta(t,s)=(1+t-s)(1+t)^{-1+1/h}(1+s)^{-1/h},$$

and h can be any positive integer. By using Lemma 2.3 and Proposition 3.3, we have

$$\|J_2^\alpha\|_{L^p(\mathbb{R}_x^n)}\leqslant CE_\rho\mathcal{M}\int_0^{t/2}(1+t-s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+2}{2}}\Theta(t,s)(1+s)^{-1/2}\,ds.$$

By noticing that

$$\int_0^{t/2}(1+t)^{-1+1/h}(1+s)^{-1/h}(1+s)^{-1/2}\,ds=(1+t)^{-1+1/h}(1+s)^{\frac{1}{2}-\frac{1}{h}}\Big|_0^{t/2}$$

$$\leqslant C(1+t)^{-1+1/h}(1+t)^{\frac{1}{2}-\frac{1}{h}}=C(1+t)^{-\frac{1}{2}},$$

we obtain

$$\|J_2^\alpha\|_{L^p(\mathbb{R}_x^n)}\leqslant CE_\rho\mathcal{M}(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}. \tag{4.12}$$

Thus, we have the following estimate on I_4^α from (4.10), (4.11) and (4.12)

$$\|I_4^\alpha\|_{L^p(\mathbb{R}_x^n)}\leqslant CE_\rho\mathcal{M}(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}. \tag{4.13}$$

In summary, by (4.4), (4.5), (4.9) and (4.13), we have

$$\begin{aligned}\left\|\partial^\alpha V_L(t)\right\|_{L^p} \leqslant{}& C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+2}{2}}\|V_0\|_{L^1}\\ &+C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}\big(\|\tilde{v}_0\|_{L^1}+\mathcal{M}^2\big)\\ &+CE_\rho\mathcal{M}(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}.\end{aligned}$$

We now summarize the L^p estimates on the low frequency component in the following theorem.

Theorem 4.1. *For $|\alpha|\leqslant k$, we have*

$$\left\|\partial^\alpha V_L(t)\right\|_{L^p}\leqslant C\big(E_0+\mathcal{M}^2\big)(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}}, \tag{4.14}$$

where $E_0=\max\{\|V_0\|_{L^1},\|\tilde{v}_0\|_{L^1},\|(V_0,U_0)\|_{H^k},\|V_t(0)\|_{H^{k-1}},E_\rho\}$.

As an immediate consequence, we have the L^2 estimate on the derivatives of order higher than kth for the low frequency component because

$$\left\|\partial_{x_i}\partial^\alpha V_L(t)\right\|_{L^2}=\left\|\xi_i\xi^\alpha\chi(\xi)\hat{V}\right\|_{L^2}\leqslant\varepsilon\left\|\xi^\alpha\chi(\xi)\hat{V}\right\|_{L^2}=\varepsilon\left\|\partial^\alpha V_L(t)\right\|_{L^2}.$$

Thus, we have the following corollary.

Corollary 4.1. *For any $|\gamma|>k$, we have*

$$\left\|\partial^\gamma V_L(t)\right\|_{L^2}\leqslant C\big(E_0+\mathcal{M}^2\big)\varepsilon(1+t)^{-\frac{n}{2}(1-\frac{1}{2})-\frac{k+1}{2}}. \tag{4.15}$$

5. Estimates on the high frequency component

In this section, we will carry out the energy estimates on the high frequency component. Recall the linearized equation (2.10),

$$V_{tt}-\Delta\big(a(x,t)V\big)+\kappa V_t=F(V,U,\bar{\rho},\bar{m}). \tag{5.1}$$

Set $\tilde{\chi}(\xi)=1-\chi(\xi)$ and $V_H(x,t)=\tilde{\chi}(D_x)V(x,t)$. By taking $\tilde{\chi}(D_x)$ on both sides of (5.1) and integrating its product with V_H and $(V_H)_t$ over $\mathbb{R}^n$ respectively, we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}V_H(V_H)_t\,dx-\int_{\mathbb{R}^n}\big((V_H)_t\big)^2\,dx-\int_{\mathbb{R}^n}V_H\Delta\tilde{\chi}(aV)\,dx+\frac{d}{dt}\int_{\mathbb{R}^n}\frac{\kappa}{2}(V_H)^2\,dx=\int_{\mathbb{R}^n}V_H\tilde{\chi}F\,dx, \tag{5.2}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^n}\frac{1}{2}\big((V_H)_t\big)^2\,dx-\int_{\mathbb{R}^n}(V_H)_t\Delta\tilde{\chi}(aV)\,dx+\int_{\mathbb{R}^n}\kappa\big((V_H)_t\big)^2\,dx=\int_{\mathbb{R}^n}(V_H)_t\tilde{\chi}F\,dx. \tag{5.3}$$

We first estimate the third term on the left-hand side of (5.2) as follows. That is,

$$-\int_{\mathbb{R}^n}V_H\Delta\tilde{\chi}(aV)\,dx=\int_{\mathbb{R}^n}a|\nabla V_H|^2\,dx-\int_{\mathbb{R}^n}V_H\nabla[\nabla\tilde{\chi},a]V\,dx,$$

where $[A,B]=A\circ B-B\circ A$ denotes the commutator. Since

$$(1+t)^{1/2}\|\nabla a\|_{L^\infty}+(1+t)\|\Delta a\|_{L^\infty}\leqslant CE_\rho, \tag{5.4}$$

where E_ρ is defined in (2.14). It is straightforward to show that

$$\int_{\mathbb{R}^n} \left|\nabla[\nabla\tilde{\chi}, a]V\right|^2 dx \leqslant CE_\rho^2\mathcal{M}^2(1+t)^{-\frac{n}{2}-3},$$

where $\mathcal{M}$ is defined in (4.7). Thus

$$\left|\int_{\mathbb{R}^n} V_H\nabla[\nabla\tilde{\chi}, a]V\,dx\right| \leqslant \eta\int_{\mathbb{R}^n} |V_H|^2\,dx + CE_\rho^2\mathcal{M}^2(1+t)^{-\frac{n}{2}-3}. \tag{5.5}$$

We now turn to estimate the second term on the left-hand side in (5.3). That is,

$$\begin{aligned}
-\int_{\mathbb{R}^n} (V_H)_t\Delta\tilde{\chi}(aV)\,dx &= \int_{\mathbb{R}^n} (\nabla V_H)_t a\nabla\tilde{\chi}(V_H)\,dx + \int_{\mathbb{R}^n} (\nabla V_H)_t[\nabla\tilde{\chi}, a]V\,dx \\
&= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} a|\nabla V_H|^2\,dx - \frac{1}{2}\int_{\mathbb{R}^n} a_t|\nabla V_H|^2\,dx - \int_{\mathbb{R}^n} (V_H)_t\nabla[\nabla\tilde{\chi}, a]V\,dx,
\end{aligned} \tag{5.6}$$

and

$$\left|\int_{\mathbb{R}^n} a_t\left|\nabla(V_H)\right|^2 dx\right| \leqslant CE_\rho^2(1+t)^{-2}\int_{\mathbb{R}^n} \left|\nabla(V_H)\right|^2 dx,$$

$$\left|\int_{\mathbb{R}^n} (V_H)_t\nabla[\nabla\tilde{\chi}, a]V\,dx\right| \leqslant \eta\int_{\mathbb{R}^n} \left|(V_H)_t\right|^2 dx + CE_\rho^2\mathcal{M}^2(1+t)^{-\frac{n}{2}-3}. \tag{5.7}$$

For $\int_{\mathbb{R}^n} V_H\tilde{\chi}F\,dx$ and $\int_{\mathbb{R}^n} (V_H)_t\tilde{\chi}F\,dx$ on the right-hand side of (5.2) and (5.3), by using Lemma 2.1 and the definition of $\mathcal{M}$ in (4.7), we have

$$\begin{gathered}
\left\|F^i\right\|_{L^2} \leqslant CE_\rho e^{-\kappa s}, \\
\left\|\partial^\gamma F^{ij}\right\|_{L^2} \leqslant C\mathcal{M}^2(1+t)^{-(n+1+\frac{|\gamma|}{2})+\frac{n}{4}}, \quad |\gamma| \leqslant k-2,
\end{gathered}$$

where F and F^i, F^{ij} defined in (2.11). It is straightforward to check that

$$\left|\int_{\mathbb{R}^n} V_H\tilde{\chi}F\,dx\right| \leqslant \eta\int_{\mathbb{R}^n} |V_H|^2\,dx + C(\eta)\left(E_\rho^2 + \mathcal{M}^4\right)\left(e^{-\kappa t} + (1+t)^{-2(n+1)+\frac{n}{2}}\right), \tag{5.8}$$

and

$$\left|\int_{\mathbb{R}^n} (V_H)_t\tilde{\chi}F\,dx\right| \leqslant \eta\int_{\mathbb{R}^n} \left|(V_H)_t\right|^2 dx + C(\eta)\left(E_\rho^2 + \mathcal{M}^4\right)\left(e^{-\kappa t} + (1+t)^{-2(n+1)+\frac{n}{2}}\right). \tag{5.9}$$

To close the energy estimate, one needs the following important fact about the high frequency component:

$$\int_{\mathbb{R}^n} |\nabla V_H|^2\,dx \geqslant \varepsilon\int_{\mathbb{R}^n} |V_H|^2\,dx. \tag{5.10}$$

This is a Poincaré-type inequality which holds only for the high frequency component in the whole space. By integrating (5.2) and (5.3) over $[0,t]$ and multiplying (5.2) by some suitably chosen constant $0<\lambda<1$, when η is small, the combination of (5.2)–(5.10) gives

$$\int_{\mathbb{R}^n} \big(|V_H|^2+\big|(V_H)_t\big|^2+|\nabla V_H|^2\big)(t)\,dx+\mu\int_0^t\int_{\mathbb{R}^n} \big(|V_H|^2+\big|(V_H)_s\big|^2+|\nabla V_H|^2\big)\,dx\,ds$$
$$\leqslant C\left(\int_{\mathbb{R}^n} \big(|V_H|^2+\big|(V_H)_t\big|^2+|\nabla V_H|^2\big)(0)\,dx+\big(E_\rho^2+\mathcal{M}^4\big)\int_0^t (1+s)^{-\frac{n}{2}-3}\,ds\right), \tag{5.11}$$

for some positive μ. Denote

$$\mathcal{F}(t)=\int_{\mathbb{R}^n} \big(|V_H|^2+\big|(V_H)_t\big|^2+|\nabla V_H|^2\big)\,dx.$$

Then the inequality (5.11) gives

$$\mathcal{F}(t)+\mu\int_0^t \mathcal{F}(s)\,ds\leqslant C\left(\mathcal{F}(0)+\big(E_\rho^2+\mathcal{M}^4\big)\int_0^t (1+s)^{-\frac{n}{2}-3}\,ds\right). \tag{5.12}$$

By using the Gronwall inequality, we have

$$\mathcal{F}(t)\leqslant Ce^{-\mu t}\left(\mathcal{F}(0)+\big(E_\rho^2+\mathcal{M}^4\big)\int_0^t e^{\mu s}(1+s)^{-\frac{n}{2}-3}\,ds\right).$$

Hence, we have

$$\big\|V_H(t)\big\|_{H^1}^2+\big\|(V_H)_t(t)\big\|_{L^2}^2\leqslant e^{-\mu t}\big(\big\|V_H(0)\big\|_{H^1}+\big\|(V_H)_t(0)\big\|_{L^2}\big)+C\big(E_\rho^2+\mathcal{M}^4\big)(1+t)^{-\frac{n}{2}-3}$$
$$\leqslant C\big(E_0^2+\mathcal{M}^4\big)(1+t)^{-\frac{n}{2}-3}, \tag{5.13}$$

where E_0 is defined in (4.14).

Next, we will derive the energy estimates on the higher order derivatives of the high frequency component, that is, $\int_{\mathbb{R}^n}|\partial^\alpha V_H|^2+|\partial^\alpha(V_H)_t|^2+|\nabla\partial^\alpha V_H|^2\,dx$ for $0<|\alpha|\leqslant k-1$. In the rest of this section, we assume $0<|\alpha|\leqslant k-1$. The estimation can be obtained by induction on $|\alpha|$. Assume that

$$\int_{\mathbb{R}^n} \big(\big|\partial^\gamma V_H\big|^2+\big|\partial^\gamma(V_H)_t\big|^2+\big|\nabla\partial^\gamma V_H\big|^2\big)\,dx\leqslant C\big(E_0^2+\mathcal{M}^4\big)(1+t)^{-\frac{n}{2}-(|\gamma|+3)} \tag{5.14}$$

holds for any multi-index γ with $|\gamma|<|\alpha|$, we want to prove

$$\int_{\mathbb{R}^n} \big(\big|\partial^\alpha V_H\big|^2+\big|\partial^\alpha(V_H)_t\big|^2+\big|\nabla\partial^\alpha V_H\big|^2\big)\,dx\leqslant C\big(E_0^2+\mathcal{M}^4\big)(1+t)^{-\frac{n}{2}-(|\alpha|+3)}. \tag{5.15}$$

Taking $\partial^\alpha\tilde{\chi}$ on (5.1) and integrating its product with $\partial^\alpha V_H$ and $\partial^\alpha(V_H)_t$ over $\mathbb{R}^n$ respectively, we have

$$\frac{d}{dt}\int_{\mathbb{R}^n} \partial^\alpha V_H \partial^\alpha (V_H)_t\,dx - \int_{\mathbb{R}^n} \left|\partial^\alpha (V_H)_t\right|^2 dx - \int_{\mathbb{R}^n} \partial^\alpha V_H \Delta \partial^\alpha \tilde{\chi}(aV)\,dx + \frac{d}{dt}\int_{\mathbb{R}^n} \frac{\kappa}{2}\left|\partial^\alpha V_H\right|^2 dx$$
$$= \int_{\mathbb{R}^n} \partial^\alpha V_H \tilde{\chi} \partial^\alpha F\,dx, \tag{5.16}$$

and

$$\frac{d}{dt}\int_{\mathbb{R}^n} \frac{1}{2}\left|\partial^\alpha (V_H)_t\right|^2 dx - \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \Delta \tilde{\chi} \partial^\alpha (aV)\,dx + \int_{\mathbb{R}^n} \kappa\left|\partial^\alpha (V_H)_t\right|^2 dx$$
$$= \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \partial^\alpha F\,dx. \tag{5.17}$$

For the third term on the left-hand side of (5.16), we have

$$-\int_{\mathbb{R}^n} \partial^\alpha V_H \Delta \tilde{\chi} \partial^\alpha (aV)\,dx = \int_{\mathbb{R}^n} a\left(\partial^\alpha \nabla V_H\right)^2 dx - \int_{\mathbb{R}^n} \partial^\alpha V_H \nabla\left[\nabla \tilde{\chi} \partial^\alpha, a\right] V\,dx. \tag{5.18}$$

Since

$$\left\|\partial_x^\beta a\right\|_{L^\infty} \leqslant C E_\rho (1+t)^{-|\beta|/2},$$

it holds that

$$\left|\int_{\mathbb{R}^n} \partial^\alpha V_H \nabla\left[\nabla \tilde{\chi} \partial^\alpha, a\right] V\,dx\right| \leqslant \eta \int_{\mathbb{R}^n} \left|\partial^\alpha V_H\right|^2 dx + C_\eta E_\rho^2 \mathcal{M}^2 (1+t)^{-\frac{n}{2}-3-|\alpha|}. \tag{5.19}$$

Similarly, for the second term on the left-hand side of (5.17), we have

$$-\int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \Delta \tilde{\chi} \partial^\alpha (aV)\,dx$$
$$= \frac{d}{dt}\int_{\mathbb{R}^n} \frac{a}{2}\left|\nabla \partial^\alpha V_H\right|^2 dx - \int_{\mathbb{R}^n} \frac{a_t}{2}\left|\nabla \partial^\alpha V_H\right|^2 dx - \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \nabla\left[\nabla \partial^\alpha \tilde{\chi}, a\right] V\,dx, \tag{5.20}$$

where

$$\left|\int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \nabla\left[\nabla \tilde{\chi} \partial^\alpha, a\right] V\,dx\right| \leqslant \eta \int_{\mathbb{R}^n} \left|\partial^\alpha (V_H)_t\right|^2 dx + C_\eta E_\rho^2 \mathcal{M}^2 (1+t)^{-\frac{n}{2}-3-|\alpha|}. \tag{5.21}$$

For the terms $\int_{\mathbb{R}^n} \partial^\alpha V_H \tilde{\chi} \partial^\alpha F\,dx$ and $\int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \partial^\alpha F\,dx$ on the right-hand side of (5.16) and (5.17), we only estimate $\int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \tilde{\chi} \partial^\alpha F\,dx$ because the estimation on $\int_{\mathbb{R}^n} \partial^\alpha V_H \tilde{\chi} \partial^\alpha F\,dx$ is easier. Notice that the estimation on the terms with derivatives of order less or equal to $|\alpha|+1$ follows directly from the definition of $\mathcal{M}$ in (4.7). Thus, we consider the terms with derivatives of order higher than $|\alpha|+1$. Firstly, by using the expression (2.10) for F, we have

$$\begin{aligned} F &= \tilde{Q} + \Delta\big(\mathcal{P}_1(\bar\rho, V)V^2\big) \\ &= \big[(R_\rho)_t + \kappa(R_\rho)\big] - (\kappa + \partial_t)(V\bar u_1)_{x_1} - \operatorname{div}\big((\bar\rho + V)_t U\big) \\ &\quad - \operatorname{div}\big((\bar\rho + V)H\big) + \Delta\big(\mathcal{P}_1(\bar\rho, V)V^2\big). \end{aligned} \tag{5.22}$$

Since (2.3) implies

$$\operatorname{div} U = -(\bar\rho + V)^{-1}\big(V_t + (\bar u + U)\cdot\nabla V + (U\cdot\nabla)\bar\rho + V\operatorname{div}\bar u - R_\rho\big), \tag{5.23}$$

substituting (5.23) in (5.22), by the definition of H defined in (2.7), we have

$$F = (\bar u + U)\cdot\nabla\big((\bar u + U)\cdot\nabla V\big) + \Delta\big(\mathcal{P}_1(\bar\rho, V)V^2\big) + \mathcal{R},$$

where $\mathcal{R}$ denotes the remainder which contains derivatives of U and V with order at most 1. Thus, $\partial^\alpha\mathcal{R}$ has derivatives with order at most $|\alpha| + 1 (\leqslant k)$. Then

$$\int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \tilde\chi \partial^\alpha F\,dx = N_1 + N_2 + N_3, \tag{5.24}$$

with

$$N_1 = \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \tilde\chi \partial^\alpha\big((\bar u + U)\cdot\nabla\big((\bar u + U)\cdot\nabla V\big)\big)\,dx,$$

$$N_2 = \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \tilde\chi \partial^\alpha \Delta\big(\mathcal{P}_1(\bar\rho, V)V^2\big)\,dx,$$

$$N_3 = \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \partial^\alpha \mathcal{R}\,dx.$$

For N_1, we have

$$\begin{aligned} N_1 &= \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \tilde\chi (\bar u + U)\cdot\nabla\big((\bar u + U)\cdot\nabla\partial^\alpha V\big)\,dx + \{\cdots\} \\ &= \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t (\bar u + U)\cdot\nabla\big((\bar u + U)\cdot\nabla\partial^\alpha V_H\big)\,dx \\ &\quad + \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \big[\big((\bar u + U)\cdot\nabla\big)^2, \tilde\chi\big]\partial^\alpha V\,dx + \{\cdots\} \\ &= -\frac{d}{dt}\int_{\mathbb{R}^n} \frac{1}{2}\big|(\bar u + U)\cdot\nabla\partial^\alpha(V_H)\big|^2\,dx \\ &\quad - \int_{\mathbb{R}^n} (\bar u + U)_t\cdot\nabla\partial^\alpha(V_H)(\bar u + U)\cdot\nabla\partial^\alpha(V_H)\,dx \\ &\quad - \int_{\mathbb{R}^n} \nabla(\bar u + U)\cdot\nabla\partial^\alpha(V_H)_t(\bar u + U)\cdot\nabla\partial^\alpha(V_H)\,dx \end{aligned}$$

$$+\int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \big[\big((\bar{u}+U)\cdot\nabla\big)^2, \tilde{\chi}\big]\partial^\alpha V\,dx + \{\cdots\}$$

$$=: -\frac{d}{dt}\int_{\mathbb{R}^n} \frac{1}{2}\big|(\bar{u}+U)\cdot\nabla\partial^\alpha (V_H)\big|^2\,dx + N_{1,1} + N_{1,2} + N_{1,3} + \{\cdots\}. \tag{5.25}$$

In the sequel of this section, we use $\{\cdots\}$ to denote the terms with derivatives of order at most $|\alpha|+1$. It is easy to see that

$$\big|N_{1,1}+N_{1,2}+N_{1,3}+\{\cdots\}\big| \leqslant C\big(E_0+\mathcal{M}^3\big)(1+t)^{-\frac{n}{2}-(|\alpha|+3)}.$$

Then for N_2, we have

$$\begin{aligned}
N_2 &= \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \tilde{\chi}\partial^\alpha \Delta\big(\mathcal{P}(\bar{\rho},V)V^2\big)\,dx \\
&= \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \tilde{\chi}\mathcal{P}'_V \partial^\alpha \Delta V\,V^2 + 2\partial^\alpha (V_H)_t \tilde{\chi}\mathcal{P}(\bar{\rho},V)V\partial^\alpha \Delta V\,dx + \{\cdots\} \\
&=: N_{2,1} + N_{2,2} + \{\cdots\}.
\end{aligned} \tag{5.26}$$

By noticing that

$$\begin{aligned}
N_{2,1} &= \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \tilde{\chi}\mathcal{P}'_V \partial^\alpha \Delta V\,V^2\,dx \\
&= \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \big(V^2\mathcal{P}'_V\big)\partial^\alpha \Delta V_H\,dx + \int_{\mathbb{R}^n} \partial^\alpha (V_H)_t \big[\tilde{\chi},\mathcal{P}'_V V^2\big]\partial^\alpha \Delta V_L\,dx \\
&= -\frac{d}{dt}\int_{\mathbb{R}^n} \big(\mathcal{P}'_V V^2\big)\big|\nabla\partial^\alpha V_H\big|^2\,dx + O_1,
\end{aligned}$$

with

$$O_1 \leqslant \eta\int_{\mathbb{R}^n} \big|\partial^\alpha (V_H)\big|^2\,dx + C_\eta\big(E_0+\mathcal{M}^3\big)(1+t)^{-\frac{n}{2}-(|\alpha|+3)}.$$

Similarly, we have

$$N_{2,2} = -2\frac{d}{dt}\int_{\mathbb{R}^n} (\mathcal{P}V)\big|\nabla\partial^\alpha V_H\big|^2\,dx + O_2, \tag{5.27}$$

with

$$O_2 \leqslant \eta\int_{\mathbb{R}^n} \big|\partial^\alpha (V_H)\big|^2\,dx + C_\eta\big(E_0+\mathcal{M}^3\big)(1+t)^{-\frac{n}{2}-(|\alpha|+3)}.$$

To close the energy estimate, we now use the fact that

$$\int_{\mathbb{R}^n} \big|\nabla_x\partial^\alpha V_H\big|^2\,dx \geqslant \epsilon\int_{\mathbb{R}^n} \big|\partial^\alpha V_H\big|^2\,dx. \tag{5.28}$$

By integrating (5.16) and (5.17) over $[0,t]$ and multiplying (5.16) by some suitably chosen constant $0<\lambda<1$, the combination of (5.17)–(5.28) gives (5.15). Therefore, we have the following estimates on the high frequency component.

Theorem 5.1. *Under the assumption of Theorem* 4.1, *we have, for* $|\alpha|\leqslant k-1$,

$$\left\|\partial^\alpha V_H\right\|_{H^1}+\left\|\partial^\alpha (V_H)_t\right\|_{L^2}\leqslant C\left(E_0+\mathcal{M}^{3/2}\right)(1+t)^{-\frac{n}{4}-\frac{|\alpha|+3}{2}}, \tag{5.29}$$

where E_0 *and* $\mathcal{M}$ *are defined in* (4.14) *and* (4.7) *respectively.*

6. Proof of Theorem 1.1

In the previous two sections, we obtain the following estimates on the low frequency component by using the approximate Green function and the high frequency component by using the energy method respectively,

$$\left\|\partial^\alpha V_L(t)\right\|_{L^p}\leqslant C\left(E_0+\mathcal{M}^2\right)(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}},\quad |\alpha|\leqslant k, \tag{6.1}$$

and

$$\left\|\partial^\alpha V_H\right\|_{H^1}+\left\|\partial^\alpha (V_H)_t\right\|_{L^2}\leqslant C\left(E_0+\mathcal{M}^{3/2}\right)(1+t)^{-\frac{n}{4}-\frac{|\alpha|+3}{2}},\quad |\alpha|\leqslant k-1. \tag{6.2}$$

It remains to combine (6.1) and (6.2) to close the a priori assumption (4.6).

Firstly, by taking $p=2$ in (6.1) and combining with (6.2), we have

$$\left\|\partial^\alpha V(t)\right\|_{L^2}\leqslant C\left(E_0+\mathcal{M}^{3/2}\right)(1+t)^{-\frac{n}{4}-\frac{|\alpha|+1}{2}},\quad |\alpha|\leqslant k. \tag{6.3}$$

Next, by using the Sobolev embedding theorem, from (6.2), we have, for $|\alpha|\leqslant k-2$,

$$\begin{aligned}\left\|\partial^\alpha V_H(t)\right\|_{L^\infty}&\leqslant\left\|\partial^\alpha V_H(t)\right\|_{H^2}\\&\leqslant\left\|\partial^\alpha V_H(t)\right\|_{L^2}+\left\|\nabla\partial^\alpha V_H(t)\right\|_{H^1}\\&\leqslant C\left(E_0+\mathcal{M}^{3/2}\right)(1+t)^{-\frac{n}{4}-\frac{|\alpha|+3}{2}}.\end{aligned} \tag{6.4}$$

Moreover, for $n=2,3$, it holds that $-\frac{n}{4}-\frac{|\alpha|+3}{2}\leqslant-\frac{n}{2}-\frac{|\alpha|+1}{2}$. Thus,

$$\left\|\partial^\alpha V_H(t)\right\|_{L^\infty}\leqslant C\left(E_0+\mathcal{M}^{3/2}\right)(1+t)^{-\frac{n}{2}-\frac{|\alpha|+1}{2}},\quad |\alpha|\leqslant k-2. \tag{6.5}$$

By Lemma 2.5, the interpolation of (6.2) and (6.5) leads to

$$\left\|\partial^\alpha V_H(t)\right\|_{L^p}\leqslant C\left(E_0+\mathcal{M}^{3/2}\right)(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}},\quad |\alpha|\leqslant k-2. \tag{6.6}$$

Combining (6.1) with (6.6) then gives

$$\left\|\partial^\alpha V(t)\right\|_{L^p}\leqslant C\left(E_0+\mathcal{M}^{3/2}\right)(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\alpha|+1}{2}},\quad |\alpha|\leqslant k-2. \tag{6.7}$$

Now, we turn to estimate U by using Eq. (2.12). Note that

$$U(x,t)=e^{-\kappa t}U(x,0)+\int_0^t e^{-\kappa(t-s)}\left(\bar\rho^{-1}\nabla(aV)+\bar H(V,U,\rho,\bar u_1)\right)(x,s)\,ds.$$

From (2.13) and the definition of $\mathcal{M}_V$ and $\mathcal{M}$ in (4.6) and (4.7) respectively, we have for $|\gamma| \leqslant k-3$,

$$\begin{cases} \left\|\partial^\gamma \bar{H}(s)\right\|_{L^p} \leqslant C\left(E_0+\mathcal{M}^2\right)(1+s)^{-(n+1+\frac{|\gamma|+1}{2})+\frac{n}{2p}}, \\ \left\|\partial^\gamma\left(\bar{\rho}^{-1}\nabla_x(aV)\right)(s)\right\|_{L^p} \leqslant C(E_\rho+\mathcal{M}_V)(1+s)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+2}{2}}. \end{cases}$$

Thus,

$$\left\|\partial^\gamma U(t)\right\|_{L^p} \leqslant C\left(E_\rho+\mathcal{M}_V+\mathcal{M}^2\right)(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+2}{2}}, \quad |\gamma| \leqslant k-3. \tag{6.8}$$

Similarly, for the L^2-norm with $|\gamma| = k-2, k-1$, we have

$$\left\|\partial^\gamma U(t)\right\|_{L^2} \leqslant C\left(E_0+\mathcal{M}_V+\mathcal{M}^2\right)(1+t)^{-\frac{n}{4}-\frac{|\gamma|+2}{2}}. \tag{6.9}$$

To close the a priori estimate when $|\gamma| = k$, by multiplying the second equation of (2.6) by $2\theta^2(t)\partial^\gamma(\partial^\gamma U_i)$, and summing them over i from 1 to n, the integration over $\mathbb{R}^n \times [0,t]$ yields

$$\begin{aligned} &\left\|\theta(t)\partial^\gamma U(t)\right\|_{L_2}^2 - \left\|\partial^\gamma U(0)\right\|_{L_2}^2 + 2\kappa\int_0^t \left\|\theta(s)\partial^\gamma U\right\|_{L_2}^2 \\ &\quad + 2\int_0^t\int_{\mathbb{R}^n} \theta^2(s)\left(\partial^\gamma\left[(\bar{\rho}+V)^{-1}\nabla_x(\mathcal{P}V)\right]\cdot\left(\partial^\gamma U\right)\right)dx\,ds \\ &= 2\int_0^t\int_{\mathbb{R}^n} \theta^2(s)\left(\partial^\gamma H\cdot\partial^\gamma U\right)dx\,ds, \end{aligned} \tag{6.10}$$

where $\theta^2(t) = (\Lambda+t)^{\frac{n}{2}+k+1}$ with a large positive constant Λ. We can write

$$\begin{aligned} &2\int_0^t\int_{\mathbb{R}^n} \theta^2(s)\left(\partial^\alpha\left[(\bar{\rho}+V)^{-1}\nabla_x(\mathcal{P}V)\right]\cdot\left(\partial^\alpha U\right)\right)dx\,ds \\ &\quad = -2\int_0^t\int_{\mathbb{R}^n} \theta^2(s)\left(\partial^\alpha\left[(\bar{\rho}+V)^{-1}(\mathcal{P}V)\right]\cdot\partial^\alpha(\operatorname{div}U)\right)dx\,ds + \{\cdots\}. \end{aligned}$$

Here and after, $\{\cdots\}$ denotes the terms which contain derivatives of U with order at most k. By applying (5.23) to the first term on the right-hand side in the above equation, we have

$$2\int_0^t\int_{\mathbb{R}^n} \theta^2(s)\left(\partial^\alpha\left[(\bar{\rho}+V)^{-1}\nabla(\mathcal{P}V)\right]\cdot\left(\partial^\alpha U\right)\right)dx\,ds = \left\|\theta(s)\left(h\partial^\alpha V\right)(s)\right\|_{L^2}^2\Big|_{s=0}^{s=t} + \{\cdots\},$$

where $h^2 = ((\bar{\rho}+V))^{-2}\mathcal{P}(\bar{\rho},V) > C_1 > 0$. By noticing $(\theta^2(s))_s \leqslant \frac{k}{\Lambda}\theta^2(s)$, we have

$$\begin{aligned} \left|\{\cdots\}\right| &\leqslant C\left(E_\rho^2+\mathcal{M}^3\right)\int_0^t (1+s)^{-n-|\alpha|-\frac{3}{2}}\theta^2(s)\,ds \\ &\leqslant C\left(E_\rho^2+\mathcal{M}^3\right). \end{aligned}$$

To illustrate the estimation on the right-hand side of (6.10), consider

$$\int_0^t \int_{\mathbb{R}^n} \theta^2(s)\partial^\alpha (U\cdot\nabla)U\cdot\partial^\alpha U\,dx\,ds = -\frac{1}{2}\int_0^t\int_{\mathbb{R}^n}\theta^2(s)\operatorname{div} U\left|\partial^\alpha U\right|^2 dx\,ds + \{\cdots\}.$$

Note that

$$\left|\int_0^t\int_{\mathbb{R}^n}\theta^2(s)\big(\partial^\alpha H\big)\cdot\big(\partial^\alpha U\big)\,dx\,ds\right| \leqslant C\big(E_\rho^2+\mathcal{M}^3\big).$$

We have

$$\theta^2(t)\left(\left\|\partial^\alpha U\right\|_{L^2}^2+\left\|\partial^\alpha V\right\|_{L^2}^2+\int_0^t\left\|\partial^\alpha U\right\|_{L^2}^2\right) \leqslant C\big(\left\|\partial^\alpha U_0\right\|_{L^2}^2+\left\|\partial^\alpha V(0)\right\|_{L^2}^2\big)+C\big(E_\rho+\mathcal{M}^3\big). \tag{6.11}$$

By combining (6.11) with (6.3) when $|\alpha|=k$, we obtain

$$\theta^2(t)\left(\left\|\partial^\alpha U\right\|_{L^2}^2+\left\|\partial^\alpha V\right\|_{L^2}^2+\int_0^t\left\|\partial^\alpha U\right\|_{L^2}^2 ds\right)\leqslant C\big(E_0^2+\mathcal{M}^3\big).$$

Thus

$$\left\|\partial^\alpha U\right\|_{L^2}^2 \leqslant C\big(E_0^2+\mathcal{M}^3\big)(1+t)^{-\frac{n}{2}-(|\alpha|+1)},\quad |\alpha|=k. \tag{6.12}$$

By the Sobolev embedding theorem, (6.9) and (6.12) yield

$$\left\|\partial^\gamma U\right\|_{L^\infty}\leqslant C\big(E_0^2+\mathcal{M}^{3/2}\big)(1+t)^{-\frac{n}{4}-\frac{|\gamma|+2}{2}},\quad |\gamma|=k-2. \tag{6.13}$$

By (6.9) for $|\gamma|=k-2$ and (6.13), the interpolation Lemma 2.5 shows that

$$\left\|\partial^\gamma U(t)\right\|_{L^p}\leqslant C\big(E_0^2+\mathcal{M}^{3/2}\big)(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+2}{2}},\quad |\gamma|=k-2. \tag{6.14}$$

Now, combining (6.3), (6.7)–(6.9), (6.12) and (6.14) gives

$$\mathcal{M}^2\leqslant C\big(E_0^2+\mathcal{M}^3\big).$$

Then if $E_0>0$ is chosen to be sufficiently small, by the continuity argument, we have $\mathcal{M}^2\leqslant CE_0^2$. Thus, it gives the following theorem.

Theorem 6.1. *Under the assumption of Theorem* 1.1, *if the initial data* (V_0,U_0) *satisfies that*

$$|\rho_+-\rho_-|+\left\|v(\cdot,0)\right\|_{L^2\cap L^1}+\left\|v_t(\cdot,0)\right\|_{L^2}+\|V_0\|_{H^k\cup L^1}+\left\|V_t(0)\right\|_{H^{k-1}}+\|U_0\|_{H^k}\leqslant\epsilon_0,$$

where $\epsilon_0 > 0$ *is a small constant, then there exists a unique global classical solution* $(V, U) \in C([0,\infty), H^k) \cap C^1((0,\infty), H^{k-1})$ *to* (2.6). *Moreover, we have*

$$\begin{cases} \left\|\partial_x^\gamma V\right\|_{L^p} \leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+1}{2}}, & |\gamma| \leqslant k-2, \\ \left\|\partial_x^\gamma V\right\|_{L^2} \leqslant C(1+t)^{-\frac{n}{4}-\frac{|\gamma|+1}{2}}, & |\gamma| = k-1, k, \\ \left\|\partial_x^\gamma U\right\|_{L^p} \leqslant C(1+t)^{-\frac{n}{2}(1-\frac{1}{p})-\frac{|\gamma|+2}{2}}, & |r| \leqslant k-2, \\ \left\|\partial_x^\gamma U\right\|_{L^2} \leqslant C(1+t)^{-\frac{n}{4}-\frac{k+1}{2}}, & |\gamma| = k-1, k. \end{cases} \tag{6.15}$$

This theorem implies (4.6) and (4.7), and then it yields the main result Theorem 1.1.

Before concluding this paper, we point out that even though the above discussion is for the space dimension $n = 2, 3$, higher-dimensional case can be considered similarly. We also would like to emphasize that the new frequency decomposition based energy method combined with Green function can be applied to the study of some hyperbolic–parabolic systems satisfying the Kawashima condition.

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