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GLOBAL SMOOTH SOLUTIONS FOR A CLASS OF QUASILINEAR HYPERBOLIC SYSTEMS WITH DISSIPATIVE TERMS

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abstract

In this paper the authors prove the existence and uniqueness of global smooth solutions to the Cauchy problem for quasilinear hyperbolic systems with some kinds of dissipative terms.

§ 1. Introduction

It is well known that for the Cauchy problem for first order quasilinear hyperbolic systems, in general singularities of solutions may appear in finite time, even if the initial data are very smooth and sufficiently small (cf. [1, 2]). However, T. Nishida proved that if the C^1 norm of the initial data is sufficiently small, then the Cauchy problem admits a unique global smooth solution on $t \geqslant 0$ for the first order quasilinear hyperbolic system in two independent variables, which is reduced from the quailinear wave equation with first order dissipative term (cf. [3]). Hsiao Ling and Li Ta-tsien discussed in [4] the following Cauchy problem for general first order quasilinear hyperbolic systems:

$$\begin{cases} \dfrac{\partial u}{\partial t}+G(u)\dfrac{\partial u}{\partial x}+f(u)=0, & (1)\\ u(x,\ 0)=u^0(x), & (2)\end{cases}$$

where $u=(u_1, \cdots, u_n)^T$, $G(u)$ and $f(u)$ are smooth matrix and vector functions of u respectively and

$$f(0)=0. \tag{3}$$

Here, the hyperbolicity of system (1) means that $G(u)$ has n real eigenvalues $\lambda_1(u)$, $\cdots$, $\lambda_n(u)$ and matrix

$$\zeta(u)=\begin{pmatrix}\zeta_1(u)\\ \vdots\\ \zeta_n(u)\end{pmatrix} \tag{4}$$

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is nonsingular, where $\zeta_l(u)=(\zeta_{l1}(u), \cdots, \zeta_{ln}(u))$ denote the left (row) eigenvectors corresponding to $\lambda_l(u)\,(l=1, \cdots, n)$. We write

$$A=(a_{lj})=\zeta(0)\nabla f(0)\zeta^{-1}(0). \tag{5}$$

It was pointed out in [4] that if matrix A is strictly row diagonal dominant, that is

$$a_{ll}>\sum_{j=1,\,j\neq l}^{n}|a_{lj}|,\ (l=1, \cdots, n), \tag{6}$$

then Cauchy problem (1) (2) admits a unique global smooth solution on $t\geqslant 0$, provided that the C^1 norm of the initial data is sufficiently small. As the authors pointed out in [4], their results do not cover Nishida's results because of assumption (6), therefore the results in [4] should be generalized. Moreover, noticing the condition of the global existence of discontinuous solutions for Cauchy prblems of quasilinear hyperbolic systems in [5], we expect naturally that Cauchy problem (1) (2) admits also a unique global smooth solution provided that A is strictly column diagonal dominant, that is

$$a_{ll}>\sum_{j=1,\,j\neq l}^{n}|a_{jl}|,\ (l=1, \cdots,n). \tag{7}$$

In the present paper we are going to discuss the related problems.

In § 2, under the assumption that all the diagonal elements of matrix A are positive, we give a sufficient condition which guarantees that there exists a diagonal matrix $\Gamma=\mathrm{diag}(\gamma_1, \cdots, \gamma_n)$, $(\gamma_i>0,\ i=1, \cdots, n)$ such that $\Gamma^{-1}A\Gamma$ is strictly row diagonal dominant. Moreover, if we assume that matrix A is weakly column (or row) diagonal dominant, then the above condition is also necessary. Thus, from the results in [4] it follows that if A is strictly column diagonal dominant, then Cauchy problem (1) (2) admits also a unique global smooth solution on $t\geqslant 0$, provided that the C^1 norm of the initial data is small.

In § 3, in order to extend T. Nishida's results to general cases, we consider the Cauchy problem for strictly hyperbolic systems of diagonal form

$$\begin{cases} \dfrac{\partial u}{\partial t}+A(u)\dfrac{\partial u}{\partial x}+f(u)=0, & (8)\\ u(x,\,0)=u^0(x), & (9)\end{cases}$$

where

$$\begin{aligned} &A(u)=\mathrm{diag}(\lambda_1(u), \cdots, \lambda_n(u)),\\ &f(u)=(f_1(u), \cdots, f_n(u))^T. \end{aligned} \tag{10}$$

Assuming

$$f(0)=0,\ \frac{\partial f_i}{\partial u_i}(0)>0 \quad (i=1, \cdots, n), \tag{11}$$

we'll prove that Cauchy problem (8) (9) with small initial data admits a unique global smooth solution on $t\geqslant 0$ provided that the C^0 norm of the solution is sufficiently small. In order to guarantee that the C^0 norm of the solution to

Cauchy problem (8) (9) is sufficiently small provided that the initial data are small, we should add some additional assumptions on $f(u)$. For some special cases which can cover the results in [3], for instance, $f(u)$ is a linear function of u and the coefficient matrix of $f(u)$ is weakly row (or column) diagonal dominant with equal positive diagonal elements, we can prove that, if the C^0 norm of initial data is sufficiently small, then the C^0 norm of the smooth solution to Cauchy problem (8) (9) is also sufficiently small. Therefore, in these cases, we obtain the existence of global smooth solutions for Cauchy problems with small initial data.

§ 2. The Case of General First Order Quasilinear Hyperbolic Systems

First of all, we introduce some definitions which will be used later on (cf. [7]).

Definition 1. *For $n\geqslant 2$, an $n\times n$ real matrix A is reducible if there exists an $n\times n$ permutation matrix P such that*

$$P^{T}AP=\begin{pmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{pmatrix}, \tag{12}$$

where A_{11} is an $r\times r$ submatrix and A_{22} is an $(n-r)\times(n-r)$ submatrix, where $1\leqslant r<n$. Otherwise, A is irreducible. For $n=1$, A is irreducible if its single element is nonzero, and reducible otherwise.

Definition 2. *An $n\times n$ real matrix $A=(a_{ij})$ is weakly row (column) diagonal dominant if*

$$\sum_{j=1,\,j\neq i}^{n}|a_{ij}|\leqslant a_{ii}\Big(\sum_{j=1,\,j\neq i}^{n}|a_{ji}|\leqslant a_{ii}\Big) \tag{13}$$

for $i=1, \cdots, n$. An $n\times n$ matrix A is strictly row (column) diagonal dominant if the strict inequality in (13) holds for $i=1, \cdots, n$. A is irreducible row (column) diagonal dominant if A is irreducible and weakly row (column) diagonal dominant with a strict inequality in (13) for at least one i.

Definition 3. *Let $A=(a_{ij})$ be an $n\times n$ real matrix. If $a_{ij}\geqslant 0\,(>0)$ for all $i, j=1, \cdots, n$, then A is called a non-negative (positive) matrix, denoted by $A\geqslant 0\,(>0)$.*

In this section we prove the following

Theorem 1. *Let $A=(a_{ij})$ be an $n\times n$ real matrix with positive diagonal elements. There exists a diagonal matrix $\Gamma=\mathrm{diag}(\gamma_1, \cdots, \gamma_n)$ with $\gamma_1, \cdots, \gamma_n>0$, such that $\Gamma^{-1}A\Gamma$ is strictly row diagonal dominant, if there exists a permutation matrix P such that*

$$F \triangleq P^T A P = \begin{pmatrix} F_{11} & F_{12} & \cdots F_{1r} \\ & F_{22} & \cdots F_{2r} \\ 0 & & \ddots \\ & & & F_{rr} \end{pmatrix}, \tag{14}$$

where all the square matrices $F_{ii}(i=1, \cdots, r)$ are irreducible column diagonal dominant. Moreover, if A is weakly column diagonal dominant, then the above sufficient condition is still necessary.

From Theorem 1 we get immediately the following

Corollary 1. 1. *If matrix A is strictly column diagonal dominant, then there must exist a diagonal matrix $\Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_n)$ with $\gamma_1, \cdots, \gamma_n > 0$, such* that $\Gamma^{-1} A \Gamma$ is strictly row diagonal dominant.

Similar to Theorem 1, we have

Theorem 1′. *Let $A=(a_{ij})$ be an $n\times n$ real matrix with positive diagonal elements. There exrsts a diagonal matrix $\Gamma=\mathrm{diag}(\gamma_1, \cdots, \gamma_n)$ with $\gamma_1, \cdots, \gamma_n>0$, such that $\Gamma^{-1} A\Gamma$ is strictly row diagonal dominant, if there exists a permutation matrix P such that* (14) *holds, where all the square matrices $F_{ii}(i=1, \cdots, r)$ are irreducible row diagonal dominant. Moreover, if A is weakly row diagonal dominant, then the above sufficient condition is still necessary.*

In order to prove Theorems 1 and 1′, we state and prove the following lemmas at first.

Lemma 1. *Let $H=(h_{ij})$ be an $n\times n$ real matrix with $h_{ij}\leqslant 0$ for $i\neq j$, then the following* 1° *and* 2° *are equivalent:*

1° *H is nonsingular and $H^{-1}\geqslant 0$;*

2° *All the eigenvalues of H have positive real parts.*

Lemma 2. *Let $H=(h_{ij})$ be an $n\times n$ real irreducible column (row) diagonal dominant matrix with positive diagonal elements, then all the eigenvalues of H have positive real parts.*

Lemma 3. *Let $H=(h_{ij})$ be an $n\times n$ real irreducible column (row) diagonal dominant matrix. If $h_{ij}\leqslant 0$ for $i\neq j$, and $h_{ii}>0$ for all $i=1, \cdots, n$, then H is nonsingular and $H^{-1}>0$.*

For the proof of the above three lemmas we refer to [7].

Lemma 4. *Let $F=(f_{ij})$ be an $n\times n$ non-negative matrix of form* (14) *with positive diagonal elements, and all $F_{ii}(i=1, \cdots, r)$ be irreducible column(row) diagonal dominant, then the matrix*

$$\widetilde{F} = \begin{pmatrix} f_{11} & -f_{12} & \cdots & -f_{1n} \\ -f_{21} & f_{22} & \cdots & -f_{2n} \\ \cdots & \cdots & \cdots & \\ -f_{n1} & -f_{n2} & \cdots & f_{nn} \end{pmatrix} = \begin{pmatrix} \widetilde{F}_{11} & -F_{12} & \cdots & -F_{1r} \\ & \widetilde{F}_{22} & \cdots & -F_{2r} \\ 0 & & \ddots & \vdots \\ & & & \widetilde{F}_{rr} \end{pmatrix} \tag{15}$$

is nonsingular. Moreover

$$\widetilde{F}^{-1}=(\widetilde{f}^{ij})\geqslant 0, \tag{16}$$

$$\widetilde{f}^{ii}>0(i=1, \cdots, n). \tag{17}$$

Proof Without loss of generality, we only discuss the case in which $F_{ii}(i=1, \cdots, r)$ are irreducible column diagonal dominant.

Since $\widetilde{F}_{ii}(i=1, \cdots, r)$ are irreducible column diagonal dominant matrices with positive diagonal elements, from Lemma 2 it follows that all the eigenvalues of $\widetilde{F}_{ii}$, then of $\widetilde{F}$, have positive real parts. Thus, by Lemma 1 we get that $\widetilde{F}$ is nonsingular and (16) holds.

By Lemma 3, we have

$$\widetilde{F}_{ii}^{-1}>0 \quad (i=1, \cdots, r).$$

Since F is a block upper tridiagonal matrix of form (14), it is easy to see that the diagonal elements consist of that of $\widetilde{F}_{ii}^{-1}(i=1, \cdots, r)$, and from this we can get (17). Lemma 4 is then proved.

The proof of Theorem 1.

Sufficiency First, we prove that under the given conditions, there must exist a diagonal matrix $D=\mathrm{diag}(d_1, \cdots, d_n)$ with positive diagonal elements such that

$$D^{-1}FD=\begin{pmatrix} f_{11} & f_{12}d_1^{-1}d_2 & \cdots & f_{1n}d_1^{-1}d_n \\ f_{21}d_2^{-1}d_1 & f_{22} & \cdots & f_{2n}d_2^{-1}d_n \\ & \cdots\cdots & & \\ f_{n1}d_n^{-1}d_1 & f_{n2}d_n^{-1}d_2 & \cdots & f_{nn} \end{pmatrix} \tag{18}$$

is a strictly row diagonal dominant marix, that is

$$f_{ii}>\sum_{\substack{j=1\\ j\neq i}}^{n}|f_{ij}|d_i^{-1}d_j, \quad i=1, \cdots, n \tag{19}$$

or

$$f_{ii}d_i-\sum_{\substack{j=1\\ j\neq i}}^{n}|f_{ij}|d_j>0, \ i=1, \cdots, n. \tag{20}$$

In fact, for any given $c_1, \cdots, c_n>0$, we consider the following linear algebraic system

$$f_{ii}d_i-\sum_{\substack{j=1\\ j\neq i}}^{n}|f_{ij}|d_j=c_i, \ i=1, \cdots, n, \tag{21}$$

the coefficient matrix of which is still denoted by $\widetilde{F}$, for the sake of simplicity. By Lemma 4, (16) and (17) hold. Therefore, system (21) admits a solution

$$d_i>0, \ i=1, \cdots, n$$

and the diagonal matrix $D=\mathrm{diag}(d_1, \cdots, d_n)$ is a desired one.

Let $\Gamma=PDP^T$, which is still a diagonal matrix with positive diagonal elements. Hence, from

$$D^{-1}FD=P^T\Gamma^{-1}A\Gamma P$$

it follows immediately that $\Gamma^{-1}A\Gamma$ is strictly row diagonal dominant. Thus, the sufficiency is proved.

Necessity Suppose that A is weakly column diagonal dominant and by means of a permutation matrix P, A can be transformed into a matrix of form(14), where all $F_{ii}(i=1, \cdots, r)$ are irreducible. If there exists a diagonal matrix

$$\Gamma=\mathrm{diag}(\gamma_1, \cdots, \gamma_n)$$

with positive diagonal elements such that $\Gamma^{-1}A\Gamma$ is strictly row diagonal dominant, then setting $D=P^T\Gamma P$, $D^{-1}FD$ must be strictly row diagonal dominant. Therefore, there exist $d_1, \cdots, d_n(>0)$ such that(20) holds.

Suppose that a submatrix F_{ll} is not irreducible column diagonal dominant. Set

$$F_{ll}=\begin{pmatrix} f_{s,s} & f_{s,s+1} & \cdots & f_{s,t} \\ f_{s+1,s} & f_{s+1,s+1} & \cdots & f_{s+1,t} \\ & \cdots\cdots & & \\ f_{t,s} & f_{t,s+1} & \cdots & f_{t,t} \end{pmatrix}, \quad s<t. \tag{22}$$

Clearly, F_{ll} is weakly column diagonal dominant and irreducible. By Definition 2 we have

$$f_{ii}-\sum_{j=s,j\neq i}^{t}|f_{ji}|=0, \quad s\leqslant i\leqslant t. \tag{23}$$

On the other hand, from(20) it follows that

$$f_{ii}d_i-\sum_{j=s,j\neq i}^{t}|f_{ij}|d_j>0, \quad s\leqslant i\leqslant t, \tag{24}$$

then

$$\sum_{i=s}^{t}\Big(f_{ii}-\sum_{j=s,j\neq i}^{t}|f_{ji}|\Big)d_i>0.$$

It contradicts (23). Thus, all $F_{ii}(i=1, \cdots, r)$ are irreducible column diagonal dominant. The proof of the necessity is then completed.

The proof of Theorem 1′ The proof of the sufficiency is just the same as in Theorem 1. Now we only discuss the necessity. Let A be weakly row diagonal dominant. If there exists a diagonal matrix Γ with positive diagonal elements such that $\Gamma^{-1}A\Gamma$ is strictly row diagonal dominant, then there must exist a diagonal matrix D of the same kind such that $D^{-1}FD$ is strictly row diagonal dominant, where F is still given by (14). If there is an irreducible submatrix F_{ll} of form (24) which is not irreducible row diagonal dominant, then instead of (23) we have

$$f_{ii}-\sum_{j=s,j\neq i}^{t}|f_{ij}|=0, \quad s\leqslant i\leqslant t. \tag{25}$$

However, in this case we still have (24). It is not difficult to show that (24) and (25) cannot simultaneously hold for $d_i>0$, $s\leqslant i\leqslant t$. The proof of Theorem 1′ is completed

Example For an $n\times n$ matrix $A=(a_{ij})$, if $a_{ij}\neq 0$ for all $i, j=1, \cdots, n$, and

$$a_{ii}=\sum_{j=1, j\neq i}^{n}|a_{ij}|, \text{ for } i=1, \cdots, n, \tag{26}$$

or

$$a_{ii}=\sum_{j=1, j\neq i}^{n}|a_{ji}|, \text{ for } i=1, \cdots, n, \tag{27}$$

then from Theorems 1 and 1′ it easily follows that there never exists any diagonal matrix Γ with positive diagonal elements such that $\Gamma^{-1}A\Gamma$ is strictly row diagonal dominant. The corresponding matrix A for the system discussed by T. Nishida in [3] belongs to this situation. In the next section, we will further discuss this situation for strictly hyperbolic systems of diagonal form.

According to Theorems 1 and 1′, from the result in [4], we get immediately thefollowing existence theorem of global smooth solutions for Cauchy problem (1) (2).

Theorem 2. *Let matrix A defined by (5) satisfies the following assumptiyns:*

$$a_{ii}>0, \ i=1, \cdots, n, \tag{28}$$

and there exists a permutation matrix P such that

$$P^TAP=\begin{pmatrix} F_{11} & F_{12} & \cdots & F_{1r} \\ & F_{22} & \cdots & F_{2r} \\ & & \ddots & \vdots \\ 0 & & & F_{rr} \end{pmatrix}, \tag{14}$$

where all $F_{ii}(i=1, \cdots, r)$ are irreducible column (or row) diagonal dominant matrices. Moreover, we assume that $A(u)$, $\zeta(u)\in C^1$, $f(u)\in C^2$, and (3) holds. Then Cauchy problem (1) (2) admits a unique global smooth solution u on $t\geqslant 0$, which decays exponentially in C^1 norm as $t\to+\infty$, provided that the C^1 norm of the initial data $u^0(x)$ is sufficiently small.

Using Theorem 2 and noticing Corollary 1.1 we immediately obtain the following

Corollary 2.1. *If matrix A defined by (5) is strictly column diagonal dominant, then the conclusion of Theorem 2 is true.*

Corollary 2. 2. *If matrix A is irreducible row (or column) diagonal dominant, then the conclusion of Theorem 2 is true.*

§ 3. The Case of Quasilinear Strictly Hyperbolic Systems of Diagonal Form

In this section, we consider Cauchy problem (8) (9). Assume

(i) System (8) is strictly hyperbolic, that is

$$\lambda_1(u)<\lambda_2(u)<\cdots<\lambda_n(u) \tag{29}$$

on the domain under consideration.

(ii)
$$f_i(0)=0,\quad \frac{\partial f_i}{\partial u_i}(0)>0,\ i=1,\cdots,n. \tag{30}$$

We have the following

Theorem 3. *Suppose that $A(u)$, $f(u)$, $u^0(x)\in C^1$ and assumptions* (i),(ii) *are satisfied. Suppose further that the smooth solution for Cauchy problem* (8)(9) *satisfies the following hypothesis*

(H): *For any given $\varepsilon>0$, there exists $\delta>0$ such that if $|u^0|_{C^0}\leqslant\delta$, then*

$$\sup_{t\geqslant0}|u(t)|_{C^0}\leqslant\varepsilon \tag{31}$$

on the domain where the classical solution exists.

Then Cauchy problem (8)(9) *admits a unique global smooth solution u on $t\geqslant 0$, provided $|u^0|_{C^1}$ is sufficiently small. Moreover, if $|u(t)|_{C^0}\leqslant D_0|u^0|_{C^0}$, then*

$$|u(t)|_{C^1}\leqslant D_1|u^0|_{C^1}\quad (t\geqslant0), \tag{32}$$

where

$$|u(t)|_{C^0}=\sup_{-\infty<x<+\infty}|u(t,x)|,$$

$$|u(t)|_{C^1}=|u(t)|_{C^0}+\left|\frac{\partial u}{\partial x}(t)\right|_{C^0},$$

and constants D_0 and D_1 are independent of t.

Proof According to the existence theorem of local classical solutions for first order quasilinear hyperbolic systems (see [6]), in order to get the existence of global smooth solutions on $t\geqslant0$, it is sufficient to prove that the first order derivatives with respect to x of the classical solution are bounded on the domain where the classical solution exists. For this aim, differentiating the first equation of system (8) with respect to x, we get

$$\frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial x}\right)+\lambda_1(u)\frac{\partial}{\partial x}\left(\frac{\partial u_1}{\partial x}\right)$$
$$=\left(-\frac{\partial f_1(u)}{\partial u_1}-\sum_{k=1}^{n}\frac{\partial\lambda_1(u)}{\partial u_k}\frac{\partial u_k}{\partial x}\right)\frac{\partial u_1}{\partial x}-\sum_{k=2}^{n}\frac{\partial f_1(u)}{\partial u_k}\frac{\partial u_k}{\partial x}. \tag{33}$$

From assumptions (ii) and (H) it follows that for any given positive number a ($a<1$ suitably small), thers exists $\varepsilon_1>0$ such that if $|u^0|_{C^0}<\varepsilon_1$, then the smooth solution u to Cauchy problem (8)(9) should satisfy

$$|u(t)|_{C^0}\leqslant a \tag{34}$$

and

$$\frac{\partial f_1(u)}{\partial u_1}\geqslant\beta>0, \tag{35}$$

where

$$\beta=\frac{1}{2}\min_{1\leqslant i\leqslant n}\left(\frac{\partial f_i(0)}{\partial u_i}\right)>0. \tag{36}$$

For the time being, we assume that on the domain where the smooth solution exists, it holds that

$$\left|\sum_{k=1}^{n} \frac{\partial \lambda_1(u)}{\partial u_k} \frac{\partial u_k}{\partial x}\right| \leqslant \frac{\beta}{2}. \tag{37}$$

Setting

$$N = -\frac{\partial f_1(u)}{\partial u_1} - \sum_{k=1}^{n} \frac{\partial \lambda_1(u)}{\partial u_k} \frac{\partial u_k}{\partial x}, \tag{38}$$

from (35) and (37) we get

$$N \leqslant -\frac{\beta}{2}. \tag{39}$$

Moreover, setting

$$\frac{d}{d_1 t} \equiv \frac{\partial}{\partial t} + \lambda_1(u) \frac{\partial}{\partial x},$$

from system (8) we obtain

$$\frac{du_k}{d_1 t} = -f_k(u) + (\lambda_1(u) - \lambda_k(u)) \frac{\partial u_k}{\partial x}, \quad k = 2, \cdots, n. \tag{40}$$

Thus, (33) and (40) give

$$\frac{d}{d_1 t}\left(\frac{\partial u_1}{\partial x}\right) = N \frac{\partial u_1}{\partial x} + \sum_{k=2}^{n} \frac{\partial f_1(u)}{\partial u_k} \cdot \frac{f_k(u) + \frac{du_k}{d_1 t}}{\lambda_k(u) - \lambda_1(u)}. \tag{41}$$

Let $x = x_1(t, \alpha)$ be the first characteristic curve passing through the point $(t, x) = (0, \alpha)$, then noticing initial condition (9), from (41) we get

$$\begin{aligned} &\frac{\partial u_1}{\partial x}(t, x_1(t, \alpha)) \\ &= \exp\left(\int_0^t N(s, x_1(s, \alpha)) ds\right) \frac{\partial u_1^0(\alpha)}{\partial \alpha} + \sum_{k=2}^{n} (P_k + Q_k), \end{aligned} \tag{42}$$

where

$$P_k = \int_0^t \frac{\partial f_1(u)}{\partial u_k} \cdot \frac{\frac{du_k}{d_1 \tau}}{\lambda_k(u) - \lambda_1(u)} (\tau, x_1(\tau, \alpha)) \exp\left(\int_\tau^t N(s, x_1(s, \alpha)) ds\right) d\tau, \tag{43}$$

$$Q_k = \int_0^t \frac{\partial f_1(u)}{\partial u_k} \cdot \frac{f_k(u)}{\lambda_k(u) - \lambda_1(u)} (\tau, x_1(\tau, \alpha)) \exp\left(\int_\tau^t N(s, x_1(s, \alpha)) ds\right) d\tau. \tag{44}$$

Set

$$h_k(u) = \frac{1}{\lambda_k(u) - \lambda_1(u)} \cdot \frac{\partial f_1(u)}{\partial u_k}, \quad k = 2, \cdots, n. \tag{45}$$

Using the mean value theorem we can rewrite (45) as

$$h_k(u) = h_k(0) + \sum_{j=1}^{n} h_k^j(u) u_j. \tag{46}$$

Thus, by integration by parts in (43), we obtain

$$\begin{aligned} P_k = {} & h_k(0) u_k(t, x_1(t, \alpha)) - h_k(0) u_k^0(\alpha) \exp\left(\int_0^t N(s, x_1(s, \alpha)) ds\right) \\ & + h_k(0) \int_0^t u_k N(\tau, x_1(\tau, \alpha)) \exp\left(\int_\tau^t N(s, x_1(s, \alpha)) ds\right) d\tau \\ & + \sum_{j=1}^{n} \int_0^t h_k^j(u) u_j \frac{d}{d_1 \tau} u_k(\tau, x_1(\tau, \alpha)) \exp\left(\int_\tau^t N(s, x_1(s, \alpha)) ds\right) d\tau. \end{aligned} \tag{47}$$

Noticing $N<0$, we have

$$\int_0^t |N(\tau, x_1(\tau, \alpha))| \cdot \exp\left(\int_\tau^t N(s, x_1(s, \alpha))ds\right)d\tau$$
$$=1-\exp\left(\int_0^t N(s, x_1(s, \alpha))ds\right)<1. \tag{48}$$

Using (40) and assumption (H) we get

$$\left|\frac{du_k}{d_1\tau}(\tau, x_1(\tau, \alpha))\right| \leqslant C_1\left(1+\left|\frac{\partial u_k}{\partial x}(\tau, x_1(\tau, \alpha))\right|\right), \tag{49}$$

here and later, $C_i (i=1, 2, \cdots)$ denote constants independent of both t and α.

Writing

$$W_k(t)=\sup_{\substack{0\leqslant\tau\leqslant t\\ -\infty<x<+\infty}}\left|\frac{\partial u_k}{\partial x}(\tau, x)\right|,$$
$$W(t)=\sum_{l=1}^n W_l(t), \tag{50}$$

from (47)—(50) and (39) it follows that

$$|P_k| \leqslant C_2 a(1+W_k(t)). \tag{51}$$

Now we estimate Q_k. Assumption (ii) gives

$$f_k(u)=\sum_{j=1}^n f_{kj}(u)u_j. \tag{52}$$

Therefore, in a similar way as estimating P_k, from (44) we obtain

$$|Q_k| \leqslant C_3 a. \tag{53}$$

Thus (42), (51) and (53) yield

$$W_1(t) \leqslant \sup_{-\infty<x<+\infty}\left|\frac{du_1^0(x)}{dx}\right| + C_4 a(1+W(t)). \tag{54}$$

Treating other $k-1$ equations in the same way, we can obtain similar estimates on $W_k (k=2, \cdots, n)$, so we have

$$W(t) \leqslant \left|\frac{du^0}{dx}\right|_{C^0} + C_5 a(1+W(t)). \tag{55}$$

Thus if a is chosen to be sufficiently small (i. e. ε_1 is sufficiently small), we can obtain

$$W(t) \leqslant 2\left|\frac{du^0}{dx}\right|_{C^0} + C_6 a, \tag{56}$$

then, $W(t)$ is uniformly bounded. Moreover, if it holds that

$$|u(t)|_{C^0} \leqslant C_7 |u^0|_{C^0},$$

then, it is easy to see that

$$W(t) \leqslant C_8 |u^0|_{C^1}, \tag{57}$$

that is, (32) holds.

Finally, we have to show that hypothesis (37) is reasonable. Since we have proved that (56) and (57) hold under this hypothesis, that is, $\left|\frac{\partial u_k}{\partial x}\right|$ can be sufficiently small provided that the C^1 norm of u^0 is sufficiently small, we can always make hypothesis (37) true if the C^1 norm of u^0 is sufficiently small. The proof of the theorem is completed.

Now we discuss which additional assumlptions should be added to $f(u)$ in order to guarantee that hypothesis (H) holds.

Let $f_i(u)$ be of the following form

$$f_i(u)=\sum_{j=1}^{n} f_{ij}u_j,\ i=1, \cdots, n, \tag{58}$$

where f_{ij} are constants, then we have

Lemma 5. *Let $f_{11}=f_{22}=\cdots=f_{nn}=a>0$. If the matrix $F=(f_{ij})$ is weakly row (or column) diagonal dominant, then the smooth solution u to Cauchy problem (8) (9) satisfies*

$$|u(t)|_{C^0}\leqslant D_0|u^0|_{C^0}, \tag{59}$$

where D_0 is independent of t.

Proof Without loss of generality, we only discuss the case where F is weakly row diagonal dominant.

Integrating the i-th equation of system (8) along the i-th characteristic curve $x=x_i(t, \alpha)$ starting from the point $(0, \alpha)$, we can get

$$|u_i(t, x_i(t, \alpha))|e^{at}\leqslant|u_i^0(\alpha)|+\int_0^t \sum_{j=1, j\neq i}^{n}|f_{ij}|\,|u_j(\tau, x_i(\tau, \alpha))|e^{a\tau}d\tau \quad (i=1, \cdots, n). \tag{60}$$

Set

$$U_i(t)=\sup_{-\infty<x<+\infty}|u_i(t, x)|,$$
$$U(t)=\max_{1\leqslant i\leqslant n}U_i(t). \tag{61}$$

Since F is weakly row diagonal dominant, it follows from (60) that

$$U_i(t)e^{at}\leqslant|u_i^0|_{C^0}+a\int_0^t U(\tau)e^{a\tau}d\tau,\ i=1, \cdots, n. \tag{62}$$

Therefore

$$U(t)e^{at}\leqslant|u^0|_{C^0}+a\int_0^t U(\tau)e^{a\tau}\,d\tau,$$

then we obtain immediately

$$U(t)\leqslant|u^0|_{C^0}. \tag{63}$$

The lemma is proved.

Theorem 4 together with Lemma 5 gives the following

Theorem 5. *Assume that system (8) is strictly hyperbolic, $A(u)$, $u^0(x)\in C^1$, and $f(u)$ is of form (58), where f_{ij} are constants and $f_{11}=f_{22}=\cdots=f_{nn}=a>0$. If matrix $F=(f_{ij})$ is weakly row (or column) diagonal dominant, then Cauchy problem (8) (9) admits a unique global smooth solntion u on $t\geqslant0$, and*

$$|u(t)|_{C^1}\leqslant D_1|u^0|_{C^1}, \tag{64}$$

where D_1 is a constant, provided that $|u^0|_{C^1}$ is sufficiently small.

Cleaarly, the result given by T. Nishida in[3] for the syftem

$$\begin{cases} \frac{\partial Z}{\partial t}+\lambda(W-Z)\,\frac{\partial Z}{\partial x}+\alpha(Z+W)=0, \\ \frac{\partial W}{\partial t}-\lambda(W-Z)\,\frac{\partial W}{\partial x}+\alpha(Z+W)=0 \end{cases} \qquad (\lambda(u)>0;\ \ \alpha>0 \text{ is a constant})$$

is a special case of our results.

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