

WELL-POSEDNESS IN GEVREY FUNCTION SPACE FOR THE PRANDTL EQUATIONS WITH NON-DEGENERATE CRITICAL POINTS

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ABSTRACT. In the paper, we study the well-posedness of the Prandtl system without monotonicity and analyticity assumption. Precisely, for any index $\sigma \in [3/2, 2]$, we obtain the local in time well-posedness in the space of Gevrey class G^σ in the tangential variable and Sobolev class in the normal variable so that the monotonicity condition on the tangential velocity is not needed to overcome the loss of tangential derivative. This answers the open question raised in the paper of D. Gérard-Varet and N. Masmoudi [*Ann. Sci. Éc. Norm. Supér.* (4) 48 (2015), no. 6, 1273-1325], in which the case $\sigma = 7/4$ is solved.

1. INTRODUCTION AND MAIN RESULTS

The Prandtl equations introduced by Prandtl in 1904 describe the behavior of the incompressible flow near a rigid wall at high Reynolds number:

$$\begin{cases} \partial_t u^P + u^P \partial_x u^P + v^P \partial_y u^P - \partial_y^2 u^P + \partial_x p = 0, & t > 0, \quad x \in \mathbb{R}, \quad y > 0, \\ \partial_x u^P + \partial_y v^P = 0, \\ u^P|_{y=0} = v^P|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = U(t, x), \\ u^P|_{t=0} = u_0^P(x, y), \end{cases} \quad (1)$$

where $u^P(t, x, y)$ and $v^P(t, x, y)$ represent the tangential and normal velocities of the boundary layer, with y being the scaled normal variable to the boundary, while $U(t, x)$ and $p(t, x)$ are the values on the boundary of the tangential velocity and pressure of the outflow satisfying the Bernoulli law

$$\partial_t U + U \partial_x U + \partial_x p = 0.$$

We refer to [14, 17] for the mathematical derivation and background of this fundamental system in the field of boundary layer.

By using the divergence free condition, one can represent v in terms of u so that the above system is reduced a scalar equation. Moreover, note that the above U and p are known functions coming from the outflow so that the Prandtl system is a degenerate parabolic mix-type equation with loss of derivative in the tangential direction x because of the term $v \partial_y u$. In fact, this is the main difficulty in the study of this boundary layer system.

Up to now, the well-posedness on the Prandtl system is achieved in various function spaces. Precisely, when the initial data satisfy the monotonic condition, that is, when the tangential velocity is monotonic with respect to y , in the classical work by Oleinik and her collaborators, they obtained the local-in-time well-posedness by using Crocco transformation. And this result together with some of her other works were well presented in the monograph [17]. Recently, Alexandre-Wang-Xu-Yang [1] and Masmoudi-Wong [15] independently obtained the well-posedness in the Sobolev space by the virtue of energy method instead of the Crocco transformation, where the key observation in their proofs is the cancellation of the loss derivative terms. On the other hand, for the initial data without the monotonicity assumption, it is natural to perform estimate in the space of analytic functions, and

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in this context, the well-posedness results were achieved by Sammartino and Caffisch, after the earlier work of Asano [2]; cf also [16, 10] for the improvement. The first result that does not require monotonicity and analyticity was established by Gérard-Varet and Masmoudi [6] in which they obtained the well-posedness in the Gevrey space $G^{7/4}$. In fact, our paper is motivated by [6] and we give an affirmative answer to an open question raised in it. Also in the very recent work of Chen-Wang-Zhang [3], the well-posedness for the linearized Prandtl equation is studied in Gevrey space G^σ for any index $1 \leq \sigma < 2$.

Recall that the Gevrey class, denoted by $G^\sigma, \sigma \geq 1$, is an intermediate function space between analytic functions and C^∞ functions. Note that the Gevrey space $G^\sigma, \sigma > 1$ contains compactly supported functions that are more physical, and this is the main difference from analytic functions. We also refer to [12] for the smoothing effects in Gevrey space under the monotonicity assumption, and the global weak solutions by Xin-Zhang [19]. On the other hand, without the monotonicity assumption on the tangential velocity field, the degeneracy may cause strong instability so that the system is ill-posed in Sobolev spaces, cf. [4, 5, 11] and references therein.

Without the assumption on monotonicity and analyticity, in the recent interesting paper [6], the authors established $G^{7/4}$ well-posedness for Prandtl equation with non-degenerate critical points with respect to the normal variable, and they also conjectured the result should be valid for G^2 . In this paper, we will give an affirmative answer to this conjecture. In fact, we show the well-posedness in all Gevrey space G^σ with $\sigma \in [3/2, 2]$ and this includes the case studied in [6]. In addition, we believe the well-posedness result can be extended, with some new technique such as subelliptic estimates, to $\sigma \in [1, 3/2]$. Finally, as the aforementioned works, the present paper also aims at giving insight on the justification of inviscid limit for the Navier-Stokes equation with physical boundary, for this, we refer to [7, 8, 13] and the references therein for the recent progress.

To have a clear presentation, we will construct a solutions u^P that is a small perturbation around a shear flow, that is, $u^P(t, x, y) = u^s(t, y) + u(t, x, y)$. For this, we suppose that the initial data u_0^P in (1) can be written as

$$u_0^P(x, y) = u_0^s(y) + u_0(x, y),$$

with u_0^s being independent of x variable. Then we reduce the original Prandtl equation (1) to the following two time evolutionary equations, one of which is the equation for the shear flow $(u^s, 0)$ with u^s solving

$$\begin{cases} \partial_t u^s - \partial_y^2 u^s = 0, \\ u^s|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u^s = 1, \\ u^s|_{t=0} = u_0^s. \end{cases} \quad (2)$$

and the another reads

$$\begin{cases} \partial_t u + (u^s + u) \partial_x u + v \partial_y (u^s + u) - \partial_y^2 u = 0, \\ u|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (3)$$

where

$$v = - \int_0^y \partial_x u(x, \tilde{y}) d\tilde{y}.$$

Note the equation (2) is the heat equation and the well-posedness problem is well studied. In this paper, we assume that the initial datum u_0^s in (2) admits non-degenerate critical points. Precisely, we impose

Assumption 1.1 (Assumption on the initial data u_0^s). *There exists a $y_0 > 0$ such that $u_0^s \in C^6(\mathbb{R}_+)$ satisfies the following properties (see Figure 1):*

(i) $\frac{du_0^s}{dy}(y_0) = 0$ and $\frac{d^2u_0^s}{dy^2}(y_0) \neq 0$. Moreover, there exist $0 < \delta < y_0/2$ and a constant c_0 such that

$$\forall y \in [y_0 - 2\delta, y_0 + 2\delta], \quad \left| \frac{d^2u_0^s}{dy^2}(y) \right| \geq c_0.$$

(ii) There exists a constant $0 < c_1 < 1$ such that

$$\forall y \in [0, y_0 - \delta] \cup [y_0 + \delta, +\infty[, \quad c_1 \langle y \rangle^{-\alpha} \leq \left| \frac{du_0^s(y)}{dy} \right| \leq c_1^{-1} \langle y \rangle^{-\alpha}$$

for some $\alpha > 1$, and that

$$\forall y \geq 0, \quad \left| \frac{d^j u_0^s(y)}{dy^j} \right| \leq c_1^{-1} \langle y \rangle^{-\alpha-1} \text{ for } 2 \leq j \leq 6.$$

(iii) The compatibility condition holds, that is, $u_0^s|_{y=0} = \partial_y^2 u_0^s|_{y=0} = 0$ and $u_0^s(y) \rightarrow 1$ as $y \rightarrow +\infty$.

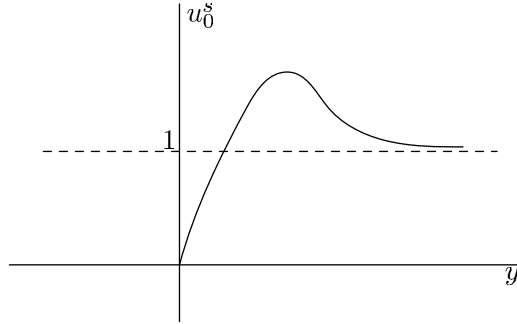


FIGURE 1. An example of u_0^s

Remark 1.2. (i) For brevity of presentation, we only consider the case when the initial datum admits one non-degenerate critical point. The result can be generalized to the case when there are several non-degenerate critical points with slight modification.

(ii) The initial datum u_0^s here is not monotonic anymore. Note that in the work [1] the monotonicity condition is required to overcome the loss of derivative in the x variable.

Proposition 1.3 (Well-posedness for the shear flow). *Let the initial data u_0^s satisfy the conditions in Assumption 1.1. Then there exists a constant $T_s > 0$ such that the heat equation (2) admits a unique solution u^s in $C([0, T_s]; C^6(\mathbb{R}_+))$. In addition, for any $t \in [0, T_s]$, we have, using the notation $\omega^s = \partial_y u^s$,*

$$\forall y \in [y_0 - \frac{7}{4}\delta, y_0 + \frac{7}{4}\delta], \quad |\partial_y \omega^s(t, y)| \geq c_0/2,$$

$$\forall y \in [0, y_0 - \frac{5}{4}\delta] \cup [y_0 + \frac{5}{4}\delta, +\infty[, \quad 2^{-1}c_1 \langle y \rangle^{-\alpha} \leq |\omega^s(t, y)| \leq 2c_1^{-1} \langle y \rangle^{-\alpha},$$

and

$$\forall y \geq 0, \quad |\partial_y^j \omega^s(t, y)| \leq 2c_1^{-1} \langle y \rangle^{-\alpha-1} \text{ for } 1 \leq j \leq 5.$$

Recall c_0, c_1, δ are the constants given in Assumption 1.1.

Observe that the solution to (2) has explicit representation by virtue of heat kernels. Then the above proposition follows from direct estimation. For brevity, we omit its proof and refer to Lemma 2.1 in the second version of [20] for detailed discussion. So it remains to solve (3), which is the main part of the paper. And we will solve the equation in the framework of Gevrey space in x and Sobolev space in y . To state the main result, we first introduce the function spaces to be used.

Definition 1.4 (Gevrey space in tangential variable). *Let α be the number given in Assumption 1.1, and let ℓ be a fixed number satisfying that*

$$\ell > 3/2, \quad \alpha \leq \ell < \alpha + \frac{1}{2}. \quad (4)$$

With each pair (ρ, σ) , $\rho > 0, \sigma \geq 1$, we associate a Banach space $X_{\rho, \sigma}$, equipped with the norm $\|\cdot\|_{\rho, \sigma}$ that consists of all the smooth functions f such that $\|f\|_{\rho, \sigma} < +\infty$, where

$$\begin{aligned} \|f\|_{\rho, \sigma} \stackrel{\text{def}}{=} & \sup_{m \geq 6} \frac{\rho^{m-5}}{[(m-6)!]^\sigma} \|\langle y \rangle^{\ell-1} \partial_x^m f\|_{L^2} + \sup_{m \geq 6} \frac{\rho^{m-5}}{[(m-6)!]^\sigma} \|\langle y \rangle^\ell \partial_x^m (\partial_y f)\|_{L^2} \\ & + \sup_{0 \leq m \leq 5} (\|\langle y \rangle^{\ell-1} \partial_x^m f\|_{L^2} + \|\langle y \rangle^\ell \partial_x^m (\partial_y f)\|_{L^2}) \\ & + \sup_{\substack{1 \leq j \leq 4 \\ i+j \geq 6}} \frac{\rho^{i+j-5}}{[(i+j-6)!]^\sigma} \|\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j (\partial_y f)\|_{L^2} + \sup_{\substack{1 \leq j \leq 4 \\ i+j \leq 5}} \|\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j (\partial_y f)\|_{L^2}. \end{aligned} \quad (5)$$

Remark 1.5. For the classical Gevrey space $G^\sigma = \cup_{L>0} G^\sigma(L)$ in x variable, $f \in G^\sigma(L)$ if the following estimates hold:

$$\forall m \geq 0, \quad \|\langle y \rangle^{\ell-1} \partial_x^m f\|_{L^2} + \|\langle y \rangle^\ell \partial_x^m (\partial_y f)\|_{L^2} \leq L^{m+1} (m!)^\sigma,$$

and

$$\forall i \geq 0, \forall 1 \leq j \leq 4, \quad \|\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j (\partial_y f)\|_{L^2} \leq L^{i+j+1} [(i+j)!]^\sigma.$$

The space $X_{\rho, \sigma}$ given in Definition 1.4 is equivalent to the classical Gevrey space G^σ in the following sense. If $f \in X_{\rho, \sigma}$ for some $\rho > 0$ then we can find a constant C such that $\|f\|_{\rho, \sigma} \leq C$. Thus direct calculation shows $f \in G^\sigma(L)$ if we choose

$$L = \frac{1}{\rho} + \sup_{0 \leq m \leq 5} (\|\langle y \rangle^{\ell-1} \partial_x^m f\|_{L^2} + \|\langle y \rangle^\ell \partial_x^m (\partial_y f)\|_{L^2}) + \sup_{\substack{1 \leq j \leq 4 \\ i+j \leq 5}} \|\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j (\partial_y f)\|_{L^2} + C.$$

Conversely, if $f \in G^\sigma(L)$, then $f \in X_{\rho, \sigma}$, provided ρ is chosen in such a way that

$$\forall m \geq 6, \quad L^{m+1} (m!)^\sigma \leq \rho^{-(m-5)} [(m-6)!]^\sigma.$$

In view of the definition $\|\cdot\|_{\rho, \sigma}$, we see the the order of y derivatives is at most 5. Then, if the equation (3) is well-posed in $X_{\rho, \sigma}$, the initial data u_0 should satisfy the following compatibility conditions, using the notation $\omega_0 = \partial_y u_0$,

$$\begin{cases} u_0|_{y=0} = \partial_y \omega_0|_{y=0} = 0, \\ \partial_y^3 \omega_0|_{y=0} = (\omega_0^s + \omega_0) \partial_x \omega_0|_{y=0}. \end{cases} \quad (6)$$

Now we state the main result in this paper as follows.

Theorem 1.6. *For $\sigma \in [3/2, 2]$, let the initial datum u_0 in (3) belong to $X_{2\rho_0, \sigma}$ for some $\rho_0 > 0$ and moreover*

$$\|u_0\|_{2\rho_0, \sigma} \leq \eta_0$$

for some $\eta_0 > 0$. Suppose that the compatibility condition (6) holds for u_0 . Then (3) admits a unique solution $u \in L^\infty([0, T]; X_{\rho, \sigma})$ for some $T > 0$ and some $0 < \rho < 2\rho_0$, provided η_0 is sufficiently small.

Remark 1.7. (i) For clear presentation, we consider the solution as a perturbation around the shear flow. In fact, the method can be applied to the general periodic case studied in [6], and we will clarify further in Section 9 why our result holds for the general case without requiring the small perturbation around a shear flow.

(ii) As pointed out in [6], it is natural, inspired by [5], to ask whether the $\sigma = 2$ is the critical Gevrey index for the well-posedness for Prandtl equation.

The methodologies. At the end of the introduction, we will present the main methodologies used in the proof.

(i) After applying ∂_x^m to the equation (3) for the velocity, the main difficulty arises from the term

$$(\partial_x^m v)(\omega^s + \omega),$$

which results in the lost of derivative in x variable. Under Oleinik's monotonicity assumption on the tangential velocity field, this can be overcome by using the cancellation introduced by AWXY [1] and Masmoudi-Wong [15]. In fact, this cancellation method works at least in the domain where $u^s + u$ admits monotonicity. Precisely, we apply ∂_x^m to the equation for the vorticity $\omega = \partial_y u$

$$\partial_t \omega + (u^s + u) \partial_x \omega + v \partial_y (\omega^s + \omega) - \partial_y^2 \omega = 0,$$

in which the most difficult term is $(\partial_x^m v)(\partial_y \omega^s + \partial_y \omega)$. To capture the cancellation, one can work on the function, introduced in [6],

$$f_m = \chi_1 \partial_x^m \omega - \chi_1 \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \partial_x^m u, \quad m \geq 1,$$

where χ_1 is a smooth function supported in the monotonic region.

(ii) As for the domain near the critical points, we do not have the monotonicity anymore. One of the new observations in this paper is that we can also apply the cancellation to the equation for the vorticity and the equation for $\partial_y \omega$

$$\partial_t (\partial_y \omega) + u \partial_x (\partial_y \omega) + v \partial_y (\partial_y \omega) - \partial_y^2 (\partial_y \omega) = -(\omega^s + \omega) \partial_x \omega + (\partial_y \omega^s + \partial_y \omega) \partial_x u,$$

by using another auxiliary function

$$h_m = \chi_2 \partial_x^m \partial_y \omega - \chi_2 \frac{\partial_y^2 \omega^s + \partial_y^2 \omega}{\partial_y \omega^s + \partial_y \omega} \partial_x^m \omega,$$

where χ_2 is a cut-off function compactly supported in the region admitting the non-degenerate critical points. However, even with this, we also have the loss of x derivative because

$$g_{m+1} \stackrel{\text{def}}{=} \partial_x^m [(\omega^s + \omega) \partial_x \omega - (\partial_y \omega^s + \partial_y \omega) \partial_x u]$$

appears in the equation for h_m . Nonetheless, we can use again the cancellation method to the equations for the velocity and the vorticity, to obtain an equation for g_{m+1} . Precisely, we apply ∂_x to the equations for velocity u and for vorticity $\omega = \partial_y u$, and then multiply respectively the obtained equations by the factors $\partial_y \omega^s + \partial_y \omega$ and $\omega^s + \omega$ respectively, and finally subtract one from another. We then obtain the equation for $g_1 \stackrel{\text{def}}{=} (\omega^s + \omega) \partial_x \omega - (\partial_y \omega^s + \partial_y \omega) \partial_x u$ as follows.

$$\left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 \right) g_1 = 2 (\partial_y^2 \omega^s + \partial_y^2 \omega) \partial_x \omega - 2 (\partial_y \omega^s + \partial_y \omega) \partial_x \partial_y \omega.$$

Note that the order of x derivative for terms on right hand side is equal to 1 that is the same as in the representation of g_1 . The above equation allows us to perform estimation on $g_{m+1} = \partial_x^m g_1$ in Gevrey norm by standard energy method.

(iii) From the above procedure, we have the upper bound, by energy method, for the auxiliary functions f_m and h_m . It remains to control the original $\partial_x^m u$ and $\partial_x^m \omega$ as well as the mixed derivatives, in terms of the auxiliary functions. This is clear when there is no cut-off functions χ_i involved, by

virtue of the Hardy-type inequality (see [15] for instance under the monotonicity assumption). In case considered in this paper, we first follow the cancellation idea used in [6], by taking L^2 inner product with $\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega}$ on both sides of the equation for $\partial_x^m \omega$, to obtain the estimate on $\chi_2 \partial_x^m \omega$. Then by using the representation of h_m , we can derive similar estimate on $\chi_2 \partial_x^m \partial_y \omega$ from those on h_m . Roughly speaking, this implies $\chi_2 \partial_x^m \partial_y \omega$ behaves similarly as the terms with m order derivatives involved, rather than the $m+1$ order of mixed derivatives in Definition 1.4. And this is the advantage of the new auxilliary function h_m introduced in this paper and this enables us to extend the well-posedness of the Prandtl system from the Gevrey index $\sigma = 7/4$ obtained in [6] to $\sigma \in [3/2, 2]$.

The rest of the paper is organized as follows. Section 2-6 are devoted to the proof of the uniform estimate in Gevrey norm for the approximate solutions to a regularized Prandtl equation. In Section 7, we will give the proof of existence of the regularized Prandtl equation and in Section 8 we will prove the main result of this paper. We explain in Section 9 why the main result in this paper holds for the general initial data rather than the small perturbations around a shear flow. The proofs of some technical lemmas will be given in the Appendix.

2. REGULARIZED PRANDTL EQUATION AND UNIFORM ESTIMATES IN GEVREY NORM

In this section as well as Sections 3-7, we will study the initial-boundary problem for the following regularized Prandtl type equation of (3) by recalling u^s given in Proposition 1.3,

$$\begin{cases} \partial_t u_\varepsilon + (u^s + u_\varepsilon) \partial_x u_\varepsilon + v_\varepsilon \partial_y (u^s + u_\varepsilon) - \partial_y^2 u_\varepsilon - \varepsilon \partial_x^2 u_\varepsilon = 0, \\ u_\varepsilon|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u_\varepsilon = 0, \\ u_\varepsilon|_{t=0} = u_0, \end{cases} \quad (7)$$

where $\varepsilon > 0$ is an arbitrarily small number and $v_\varepsilon = -\int_0^y \partial_x u_\varepsilon(x, \tilde{y}) d\tilde{y}$. We remark the regularized equation above shares the same compatibility condition (6) as the original one (3).

The existence of solutions to (7) will be given in Section 7, where the life span T_ε^* may depend on the ε . Thus in order to obtain the solution to the original equation by letting $\varepsilon \rightarrow 0$, we need an uniform estimate, for example, in the Gevrey norm for u_ε , that will be stated in this section with the proof given in Sections 3-6. To simplify the notations, we will use the notations $\omega_\varepsilon = \partial_y u_\varepsilon$ and $\omega^s = \partial_y u^s$ from now on.

Throughout the paper, we will work on those solutions u_ε that the properties listed in Proposition 1.3 for u^s can be preserved by $u^s + u_\varepsilon$. Precisely, we suppose that the solution $u_\varepsilon \in L^\infty([0, T]; X_{\rho_0, \sigma})$ to (7) has the following properties. For any $t \in [0, T]$ and any $x \in \mathbb{R}$, we have

$$\begin{cases} |\partial_y \omega^s(t, y) + \partial_y \omega_\varepsilon(t, x, y)| \geq \frac{c_0}{4}, \quad \text{if } y \in [y_0 - \frac{7}{4}\delta, y_0 + \frac{7}{4}\delta], \\ 4^{-1} c_1 \langle y \rangle^{-\alpha} \leq |\omega^s(t, y) + \omega_\varepsilon(t, x, y)| \leq 4c_1^{-1} \langle y \rangle^{-\alpha}, \quad \text{if } y \in [0, y_0 - \frac{5}{4}\delta] \cup [y_0 + \frac{5}{4}\delta, +\infty[, \\ |\partial_y \omega^s(t, y) + \partial_y \omega_\varepsilon(t, x, y)| \leq 4c_1^{-1} \langle y \rangle^{-\alpha-1} \quad \text{for } y \geq 0, \\ \sum_{1 \leq j \leq 2} \left(\|\langle y \rangle^{\ell-1} \partial_x^j u_\varepsilon\|_{L^\infty} + \|\partial_x^{j-1} v_\varepsilon\|_{L^\infty} + \|\langle y \rangle^\ell \partial_x^j \omega_\varepsilon\|_{L^\infty} \right) + \sum_{1 \leq i, j \leq 2} \|\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega_\varepsilon\|_{L^\infty} \leq 1, \end{cases} \quad (8)$$

where c_0, c_1 and δ are the constants given in Assumption 1.1.

According to the properties (8) above, we can divide the normal direction $y \geq 0$ into two parts, one is near the critical points of $u^s + u_\varepsilon$, and another one is away from the critical points where $u^s + u_\varepsilon$ admits the monotonicity condition. That is, we can find two non-negative C^∞ smooth functions χ_1 and χ_2 depending only on y such that

$$0 \leq \chi_1 \leq 1, \quad \chi_1 \equiv 1 \quad \text{on }]-\infty, y_0 - \frac{3}{2}\delta] \cup [y_0 + \frac{3}{2}\delta, +\infty[, \quad \chi_1 \equiv 0 \quad \text{on } [y_0 - \frac{5}{4}\delta, y_0 + \frac{5}{4}\delta], \quad (9)$$

and

$$0 \leq \chi_2 \leq 1, \quad \chi_2 \equiv 1 \text{ on } \left[y_0 - \frac{3}{2}\delta, y_0 + \frac{3}{2}\delta \right], \quad \text{supp } \chi_2 \subset \left[y_0 - \frac{7}{4}\delta, y_0 + \frac{7}{4}\delta \right]. \quad (10)$$

From the properties listed in (8), it follows that $|\omega^s + \omega| > 0$ on $\text{supp } \chi_1$, and $|\partial_y \omega^s + \partial_y \omega| > 0$ on $\text{supp } \chi_2$. Moreover,

$$\chi'_1 = \chi'_1 \chi_2, \quad \chi'_2 = \chi'_2 \chi_1, \quad \text{and } (1 - \chi_2) = (1 - \chi_2) \chi_1, \quad (11)$$

because $\chi_2 \equiv 1$ on $\text{supp } \chi'_1$, $\chi_1 \equiv 1$ on $\text{supp } \chi'_2$, and $\chi_1 \equiv 1$ on $\text{supp } (1 - \chi_2)$. Here and throughout the paper, f' and f'' stand for the first and the second order derivatives of f .

Definition 2.1. Let χ_1 and χ_2 given above and let u_ε satisfy the properties (8). For $m \geq 1$, we define three auxilliary functions $f_{m,\varepsilon}$, $h_{m,\varepsilon}$ and $g_{m,\varepsilon}$ according to the cancellation property:

$$f_{m,\varepsilon} = \chi_1 \partial_x^m \omega_\varepsilon - \chi_1 \frac{\partial_y \omega^s + \partial_y \omega_\varepsilon}{\omega^s + \omega_\varepsilon} \partial_x^m u_\varepsilon = \chi_1 (\omega^s + \omega_\varepsilon) \partial_y \left(\frac{\partial_x^m u_\varepsilon}{\omega^s + \omega_\varepsilon} \right), \quad (12)$$

$$h_{m,\varepsilon} = \chi_2 \partial_x^m \partial_y \omega_\varepsilon - \chi_2 \frac{\partial_y^2 \omega^s + \partial_y^2 \omega_\varepsilon}{\partial_y \omega^s + \partial_y \omega_\varepsilon} \partial_x^m \omega_\varepsilon, \quad (13)$$

and

$$g_{m,\varepsilon} = \partial_x^{m-1} \left((\omega^s + \omega_\varepsilon) \partial_x \omega_\varepsilon - (\partial_y \omega^s + \partial_y \omega_\varepsilon) \partial_x u_\varepsilon \right). \quad (14)$$

Definition 2.2. Let $X_{\rho,\sigma}$ be given in Definition 1.4, equipped with the norm $\|\cdot\|_{\rho,\sigma}$ defined by (5). Let χ_1, χ_2 be given by (9)-(10), and let u_ε satisfy the properties listed in (8). We will use the notation $|\cdot|_{\rho,\sigma}$ which is given by

$$\begin{aligned} |u_\varepsilon|_{\rho,\sigma} &= \|u_\varepsilon\|_{\rho,\sigma} + \sup_{1 \leq m \leq 5} \left(m \|g_{m,\varepsilon}\|_{L^2} + \|\langle y \rangle^\ell f_{m,\varepsilon}\|_{L^2} + \|h_{m,\varepsilon}\|_{L^2} + \|\chi_2 \partial_y \partial_x^m \omega_\varepsilon\|_{L^2} \right) \\ &\quad + \sup_{m \geq 6} \frac{\rho^{m-5}}{[(m-6)!]^\sigma} \left(m \|g_{m,\varepsilon}\|_{L^2} + \|\langle y \rangle^\ell f_{m,\varepsilon}\|_{L^2} + \|h_{m,\varepsilon}\|_{L^2} + \|\chi_2 \partial_y \partial_x^m \omega_\varepsilon\|_{L^2} \right). \end{aligned}$$

Similarly we can define $|u_0|_{\rho,\sigma}$.

Remark 2.3. (i) Observe there is an extra factor m in front of the term $\|g_{m,\varepsilon}\|_{L^2}$ in the definition of the norm $|\cdot|_{\rho,\sigma}$.

(ii) Direct calculation shows that

$$\|u_\varepsilon\|_{\rho,\sigma} \leq |u_\varepsilon|_{\rho,\sigma} \leq C_{\rho,\rho^*} \left(\|u_\varepsilon\|_{\rho^*,\sigma} + \|u_\varepsilon\|_{\rho^*,\sigma}^2 \right) \quad (15)$$

for any $\rho < \rho^*$, with C_{ρ,ρ^*} being a constant depending only on the difference $\rho^* - \rho$.

Theorem 2.4 (uniform estimates in Gevrey space). Let $3/2 \leq \sigma \leq 2$. Let $u_\varepsilon \in L^\infty([0, T]; X_{\rho_0,\sigma})$ be a solution to (7) such that the properties listed in (8) hold. Then there exists a constant $C_* > 1$, independent of ε and the solution u_ε , such that the estimate

$$|u_\varepsilon(t)|_{\rho,\sigma}^2 \leq C_* |u_0|_{\rho,\sigma}^2 + C_* \int_0^t \left(|u_\varepsilon(s)|_{\rho,\sigma}^2 + |u_\varepsilon(s)|_{\rho,\sigma}^4 \right) ds + C_* \int_0^t \frac{|u_\varepsilon(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \quad (16)$$

holds for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$, and for any $t \in [0, \tilde{T}]$, where $[0, \tilde{T}]$ is the maximal interval of existence for $|u_\varepsilon|_{\tilde{\rho},\sigma} < +\infty$.

The above theorem is the key part of the paper, and its proof follows from the discussion in Sections 3 to 6.

3. PROOF OF THEOREM 2.4: UNIFORM ESTIMATE ON g_m

This section along with Sections 4-6 are devoted to proving Theorem 2.4, the uniform estimates for the approximate solutions u^ε . To simplify the notations, we will remove in the following discussion the subscript ε in $u_\varepsilon, \omega_\varepsilon$ if no confusion occurs. Similarly, we write f_m, h_m and g_m for the auxilliary functions $f_{m,\varepsilon}, h_{m,\varepsilon}$ and $g_{m,\varepsilon}$ defined in (12)-(14). Moreover, we will use the capital letter C to denote some generic constants, which may vary from line to line that depend only on the constants c_j, δ, ρ_0 and α in Assumption 1.1 as well as on the Sobolev embedding constants, but are independent of ε and the order m of derivatives.

We begin with a uniform estimate on $g_m = g_{m,\varepsilon}$ with $g_{m,\varepsilon}$ defined by (14), that is,

$$g_m = \partial_x^{m-1} \left((\omega^s + \omega) \partial_x \omega - (\partial_y \omega^s + \partial_y \omega) \partial_x u \right), \quad m \geq 1. \quad (17)$$

The main result in this section can be stated as follows.

Proposition 3.1. *Let $m \geq 6$ and let $u \in L^\infty([0, T]; X_{\rho_0, \sigma})$ be a solution to (7) under the assumptions in Theorem 2.4. Then for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} \leq \rho_0$ and for any small $t \in [0, T]$, we have*

$$\begin{aligned} m^2 \|g_m(t)\|_{L^2}^2 &\leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho, \sigma}^2 + C \int_0^t \left(\varepsilon \|\partial_x^{m+1} u\|_{L^2}^2 + \varepsilon \|\partial_x^{m+1} \omega\|_{L^2}^2 \right) ds \\ &+ Cm^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

3.1. Preliminaries. Before proving Proposition 3.1, we first list some inequalities used throughout the paper.

Lemma 3.2 (Some inequalities). *(i) Given any non-negative integers p and q , we have*

$$p!q! \leq (p+q)!$$

(ii) We have $|\cdot|_{\rho, \sigma} \leq |\cdot|_{\tilde{\rho}, \sigma}$ for $\rho \leq \tilde{\rho}$.

(iii) For any integer $k \geq 1$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} \leq 1$, we have

$$k \left(\frac{\rho}{\tilde{\rho}} \right)^k \leq \frac{1}{\tilde{\rho}} k \left(\frac{\rho}{\tilde{\rho}} \right)^k \leq \frac{1}{\tilde{\rho} - \rho}. \quad (18)$$

(iv) Let χ_2 be given in (10) and let $\sigma \geq 1$. Then for any $0 < r \leq 1$, we have

$$\left\| \langle y \rangle^{\ell-1} \partial_x^m u \right\|_{L^2} + \left\| \langle y \rangle^\ell \partial_x^m \omega \right\|_{L^2} \leq \begin{cases} \frac{[(m-6)!]^\sigma}{r^{(m-5)}} |u|_{r, \sigma}, & \text{if } m \geq 6, \\ |u|_{r, \sigma}, & \text{if } 0 \leq m \leq 5, \end{cases} \quad (19)$$

and

$$m \|g_m\|_{L^2} + \left\| \langle y \rangle^\ell f_m \right\|_{L^2} + \|h_m\|_{L^2} + \|\chi_2 \partial_y \partial_x^m \omega\|_{L^2} \leq \begin{cases} \frac{[(m-6)!]^\sigma}{r^{(m-5)}} |u|_{r, \sigma}, & \text{if } m \geq 6, \\ |u|_{r, \sigma}, & \text{if } 1 \leq m \leq 5, \end{cases} \quad (20)$$

and

$$\left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right\|_{L^2} \leq \begin{cases} \frac{[(i+j-6)!]^\sigma}{r^{(i+j-5)}} |u|_{r, \sigma}, & \text{if } i+j \geq 6 \text{ and } 1 \leq j \leq 4, \\ |u|_{r, \sigma}, & \text{if } 0 \leq i+j \leq 5 \text{ and } 1 \leq j \leq 4. \end{cases} \quad (21)$$

(v) Let $\sigma \geq 1$ and let $m \geq 7$. Then for any $0 < r \leq 1$ we have

$$\|\partial_x^{m-1} \partial_y \omega\|_{L^2} \leq \|\partial_y f_{m-1}\|_{L^2} + \frac{C[(m-7)!]^\sigma}{r^{m-6}} |u|_{r, \sigma} \leq \|\partial_y f_{m-1}\|_{L^2} + Cm^{-\sigma} \frac{[(m-6)!]^\sigma}{r^{m-5}} |u|_{r, \sigma}. \quad (22)$$

Proof. The first statement (i) is clear. The second and the fourth statements (ii) and (iv) follow directly from the definition of $|\cdot|_{\rho,\sigma}$ (See Definition 2.2). As for (iii), we have for any $k \geq 1$ and any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} \leq 1$,

$$\frac{1}{1 - \frac{\rho}{\tilde{\rho}}} = \sum_{j=0}^{\infty} \left(\frac{\rho}{\tilde{\rho}}\right)^j \geq k \left(\frac{\rho}{\tilde{\rho}}\right)^k,$$

from which the desired inequalities follow.

Now we prove (v). In view of (11) we see $\chi_1 \equiv 1$ on $\text{supp } 1 - \chi_2$ so that

$$\begin{aligned} \|\partial_x^{m-1} \partial_y \omega\|_{L^2} &\leq \|(1 - \chi_2) \partial_y (\chi_1 \partial_x^{m-1} \omega)\|_{L^2} + \|\chi_2 \partial_y \partial_x^{m-1} \omega\|_{L^2} \\ &\leq \|(1 - \chi_2) \partial_y f_{m-1}\|_{L^2} + \left\| (1 - \chi_2) \partial_y \left(\chi_1 \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \partial_x^{m-1} u \right) \right\|_{L^2} + \|\chi_2 \partial_y \partial_x^{m-1} \omega\|_{L^2} \\ &\leq \|\partial_y f_{m-1}\|_{L^2} + C \|\partial_x^{m-1} u\|_{L^2} + C \|\partial_x^{m-1} \omega\|_{L^2} + \|\chi_2 \partial_y \partial_x^{m-1} \omega\|_{L^2}. \end{aligned}$$

In the above, the second inequality uses (12), the definition of f_m . This along with (19) and (20) yield

$$\|\partial_x^{m-1} \partial_y \omega\|_{L^2} \leq \|\partial_y f_{m-1}\|_{L^2} + \frac{C [(m-7)!]^\sigma}{r^{m-6}} |u|_{r,\sigma} \leq \|\partial_y f_{m-1}\|_{L^2} + C m^{-\sigma} \frac{[(m-6)!]^\sigma}{r^{m-5}} |u|_{r,\sigma}.$$

The proof is then completed. \square

Let g_m be given in (17), and we define its key component \tilde{g}_m by setting

$$\tilde{g}_m = (\omega^s + \omega) \partial_x^m \omega - (\partial_y \omega^s + \partial_y \omega) \partial_x^m u, \quad m \geq 1. \quad (23)$$

The next lemma is concerned with the difference $g_m - \tilde{g}_m$.

Lemma 3.3. *Let $m \geq 6$ and let $1 \leq \sigma \leq 2$. We have*

$$\|g_m - \tilde{g}_m\|_{L^2} \leq C \|\partial_y f_{m-1}\|_{L^2} + C m^{1-\sigma} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-5}} |u|_{\tilde{\rho},\sigma} + C m^{2-2\sigma} \frac{[(m-6)!]^\sigma}{\rho^{m-5}} |u|_{\rho,\sigma}^2.$$

Proof. First, direct calculation shows

$$g_m - \tilde{g}_m = \sum_{j=1}^{m-1} \binom{m-1}{j} (\partial_x^j \omega) \partial_x^{m-j} \omega - \sum_{j=1}^{m-1} \binom{m-1}{j} (\partial_x^j \partial_y \omega) \partial_x^{m-j} u. \quad (24)$$

Thus

$$\|g_m - \tilde{g}_m\|_{L^2} \leq \sum_{j=1}^{m-1} \frac{(m-1)!}{j!(m-1-j)!} \|(\partial_x^j \omega) \partial_x^{m-j} \omega\|_{L^2} + \sum_{j=1}^{m-1} \frac{(m-1)!}{j!(m-1-j)!} \|(\partial_x^j \partial_y \omega) \partial_x^{m-j} u\|_{L^2}.$$

We first handle the second term on the right side of the above estimate, and write

$$\sum_{j=1}^{m-1} \frac{(m-1)!}{j!(m-1-j)!} \|(\partial_x^j \partial_y \omega) \partial_x^{m-j} u\|_{L^2} \leq R_1 + R_2,$$

where

$$R_1 = \sum_{j=1}^{[m/2]} \frac{(m-1)!}{j!(m-1-j)!} \|\partial_x^j \partial_y \omega\|_{L^\infty} \|\partial_x^{m-j} u\|_{L^2}$$

with $[m/2]$ standing for the largest integer less than or equal to $m/2$, and

$$R_2 = \sum_{j=[m/2]+1}^{m-1} \frac{(m-1)!}{j!(m-1-j)!} \|\partial_x^j \partial_y \omega\|_{L^2} \|\partial_x^{m-j} u\|_{L^\infty}.$$

To estimate R_2 , we use (19) and (21) along with the Sobolev inequality (see Lemma A.1 in Appendix), to compute

$$\begin{aligned} R_2 \leq & \|\partial_x^{m-1}\partial_y\omega\|_{L^2}\|\partial_x u\|_{L^\infty} + m\|\partial_x^{m-2}\partial_y\omega\|_{L^2}\|\partial_x^2 u\|_{L^\infty} + \sum_{j=m-4}^{m-3} \frac{(m-1)!}{j!(m-1-j)!} \frac{[(j-5)!]^\sigma}{\rho^{j-4}} |u|_{\rho,\sigma}^2 \\ & + \sum_{j=[m/2]+1}^{m-5} \frac{(m-1)!}{j!(m-1-j)!} \frac{[(j-5)!]^\sigma}{\rho^{j-4}} \frac{[(m-j-5)!]^\sigma}{\rho^{m-j-4}} |u|_{\rho,\sigma}^2. \end{aligned}$$

Moreover, by (21) and the last inequality in (8), we have

$$m\|\partial_x^{m-2}\partial_y\omega\|_{L^2}\|\partial_x^2 u\|_{L^\infty} \leq Cm^{1-\sigma} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-5}} |u|_{\tilde{\rho},\sigma}.$$

Direct computation gives

$$\sum_{j=m-4}^{m-3} \frac{(m-1)!}{j!(m-1-j)!} \frac{[(j-5)!]^\sigma}{\rho^{j-4}} |u|_{\rho,\sigma}^2 \leq Cm^{2-2\sigma} \frac{[(m-6)!]^\sigma}{\rho^{m-5}} |u|_{\rho,\sigma}^2,$$

and, using the statement (i) in Lemma 3.2,

$$\begin{aligned} & \sum_{j=[m/2]+1}^{m-5} \frac{(m-1)!}{j!(m-1-j)!} \frac{[(j-5)!]^\sigma}{\rho^{j-4}} \frac{[(m-j-5)!]^\sigma}{\rho^{m-j-4}} |u|_{\rho,\sigma}^2 \\ \leq & C \frac{1}{\rho^{m-5}} |u|_{\rho,\sigma}^2 \sum_{j=[m/2]+1}^{m-5} \frac{(m-1)![(j-5)!]^{\sigma-1} [(m-j-5)!]^{\sigma-1}}{j^5(m-j)^4} \\ \leq & C \frac{1}{\rho^{m-5}} |u|_{\rho,\sigma}^2 \sum_{j=[m/2]+1}^{m-3} \frac{(m-6)![(m-10)!]^{\sigma-1} m^5}{j^5(m-j)^4} \\ \leq & Cm^{4-4\sigma} \frac{[(m-6)!]^\sigma}{\rho^{m-5}} |u|_{\rho,\sigma}^2. \end{aligned}$$

Finally, for any small $\kappa > 0$, we use (22) and the last inequality in (8) to obtain

$$\|\partial_x^{m-1}\partial_y\omega\|_{L^2}\|\partial_x u\|_{L^\infty} \leq C\|\partial_y f_{m-1}\|_{L^2} + Cm^{-\sigma} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-5}} |u|_{\tilde{\rho},\sigma}.$$

Combining the inequalities above we conclude

$$R_2 \leq C\|\partial_y f_{m-1}\|_{L^2} + Cm^{1-\sigma} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-5}} |u|_{\tilde{\rho},\sigma} + Cm^{2-2\sigma} \frac{[(m-6)!]^\sigma}{\rho^{m-5}} |u|_{\rho,\sigma}^2.$$

The estimation on R_1 is similar as above with simpler so that we omit it for brevity. Then we have

$$R_1 \leq Cm^{1-\sigma} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-5}} |u|_{\tilde{\rho},\sigma} + Cm^{2-2\sigma} \frac{[(m-6)!]^\sigma}{\rho^{m-5}} |u|_{\rho,\sigma}^2.$$

Thus the desired estimate follows and the proof of Lemma 3.3 is completed. \square

3.2. Proof of Proposition 3.1. The rest of this section is devoted to proving Proposition 3.1 by energy method, and the proof is inspired by the arguments used in [6]. To do so, we first write the

equation for g_m as follows with its derivation given in the Appendix (see Lemma B.3 in the Appendix).

$$\begin{aligned}
& \left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) g_m \\
= & - \sum_{j=1}^{m-1} \binom{m-1}{j} (\partial_x^j u) g_{m-j+1} - \sum_{j=1}^{m-1} \binom{m-1}{j} (\partial_x^j v) \partial_y g_{m-j} \\
& + 2 \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^j \partial_y^2 \omega^s + \partial_x^j \partial_y^2 \omega) \partial_x^{m-j} \omega + 2\varepsilon \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^{j+1} \partial_y \omega) \partial_x^{m-j+1} u \\
& - 2 \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega) \partial_x^{m-j} \partial_y \omega - 2\varepsilon \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^{j+1} \omega) \partial_x^{m-j+1} \omega,
\end{aligned}$$

Moreover, observe that $\partial_y \omega^s|_{y=0} = \partial_y \omega|_{y=0} = 0$ and then

$$\partial_y g_m|_{y=0} = 0.$$

Thus multiplying both sides by $m^2 g_m$ and then taking integration over \mathbb{R}_+^2 , we have

$$\frac{1}{2} m^2 \|g_m(t)\|_{L^2}^2 + m^2 \int_0^t \|\partial_y g_m(s)\|_{L^2}^2 ds + \varepsilon m^2 \int_0^t \|\partial_x g_m\|_{L^2}^2 ds \leq \frac{1}{2} m^2 \|g_m(0)\|_{L^2}^2 + \sum_{1 \leq i \leq 6} \mathcal{P}_i \quad (25)$$

with

$$\begin{aligned}
\mathcal{P}_1 &= - \int_0^t \left(\sum_{j=1}^{m-1} \binom{m-1}{j} (\partial_x^j u) g_{m-j+1}, m^2 g_m \right)_{L^2} ds, \\
\mathcal{P}_2 &= - \int_0^t \left(\sum_{j=1}^{m-1} \binom{m-1}{j} (\partial_x^j v) \partial_y g_{m-j}, m^2 g_m \right)_{L^2} ds, \\
\mathcal{P}_3 &= \int_0^t \left(2 \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^j \partial_y^2 \omega^s + \partial_x^j \partial_y^2 \omega) \partial_x^{m-j} \omega, m^2 g_m \right)_{L^2} ds, \\
\mathcal{P}_4 &= - \int_0^t \left(2 \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega) \partial_x^{m-j} \partial_y \omega, m^2 g_m \right)_{L^2} ds, \\
\mathcal{P}_5 &= \int_0^t \left(2\varepsilon \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^{j+1} \partial_y \omega) \partial_x^{m-j+1} u, m^2 g_m \right)_{L^2} ds, \\
\mathcal{P}_6 &= - \int_0^t \left(2\varepsilon \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^{j+1} \omega) \partial_x^{m-j+1} \omega, m^2 g_m \right)_{L^2} ds.
\end{aligned}$$

In the following lemmas, we will estimate $\mathcal{P}_i, 1 \leq i \leq 6$ respectively.

Lemma 3.4 (Estimate on \mathcal{P}_3). *Let $3/2 \leq \sigma \leq 2$. Then for any small $\kappa > 0$ and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho}$, we have*

$$\begin{aligned}
\mathcal{P}_3 &\leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + C \kappa^{-1} m^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds \\
&\quad + \frac{C \kappa^{-1} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho}(s) - \rho} ds \right).
\end{aligned}$$

Proof. We write

$$\mathcal{P}_3 = 2 \int_0^t \left(\sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^j \partial_y^2 \omega^s + \partial_x^j \partial_y^2 \omega) \partial_x^{m-j} \omega, m^2 g_m \right)_{L^2} dt \leq J_1 + J_2 + J_3 + J_4$$

with

$$\begin{aligned} J_1 &= 2m \int_0^t \|\partial_y^2 \omega^s + \partial_y^2 \omega\|_{L^\infty} \|\partial_x^m \omega\|_{L^2} (m \|g_m\|_{L^2}) dt, \\ J_2 &= 2m(m-1) \int_0^t \|\partial_x \partial_y^2 \omega\|_{L^\infty} \|\partial_x^{m-1} \omega\|_{L^2} (m \|g_m\|_{L^2}) dt, \\ J_3 &= 2m \sum_{j=2}^{[m/2]} \frac{(m-1)!}{j!(m-1-j)!} \int_0^t \|\partial_x^j \partial_y^2 \omega\|_{L^\infty} \|\partial_x^{m-j} \omega\|_{L^2} (m \|g_m\|_{L^2}) dt, \\ J_4 &= 2 \sum_{j=[m/2]+1}^{m-1} \binom{m-1}{j} \int_0^t \left((\partial_x^j \partial_y^2 \omega) \partial_x^{m-j} \omega, m^2 g_m \right)_{L^2}. \end{aligned}$$

Estimate on J_1 and J_2 : For J_1 , we use the third estimate in (8) as well as (20) to obtain

$$J_1 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u(s)|_{\tilde{\rho}, \sigma}^2 \frac{m\rho^{2(m-5)}}{\tilde{\rho}^{2(m-5)}} ds \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho}(s) - \rho} ds,$$

where the last inequality follows from (18) in Lemma 3.2. Similarly,

$$J_2 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u(s)|_{\tilde{\rho}, \sigma}^2 \frac{m^{2-\sigma} \rho^{2(m-5)}}{\tilde{\rho}^{2(m-5)}} ds \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho}(s) - \rho} ds,$$

where the last inequality follows from the fact that $\sigma \geq 3/2 \geq 1$.

Estimate on J_3 : By using the statement (iv) in Lemma 3.2 as well as the Sobolev inequality (see Lemma A.1 in the Appendix), we have

$$\begin{aligned} J_3 &\leq Cm \sum_{j=2}^{[m/2]} \frac{(m-1)!}{j!(m-1-j)!} \frac{[(j-2)!]^\sigma}{\rho^{j-1}} \frac{[(m-j-6)!]^\sigma}{\rho^{m-j-5}} \frac{[(m-6)!]^\sigma}{\rho^{m-5}} \int_0^t |u(s)|_{\rho, \sigma}^3 ds \\ &\leq Cm \frac{[(m-6)!]^{\sigma+1}}{\rho^{2(m-5)}} \sum_{j=2}^{[m/2]} \frac{m^5 [(j-2)!]^{\sigma-1} [(m-j-6)!]^{\sigma-1}}{j^2 (m-j)^5} \int_0^t |u(s)|_{\rho, \sigma}^3 ds \\ &\leq Cm [(m-8)!]^{\sigma-1} \frac{[(m-6)!]^{\sigma+1}}{\rho^{2(m-5)}} \int_0^t |u(s)|_{\rho, \sigma}^3 ds \sum_{j=2}^{[m/2]} \frac{1}{j^2} \quad (\text{using (i) in Lemma 3.2}) \\ &\leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u(s)|_{\rho, \sigma}^3 ds, \end{aligned}$$

where in the last inequality, we have used $m^{-2(\sigma-1)} \leq m^{-1}$ because $\sigma \geq 3/2$.

Estimate on J_4 : By integration by parts, for any small $\kappa > 0$, we have

$$\begin{aligned}
J_4 &= -2 \sum_{j=[m/2]+1}^{m-1} \binom{m-1}{j} \int_0^t \left((\partial_x^j \partial_y \omega) \partial_x^{m-j} \omega, m^2 \partial_y g_m \right)_{L^2} \\
&\quad -2 \sum_{j=[m/2]+1}^{m-1} \binom{m-1}{j} \int_0^t \left((\partial_x^j \partial_y \omega) \partial_x^{m-j} \partial_y \omega, m^2 g_m \right)_{L^2} \\
&\leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + C \kappa^{-1} m^2 \int_0^t \|\partial_x^{m-1} \partial_y \omega\|_{L^2}^2 \|\partial_x \omega\|_{L^\infty}^2 ds \\
&\quad + C \kappa^{-1} m^2 \int_0^t \left[\sum_{j=[m/2]+1}^{m-2} \binom{m-1}{j} \|\partial_x^j \partial_y \omega\|_{L^2} \|\partial_x^{m-j} \omega\|_{L^\infty} \right]^2 ds \\
&\quad + 2m \sum_{j=[m/2]+1}^{m-1} \binom{m-1}{j} \int_0^t \|\partial_x^j \partial_y \omega\|_{L^2} \|\partial_x^{m-j} \partial_y \omega\|_{L^\infty} (m \|g_m\|_{L^2}) ds \\
&\stackrel{\text{def}}{=} \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + J_{4,1} + J_{4,2} + J_{4,3}.
\end{aligned}$$

Now we use the statements (iii) and (iv) in Lemma 3.2 to get by repeating the arguments used for the terms J_1 - J_3 ,

$$J_{4,3} \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho,\sigma}^3 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}(s) - \rho} ds \right),$$

and

$$J_{4,2} \leq \frac{C \kappa^{-1} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho,\sigma}^4 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}(s) - \rho} ds \right).$$

It remains to treat $J_{4,1}$. To do so, we use (22) and the last inequality in (8) to obtain by using $\sigma \geq 3/2 > 1$,

$$J_{4,1} \leq C \kappa^{-1} m^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds + \frac{C \kappa^{-1} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u|_{\rho,\sigma}^2 ds.$$

This along with the estimates on $J_{4,2}$ and $J_{4,3}$ given above imply that for any small $\kappa > 0$,

$$\begin{aligned}
J_4 &\leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + C \kappa^{-1} m^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds \\
&\quad + \frac{C \kappa^{-1} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho}(s) - \rho} ds \right).
\end{aligned}$$

Then combining the estimates on the terms J_1 - J_4 , the desired estimate follows. Thus the proof of the lemma is completed. \square

Lemma 3.5 (Estimate on \mathcal{P}_4). *Let $3/2 \leq \sigma \leq 2$. Then for any small $\kappa > 0$, and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho}$, we have*

$$\begin{aligned}
\mathcal{P}_4 &\leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + C \kappa^{-1} m^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds \\
&\quad + \frac{C \kappa^{-1} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right).
\end{aligned}$$

Proof. Let χ_2 be the function given in (10). We can decompose \mathcal{P}_4 by

$$\begin{aligned}
\mathcal{P}_4 &= - \int_0^t \left(2 \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega) \partial_x^{m-j} \partial_y \omega, m^2 g_m \right)_{L^2} ds, \\
&= -2 \sum_{j=0}^{m-1} \binom{m-1}{j} \int_0^t \left(\chi_2 (\partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega) \partial_x^{m-j} \partial_y \omega, m^2 g_m \right)_{L^2} ds \\
&\quad -2 \sum_{j=0}^{m-1} \binom{m-1}{j} \int_0^t \left((1 - \chi_2) (\partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega) \partial_x^{m-j} \partial_y \omega, m^2 g_m \right)_{L^2} ds \\
&\stackrel{\text{def}}{=} S_1 + S_2,
\end{aligned}$$

Estimate on S_1 : Note that $S_1 \leq S_{1,1} + S_{1,2}$ with

$$\begin{aligned}
S_{1,1} &= 2m \sum_{j=0}^{[m/2]} \frac{(m-1)!}{j!(m-1-j)!} \int_0^t \left\| \partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega \right\|_{L^\infty} \left\| \chi_2 \partial_x^{m-j} \partial_y \omega \right\|_{L^2} \left(m \|g_m\|_{L^2} \right) ds, \\
S_{1,2} &= 2m \sum_{j=[m/2]+1}^{m-1} \frac{(m-1)!}{j!(m-1-j)!} \int_0^t \left\| \chi_2 \partial_x^j \partial_y \omega \right\|_{L^2} \left\| \partial_x^{m-j} \partial_y \omega \right\|_{L^\infty} \left(m \|g_m\|_{L^2} \right) ds.
\end{aligned}$$

Moreover, we use (20) and (21) in Lemma 3.2 and the Sobolev inequality (see Lemma A.1 in the Appendix), by following the arguments used for the terms J_1 - J_3 in Lemma 3.4, to obtain

$$\begin{aligned}
S_{1,1} &\leq Cm \sum_{j=3}^{[m/2]} \frac{(m-1)!}{j!(m-1-j)!} \frac{[(j-3)!]^\sigma}{\rho^{j-2}} \frac{[(m-j-6)!]^\sigma}{\rho^{m-j-5}} \frac{[(m-6)!]^\sigma}{\rho^{m-5}} \int_0^t |u(s)|_{\rho,\sigma}^3 ds \\
&\quad + Cm \sum_{0 \leq j \leq 2} \frac{(m-1)!}{j!(m-1-j)!} \int_0^t \frac{[(m-j-6)!]^\sigma}{\tilde{\rho}^{m-j-5}} \frac{[(m-6)!]^\sigma}{\tilde{\rho}^{m-5}} \left\| \partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega \right\|_{L^\infty} |u(s)|_{\tilde{\rho},\sigma}^2 ds \\
&\leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho,\sigma}^3 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right).
\end{aligned}$$

Similar estimate holds for $S_{1,2}$. Thus, we conclude that

$$S_1 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho,\sigma}^4 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^3}{\tilde{\rho} - \rho} ds \right). \quad (26)$$

Estimate on S_2 : Write $S_2 = S_{2,1} + S_{2,2}$ with

$$\begin{aligned}
S_{2,1} &= -2m^2 \sum_{j=0}^{[m/2]} \binom{m-1}{j} \int_0^t \left((1 - \chi_2) (\partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega) \partial_x^{m-j} \partial_y \omega, g_m \right)_{L^2} ds, \\
S_{2,2} &= -2m^2 \sum_{j=[m/2]+1}^{m-1} \binom{m-1}{j} \int_0^t \left((1 - \chi_2) (\partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega) \partial_x^{m-j} \partial_y \omega, g_m \right)_{L^2} ds.
\end{aligned}$$

Following the arguments for J_1 - J_3 in Lemma 3.4, we have

$$S_{2,2} \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho,\sigma}^3 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \quad (27)$$

As for $S_{2,1}$, integration by parts yields

$$\begin{aligned}
S_{2,1} &= 2m^2 \int_0^t \left((1 - \chi_2) (\partial_y \omega^s + \partial_y \omega) \partial_x^m \omega, \partial_y g_m \right)_{L^2} ds \\
&+ 2m^2 \sum_{j=1}^{[m/2]} \binom{m-1}{j} \int_0^t \left((1 - \chi_2) (\partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega) \partial_x^{m-j} \omega, \partial_y g_m \right)_{L^2} ds \\
&+ 2m^2 \sum_{j=0}^{[m/2]} \binom{m-1}{j} \int_0^t \left(\left[\partial_y \left((1 - \chi_2) (\partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega) \right) \right] \partial_x^{m-j} \omega, g_m \right)_{L^2} ds,
\end{aligned} \tag{28}$$

where the last two terms on the right side of (28) are bounded above by

$$\kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^3 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right),$$

for any small $\kappa > 0$. This can be derived from a similar calculation as in Lemma 3.4, observing $3/2 \leq \sigma \leq 2$. It remains to treat the first term on the right side of (28), for this, we claim that

$$\begin{aligned}
&2m^2 \int_0^t \left((1 - \chi_2) (\partial_y \omega^s + \partial_y \omega) \partial_x^m \omega, \partial_y g_m \right)_{L^2} ds \\
&\leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + C\kappa^{-1} m^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds \\
&+ \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right).
\end{aligned} \tag{29}$$

To confirm this, we use the fact that $|\omega^s + \omega| \geq c_1/4$ on $\text{supp}(1 - \chi_2)$ to write, in view of (23),

$$\partial_x^m \omega = \frac{\tilde{g}_m}{\omega^s + \omega} + \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \partial_x^m u \quad \text{for } y \in \text{supp}(1 - \chi_2).$$

As a result, for any $\kappa > 0$, we use (8) to have

$$\begin{aligned}
&2m^2 \int_0^t \left((1 - \chi_2) (\partial_y \omega^s + \partial_y \omega) \partial_x^m \omega, \partial_y g_m \right)_{L^2} dt \\
&= 2m^2 \int_0^t \left((1 - \chi_2) (\partial_y \omega^s + \partial_y \omega) \frac{\tilde{g}_m}{\omega^s + \omega}, \partial_y g_m \right)_{L^2} dt \\
&+ 2m^2 \int_0^t \left((1 - \chi_2) (\partial_y \omega^s + \partial_y \omega) \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \partial_x^m u, \partial_y g_m \right)_{L^2} dt \\
&\leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + C\kappa^{-1} m^2 \int_0^t \|(g_m - \tilde{g}_m)\|_{L^2}^2 ds + C\kappa^{-1} m^2 \int_0^t \|g_m\|_{L^2}^2 ds \\
&- 2m^2 \int_0^t \left((\partial_x^m u) \partial_y \left[(1 - \chi_2) (\partial_y \omega^s + \partial_y \omega) \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \right], g_m \right)_{L^2} dt \\
&- 2m^2 \int_0^t \left((1 - \chi_2) (\partial_y \omega^s + \partial_y \omega) \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \partial_x^m \omega, g_m \right)_{L^2} dt,
\end{aligned}$$

where the last three terms on the right of the above inequality are bounded above by

$$\frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho,\sigma}^2 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right)$$

by using the statements (iii) and (iv) in Lemma 3.2. Furthermore, by Lemma 3.3 we have

$$\begin{aligned} m^2 \int_0^t \|(g_m - \tilde{g}_m)\|_{L^2}^2 &\leq C m^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u(s)|_{\tilde{\rho},\sigma}^2 \frac{m^{2+2-2\sigma} \rho^{2(m-5)}}{\tilde{\rho}^{2(m-5)}} ds \\ &\quad + \frac{C m^{2+4-4\sigma} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u(s)|_{\rho,\sigma}^4 ds \\ &\leq C m^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho,\sigma}^4 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

where for the last inequality we again use the fact that $\sigma \geq 3/2$ and (18). Combining these estimates gives (29). Consequently, in view of (28) we conclude

$$\begin{aligned} S_{2,1} &\leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + C \kappa^{-1} m^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds \\ &\quad + \frac{C \kappa^{-1} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

which along with (27) yields

$$\begin{aligned} S_2 &\leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + C \kappa^{-1} m^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds \\ &\quad + \frac{C \kappa^{-1} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Combining the estimates on S_1 and S_2 yields the desired estimate, and this completes the proof of the lemma. \square

Lemma 3.6 (Estimate on \mathcal{P}_1 and \mathcal{P}_2). *Let $3/2 \leq \sigma \leq 2$. Then for any small $\kappa > 0$, and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho}$, we have*

$$\mathcal{P}_1 + \mathcal{P}_2 \leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + \frac{C \kappa^{-1} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^3 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right).$$

Proof. Following the argument in Lemma 3.4, we can obtain that

$$\begin{aligned} \mathcal{P}_1 &= - \int_0^t \left(\sum_{j=1}^{m-1} \binom{m-1}{j} (\partial_x^j u) g_{m-j+1}, m^2 g_m \right)_{L^2} ds \\ &\leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho,\sigma}^3 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

It remains to treat \mathcal{P}_2 . Firstly, integration by parts gives,

$$\begin{aligned} \mathcal{P}_2 &= - \int_0^t \left(\sum_{j=1}^{m-1} \binom{m-1}{j} (\partial_x^j v) \partial_y g_{m-j}, m^2 g_m \right)_{L^2} ds \\ &= - \int_0^t \left((\partial_x^{m-1} v) \partial_y g_1, m^2 g_m \right)_{L^2} ds + m^2 \int_0^t \left(\sum_{j=1}^{m-2} \binom{m-1}{j} (\partial_x^j v) g_{m-j}, \partial_y g_m \right)_{L^2} ds \\ &\quad - m^2 \int_0^t \left(\sum_{j=1}^{m-2} \binom{m-1}{j} (\partial_x^{j+1} u) g_{m-j}, g_m \right)_{L^2} ds. \end{aligned}$$

Moreover, by observing $m^{4-2\sigma} \leq m$ due to $\sigma \geq 3/2$, for any small $\kappa > 0$ we have

$$\begin{aligned} & m^2 \int_0^t \left(\sum_{j=1}^{m-2} \binom{m-1}{j} (\partial_x^j v) g_{m-j}, \partial_y g_m \right)_{L^2} ds \\ & \leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + \kappa^{-1} m^2 \int_0^t \left[\sum_{j=1}^{m-2} \frac{(m-1)!}{j!(m-j)!} \|(\partial_x^j v) g_{m-j}\|_{L^2} \right]^2 ds \\ & \leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho,\sigma}^4 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

where for the last inequality we have again used the argument for Lemma 3.4. Similarly, we have

$$\begin{aligned} & - \int_0^t \left((\partial_x^{m-1} v) \partial_y g_1, m^2 g_m \right)_{L^2} ds - m^2 \int_0^t \left(\sum_{j=1}^{m-2} \binom{m-1}{j} (\partial_x^{j+1} u) g_{m-j}, g_m \right)_{L^2} ds \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t \left(|u(s)|_{\rho,\sigma}^3 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Thus we obtain

$$\mathcal{P}_2 \leq \kappa m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^3 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right).$$

This along with the upper bound for \mathcal{P}_1 completes the proof of the lemma. \square

Lemma 3.7 (Estimate on \mathcal{P}_5 and \mathcal{P}_6). *Let $3/2 \leq \sigma \leq 2$. Then for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho}$ and for any small $\kappa > 0$, we have*

$$\begin{aligned} \mathcal{P}_5 + \mathcal{P}_6 & \leq \kappa \varepsilon m^2 \int_0^t \left(\|\partial_x g_m\|_{L^2}^2 + \|\partial_y g_m\|_{L^2}^2 \right) ds + \kappa^{-1} \varepsilon \int_0^t \left(\|\partial_x^{m+1} u\|_{L^2}^2 + \|\partial_x^{m+1} \omega\|_{L^2}^2 \right) ds \\ & \quad + \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Proof. We only need to handle \mathcal{P}_5 , because the estimation on \mathcal{P}_6 is similar so that we omit it for brevity. Integrating by parts yields, for any $\kappa > 0$,

$$\begin{aligned} \mathcal{P}_5 & = \int_0^t \left(2\varepsilon \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^{j+1} \partial_y \omega) \partial_x^{m-j+1} u, m^2 g_m \right)_{L^2} ds \\ & = -2\varepsilon m^2 \int_0^t \left(\sum_{j=0}^{[m/2]} \frac{(m-1)!}{j!(m-1-j)!} (\partial_x^{j+1} \partial_y \omega) \partial_x^{m-j} u, \partial_x g_m \right)_{L^2} ds \\ & \quad - 2\varepsilon m^2 \int_0^t \left(\sum_{j=0}^{[m/2]} \frac{(m-1)!}{j!(m-1-j)!} (\partial_x^{j+2} \partial_y \omega) \partial_x^{m-j} u, g_m \right)_{L^2} ds \\ & \quad - 2\varepsilon m^2 \int_0^t \left(\sum_{j=[m/2]+1}^{m-1} \frac{(m-1)!}{j!(m-1-j)!} (\partial_x^{j+1} \omega) \partial_x^{m-j+1} u, \partial_y g_m \right)_{L^2} ds \\ & \quad - 2\varepsilon m^2 \int_0^t \left(\sum_{j=[m/2]+1}^{m-1} \frac{(m-1)!}{j!(m-1-j)!} (\partial_x^{j+1} \omega) \partial_x^{m-j+1} \omega, g_m \right)_{L^2} ds \\ & \stackrel{\text{def}}{=} K_1 + K_2 + K_3 + K_4. \end{aligned}$$

Moreover, following the arguments used in Lemma 3.4 for J_3 , we see the

$$K_2 + K_4 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho,\sigma}^3 dt + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} dt \right).$$

As for K_3 we have, for any $\kappa > 0$,

$$\begin{aligned} K_3 &\leq 2\kappa\epsilon m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + \kappa^{-1}\epsilon m^2 \int_0^t \|\partial_x^2 u\|_{L^\infty}^2 \|\partial_x^m \omega\|_{L^2}^2 ds \\ &\quad + \kappa^{-1}\epsilon m^2 \int_0^t \left[\sum_{j=[m/2]+1}^{m-2} \frac{(m-1)!}{j!(m-1-j)!} \|\partial_x^{m-j+1} u\|_{L^\infty} \|\partial_x^{j+1} \omega\|_{L^2} \right]^2 ds, \end{aligned}$$

where the last term was bounded above by

$$\frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho,\sigma}^4 dt + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} dt \right),$$

which can be derived similarly as the terms J_1 - J_4 in Lemma 3.4. On the other hand, for the second term above, we use the interpolation inequality to obtain, observing the fact that $3/2 \leq \sigma \leq 2$ as well as the last inequality in (8),

$$\begin{aligned} \kappa^{-1}\epsilon m^2 \int_0^t \|\partial_x^2 u\|_{L^\infty}^2 \|\partial_x^m \omega\|_{L^2}^2 ds &\leq \kappa^{-1}\epsilon m^2 \int_0^t \left(m^{-2} \|\partial_x^{m+1} \omega\|_{L^2}^2 + m^2 \|\partial_x^{m-1} \omega\|_{L^2}^2 \right) ds \\ &\leq \kappa^{-1}\epsilon \int_0^t \|\partial_x^{m+1} \omega\|_{L^2}^2 ds + \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds. \end{aligned}$$

Thus, combining the estimates above we obtain the upper bound for K_3 , that is,

$$\begin{aligned} K_3 &\leq 2\kappa\epsilon m^2 \int_0^t \|\partial_y g_m\|_{L^2}^2 ds + \kappa^{-1}\epsilon \int_0^t \|\partial_x^{m+1} \omega\|_{L^2}^2 ds \\ &\quad + \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t \left(|u(s)|_{\rho,\sigma}^3 + |u(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

The estimation on K_1 is similar, and we have

$$\begin{aligned} K_1 &\leq \kappa\epsilon m^2 \int_0^t \|\partial_x g_m\|_{L^2}^2 ds + \kappa^{-1}\epsilon \int_0^t \|\partial_x^{m+1} u\|_{L^2}^2 ds \\ &\quad + \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t \left(|u(s)|_{\rho,\sigma}^3 + |u(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Then the upper bound for \mathcal{P}_5 follows. Similar argument works for \mathcal{P}_6 . Then the proof is then completed. \square

Completion of the proof of Proposition 3.1. Combining (25) and the estimates in lemmas 3.4-3.7, we have, for any $\kappa > 0$,

$$\begin{aligned} &\frac{1}{2}m^2 \|g_m(t)\|_{L^2}^2 + m^2 \int_0^t \|\partial_y g_m(s)\|_{L^2}^2 ds + \epsilon m^2 \int_0^t \|\partial_x g_m\|_{L^2}^2 ds \\ &\leq \frac{1}{2}m^2 \|g_m(0)\|_{L^2}^2 + 4\kappa m^2 \int_0^t \|\partial_y g_m(s)\|_{L^2}^2 ds + \kappa\epsilon m^2 \int_0^t \|\partial_x g_m\|_{L^2}^2 ds \\ &\quad + \kappa^{-1}\epsilon \int_0^t \left(\|\partial_x^{m+1} u\|_{L^2}^2 + \|\partial_x^{m+1} \omega\|_{L^2}^2 \right) ds + C\kappa^{-1}m^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds \\ &\quad + \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t \left(|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4 \right) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

The second and third terms on the right sides can be absorbed provided $\kappa \leq 1/4$. Moreover, recalling the definition of $|\cdot|_{\rho,\sigma}$ (see Definition 2.2), we have

$$m^2 \|g_m(0)\|_{L^2}^2 \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2.$$

Thus the desired estimate in Proposition 3.1 follows and the proof is completed. \square

4. PROOF OF THEOREM 2.4: UNIFORM ESTIMATES AWAY FROM THE CRITICAL POINT

In this section, we will perform estimates in the domain where $u^s + u$ admits monotonicity, and derive uniform upper bound for f_m appearing the definition of $|\cdot|$ (see Definition 2.2). Recall f_m is defined in (12), that is,

$$f_m = \chi_1 \partial_x^m \omega - \chi_1 \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \partial_x^m u = \chi_1 (\omega^s + \omega) \partial_y \left(\frac{\partial_x^m u}{\omega^s + \omega} \right), \quad m \geq 1, \quad (30)$$

with χ_1 given in (9). Moreover, we denote \tilde{f}_m the main component of f_m by

$$\tilde{f}_m = \chi_1' \partial_x^m \omega - \chi_1' \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \partial_x^m u = \chi_1' (\omega^s + \omega) \partial_y \left(\frac{\partial_x^m u}{\omega^s + \omega} \right), \quad m \geq 1. \quad (31)$$

The main result in this section is the following proposition.

Proposition 4.1. *Let $m \geq 6$ and $u \in L^\infty([0, T]; X_{\rho_0, \sigma})$ be a solution to (7) under the assumptions in Theorem 2.4. Then we have, for any $t \in [0, T]$, and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} \leq \rho_0$,*

$$\begin{aligned} & \left\| \langle y \rangle^\ell f_m(t) \right\|_{L^2}^2 + \left\| \tilde{f}_m(t) \right\|_{L^2}^2 + \int_0^t \left\| \langle y \rangle^\ell \partial_y f_m \right\|_{L^2}^2 ds + \varepsilon \int_0^t \left(\left\| \langle y \rangle^\ell \partial_x f_m \right\|_{L^2}^2 + \left\| \partial_x \tilde{f}_m \right\|_{L^2}^2 \right) ds \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Before presenting the proof of the above proposition, we give an immediate corollary.

Corollary 4.2. *Let $m \geq 6$ and $u \in L^\infty([0, T]; X_{\rho_0, \sigma})$ be a solution to (7) under the assumptions in Theorem 2.4. Then we have*

$$\begin{aligned} m^2 \|g_m(t)\|_{L^2}^2 & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + C \int_0^t \left(\varepsilon \|\partial_x^{m+1} u\|_{L^2}^2 + \varepsilon \|\partial_x^{m+1} \omega\|_{L^2}^2 \right) ds \\ & \quad + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

and

$$\begin{aligned} & m^{2\sigma-1} \int_0^t \|g_m(s) - \tilde{g}_m(s)\|_{L^2}^2 ds \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Proof of the corollary. The first inequality follows from Proposition 3.1 and Proposition 4.1, and the second one holds because Lemma 3.3 and Proposition 4.1 as well as the fact that $\sigma \geq 3/2$, since applying Proposition 4.1 for $m-1$ gives

$$\begin{aligned} & m^2 \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds \leq m^{2\sigma-1} \int_0^t \|\partial_y f_{m-1}\|_{L^2}^2 ds \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

The proof is completed. \square

The rest of this section is devote to proving Proposition 4.1 by the following lemmas and the main tool used here is the cancellation property observed in [15].

Lemma 4.3. *The functions f_m and \tilde{f}_m defined in (30) and (31) satisfy the following equations:*

$$\left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) f_m = F_{m,\varepsilon}, \quad (32)$$

and

$$\left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) \tilde{f}_m = \tilde{F}_{m,\varepsilon}, \quad (33)$$

where

$$F_{m,\varepsilon} = -\chi_1 \sum_{k=1}^m \binom{m}{k} \left(\partial_x^k u \right) \partial_x^{m-k+1} \omega - \chi_1 \sum_{k=1}^{m-1} \binom{m}{k} \left(\partial_x^k v \right) \partial_x^{m-k} \partial_y \omega \quad (34)$$

$$+ \chi_1 a \sum_{k=1}^m \binom{m}{k} \left(\partial_x^k u \right) \partial_x^{m-k+1} u + \chi_1 a \sum_{k=1}^{m-1} \binom{m}{k} \left(\partial_x^k v \right) \partial_x^{m-k} \omega \quad (35)$$

$$+ \chi_1' v \partial_x^m \omega - 2\chi_1' \partial_x^m \partial_y \omega - \chi_1'' \partial_x^m \omega - a \left(\chi_1' v \partial_x^m u - 2\chi_1' \partial_x^m \omega - \chi_1'' \partial_x^m u \right) \quad (36)$$

$$+ \left[\partial_x \omega - (\partial_x u) a - 2a \partial_y a - 2\varepsilon \frac{\partial_x \omega}{\omega^s + \omega} \partial_x a \right] \chi_1 \partial_x^m u \quad (37)$$

$$+ 2\chi_1 (\partial_y a) \partial_x^m \omega + 2\chi_1' (\partial_y a) \partial_x^m u + 2\varepsilon \chi_1 (\partial_x a) \partial_x^{m+1} u \quad (38)$$

with

$$a = \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega},$$

and the representation of $\tilde{F}_{m,\varepsilon}$ is quite similar to $F_{m,\varepsilon}$, with the functions χ_1, χ_1' and χ_1'' in (34)-(38) replaced by χ_1', χ_1'' and χ_1''' respectively. Moreover,

$$\partial_y f_m|_{y=0} = 0. \quad (39)$$

Proof. This proof is based on direct calculation that will be sketched in the Appendix (see Lemma B.1). \square

In the next two lemmas, we will derive the energy estimates on f_m and \tilde{f}_m , starting from the equations (32) and (33).

Lemma 4.4. *We have*

$$\frac{1}{2} \frac{d}{dt} \left\| \langle y \rangle^\ell f_m \right\|_{L^2}^2 + \frac{1}{2} \left\| \langle y \rangle^\ell \partial_y f_m \right\|_{L^2}^2 + \varepsilon \left\| \langle y \rangle^\ell \partial_x f_m \right\|_{L^2}^2 \leq \left(\langle y \rangle^\ell F_{m,\varepsilon}, \langle y \rangle^\ell f_m \right)_{L^2} + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u|_{\rho,\sigma}^2.$$

The above estimate also holds when $F_{m,\varepsilon}$ and f_m are replaced respectively by $\tilde{F}_{m,\varepsilon}$ and \tilde{f}_m .

Proof. We multiply both sides of (32) by $\langle y \rangle^{2\ell} f_m$ and then take integration over \mathbb{R}_+^2 ; Integrating by parts with the boundary condition (39) gives

$$\begin{aligned} \left(\langle y \rangle^\ell F_{m,\varepsilon}, \langle y \rangle^\ell f_m \right)_{L^2} &= \int_{\mathbb{R}_+^2} \left((\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2) f_m \right) \langle y \rangle^{2\ell} f_m \, dx dy \\ &= \frac{1}{2} \frac{d}{dt} \left\| \langle y \rangle^\ell f_m \right\|_{L^2}^2 + \left\| \langle y \rangle^\ell \partial_y f_m \right\|_{L^2}^2 + \varepsilon \left\| \langle y \rangle^\ell \partial_x f_m \right\|_{L^2}^2 \\ &\quad + \int_{\mathbb{R}_+^2} \left(\partial_y \langle y \rangle^{2\ell} \right) (\partial_y f_m) f_m \, dx dy - \int_{\mathbb{R}_+^2} v \left(\partial_y \langle y \rangle^\ell \right) \langle y \rangle^\ell f_m^2 \, dx dy. \end{aligned}$$

Moreover, as for the last two terms on the right side, using the last inequality in (8) as well as (20), we have

$$\begin{aligned} & \left| \int_{\mathbb{R}_+^2} \left(\partial_y \langle y \rangle^{2\ell} \right) (\partial_y f_m) f_m dx dy \right| + \left| \int_{\mathbb{R}_+^2} v \left(\partial_y \langle y \rangle^\ell \right) \langle y \rangle^\ell f_m^2 dx dy \right| \\ & \leq \frac{1}{2} \left\| \langle y \rangle^\ell \partial_y f_m \right\|_{L^2}^2 + C \left\| \langle y \rangle^\ell f_m \right\|^2 \leq \frac{1}{2} \left\| \langle y \rangle^\ell \partial_y f_m \right\|_{L^2}^2 + \frac{C [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u|_{\rho, \sigma}^2. \end{aligned}$$

Combining the above equalities gives the desired estimate and then completes the proof of the lemma. \square

Lemma 4.5. *Let $3/2 \leq \sigma \leq 2$. We have, for any $\kappa > 0$, and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho}$,*

$$\left(\langle y \rangle^\ell F_{m, \ell}, \langle y \rangle^\ell f_m \right)_{L^2} \leq \kappa \varepsilon \left\| \langle y \rangle^\ell \partial_x f_m \right\|_{L^2}^2 + \frac{C \kappa^{-1} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|u|_{\rho, \sigma}^2 + |u|_{\rho, \sigma}^4 + \frac{|u|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} \right).$$

The above estimate also holds with $F_{m, \varepsilon}$ and f_m replaced by $\tilde{F}_{m, \varepsilon}$ and \tilde{f}_m respectively.

Proof. We only need prove the first statement. To do so, we estimate term by term in the representation of $F_{m, \varepsilon}$.

Estimate on the terms in (34)-(36) : We apply similar arguments as for J_1 - J_3 used in Lemma 3.4 to obtain that

$$\begin{aligned} & \left(\langle y \rangle^\ell \chi_1 \sum_{k=1}^{m-1} \binom{m}{k} \left(\partial_x^k v \right) \partial_x^{m-k} \partial_y \omega, \langle y \rangle^\ell f_m \right)_{L^2} + \left(\langle y \rangle^\ell \chi_1 \sum_{k=1}^m \binom{m}{k} \left(\partial_x^k u \right) \partial_x^{m-k+1} \omega, \langle y \rangle^\ell f_m \right)_{L^2} \\ & \leq \frac{C [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|u|_{\rho, \sigma}^3 + \frac{|u|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} \right). \end{aligned}$$

This gives the upper bound for the terms in (34). Similarly, observe $|a(t, x, y)| \leq C \langle y \rangle^{-1}$ and thus

$$\begin{aligned} & \left(\langle y \rangle^\ell \chi_1 a \sum_{k=1}^m \binom{m}{k} \left(\partial_x^k u \right) \partial_x^{m-k+1} u + \langle y \rangle^\ell \chi_1 a \sum_{k=1}^{m-1} \binom{m}{k} \left(\partial_x^k v \right) \partial_x^{m-k} \omega, \langle y \rangle^\ell f_m \right)_{L^2} \\ & \leq \frac{C [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|u|_{\rho, \sigma}^3 + \frac{|u|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} \right). \end{aligned}$$

This gives the estimates on the terms in (35). Furthermore, observing $\chi_1' \partial_x^m \partial_y \omega = \chi_1' \chi_2 \partial_x^m \partial_y \omega$ due to (11) and thus using (20), we obtain

$$\begin{aligned} & \left(\langle y \rangle^\ell \left[\chi_1' v \partial_x^m \omega - 2 \chi_1' \partial_x^m \partial_y \omega - \chi_1'' \partial_x^m \omega - a \left(\chi_1' v \partial_x^m u - 2 \chi_1' \partial_x^m \omega - \chi_1'' \partial_x^m u \right) \right], \langle y \rangle^\ell f_m \right)_{L^2} \\ & \leq \frac{C [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u|_{\rho, \sigma}^2. \end{aligned}$$

This gives the upper bound for the terms in (36).

Estimate on the terms in (37)-(38): As for the term in (37), we can verify that, for any $y \in \text{supp } \chi_1$,

$$\left| \frac{\partial_x \omega}{\omega^s + \omega} (t, x, y) \right| \leq C \langle y \rangle^\alpha |\partial_x \omega(t, x, y)| \leq C \langle y \rangle^\ell |\partial_x \omega(t, x, y)| \leq C$$

due to the fact that $\alpha \leq \ell$ and the last inequality in (8). Similarly using (8) gives, for any $y \in \text{supp } \chi_1$,

$$|a(t, x, y)| + |\partial_x a(t, x, y)| + |\partial_x^2 a(t, x, y)| \leq C \langle y \rangle^{-1},$$

and

$$|\partial_y a(t, x, y)| \leq C \langle y \rangle^{-1} \left(1 + |u|_{\rho, \sigma}\right).$$

Hence, we have

$$\left(\langle y \rangle^\ell \left[\partial_x \omega - (\partial_x u) a - 2\varepsilon \frac{\partial_x \omega}{\omega^s + \omega} \partial_x a \right] \chi_1 \partial_x^m u, \langle y \rangle^\ell f_m \right)_{L^2} \leq C \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u|_{\rho, \sigma}^2,$$

and, for any $\kappa > 0$,

$$\begin{aligned} & \left(\langle y \rangle^\ell \left[2\chi_1 (\partial_y a) \partial_x^m \omega + 2\chi_1' (\partial_y a) \partial_x^m u + 2\varepsilon \chi_1 (\partial_x a) \partial_x^{m+1} u \right], \langle y \rangle^\ell f_m \right)_{L^2} \\ & \leq \kappa \varepsilon \left\| \langle y \rangle^\ell \partial_x f_m \right\|_{L^2}^2 + \frac{C \kappa^{-1} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|u|_{\rho, \sigma}^2 + |u|_{\rho, \sigma}^3 \right). \end{aligned}$$

This gives the upper bound for the terms in (37)-(38). The proof of Lemma 4.5 is completed. \square

Completion of the proof of Proposition 4.1. Combining Lemma 4.4 and Lemma 4.5, we have for any $\kappa > 0$,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\| \langle y \rangle^\ell f_m \right\|_{L^2}^2 + \frac{1}{2} \left\| \langle y \rangle^\ell \partial_y f_m \right\|_{L^2}^2 + \varepsilon \left\| \langle y \rangle^\ell \partial_x f_m \right\|_{L^2}^2 \\ & \leq \kappa \varepsilon \left\| \langle y \rangle^\ell \partial_x f_m \right\|_{L^2}^2 + \frac{C \kappa^{-1} [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(|u|_{\rho, \sigma}^2 + |u|_{\rho, \sigma}^4 + \frac{|u|_{\rho, \sigma}^2}{\tilde{\rho} - \rho} \right). \end{aligned}$$

Letting κ be small sufficiently and then taking integration over $[0, t]$ yields the estimate on f_m as stated in Proposition 4.1 because

$$\left\| \langle y \rangle^\ell f_m(0) \right\|_{L^2}^2 \leq \frac{C [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho, \sigma}^2.$$

The estimation on \tilde{f}_m is the same as that of f_m . The proof is thus then completed. \square

5. PROOF OF THEOREM 2.4: UNIFORM ESTIMATES NEAR THE CRITICAL POINT

Here we will perform the estimation, by virtue of the cut-off function χ_2 introduced in(10), in the domain that contains the non-degenerate critical point. Precisely, in this part we will work on the terms h_m and $\chi_2 \partial_y \partial_x^m \omega$, recalling

$$h_m = \chi_2 \partial_x^m \partial_y \omega - \chi_2 \frac{\partial_y^2 \omega^s + \partial_y^2 \omega}{\partial_y \omega^s + \partial_y \omega} \partial_x^m \omega, \quad m \geq 1. \quad (40)$$

The main result can be stated as follows.

Proposition 5.1. *Let $m \geq 6$ and let $u \in L^\infty([0, T]; X_{\rho_0, \sigma})$ be the solution to (7) under the assumptions in Theorem 2.4. Then we have, for any $t \in [0, T]$, and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$,*

$$\begin{aligned} & \|h_m\|_{L^2}^2 + \|\chi_2 \partial_y \partial_x^m \omega\|_{L^2}^2 \\ & \leq \frac{C [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho, \sigma}^2 + \frac{C [(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

We will prove the above proposition through the following subsections. As a preliminary we first estimate $\chi_2 \partial_x^m \omega$ in Subsection 5.1. The estimation on h_m and $\chi_2 \partial_y \partial_x^m \omega$ is given in Subsection 5.2.

5.1. Uniform upper bound for $\chi_2 \partial_x^m \omega$. Here we estimate $\chi_2 \partial_x^m \omega$, following the same cancellation method used in [6]. The main result can be stated as follows.

Proposition 5.2. *Let χ_2 be the cut-off function given in (10), and let $u \in L^\infty([0, T]; X_{\rho_0, \sigma})$ be the solution to (7) under the assumptions in Theorem 2.4. We have, for any $t \in [0, T]$, and for any $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho}$,*

$$\begin{aligned} & \|\chi_2 \partial_x^m \omega(t)\|_{L^2}^2 + \int_0^t \|\chi_2 \partial_y \partial_x^m \omega\|_{L^2}^2 + \varepsilon \int_0^t \|\chi_2 \partial_x^{m+1} \omega\|_{L^2}^2 \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho, \sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

The proof follows from the same strategy as in [6]. The key part is to estimate the term

$$|(\chi_2' \partial_x^m v, \chi_2 \partial_x^m u)_{L^2}|.$$

Before presenting the proof of Proposition 5.2, we first recall the upper bound for the term above, established in [6] by virtue of a crucial representations of $\partial_x^m u$ in terms of \hat{g}_m (see [6, Lemma 3]), with \hat{g}_m defined by

$$\hat{g}_m = \left(\psi(\omega^s + \omega) + 1 - \psi \right) \left(\partial_x^m \omega - \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \partial_x^m u \right) = \left(\psi + \frac{1 - \psi}{\omega^s + \omega} \right) (\omega^s + \omega)^2 \partial_y \left(\frac{\partial_x^m u}{\omega^s + \omega} \right), \quad (41)$$

where $m \geq 1$ and $\psi(y) \in C_0^\infty(\mathbb{R})$ is a given function such that $\psi \equiv 1$ in $[0, y_0 + 2\delta]$. Precisely, by implicit function theorem, if the level set $\{(x, y); \omega^s + \omega = 0\}$ of $\omega^s + \omega$ is non-empty and it is a curve in \mathbb{R}_+^2 denoted by $y = \gamma(x)$. Then $\partial_x^m u$ can be represented as

$$\partial_x^m u(t, x, y) = (\omega^s(t, x, y) + \omega(t, x, y)) \int_0^y \left(\psi + \frac{1 - \psi}{\omega^s + \omega} \right)^{-1} \frac{\hat{g}_m}{(\omega^s + \omega)^2} dy,$$

for $y < \gamma(x)$, and for $y > \gamma(x)$

$$\partial_x^m u(t, x, y) = (\omega^s(t, x, y) + \omega(t, x, y)) \left[\int_{y_0+2\delta}^y \left(\psi + \frac{1 - \psi}{\omega^s + \omega} \right)^{-1} \frac{\hat{g}_m}{(\omega^s + \omega)^2} dy + \beta(t, x) \right]$$

with $\beta(t, x) = \partial_x^m u(t, x, y_0 + 2\delta) / (\omega^s(t, y_0 + 2\delta) + \omega(t, x, y_0 + 2\delta))$. By virtue of the above representations we can derive that, cf. [6, Lemma 6],

$$|(\chi_2' \partial_x^m v, \chi_2 \partial_x^m u)_{L^2}| \leq C \|\hat{g}_m\|_{L^2(\mathbb{R}_x \times \{0 \leq y \leq y_0 + 2\delta\})} \|\partial_x^{m+1} \omega\|_{L^2} + C \|\partial_x^m \omega\|_{L^2},$$

and thus

$$|(\chi_2' \partial_x^m v, \chi_2 \partial_x^m u)_{L^2}| \leq C \|\tilde{g}_m\|_{L^2} \|\partial_x^{m+1} \omega\|_{L^2} + C \|\partial_x^m \omega\|_{L^2}, \quad (42)$$

since $\hat{g}_m = \tilde{g}_m$ for $y \in [0, y_0 + 2\delta]$.

The rest is for the proof of Proposition 5.2. We first have the equation for $\chi_2 \partial_x^m \omega$:

$$\begin{aligned} & \left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) \chi_2 \partial_x^m \omega \\ & = -\chi_2 \sum_{k=1}^m \binom{m}{k} \left(\partial_x^k u \right) \partial_x^{m-k+1} \omega - \chi_2 (\partial_y \omega^s + \partial_y \omega) \partial_x^m v - \chi_2 \sum_{k=1}^{m-1} \binom{m}{k} \left(\partial_x^k v \right) \partial_x^{m-k} \partial_y \omega \\ & \quad + \chi_2' v \partial_x^m \omega - \chi_2'' \partial_x^m \omega - 2\chi_2' \partial_x^m \partial_y \omega. \end{aligned}$$

This can be derived directly from the equation of the vorticity ω . In view of (8), we see $|\partial_y \omega^s + \partial_y \omega| \geq c_0/4$ on $\text{supp } \chi_2$, and without loss of generality, we can assume $|\partial_y \omega^s + \partial_y \omega| = -(\partial_y \omega^s + \partial_y \omega)$ on

supp χ_2 . This enables us to take L^2 inner product on both sides of the above equation with the function

$$-\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega}.$$

This gives

$$\begin{aligned} & \left(\left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) \chi_2 \partial_x^m \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} \\ &= (\chi_2 \partial_x^m v, \chi_2 \partial_x^m \omega)_{L^2} - \left(\chi_2 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} \\ & \quad - \left(\chi_2 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} \\ & \quad + \left(\chi_2' v \partial_x^m \omega - \chi_2'' \partial_x^m \omega - 2\chi_2' \partial_x^m \partial_y \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2}. \end{aligned} \quad (43)$$

As for the last three terms on the right side of the above equation, we follow the argument used in Lemma 4.5 to get

$$\begin{aligned} & \left| \int_0^t \left(\chi_2 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} ds \right| \\ & \quad + \left| \int_0^t \left(\chi_2 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} ds \right| \\ & \quad + \left| \int_0^t \left(\chi_2' v \partial_x^m \omega - \chi_2'' \partial_x^m \omega - 2\chi_2' \partial_x^m \partial_y \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} ds \right| \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t |u(s)|_{\rho, \sigma}^3 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned} \quad (44)$$

In the following two lemmas, we will estimate the term on the left hand side of (43) and the first term on the right side respectively.

Lemma 5.3. *We have*

$$\begin{aligned} & \|\chi_2 \partial_x^m \omega(t)\|_{L^2}^2 + \int_0^t \|\chi_2 \partial_y \partial_x^m \omega\|_{L^2}^2 + \varepsilon \int_0^t \|\chi_2 \partial_x^{m+1} \omega\|_{L^2}^2 \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho, \sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u(s)|_{\rho, \sigma}^2 ds \\ & \quad + C \left| \int_0^t \left(\left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) \chi_2 \partial_x^m \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} ds \right|. \end{aligned}$$

Proof. Direct computation shows

$$\begin{aligned} & \left(\partial_t \chi_2 \partial_x^m \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} \\ &= \frac{1}{2} \frac{d}{dt} \left\| (-\partial_y \omega^s - \partial_y \omega)^{-1/2} \chi_2 \partial_x^m \omega \right\|_{L^2}^2 - \frac{1}{2} \left(\chi_2 \partial_x^m \omega, \frac{\partial_t \partial_y \omega^s + \partial_t \partial_y \omega}{(\partial_y \omega^s + \partial_y \omega)^2} \chi_2 \partial_x^m \omega \right)_{L^2}. \end{aligned}$$

Then integration by parts gives

$$\begin{aligned}
& \left(\left((u^s + u) \partial_x + v \partial_y \right) \chi_2 \partial_x^m \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} \\
&= -\frac{1}{2} \left(\chi_2 \partial_x^m \omega, \frac{\left((u^s + u) \partial_x + v \partial_y \right) \partial_y \omega + v \partial_y^2 \omega^s}{(\partial_y \omega^s + \partial_y \omega)^2} \chi_2 \partial_x^m \omega \right)_{L^2}, \\
& \left(-\partial_y^2 \chi_2 \partial_x^m \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} \\
&= \left\| (-\partial_y \omega^s - \partial_y \omega)^{-1/2} \partial_y (\chi_2 \partial_x^m \omega) \right\|_{L^2}^2 - \frac{1}{2} \left(\chi_2 \partial_x^m \omega, \frac{\partial_y^3 \omega^s + \partial_y^3 \omega}{(\partial_y \omega^s + \partial_y \omega)^2} \chi_2 \partial_x^m \omega \right)_{L^2} \\
&+ \left(\chi_2 \partial_x^m \omega, \frac{(\partial_y^2 \omega^s + \partial_y^2 \omega)^2}{(\partial_y \omega^s + \partial_y \omega)^3} \chi_2 \partial_x^m \omega \right)_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
& \left(-\varepsilon \partial_x^2 \chi_2 \partial_x^m \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} \\
&= \varepsilon \left\| (-\partial_x^2 u^s - \partial_y \omega)^{-1/2} \chi_2 \partial_x^{m+1} \omega \right\|_{L^2}^2 - \frac{1}{2} \left(\chi_2 \partial_x^m \omega, \frac{\varepsilon \partial_x^2 \partial_y \omega}{(\partial_y \omega^s + \partial_y \omega)^2} \chi_2 \partial_x^m \omega \right)_{L^2} \\
&+ \left(\chi_2 \partial_x^m \omega, \frac{\varepsilon (\partial_x \partial_y \omega)^2}{(\partial_y \omega^s + \partial_y \omega)^3} \chi_2 \partial_x^m \omega \right)_{L^2}.
\end{aligned}$$

Moreover, it follows from the equation of the vorticity that

$$\begin{aligned}
& \partial_t \partial_y \omega + \left((u^s + u) \partial_x + v \partial_y \right) \partial_y \omega + v \partial_y^2 \omega^s - \partial_y^3 \omega - \varepsilon \partial_x^2 \partial_y \omega \\
&= -(\omega^s + \omega) \partial_x \omega + (\partial_y \omega^s + \partial_y \omega) \partial_x u.
\end{aligned}$$

Hence, combining these estimates gives

$$\begin{aligned}
& \left(\left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) \chi_2 \partial_x^m \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} \\
&= \left(\left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) \chi_2 \partial_x^m \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} \\
&= \frac{1}{2} \frac{d}{dt} \left\| (-\partial_y \omega^s - \partial_y \omega)^{-1/2} \chi_2 \partial_x^m \omega \right\|_{L^2}^2 + \left\| (-\partial_y \omega^s - \partial_y \omega)^{-1/2} \partial_y (\chi_2 \partial_x^m \omega) \right\|_{L^2}^2 \\
&+ \varepsilon \left\| (-\partial_x^2 u^s - \partial_y \omega)^{-1/2} \chi_2 \partial_x^{m+1} \omega \right\|_{L^2}^2 - \left(\chi_2 \partial_x^m \omega, \frac{\partial_y^3 \omega^s + \partial_y^3 \omega + \varepsilon \partial_x^2 \partial_y \omega}{(\partial_y \omega^s + \partial_y \omega)^2} \chi_2 \partial_x^m \omega \right)_{L^2} \\
&- \frac{1}{2} \left(\chi_2 \partial_x^m \omega, \frac{(\partial_x u) (\partial_y \omega^s + \partial_y \omega) - (\omega^s + \omega) \partial_x \omega}{(\partial_y \omega^s + \partial_y \omega)^2} \chi_2 \partial_x^m \omega \right)_{L^2} \\
&+ \left(\chi_2 \partial_x^m \omega, \frac{(\partial_y^2 \omega^s + \partial_y^2 \omega)^2}{(\partial_y \omega^s + \partial_y \omega)^3} \chi_2 \partial_x^m \omega \right)_{L^2} + \left(\chi_2 \partial_x^m \omega, \frac{\varepsilon (\partial_x \partial_y \omega)^2}{(\partial_y \omega^s + \partial_y \omega)^3} \chi_2 \partial_x^m \omega \right)_{L^2},
\end{aligned}$$

with the modulus of the last four terms on the right side bounded above by

$$\frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u|_{\rho, \sigma}^2$$

due to the inequalities in (8). Thus by integrating both sides over $[0, t]$, we have

$$\begin{aligned} & \left\| (-\partial_y \omega^s - \partial_y \omega)^{-1/2} \chi_2 \partial_x^m \omega(t) \right\|_{L^2}^2 + \int_0^t \left\| (-\partial_y \omega^s - \partial_y \omega)^{-1/2} \chi_2 \partial_y \partial_x^m \omega \right\|_{L^2}^2 ds \\ & \quad + \int_0^t \varepsilon \left\| (-\partial_x^2 u^s - \partial_y \omega)^{-1/2} \chi_2 \partial_x^{m+1} \omega \right\|_{L^2}^2 ds \\ & \leq \left\| (-\partial_y \omega^s(0) - \partial_y \omega(0))^{-1/2} \chi_2 \partial_x^m \omega(0) \right\|_{L^2}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u(s)|_{\rho, \sigma}^2 ds \\ & \quad + 2 \left| \int_0^t \left((\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2) \chi_2 \partial_x^m \omega, -\frac{\chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \right)_{L^2} ds \right|. \end{aligned}$$

Observe that $(-\partial_y \omega^s - \partial_y \omega)^{-1/2} \geq \sqrt{c_0}/2$ on $\text{supp } \chi_2$ and that the first term on the right side is bounded above by

$$\frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho, \sigma}^2.$$

Then the estimate in Lemma 5.3 follows. The proof is completed. \square

Lemma 5.4. *We have*

$$\begin{aligned} & \left| \int_0^t (\chi_2 \partial_x^m v, \chi_2 \partial_x^m \omega)_{L^2} ds \right| \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho, \sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|u(s)|_{\rho, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Proof. Integrating by parts gives

$$(\chi_2 \partial_x^m v, \chi_2 \partial_x^m \omega)_{L^2} = -2 (\chi_2 \partial_x^m v, \chi_2' \partial_x^m u)_{L^2} + (\chi_2 \partial_x^{m+1} u, \chi_2 \partial_x^m \omega)_{L^2} = -2 (\chi_2 \partial_x^m v, \chi_2' \partial_x^m u)_{L^2}.$$

Moreover, in view of (42), we have

$$\begin{aligned} |(\chi_2 \partial_x^m v, \chi_2' \partial_x^m u)_{L^2}| & \leq C \|\tilde{g}_m\|_{L^2} \|\partial_x^{m+1} \omega\|_{L^2} + C \|\partial_x^m \omega\|_{L^2}^2 \\ & \leq C \|g_m\|_{L^2} \|\partial_x^{m+1} \omega\|_{L^2} + C \|g_m - \tilde{g}_m\|_{L^2} \|\partial_x^{m+1} \omega\|_{L^2} + C \|\partial_x^m \omega\|_{L^2}^2. \end{aligned}$$

Thus

$$\begin{aligned} & \left| \int_0^t (\chi_2 \partial_x^m v, \chi_2 \partial_x^m \omega)_{L^2} ds \right| \\ & \leq C \int_0^t \|g_m\|_{L^2} \|\partial_x^{m+1} \omega\|_{L^2} ds + C \int_0^t \|\partial_x^m \omega\|_{L^2}^2 ds + C \int_0^t \|g_m - \tilde{g}_m\|_{L^2} \|\partial_x^{m+1} \omega\|_{L^2} ds. \end{aligned} \tag{45}$$

On the other hand, since $\sigma \leq 2$, we can use (19) and (20) as well as the statements (ii)-(iii) in Lemma 3.2 to get

$$\begin{aligned} \int_0^t \|g_m\|_{L^2} \|\partial_x^{m+1} \omega\|_{L^2} ds & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u|_{\rho, \sigma} |u|_{\tilde{\rho}, \sigma} \frac{m^{\sigma-1} \rho^{m-5}}{\tilde{\rho}^{m-4}} ds \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t \frac{|u(s)|_{\rho, \sigma}^2}{\tilde{\rho} - \rho} ds, \end{aligned}$$

and

$$\int_0^t \|\partial_x^m \omega\|_{L^2}^2 ds \leq \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u|_{\rho, \sigma}^2 ds.$$

Moreover, by the second inequality in Corollary 4.2 we have

$$\begin{aligned} & \int_0^t \|\tilde{g}_m - g_m\|_{L^2} \|\partial_x^{m+1}\omega\|_{L^2} ds \\ & \leq m^{2\sigma-1} \int_0^t \|\tilde{g}_m - g_m\|_{L^2}^2 ds + m^{-2\sigma+1} \int_0^t \|\partial_x^{m+1}\omega\|_{L^2}^2 ds \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

where in the last inequality we have used the statement (iii) in Lemma 3.2. Inserting these inequalities into (45) gives the desired estimate. Thus the proof of Lemma 5.4 is completed. \square

Completion of the proof of Proposition 5.2. In view of (43), we combine (44) and the estimates in Lemmas 5.3-5.4 to conclude

$$\begin{aligned} & \|\chi_2 \partial_x^m \omega(t)\|_{L^2}^2 + \int_0^t \|\chi_2 \partial_y \partial_x^m \omega\|_{L^2}^2 + \varepsilon \int_0^t \|\chi_2 \partial_x^{m+1} \omega\|_{L^2}^2 \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\rho,\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

This is just the estimate in Proposition 5.2. The proof is then completed. \square

5.2. Proof of Proposition 5.1. (uniform estimate for h_m and $\chi_2 \partial_y \partial_x^m \omega$). This part is devoted to proving Proposition 5.1. We begin with the estimation on h_m . Note that h_m solves the equation (see Lemma B.2 in the Appendix for its derivation):

$$\begin{aligned} & (\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2) h_m \\ & = P_m + 2(\partial_y b) \partial_y (\chi_2 \partial_x^m \omega) + 2\varepsilon (\partial_x b) \partial_x (\chi_2 \partial_x^m \omega) \\ & \quad + \chi_2 b \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \omega + \chi_2 b \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y \omega \\ & \quad - \chi_2' b v \partial_x^m \omega + b \chi_2'' \partial_x^m \omega + 2b \chi_2' \partial_x^m \partial_y \omega \\ & \quad - \chi_2 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \partial_y \omega - \chi_2 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y^2 \omega \\ & \quad + \chi_2' v \partial_x^m \partial_y \omega - \chi_2'' \partial_x^m \partial_y \omega - 2\chi_2' \partial_x^m \partial_y^2 \omega - \chi_2 g_{m+1}, \end{aligned} \tag{46}$$

where $b = \frac{\partial_y^2 \omega^s + \partial_y^2 \omega}{\partial_y \omega^s + \partial_y \omega}$ and

$$\begin{aligned} P_m & = \frac{2\left((\omega^s + \omega) \partial_x \partial_y \omega - (\partial_x u) (\partial_y^2 \omega^s + \partial_y^2 \omega)\right) \chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \\ & \quad - \frac{\left(\omega \partial_x \omega - (\partial_x u) (\partial_y \omega^s + \partial_y \omega)\right) (\partial_y^2 \omega^s + \partial_y^2 \omega) \chi_2 \partial_x^m \omega}{(\partial_y \omega^s + \partial_y \omega)^2} \\ & \quad - \frac{2\left((\partial_y^3 \omega^s + \partial_y^3 \omega) (\partial_y^2 \omega^s + \partial_y^2 \omega) + \varepsilon (\partial_x \partial_y^2 \omega) \partial_x \partial_y \omega\right) \chi_2 \partial_x^m \omega}{(\partial_y \omega^s + \partial_y \omega)^2} \\ & \quad + \frac{2\left((\partial_y^2 \omega^s + \partial_y^2 \omega)^2 + \varepsilon (\partial_x \partial_y \omega)^2\right) (\partial_y^2 \omega^s + \partial_y^2 \omega) \chi_2 \partial_x^m \omega}{(\partial_y \omega^s + \partial_y \omega)^3}. \end{aligned}$$

Clearly,

$$\begin{aligned}
& \frac{1}{2} \|h_m(t)\|_{L^2}^2 + \int_0^t \|\partial_y h_m(s)\|_{L^2}^2 ds + \varepsilon \int_0^t \|\partial_x h_m(s)\|_{L^2}^2 ds \\
& \leq \frac{1}{2} \|h_m(0)\|_{L^2}^2 + \int_0^t \left((\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2) h_m, h_m \right)_{L^2} ds \\
& \leq \frac{1}{2} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho, \sigma}^2 + \int_0^t \left((\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2) h_m, h_m \right)_{L^2} ds.
\end{aligned} \tag{47}$$

It remains to estimate the terms on the right hand side of the last inequality that will be given in the following two lemmas.

Lemma 5.5. *We have, for any small $\kappa > 0$,*

$$\begin{aligned}
& \left| \int_0^t \left(P_m + 2(\partial_y b) \partial_y (\chi_2 \partial_x^m \omega) + 2\varepsilon (\partial_x b) \partial_x (\chi_2 \partial_x^m \omega), h_m \right)_{L^2} ds \right| \\
& + \left| \int_0^t \left(\chi_2 b \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \omega + \chi_2 b \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y \omega, h_m \right)_{L^2} ds \right| \\
& + \left| \int_0^t \left(-\chi_2 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \partial_y \omega - \chi_2 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y^2 \omega, h_m \right)_{L^2} ds \right| \\
& \leq \kappa \int_0^t \|\partial_y h_m\|_{L^2}^2 + \kappa \varepsilon \int_0^t \|\partial_x h_m\|_{L^2}^2 + \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u|_{\rho, \sigma}^2 + |u|_{\rho, \sigma}^3) ds + \int_0^t \frac{|u|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right).
\end{aligned}$$

Proof. Since the proof is similar to the one for Lemma 4.5, we omit it for brevity. \square

Lemma 5.6. *Let $\sigma \leq 2$. We have, for any small $\kappa > 0$,*

$$\begin{aligned}
& \left| \int_0^t \left(-\chi_2' b v \partial_x^m \omega + b \chi_2'' \partial_x^m \omega + 2b \chi_2' \partial_x^m \partial_y \omega, h_m \right)_{L^2} ds \right| \\
& + \left| \int_0^t \left(\chi_2' v \partial_x^m \partial_y \omega - \chi_2'' \partial_x^m \partial_y \omega - 2\chi_2' \partial_x^m \partial_y^2 \omega - \chi_2 g_{m+1}, h_m \right)_{L^2} ds \right| \\
& \leq \kappa \int_0^t \|\partial_y h_m\|_{L^2}^2 + C\kappa^{-1} \left(\int_0^t \|\partial_y f_m\|_{L^2}^2 + \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u|_{\rho, \sigma}^2 ds + \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right).
\end{aligned}$$

Proof. It is clear that

$$\left| \int_0^t \left(-\chi_2' b v \partial_x^m \omega + b \chi_2'' \partial_x^m \omega, h_m \right)_{L^2} \right| \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u|_{\rho, \sigma}^2 ds.$$

Moreover, integrating by parts yields, for any $\kappa > 0$,

$$\begin{aligned}
\left| \int_0^t (2b \chi_2' \partial_x^m \partial_y \omega, h_m)_{L^2} \right| & \leq \left| \int_0^t ([\partial_y (2b \chi_2')] \partial_x^m \omega, h_m)_{L^2} \right| + \left| \int_0^t (2b \chi_2' \partial_x^m \omega, \partial_y h_m)_{L^2} \right| \\
& \leq \kappa \int_0^t \|\partial_y h_m\|_{L^2}^2 ds + \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u|_{\rho, \sigma}^2 ds.
\end{aligned}$$

Similarly

$$\left| \int_0^t (\chi_2' v \partial_x^m \partial_y \omega - \chi_2'' \partial_x^m \partial_y \omega, h_m)_{L^2} \right| \leq \kappa \int_0^t \|\partial_y h_m\|_{L^2}^2 ds + \frac{C\kappa^{-1}[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u|_{\rho, \sigma}^2 ds,$$

and

$$\begin{aligned} \left| \int_0^t (2\chi_2' \partial_x^m \partial_y^2 \omega, h_m)_{L^2} \right| &\leq \kappa \int_0^t \|\partial_y h_m\|_{L^2}^2 ds + C\kappa^{-1} \left(\int_0^t \|\partial_y \partial_x^m \omega\|_{L^2}^2 ds + \int_0^t \|h_m\|_{L^2}^2 ds \right) \\ &\leq \kappa \int_0^t \|\partial_y h_m\|_{L^2}^2 ds + C\kappa^{-1} \left(\int_0^t \|\partial_y f_m\|_{L^2}^2 ds + \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u|_{\rho,\sigma}^2 ds \right), \end{aligned}$$

where in the last inequality we have used the first estimate in (22). Finally, we use (20) and the statements (ii)-(iii) in Lemma 3.2, to get by noticing $\sigma \leq 2$ that

$$\begin{aligned} \left| \int_0^t (\chi_2 g_{m+1}, h_m)_{L^2} \right| &\leq \int_0^t \|g_{m+1}\|_{L^2} \|h_m\|_{L^2} ds \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t |u|_{\rho,\sigma} |u|_{\tilde{\rho},\sigma} \frac{m^{\sigma-1} \rho^{m-5}}{\tilde{\rho}^{m-4}} ds \\ &\leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds. \end{aligned}$$

Combining these inequalities gives the estimate as desired. The proof is completed. \square

Completion of the proof of Proposition 5.1. In view of (47) and (46), we combine the estimates in Lemma 5.5-5.6 to obtain that by choosing κ being sufficiently small,

$$\begin{aligned} &\|h_m(t)\|_{L^2}^2 + \int_0^t \|\partial_y h_m(s)\|_{L^2}^2 ds + \varepsilon \int_0^t \|\partial_x h_m(s)\|_{L^2}^2 ds \\ &\leq C \int_0^t \|\partial_y f_m\|_{L^2}^2 ds + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 \\ &\quad + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + u(s)_{\rho,\sigma}^3) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right) \\ &\leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right), \end{aligned}$$

where in the last inequality we have used Proposition 4.1. This gives the upper bound for h_m . Moreover, in view of (40),

$$\|\chi_2 \partial_x^m \partial_y \omega(t)\|_{L^2}^2 \leq \|h_m(t)\|_{L^2}^2 + C \|\chi_2 \partial_x^m \omega(t)\|_{L^2}^2.$$

Finally, we use Proposition 5.2 and the estimate on h_m to get

$$\begin{aligned} &\|\chi_2 \partial_x^m \partial_y \omega(t)\|_{L^2}^2 \\ &\leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

The upper bound for $\chi_2 \partial_x^m \partial_y \omega$ follows. Thus we complete the proof of Proposition 5.1. \square

6. COMPLETENESS OF THE PROOF OF THEOREM 2.4: UNIFORM ESTIMATES FOR $\|u\|_{\rho,\sigma}$

To complete the proof of Theorem 2.4, it remains to estimate $\|u\|_{\rho,\sigma}$ that is given in Definition 1.4. We will perform estimates on tangential derivatives and mixed derivatives of u and ω respectively in the following two subsections. In the last subsection we will give the proof of Theorem 2.4 by combining all the estimates obtained in the previous sections.

6.1. Estimate on tangential derivatives. The main estimate in this subsection can be stated as follows.

Proposition 6.1. *Let $u \in L^\infty([0, T]; X_{\rho_0, \sigma})$ be the solution to (7) under the assumptions in Theorem 2.4. Then for any $m \geq 6$, for any $t \in [0, T]$, and for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} \leq \rho_0$, we have*

$$\begin{aligned} & \left\| \langle y \rangle^{\ell-1} \partial_x^m u \right\|_{L^2}^2 + \left\| \langle y \rangle^\ell \partial_x^m \omega \right\|_{L^2}^2 + \varepsilon \int_0^t \left(\left\| \langle y \rangle^{\ell-1} \partial_x^{m+1} u \right\|_{L^2}^2 + \left\| \langle y \rangle^\ell \partial_x^{m+1} \omega \right\|_{L^2}^2 \right) ds \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho, \sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

As a preliminary to prove the above proposition, we need the following

Lemma 6.2. *Let χ_1, χ_2 be given in (9) and (10), and let u satisfy the condition (8). Then*

$$\left\| \chi_1 \langle y \rangle^{\ell-1} \partial_x^m u \right\|_{L^2} + \left\| \chi_1 \langle y \rangle^\ell \partial_x^m \omega \right\|_{L^2} \leq C \left\| \langle y \rangle^\ell f_m \right\|_{L^2} + C \left\| \tilde{f}_m \right\|_{L^2} + C \left\| \chi_2 \partial_x^m \omega \right\|_{L^2}, \quad (48)$$

and

$$\begin{aligned} & \left\| \chi_1 \langle y \rangle^{\ell-1} \partial_x^{m+1} u \right\|_{L^2} + \left\| \chi_1 \langle y \rangle^\ell \partial_x^{m+1} \omega \right\|_{L^2} \\ & \leq C \left\| \langle y \rangle^\ell \partial_x f_m \right\|_{L^2} + C \left\| \partial_x \tilde{f}_m \right\|_{L^2} + C \left\| \chi_2 \partial_x^{m+1} \omega \right\|_{L^2} + C \left\| \langle y \rangle^{\ell-1} \partial_x^m u \right\|_{L^2}, \end{aligned} \quad (49)$$

with f_m and \tilde{f}_m defined in (30) and (31).

Proof. by using the fact that

$$\chi_1 |\omega^s + \omega| \sim \chi_1 (1 + y)^{-\alpha}$$

due to (8), integration by parts gives

$$\begin{aligned} & \left\| (1 + y)^{\ell-1} \chi_1 \partial_x^m u \right\|_{L^2}^2 \\ & \leq C \int_{\mathbb{R}_x} \int_0^{+\infty} (1 + y)^{2\ell-2-2\alpha} \left(\frac{\chi_1 \partial_x^m u}{\omega^s + \omega} \right)^2 dy dx \\ & = \frac{C}{2\ell - 1 - 2\alpha} \int_{\mathbb{R}_x} \int_0^{+\infty} \left[\partial_y (1 + y)^{2\ell-1-2\alpha} \right] \left(\frac{\chi_1 \partial_x^m u}{\omega^s + \omega} \right)^2 dy dx \\ & = -\frac{2C}{2\ell - 1 - 2\alpha} \int_{\mathbb{R}_x} \int_0^{+\infty} (1 + y)^{2\ell-1-2\alpha} \left(\frac{\chi_1 \partial_x^m u}{\omega^s + \omega} \right) \left[\partial_y \left(\frac{\chi_1 \partial_x^m u}{\omega^s + \omega} \right) \right] dy dx \\ & \leq C \int_{\mathbb{R}_x} \int_0^{+\infty} \left| (1 + y)^{\ell-1} \chi_1 \partial_x^m u \right| \times \left| (1 + y)^\ell (\omega^s + \omega) \partial_y \left(\frac{\chi_1 \partial_x^m u}{\omega^s + \omega} \right) \right| dy dx \\ & \leq C \left\| (1 + y)^{\ell-1} \chi_1 \partial_x^m u \right\|_{L^2} \left\| \langle y \rangle^\ell (\omega^s + \omega) \partial_y \left(\frac{\chi_1 \partial_x^m u}{\omega^s + \omega} \right) \right\|_{L^2}. \end{aligned}$$

This implies

$$\left\| (1 + y)^{\ell-1} \chi_1 \partial_x^m u \right\|_{L^2} \leq C \left\| \langle y \rangle^\ell (\omega^s + \omega) \partial_y \left(\frac{\chi_1 \partial_x^m u}{\omega^s + \omega} \right) \right\|_{L^2}.$$

Moreover, for the term on the right side of the above inequality, we have in view of the definition of f_m given in (30) that

$$\left\| \langle y \rangle^\ell (\omega^s + \omega) \partial_y \left(\frac{\chi_1 \partial_x^m u}{\omega^s + \omega} \right) \right\|_{L^2} \leq \left\| \langle y \rangle^\ell f_m \right\|_{L^2} + \left\| \chi_1' \langle y \rangle^\ell \partial_x^m u \right\|_{L^2} \leq \left\| \langle y \rangle^\ell f_m \right\|_{L^2} + C \left\| \chi_1' \partial_x^m u \right\|_{L^2}.$$

Thus

$$\left\| (1 + y)^{\ell-1} \chi_1 \partial_x^m u \right\|_{L^2} \leq C \left\| \langle y \rangle^\ell f_m \right\|_{L^2} + C \left\| \chi_1' \partial_x^m u \right\|_{L^2}. \quad (50)$$

Next we estimate the last term in (50). Observe that the condition (8) implies

$$\left| \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \right| \geq \tilde{c} > 0 \quad \text{on } \text{supp } \chi'_1,$$

for some constant \tilde{c} depending only on the constants c_0, c_1, δ and α in Assumption 1.1. Then

$$\|\chi'_1 \partial_x^m u\|_{L^2} \leq C \left\| \chi'_1 \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \partial_x^m u \right\|_{L^2} \leq C \|\tilde{f}_m\|_{L^2} + C \|\chi'_1 \partial_x^m \omega\|_{L^2} \leq C \|\tilde{f}_m\|_{L^2} + C \|\chi_2 \partial_x^m \omega\|_{L^2},$$

where for the second inequality we have used (31), the definition of \tilde{f}_m , and the last inequality follows from (11). Now we combine the above estimates with (50) to obtain

$$\left\| \langle y \rangle^{\ell-1} \chi_1 \partial_x^m u \right\|_{L^2} \leq C \left\| \langle y \rangle^\ell f_m \right\|_{L^2} + C \|\tilde{f}_m\|_{L^2} + C \|\chi_2 \partial_x^m \omega\|_{L^2}, \quad (51)$$

that yields the upper bound for the first term in (48). On the other hand, note that

$$\left| \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \right| \leq C \langle y \rangle^{-1} \quad \text{on } \text{supp } \chi_1$$

because of (8), and then

$$\left\| \langle y \rangle^\ell \chi_1 \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \partial_x^m u \right\|_{L^2} \leq C \left\| \langle y \rangle^{\ell-1} \chi_1 \partial_x^m u \right\|_{L^2},$$

that along with (30) and (51) yield

$$\left\| \chi_1 \langle y \rangle^\ell \partial_x^m \omega \right\|_{L^2} \leq \left\| \langle y \rangle^\ell f_m \right\|_{L^2} + C \left\| \langle y \rangle^{\ell-1} \chi_1 \partial_x^m u \right\|_{L^2} \leq C \left\| \langle y \rangle^\ell f_m \right\|_{L^2} + C \|\tilde{f}_m\|_{L^2} + C \|\chi_2 \partial_x^m \omega\|_{L^2}.$$

The upper bound for the second term in (48) follows. We have proven (48).

It remains to prove the second statement (49). Note that

$$\partial_x f_m = f_{m+1} - \chi_1 \left[\partial_x \left(\frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \right) \right] \partial_x^m u,$$

and moreover, direct calculation gives

$$\left\| \chi_1 \langle y \rangle \partial_x \left(\frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega} \right) \right\|_{L^\infty} \leq C \|\langle y \rangle^\alpha \partial_x \omega\|_{L^\infty} + \|\langle y \rangle^{1+\alpha} \partial_x \partial_y \omega\|_{L^\infty} \leq C$$

because of (8) and the fact that $\alpha \leq \ell$. Thus

$$\left\| \langle y \rangle^\ell f_{m+1} \right\|_{L^2} \leq \left\| \langle y \rangle^\ell \partial_x f_m \right\|_{L^2} + C \left\| \langle y \rangle^{\ell-1} \partial_x^m u \right\|_{L^2}.$$

Similar estimate holds for $\|\tilde{f}_{m+1}\|_{L^2}$. This estimate and (48) give the second statement (49) in Lemma 6.2. The proof is then completed. \square

Proof of Proposition 6.1. In view of Lemma 6.2 we have

$$\begin{aligned} & \left\| \chi_1 \langle y \rangle^{\ell-1} \partial_x^m u \right\|_{L^2}^2 + \left\| \chi_1 \langle y \rangle^\ell \partial_x^m \omega \right\|_{L^2}^2 \\ & \quad + \varepsilon \int_0^t \left(\left\| \chi_1 \langle y \rangle^{\ell-1} \partial_x^{m+1} u \right\|_{L^2}^2 + \left\| \chi_1 \langle y \rangle^\ell \partial_x^{m+1} \omega \right\|_{L^2}^2 \right) ds \\ & \leq \left\| \langle y \rangle^\ell f_m \right\|_{L^2}^2 + C \|\tilde{f}_m\|_{L^2}^2 + C \|\chi_2 \partial_x^m \omega\|_{L^2}^2 \\ & \quad + \varepsilon C \int_0^t \left(\left\| \langle y \rangle^\ell \partial_x f_m \right\|_{L^2}^2 + \left\| \partial_x \tilde{f}_m \right\|_{L^2}^2 + \|\chi_2 \partial_x^{m+1} \omega\|_{L^2}^2 + \left\| \langle y \rangle^{\ell-1} \partial_x^m u \right\|_{L^2}^2 \right) ds. \end{aligned}$$

Moreover, the terms on the right side of the above inequality are bounded above by

$$\frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho, \sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds \right)$$

by using Proposition 4.1 and Proposition 5.2. Thus

$$\begin{aligned} & \left\| \chi_1 \langle y \rangle^{\ell-1} \partial_x^m u \right\|_{L^2}^2 + \left\| \chi_1 \langle y \rangle^\ell \partial_x^m \omega \right\|_{L^2}^2 \\ & \quad + \varepsilon \int_0^t \left(\left\| \chi_1 \langle y \rangle^{\ell-1} \partial_x^{m+1} u \right\|_{L^2}^2 + \left\| \chi_1 \langle y \rangle^\ell \partial_x^{m+1} \omega \right\|_{L^2}^2 \right) ds \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Note that $1 \leq \chi_1 + \chi_2$ and $\langle y \rangle^\ell$ is equivalent to a constant on $\text{supp } \chi_2$. Then combining the above inequality and Proposition 5.2, we obtain

$$\begin{aligned} & \left\| \chi_1 \langle y \rangle^{\ell-1} \partial_x^m u \right\|_{L^2}^2 + \left\| \langle y \rangle^\ell \partial_x^m \omega \right\|_{L^2}^2 + \varepsilon \int_0^t \left(\left\| \chi_1 \langle y \rangle^{\ell-1} \partial_x^{m+1} u \right\|_{L^2}^2 + \left\| \langle y \rangle^\ell \partial_x^{m+1} \omega \right\|_{L^2}^2 \right) ds \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned} \quad (52)$$

Moreover, by Poincaré inequality we have, for $j = m$ and $j = m + 1$,

$$\left\| \chi_2 \langle y \rangle^{\ell-1} \partial_x^j u \right\|_{L^2} \leq C \left\| \chi_2 \partial_x^j u \right\|_{L^2} \leq C \left\| \partial_y (\chi_2 \partial_x^j u) \right\|_{L^2} \leq C \left\| \chi_1 \partial_x^j u \right\|_{L^2} + C \left\| \partial_x^j \omega \right\|_{L^2},$$

when in the last inequality we have used the fact that $\chi_2' = \chi_1 \chi_2'$ by (11). This and (52) give

$$\begin{aligned} & \left\| \chi_2 \langle y \rangle^{\ell-1} \partial_x^m u \right\|_{L^2}^2 + \varepsilon \int_0^t \left\| \chi_2 \langle y \rangle^{\ell-1} \partial_x^{m+1} u \right\|_{L^2}^2 ds \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Consequently, we combine the above inequality and (52) to conclude, by using again the fact that $1 \leq \chi_1 + \chi_2$,

$$\begin{aligned} & \left\| \langle y \rangle^{\ell-1} \partial_x^m u \right\|_{L^2}^2 + \left\| \langle y \rangle^\ell \partial_x^m \omega \right\|_{L^2}^2 + \varepsilon \int_0^t \left(\left\| \langle y \rangle^{\ell-1} \partial_x^{m+1} u \right\|_{L^2}^2 + \left\| \langle y \rangle^\ell \partial_x^{m+1} \omega \right\|_{L^2}^2 \right) ds \\ & \leq \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(\int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds + \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds \right). \end{aligned}$$

Thus we get the desired estimate in Proposition 6.1 and this completes the proof. \square

6.2. Estimate on the mixed derivatives. For the mixed derivatives $\partial_x^i \partial_y^j \omega$ of vorticity, we have

Proposition 6.3. *Let $u \in L^\infty([0, T]; X_{\rho_0, \sigma})$ be a solution to (7) under the assumptions in Theorem 2.4. Then we have, for any pair (i, j) with $1 \leq j \leq 4$ and $i + j \geq 6$, for any $t \in [0, T]$, and for any $\rho > 0$,*

$$\left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega(t) \right\|_{L^2}^2 \leq \frac{C[(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} |u_0|_{\rho,\sigma}^2 + \frac{C[(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t (|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4) ds.$$

The proof of the above proposition can be obtained by similar argument used in the previous sections, and the main difference arises from the boundary values since higher derivatives in y are involved when we perform integration by parts. So we first calculate $\partial_y^j \omega|_{y=0}$. Firstly, we have

$$\partial_y \omega^s(t, 0) = \partial_y \omega(t, x, 0) = 0.$$

Then by the equation of vorticity, we obtain that

$$\partial_y^3 \omega|_{y=0} = \partial_y (\partial_t \omega + (u^s + u) \partial_x \omega + v \partial_y (\omega^s + \omega) - \varepsilon \partial_x^2 \omega)|_{y=0} = (\omega^s + \omega) \partial_x \omega|_{y=0},$$

and direct computation yields

$$\partial_y^5 \omega|_{y=0} = -(\partial_y^2 \omega^s + \partial_y^2 \omega) \partial_x \omega|_{y=0} + 4(\omega^s + \omega) \partial_x \partial_y^2 \omega|_{y=0} - 2\varepsilon (\partial_x \omega) \partial_x^2 \omega|_{y=0}. \quad (53)$$

We apply $\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j$ to the equation for vorticity ω to have

$$\left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega = A_{i,j} \quad (54)$$

with

$$\begin{aligned} A_{i,j} &= -\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j (v \partial_y \omega^s) - \langle y \rangle^{\ell+1} \sum_{\substack{k+q \geq 1 \\ k \leq i, q \leq j}} \binom{i}{k} \binom{j}{q} \left(\partial_x^k \partial_y^q (u^s + u) \right) \partial_x^{i-k+1} \partial_y^{j-q} \omega \\ &\quad - \langle y \rangle^{\ell+1} \sum_{\substack{k+q \geq 1 \\ k \leq i, q \leq j}} \binom{i}{k} \binom{j}{q} \left(\partial_x^k \partial_y^q v \right) \partial_x^{i-k} \partial_y^{j-q+1} \omega \\ &\quad + v \left(\partial_y \langle y \rangle^{\ell+1} \right) \partial_x^i \partial_y^j \omega - \left(\partial_y^2 \langle y \rangle^{\ell+1} \right) \partial_x^i \partial_y^j \omega - 2 \left(\partial_y \langle y \rangle^{\ell+1} \right) \partial_x^i \partial_y^{j+1} \omega \\ &\stackrel{\text{def}}{=} A_{i,j,1} + A_{i,j,2} + A_{i,j,3} + A_{i,j,4} + A_{i,j,5} + A_{i,j,6}. \end{aligned}$$

We will estimate the terms on both sides of (54) in the following lemmas.

Lemma 6.4. *Let $1 \leq j \leq 4$ and $i + j \geq 6$. Then we have*

$$\begin{aligned} &\int_0^t \left((\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2) \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \\ &\geq \frac{1}{2} \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right\|_{L^2}^2 + \frac{1}{2} \int_0^t \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^{j+1} \omega \right\|_{L^2}^2 - \frac{1}{2} \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega(0) \right\|_{L^2}^2 \\ &\quad - \frac{C [((i+j)-6)!]^{2\sigma}}{\rho^{2((i+j)-5)}} \int_0^t \left(|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4 \right) ds. \end{aligned}$$

Proof. We only need to discuss the boundary terms when we use integration by parts:

$$\begin{aligned} &\int_0^t \left(-\partial_y^2 \left(\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right), \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \\ &= \int_0^t \left\| \partial_y \left(\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right) \right\|_{L^2}^2 ds + \int_0^t \int_{\mathbb{R}} \partial_y \left(\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right) \Big|_{y=0} \partial_x^i \partial_y^j \omega(t, x, 0) dx dt \\ &\geq \int_0^t \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^{j+1} \omega \right\|_{L^2}^2 ds + \int_0^t \int_{\mathbb{R}} \left(\partial_x^i \partial_y^j \omega(t, x, 0) \right) \partial_x^i \partial_y^{j+1} \omega(t, x, 0) dx ds \\ &\quad - \frac{C [((i+j)-6)!]^{2\sigma}}{\rho^{2((i+j)-5)}} \int_0^t |u(s)|_{\rho,\sigma}^2 ds, \end{aligned}$$

where we have used (21) in the last inequality. Note that the boundary value is well-defined in view of (53). Thus the estimate in Lemma 6.4 follows by standard energy method if we can show that, for any $\kappa > 0$, $1 \leq j \leq 4$ and $i \geq 0$ with $i + j \geq 6$ that

$$\begin{aligned} &\left| \int_0^t \int_{\mathbb{R}} \left(\partial_x^i \partial_y^j \omega(s, x, 0) \right) \partial_x^i \partial_y^{j+1} \omega(s, x, 0) dx ds \right| \\ &\leq \kappa \int_0^t \left\| \partial_x^i \partial_y^{j+1} \omega \right\|_{L^2}^2 ds + \frac{C \kappa^{-1} [((i+j)-6)!]^{2\sigma}}{\rho^{2((i+j)-5)}} \int_0^t |u(s)|_{\rho,\sigma}^4 ds. \end{aligned} \quad (55)$$

Note that (55) holds obviously for $j = 1$ because $\partial_y \omega(t, x, 0) = 0$. It remains to consider the cases when $j = 2, 3, 4$.

The case when $j = 4$: Recall, in view of (53),

$$\partial_y^5 \omega|_{y=0} - (\partial_y^2 \omega^s + \partial_y^2 \omega) \partial_x \omega|_{y=0} + 4(\omega^s + \omega) \partial_x \partial_y^2 \omega|_{y=0} - 2\varepsilon (\partial_x \omega) \partial_x^2 \omega|_{y=0}.$$

Then direct computation gives, using the argument in Lemma 3.4 as well as the Sobolev inequality (see Lemma A.1 in the Appendix),

$$\|\partial_x^i \partial_y^5 \omega(t, x, 0)\|_{L_x^2} \leq \frac{C [(i+4)-6]!^\sigma}{\rho^{(i+4)-5}} |u|_{\rho, \sigma}.$$

Hence, for any small $\kappa > 0$,

$$\begin{aligned} & \left| \int_0^t \int_{\mathbb{R}} \left(\partial_x^i \partial_y^4 \omega(s, x, 0) \right) \partial_x^i \partial_y^5 \omega(s, x, 0) dx ds \right| \\ & \leq \kappa \int_0^t \|\partial_x^i \partial_y^4 \omega(s, x, 0)\|_{L_x^2}^2 ds + \kappa^{-1} C \int_0^t \|\partial_x^i \partial_y^5 \omega(s, x, 0)\|_{L_x^2}^2 ds \\ & \leq \kappa C \int_0^t \|\partial_x^i \partial_y^5 \omega\|_{L^2}^2 ds + \frac{C \kappa^{-1} [(i+4)-6]!^{2\sigma}}{\rho^{2((i+4)-5)}} \int_0^t |u(s)|_{\rho, \sigma}^4 ds. \end{aligned}$$

Thus we obtain (55) for $j = 4$.

The case when $2 \leq j \leq 3$: The estimation on

$$\left| \int_0^t \int_{\mathbb{R}} \left(\partial_x^i \partial_y^j \omega(s, x, 0) \right) \partial_x^i \partial_y^{j+1} \omega(s, x, 0) dx ds \right|$$

for $j = 2$ and $j = 3$ is simpler than the case $j = 4$, since only lower order derivatives are involved. And thus for brevity we omit the details. The proof is then completed. \square

Lemma 6.5 (Estimate on $A_{i,j,1}$). *Under the same assumption as in Proposition 6.3, we have*

$$\left| \int_0^t \left(-\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j (v \partial_y \omega^s), \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \leq \frac{C [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t \left(|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^3 \right) ds.$$

Proof. Using $\partial_y v = -\partial_x u$, we have

$$\begin{aligned} & \left| \int_0^t \left(-\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j (v \partial_y \omega^s), \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \\ & \leq \left| \int_0^t \left(\langle y \rangle^{\ell+1} (\partial_x^i v) \partial_y^{j+1} \omega^s, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \\ & \quad + \left| \int_0^t \left(\sum_{q=1}^j \langle y \rangle^{\ell+1} (\partial_x^{i+1} \partial_y^{q-1} u) \partial_y^{j-q+1} \omega^s, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right|. \end{aligned}$$

Moreover, note that for $1 \leq j \leq 4$ we have $\langle y \rangle^{\ell+1} \langle y \rangle^{-\alpha-j-1} \in L^2(\mathbb{R}_+)$ because of $\ell < \alpha + 1/2$, and thus

$$\left\| \langle y \rangle^{\ell+1} \partial_y^{j+1} \omega^s \right\|_{L^2(\mathbb{R}_+)} \leq C.$$

Consequently,

$$\begin{aligned} \left| \int_0^t \left(\langle y \rangle^{\ell+1} (\partial_x^i v) \partial_y^{j+1} \omega^s, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| & \leq \frac{C [(i+1-6)!]^\sigma [(i+j-6)!]^\sigma}{\rho^{i+1-5} \rho^{i+j-5}} \int_0^t |u|_{\rho, \sigma}^2 ds \\ & \leq \frac{C [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t |u|_{\rho, \sigma}^2 ds. \end{aligned}$$

Direct calculation also shows

$$\left| \int_0^t \left(\sum_{q=1}^j \langle y \rangle^{\ell+1} (\partial_x^{i+1} \partial_y^{q-1} u) \partial_y^{j-q+1} \omega^s, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \leq \frac{C [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t |u|_{\rho, \sigma}^3 ds.$$

Then the desired estimate follows and we complete the proof. \square

Lemma 6.6 (Estimate on $A_{i,j,3}$ and $A_{i,j,6}$). *Under the same assumption as Proposition 6.3, we have, for any $\kappa > 0$,*

$$\begin{aligned} & \left| \int_0^t \left(A_{i,j,3}, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| + \left| \int_0^t \left(A_{i,j,6}, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \\ & \leq \kappa \int_0^t \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^{j+1} \omega \right\|_{L^2}^2 ds + \frac{C \kappa^{-1} [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t \left(|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4 \right) ds. \end{aligned}$$

Proof. We decompose $A_{i,j,3}$ as follows by using $\partial_y v = -\partial_x u$,

$$\begin{aligned} A_{m,j,3} &= -\langle y \rangle^{\ell+1} \sum_{1 \leq k \leq i} \binom{i}{k} (\partial_x^k v) \partial_x^{i-k} \partial_y^{j+1} \omega \\ &\quad + \langle y \rangle^{\ell+1} \sum_{\substack{q \geq 1 \\ k \leq i, q \leq j}} \binom{i}{k} \binom{j}{q} (\partial_x^{k+1} \partial_y^{q-1} u) \partial_x^{i-k} \partial_y^{j-q+1} \omega. \end{aligned}$$

Following the similar argument as in Lemma 3.4, we have

$$\begin{aligned} & \left| \int_0^t \left(\langle y \rangle^{\ell+1} \sum_{\substack{q \geq 1 \\ k \leq i, q \leq j}} \binom{i}{k} \binom{j}{q} (\partial_x^{k+1} \partial_y^{q-1} u) \partial_x^{i-k} \partial_y^{j-q+1} \omega, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \\ & \leq \frac{C [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t |u|_{\rho, \sigma}^3 ds. \end{aligned} \tag{56}$$

Next, we will prove that, for any small $\kappa > 0$,

$$\begin{aligned} & \left| \int_0^t \left(-\langle y \rangle^{\ell+1} \sum_{1 \leq k \leq i} \binom{i}{k} (\partial_x^k v) \partial_x^{i-k} \partial_y^{j+1} \omega, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \\ & \leq \kappa \int_0^t \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^{j+1} \omega \right\|_{L^2}^2 ds + \frac{C \kappa^{-1} [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t \left(|u|_{\rho, \sigma}^3 + |u|_{\rho, \sigma}^4 \right) ds. \end{aligned} \tag{57}$$

To do so, integration by parts gives

$$\begin{aligned} & \left| \int_0^t \left(-\langle y \rangle^{\ell+1} \sum_{1 \leq k \leq i} \binom{i}{k} (\partial_x^k v) \partial_x^{i-k} \partial_y^{j+1} \omega, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \\ & \leq \left| \int_0^t \left(\langle y \rangle^{\ell+1} \sum_{1 \leq k \leq i} \binom{i}{k} (\partial_x^k v) \partial_x^{i-k} \partial_y^j \omega, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^{j+1} \omega \right)_{L^2} ds \right| \\ & \quad + \left| \int_0^t \left(\langle y \rangle^{\ell+1} \sum_{1 \leq k \leq i} \binom{i}{k} (\partial_x^{k+1} u) \partial_x^{i-k} \partial_y^j \omega, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \\ & \quad + \left| \int_0^t \left((\partial_y \langle y \rangle^{2\ell+2}) \sum_{1 \leq k \leq i} \binom{i}{k} (\partial_x^k v) \partial_x^{i-k} \partial_y^j \omega, \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right|. \end{aligned} \tag{58}$$

Moreover, as in Lemma 3.4 and Lemma 4.5, we can prove that the first term on the right side of (58) is bounded above by

$$\kappa \int_0^t \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^{j+1} \omega \right\|_{L^2}^2 ds + \frac{C \kappa^{-1} [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t |u|_{\rho, \sigma}^4 ds,$$

and the last two terms are bounded above by

$$\frac{C [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t |u|_{\rho,\sigma}^3 ds.$$

Thus combining the above estimate, (57) follows. This and (56) give

$$\begin{aligned} & \left| \int_0^t \left(A_{i,j,3}, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \\ & \leq \kappa \int_0^t \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^{j+1} \omega \right\|_{L^2}^2 ds + \frac{C \kappa^{-1} [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t \left(|u|_{\rho,\sigma}^3 + |u|_{\rho,\sigma}^4 \right) ds. \end{aligned}$$

Similarly,

$$\left| \int_0^t \left(A_{i,j,6}, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \leq \kappa \int_0^t \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^{j+1} \omega \right\|_{L^2}^2 ds + \frac{C \kappa^{-1} [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t |u|_{\rho,\sigma}^2 ds.$$

Thus the proof is completed. \square

Lemma 6.7. *Let $1 \leq j \leq 4$ and $i+j \geq 6$. Then we have*

$$\left| \int_0^t \left(A_{i,j,2} + A_{i,j,4} + A_{i,j,5}, \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right)_{L^2} ds \right| \leq \frac{C [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t \left(|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^3 \right) ds.$$

Proof. Since there is no $j+1$ order derivative in y involved, the proof is straightforward so that we omit the detail for brevity. \square

Completion of the proof of Proposition 6.3. Let $1 \leq j \leq 4$ and $i+j \geq 6$. In view of (54), we combine the estimates in Lemmas 6.4-6.7 to conclude by choosing κ sufficiently small that

$$\begin{aligned} & \frac{1}{2} \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega(t) \right\|_{L^2}^2 + \frac{1}{4} \int_0^t \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^{j+1} \omega \right\|_{L^2}^2 \\ & \leq \frac{1}{2} \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega(0) \right\|_{L^2}^2 + \frac{C [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} \int_0^t \left(|u|_{\rho,\sigma}^2 + |u|_{\rho,\sigma}^4 \right) ds. \end{aligned}$$

Note that

$$\left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega(0) \right\|_{L^2}^2 \leq \frac{C [(i+j-6)!]^{2\sigma}}{\rho^{2(i+j-5)}} |u_0|_{\rho,\sigma}^2.$$

Then the desired estimate in Proposition 6.3 follows. The proof is then completed. \square

6.3. The proof of Theorem 2.4. By Proposition 6.1 and Proposition 6.3, we have

$$\begin{aligned} & \left[\sup_{m \geq 6} \frac{\rho^{m-5}}{[(m-6)!]^\sigma} \left(\left\| \langle y \rangle^{\ell-1} \partial_x^m u(t) \right\|_{L^2} + \left\| \langle y \rangle^\ell \partial_x^m \omega(t) \right\|_{L^2} \right) \right]^2 \\ & \quad + \left[\sup_{\substack{1 \leq j \leq 4 \\ i+j \geq 6}} \frac{\rho^{i+j-5}}{[(i+j-6)!]^\sigma} \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega(t) \right\|_{L^2} \right]^2 \\ & \leq C |u_0|_{\rho,\sigma}^2 + C \int_0^t \left(|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4 \right) ds + C \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds. \end{aligned}$$

From Proposition 4.1 and Proposition 5.1, it follows that

$$\begin{aligned} & \left[\sup_{m \geq 6} \frac{\rho^{m-5}}{[(m-6)!]^\sigma} \left(\left\| \langle y \rangle^\ell f_m(t) \right\|_{L^2} + \|h_m(t)\|_{L^2} + \|\chi_2 \partial_y \partial_x^m \omega(t)\|_{L^2} \right) \right]^2 \\ & \leq C |u_0|_{\rho,\sigma}^2 + C \int_0^t \left(|u(s)|_{\rho,\sigma}^2 + |u(s)|_{\rho,\sigma}^4 \right) ds + C \int_0^t \frac{|u(s)|_{\tilde{\rho},\sigma}^2}{\tilde{\rho} - \rho} ds. \end{aligned}$$

Moreover, we combine the first estimate in Corollary 4.2 and Proposition 6.1 to have

$$\left[\sup_{m \geq 6} \frac{\rho^{m-5}}{[(m-6)!]^\sigma} (m \|g_m(t)\|_{L^2}) \right]^2 \leq C |u_0|_{\rho, \sigma}^2 + C \int_0^t (|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4) ds + C \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds.$$

Finally, direct computation gives

$$\begin{aligned} & \left[\sup_{m \leq 5} \left(\|\langle y \rangle^{\ell-1} \partial_x^m u(t)\|_{L^2} + \|\langle y \rangle^\ell \partial_x^m \omega(t)\|_{L^2} \right) \right]^2 + \left[\sup_{\substack{1 \leq j \leq 4 \\ i+j \leq 5}} \|\langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega(t)\|_{L^2} \right]^2 \\ & + \left[\sup_{1 \leq m \leq 5} \left(m \|g_m(t)\|_{L^2} + \|\langle y \rangle^\ell f_m(t)\|_{L^2} + \|h_m(t)\|_{L^2} + \|\chi_2 \partial_y \partial_x^m \omega(t)\|_{L^2} \right) \right]^2 \\ & \leq C |u_0|_{\rho, \sigma}^2 + C \int_0^t (|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4) ds + C \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds. \end{aligned}$$

Combining these inequalities yields

$$|u(t)|_{\rho, \sigma}^2 \leq C |u_0|_{\rho, \sigma}^2 + C \int_0^t (|u(s)|_{\rho, \sigma}^2 + |u(s)|_{\rho, \sigma}^4) ds + C \int_0^t \frac{|u(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds.$$

This completes the proof of Theorem 2.4.

7. EXISTENCE FOR THE REGULARIZED PRANDTL EQUATION

In this section, we study the existence of the regularized Prandtl equation introduced in Section 2:

$$\begin{cases} \partial_t u_\varepsilon + (u^s + u_\varepsilon) \partial_x u_\varepsilon + v_\varepsilon \partial_y (u^s + u_\varepsilon) - \partial_y^2 u_\varepsilon - \varepsilon \partial_x^2 u_\varepsilon = 0, \\ u_\varepsilon|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u_\varepsilon = 0, \\ u_\varepsilon|_{t=0} = u_0, \end{cases} \quad (59)$$

with $v_\varepsilon = -\int_0^y \partial_x u_\varepsilon(x, \tilde{y}) d\tilde{y}$. This is a nonlinear parabolic equation. The main result can be stated as follows.

Theorem 7.1 (Existence for the regularized Prandtl equation). *Let $\rho_0 > 0, \sigma \geq 1$ be two given constants. Suppose the initial datum $u_0 \in X_{2\rho_0, \sigma}$ satisfies the compatibility condition (6). Then there exists $T_\varepsilon^* > 0$, such that the regularized Prandtl equation (59) admits a unique solution $u_\varepsilon \in L^\infty([0, T_\varepsilon^*]; X_{3\rho_0/2, \sigma})$.*

Sketch of the proof of Theorem 7.1. We will use iteration to prove the existence. Since (59) is a parabolic equation, then we can apply the standard energy estimate in Gevrey norms.

Step (i). We first choose $u_j, j \geq 0$, as follows. Let u_0 be the initial datum in (59) and let u_j be the solution to the linear parabolic equation

$$\begin{cases} \partial_t u_j - \partial_y^2 u_j - \varepsilon \partial_x^2 u_j = -(u^s + u_{j-1}) \partial_x u_{j-1} - v_{j-1} \partial_y (u^s + u_{j-1}), \\ u_j|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u_j = 0, \\ u_j|_{t=0} = u_0, \end{cases}$$

where $v_j = -\int_0^y \partial_x u_j(x, \tilde{y}) d\tilde{y}$. Note that the existence of solutions to the above linear initial-boundary problem is guaranteed by using the heat kernel

$$E_0(t, x, y) = \frac{1}{4\pi t \sqrt{\varepsilon}} e^{-x^2/(4t\varepsilon)} e^{-y^2/(4t)}.$$

Indeed, define two heat operators M_1 and M_2 by

$$\begin{aligned} M_1(t)f &= \int_{\mathbb{R}} d\tilde{x} \int_0^{+\infty} d\tilde{y} \left[E_0(t, x - \tilde{x}, y - \tilde{y}) - E_0(t, x - \tilde{x}, y + \tilde{y}) \right] f(\tilde{x}, \tilde{y}), \\ M_2f &= \int_0^t M_1(t-s)f(s)ds. \end{aligned}$$

then we have

$$u_j = M_1u_0 - M_2 \left((u^s + u_{j-1}) \partial_x u_{j-1} + v_{j-1} \partial_y (u^s + u_{j-1}) \right).$$

Step (ii). Now we consider the difference

$$\xi_0 = u_0, \quad \xi_j = u_j - u_{j-1}, \quad \zeta_j = v_j - v_{j-1}, \quad j \geq 1.$$

Then

$$\partial_t \xi_1 - \partial_y^2 \xi_1 - \varepsilon \partial_x^2 \xi_1 = \partial_y^2 u_0 + \varepsilon \partial_x^2 u_0 - (u^s + u_0) \partial_x u_0 - v_0 \partial_y (u^s + u_0), \quad (60)$$

and for $j \geq 2$ we have

$$\partial_t \xi_j - \partial_y^2 \xi_j - \varepsilon \partial_x^2 \xi_j = - (u^s + u_{j-1}) \partial_x \xi_{j-1} - \xi_j \partial_x u_{j-2} - \zeta_{j-1} \partial_y (u^s + u_{j-1}) - v_{j-2} \partial_y \xi_{j-1}. \quad (61)$$

In view of equation (60), the estimation on ξ_1 follows from the classical Gevrey regularity theorem for parabolic equation. And we conclude, for some $T > 0$ independent of ε ,

$$\sup_{t \in [0, T]} \|\xi_1(t)\|_{3\rho_0/2, \sigma} \leq C \|u_0\|_{2\rho_0, \sigma}. \quad (62)$$

Note that the higher order derivatives are involved in the initial datum u_0 on the right side of (60). This can be overcome by reducing the initial Gevrey radius $2\rho_0$ to a smaller one, saying $3\rho_0/2$ for instance.

Now we consider the case $j \geq 2$. Applying $\langle y \rangle^{\ell-1} \partial_x^m$ to the above equation (61) we have

$$\begin{aligned} (\partial_t - \partial_y^2 - \varepsilon \partial_x^2) \langle y \rangle^{\ell-1} \partial_x^m \xi_j &= -2 \left(\partial_y \langle y \rangle^{\ell-1} \right) \partial_y \partial_x^m \xi_j - \left(\partial_y^2 \langle y \rangle^{\ell-1} \right) \partial_x^m \xi_j \\ &\quad - \langle y \rangle^{\ell-1} \partial_x^m \left((u^s + u_{j-1}) \partial_x \xi_{j-1} + \xi_j \partial_x u_{j-2} + \zeta_{j-1} \partial_y (u^s + u_{j-1}) + v_{j-2} \partial_y \xi_{j-1} \right). \end{aligned} \quad (63)$$

Moreover for the terms on right side, direct computation yields

$$\begin{aligned} &\left| \left(\langle y \rangle^{\ell-1} \partial_x^m \left((u^s + u_{j-1}) \partial_x \xi_{j-1} + \xi_j \partial_x u_{j-2} + \zeta_{j-1} \partial_y (u^s + u_{j-1}) + v_{j-2} \partial_y \xi_{j-1} \right), \langle y \rangle^{\ell-1} \partial_x^m \xi_j \right)_{L^2} \right| \\ &\quad + \left| \left(-2 \left(\partial_y \langle y \rangle^{\ell-1} \right) \partial_y \partial_x^m \xi_j - \left(\partial_y^2 \langle y \rangle^{\ell-1} \right) \partial_x^m \xi_j, \langle y \rangle^{\ell-1} \partial_x^m \xi_j \right)_{L^2} \right| \\ &\leq \frac{\varepsilon}{2} \left\| \langle y \rangle^{\ell-1} \partial_x^{m+1} \xi_j \right\|_{L^2}^2 + \frac{1}{2} \left\| \langle y \rangle^{\ell-1} \partial_y \partial_x^m \xi_j \right\|_{L^2}^2 + C \left\| \langle y \rangle^{\ell-1} \partial_x^m \xi_j \right\|_{L^2}^2 \\ &\quad + C \varepsilon^{-1} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \left(1 + \|u_{j-1}\|_{\rho, \sigma}^2 + \|u_{j-2}\|_{\rho, \sigma}^2 \right) \|\xi_{j-1}\|_{\rho, \sigma}^2. \end{aligned}$$

Thus for $m \geq 6$, we can apply energy method and the Gronwall inequality to (63) to obtain by noting that $\xi_j|_{t=0} = 0$,

$$\begin{aligned} &\left\| \langle y \rangle^{\ell-1} \partial_x^m \xi_j(t) \right\|_{L^2}^2 + \int_0^t \left\| \langle y \rangle^{\ell-1} \partial_y \partial_x^m \xi_j \right\|_{L^2}^2 + \varepsilon \int_0^t \left\| \langle y \rangle^{\ell-1} \partial_x^{m+1} \xi_j \right\|_{L^2}^2 \\ &\leq C \varepsilon^{-1} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t \left(1 + \|u_{j-1}(s)\|_{\rho, \sigma}^2 + \|u_{j-2}(s)\|_{\rho, \sigma}^2 \right) \|\xi_{j-1}(s)\|_{\rho, \sigma}^2 ds. \end{aligned}$$

The upper bound estimate for $m \leq 5$ is straightforward, and we have

$$\sup_{m \leq 5} \left\| \langle y \rangle^{\ell-1} \partial_x^m \xi_j(t) \right\|_{L^2}^2 \leq C \varepsilon^{-1} \int_0^t \left(1 + \|u_{j-1}(s)\|_{\rho, \sigma}^2 + \|u_{j-2}(s)\|_{\rho, \sigma}^2 \right) \|\xi_{j-1}(s)\|_{\rho, \sigma}^2 ds.$$

Similarly, for $m \geq 6$,

$$\begin{aligned} & \left\| \langle y \rangle^\ell \partial_x^m \partial_y \xi_j(t) \right\|_{L^2}^2 + \int_0^t \left\| \langle y \rangle^\ell \partial_x^m \partial_y^2 \xi_j \right\|_{L^2}^2 + \varepsilon \int_0^t \left\| \langle y \rangle^\ell \partial_x^{m+1} \partial_y \xi_j \right\|_{L^2}^2 \\ & \leq C\varepsilon^{-1} \frac{[(m-6)!]^{2\sigma}}{\rho^{2(m-5)}} \int_0^t \left(1 + \|u_{j-1}(s)\|_{\rho,\sigma}^2 + \|u_{j-2}(s)\|_{\rho,\sigma}^2 \right) \|\xi_{j-1}(s)\|_{\rho,\sigma}^2 ds, \end{aligned}$$

and for $1 \leq q \leq 4$ and $p+q \geq 6$, repeating the argument used in Proposition 6.3 yields

$$\begin{aligned} & \left\| \langle y \rangle^\ell \partial_x^p \partial_y^q \partial_y \xi_j(t) \right\|_{L^2}^2 + \int_0^t \left\| \langle y \rangle^\ell \partial_x^p \partial_y^{q+1} \partial_y \xi_j \right\|_{L^2}^2 + \varepsilon \int_0^t \left\| \langle y \rangle^\ell \partial_x^{p+1} \partial_y^q \partial_y \xi_j \right\|_{L^2}^2 \\ & \leq C\varepsilon^{-1} \frac{[(p+q-6)!]^{2\sigma}}{\rho^{2(p+q-5)}} \int_0^t \left(1 + \|u_{j-1}(s)\|_{\rho,\sigma}^2 + \|u_{j-2}(s)\|_{\rho,\sigma}^2 \right) \|\xi_{j-1}(s)\|_{\rho,\sigma}^2 ds. \end{aligned}$$

And the above two estimates for $m \leq 5$ and $p+q \leq 5$ are also straightforward. Combining the above inequalities we conclude

$$\forall j \geq 2, \quad \|\xi_j(t)\|_{\rho,\sigma}^2 \leq C\varepsilon^{-1} \int_0^t \left(1 + \|u_{j-1}(s)\|_{\rho,\sigma}^2 + \|u_{j-2}(s)\|_{\rho,\sigma}^2 \right) \|\xi_{j-1}(s)\|_{\rho,\sigma}^2 ds. \quad (64)$$

The above estimate and (62) enable us to use induction on j to conclude that there exists a constant M , depending only on $\|u_0\|_{2\rho_0,\sigma}$, such that

$$\forall j \geq 0, \quad \|u_j(s)\|_{3\rho_0/2,\sigma} \leq M, \quad \text{and} \quad \sup_{0 \leq t \leq T_\varepsilon^*} \|\xi_j(t)\|_{3\rho_0/2,\sigma}^2 \leq C2^{-j+1} \|u_0\|_{2\rho_0,\sigma}^2,$$

provided $T_\varepsilon^* \leq \frac{\varepsilon}{2C(2M^{2+1})}$ with C the constant in (64). This implies $u_j, j \geq 0$, is a Cauchy sequence in the Banach space $L^\infty([0, T_\varepsilon^*]; X_{3\rho_0/2,\sigma})$, with T_ε^* depending only on ε but independent of j . Thus the limit u_ε of the Cauchy sequence u_j in $L^\infty([0, T_\varepsilon^*]; X_{3\rho_0/2,\sigma})$ solves the initial-boundary problem (59). The proof is thus complete. \square

8. PROOF OF THE MAIN RESULT THEOREM 1.6

In this section, we will prove the main result Theorem 1.6.

8.1. Proof of Theorem 1.6: existence. Here we will adopt the idea of abstract Cauchy-Kovalevskaya theorem to prove the existence of solution to equation (3), by virtue of the uniform estimate established in Theorem 2.4. Let the initial data $u_0 \in X_{2\rho_0,\sigma}$ satisfy the assumptions listed in Theorem 1.6. Then by Theorem 7.1, we can find a solution $u_\varepsilon \in L^\infty([0, T_\varepsilon^*]; X_{3\rho_0/2,\sigma})$ to the regularized equation (7). In the following discussions we will remove the ε -dependence of the lifespan and derive an uniform upper bound for u_ε .

Step (i). We begin with the construction of two constants R and λ , which depend only on the initial datum u_0 and the constants C_*, c_j given respectively in Theorem 2.4 and Assumption 1.1, as well as the constants in the Sobolev imbedding inequalities. First, in view of (15), we can find a constant $\hat{C} \geq 1$, depending only on ρ_0 , such that

$$\|u_0\|_{\rho_0,\sigma} \leq \hat{C} \left(\|u_0\|_{2\rho_0,\sigma} + \|u_0\|_{2\rho_0,\sigma}^2 \right). \quad (65)$$

And by Sobolev inequalities and the definition of $|\cdot|_{\rho,\sigma}$ (see Definition 2.2), we deduce that, for any $t \geq 0$ and for any $(x, y) \in \mathbb{R}_+^2$,

$$\begin{aligned} & \sum_{1 \leq j \leq 2} \left(\left\| \langle y \rangle^{\ell-1} \partial_x^j u_\varepsilon(t) \right\|_{L^\infty} + \left\| \partial_x^{j-1} v_\varepsilon(t) \right\|_{L^\infty} + \left\| \langle y \rangle^\ell \partial_x^j \omega_\varepsilon(t) \right\|_{L^\infty} \right) \\ & + \sum_{1 \leq i, j \leq 2} \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega_\varepsilon(t) \right\|_{L^\infty} + \left\| \langle y \rangle^\ell \omega_\varepsilon(t) \right\|_{L^\infty} + \left\| \langle y \rangle^{\ell+1} \partial_y \omega_\varepsilon(t) \right\|_{L^\infty} \leq \tilde{C} |u(t)|_{\rho_0/2,\sigma}, \end{aligned} \quad (66)$$

with \tilde{C} being a constant depending only on the Sobolev imbedding constants but independent of ε . Let $C_* \geq 1$ be the constant given in Theorem 2.4 and let \hat{C}, \tilde{C} be the constants given in (65)-(66). Now we take two positive constants $R > 0, \lambda > 0$ such that

$$R \geq 4C_*\hat{C} \left(\|u_0\|_{2\rho_0, \sigma} + \|u_0\|_{2\rho_0, \sigma}^2 \right), \quad \text{and} \quad \sqrt{2}R\tilde{C} \leq \frac{1}{4} \min \{c_0, c_1\}, \quad (67)$$

and

$$\frac{\sqrt{5C_* + C_*R^2}}{\sqrt{\lambda}} = \frac{1}{2}, \quad (68)$$

recalling c_0, c_1 are the constants given in Assumption 1.1. We remark that the above R indeed exist, provided

$$\|u_0\|_{2\rho_0, \sigma} + \|u_0\|_{2\rho_0, \sigma}^2 < \frac{1}{16\sqrt{2}\tilde{C}\hat{C}C_*} \min \{c_0, c_1\}.$$

In the following discussion we will let R and λ be fixed so that (67) and (68) hold.

Step (ii). We define a function $T \rightarrow \|u_\varepsilon\|_{(\lambda, T)}$ by setting

$$\|u_\varepsilon\|_{(\lambda, T)} \stackrel{\text{def}}{=} \sup_{\rho, t} \left(\frac{\rho_0 - \rho - \lambda t}{\rho_0 - \rho} \right)^{1/2} |u_\varepsilon(t)|_{\rho, \sigma}, \quad (69)$$

where the supremum is taken over all pairs (ρ, t) such that $\rho > 0, 0 \leq t \leq T$ and $\rho + \lambda t < \rho_0$. Note that the above function is well-defined over the interval $[0, P_\varepsilon[$ with P_ε given by

$$P_\varepsilon = \sup \left\{ T \in [0, \rho_0/\lambda[; \|u_\varepsilon\|_{(\lambda, T)} < +\infty \right\}.$$

Note that $P_\varepsilon \geq T_\varepsilon^*$ because of (15) and by recalling $[0, T_\varepsilon^*]$ is the interval of the existence for $u_\varepsilon \in X_{3\rho_0/2, \sigma}$. It is clear that

$$T \rightarrow \|u_\varepsilon\|_{(\lambda, T)}$$

is a increasing function of T . Moreover, we have

$$\|u_\varepsilon\|_{(\lambda, 0)} = \sup_{\rho \in]0, \rho_0[} |u_\varepsilon(0)|_{\rho, \sigma} \leq |u_0|_{\rho_0, \sigma} \leq \hat{C} \left(\|u_0\|_{2\rho_0, \sigma} + \|u_0\|_{2\rho_0, \sigma}^2 \right) < R, \quad (70)$$

where in the second inequality we have used (65) and the last one follows from (67).

Step (iii). In this step, recalling R is given in Step (i) and P_ε is defined in the previous step, we will show that

$$\forall 0 \leq T < \min \left\{ \rho_0/(4\lambda), P_\varepsilon \right\}, \quad \|u_\varepsilon\|_{(\lambda, T)} \leq R. \quad (71)$$

To confirm this, suppose on the contrary to (71) that $\|u_\varepsilon\|_{(\lambda, t_\varepsilon)} > R$ for some $t_\varepsilon < \min \{ \rho_0/(4\lambda), P_\varepsilon \}$. Then in view of (70), we can find some $T_\varepsilon \in]0, t_\varepsilon[\subset [0, \rho_0/(4\lambda)]$ such that

$$\|u_\varepsilon\|_{(\lambda, T_\varepsilon)} = R, \quad (72)$$

since $T \rightarrow \|u_\varepsilon\|_{(\lambda, T)}$ is a increasing function of T . Thus, observing $T_\varepsilon < \rho_0/(4\lambda)$,

$$\forall t \in [0, T_\varepsilon], \quad \frac{\sqrt{2}}{2} |u(t)|_{\rho_0/2, \sigma} \leq \left(\frac{\rho_0 - \rho_0/2 - \lambda t}{\rho_0 - \rho_0/2} \right)^{1/2} |u(t)|_{\rho_0/2, \sigma} \leq \|u\|_{(\lambda, T_\varepsilon)} = R.$$

As a result, for any $t \in [0, T_\varepsilon]$,

$$\begin{aligned} & \sum_{1 \leq j \leq 2} \left(\left\| \langle y \rangle^{\ell-1} \partial_x^j u_\varepsilon \right\|_{L^\infty} + \left\| \partial_x^{j-1} v_\varepsilon \right\|_{L^\infty} + \left\| \langle y \rangle^\ell \partial_x^j \omega_\varepsilon \right\|_{L^\infty} \right) \\ & + \sum_{1 \leq i, j \leq 2} \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega_\varepsilon \right\|_{L^\infty} + \left\| \langle y \rangle^\ell \omega_\varepsilon \right\|_{L^\infty} + \left\| \langle y \rangle^{\ell+1} \partial_y \omega_\varepsilon \right\|_{L^\infty} \leq \frac{1}{4} \min \{c_0, c_1\} \end{aligned}$$

because (66) and (67), so that the property (8) holds by u_ε for all $t \in [0, T_\varepsilon]$ due to the fact that $\alpha \leq \ell$.

In the following argument, we let (ρ, t) be an arbitrary pair which is fixed at moment and satisfies that $\rho > 0$, $t \in [0, T_\varepsilon]$ and $\rho + \lambda t < \rho_0$. Then we have, in view of (69),

$$\forall 0 \leq s \leq t, \quad |u_\varepsilon(s)|_{\rho, \sigma} \leq \|u_\varepsilon\|_{(\lambda, T_\varepsilon)} \left(\frac{\rho_0 - \rho - \lambda s}{\rho_0 - \rho} \right)^{-1/2}. \quad (73)$$

Furthermore, we take in particular such a $\tilde{\rho}(s)$ that

$$\tilde{\rho}(s) = \frac{\rho_0 + \rho - \lambda s}{2}.$$

Then direct calculation shows that

$$\forall 0 \leq s \leq t, \quad \rho < \tilde{\rho}(s) \quad \text{and} \quad \tilde{\rho}(s) + \lambda s < \rho_0, \quad (74)$$

and

$$\forall 0 \leq s \leq t, \quad \tilde{\rho}(s) - \rho = \frac{\rho_0 - \rho - \lambda s}{2} = \rho_0 - \tilde{\rho}(s) - \lambda s, \quad \rho_0 - \tilde{\rho}(s) \leq \rho_0 - \rho. \quad (75)$$

The inequalities in (74) imply

$$\forall 0 \leq s \leq t, \quad |u_\varepsilon(s)|_{\tilde{\rho}(s), \sigma} \leq \|u_\varepsilon\|_{(\lambda, T_\varepsilon)} \left(\frac{\rho_0 - \tilde{\rho}(s) - \lambda s}{\rho_0 - \tilde{\rho}(s)} \right)^{-1/2} \leq \|u_\varepsilon\|_{(\lambda, T_\varepsilon)} \left(\frac{2(\rho_0 - \rho)}{\rho_0 - \rho - \lambda s} \right)^{1/2}, \quad (76)$$

where the last inequality follows from (75).

Now we apply Theorem 2.4 to the pair $(\rho, \tilde{\rho}(s))$ given above to have for any $t \in [0, T_\varepsilon]$,

$$|u_\varepsilon(t)|_{\rho, \sigma}^2 \leq C_* |u_0|_{\rho, \sigma}^2 + C_* \int_0^t \left(|u_\varepsilon(s)|_{\rho, \sigma}^2 + |u_\varepsilon(s)|_{\rho, \sigma}^4 \right) ds + C_* \int_0^t \frac{|u_\varepsilon(s)|_{\tilde{\rho}(s), \sigma}^2}{\tilde{\rho}(s) - \rho} ds.$$

Moreover, we insert (73) and (76) into the above inequality to obtain, using (75) as well,

$$\begin{aligned} |u_\varepsilon(t)|_{\rho, \sigma}^2 &\leq C_* |u_0|_{\rho, \sigma}^2 + C_* \|u_\varepsilon\|_{(\lambda, T_\varepsilon)}^2 \int_0^t \frac{\rho_0 - \rho}{\rho_0 - \rho - \lambda s} ds + C_* \|u_\varepsilon\|_{(\lambda, T_\varepsilon)}^4 \int_0^t \frac{(\rho_0 - \rho)^2}{(\rho_0 - \rho - \lambda s)^2} ds \\ &\quad + C_* \|u_\varepsilon\|_{(\lambda, T_\varepsilon)}^2 \int_0^t \frac{2^2 (\rho_0 - \rho)}{(\rho_0 - \rho - \lambda s)^2} ds \\ &\leq C_* |u_0|_{\rho, \sigma}^2 + \frac{(5C_* + C_* R^2) \|u_\varepsilon\|_{(\lambda, T_\varepsilon)}^2}{\lambda} \left(\frac{\rho_0 - \rho - \lambda t}{\rho_0 - \rho} \right)^{-1}, \end{aligned}$$

where in the last inequality we have used (72) and the fact that

$$\frac{\rho_0 - \rho}{\rho_0 - \rho - \lambda s} \leq \frac{(\rho_0 - \rho)^2}{(\rho_0 - \rho - \lambda s)^2} \leq \frac{(\rho_0 - \rho)}{(\rho_0 - \rho - \lambda s)^2}.$$

Then multiplying both sides by the fact $(\rho_0 - \rho - \lambda t) / (\rho_0 - \rho)$ implies, observing (ρ, t) is an arbitrary pair with $\rho > 0$, $t \in [0, T_\varepsilon]$ and $\rho + \lambda t < \rho_0$,

$$\|u_\varepsilon\|_{(\lambda, T_\varepsilon)} \leq \sqrt{C_*} \sup_{\rho, t} \left(\frac{\rho_0 - \rho - \lambda t}{\rho_0 - \rho} \right)^{1/2} |u_0|_{\rho, \sigma} + \frac{\sqrt{5C_* + C_* R^2}}{\sqrt{\lambda}} \|u_\varepsilon\|_{(\lambda, T_\varepsilon)} \leq C_* |u_0|_{\rho_0, \sigma} + \frac{1}{2} \|u_\varepsilon\|_{(\lambda, T_\varepsilon)}.$$

Here the last inequality holds because of (68) and the fact that $C_* \geq 1$. Then we conclude

$$\|u_\varepsilon\|_{(\lambda, T_\varepsilon)} \leq 2C_* |u_0|_{\rho_0, \sigma} \leq 2C_* \hat{C} \left(\|u_0\|_{2\rho_0, \sigma} + \|u_0\|_{2\rho_0, \sigma}^2 \right) \leq R/2,$$

where the second inequality follows from (65) and in the last inequality we have used (67). This contradicts (72) so that (71) holds.

Step (iv). We conclude that $P_\varepsilon > \rho_0/(4\lambda)$, otherwise, it follows from (71) that for any $T \in [0, P_\varepsilon[$ we have $\|u_\varepsilon\|_{(\lambda, T)} \leq R$, which contradicts to the definition of P_ε . Consequently, we can rewrite (71) as

$$\forall 0 \leq T \leq \rho_0/(4\lambda), \quad \|u_\varepsilon\|_{(\lambda, T)} \leq R.$$

Thus

$$\forall t \in [0, \rho_0/(4\lambda)], \quad \frac{\sqrt{2}}{2} |u_\varepsilon(t)|_{\rho_0/2, \sigma} \leq \left(\frac{\rho_0 - \rho_0/2 - \lambda t}{\rho_0 - \rho_0/2} \right)^{1/2} |u(t)|_{\rho_0/2, \sigma} \leq \|u_\varepsilon\|_{(\lambda, \rho_0/(4\lambda))} \leq R.$$

This gives

$$\forall \varepsilon > 0, \quad u_\varepsilon \in L^\infty([0, \rho_0/(4\lambda)]; X_{\rho_0/2, \sigma}) \quad \text{and} \quad \|u_\varepsilon(t)\|_{\rho_0/2, \sigma} \leq |u_\varepsilon(t)|_{\rho_0/2, \sigma} \leq \sqrt{2}R.$$

Now let $\varepsilon \rightarrow 0$ and we have, by compactness arguments, the limit u of u_ε solves the equation (3). We complete the existence part of Theorem 1.6.

8.2. Proof of Theorem 1.6: uniqueness. Let $u_1, u_2 \in L^\infty([0, \rho_0/(4\lambda)]; X_{\rho_0/2, \sigma})$ be two solutions to the Prandtl equation (3), and let $v_j = -\int_0^y \partial_x u_j(x, \tilde{y}) d\tilde{y}$. Then the differences

$$u \stackrel{\text{def}}{=} u_1 - u_2, \quad v \stackrel{\text{def}}{=} v_1 - v_2,$$

satisfy the following initial boundary problem, using the notation $\omega = \partial_y u$ and $\omega_j = \partial_y u_j$ as before,

$$\begin{cases} \partial_t u + (u^s + u_1) \partial_x u + v_1 \partial_y u + u \partial_x u_2 + v(\omega^s + \omega_2) - \partial_y^2 u = 0, \\ u|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = 0, \\ u|_{t=0} = 0. \end{cases} \quad (77)$$

Moreover, we have the equations for ω and $\partial_y \omega$:

$$\partial_t \omega + (u^s + u_1) \partial_x \omega + v_1 \partial_y \omega - \partial_y^2 \omega + u \partial_x \omega_2 + v(\partial_y \omega^s + \partial_y \omega_2) = 0, \quad (78)$$

and

$$\begin{aligned} & \partial_t(\partial_y \omega) + (u^s + u_1) \partial_x(\partial_y \omega) + v_1 \partial_y(\partial_y \omega) - \partial_y^2(\partial_y \omega) + v(\partial_y^2 \omega^s + \partial_y^2 \omega_2) \\ &= - \left[(\omega^s + \omega_1) \partial_x \omega - (\partial_y \omega^s + \partial_y \omega_2) \partial_x u \right] + (\partial_x u_1) \partial_y \omega - \omega \partial_x \omega_2 - u \partial_x \partial_y \omega_2. \end{aligned} \quad (79)$$

Now we apply ∂_x^m to the three equations above, and then we have, as in the previous sections, several terms have loss of x derivative. Precisely, $(\partial_x^m v)(\omega^s + \omega_2)$ is involved in the equation for $\partial_x^m u$, and $(\partial_x^m v)(\partial_y \omega^s + \partial_y \omega_2)$ in the equation for $\partial_x^m \omega$, and meanwhile two terms $(\partial_x^m v)(\partial_y^2 \omega^s + \partial_y^2 \omega_2)$ and $\partial_x^m [(\omega^s + \omega_1) \partial_x \omega - (\partial_y \omega^s + \partial_y \omega_2) \partial_x u]$ in the equation for $\partial_x^m \partial_y \omega$. To overcome the degeneracy, we just follow the same strategy as in Sections 3-6, with f_m, h_m and g_m therein replaced respectively by

$$\begin{aligned} f_m^* &= \chi_1 \partial_x^m \omega - \chi_1 \frac{\partial_y \omega^s + \partial_y \omega_2}{\omega^s + \omega_2} \partial_x^m u, \\ h_m^* &= \chi_2 \partial_x^m \partial_y \omega - \chi_2 \frac{\partial_y^2 \omega^s + \partial_y^2 \omega_2}{\partial_y \omega^s + \partial_y \omega_2} \partial_x^m \omega, \\ g_m^* &= \partial_x^{m-1} \left[(\omega^s + \omega_1) \partial_x \omega - (\partial_y \omega^s + \partial_y \omega_2) \partial_x u \right]. \end{aligned}$$

Then just repeating the argument in the Sections 3-6, with slight modification, we can obtain, observing $u|_{t=0} = 0$,

$$\|u(t)\|_{\rho, \sigma}^2 \leq C \int_0^t \frac{\left(1 + |u_1(s)|_{\rho/2, \sigma}^2 + |u_2(s)|_{\rho/2, \sigma}^2 + |u_1(s)|_{\rho/2, \sigma}^4 + |u_2(s)|_{\rho/2, \sigma}^4\right) \|u(s)\|_{\tilde{\rho}(s), \sigma}^2}{\tilde{\rho}(s) - \rho} ds, \quad (80)$$

where the definition of $\|u\|_{\rho,\sigma}$ is similar as $|u|_{\rho,\sigma}$ by just replacing respectively the summations

$$\sup_{\substack{1 \leq j \leq 4 \\ i+j \geq 6}} \quad \text{and} \quad \sup_{\substack{1 \leq j \leq 4 \\ i+j \leq 5}}$$

in Definition 1.4 by

$$\sup_{\substack{1 \leq j \leq 2 \\ i+j \geq 6}} \quad \text{and} \quad \sup_{\substack{1 \leq j \leq 2 \\ i+j \leq 5}}.$$

Now we emphasize the difference between (80) and (16). Note that we work on $\|u\|_{\rho,\sigma}$ instead of $|u|_{\rho,\sigma}$ because we lose y -derivative for the term $v(\partial_y^2 \omega^s + \partial_y^2 \omega_2)$ in (79). So we have to reduce the order of y derivatives from 4 to 2. Moreover, observe that we also lose x derivatives for u_1 and u_2 in equations (77)-(79) and this can be overcome by reducing the Gevrey radius ρ to $\rho/2$. Then by virtue of (80), we can follow the argument used in the existence part to conclude

$$\sup_{\rho,t} \left(\frac{\rho_0/2 - \rho - \lambda t}{\rho_0/2 - \rho} \right)^{1/2} \|u(s)\|_{\rho,\sigma} = 0,$$

where the supremum is taken over all pairs (ρ, t) such that $\rho > 0$ and $\rho + \lambda t < \rho_0/2$. And thus $u \equiv 0$ and the uniqueness follows.

9. GENERAL INITIAL DATA

In this section, we will clarify why the above result holds for general initial data without requiring the small perturbations around a shear flow. Precisely, we consider the Prandtl equation in $\Omega \times \mathbb{R}_+$ with Ω the whole space \mathbb{R} or the torus \mathbb{T} , that is,

$$\begin{cases} \partial_t u^P + u^P \partial_x u^P + v^P \partial_y u - \partial_y^2 u^P + \partial_x p = 0, & t > 0, \quad x \in \Omega, \quad y > 0, \\ \partial_x u^P + \partial_y v^P = 0, \\ u^P|_{y=0} = v^P|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = U(t, x), \\ u^P|_{t=0} = u_0^P(x, y). \end{cases} \quad (81)$$

Without loss generality, we suppose that $U \equiv 0$, and thus $\partial_x p \equiv 0$ by Bernoulli law.

To investigate the well-posedness in Gevrey class for the above Prandtl equation, there are two main ingredients, one is about the existence of approximate solution for the regularized Prandtl equation

$$\begin{cases} \partial_t u^P + u^P \partial_x u^P + v^P \partial_y u - \partial_y^2 u^P - \varepsilon \partial_x^2 u^P = 0, & t > 0, \quad x \in \Omega, \quad y > 0, \\ \partial_x u^P + \partial_y v^P = 0, \\ u^P|_{y=0} = v^P|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} u = 0, \\ u^P|_{t=0} = u_0^P(x, y), \end{cases} \quad (82)$$

where and in the following discussion, we will use u^P and v^P instead of u_ε^P and v_ε^P , by omitting ε for simpler presentation. And another ingredient is the uniform estimate for the approximate solution, which is the main concern of this paper, cf. Sections 2-6. We will explain why we do not need the small perturbation in obtaining the uniform estimate, and the requirement on the initial data is only for the construction of approximate solution.

Suppose that the initial-boundary problem (82) admits a solution (u^P, v^P) in the interval $[0, T]$ satisfying the properties listed below. That is, given $y_0 > 0$, there are three large constants C_1, C_2, C_3 and a positive number $\delta_0 \in [0, y_0/2]$ and two positive numbers ℓ, α with $\ell > 3/2$ and $\alpha + \frac{1}{2} > \ell$, such

that for any $t \in [0, T]$ and any $x \in \Omega$, we have, using the notation $\omega = \partial_y u^P$,

$$\begin{cases} |\partial_y \omega(t, x, y)| \geq \frac{1}{4C_1}, & \text{if } y \in [y_0 - \frac{7}{4}\delta_0, y_0 + \frac{7}{4}\delta_0], \\ 4^{-1}C_2^{-1} \langle y \rangle^{-\alpha} \leq |\omega(t, x, y)| \leq 4C_2 \langle y \rangle^{-\alpha}, & \text{if } y \in [0, y_0 - \frac{5}{4}\delta_0] \cup [y_0 + \frac{5}{4}\delta_0, +\infty[, \\ |\partial_y \omega(t, x, y)| \leq 4C_2 \langle y \rangle^{-\alpha-1} & \text{for } y \geq 0, \\ \sum_{1 \leq j \leq 2} \left(\left\| \langle y \rangle^{\ell-1} \partial_x^j u^P \right\|_{L^\infty} + \left\| \partial_x^{j-1} v^P \right\|_{L^\infty} + \left\| \langle y \rangle^\ell \partial_x^j \omega \right\|_{L^\infty} \right) + \sum_{1 \leq i, j \leq 2} \left\| \langle y \rangle^{\ell+1} \partial_x^i \partial_y^j \omega \right\|_{L^\infty} \leq C_3. \end{cases} \quad (83)$$

Let $(X_{\rho, \sigma}, \|\cdot\|_{\rho, \sigma})$ be the Gevrey space in the tangential variable $x \in \Omega$ introduced in Definition 1.4, with the L^2 norm therein taken over $\Omega \times \mathbb{R}_+$. Similarly, as in Definition 2.2, we can define $|u^P|_{\rho, \sigma}$ with the auxilliary functions therein replaced respectively by the following new ones:

$$\begin{aligned} f_m^P &= \chi_1 \partial_x^m \omega - \chi_1 \frac{\partial_y \omega}{\omega} \partial_x^m u^P, \\ h_m^P &= \chi_2 \partial_x^m \partial_y \omega - \chi_2 \frac{\partial_y^2 \omega}{\partial_y \omega} \partial_x^m \omega, \\ g_m^P &= \partial_x^{m-1} (\omega \partial_x \omega - (\partial_y \omega) \partial_x u^P). \end{aligned}$$

Here $\chi_i, i = 1, 2$, are given in (9) and (10).

Theorem 9.1 (uniform estimates in Gevrey space). *Let $3/2 \leq \sigma \leq 2$. Let the initial datum $u_0^P \in X_{2\rho_0, \sigma}$ and let $u^P \in L^\infty([0, T]; X_{\rho_0, \sigma})$ be a solution to (82) such that the properties listed in (83) hold. Then there exists a constant $\tilde{C}_* > 1$, independent of ε , such that the estimate*

$$|u^P(t)|_{\rho, \sigma}^2 \leq \tilde{C}_* |u_0^P|_{\rho, \sigma}^2 + \tilde{C}_* \int_0^t \left(|u^P(s)|_{\rho, \sigma}^2 + |u^P(s)|_{\rho, \sigma}^4 \right) ds + \tilde{C}_* \int_0^t \frac{|u^P(s)|_{\tilde{\rho}, \sigma}^2}{\tilde{\rho} - \rho} ds$$

holds for any pair $(\rho, \tilde{\rho})$ with $0 < \rho < \tilde{\rho} < \rho_0$, and for any $t \in [0, \tilde{T}]$, where $[0, \tilde{T}]$ is the maximal interval of existence for $|u^P(t)|_{\tilde{\rho}, \sigma} < +\infty$.

Proof. The proof is similar to that of Theorem 2.4 by following the argument in Sections 3-6 with slight modification. So we omit it. \square

Remark 9.2. To prove the above theorem we only require that the initial datum $u_0^P \in X_{2\rho_0, \sigma}$ and satisfies the conditions in (83). Hence, we do not need the additional assumption that the initial datum is the small perturbation of a shear flow.

The remaining ingredient in the proof is to construct solution to (82) satisfying the properties listed in (83). In fact, this together with the uniform estimate given in Theorem 9.1 enables us to repeat the argument in Section 8 to conclude the well-posedness in Gevrey space to the original Prandtl equation (81). For this, it is not difficult to construct solution to (82) because it is a parabolic initial-boundary problem. The key point is to prove the properties listed in (83) are preserved in time by supposing that they hold initially. It is clear that these properties are indeed preserved with time for the shear flows since they satisfy the heat equation with initial-boundary conditions (see Proposition 1.3), and thus so are for the solutions to Prandtl equation by small perturbation. For the general initial data rather than the small perturbation around a shear flow, the existence of such approximate solutions that satisfy (83) is proven by Gérard-Varet and Masmoudi [6, Section 4] where they use the maximum principle so that the assumptions in Theorem 9.1 hold. This enables us to conclude that the result obtained by Gérard-Varet and Masmoudi [6] still holds when the Gevrey index $7/4$ therein is replaced by any $\sigma \in [3/2, 2]$, and there is no additional assumption required.

APPENDIX A. SOBOLEV INEQUALITY

Lemma A.1. *For any $h \in H^2(\mathbb{R}_+^2) \cap C^2(\mathbb{R}_+^2)$, we have*

$$\|h\|_{L^\infty(\mathbb{R}_+^2)} \leq \sqrt{2} (\|h\|_{L^2} + \|\partial_x h\|_{L^2} + \|\partial_y h\|_{L^2} + \|\partial_x \partial_y h\|_{L^2}).$$

Proof. We begin with the 1D Sobolev inequality:

$$\|f\|_{L^\infty(\Omega)} \leq \|f\|_{L^2(\Omega)} + \|f'\|_{L^2(\Omega)}, \quad \Omega = \mathbb{R}_+ \text{ or } \mathbb{R}. \quad (84)$$

To see this, let $\omega = \mathbb{R}_+$ and let $r \in \mathbb{R}_+$. By mean value Theorem, we can find a $\xi \in [r, r+1]$ such that

$$\int_r^{r+1} f(\tilde{r}) d\tilde{r} = f(\xi).$$

Moreover

$$|f(r) - f(\xi)| = \left| \int_\xi^r f'(\tilde{r}) d\tilde{r} \right| \leq \|f'\|_{L^2(\mathbb{R}_+)}.$$

Thus

$$|f(r)| \leq |f(\xi)| + |f(r) - f(\xi)| \leq \|f\|_{L^2(\mathbb{R}_+)} + \|f'\|_{L^2(\mathbb{R}_+)},$$

which implies, taking the supremum over $r \in \mathbb{R}_+$,

$$\|f\|_{L^\infty(\mathbb{R}_+)} \leq \|f\|_{L^2(\mathbb{R}_+)} + \|f'\|_{L^2(\mathbb{R}_+)}.$$

Similarly, the above estimate also holds with \mathbb{R}_+ replaced by \mathbb{R} . Then (84) follows.

Now we use (84) to prove Lemma A.1. For any $(x, y) \in \mathbb{R}_+^2$, we have

$$\begin{aligned} |h(x, y)| &\leq \left(\int_{\mathbb{R}_+} |h(x, \tilde{y})|^2 d\tilde{y} \right)^{1/2} + \left(\int_{\mathbb{R}_+} |(\partial_y h)(x, \tilde{y})|^2 d\tilde{y} \right)^{1/2} \\ &\leq \left(2 \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}} |h(\tilde{x}, \tilde{y})|^2 d\tilde{x} + \int_{\mathbb{R}} |(\partial_x h)(\tilde{x}, \tilde{y})|^2 d\tilde{x} \right) d\tilde{y} \right)^{1/2} \\ &\quad + \left(2 \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}} |(\partial_y h)(\tilde{x}, \tilde{y})|^2 d\tilde{x} + \int_{\mathbb{R}} |(\partial_x \partial_y h)(\tilde{x}, \tilde{y})|^2 d\tilde{x} \right) d\tilde{y} \right)^{1/2}. \end{aligned}$$

Taking the supremum over $(x, y) \in \mathbb{R}_+^2$, we obtain the desired estimate in Lemma A.1. \square

APPENDIX B. AUXILLIARY FUNCTIONS

Lemma B.1 (Equation for f_m). *Let f_m be given in (30). Then we have*

$$\begin{aligned} &(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2) f_m \\ &= -\chi_1 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \omega - \chi_1 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y \omega \\ &\quad + \chi_1 a \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} u + \chi_1 a \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \omega \\ &\quad + \chi_1' v \partial_x^m \omega - 2\chi_1' \partial_x^m \partial_y \omega - \chi_1'' \partial_x^m \omega - a (\chi_1' v \partial_x^m u - 2\chi_1' \partial_x^m \omega - \chi_1'' \partial_x^m u) \\ &\quad + \left[\partial_x \omega - (\partial_x u) a - 2a \partial_y a - 2\varepsilon \frac{\partial_x \omega}{\omega^s + \omega} \partial_x a \right] \chi_1 \partial_x^m u \\ &\quad + 2\chi_1 (\partial_y a) \partial_x^m \omega + 2\chi_1' (\partial_y a) \partial_x^m u + 2\varepsilon \chi_1 (\partial_x a) \partial_x^{m+1} u. \end{aligned}$$

where $a = \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega}$.

Proof. Observe that $\partial_x u + \partial_y v = 0$ and then it follows from the equation

$$\partial_t u + (u^s + u) \partial_x u + v (\omega^s + \omega) - \partial_y^2 u - \varepsilon \partial_x^2 u = 0, \quad (85)$$

that $\omega = \partial_y u$ satisfies

$$\partial_t \omega + (u^s + u) \partial_x \omega + v (\partial_y \omega^s + \partial_y \omega) - \partial_y^2 \omega - \varepsilon \partial_x^2 \omega = 0. \quad (86)$$

We apply the operator ∂_x^m to the two equations above and then multiply the resulting equations by $\chi_1(y)$; this gives

$$\begin{aligned} & \left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) \chi_1 \partial_x^m u + \chi_1 (\partial_x^m v) (\omega^s + \omega) \\ &= -\chi_1 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} u - \chi_1 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \omega \\ & \quad + \chi_1' v \partial_x^m u - 2\chi_1' \partial_x^m \omega - \chi_1'' \partial_x^m u, \end{aligned} \quad (87)$$

and

$$\begin{aligned} & \left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) \chi_1 \partial_x^m \omega + \chi_1 (\partial_x^m v) (\partial_y \omega^s + \partial_y \omega) \\ &= -\chi_1 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \omega - \chi_1 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y \omega \\ & \quad + \chi_1' v \partial_x^m \omega - 2\chi_1' \partial_x^m \partial_y \omega - \chi_1'' \partial_x^m \omega. \end{aligned} \quad (88)$$

Observe $|\omega^s + \omega| > 0$ on $\text{supp} \chi_1$ and then we can multiply both sides of (87) by the factor

$$a = \frac{\partial_y \omega^s + \partial_y \omega}{\omega^s + \omega},$$

and then subtract the resulting equation by (88). Then the function f_m , defined in (30) solves

$$\begin{aligned} & \left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) f_m \\ &= -\chi_1 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \omega - \chi_1 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y \omega \\ & \quad + \chi_1 a \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} u + \chi_1 a \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \omega \\ & \quad + \chi_1' v \partial_x^m \omega - 2\chi_1' \partial_x^m \partial_y \omega - \chi_1'' \partial_x^m \omega - a (\chi_1' v \partial_x^m u - 2\chi_1' \partial_x^m \omega - \chi_1'' \partial_x^m u) \\ & \quad - \left[\partial_t a + (u^s + u) \partial_x a + v \partial_y a - \partial_y^2 a - \varepsilon \partial_x^2 a \right] \chi_1 \partial_x^m u \\ & \quad + 2\chi_1 (\partial_y a) \partial_x^m \omega + 2\chi_1' (\partial_y a) \partial_x^m u + 2\varepsilon \chi_1 (\partial_x a) \partial_x^{m+1} u. \end{aligned}$$

On the other hand we notice that, for any $y \in \text{supp} \chi_1$,

$$\partial_t a + (u^s + u) \partial_x a + v \partial_y a - \partial_y^2 a - \varepsilon \partial_x^2 a = -\partial_x \omega + (\partial_x u) a + 2a \partial_y a + 2\varepsilon \frac{\partial_x \omega}{\omega^s + \omega} \partial_x a.$$

Then combining the above equations completes the proof. \square

Lemma B.2 (Equation for h_m). *Let h_m be given in (40). Then we have*

$$\begin{aligned}
& \left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) h_m \\
= & P_m + 2 (\partial_y b) \partial_y (\chi_2 \partial_x^m \omega) + 2\varepsilon (\partial_x b) \partial_x (\chi_2 \partial_x^m \omega) \\
& + \chi_2 b \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \omega + \chi_2 b \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y \omega \\
& - b \chi_2' v \partial_x^m \omega + b \chi_2'' \partial_x^m \omega + 2b \chi_2' \partial_x^m \partial_y \omega \\
& - \chi_2 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \partial_y \omega - \chi_2 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y^2 \omega \\
& + \chi_2' v \partial_x^m \partial_y \omega - \chi_2'' \partial_x^m \partial_y \omega - 2\chi_2' \partial_x^m \partial_y^2 \omega - \chi_2 g_{m+1},
\end{aligned}$$

where $b = \frac{\partial_y^2 \omega^s + \partial_y^2 \omega}{\partial_y \omega^s + \partial_y \omega}$ and

$$\begin{aligned}
P_m = & \frac{2 \left((\omega^s + \omega) \partial_x \partial_y \omega - (\partial_x u) (\partial_y^2 \omega^s + \partial_y^2 \omega) \right) \chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \\
& - \frac{\left(\omega \partial_x \omega - (\partial_x u) (\partial_y \omega^s + \partial_y \omega) \right) (\partial_y^2 \omega^s + \partial_y^2 \omega) \chi_2 \partial_x^m \omega}{(\partial_y \omega^s + \partial_y \omega)^2} \\
& - \frac{2 \left((\partial_y^3 \omega^s + \partial_y^3 \omega) (\partial_y^2 \omega^s + \partial_y^2 \omega) + \varepsilon (\partial_x \partial_y^2 \omega) \partial_x \partial_y \omega \right) \chi_2 \partial_x^m \omega}{(\partial_y \omega^s + \partial_y \omega)^2} \\
& + \frac{2 \left((\partial_y^2 \omega^s + \partial_y^2 \omega)^2 + \varepsilon (\partial_x \partial_y \omega)^2 \right) (\partial_y^2 \omega^s + \partial_y^2 \omega) \chi_2 \partial_x^m \omega}{(\partial_y \omega^s + \partial_y \omega)^3}.
\end{aligned}$$

Proof. Observe $\omega = \partial_y u$ and $\partial_y \omega$ solve the following equations:

$$\partial_t \omega + (u^s + u) \partial_x \omega + v (\partial_y \omega^s + \partial_y \omega) - \partial_y^2 \omega - \varepsilon \partial_x^2 \omega = 0, \quad (89)$$

and

$$\partial_t (\partial_y \omega) + (u^s + u) \partial_x (\partial_y \omega) + v (\partial_y^2 \omega^s + \partial_y^2 \omega) - \partial_y^2 (\partial_y \omega) - \varepsilon \partial_x^2 (\partial_y \omega) = -g_1, \quad (90)$$

by recalling $g_1 = (\omega^s + \omega) \partial_x \omega - (\partial_y \omega^s + \partial_y \omega) \partial_x u$. Now we perform $\chi_2 \partial_x^m$, $m \geq 1$, on both sides of (89)-(90), to obtain that

$$\begin{aligned}
& \left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) \chi_2 \partial_x^m \omega + \chi_2 (\partial_y \omega^s + \partial_y \omega) \partial_x^m v \\
= & -\chi_2 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \omega - \chi_2 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y \omega \\
& + \chi_2' v \partial_x^m \omega - \chi_2'' \partial_x^m \omega - 2\chi_2' \partial_x^m \partial_y \omega,
\end{aligned}$$

and

$$\begin{aligned}
& \left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) \chi_2 \partial_x^m \partial_y \omega + \chi_2 (\partial_y^2 \omega^s + \partial_y^2 \omega) \partial_x^m v \\
= & -\chi_2 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \partial_y \omega - \chi_2 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y^2 \omega \\
& + \chi_2' v \partial_x^m \partial_y \omega - \chi_2'' \partial_x^m \partial_y \omega - 2\chi_2' \partial_x^m \partial_y^2 \omega - \chi_2 g_{m+1}.
\end{aligned}$$

Now we multiply the first equation by $(\partial_y^2 \omega^s + \partial_y^2 \omega)/(\partial_y \omega^s + \partial_y \omega)$, and then subtract the obtained equation by the second equation. This gives the equation for h_m :

$$\begin{aligned}
& (\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2) h_m \\
= & (-\partial_t b - (u^s + u) \partial_x b - v \partial_y b + \partial_y^2 b + \varepsilon \partial_x^2 b) \chi_2 \partial_x^m \omega \\
& + 2(\partial_y b) \partial_y (\chi_2 \partial_x^m \omega) + 2\varepsilon (\partial_x b) \partial_x (\chi_2 \partial_x^m \omega) \\
& + \chi_2 b \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \omega + \chi_2 b \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y \omega \\
& - b \chi_2' v \partial_x^m \omega + b \chi_2'' \partial_x^m \omega + 2b \chi_2' \partial_x^m \partial_y \omega \\
& - \chi_2 \sum_{k=1}^m \binom{m}{k} (\partial_x^k u) \partial_x^{m-k+1} \partial_y \omega - \chi_2 \sum_{k=1}^{m-1} \binom{m}{k} (\partial_x^k v) \partial_x^{m-k} \partial_y^2 \omega \\
& + \chi_2' v \partial_x^m \partial_y \omega - \chi_2'' \partial_x^m \partial_y \omega - 2\chi_2' \partial_x^m \partial_y^2 \omega - \chi_2 g_{m+1}.
\end{aligned}$$

Finally, we use the equation (90) to compute

$$\begin{aligned}
& [-\partial_t b - (u^s + u) \partial_x b - v \partial_y b + \partial_y^2 b + \varepsilon \partial_x^2 b] \chi_2 \partial_x^m \omega \\
= & \frac{2 \left((\omega^s + \omega) \partial_x \partial_y \omega - (\partial_x u) (\partial_y^2 \omega^s + \partial_y^2 \omega) \right) \chi_2 \partial_x^m \omega}{\partial_y \omega^s + \partial_y \omega} \\
& - \frac{\left(\omega \partial_x \omega - (\partial_x u) (\partial_y \omega^s + \partial_y \omega) \right) (\partial_y^2 \omega^s + \partial_y^2 \omega) \chi_2 \partial_x^m \omega}{(\partial_y \omega^s + \partial_y \omega)^2} \\
& - \frac{2 \left((\partial_y^3 \omega^s + \partial_y^3 \omega) (\partial_y^2 \omega^s + \partial_y^2 \omega) + \varepsilon (\partial_x \partial_y^2 \omega) \partial_x \partial_y \omega \right) \chi_2 \partial_x^m \omega}{(\partial_y \omega^s + \partial_y \omega)^2} \\
& + \frac{2 \left((\partial_y^2 \omega^s + \partial_y^2 \omega)^2 + \varepsilon (\partial_x \partial_y \omega)^2 \right) (\partial_y^2 \omega^s + \partial_y^2 \omega) \chi_2 \partial_x^m \omega}{(\partial_y \omega^s + \partial_y \omega)^3}.
\end{aligned}$$

Then combining the three equations above we obtain the desired equation of h_m . □

Lemma B.3 (Equation for g_m). *Let g_m be given in (17). Then we have*

$$\begin{aligned}
& (\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2) g_m \\
= & - \sum_{j=1}^{m-1} \binom{m-1}{j} (\partial_x^j u) g_{m-j+1} - \sum_{j=1}^{m-1} \binom{m-1}{j} (\partial_x^j v) \partial_y g_{m-j} \\
& + 2 \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^j \partial_y^2 \omega^s + \partial_x^j \partial_y^2 \omega) \partial_x^{m-j} \omega + 2\varepsilon \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^{j+1} \partial_y \omega) \partial_x^{m-j+1} u \\
& - 2 \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^j \partial_y \omega^s + \partial_x^j \partial_y \omega) \partial_x^{m-j} \partial_y \omega - 2\varepsilon \sum_{j=0}^{m-1} \binom{m-1}{j} (\partial_x^{j+1} \omega) \partial_x^{m-j+1} \omega.
\end{aligned}$$

Proof. It follows from the equations for velocity and vorticity that

$$(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2) \partial_x u + (\omega^s + \omega) \partial_x v = -(\partial_x u) \partial_x u,$$

and

$$(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2) \partial_x \omega + (\partial_y \omega^s + \partial_y \omega) \partial_x v = -(\partial_x u) \partial_x \omega.$$

Now we multiply the first equation above by $\partial_y \omega^s + \partial_y \omega$ and the second one by $\omega^s + \omega$, and then subtract one from the other to have

$$\begin{aligned} & \left(\partial_t + (u^s + u) \partial_x + v \partial_y - \partial_y^2 - \varepsilon \partial_x^2 \right) g_1 \\ &= -(\partial_x u) g_1 - \left[\partial_t \partial_y \omega + (u^s + u) \partial_x \partial_y \omega + v \partial_y (\partial_y \omega^s + \partial_y \omega) - \partial_y^3 \omega - \varepsilon \partial_x^2 \partial_y \omega \right] \partial_x u \\ & \quad + 2(\partial_y^2 \omega^s + \partial_y^2 \omega) \partial_x \omega + 2\varepsilon (\partial_x \partial_y \omega) \partial_x^2 u - 2(\partial_y \omega^s + \partial_y \omega) \partial_x \partial_y \omega - 2\varepsilon (\partial_x \omega) \partial_x^2 \omega \\ &= 2(\partial_y^2 \omega^s + \partial_y^2 \omega) \partial_x \omega + 2\varepsilon (\partial_x \partial_y \omega) \partial_x^2 u - 2(\partial_y \omega^s + \partial_y \omega) \partial_x \partial_y \omega - 2\varepsilon (\partial_x \omega) \partial_x^2 \omega, \end{aligned}$$

where in the last equality we have used the fact that

$$\partial_t \partial_y \omega + (u^s + u) \partial_x \partial_y \omega + v \partial_y (\partial_y \omega^s + \partial_y \omega) - \partial_y^3 \omega - \varepsilon \partial_x^2 \partial_y \omega = -g_1.$$

Then applying ∂_x^{m-1} to the equation yields the equation for g_m . \square

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