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A New Glimm Functional and Convergence Rate of Glimm Scheme for General Systems of Hyperbolic Conservation Laws

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Abstract

In this paper, we introduce a new Glimm functional for general systems of hyperbolic conservation laws. This new functional is consistent with the classical Glimm functional for the case when each characteristic field is either genuinely nonlinear or linearly degenerate, so that it can be viewed as "optimal" in some sense. With this new functional, the consistency of the Glimm scheme is proved clearly for general systems. Moreover, the convergence rate of the Glimm scheme is shown to be the same as the one obtained in Bressan, Marson (Arch Ration Mech Anal 142(2):155–176, 1998) for systems with each characteristic field being genuinely nonlinear or linearly degenerate.

1. Introduction

There have been extensive studies on the mathematical theory for the systems of hyperbolic conservation laws. One of the typical examples of these systems is the compressible Euler equations for fluid dynamics. As for the Cauchy problem, the celebrated paper [13] by Glimm in 1965 established the global existence of weak solutions with small total variation under the assumption that each characteristic field is either genuinely nonlinear or linearly degenerate. Even though the system of compressible Euler equations for gas dynamics satisfies this assumption, there are many other physical systems such as those arising from elasticity and magneto-hydrodynamics whose characteristic fields do not all satisfy this assumption. To extend the Glimm theory to general systems, the key step is to redefine of the Glimm functional for wave interactions in the same characteristic family. For this purpose, a cubic functional was introduced by Liu in [21], and was elaborated by Liu and Yang in [23]. It employs an effective angle between two waves in the same family. This improvement is successful for establishing existence, but it is less satisfactory for consistency and convergence rate analysis. Therefore, the purpose of this paper is to introduce a new Glimm functional for wave interactions in the same family for general systems, so that the Glimm theory can now be presented in an elegant way.

Consider the Cauchy problem for a system of hyperbolic conservation laws

$$\begin{cases} u_t + f(u)_x = 0, & t \geqq 0,\ -\infty < x < \infty, \\ u(x,0) = u_0(x), & -\infty < x < \infty, \end{cases} \tag{1.1}$$

where $u \in \mathbb{R}^n$, $f : \ \Omega \mapsto \mathbb{R}^n$ is a smooth vector field with $\Omega \subset \mathbb{R}^n$ being an open set. Denote $A(u) = Df(u)$ the $n \times n$ Jacobian matrix of the flux function f. One of the features of systems of this type is that in general discontinuities will form in finite time no matter how smooth the initial data. This leads to the study of shock waves for which many theories have been developed, refer to [2,5,8,10–14,16,18,25,26] and references therein. In the framework of solutions with small total variation, the global existence was established in the fundamental work of Glimm by introducing the Glimm scheme which uses as building blocks the solutions to the Riemann problems solved by Lax. The stability of solutions in the $\mathbf{L}^1$ norm was obtained much later, refer to [2,6,8,19,23] and the references therein. We also mention the recent breakthrough by Bianchini and Bressan on constructing solutions for this class for hyperbolic systems by the method of vanishing viscosity.

For later reference, we now introduce some notations. As usual, the system (1.1) is called strictly hyperbolic if for every $u \in \Omega$ the matrix $A(u)$ has n real distinct eigenvalues denoted by

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u).$$

Corresponding to these eigenvalues, there are n linearly independent right eigenvectors

$$r_1(u), r_2(u), \ldots, r_n(u).$$

To capture the nonlinearity of these characteristic fields, the following definition is from [18].

Definition 1.1. For each $i \in \{1, 2, \ldots, n\}$, the i-th characteristic field is called genuinely nonlinear, if

$$\nabla\lambda_i \cdot r_i \neq 0, \ \text{for all } u \in \Omega, \tag{1.2}$$

while the i-th characteristic field is called linearly degenerate, if

$$\nabla\lambda_i \cdot r_i \equiv 0, \ \text{for all } u \in \Omega. \tag{1.3}$$

Definition 1.2. A function $u : [0,\infty) \times \mathbb{R} \longmapsto \mathbb{R}^n$ is a weak solution of the problem (1.1), if u is a bounded measurable function and

$$\iint_{t \geqq 0} [u\phi_t + f(u)\phi_x]\, \mathrm{d}x\, \mathrm{d}t + \int_{t=0} u_0(x)\phi(x,0)\mathrm{d}x = 0, \tag{1.4}$$

holds for any smooth function ϕ with compact support in $\{(x,t) | (x,t) \in \mathbb{R}^2\}$.

To solve the Cauchy problem (1.1), Glimm introduces a scheme for constructing the solution to systems under the following assumption [13]:

(A) Each characteristic field is either genuinely nonlinear or linearly degenerate.

There are two ingredients in the Glimm scheme. The first one is the Glimm functional, which we will discuss in the next section. The other is the approximation of the initial data by piecewise constant functions and the solution of Riemann problems locally in space and time. Here, the Riemann problem is the problem (1.1) when the initial data is given by:

$$u_0(x) = \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{if } x > 0, \end{cases} \tag{1.5}$$

where $u^\pm$ are constant vectors.

The Glimm functional is used to guarantee that the total variation of the solution is bounded by the total variation of the initial data, so that the solutions to the Riemann problems solved locally in space and time can be used as building blocks for the construction of the approximate solution. In addition, the uniform boundedness in the total variation of the approximate solutions leads to the convergence to the global entropy solution as the grid size tends to zero.

Now let us briefly recall the Glimm scheme. Divide the (x, t) plane into rectangles with grid sizes r and s satisfying the CFL condition, that is, $\frac{r}{s} > \sup_i |\lambda_i(u)|$ for all u under consideration and pick a sequence of random numbers $\{\theta_m\}_{m=1}^\infty$. Then we construct the approximate solution $u_{\theta,r}(x, t)$ inductively:

– At $t = 0$, let $u_{\theta,r}(x, 0) = u_0(ir), \quad \text{for } (i-1)r < x < (i+1)r, \quad i \text{ odd}.$
– Suppose $u_{\theta,r}(x, t)$ is defined for $t < js$, then

$$u_{\theta,r}(x, js) = u_{\theta,r}((i + (2\theta_j - 1))r, js - 0), \ (i-1)r < x < (i+1)r, i + j \text{ odd}.$$

Note that $u_{\theta,r}(x, js)$ is a piecewise constant function with possible jumps at $x = ir$, where $i + j$ even.
– Now for $i + j$ even, in $js \leqq t < (j+1)s$, $(i-1)r < x < (i+1)r$, define $u_{\theta,r}(x, t)$ as the solution to the Riemann problem

$$\begin{cases} u_t + f(u)_x = 0, & (i-1)r < x < (i+1)r, \ js \leqq t, \\ u(x, js) = u_{\theta,r}(x, js) & (i-1)r < x < (i+1)r, i + j \text{ even}. \end{cases}$$

Then the approximate solution can be defined up to $t < (j+1)s$.

The Glimm scheme converges with "probability one". To secure deterministic convergence, a wave tracing argument was introduced in [20]. It was shown that the sequence of approximate solutions converges as long as the random sequence is chosen to be equidistributed defined later in Definition 1.3.

In the deterministic version of the Glimm scheme, with the wave tracing argument, physical waves are divided into virtual waves which can be either traced back to the origin, or may be canceled or may be created in a short time interval. The wave pattern is greatly simplified if we keep only those waves that can be traced back. Moreover, it can be further simplified if we replace each of them by the corresponding one at the initial time, so that it has the same strength and propagation speed in a small time interval. If the random sequence is equidistributed, the error due to this simplification can be controlled by the Glimm functional times a small factor related to the grid size, and this factor converges to zero in $\mathbf{L}^1$ norm when the grid size tends to zero. Here the equidistributed sequence is defined as follows.

Definition 1.3. A sequence $\{\theta_i\}_{i=0}^{\infty}$ in [0, 1] is called equidistributed if

$$A(N, I) \equiv |\tfrac{B(N,I)}{N} - |I|| \to 0, \text{ as } N \to \infty,$$

for any subinterval I of [0, 1]. Here $B(N, I)$ denotes the number of i, $1 \leqq i \leqq N$, such that $\theta_i \in I$ and $|I|$ is the length of I.

The equidistributed sequence leads to a clear description of the structure of the weak solution through the wave tracing argument. As one step further, to study the convergence rate of the Glimm scheme as the grid size tends to zero for general entropy solutions, the following sequence is used, refer to [7].

Lemma 1.1. *Let*

$$D_{m,n} = \sup_{\lambda\in[0,1]} \left| \lambda - \frac{1}{n-m} \sum_{m\leqq l<n} \chi_{[0,\lambda]}(\theta_l) \right|, \tag{1.6}$$

then there exists a sequence $\{\theta_l\}_{l\geqq 0} \subset [0, 1]$ *such that*

$$D_{m,n} \leqq O(1)\frac{1+\ln(n-m)}{n-m} \quad \forall\, n > m \geqq 1. \tag{1.7}$$

By applying the new Glimm functional to the study of the convergence rate, we need to use the L^1 stability of the standard Riemann semigroup generated by (1.1), denoted by $\{S_t; t \geqq 0\}$. The first breakthrough on the L^1 stability of the weak solutions to (1.1) was made in [4] for 2×2 systems, and it was completed in [6,8,23] for systems satisfying the condition (A). The L^1 stability of entropy solutions for general hyperbolic conservation laws was later proved in [2] through the vanishing viscosity argument. In fact, [2] considers the Cauchy problem for the hyperbolic system with artificial viscosity

$$u_t + A(u)u_x = \epsilon u_{xx}, \quad u(0, x) = u_0(x). \tag{1.8}$$

Assume that the matrix $A(u)$ is strictly hyperbolic, smoothly depending on u in a neighborhood of a compact set $K \subset \Omega \subset \mathbb{R}^n$. Then there exist constants c, L, L' and $\delta > 0$ such that the following holds. If

$$T.V.u_0 < \delta, \quad \lim_{x\to-\infty} u_0(x) \in K,$$

where $T.V.$ means the total variation in x variable, then for each $\epsilon > 0$ the Cauchy problem (1.8) has a unique solution, defined for all $t \geqq 0$, denoted by $u^\epsilon = u^\epsilon(t,x) = S_t^\epsilon(u_0)$. In addition,

$$
\begin{aligned}
&T.V.S_t^\epsilon u_0 \leqq CT.V.u_0,\\
&\|S_t^\epsilon u_0 - S_t^\epsilon v_0\|_{L^1} \leqq L\|u_0 - v_0\|_{L^1},\\
&\|S_t^\epsilon u_0 - S_s^\epsilon u_0\|_{L^1} \leqq L'(|t-s| + |\sqrt{\epsilon t} - \sqrt{\epsilon s}|).
\end{aligned}
$$

Moreover, when $\epsilon \to 0+$, the solution u^ϵ converges to the trajectory of a semigroup S_t such that

$$\|S_t u_0 - S_s v_0\|_{L^1} \leqq L\|u_0 - v_0\|_{L^1} + L'|t-s|.$$

This vanishing viscosity limit can be regarded as the unique vanishing viscosity solution of the hyperbolic Cauchy problem

$$u_t + A(u)u_x = 0, \quad u(0,x) = u_0(x). \tag{1.9}$$

In the conservative case when $A(u) = Df(u)$, every vanishing viscosity solution is a weak solution of (1.1) satisfying the entropy condition.

Furthermore, under the condition (A), the vanishing viscosity solution coincides with the unique limit of the Glimm and front-tracking approximation. In this paper, we will not touch the L^1 stability or the uniqueness of the weak solutions to the general hyperbolic conservation laws through the Glimm scheme. Instead, we will use that for any two nearby initial data, the unique semigroup generated by the Glimm scheme S_t satisfies, refer to [1], that

$$\|S_t\bar{u} - S_t\bar{v}\|_{\mathbf{L}^1} \leqq L\|\bar{u} - \bar{v}\|_{\mathbf{L}^1}, \quad \forall \bar{u}, \bar{v} \in \mathcal{D}, t \geqq 0, \tag{1.10}$$

for some uniform constant L.

Based on the L^1 stability (1.10), under the condition (A), the convergence rate is shown to be $o(1)\sqrt{s}|\ln(s)|$, refer to [7]. Here s is the grid size in the Glimm scheme. And this convergence rate will be shown to be the same for general hyperbolic conservation laws by using the new functional introduced in this paper.

For general systems, the solution to the Riemann problem has different structure so that the Cauchy problem exhibits richer nonlinear phenomena. To estimate the wave interactions, one uses the same Glimm functional for waves in different families but a different one for waves in the same family. For this purpose, a cubic functional was introduced in [21] and was elaborated in [24] in order to take care of the wave interactions globally. The functional used in [24] is defined by the product of the strengths of two interacting waves and their effective "interaction" angle. Based on this improvement, the complete existence theory with the wave tracing argument for general systems was obtained in [24] under the assumption:

(B) For each characteristic field, the linear degeneracy manifold $LD_i \equiv \{u : \nabla\lambda_i(u) \cdot r_i(u) = 0\}$ is either the whole space or it consists of a finite number of smooth manifolds of codimension one, which are transversal to the characteristic vector $r_i(u)$.

Even though the improved Glimm functional used in [24] is effective in the study of the existence of entropy solutions, it is not satisfactory in proving the consistency and the convergence rate of the Glimm scheme. In fact, the consistency of the Glimm scheme was proved in [24] by carefully and artificially dividing the waves into groups according to their wave strength in comparison with the grid size to some power, and the convergence rate of the Glimm scheme was shown to be $o(1)s^{\frac{1}{4}}|\ln s|$ in [28], and then $o(1)s^{\frac{1}{3}}|\ln s|$ in [15] which are slower than the one given in [7] under the condition (A).

The new Glimm functional for the wave interactions in the same family is optimal in the following sense. First, it yields a clear and complete proof of the consistency of the Glimm scheme. Then it leads to the proof of the same order of convergence rate for the general systems as for those under the condition (A). Finally, it will be shown that it has the same decay effect as the classical one introduced by Glimm when the assumption of genuine nonlinearity is imposed. Therefore, the Glimm scheme for general systems can be analyzed satisfactorily without any artificial adjustment.

The convergence rate of the Glimm scheme can be stated as follows.

Theorem 1.1. *Let $\{\theta_m\}_{m=1}^{\infty}$ be a sequence of numbers in $[0,1]$ satisfying (1.7). Given any initial condition $\bar{u}$ with small total variation, let $u(\cdot,t) = S_t\bar{u}$ be the unique solution of (1.1), and let u^s be the corresponding Glimm approximate solution with grid size s in the time direction, generated by the sampling sequence $\{\theta_m\}_{m=1}^{\infty}$. Then for every $T \geqq 0$,*

$$\lim_{s\to 0} \frac{\|u^s(\cdot,T) - u(\cdot,T)\|_{L^1}}{s^{\frac{1}{2}}|\ln s|} = 0. \tag{1.11}$$

The limit is uniform with respect to $\bar{u}$, as long as $T.V.\bar{u}$ remains uniformly small.

Finally in the introduction, we mention the corresponding result on the convergence rate for the vanishing viscosity approach. Let the system (1.1) be strictly hyperbolic and the condition (A) hold. Then, given any initial data $u(0,x) = u_0(x)$ with small total variation, for every $\tau > 0$ the corresponding solutions u, u^ϵ of (1.1) and (1.8) were shown to satisfy the following estimate in [9]

$$\left\|u^\epsilon(\tau,\cdot) - u(\tau,\cdot)\right\|_{L^1} = O(1)\cdot(1+\tau)\sqrt{\epsilon}|\ln\epsilon| T.V.u_0(x). \tag{1.12}$$

This convergence rate is "optimal" in the sense that even for a scalar conservation law, the method of Kuznetsov in [17] shows that the convergence rate is $O(1)\cdot\epsilon^{1/2}$ which is sharp, refer to [27]. The factor $|\ln\epsilon|$ comes from the interaction of waves in different families in the system which does not exist for scalar equation.

The rest of the paper will be organized as follows. In the next section, the new functional is introduced together with some preliminaries on the wave interaction estimates in the wave tracing argument. The non-increasing property of the new Glimm functional and its application to the consistency of the Glimm scheme will be proved in Section 3. And the convergence rate of the Glimm scheme, stated in Theorem 1.1, will be proved in the last section.

2. Wave tracing and new Glimm functional

The Riemann problem under the general condition (B) is much more complicated than under the condition (A). The Lax entropy condition used under the condition (A) should be replaced by the following Liu entropy condition under the condition (B).

Definition 2.1. [22] A discontinuity (u_-, u_+) is admissible if

$$\sigma(u_-, u_+) \leqq \sigma(u_-, u), \tag{2.1}$$

for any state u on the Hugoniot curve $H(u_-)$ between u_- and u_+, where $H(u_-) \equiv \{u : \sigma(u_- - u) = f(u_-) - f(u)\}$.

Corresponding to the n characteristic fields of the system, there are n Hugoniot curves. Any state u on the i-th Hugoniot curve $H_i(u_0)$ is connected to u_0 by an i-th shock wave, if the above entropy condition is satisfied. We denote $H_i(\alpha)(u_0)$ the state which can be connected to u_0 by an i-th shock wave with strength α. Note that the shock wave described here includes the case of contact discontinuity.

Another basic wave pattern used for solving the Riemann problem is called a rarefaction wave. For each characteristic field, the state $R_i(\alpha)(u_0)(i = 1, 2, \ldots, n)$ is connected to u_0 by an i-th rarefaction wave with strength α, if

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\alpha} R_i(\alpha)(u_0) = r_i(R_i(\alpha)(u_0)), \quad \lambda_i(u) \text{ is monotone increasing,} \\ R_i(0)(u_0) = u_0. \end{cases}$$

Here the wave strength α is used as a parameter along the rarefaction wave curve.

By the implicit function theorem, the Riemann problem for general systems is solved by piecing together waves in different families. And with the Liu entropy condition, each wave in the i-th family, may be the composition of several i-th admissible shocks and rarefaction waves, refer to [3].

Suppose in the k-wave (u_l, u_r), the k-th elementary waves (u_k^{h-1}, u_k^h), $h = 1, 2, \ldots, n_k$, are defined as

$$\begin{aligned} &u_k^0 = u_l, \quad u_k^{n_k} = u_r, \\ &u_k^h = \begin{cases} R_k(\alpha_k^h)(u_k^{h-1}), & h \text{ is odd}, \\ H_k(\alpha_k^h)(u_k^{h-1}) & h \text{ is even}, \end{cases} \quad (h = 1, 2, \ldots, n_k). \end{aligned} \tag{2.2}$$

Notice that the strength of α_k^h can be zero in the following discussion. Then due to Definition 2.1, these elementary waves satisfy the following monotonicity property:

$$\begin{aligned} &\lambda_k(u_k^{2p}) < \lambda_k(u_k^{2p+1}) = \sigma_k(u_k^{2p+1}, u_k^{2p+2}), \quad \text{if } \alpha_k^{2p+1} \neq 0, \\ &\sigma_k(u_k^{2p+1}, u_k^{2p+2}) = \lambda_k(u_k^{2p+2}) < \lambda_k(u_k^{2p+3}), \quad \text{if } \alpha_k^{2p+3} \neq 0, \\ &2p+1, 2p+3 \in \{1, 2, \ldots, n_k\}. \end{aligned} \tag{2.3}$$

We can construct the wave curve $W_i(s, u_0)$ as the curve consisting of all the end states that can be connected to u_0 by admissible shocks, rarefaction waves or combinations thereof of the i-th family. Here s is a non-degenerate parameter along the curve. Up to a linear transformation, this parameter can be chosen as the i-th component of u, that is u^i. Then we have the following regularity result.

Lemma 2.1. [3] *Under the assumption (B), the admissible i-th curve $W_i(s, u_0)$ has Lipschitz continuous first order derivatives.*

To define the approximate solutions, we use the deterministic version of the Glimm scheme [13,20]. The approximate solution will be well defined provided that a uniform bound on the total variation is obtained. For this purpose, one has to investigate the wave interactions. In [21,24], the following Glimm type functional is defined.

$$F_o(J) \equiv L(J) + MQ_o(J),$$

where the subscript "o" refers to the previous one, in contrast to the new one we shall define later. In the above definition,

$$\begin{aligned} L(J) &= \sum\{|\alpha| : \alpha \text{ any wave crossing J}\}, \quad Q_o(J) = Q_d(J) + Q_{os}(J),\\ Q_d(J) &= \sum\{|\alpha|\,|\beta| : \text{ interacting waves } \alpha \text{ and } \beta \text{ of distinct}\\ &\qquad \text{characteristic field crossing J}\},\\ Q_{os}(J) &= \sum_{i=1}^{n} Q_{os}^i,\\ Q_{os}^i &= \sum\{|\alpha|\,|\beta| \max\{-\Theta(\alpha,\beta), 0\} : \alpha \text{ and } \beta \text{ i-waves crossing J},\\ &\qquad \alpha \text{ to the left of } \beta\}. \end{aligned} \tag{2.4}$$

Here $|\alpha|$ is the wave strength of a wave α, M is a sufficiently large constant, J is any space-like curve. An i-wave α_i on the left and a j-wave β_j on the right are said to be approaching, if $i > j$. And $\Theta(\alpha, \beta)$, called the effective angle between waves α and β of the same i-th family, is defined as follows:

$$\Theta(\alpha, \beta) \equiv \theta_\alpha^+ + \theta_\beta^- + \sum \theta_\gamma. \tag{2.5}$$

Here $\theta_\alpha^+ \leqq 0$ represents the value of λ_i at the right state of α minus its wave speed if α is a shock and is set to be zero if it is a rarefaction wave. Similarly, $\theta_\beta^- \leqq 0$ denotes the difference between the speed of β and the value of λ_i at its left end state. θ_γ is the value of λ_i at the right state of the wave γ minus that of the left state. The sum $\sum \theta_\gamma$ is over the i-waves γ between α and β. When $\Theta(\alpha, \beta)$ is positive, the two waves will not likely meet; when $\Theta(\alpha, \beta)$ is negative, the two waves may eventually meet and interact.

In the deterministic version of the Glimm scheme, all the waves in the solution are partitioned into small subwaves as follows.

Definition 2.2. [24] Let $u_r \in W_i(u_l)$ so that u_l is connected to u_r by i-discontinuities (u_{j-1}, u_j), and i-rarefaction waves (u_j, u_{j+1}), j odd, $1 \leqq j \leqq m-1$, $u_0 = u_l$ and $u_m = u_r$. A set of vectors $\{v_0, v_1, \ldots, v_p\}$ is a partition of (u_l, u_r) if

(i) $v_0 = u_l, v_p = u_r, v_{k-1}^i \leqq v_k^i, k = 1, 2, \ldots, p,$
(ii) $\{u_0, u_1, \ldots, u_m\} \subset \{v_0, v_1, \ldots, v_p\},$
(iii) $v_k \in R_i(u_j)$, j odd, if $u_j^i < v_k^i < u_{j+1}^i,$

(iv) $v_k \in D_i(u_{j-1}, u_j)$, j odd, if $u^i_{j-1} < v^i_k < u^i_j$. Here

$$D_i(u_l, u_r) \equiv \{u : (u - u_l)\sigma(u_l, u_r) - (f(u) - f(u_l)) = c(u)r_i(u) \text{ for some scalar c(u)}\}.$$

Then set

(1) $y_k \equiv v_k - v_{k-1}$,
(2) $\lambda_{i,k} \equiv \lambda_i(v_{k-1})$ and
$[\lambda_i]_k \equiv [\lambda_i](v_{k-1}, v_k) \equiv \lambda_i(v_k) - \lambda_i(v_{k-1}) > 0$ if (iii) holds,
(3) $\lambda_{i,k} \equiv \sigma(u_{j-1}, u_j)$ and $[\lambda_i]_k \equiv [\lambda_i](v_{k-1}, v_k) \equiv 0$ if (iv) holds.

In the following, we always assume that a rarefaction wave is divided into several small rarefaction shocks with strength less than the grid sizes of the Glimm scheme. Then the shock waves and rarefaction waves can be treated similarly after the wave partition. Due to the regularity of the composite wave curve, that is Lemma 2.1, such a partition is stable under perturbations in the following sense.

Lemma 2.2. [24] *Suppose that $u_r \in W_i(u_l)$, $\bar{u}_r \in W_i(\bar{u}_l)$, with $u^i_r - u^i_l = \bar{u}^i_r - \bar{u}^i_l \equiv \alpha > 0$, and $|u_l - \bar{u}_l| \equiv \beta$. Then there exist partitions $\{v_0, v_1, \ldots, v_p\}$ and $\{\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_p\}$ for the i-waves (u_l, u_r) and $(\bar{u}_l, \bar{u}_r)$ respectively such that $\bar{v}^i_k - \bar{v}^i_0 = v^i_k - v^i_0$, $k = 1, 2, \ldots, p$, and the following holds:*

(i) $\sum_{k=1}^{p} |y_k - \bar{y}_k| = 0(1)\alpha\beta$,
(ii) $|\lambda_{i,k} - \bar{\lambda}_{i,k}| = 0(1)\beta$, $k = 1, 2, \ldots, p$,
(iii) *Let $\Theta^+(u_l, u_r)$ represent the value of λ_i at the right state u_r minus the wave speed of the right-most i-wave in (u_l, u_r). Similar definition holds for $\Theta^-(u_l, u_r)$. Then*

$$|\Theta^-(u_l, u_r) - \Theta^-(\bar{u}_l, \bar{u}_r)| + |\Theta^+(u_l, u_r) - \Theta^+(\bar{u}_l, \bar{u}_r)| = 0(1)\alpha\beta.$$

Moreover, the index set $\{1, 2, \ldots, p\}$ can be written as a disjoint union of subsets I, II and III such that

(iv) *For $k \in I$ corresponding to rarefaction waves, both v_k and $\bar{v}_k$ are of type (iii) in Definition 2.2 and*

$$\sum_{k \in I} |[\lambda_i]_k - [\bar{\lambda}_i]_k| = 0(1)\alpha\beta.$$

(v) *For $k \in II$ corresponding to discontinuities, both v_k and $\bar{v}_k$ are of the type (iv) in Definition 2.2.*
(vi) *For $k \in III$ corresponding to wave of mixed types, v_k and $\bar{v}_k$ are of different type and*

$$\sum_{k \in III} |[\lambda_i]_k + [\bar{\lambda}_i]_k| = 0(1)\alpha\beta.$$

Here $\Theta^+(u_l, u_r)$ represents the value of λ_i at the right state u_r minus the wave speed of the rightmost i-wave in (u_l, u_r). Similar definition holds for $\Theta^-(u_l, u_r)$.

This lemma describes the C^2 like dependency of the Riemann problem on the end states. Then the effect of wave interaction can be estimated by the Glimm functional and the cancelation as in [24].

Lemma 2.3. [24] *Let u_l, u_m and u_r be three nearby states and (u_{i-1}, u_i) (v_{i-1}, v_i), $i = 1, 2, \ldots, n$, be i-waves in the Riemann problem (u_l, u_m) and (u_m, u_r) respectively with the partition defined in Definition 2.2. Here, rarefaction waves are divided into small rarefaction shocks with strength less than the grid size s in t direction. Then the wave partition of the i-wave (w_{i-1}, w_i), $i = 1, 2, \ldots, n$, in the Riemann problem (u_l, u_r) is the linear superposition of the above two solutions modulo the nonlinear effect of the order s, $Q(u_l, u_m, u_r)$ and $\delta C(u_l, u_m, u_r)$, where $\delta = |u_m - u_l| + |u_r - u_m|$. In other words,*

$$\gamma_i = \alpha_i + \beta_i + O(1)(\delta C(u_l, u_m, u_r) + Q_o(u_l, u_m, u_r) + s), \tag{2.6}$$

$$\eta(\gamma_i) = \eta(\alpha_i) + \eta(\beta_i) + O(1)(\delta C(u_l, u_m, u_r) + Q_o(u_l, u_m, u_r) + s), \tag{2.7}$$

with

$$\begin{aligned}
\alpha_i &= \sum_{k=1}^{n_{\alpha_i}} \alpha_{i,k} = u_i^i - u_{i-1}^i \ \ \beta_i = \sum_{k=1}^{n_{\beta_i}} \beta_{i,k} = v_i^i - v_{i-1}^i, \ \textit{and} \\
\gamma_i &= w_i^i - w_{i-1}^i, \\
\eta(\alpha_i) &= \sum_{k=1}^{n_{\alpha_i}} \eta(\alpha_{i,k}), \ \textit{with} \ \ \eta(\alpha_{i,k}) = \alpha_{i,k}\lambda_{i,k}, \\
&\qquad \textit{similar definition for } \eta(\beta_i) \textit{ and } \eta(\gamma_i), \\
C(u_l, u_m, u_r) &\equiv \sum_{i=1}^{n} C^i(u_l, u_m, u_r) = \frac{1}{2}||\gamma_i| - |\alpha_i| - |\beta_i||,
\end{aligned}$$

for some constants n_{α_i} and n_{β_i}, $i = 1, 2 \cdots, n$. Each $\alpha_{i,k} = (u_{i,k-1}, u_{i,k})$ and $\beta_{i,k} = (v_{i,k-1}, v_{i,k})$ is a shock or a rarefaction shock. $C(u_l, u_m, u_r)$ measures the amount of cancelation.

The above wave interaction estimate is crucial in the study of conservation laws under general assumption (B). Compared with the corresponding estimate under condition (A), the error is bounded by terms at least of cubic rather than quadratic order. This looks better when the total variation of the approximate solutions is small. On the other hand, the decrease of Q_o after the wave interaction in the same family is much less than the decrease in the classical Glimm functional under the condition (A). This fact causes difficulty in the proof of consistency and wave tracing argument. For example, one has to divide waves into two groups by checking whether the total strength of the waves involved is greater or not than a prechosen small constant. We include the following estimate from [24] by using the old Glimm functional for comparison with Theorem 2.2 given later.

Lemma 2.4. [24] *Let ϵ be a constant with $\frac{1}{2} < \epsilon < 1$. The waves in an approximate solution in a given time zone $\Lambda = \{(x, t) : -\infty < x < \infty, Ms \leqq t < (M+N)s\}$,* *can be partitioned into subwaves of categories I, II or III with the following properties:*

(i). *The subwaves in I are surviving. Given a subwave $\alpha(t)$, $Ms \leqq t < (M+N)s$, in I, write $\alpha \equiv \alpha(Ms)$ and denote by $|\alpha(t)|$ its strength at time t, by $[\sigma(\alpha)]$ the variation of its speed and by $[\alpha]$ the variation of the jump of the states across it over the time interval $Ms \leqq t < (M+N)s$. Then*

$$\sum_{\alpha \in I}([\alpha] + |\alpha(Ms)|[\sigma(\alpha)]) = O(1)(D_o(\Lambda)(Ns)^{-\epsilon} + T.V.N^{1+\epsilon}s^{\epsilon} + s).$$

(ii). *A subwave $\alpha(t)$ in II has positive initial strength $|\alpha(Ms)| > 0$, but is canceled in the zone Λ, $|\alpha((M+N)s)| = 0$. Moreover, the total strength and variation of the wave speed satisfy*

$$\sum_{\alpha \in II}([\alpha] + |\alpha(t)|) = O(1)(D_o(\Lambda) + s), \;\; Ms \leqq t < (M+N)s,$$

$$\sum_{\alpha \in II}([\alpha] + |\alpha(Ms)|[\sigma(\alpha)]) \leqq 0(1)(D_o(\Lambda)(Ns)^{-\epsilon} + T.V.N^{1+\epsilon}s^{\epsilon} + s).$$

(iii). *A subwave in III has zero initial strength $|\alpha(Ms)| = 0$, and is created in the zone Λ, $|\alpha((M+N)s)| \neq 0$. Moreover, the total strength and variation of the wave speed satisfy*

$$\sum_{\alpha \in III}([\alpha] + |\alpha(t)|) = O(1)(D_o(\Lambda) + s), \;\; Ms \leqq t < (M+N)s,$$

$$\sum_{\alpha \in III}([\alpha] + |\alpha((M+N)s)|[\sigma(\alpha)]) \leqq 0(1)(D_o(\Lambda)(Ns)^{-\epsilon} + T.V.N^{1+\epsilon}s^{\epsilon} + s).$$

Here $D_o(\Lambda) = F_o(Ms) - F_o((M+N)s)$, and $T.V. = Tot.Var.\{u_0(x)\}$. And $F(t)$ is the Glimm functional on the space-like curve at time t.

Moreover, the functional defined in (2.4) cannot be reduced to the one defined in [13] even if each characteristic field is genuinely nonlinear or linearly degenerate because these two functionals are not of the same order.

To overcome these difficulties, we define a new Glimm functional as follows.

$$F(J) \equiv L(J) + MQ_n(J), \tag{2.8}$$

where

$$\begin{aligned} Q_n(J) &= Q_d(J) + Q_{ns}(J), \quad Q_{ns}(J) = \sum_{i=1}^{n} Q^i_{ns}, \\ Q^i_{ns} &= \sum\{|\alpha|\,|\beta| \frac{\max\{-\Theta(\alpha,\beta),0\}}{t.v.(\alpha,\beta)_i} : \alpha \text{ and } \beta \text{ } i\text{-waves crossing } J, \\ &\qquad \alpha \text{ to the left of } \beta\}, \end{aligned} \tag{2.9}$$

$$\text{where } t.v.(\alpha,\beta)_i = \sum\{|\gamma| : \gamma \text{ any } i\text{-wave crossing } J \text{ and lying between } \alpha \text{ and } \beta, \text{ including } \alpha \text{ and } \beta\}.$$

Q_d, $L(J)$ and M are defined as before.

For this new functional, we shall first prove that F is non-increasing so that the uniform bound on the total variation of the approximate solution follows.

Theorem 2.1. *J_1 and J_2 are two space-like curves and J_2 is the immediate successor of J_1. When $F(J_1)$ is sufficiently small, the following estimate holds:*

$$F(J_2) - F(J_1) \leqq 0. \tag{2.10}$$

The proof of this theorem will be given in the next section.

By the Lipschitz dependency of the wave speed on the left and right states, if the characteristic field is genuinely nonlinear, one can check that the decay rate of the new functional is the same as the classical Glimm functional through wave interactions. Thus, it gives a better way for controlling the error of wave tracing argument than the old Glimm functional in the general setting. Furthermore, we have the following clear estimate for the wave tracing argument.

Theorem 2.2. *The waves in an approximate solution in a given a time zone $\Lambda = \{(x,t) : -\infty < x < \infty, Ms \leqq t < (M+N)s\}$, can be partitioned into subwaves of categories I, II or III with the following properties:*

(i). *The subwaves in I are surviving. Given a subwave $\alpha(t)$, $Ms \leqq t < (M+N)s$, in I, write $\alpha \equiv \alpha(Ms)$ and denote by $|\alpha(t)|$ its strength at time t, by $[\sigma(\alpha)]$ the variation of its speed and by $[\alpha]$ the variation of the jump of the states across it over the time interval $Ms \leqq t < (M+N)s$. Then*

$$\sum_{\alpha\in I}([\alpha] + |\alpha(Ms)|[\sigma(\alpha)]) = O(1)(D(\Lambda) + s).$$

(ii). *A subwave $\alpha(t)$ in II has positive initial strength $|\alpha(Ms)| > 0$, but is canceled in the zone Λ, $|\alpha((M+N)s)| = 0$. Moreover, the total strength and variation of the wave speed satisfy*

$$\sum_{\alpha\in II}([\alpha] + |\alpha(t)|) = O(1)(D(\Lambda) + s),\ \ Ms \leqq t < (M+N)s,$$

$$\sum_{\alpha\in II}([\alpha] + |\alpha(Ms)|[\sigma(\alpha)]) \leqq 0(1)(D(\Lambda) + s).$$

(iii). *A subwave in III has zero initial strength $|\alpha(Ms)| = 0$, and is created in the zone Λ, $|\alpha((M+N)s)| \neq 0$. Moreover, the total strength and variation of the wave speed satisfy*

$$\sum_{\alpha\in III}([\alpha] + |\alpha(t)|) = O(1)(D(\Lambda) + s),\ \ Ms \leqq t < (M+N)s,$$

$$\sum_{\alpha\in III}([\alpha] + |\alpha((M+N)s)|[\sigma(\alpha)]) \leqq 0(1)(D(\Lambda) + s).$$

Here $D(\Lambda) = F(Ms) - F((M+N)s)$, and $T.V. = Tot.Var.\{u_0(x)\}$. And $F(t)$ is the new Glimm functional on the space-like curve at time t.

The proof of Theorem 2.2 will be given in the last section.

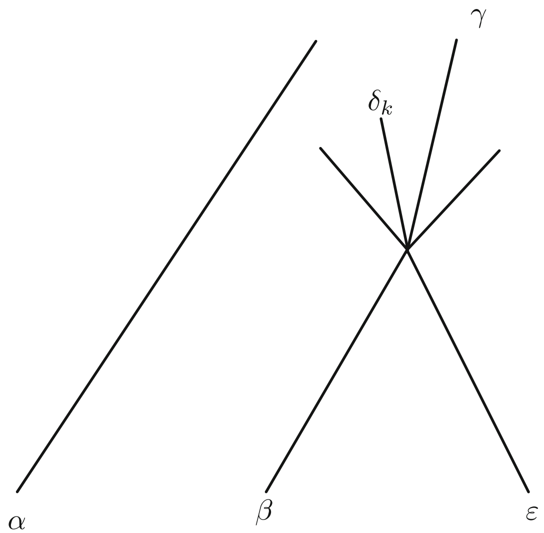

Fig. 1. Case I

3. Wave interaction estimates

In this section we show that the new Glimm functional is decreasing due to the wave interactions. To do so, we check two typical cases first. All the other cases can be dealt by a similar argument.

Proof of Theorem 2.1. **Case (I)**(refer to Fig.1): Suppose that there are three waves: α, β, ϵ are i-shocks or rarefaction shocks. β interacts with ϵ at time t without any cancelation:

$$\beta + \epsilon \to \gamma + \sum_{k \neq i} \delta_k.$$

From the definition of effective angle, $\Theta(\beta, \epsilon) \leqq 0$. Since there is no cancelation, for simplicity, we assume $\alpha, \beta, \epsilon \geqq 0$, that is, the waves are in the same direction.

By the monotonicity property (2.3), we also assume, without loss of generality, that the generated i-wave γ is a single shock and

$$\Theta(\alpha, \epsilon) \leqq 0.$$

From the standard wave interaction estimate, we know the change in the functional L crossing the time t is

$$\Delta L \equiv L(t+) - L(t-) = O(1)\beta\epsilon(-\Theta(\beta, \epsilon)). \tag{3.1}$$

We only need to estimate the change of Q_{ns} because the estimation on other parts are the same as those for the classical Glimm functional.

Before the wave interaction at time t, the potential wave interaction Q_{ns} is

$$Q_{ns}(t-) = \frac{\alpha\epsilon(-\Theta(\alpha, \epsilon))}{t.v.(\alpha, \epsilon)_i} + \frac{\beta\epsilon(-\Theta(\beta, \epsilon))}{t.v.(\beta, \epsilon)_i} + \frac{\alpha\beta(\max\{-\Theta(\alpha, \beta), 0\})}{t.v.(\alpha, \beta)_i}.$$

After the wave interaction, it becomes

$$Q_{ns}(t+) = \frac{\alpha\gamma(\max\{-\Theta(\alpha,\gamma),0\})}{t.v.(\alpha,\gamma)_i}.$$

Then

$$\begin{aligned}\Delta Q_{ns} &\equiv Q_{ns}(t+) - Q_{ns}(t-)\\ &= -\frac{\beta\epsilon(-\Theta(\beta,\epsilon))}{t.v.(\beta,\epsilon)_i} + \Bigg[\frac{\alpha\gamma(\max\{-\Theta(\alpha,\gamma),0\})}{t.v.(\alpha,\gamma)_i} - \frac{\alpha\epsilon(-\Theta(\alpha,\epsilon))}{t.v.(\alpha,\epsilon)_i}\\ &\quad - \frac{\alpha\beta(\max\{-\Theta(\alpha,\beta),0\})}{t.v.(\alpha,\beta)_i}\Bigg]\\ &= I + II.\end{aligned}$$

We may assume $\Theta(\alpha,\gamma) \leqq 0$. Otherwise it is easy to see that

$$\Delta Q_{ns} \leqq -\frac{\beta\epsilon(-\Theta(\beta,\epsilon))}{t.v.(\beta,\epsilon)_i}.$$

By Lemma 2.3, we have

$$\begin{aligned}\gamma &= \beta + \epsilon + O(1)\beta\epsilon(-\Theta(\beta,\epsilon)),\\ \sigma(\gamma)\gamma &= \sigma(\beta)\beta + \sigma(\epsilon)\epsilon + O(1)\beta\epsilon(-\Theta(\beta,\epsilon)).\end{aligned}$$

Thus,

$$\begin{aligned}t.v.(\alpha,\epsilon)_i &= t.v.(\alpha,\beta)_i + \epsilon,\\ t.v.(\alpha,\gamma)_i &= t.v.(\alpha,\beta)_i + \epsilon + O(1)\beta\epsilon(-\Theta(\beta,\epsilon)),\\ -\Theta(\alpha,\gamma) &= -\Theta(\alpha,\beta) + \sigma(\beta) - \sigma(\gamma),\\ -\Theta(\alpha,\epsilon) &= -\Theta(\alpha,\beta) + \sigma(\beta) - \sigma(\epsilon).\end{aligned}$$

This implies that

$$\begin{aligned}II &\leqq \frac{\alpha\gamma(-\Theta(\alpha,\gamma))}{t.v.(\alpha,\epsilon)_i}\left[1 + O(1)\frac{\beta\epsilon(-\Theta(\beta,\epsilon))}{t.v.(\alpha,\epsilon)_i}\right] - \frac{\alpha\epsilon(-\Theta(\alpha,\epsilon))}{t.v.(\alpha,\epsilon)_i}\\ &\quad - \frac{\alpha\beta(\max\{-\Theta(\alpha,\beta),0\})}{t.v.(\alpha,\beta)_i}\\ &= \frac{\alpha[\gamma(-\Theta(\alpha,\gamma)) - \epsilon(-\Theta(\alpha,\epsilon))]}{t.v.(\alpha,\epsilon)_i} + O(1)\frac{\alpha[\beta\epsilon(-\Theta(\beta,\epsilon))]}{t.v.(\alpha,\epsilon)_i}\\ &\quad - \frac{\alpha\beta(\max\{-\Theta(\alpha,\beta),0\})}{t.v.(\alpha,\beta)_i}\\ &= \frac{\alpha\left[\gamma(-\Theta(\alpha,\beta) + \sigma(\beta) - \sigma(\gamma)) - \epsilon(-\Theta(\alpha,\beta) + \sigma(\beta) - \sigma(\epsilon))\right]}{t.v.(\alpha,\epsilon)_i}\\ &\quad - \frac{\alpha\beta(\max\{-\Theta(\alpha,\beta),0\})}{t.v.(\alpha,\beta)_i} + O(1)\frac{\alpha[\beta\epsilon(-\Theta(\beta,\epsilon))]}{t.v.(\alpha,\epsilon)_i}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha\,[\beta(-\Theta(\alpha,\beta)) + O(1)\beta\epsilon(-\Theta(\beta,\epsilon))]}{t.v.(\alpha,\epsilon)_i} \\
&\quad - \frac{\alpha\beta(\max\{-\Theta(\alpha,\beta),0\})}{t.v.(\alpha,\beta)_i} + O(1)\frac{\alpha[\beta\epsilon(-\Theta(\beta,\epsilon))]}{t.v.(\alpha,\epsilon)_i} \\
&\leqq O(1)\frac{\alpha[\beta\epsilon(-\Theta(\beta,\epsilon))]}{t.v.(\beta,\epsilon)_i}.
\end{aligned}
$$

In the last inequality, we have used the fact that $t.v.(\alpha,\epsilon)_i \geqq t.v.(\alpha,\beta)_i$ and $t.v.(\alpha,\epsilon)_i \geqq t.v.(\beta,\epsilon)_i$.

Therefore, when the total variation of the approximate solution is sufficiently small, we deduce that

$$
\begin{aligned}
\Delta Q_{ns} &\leqq -\frac{[\beta\epsilon(-\Theta(\beta,\epsilon))]}{t.v.(\beta,\epsilon)_i} + O(1)\frac{\alpha[\beta\epsilon(-\Theta(\beta,\epsilon))]}{t.v.(\beta,\epsilon)_i} \\
&\leqq -\frac{1}{2}\frac{[\beta\epsilon(-\Theta(\beta,\epsilon))]}{t.v.(\beta,\epsilon)_i}.
\end{aligned} \tag{3.2}
$$

Combined with (3.1), this implies that F is decreasing for suitably chosen constant M

$$\Delta F \leqq 0.$$

Case (II)(refer to Fig.2): Assume that α, β are i-waves and γ is a j-wave ($i > j$). Without loss of generality, we assume that they are shocks or rarefaction shocks. Furthermore, we assume that α, β do not interact with each other before the time t, but after the interaction at time t

$$\beta + \gamma \longrightarrow \overline{\beta} + \overline{\gamma} + \sum_{k\neq i,j} \delta_k,$$

$\overline{\beta}$ interacts with α. Again we may assume that no cancelation happens and α, β, $\gamma \geqq 0$.

In this case, the standard wave interaction estimates yield

$$\Delta L \equiv L(t+) - L(t-) = O(1)|\beta||\gamma|.$$

Again we need to estimate the change of Q_n.

Before the wave interaction at time t, the potential wave interaction Q_n is

$$Q_n(t-) = \beta\gamma + \alpha\gamma.$$

After the wave interaction, the i-wave $\bar{\beta}$ may be a composite wave. We partition it into $\bar{\beta}_1, \ldots, \bar{\beta}_m$ as in Lemma 2.3 with intermediate states $\bar{u}_l = \bar{u}_0, \bar{u}_1, \ldots, \bar{u}_m = \bar{u}_r$. The wave interaction potential Q_n now becomes

$$Q_n(t+) = \alpha\overline{\gamma} + \sum_{k=1}^{m} \frac{\overline{\beta}_k \alpha(\max\{-\Theta(\alpha,\overline{\beta}_k),0\})}{t.v.(\alpha,\overline{\beta}_k)_i} + \sum_{k<i,k\neq j} \alpha\delta_k.$$

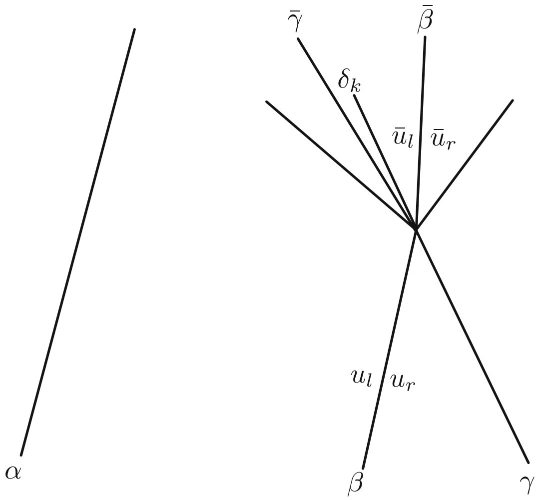

Fig. 2. Case II

Then the change in Q_n is

$$\Delta Q_n = Q_n(t+) - Q_n(t-) = -\beta\gamma + O(1)\alpha\beta\gamma + \sum_{k=1}^{m} \frac{\overline{\beta}_k \alpha(\max\{-\Theta(\alpha, \overline{\beta}_k), 0\})}{t.v.(\alpha, \overline{\beta}_k)_i}.$$

At the first glance, from the Lipschitz continuity of wave speed on the end states, the third term on the right-hand side can only be bounded by γ. But in fact, with the regularity of composite wave curves, that is Lemma 2.1, we can have a better estimate. By Lemma 2.2, we partition β into small subwaves $\beta_1, \dots, \beta_m$ corresponding to $\bar{\beta}_1, \dots, \bar{\beta}_m$. Then we can write $\Theta(\alpha, \overline{\beta}_k)$ in terms of $\Theta(\alpha, \beta_k)$:

$$-\Theta(\alpha, \overline{\beta}_k) = -\Theta(\alpha, \beta_k) + \Theta^-(u_l, u_r) + \sum_{l=1}^{k-1} [\lambda_i]_l + [\lambda_{i,k} - \lambda_i(u_{k-1})]$$
$$-\Theta^-(\bar{u}_l, \bar{u}_r) - \sum_{l=1}^{k-1} [\bar{\lambda}_i]_l - [\bar{\lambda}_{i,k} - \lambda_i(\bar{u}_{k-1})].$$

As α, β do not interact with each other, by definition, we have

$$-\Theta(\alpha, \beta) \leqq 0.$$

Moreover, since we assume that β is an i-th simple wave, from the Definition 2.2, it follows that $-\Theta(\alpha, \beta_k) = -\Theta(\alpha, \beta) \leqq 0$. Then by applying the Lemma 2.2 and observing that $|u_l - \bar{u}_l| = O(1)\gamma$, the effective angle can be estimated as follows:

$$\max\{-\Theta(\alpha, \overline{\beta}_k), 0\} \leqq O(1)\beta\gamma.$$

Therefore,

$$\Delta Q_n \leqq -\beta\gamma + O(1)\alpha\beta\gamma \leqq -\frac{1}{2}\beta\gamma. \tag{3.3}$$

Similar to case (I), this implies that F is decreasing for suitably chosen constant M, that is

$$\Delta F \leqq 0.$$

When cancelation occurs, we can prove F is non-increasing in a straightforward way because the amount of cancelation is of first order. So for the general case, we can regard the problem as the superposition of case (I) and (II). By using (3.2) and (3.3) repeatedly, we can show that F is non-increasing after the wave interaction, under the condition that the total variation of the approximate solution is sufficiently small. Thus Theorem 2.1 follows.

4. Consistency and convergence rate

With the above wave interaction estimates, we can prove Theorem 2.2 on the wave tracing argument.

Proof of Theorem 2.2. It is obvious that $|\alpha(Ms)|[\sigma(\alpha)]$ at time t can be controlled by $O(1)(Q_n(\Lambda) + C(\Lambda))$ if the wave interaction is between waves of different families or cancelation occurs. We only need to consider the interaction of waves of the same family and with the same sign.

Consider the interaction of two Riemann problems:

$$(u_l, u_m) + (u_m, u_r) \longrightarrow (u_l, u_r).$$

We can assume that (u_l, u_m) and (u_m, u_r) are connected by several k simple waves $\alpha = (\alpha_k^1, \ldots, \alpha_k^{n_1})$ and $\beta = (\beta_k^1, \ldots, \beta_k^{n_2})$ respectively. Furthermore, from the monotonicity property (2.3), we may assume without loss of generality that after interaction (u_l, u_r) is resolved by a single k-shock γ.

We have the following estimates from the Lemma 2.3:

$$\gamma = \sum_{i=1}^{n_1} \alpha_k^i + \sum_{i=1}^{n_2} \beta_k^i + O(1) Q_{os}(\alpha, \beta),$$

$$\sigma(\gamma)\gamma = \sum_{i=1}^{n_1} \sigma(\alpha_k^i)\alpha_k^i + \sum_{i=1}^{n_2} \sigma(\beta_k^i)\beta_k^i + O(1) Q_{os}(\alpha, \beta),$$

$$\sigma(\alpha_k^1) \leqq \sigma(\alpha_k^2) \leqq \ldots \leqq \sigma(\alpha_k^{n_1}) \leqq \sigma(\gamma) \leqq \sigma(\beta_k^1) \leqq \sigma(\beta_k^2) \leqq \ldots \leqq \sigma(\beta_k^{n_2}).$$

Then we can estimate the variation of the speeds $[\sigma(\alpha_k^i)]$ and $[\sigma(\beta_k^i)]$ as follows:

$$\begin{aligned} &\sum_{i=1}^{n_1} \alpha_k^i [\sigma(\alpha_k^i)] + \sum_{i=1}^{n_2} \beta_k^i [\sigma(\beta_k^i)] \\ &\quad = \sum_{i=1}^{n_1} \alpha_k^i (\sigma(\alpha_k^i) - \sigma(\gamma)) + \sum_{i=1}^{n_2} \beta_k^i (\sigma(\beta_k^i) - \sigma(\gamma)) \\ &\quad \equiv I + II. \end{aligned}$$

Hence

$$
\begin{aligned}
I &= \sum_{i=1}^{n_1} \alpha_k^i \left\{ \sigma(\alpha_k^i) - \frac{1}{\gamma} \left[\sum_{i=1}^{n_1} \sigma(\alpha_k^i)\alpha_k^i + \sum_{i=1}^{n_2} \sigma(\beta_k^i)\beta_k^i + O(1)Q_{os}(\alpha, \beta) \right] \right\} \\
&= \sum_{i=1}^{n_1} \frac{\alpha_k^i}{t.v.(\alpha, \beta)_i} \left[\sum_{j=1}^{n_1} \alpha_k^j (\sigma(\alpha_k^i) - \sigma(\alpha_k^j)) \right. \\
&\qquad \left. + \sum_{l=1}^{n_2} \beta_k^l (\sigma(\alpha_k^i) - \sigma(\beta_k^l)) + O(1)Q_{os}(\alpha, \beta) \right] \\
&= \frac{[\sum_{i,j=1}^{n_1} \alpha_k^i \alpha_k^j (\sigma(\alpha_k^i) - \sigma(\alpha_k^j)) + \sum_{i=1}^{n_1} \sum_{l=1}^{n_2} \alpha_k^i \beta_k^l (\sigma(\alpha_k^i) - \sigma(\beta_k^l))]}{t.v.(\alpha, \beta)_i} \\
&\quad + \frac{\sum_{i=1}^{n_1} \alpha_k^i O(1) Q_{os}(\alpha, \beta)}{t.v.(\alpha, \beta)_i} \\
&= \frac{[\sum_{i=1}^{n_1} \sum_{l=1}^{n_2} \alpha_k^i \beta_k^l (\sigma(\alpha_k^i) - \sigma(\beta_k^l))]}{t.v.(\alpha, \beta)_i} + \frac{\sum_{i=1}^{n_1} \alpha_k^i O(1) Q_{os}(\alpha, \beta)}{t.v.(\alpha, \beta)_i},
\end{aligned}
$$

since the summation $\sum_{i,j=1}^{n_1} \alpha_k^i \alpha_k^j (\sigma(\alpha_k^i) - \sigma(\alpha_k^j)) = 0$.

Similarly,

$$
II = \frac{[\sum_{i=1}^{n_1} \sum_{l=1}^{n_2} \alpha_k^i \beta_k^l (\sigma(\alpha_k^i) - \sigma(\beta_k^l))]}{t.v.(\alpha, \beta)_i} + \frac{\sum_{i=1}^{n_1} \alpha_k^i O(1) Q_{os}(\alpha, \beta)}{t.v.(\alpha, \beta)_i}.
$$

Therefore, we have

$$
I + II = \left(2 + \sum_{i=1}^{n_1} \alpha_k^i \right) \frac{Q_{os}(\alpha, \beta)}{t.v.(\alpha, \beta)_i} = O(1) Q_{ns}(\alpha, \beta). \tag{4.1}
$$

Then Theorem 2.2 follows.

With the help of Theorem 2.2, we can prove the consistency of Glimm scheme in a clear fashion, that is we can show that the following term

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \int_0^{\infty} (u\phi_t + f(u)\phi_x)(x, t) \mathrm{d}x \mathrm{d}t + \int_{-\infty}^{\infty} (u\phi)(x, 0) \mathrm{d}x \\
&\quad = \sum_{k=0}^{MN} \int_{-\infty}^{\infty} (u(x, ks + 0) - u(x, ks - 0)) \phi(x, ks) \mathrm{d}x,
\end{aligned} \tag{4.2}
$$

vanishes as the grid size r tends to zero. Here $\phi(x, t)$ is the test function with compact support, $\phi(x, t) = 0,\ t > T = MNs$.

As in [23] we may start with the simple example of a single shock. Denote by $x = x(k)r$ the location of the shock at time $t = ks$. We have

$$
\begin{aligned}
&\int_{-\infty}^{\infty} (u(x, ks + 0) - u(x, ks - 0)) \phi(x, ks) \mathrm{d}x \\
&\quad = \begin{cases} \int_{x(k)r}^{x(k)r + \sigma s} (u_+ - u_-) \phi(x, ks) \mathrm{d}x, & \text{if } \theta_k r > \sigma s, \\ \int_{x(k)r + \sigma s}^{(x(k)+1)r} (u_- - u_+) \phi(x, ks) \mathrm{d}x, & \text{if } \theta_k r < \sigma s. \end{cases}
\end{aligned}
$$

If the test function is a constant ϕ_0 then the error becomes, for the interval $I = (0, \sigma s/r)$,

$$\begin{aligned}
&\sum_{k=0}^{MN} \int_{-\infty}^{\infty} (u(x, ks+0) - u(x, ks-0))\phi(x, ks)\mathrm{d}x \\
&\quad = \phi_0(u_+ - u_-)(B(MN, I)(r - \sigma s) - B(MN, I^c)\sigma s) \\
&\quad = \phi_0(u_+ - u_-)T(B(MN, I)(\frac{r}{s} - \sigma) - (MN - B(MN, I))\sigma)\frac{1}{MN} \\
&\quad = \phi_0(u_+ - u_-)T\left(\frac{B(MN, I)}{MN} - \sigma\frac{s}{r}\right)\frac{r}{s},
\end{aligned}$$

which tends to zero as $MN \to \infty$ when the random sequence is equidistributed as in (1.6).

The fact that the test function $\phi(x, t)$ is not constant induces an error of the order $O(1)LNs = O(1)LT/M$, if we divide the time zone $0 \leqq t < T = MNs$ into M small time zones $N(l-1)s \leqq t < Nls$, $l = 1, 2, \ldots, M$. Here L is the Lipschitz constant of $\phi(x, t)$. Then this error tends to zero as $M \to \infty$.

For a general solution, the speed of the shocks or rarefaction shocks is changing. The variation of the speed has been discussed in Theorem 2.2, which says that for a surviving wave α, its strength $|\alpha|$ times the variation of its speed in a time zone Λ_l is of the order of $D(\Lambda_l) + s$. Thus the error contributed by surviving subwaves in a given time zone Λ_l is $O(1)(D(\Lambda_l) + s)Ns$. The total error of this kind over $0 \leqq t < T$ is then $E_1 = O(1)(D(t \geqq 0)\frac{T}{M} + sT)$. Similarly, the error contributed by the canceled subwaves in $0 \leqq t \leqq T$ is

$$E_2 = O(1)(D(t \geqq 0) + sT).$$

Thus the total error is of the form

$$E = O(1)\left[(B(N, I)/N - |I|)T + sT + D(t \geqq 0)\frac{T}{M}\right],$$

which tends to zero as $M, N \to \infty$. This means that the approximate solutions constructed by the deterministic version of the Glimm scheme converge to a weak solution of the Cauchy problem as the grid size tends to zero.

Finally, we prove Theorem 1.1 on the convergence rate of the Glimm scheme. In fact, after the preparation in the previous sections, the proof can follow the argument in [7] for systems satisfying condition (A). For completeness, we outline the proof as follows by using the new Glimm functional.

Proof of Theorem 1.1. Consider the approximate solution up to time $T = \bar{m}s + s'$, where $s' \in [0, \epsilon)$, s is the grid size in x and t. Pick a constant $\delta \gg s$. Divide $[0, T]$ into finitely many intervals $J_i \equiv [m_i s, m_{i+1}s]$, $i = 0, 1, \ldots$. Let $m_0 = 0$. Construct J_i and a subset E of $\mathbb{N}$ inductively as follows.

1. For each $i \in E$, we have $F(m_i s) - F((m_i + 1)s) \leqq \delta$, and m_{i+1} is the largest integer such that $F(m_i s) - F((m_{i+1}s)) \leqq \delta$ and $m_{i+1}s - m_i s \leqq \delta$.
2. On the other hand, for each $i \in E^c$, we have $F(m_i s) - F((m_i + 1)s) > \delta$. Then denote $m_{i+1} = m_i + 1$.

Then there exists some finite number $\mu \leqq \bar{m}$ such that $m_\mu = \bar{m}$. Since T is fixed and $F(t)$ is uniformly bounded,

$$\#E \leqq O(1)\frac{1}{\delta},\ \#E^c \leqq O(1)\frac{1}{\delta}. \tag{4.3}$$

For each interval $J_i, i \in E, i \leqq \mu$, define an auxiliary piecewise constant function $w(x,t)$ with the following property:

- Every subwave $\alpha(t)$ in u corresponds to a wave front in w with the same initial and final position.
- The wave front in w has constant wave strength and speed.

Applying the wave tracing result Theorem 2.2,

$$\begin{aligned}
&\|w(\cdot, m_{i+1}s) - S_{m_{i+1}s - m_i s} w(\cdot, m_i s)\|_{L_1} \\
&\quad = O(1)(m_{i+1}s - m_i s)\{(F(m_i s) - F(m_{i+1}s))\} \\
&\qquad + O(1)(m_{i+1}s - m_i s)\left\{s + \frac{1 + \ln(m_{i+1} - m_i)}{m_{i+1} - m_i}\right\}.
\end{aligned} \tag{4.4}$$

From the structure of w, we also obtain

$$\|u(\cdot, m_{i+1}s) - w(\cdot, m_{i+1}s)\|_{L_1} = 0(1)(F(m_i s) - F(m_{i+1}s))(m_{i+1}s - m_i s), \tag{4.5}$$

Thus, by combining (4.4) and (4.5), for each $i \in E$, we have

$$\begin{aligned}
&\|u(\cdot, m_{i+1}s) - S_{m_{i+1}s - m_i s} u(\cdot, m_i s)\|_{L_1} \\
&\quad \leqq O(1)(m_{i+1}s - m_i s)\left\{(F(m_i s) - F(m_{i+1}s)) \right. \\
&\qquad \left. + \left[s + \frac{1 + \ln(m_{i+1} - m_i)}{m_{i+1} - m_i}\right]\right\}.
\end{aligned} \tag{4.6}$$

For each $i \in E^c$, by the Lipschitz continuity of approximate solution constructed by the Glimm scheme, we have

$$\|u(\cdot, m_{i+1}s) - S_{m_{i+1}s - m_i s} u(\cdot, m_i s)\|_{L_1} \leqq O(1)s \tag{4.7}$$

By applying these two estimates, (4.3), the construction of J_i and the property of standard Riemann semigroup, the L_1 distance between $u(x,T)$ and $S_T u(x,0)$ can be controlled by

$$\begin{aligned}
&\|u(\cdot, T) - S_T u(\cdot, 0)\|_{L_1} \\
&\quad \leqq \sum_{i=0}^{\mu-1} \|S_{T - m_{i+1}s} u(\cdot, m_{i+1}s) - S_{T - m_i s} u(\cdot, m_i s)\|_{L_1} \\
&\qquad + \|u(\cdot, T) - S_{T - \bar{m}s} u(\cdot, \bar{m}s)\|_{L_1} \\
&\quad \leqq L \sum_{i=0}^{\mu-1} \|u(\cdot, m_{i+1}s) - S_{m_{i+1}s - m_i s} u(\cdot, m_i s)\|_{L_1} + O(1)s'
\end{aligned}$$

$$\leqq O(1) \sum_{i \in E} \left\{ (m_{i+1}s - m_i s) \left\{ (F(m_i s) - F(m_{i+1}s)) \right.\right.$$
$$\left.\left. + \left[s + \frac{1 + \ln(m_{i+1} - m_i)}{m_{i+1} - m_i} \right] \right\}\right\} + O(1) \sum_{i \in E^c} s$$
$$\leqq O(1) \left\{ \delta + s + \frac{s}{\delta} \left(2 + |\ln \frac{\delta}{s}| \right) \right\}. \tag{4.8}$$

Let $\delta = s^k \ln(|\ln s|),\ k \in (0, 1)$. Then simple a computation yields that when $k = \frac{1}{2}$, the convergence rate is $o(1)s^{\frac{1}{2}} |\ln s|$. This completes the proof. □

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