

# MHD BOUNDARY LAYERS THEORY IN SOBOLEV SPACES WITHOUT MONOTONICITY. I. WELL-POSEDNESS THEORY

CHENG-JIE LIU, FENG XIE, AND TONG YANG

**ABSTRACT.** We study the well-posedness theory for the MHD boundary layer. The boundary layer equations are governed by the Prandtl type equations that are derived from the incompressible MHD system with non-slip boundary condition on the velocity and perfectly conducting condition on the magnetic field. Under the assumption that the initial tangential magnetic field is not zero, we establish the local-in-time existence, uniqueness of solution for the nonlinear MHD boundary layer equations. Compared with the well-posedness theory of the classical Prandtl equations for which the monotonicity condition of the tangential velocity plays a crucial role, this monotonicity condition is not needed for MHD boundary layer. This justifies the physical understanding that the magnetic field has a stabilizing effect on MHD boundary layer in rigorous mathematics.

## CONTENTS

1. Introduction and Main Result	1
2. Preliminaries	5
3. A priori estimates	7
3.1. Weighted $H_t^m$ -estimates with normal derivatives	8
3.2. Weighted $H_t^m$ -estimates only in tangential variables	16
3.3. Closeness of the a priori estimates	23
4. Local-in-time existence and uniqueness	26
4.1. Existence	26
4.2. Uniqueness	28
5. A coordinate transformation	30
Appendix A. Some inequalities	31
References	32

## 1. INTRODUCTION AND MAIN RESULT

One important problem about Magnetohydrodynamics(MHD) is to understand the high Reynolds numbers limits in a domain with boundary. In this paper, we consider the following initial boundary value problem for the two dimensional (2D) viscous MHD equations (cf. [4, 5, 7, 37]) in a periodic domain  $\{(t, x, y) : t \in [0, T], x \in \mathbb{T}, y \in \mathbb{R}_+\}$  :

$$\begin{cases} \partial_t \mathbf{u}^\epsilon + (\mathbf{u}^\epsilon \cdot \nabla) \mathbf{u}^\epsilon - (\mathbf{H}^\epsilon \cdot \nabla) \mathbf{H}^\epsilon + \nabla p^\epsilon = \mu \epsilon \Delta \mathbf{u}^\epsilon, \\ \partial_t \mathbf{H}^\epsilon - \nabla \times (\mathbf{u}^\epsilon \times \mathbf{H}^\epsilon) = \kappa \epsilon \Delta \mathbf{H}^\epsilon, \\ \nabla \cdot \mathbf{u}^\epsilon = 0, \quad \nabla \cdot \mathbf{H}^\epsilon = 0. \end{cases} \quad (1.1)$$

Here, we assume the viscosity and resistivity coefficients have the same order of a small parameter  $\epsilon$ .  $\mathbf{u}^\epsilon = (u_1^\epsilon, u_2^\epsilon)$  denotes the velocity vector,  $\mathbf{H}^\epsilon = (h_1^\epsilon, h_2^\epsilon)$  denotes the magnetic field, and  $p^\epsilon = \tilde{p}^\epsilon + \frac{|\mathbf{H}^\epsilon|^2}{2}$  denotes the total pressure with  $\tilde{p}^\epsilon$  the pressure of the fluid. On the boundary, the non-slip boundary condition is imposed on velocity field

$$\mathbf{u}^\epsilon|_{y=0} = \mathbf{0}, \quad (1.2)$$

---

2000 *Mathematics Subject Classification.* 76N20, 35A07, 35G31, 35M33.

*Key words and phrases.* Prandtl type equations, MHD, well-posedness, Sobolev space, non-monotone condition.

and the perfectly conducting boundary condition on magnetic field

$$h_2^\epsilon|_{y=0} = \partial_y h_1^\epsilon|_{y=0} = 0. \quad (1.3)$$

The formal limiting system of (1.1) yields the ideal MHD equations when  $\epsilon$  tends to zero. However, there is a mismatch in the tangential velocity between the equations (1.1) and the limiting equations on the boundary  $y = 0$ . This is why a boundary layer forms in the vanishing viscosity and resistivity limit process. To find out the terms in (1.1) whose contribution is essential for the boundary layer, we use the same scaling as the one used in [32],

$$t = t, \quad x = x, \quad \tilde{y} = \epsilon^{-\frac{1}{2}}y,$$

then, set

$$\begin{cases} u_1(t, x, \tilde{y}) = u_1^\epsilon(t, x, y), \\ u_2(t, x, \tilde{y}) = \epsilon^{-\frac{1}{2}}u_2^\epsilon(t, x, y), \end{cases} \quad \begin{cases} h_1(t, x, \tilde{y}) = h_1^\epsilon(t, x, y), \\ h_2(t, x, \tilde{y}) = \epsilon^{-\frac{1}{2}}h_2^\epsilon(t, x, y), \end{cases}$$

and

$$p(t, x, \tilde{y}) = p^\epsilon(t, x, y).$$

Then by taking the leading order, the equations (1.1) are reduced to

$$\begin{cases} \partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 - h_1 \partial_x h_1 - h_2 \partial_y h_1 + \partial_x p = \mu \partial_y^2 u_1, \\ \partial_y p = 0, \\ \partial_t h_1 + \partial_y (u_2 h_1 - u_1 h_2) = \kappa \partial_y^2 h_1, \\ \partial_t h_2 - \partial_x (u_2 h_1 - u_1 h_2) = \kappa \partial_y^2 h_2, \\ \partial_x u_1 + \partial_y u_2 = 0, \quad \partial_x h_1 + \partial_y h_2 = 0, \end{cases} \quad (1.4)$$

in  $\{t > 0, x \in \mathbb{T}, y \in \mathbb{R}^+\}$ , where we have replaced  $\tilde{y}$  by  $y$  for simplicity of notations.

The second equation of (1.4) implies that the leading order of boundary layers for the total pressure  $p^\epsilon(t, x, y)$  is invariant across the boundary layer, and should be matched to the outflow pressure  $P(t, x)$  on top of boundary layer, that is, the trace of pressure of ideal MHD flow. Consequently, we have

$$p(t, x, y) \equiv P(t, x).$$

It is worth noting that the pressure  $\tilde{p}^\epsilon$  of the fluid may have the leading order of boundary layers because of the appearance of the boundary layer for magnetic field. It is different from the general fluid in the absence of magnetic field, for which the leading boundary layer for the pressure of the fluid always vanishes.

The tangential component  $u_1(t, x, y)$  of velocity field, respectively  $h_1(t, x, y)$  of magnetic field, should match the outflow tangential velocity  $U(t, x)$ , respectively the outflow tangential magnetic field  $H(t, x)$ , on the top of boundary layer, that is,

$$u_1(t, x, y) \rightarrow U(t, x), \quad h_1(t, x, y) \rightarrow H(t, x), \quad \text{as } y \rightarrow +\infty, \quad (1.5)$$

where  $U(t, x)$  and  $H(t, x)$  are the trace of tangential velocity and magnetic field respectively. Therefore, we have the following ‘‘matching’’ condition:

$$U_t + UU_x - HH_x + P_x = 0, \quad H_t + UH_x - HU_x = 0, \quad (1.6)$$

which shows that (1.5) is consistent with the first and third equations of (1.4). Moreover, on the boundary  $\{y = 0\}$ , the boundary conditions (1.2) and (1.3) give

$$u_1|_{y=0} = u_2|_{y=0} = \partial_y h_1|_{y=0} = h_2|_{y=0} = 0. \quad (1.7)$$

On the other hand, it is noted that equation (1.4)<sub>4</sub> is a direct consequence of equations (1.4)<sub>3</sub>,  $\partial_x h_1 + \partial_y h_2 = 0$  in (1.4)<sub>5</sub> and the boundary condition (1.7). Hence, we only need to study the following initial-boundary value problem of the MHD boundary layer equations in  $\{t \in [0, T], x \in \mathbb{T}, y \in \mathbb{R}^+\}$ ,

$$\begin{cases} \partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 - h_1 \partial_x h_1 - h_2 \partial_y h_1 = \mu \partial_y^2 u_1 - P_x, \\ \partial_t h_1 + \partial_y (u_2 h_1 - u_1 h_2) = \kappa \partial_y^2 h_1, \\ \partial_x u_1 + \partial_y u_2 = 0, \quad \partial_x h_1 + \partial_y h_2 = 0, \\ u_1|_{t=0} = u_{10}(x, y), \quad h_1|_{t=0} = h_{10}(x, y), \\ (u_1, u_2, \partial_y h_1, h_2)|_{y=0} = \mathbf{0}, \quad \lim_{y \rightarrow +\infty} (u_1, h_1) = (U, H)(t, x). \end{cases} \quad (1.8)$$

The aim of this paper is to show the local well-posedness of the system (1.8) with non-zero tangential component of that magnetic field, that is, without loss of generality, by assuming

$$h_1(t, x, y) > 0. \quad (1.9)$$

Let us first introduce some weighted Sobolev spaces for later use. Denote

$$\Omega := \{(x, y) : x \in \mathbb{T}, y \in \mathbb{R}_+\}.$$

For any  $l \in \mathbb{R}$ , denote by  $L_l^2(\Omega)$  the weighted Lebesgue space with respect to the spatial variables:

$$L_l^2(\Omega) := \left\{ f(x, y) : \Omega \rightarrow \mathbb{R}, \|f\|_{L_l^2(\Omega)} := \left( \int_{\Omega} \langle y \rangle^{2l} |f(x, y)|^2 dx dy \right)^{\frac{1}{2}} < +\infty \right\}, \quad \langle y \rangle = 1 + y,$$

and then, for any given  $m \in \mathbb{N}$ , denote by  $H_l^m(\Omega)$  the weighted Sobolev spaces:

$$H_l^m(\Omega) := \left\{ f(x, y) : \Omega \rightarrow \mathbb{R}, \|f\|_{H_l^m(\Omega)} := \left( \sum_{m_1+m_2 \leq m} \|\langle y \rangle^{l+m_2} \partial_x^{m_1} \partial_y^{m_2} f\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} < +\infty \right\}.$$

Now, we can state the main result as follows.

**Theorem 1.1.** *Let  $m \geq 5$  be a integer, and  $l \geq 0$  a real number. Assume that the outer flow  $(U, H, P_x)(t, x)$  satisfies that for some  $T > 0$ ,*

$$M_0 := \sum_{i=0}^{2m+2} \left( \sup_{0 \leq t \leq T} \|\partial_t^i(U, H, P)(t, \cdot)\|_{H^{2m+2-i}(\mathbb{T}_x)} + \|\partial_t^i(U, H, P)\|_{L^2(0, T; H^{2m+2-i}(\mathbb{T}_x))} \right) < +\infty. \quad (1.10)$$

Also, we suppose the initial data  $(u_{10}, h_{10})(x, y)$  satisfies

$$\left( u_{10}(x, y) - U(0, x), h_{10}(x, y) - H(0, x) \right) \in H_l^{3m+2}(\Omega), \quad (1.11)$$

and the compatibility conditions up to  $m$ -th order. Moreover, there exists a sufficiently small constant  $\delta_0 > 0$  such that

$$\left| \langle y \rangle^{l+1} \partial_y^i(u_{10}, h_{10})(x, y) \right| \leq (2\delta_0)^{-1}, \quad h_{10}(x, y) \geq 2\delta_0, \quad \text{for } i = 1, 2, (x, y) \in \Omega. \quad (1.12)$$

Then, there exist a positive time  $0 < T_* \leq T$  and a unique solution  $(u_1, u_2, h_1, h_2)$  to the initial boundary value problem (1.8), such that

$$(u_1 - U, h_1 - H) \in \bigcap_{i=0}^m W^{i, \infty} \left( 0, T_*; H_l^{m-i}(\Omega) \right), \quad (1.13)$$

and

$$\begin{aligned} (u_2 + U_x y, h_2 + H_x y) &\in \bigcap_{i=0}^{m-1} W^{i, \infty} \left( 0, T_*; H_{-1}^{m-1-i}(\Omega) \right), \\ (\partial_y u_2 + U_x, \partial_y h_2 + H_x) &\in \bigcap_{i=0}^{m-1} W^{i, \infty} \left( 0, T_*; H_l^{m-1-i}(\Omega) \right). \end{aligned} \quad (1.14)$$

Moreover, if  $l > \frac{1}{2}$ ,

$$(u_2 + U_x y, h_2 + H_x y) \in \bigcap_{i=0}^{m-1} W^{i, \infty} \left( 0, T_*; L^\infty(\mathbb{R}_{y,+}; H^{m-1-i}(\mathbb{T}_x)) \right). \quad (1.15)$$

*Remark 1.1.* Note that the regularity assumption on the outflow  $(U, H, P)$  and the initial data  $(u_{10}, h_{10})$  is not optimal. Here, we need the regularity to simplify the construction of approximate solution, cf. Section 4. One may relax the regularity requirement by using other approximations.

We now review some related works to the problem studied in this paper. First of all, the study on fluid around a rigid body with high Reynolds numbers is an important problem in both physics and mathematics. The classical work can be traced back to Prandtl in 1904 about the derivation of the Prandtl equations for boundary layers from the incompressible Navier-Stokes equations with non-slip boundary condition, cf. [33]. About sixty years after its derivation, the first systematic work in rigorous mathematics was achieved by Oleinik, cf. [31], in which she showed that under the monotonicity condition on the tangential velocity field in the normal direction to the boundary, local in time well-posedness of the Prandtl system can be justified

in 2D by using the Crocco transformation. This result together with some extensions are presented in Oleinik-Samokhin's classical book [32]. Recently, this well-posedness result was proved by using simply energy method in the framework of Sobolev spaces in [1] and [30] independently by taking care of the cancellation in the convection terms to overcome the loss of derivative in the tangential direction. Moreover, by imposing an additional favorable condition on the pressure, a global in time weak solution was obtained in [39]. Some three space dimensional cases were studied for both classical and weak solutions in [23, 24]. Since Oleinik's classical work, the necessity of the monotonicity condition on the velocity field for well-posedness remained as a question until 1980s when Caffisch and Sammartino [35, 36] obtained the well-posedness in the framework of analytic functions without this condition, cf. [18–20, 28, 29, 41] and the references therein. And recently, the analyticity condition can be further relaxed to Gevrey regularity, cf. [10, 11, 21, 22].

When the monotonicity condition is violated, separation of the boundary layer is expected and observed for classical fluid. For this, E-Engquist constructed a finite time blowup solution to the Prandtl equations in [8]. Recently, when the background shear flow has a non-degenerate critical point, some interesting ill-posedness (or instability) phenomena of solutions to both the linear and nonlinear Prandtl equations around the shear flow are studied, cf. [9, 12, 14, 15, 25, 27] and the references therein. All these results show that the monotonicity assumption on the tangential velocity is essential for the well-posedness except in the framework of analytic functions or Gevrey functions.

On the other hand, for electrically conducting fluid such as plasmas and liquid metals, the system of magnetohydrodynamics (denoted by MHD) is a fundamental system to describe the movement of fluid under the influence of electro-magnetic field. The study on the MHD was initiated by Alfvén [2] who showed that the magnetic field can induce current in a moving conductive fluid with a new propagation mechanism along the magnetic field, called Alfvén waves.

For plasma, the boundary layer equations can be derived from the fundamental MHD system and they are more complicated than the classical Prandtl system because of the coupling of the magnetic field with velocity field through the Maxwell equations. On the other hand, in physics, it is believed that the magnetic field has a stabilizing effect on the boundary layer that could provide a mechanism for containment of, for example, the high temperature gas. If the magnetic field is transversal to the boundary, there are extensive discussions on the so called Hartmann boundary layer, cf. [5, 16, 17]. In addition, there are works on the stability of boundary layers with minimum Reynolds number for flow with different structures to reveal the difference from the classical boundary layers without electro-magnetic field, cf. [3, 6, 34].

In terms of mathematical derivation when the non-slip boundary condition for the velocity is present, the boundary layer systems that capture the leading order of fluid variables around the boundary depend on three physical parameters, magnetic Reynolds number, Reynolds number and their ratio called magnetic Prandtl number. When the Reynolds number tends to infinity while the magnetic Reynolds number is fixed, the derived boundary layer system is similar to the Prandtl system for classical fluid and its well-posedness was discussed in Oleinik-Samokhin's book [32], for which the monotonicity condition on the velocity field is needed. When the Reynolds number is fixed while the magnetic Reynolds number tends to infinity that corresponds to infinite magnetic Prandtl number, the boundary layer system is similar to inviscid Prandtl system and the monotonicity condition on the velocity field is not needed for well-posedness. The case with finite magnetic Prandtl number when both the Reynolds number and magnetic Reynolds number tend to infinity at the same rate, the boundary layer system is totally different from the classical Prandtl system, and this is the system to be discussed in this paper. Note that for this system, there are no any mathematical well-posedness results obtained so far in the Sobolev spaces. Furthermore, we mention that in [38], the authors establish the vanishing viscosity limit for the MHD system in a bounded smooth domain of  $\mathbb{R}^d$ ,  $d = 2, 3$  with a slip boundary condition, while the leading order of boundary layers for both velocity and magnetic field vanishes because of the slip boundary conditions.

Precisely, in this paper, to capture the stabilizing effect of the magnetic field, we establish the well-posedness theory for the problem (1.8) without any monotonicity assumption on the tangential velocity. The only essential condition is that the background tangential magnetic field has a lower positive bound. Hence, the result in this paper enriches the classical local well-posedness results of the classical Prandtl equations. In the same time, it is in agreement with the general physical understanding that the magnetic field stabilizes the boundary layer.

The rest of the paper is organized as follows. Some preliminaries are given in Section 2. In Section 3, we establish the a priori energy estimates for the nonlinear problem (1.8). The local-in-time existence and

uniqueness of the solution to (1.8) in Sobolev space are given in Section 4. In Section 5, we introduce another method for the study on the well-posedness theory for (1.8) by using a nonlinear coordinate transform in the spirit of Crocco transformation for the classical Prandtl system. Finally, some technical proof of a lemma is given in the Appendix.

## 2. PRELIMINARIES

Firstly, we introduce some notations. Use the tangential derivative operator

$$\partial_\tau^\beta = \partial_t^{\beta_1} \partial_x^{\beta_2}, \quad \text{for } \beta = (\beta_1, \beta_2) \in \mathbb{N}^2, \quad |\beta| = \beta_1 + \beta_2,$$

and then denote the derivative operator (in both time and space) by

$$D^\alpha = \partial_\tau^\beta \partial_y^k, \quad \text{for } \alpha = (\beta_1, \beta_2, k) \in \mathbb{N}^3, \quad |\alpha| = |\beta| + k.$$

Set  $e_i \in \mathbb{N}^2, i = 1, 2$  and  $E_j \in \mathbb{N}^3, j = 1, 2, 3$  by

$$e_1 = (1, 0) \in \mathbb{N}^2, \quad e_2 = (0, 1) \in \mathbb{N}^2, \quad E_1 = (1, 0, 0) \in \mathbb{N}^3, \quad E_2 = (0, 1, 0) \in \mathbb{N}^3, \quad E_3 = (0, 0, 1) \in \mathbb{N}^3,$$

and denote by  $\partial_y^{-1}$  the inverse of derivative  $\partial_y$ , i.e.,  $(\partial_y^{-1} f)(y) := \int_0^y f(z) dz$ . Moreover, we use the notation  $[\cdot, \cdot]$  to denote the commutator, and denote a nondecreasing polynomial function by  $\mathcal{P}(\cdot)$ , which may differ from line to line.

For  $m \in \mathbb{N}$ , define the function spaces  $\mathcal{H}_t^m$  of measurable functions  $f(t, x, y) : [0, T] \times \Omega \rightarrow \mathbb{R}$ , such that for any  $t \in [0, T]$ ,

$$\|f(t)\|_{\mathcal{H}_t^m} := \left( \sum_{|\alpha| \leq m} \|\langle y \rangle^{l+k} D^\alpha f(t, \cdot)\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}} < +\infty. \quad (2.1)$$

The following inequalities will be used frequently in this paper.

**Lemma 2.1.** *For proper functions  $f, g, h$ , the following holds.*

i) If  $\lim_{y \rightarrow +\infty} (fg)(x, y) = 0$ , then

$$\left| \int_{\mathbb{T}_x} (fg)|_{y=0} dx \right| \leq \|\partial_y f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \|\partial_y g\|_{L^2(\Omega)}. \quad (2.2)$$

In particular, if  $\lim_{y \rightarrow +\infty} f(x, y) = 0$ , then

$$\|f|_{y=0}\|_{L^2(\mathbb{T}_x)} \leq \sqrt{2} \|f\|_{L^2(\Omega)}^{\frac{1}{2}} \|\partial_y f\|_{L^2(\Omega)}^{\frac{1}{2}}. \quad (2.3)$$

ii) For  $l \in \mathbb{R}$  and an integer  $m \geq 3$ , any  $\alpha = (\beta, k) \in \mathbb{N}^3, \tilde{\alpha} = (\tilde{\beta}, \tilde{k}) \in \mathbb{N}^3$  with  $|\alpha| + |\tilde{\alpha}| \leq m$ ,

$$\|(D^\alpha f \cdot D^{\tilde{\alpha}} g)(t, \cdot)\|_{L^2_{l+k+\tilde{k}}(\Omega)} \leq C \|f(t)\|_{\mathcal{H}_t^m} \|g(t)\|_{\mathcal{H}_t^m}, \quad \forall l_1, l_2 \in \mathbb{R}, \quad l_1 + l_2 = l. \quad (2.4)$$

iii) For any  $\lambda > \frac{1}{2}, \tilde{\lambda} > 0$ ,

$$\|\langle y \rangle^{-\lambda} (\partial_y^{-1} f)(y)\|_{L_y^2(\mathbb{R}_+)} \leq \frac{2}{2\lambda - 1} \|\langle y \rangle^{1-\lambda} f(y)\|_{L_y^2(\mathbb{R}_+)}, \quad \|\langle y \rangle^{-\tilde{\lambda}} (\partial_y^{-1} f)(y)\|_{L_y^\infty(\mathbb{R}_+)} \leq \frac{1}{\tilde{\lambda}} \|\langle y \rangle^{1-\tilde{\lambda}} f(y)\|_{L_y^\infty(\mathbb{R}_+)}, \quad (2.5)$$

and then, for  $l \in \mathbb{R}$ , an integer  $m \geq 3$ , and any  $\alpha = (\beta, k) \in \mathbb{N}^3, \tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2) \in \mathbb{N}^2$  with  $|\alpha| + |\tilde{\beta}| \leq m$ ,

$$\|(D^\alpha g \cdot \partial_\tau^{\tilde{\beta}} \partial_y^{-1} h)(t, \cdot)\|_{L^2_{l+k}(\Omega)} \leq C \|g(t)\|_{\mathcal{H}_t^m} \|h(t)\|_{\mathcal{H}_t^{m-\lambda}}. \quad (2.6)$$

In particular, for  $\lambda = 1$ ,

$$\|\langle y \rangle^{-1} (\partial_y^{-1} f)(y)\|_{L_y^2(\mathbb{R}_+)} \leq 2 \|f\|_{L_y^2(\mathbb{R}_+)}, \quad \|(D^\alpha g \cdot \partial_\tau^{\tilde{\beta}} \partial_y^{-1} h)(t, \cdot)\|_{L^2_{l+k}(\Omega)} \leq C \|g(t)\|_{\mathcal{H}_t^m} \|h(t)\|_{\mathcal{H}_t^0}. \quad (2.7)$$

iv) For any  $\lambda > \frac{1}{2}$ ,

$$\|(\partial_y^{-1} f)(y)\|_{L_y^\infty(\mathbb{R}_+)} \leq C \|f\|_{L_{y,\lambda}^2(\mathbb{R}_+)}, \quad (2.8)$$

and then, for  $l \in \mathbb{R}$ , an integer  $m \geq 2$ , and any  $\alpha = (\beta, k) \in \mathbb{N}^3, \tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2) \in \mathbb{N}^2$  with  $|\alpha| + |\tilde{\beta}| \leq m$ ,

$$\|(D^\alpha f \cdot \partial_\tau^{\tilde{\beta}} \partial_y^{-1} g)(t, \cdot)\|_{L^2_{l+k}(\Omega)} \leq C \|f(t)\|_{\mathcal{H}_t^m} \|g(t)\|_{\mathcal{H}_t^m}. \quad (2.9)$$

To overcome the technical difficulty originated from the boundary terms at  $\{y = +\infty\}$ , we introduce an auxiliary function  $\phi(y) \in C^\infty(\mathbb{R}_+)$  satisfying that

$$\phi(y) = \begin{cases} y, & y \geq 2R_0, \\ 0, & 0 \leq y \leq R_0 \end{cases}$$

for some constant  $R_0 > 0$ . Then, set the new unknowns:

$$\begin{aligned} u(t, x, y) &:= u_1(t, x, y) - U(t, x)\phi'(y), & v(t, x, y) &:= u_2(t, x, y) + U_x(t, x)\phi(y), \\ h(t, x, y) &:= h_1(t, x, y) - H(t, x)\phi'(y), & g(t, x, y) &:= h_2(t, x, y) + H_x(t, x)\phi(y). \end{aligned} \quad (2.10)$$

Choose the above construction for  $(u, v, h, g)$  to ensure the divergence free conditions and homogenous boundary conditions, i.e.,

$$\begin{aligned} \partial_x u + \partial_y v &= 0, & \partial_x h + \partial_y g &= 0, \\ (u, v, \partial_y h, g)|_{y=0} &= \mathbf{0}, & \lim_{y \rightarrow +\infty} (u, h) &= \mathbf{0}, \end{aligned}$$

which implies that  $v = -\partial_y^{-1}\partial_x u$  and  $g = -\partial_y^{-1}\partial_x h$ . And it is easy to get that

$$(u, h)(t, x, y) = (u_1(t, x, y) - U(t, x), h_1(t, x, y) - H(t, x)) + (U(t, x)(1 - \phi'(y)), H(t, x)(1 - \phi'(y))),$$

which implies that by the construction of  $\phi(y)$ ,

$$\|(u, h)(t)\|_{\mathcal{H}_t^m} - CM_0 \leq \|(u_1 - U, h_1 - H)(t)\|_{\mathcal{H}_t^m} \leq \|(u, h)(t)\|_{\mathcal{H}_t^m} + CM_0. \quad (2.11)$$

By using the new unknowns  $(u, v, h, g)$  given by (2.10), we can reformulate the original problem (1.8) to the following:

$$\begin{cases} \partial_t u + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h - \mu\partial_y^2 u \\ \quad + U_x\phi'u + U\phi''v - H_x\phi'h - H\phi''g = r_1, \\ \partial_t h + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]h - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]u - \kappa\partial_y^2 h \\ \quad + H_x\phi'u + H\phi''v - U_x\phi'h - U\phi''g = r_2, \\ \partial_x u + \partial_y v = 0, & \partial_x h + \partial_y g = 0, \\ (u, v, \partial_y h, g)|_{y=0} = \mathbf{0}, \\ (u, h)|_{t=0} = (u_{10}(x, y) - U(0, x)\phi'(y), h_{10}(x, y) - H(0, x)\phi'(y)) \triangleq (u_0, h_0)(x, y), \end{cases} \quad (2.12)$$

where

$$\begin{cases} r_1 = U_t[(\phi')^2 - \phi\phi'' - \phi'] + P_x[(\phi')^2 - \phi\phi'' - 1] + \mu U\phi^{(3)}, \\ r_2 = H_t[(\phi')^2 + \phi\phi'' - \phi'] + \kappa H\phi^{(3)}. \end{cases} \quad (2.13)$$

Note that we have used the divergence free conditions in obtaining the equations of  $(u, h)$  in (2.12), and the relations (1.6) in the calculation of (2.13). It is worth noting that by substituting (2.10) into the second equation of (1.8) directly, there is another equivalent form for the equation of  $h$ , which may be convenient for use in some situations:

$$\partial_t h + \partial_y [(v - U_x\phi)(h + H\phi') - (u + U\phi')(g - H_x\phi)] - \kappa\partial_y^2 h = -H_t\phi' + \kappa H\phi^{(3)}. \quad (2.14)$$

By the choice of  $\phi(y)$ , it is easy to get that

$$\begin{aligned} r_1(t, x, y), r_2(t, x, y) &\equiv 0, & y &\geq 2R_0, \\ r_1(t, x, y) &\equiv -P_x(t, x), & r_2(t, x, y) &\equiv 0, & 0 \leq y \leq R_0, \end{aligned} \quad (2.15)$$

and then for any  $t \in [0, T]$ ,  $\lambda \geq 0$  and  $|\alpha| \leq m$ , by virtue of (1.10),

$$\|\langle y \rangle^\lambda D^\alpha r_1(t)\|_{L^2(\Omega)}, \|\langle y \rangle^\lambda D^\alpha r_2(t)\|_{L^2(\Omega)} \leq C \sum_{|\beta| \leq |\alpha|+1} \|\partial_\tau^\beta (U, H, P_x)(t)\|_{L^2(\mathbb{T}_x)} \leq CM_0. \quad (2.16)$$

Furthermore, similar to (2.11) we have that for the initial data:

$$\|(u_0, h_0)\|_{H_t^{2m}(\Omega)} - CM_0 \leq \|(u_{10}(x, y) - U(0, x), h_{10} - H(0, x))\|_{H_t^{2m}(\Omega)} \leq \|(u_0, h_0)\|_{H_t^{2m}(\Omega)} + CM_0. \quad (2.17)$$

Finally, from the transformation (2.10), and the relations (2.11) and (2.17), it is easy to know that Theorem 1.1 is a corollary of the following result.

**Theorem 2.2.** *Let  $m \geq 5$  be a integer,  $l \geq 0$  a real number, and  $(U, H, P_x)(t, x)$  satisfies the hypotheses given in Theorem 1.1. In addition, assume that for the problem (2.12), the initial data  $(u_0(x, y), h_0(x, y)) \in H_1^{3m+2}(\Omega)$ , and the compatibility conditions up to  $m$ -th order. Moreover, there exists a sufficiently small constant  $\delta_0 > 0$ , such that*

$$|\langle y \rangle^{l+1} \partial_y^i (u_0, h_0)(x, y)| \leq (2\delta_0)^{-1}, \quad h_0(x, y) + H(0, x)\phi'(y) \geq 2\delta_0, \quad \text{for } i = 1, 2, (x, y) \in \Omega. \quad (2.18)$$

*Then, there exist a time  $0 < T_* \leq T$  and a unique solution  $(u, v, h, g)$  to the initial boundary value problem (2.12), such that*

$$(u, h) \in \bigcap_{i=0}^m W^{i, \infty} \left( 0, T_*; H_1^{m-i}(\Omega) \right), \quad (2.19)$$

and

$$(v, g) \in \bigcap_{i=0}^{m-1} W^{i, \infty} \left( 0, T_*; H_{-1}^{m-1-i}(\Omega) \right), \quad (\partial_y v, \partial_y g) \in \bigcap_{i=0}^{m-1} W^{i, \infty} \left( 0, T_*; H_1^{m-1-i}(\Omega) \right). \quad (2.20)$$

Moreover, if  $l > \frac{1}{2}$ ,

$$(v, g) \in \bigcap_{i=0}^{m-1} W^{i, \infty} \left( 0, T_*; L^\infty(\mathbb{R}_{y,+}; H^{m-1-i}(\mathbb{T}_x)) \right). \quad (2.21)$$

Therefore, our main task is to show the above Theorem 2.2, and its proof will be given in the following two sections.

### 3. A PRIORI ESTIMATES

In this section, we will establish a priori estimates for the nonlinear problem (2.12).

**Proposition 3.1.** *[Weighted estimates for  $D^m(u, h)$ ]*

*Let  $m \geq 5$  be a integer,  $l \geq 0$  be a real number, and the hypotheses for  $(U, H, P_x)(t, x)$  given in Theorem 1.1 hold. Assume that  $(u, v, h, g)$  is a classical solution to the problem (2.12) in  $[0, T]$ , satisfying that  $(u, h) \in L^\infty(0, T; \mathcal{H}_1^m)$ ,  $(\partial_y u, \partial_y h) \in L^2(0, T; \mathcal{H}_1^m)$ , and for sufficiently small  $\delta_0$ :*

$$h(t, x, y) + H(t, x)\phi'(y) \geq \delta_0, \quad \langle y \rangle^{l+1} \partial_y^i (u, h)(t, x, y) \leq \delta_0^{-1}, \quad i = 1, 2, (t, x, y) \in [0, T] \times \Omega. \quad (3.1)$$

*Then, it holds that for small time,*

$$\begin{aligned} \sup_{0 \leq s \leq t} \|(u, h)(s)\|_{\mathcal{H}_1^m} &\leq \delta_0^{-4} \left( \mathcal{P}(M_0 + \|(u_0, h_0)\|_{H_1^{2m}(\Omega)}) + CM_0^6 t \right)^{\frac{1}{2}} \\ &\cdot \left\{ 1 - C\delta_0^{-24} \left( \mathcal{P}(M_0 + \|(u_0, h_0)\|_{H_1^{2m}(\Omega)}) + CM_0^6 t \right)^2 t \right\}^{-\frac{1}{4}}. \end{aligned} \quad (3.2)$$

*Also, we have that for  $i = 1, 2$ ,*

$$\begin{aligned} \|\langle y \rangle^{l+1} \partial_y^i (u, h)(t)\|_{L^\infty(\Omega)} &\leq \|\langle y \rangle^{l+1} \partial_y^i (u_0, h_0)\|_{L^\infty(\Omega)} + C\delta_0^{-4} t \left( \mathcal{P}(M_0 + \|(u_0, h_0)\|_{H_1^{2m}(\Omega)}) + CM_0^6 t \right)^{\frac{1}{2}} \\ &\cdot \left\{ 1 - C\delta_0^{-24} \left( \mathcal{P}(M_0 + \|(u_0, h_0)\|_{H_1^{2m}(\Omega)}) + CM_0^6 t \right)^2 t \right\}^{-\frac{1}{4}}, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} h(t, x, y) &\geq h_0(x, y) - C\delta_0^{-4} t \left( \mathcal{P}(M_0 + \|(u_0, h_0)\|_{H_1^{2m}(\Omega)}) + CM_0^6 t \right)^{\frac{1}{2}} \\ &\cdot \left\{ 1 - C\delta_0^{-24} \left( \mathcal{P}(M_0 + \|(u_0, h_0)\|_{H_1^{2m}(\Omega)}) + CM_0^6 t \right)^2 t \right\}^{-\frac{1}{4}}. \end{aligned} \quad (3.4)$$

The proof of Proposition 3.1 will be given in the following two subsections. More precisely, we will obtain the weighted estimates for  $D^\alpha(u, h)$  for  $\alpha = (\beta, k) = (\beta_1, \beta_2, k)$ , satisfying  $|\alpha| = |\beta| + k \leq m$ ,  $|\beta| \leq m - 1$ , in the first subsection, and the weighted estimates for  $\partial_\tau^\beta(u, h)$  for  $|\beta| = m$  in the second subsection.

### 3.1. Weighted $H_l^m$ -estimates with normal derivatives.

The weighted estimates on  $D^\alpha(u, h)$  with  $|\alpha| = |\beta| + k \leq m$ ,  $|\beta| \leq m - 1$  can be obtained by the standard energy method because one order tangential regularity loss is allowed. That is, we have the following estimates:

**Proposition 3.2.** *[Weighted estimates for  $D^\alpha(u, h)$  with  $|\alpha| \leq m, |\beta| \leq m - 1$ ]*

Let  $m \geq 5$  be a integer,  $l \geq 0$  be a real number, and the hypotheses for  $(U, H, P_x)(t, x)$  given in Theorem 1.1 hold. Assume that  $(u, v, h, g)$  is a classical solution to the problem (2.12) in  $[0, T]$ , and satisfies  $(u, h) \in L^\infty(0, T; \mathcal{H}_l^m)$ ,  $(\partial_y u, \partial_y h) \in L^2(0, T; \mathcal{H}_l^m)$ . Then, there exists a positive constant  $C$ , depending on  $m, l$  and  $\phi$ , such that for any small  $0 < \delta_1 < 1$ ,

$$\begin{aligned} & \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \left( \frac{d}{dt} \|D^\alpha(u, h)(t)\|_{L_{i+k}^2(\Omega)}^2 + \mu \|D^\alpha \partial_y u(t)\|_{L_{i+k}^2(\Omega)}^2 + \kappa \|D^\alpha \partial_y h(t)\|_{L_{i+k}^2(\Omega)}^2 \right) \\ & \leq \delta_1 C \|(\partial_y u, \partial_y h)(t)\|_{\mathcal{H}_0^m}^2 + C \delta_1^{-1} \|(u, h)(t)\|_{\mathcal{H}_l^m}^2 (1 + \|(u, h)(t)\|_{\mathcal{H}_l^m}^2) + \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(r_1, r_2)(t)\|_{L_{i+k}^2(\Omega)}^2 \\ & + C \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H, P)(t)\|_{L^2(\mathbb{T}_x)}^2. \end{aligned} \quad (3.5)$$

**Proof.** Applying the operator  $D^\alpha = \partial_\tau^\beta \partial_y^k$  for  $\alpha = (\beta, k) = (\beta_1, \beta_2, k)$ , satisfying  $|\alpha| = |\beta| + k \leq m$ ,  $|\beta| \leq m - 1$ , to the first two equations of (2.12), it yields that

$$\begin{cases} \partial_t D^\alpha u = D^\alpha r_1 + \mu \partial_y^2 D^\alpha u - D^\alpha \left\{ [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h \right. \\ \quad \left. + U_x\phi'u + U\phi''v - H_x\phi'h - H\phi''g \right\}, \\ \partial_t D^\alpha h = D^\alpha r_2 + \kappa \partial_y^2 D^\alpha h - D^\alpha \left\{ [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]h - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]u \right. \\ \quad \left. + H_x\phi'u + H\phi''v - U_x\phi'h - U\phi''g \right\}. \end{cases} \quad (3.6)$$

Multiplying (3.6)<sub>1</sub> by  $\langle y \rangle^{2l+2k} D^\alpha u$ , (3.6)<sub>2</sub> by  $\langle y \rangle^{2l+2k} D^\alpha h$  respectively, and integrating them over  $\Omega$ , with respect to the spatial variables  $x$  and  $y$ , we obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\langle y \rangle^{l+k} D^\alpha(u, h)(t)\|_{L^2(\Omega)}^2 &= \int_\Omega \left( D^\alpha r_1 \cdot \langle y \rangle^{2l+2k} D^\alpha u + D^\alpha r_2 \cdot \langle y \rangle^{2l+2k} D^\alpha h \right) dx dy \\ &+ \mu \int_\Omega (\partial_y^2 D^\alpha u \cdot \langle y \rangle^{2l+2k} D^\alpha u) dx dy + \kappa \int_\Omega (\partial_y^2 D^\alpha h \cdot \langle y \rangle^{2l+2k} D^\alpha h) dx dy \\ &- \int_\Omega \left( I_1 \cdot \langle y \rangle^{2l+2k} D^\alpha u + I_2 \cdot \langle y \rangle^{2l+2k} D^\alpha h \right) dx dy, \end{aligned} \quad (3.7)$$

where

$$\begin{cases} I_1 = D^\alpha \left\{ [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h \right. \\ \quad \left. + U_x\phi'u + U\phi''v - H_x\phi'h - H\phi''g \right\}, \\ I_2 = D^\alpha \left\{ [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]h - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]u \right. \\ \quad \left. + H_x\phi'u + H\phi''v - U_x\phi'h - U\phi''g \right\}. \end{cases} \quad (3.8)$$

First of all, it is easy to get that by virtue of (2.16),

$$\begin{aligned} & \int_\Omega \left( D^\alpha r_1 \cdot \langle y \rangle^{2l+2k} D^\alpha u + D^\alpha r_2 \cdot \langle y \rangle^{2l+2k} D^\alpha h \right) dx dy \\ & \leq \frac{1}{2} \|D^\alpha(u, h)(t)\|_{L_{i+k}^2(\Omega)}^2 + \frac{1}{2} \|D^\alpha(r_1, r_2)(t)\|_{L_{i+k}^2(\Omega)}^2. \end{aligned} \quad (3.9)$$

Next, we assume that the following two estimates holds, which will be proved later: for any small  $0 < \delta_1 < 1$ ,

$$\mu \int_\Omega (\partial_y^2 D^\alpha u \cdot \langle y \rangle^{2l+2k} D^\alpha u) dx dy + \kappa \int_\Omega (\partial_y^2 D^\alpha h \cdot \langle y \rangle^{2l+2k} D^\alpha h) dx dy$$



$$\begin{aligned}
&\leq -\frac{\mu}{2}\|D^\alpha \partial_y u(t)\|_{L^2_{l+k}(\Omega)}^2 - \frac{\kappa}{2}\|D^\alpha \partial_y h(t)\|_{L^2_{l+k}(\Omega)}^2 + \delta_1 \|(\partial_y u, \partial_y h)(t)\|_{\mathcal{H}_0^m}^2 \\
&\quad + C\delta_1^{-1}\|(u, h)(t)\|_{\mathcal{H}_1^m}^2 (1 + \|(u, h)(t)\|_{\mathcal{H}_1^m}^2) + C \sum_{|\beta| \leq m-1} \|\partial_\tau^\beta P_x(t)\|_{L^2(\mathbb{T}_x)}^2,
\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
&-\int_{\Omega} \left( I_1 \cdot \langle y \rangle^{2l+2k} D^\alpha u + I_2 \cdot \langle y \rangle^{2l+2k} D^\alpha h \right) dx dy \\
&\leq C \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta (U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_1^m} \right) \|(u, h)(t)\|_{\mathcal{H}_1^m}^2.
\end{aligned} \tag{3.11}$$

At the moment, by plugging the above inequalities (3.9)-(3.11) into (3.7), and summing over  $\alpha$ , we obtain that there exists a constant  $C_m > 0$ , depending only on  $m$ , such that

$$\begin{aligned}
&\sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \left( \frac{d}{dt} \|D^\alpha (u, h)(t)\|_{L^2_{l+k}(\Omega)}^2 + \mu \|D^\alpha \partial_y u(t)\|_{L^2_{l+k}(\Omega)}^2 + \kappa \|D^\alpha \partial_y h(t)\|_{L^2_{l+k}(\Omega)}^2 \right) \\
&\leq \delta_1 C_m \|(\partial_y u, \partial_y h)(t)\|_{\mathcal{H}_0^m}^2 + C\delta_1^{-1} \|(u, h)(t)\|_{\mathcal{H}_1^m}^2 (1 + \|(u, h)(t)\|_{\mathcal{H}_1^m}^2) + \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha (r_1, r_2)(t)\|_{L^2_{l+k}(\Omega)}^2 \\
&\quad + C \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta (U, H, P)(t)\|_{L^2(\mathbb{T}_x)}^2,
\end{aligned} \tag{3.12}$$

which implies the estimate (3.5) immediately.

Now, it remains to show the estimates (3.10) and (3.11) that will be given as follows.

**Proof of (3.10).** In this part, we will first handle the term  $\mu \int_{\Omega} (\partial_y^2 D^\alpha u \cdot \langle y \rangle^{2l+2k} D^\alpha u) dx dy$ , and the term  $\kappa \int_{\Omega} (\partial_y^2 D^\alpha h \cdot \langle y \rangle^{2l+2k} D^\alpha h) dx dy$  can be estimated similarly. By integration by parts, we have

$$\begin{aligned}
\mu \int_{\Omega} (\partial_y^2 D^\alpha u \cdot \langle y \rangle^{2l+2k} D^\alpha u) dx dy &= -\mu \|\langle y \rangle^{l+k} \partial_y D^\alpha u(t)\|_{L^2(\Omega)}^2 + 2(l+k)\mu \int_{\Omega} (\langle y \rangle^{2l+2k-1} \partial_y D^\alpha u \cdot D^\alpha u) dx dy \\
&\quad + \mu \int_{\mathbb{T}_x} (\partial_y D^\alpha u \cdot D^\alpha u)|_{y=0} dx.
\end{aligned} \tag{3.13}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
&2(l+k)\mu \int_{\Omega} (\langle y \rangle^{2l+2k-1} \partial_y D^\alpha u \cdot D^\alpha u) dx dy \\
&\leq \frac{\mu}{14} \|\langle y \rangle^{l+k} \partial_y D^\alpha u(t)\|_{L^2(\Omega)}^2 + 14\mu(l+k)^2 \|\langle y \rangle^{l+k} D^\alpha u(t)\|_{L^2(\Omega)}^2,
\end{aligned} \tag{3.14}$$

which implies that by plugging (3.14) into (3.13),

$$\begin{aligned}
&\mu \int_{\Omega} (\partial_y^2 D^\alpha u \cdot \langle y \rangle^{2l+2k} D^\alpha u) dx dy \\
&\leq -\frac{13\mu}{14} \|\langle y \rangle^{l+k} \partial_y D^\alpha u(t)\|_{L^2(\Omega)}^2 + C\|u(t)\|_{\mathcal{H}_1^m}^2 + \mu \int_{\mathbb{T}_x} (\partial_y D^\alpha u \cdot D^\alpha u)|_{y=0} dx.
\end{aligned} \tag{3.15}$$

The last term in (3.15), that is, the boundary integral  $\mu \int_{\mathbb{T}_x} (\partial_y D^\alpha u \cdot D^\alpha u)|_{y=0} dx$  is treated in the following two cases.

**Case 1:**  $|\alpha| \leq m-1$ . By the inequality (2.2), we obtain that for any small  $0 < \delta_1 < 1$ ,

$$\begin{aligned}
\left| \mu \int_{\mathbb{T}_x} (\partial_y D^\alpha u \cdot D^\alpha u)|_{y=0} dx \right| &\leq \mu \|\partial_y^2 D^\alpha u(t)\|_{L^2(\Omega)} \|D^\alpha u(t)\|_{L^2(\Omega)} + \mu \|\partial_y D^\alpha u(t)\|_{L^2(\Omega)}^2 \\
&\leq \delta_1 \|\partial_y^2 D^\alpha u(t)\|_{L^2(\Omega)}^2 + \frac{\mu^2}{4\delta_1} \|D^\alpha u(t)\|_{L^2(\Omega)}^2 + \mu \|\partial_y D^\alpha u(t)\|_{L^2(\Omega)}^2 \\
&\leq \delta_1 \|\partial_y u(t)\|_{\mathcal{H}_0^m} + C\delta_1^{-1} \|u(t)\|_{\mathcal{H}_0^m}^2.
\end{aligned} \tag{3.16}$$

**Case 2:**  $|\alpha| = |\beta| + k = m$ . It implies that  $k \geq 1$  from  $|\beta| \leq m-1$ . Then, denote by  $\gamma \triangleq \alpha - E_3 = (\beta, k-1)$  with  $|\gamma| = |\beta| + k - 1 = m-1$ , the first equation in (2.12) reads

$$\mu \partial_y D^\alpha u = \mu \partial_y^2 D^\gamma u = D^\gamma \left\{ \partial_t u + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h \right\}$$

$$+ U_x \phi' u + U \phi'' v - H_x \phi' h - H \phi'' g - r_1 \}.$$

Then, combining (2.15) with the fact  $\phi \equiv 0$  for  $y \leq R_0$ , it yields that at  $y = 0$ ,

$$\begin{aligned} \mu \partial_y D^\alpha u &= D^\gamma \left[ \partial_t u + (u \partial_x + v \partial_y) u - (h \partial_x + g \partial_y) h + P_x \right] \\ &= D^\gamma P_x + D^{\gamma+E_1} u + D^\gamma (u \partial_x u - h \partial_x h) + D^\gamma (v \partial_y u - g \partial_y h). \end{aligned} \quad (3.17)$$

It is easy to get that by (2.3),

$$\begin{aligned} \left| \int_{\mathbb{T}_x} (D^\gamma P_x \cdot D^\alpha u)|_{y=0} dx \right| &\leq \|D^\gamma P_x(t)\|_{L^2(\mathbb{T}_x)} \|D^\alpha u(t)|_{y=0}\|_{L^2(\mathbb{T}_x)} \\ &\leq \sqrt{2} \|D^\gamma P_x(t)\|_{L^2(\mathbb{T}_x)} \|D^\alpha u(t)\|_{L^2(\Omega)}^{\frac{1}{2}} \|D^\alpha \partial_y u(t)\|_{L^2(\Omega)}^{\frac{1}{2}} \\ &\leq \frac{\mu}{14} \|D^\alpha \partial_y u(t)\|_{L^2(\Omega)}^2 + C \|u(t)\|_{\mathcal{H}_0^m}^2 + C \|D^\gamma P_x(t)\|_{L^2(\mathbb{T}_x)}^2, \end{aligned} \quad (3.18)$$

provided  $|\alpha| = m$ . Also, by (2.2) and  $|\gamma + E_1| = m$ ,

$$\begin{aligned} \left| \int_{\mathbb{T}_x} (D^{\gamma+E_1} u \cdot D^\alpha u)|_{y=0} dx \right| &\leq \|D^{\gamma+E_1} \partial_y u(t)\|_{L^2(\Omega)} \|D^\alpha u(t)\|_{L^2(\Omega)} + \|D^{\gamma+E_1} u(t)\|_{L^2(\Omega)} \|D^\alpha \partial_y u(t)\|_{L^2(\Omega)} \\ &\leq \frac{\delta_1}{3} \|D^{\gamma+E_1} \partial_y u(t)\|_{L^2(\Omega)}^2 + \frac{\mu}{14} \|D^\alpha \partial_y u(t)\|_{L^2(\Omega)}^2 + C \delta_1^{-1} \|u(t)\|_{\mathcal{H}_0^m}^2. \end{aligned} \quad (3.19)$$

Hence, as we know  $D^\gamma (u \partial_x u) = \sum_{\tilde{\gamma} \leq \gamma} \binom{\gamma}{\tilde{\gamma}} (D^{\tilde{\gamma}} u \cdot D^{\gamma-\tilde{\gamma}+E_2} u)$ , it follows that

$$\begin{aligned} \left| \int_{\mathbb{T}_x} (D^\gamma (u \partial_x u) \cdot D^\alpha u)|_{y=0} dx \right| &\leq C \sum_{\tilde{\gamma} \leq \gamma} \left\{ \|\partial_y (D^{\tilde{\gamma}} u \cdot D^{\gamma-\tilde{\gamma}+E_2} u)\|_{L^2(\Omega)} \|D^\alpha u\|_{L^2(\Omega)} \right. \\ &\quad \left. + \|D^{\tilde{\gamma}} u \cdot D^{\gamma-\tilde{\gamma}+E_2} u\|_{L^2(\Omega)} \|D^\alpha \partial_y u\|_{L^2(\Omega)} \right\}. \end{aligned} \quad (3.20)$$

Then, by using (2.4) and note that  $|\gamma| = m - 1 \geq 3$ , we have

$$\begin{aligned} \|\partial_y (D^{\tilde{\gamma}} u \cdot D^{\gamma-\tilde{\gamma}+E_2} u)\|_{L^2(\Omega)} &\leq \|D^{\tilde{\gamma}} \partial_y u \cdot D^{\gamma-\tilde{\gamma}+E_2} u\|_{L^2(\Omega)} + \|D^{\tilde{\gamma}} u \cdot D^{\gamma-\tilde{\gamma}+E_2} \partial_y u\|_{L^2(\Omega)} \\ &\leq C \|\partial_y u(t)\|_{\mathcal{H}_0^{m-1}} \|\partial_x u(t)\|_{\mathcal{H}_0^{m-1}} + C \|u(t)\|_{\mathcal{H}_0^{m-1}} \|\partial_{xy}^2 u(t)\|_{\mathcal{H}_0^{m-1}} \\ &\leq C \|u(t)\|_{\mathcal{H}_0^m} \|\partial_y u(t)\|_{\mathcal{H}_0^m} + C \|u(t)\|_{\mathcal{H}_0^m}^2, \end{aligned}$$

and

$$\|D^{\tilde{\gamma}} u \cdot D^{\gamma-\tilde{\gamma}+E_2} u\|_{L^2(\Omega)} \leq C \|u(t)\|_{\mathcal{H}_0^m} \|u(t)\|_{\mathcal{H}_0^m} \leq C \|u(t)\|_{\mathcal{H}_0^m}^2.$$

Substituting the above two inequalities into (3.20) gives

$$\begin{aligned} &\left| \int_{\mathbb{T}_x} (D^\gamma (u \partial_x u) \cdot D^\alpha u)|_{y=0} dx \right| \\ &\leq C \sum_{\tilde{\gamma} \leq \gamma} \left( (\|u(t)\|_{\mathcal{H}_0^m} \|\partial_y u(t)\|_{\mathcal{H}_0^m} + \|u(t)\|_{\mathcal{H}_0^m}^2) \|D^\alpha u\|_{L^2(\Omega)} + \|u(t)\|_{\mathcal{H}_0^m}^2 \|\partial_y D^\alpha u\|_{L^2(\Omega)} \right) \\ &\leq \frac{\delta_1}{3} \|\partial_y u(t)\|_{\mathcal{H}_0^m}^2 + \frac{\mu}{14} \|D^\alpha \partial_y u(t)\|_{L^2(\Omega)}^2 + C \delta_1^{-1} \|u(t)\|_{\mathcal{H}_0^m}^4 + C \|u(t)\|_{\mathcal{H}_0^m}^2. \end{aligned} \quad (3.21)$$

Similarly, we have

$$\begin{aligned} &\left| \int_{\mathbb{T}_x} (D^\gamma (h \partial_x h) \cdot D^\alpha u)|_{y=0} dx \right| \\ &\leq C (\|h(t)\|_{\mathcal{H}_0^m} \|\partial_y h(t)\|_{\mathcal{H}_0^m} + C \|h(t)\|_{\mathcal{H}_0^m}^2) \|D^\alpha u\|_{L^2(\Omega)} + C \|h(t)\|_{\mathcal{H}_0^m}^2 \|\partial_y D^\alpha u\|_{L^2(\Omega)} \\ &\leq \frac{\delta_1}{3} \|(\partial_y u, \partial_y h)(t)\|_{\mathcal{H}_0^m}^2 + \frac{\mu}{14} \|D^\alpha \partial_y u(t)\|_{L^2(\Omega)}^2 + C \delta_1^{-1} \|(u, h)(t)\|_{\mathcal{H}_0^m}^4 + C \|(u, h)(t)\|_{\mathcal{H}_0^m}^2. \end{aligned} \quad (3.22)$$

We now turn to control the integral  $\left| \int_{\mathbb{T}_x} (D^\gamma (v \partial_y u) \cdot D^\alpha u) \Big|_{y=0} dx \right|$ . Recall that  $D^\gamma = \partial_\tau^\beta \partial_y^{k-1}$ , by the boundary condition  $v|_{y=0} = 0$  and divergence free condition  $u_x + v_y = 0$ , we obtain that on  $\{y = 0\}$ ,

$$\begin{aligned} D^\gamma (v \partial_y u) &= \partial_\tau^\beta \left( v \partial_y^k u + \sum_{i=1}^{k-1} \binom{k-1}{i} \partial_y^i v \cdot \partial_y^{k-i} u \right) = \sum_{j=0}^{k-2} \binom{k-1}{j+1} \partial_\tau^\beta \left[ -\partial_y^j \partial_x u \cdot \partial_y^{k-j-1} u \right] \\ &= - \sum_{\substack{\tilde{\beta} \leq \beta \\ 0 \leq j \leq k-2}} \binom{k-1}{j+1} \binom{\beta}{\tilde{\beta}} \left( \partial_\tau^{\tilde{\beta}+e_2} \partial_y^j u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{k-j-1} u \right), \end{aligned}$$

where we denote  $\binom{j}{i} = 0$  for  $i > j$ . Note that the right-hand side of the above equality vanishes when  $k = 1$ , and we only need to consider the case  $k \geq 2$ . Thus, from the above expression for  $D^\gamma (v \partial_y u)$  at  $y = 0$ , we obtain that by (2.2),

$$\begin{aligned} \left| \int_{\mathbb{T}_x} (D^\gamma (v \partial_y u) \cdot D^\alpha u) \Big|_{y=0} dx \right| &\leq C \sum_{\substack{\tilde{\beta} \leq \beta \\ 0 \leq j \leq k-2}} \left\{ \left\| \partial_y (\partial_\tau^{\tilde{\beta}+e_2} \partial_y^j u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{k-j-1} u) \right\|_{L^2(\Omega)} \left\| D^\alpha u \right\|_{L^2(\Omega)} \right. \\ &\quad \left. + \left\| \partial_\tau^{\tilde{\beta}+e_2} \partial_y^j u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{k-j-1} u \right\|_{L^2(\Omega)} \left\| D^\alpha \partial_y u \right\|_{L^2(\Omega)} \right\}. \end{aligned} \quad (3.23)$$

As  $0 \leq j \leq k-2$ , it follows that by (2.4),

$$\begin{aligned} \left\| \partial_y (\partial_\tau^{\tilde{\beta}+e_2} \partial_y^j u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{k-j-1} u) \right\|_{L^2(\Omega)} &\leq \left\| \partial_\tau^{\tilde{\beta}+e_2} \partial_y^{j+1} u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{k-j-1} u \right\|_{L^2(\Omega)} + \left\| \partial_\tau^{\tilde{\beta}+e_2} \partial_y^j u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{k-j} u \right\|_{L^2(\Omega)} \\ &\leq C \left\| \partial_y u(t) \right\|_{\mathcal{H}_0^{m-1}} \left\| \partial_y u(t) \right\|_{\mathcal{H}_0^{m-1}} + C \left\| \partial_x u(t) \right\|_{\mathcal{H}_0^{m-1}} \left\| \partial_y u(t) \right\|_{\mathcal{H}_0^{m-1}} \\ &\leq C \left\| u(t) \right\|_{\mathcal{H}_0^m}^2, \end{aligned}$$

and

$$\left\| \partial_\tau^{\tilde{\beta}+e_2} \partial_y^j u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{k-j-1} u \right\|_{L^2(\Omega)} \leq C \left\| u(t) \right\|_{\mathcal{H}_0^m} \left\| u(t) \right\|_{\mathcal{H}_0^m} \leq C \left\| u(t) \right\|_{\mathcal{H}_0^m}^2,$$

provided that  $|\beta| + k = |\alpha| = m$ . Substituting the above two inequalities into (3.23) gives

$$\begin{aligned} \left| \int_{\mathbb{T}_x} (D^\gamma (v \partial_y u) \cdot D^\alpha u) \Big|_{y=0} dx \right| &\leq C \sum_{\substack{\tilde{\beta} \leq \beta \\ 0 \leq j \leq k-2}} \left\{ \left\| u(t) \right\|_{\mathcal{H}_0^m}^2 \left\| D^\alpha u \right\|_{L^2(\Omega)} + \left\| u(t) \right\|_{\mathcal{H}_0^m}^2 \left\| \partial_y D^\alpha u \right\|_{L^2(\Omega)} \right\} \\ &\leq \frac{\mu}{14} \left\| D^\alpha \partial_y u(t) \right\|_{L^2(\Omega)}^2 + C \left\| u(t) \right\|_{\mathcal{H}_0^m}^4 + C \left\| u(t) \right\|_{\mathcal{H}_0^m}^2. \end{aligned} \quad (3.24)$$

Similarly, we can obtain

$$\begin{aligned} \left| \int_{\mathbb{T}_x} (D^\gamma (g \partial_y h) \cdot D^\alpha u) \Big|_{y=0} dx \right| &\leq C \left\| h(t) \right\|_{\mathcal{H}_0^m}^2 \left\| D^\alpha u \right\|_{L^2(\Omega)} + C \left\| h(t) \right\|_{\mathcal{H}_0^m}^2 \left\| \partial_y D^\alpha u \right\|_{L^2(\Omega)} \\ &\leq \frac{\mu}{14} \left\| D^\alpha \partial_y u(t) \right\|_{L^2(\Omega)}^2 + C \left\| (u, h)(t) \right\|_{\mathcal{H}_0^m}^4 + C \left\| (u, h)(t) \right\|_{\mathcal{H}_0^m}^2. \end{aligned} \quad (3.25)$$

Therefore, from (3.17) and combining the estimates (3.18), (3.19), (3.21), (3.22), (3.24) and (3.25), we have that when  $|\alpha| = |\beta| + k = m$  with  $|\beta| \leq m-1$ ,

$$\begin{aligned} \left| \int_{\mathbb{T}_x} (\mu \partial_y D^\alpha u \cdot D^\alpha u) \Big|_{y=0} dx \right| &\leq \delta_1 \left\| (\partial_y u, \partial_y h)(t) \right\|_{\mathcal{H}_0^m}^2 + \frac{3\mu}{7} \left\| D^\alpha \partial_y u(t) \right\|_{L^2(\Omega)}^2 + C \delta_1^{-1} \left\| (u, h)(t) \right\|_{\mathcal{H}_0^m}^4 \\ &\quad + C \delta_1^{-1} \left\| (u, h)(t) \right\|_{\mathcal{H}_0^m}^2 + C \left\| D^\gamma P_x(t) \right\|_{L^2(\mathbb{T}_x)}^2. \end{aligned} \quad (3.26)$$

Combining (3.16) with (3.26), it implies that for  $|\alpha| = |\beta| + k \leq m, |\beta| \leq m-1$ ,

$$\begin{aligned} &\left| \int_{\mathbb{T}_x} (\mu \partial_y D^\alpha u \cdot D^\alpha u) \Big|_{y=0} dx \right| \\ &\leq \delta_1 \left\| (\partial_y u, \partial_y h)(t) \right\|_{\mathcal{H}_0^m}^2 + \frac{3\mu}{7} \left\| D^\alpha \partial_y u(t) \right\|_{L^2(\Omega)}^2 + C \delta_1^{-1} \left\| (u, h)(t) \right\|_{\mathcal{H}_0^m}^2 (1 + \left\| (u, h)(t) \right\|_{\mathcal{H}_0^m}^2) \\ &\quad + C \sum_{|\beta| \leq m-1} \left\| \partial_\tau^\beta P_x(t) \right\|_{L^2(\mathbb{T}_x)}^2. \end{aligned} \quad (3.27)$$

Then, plugging the above estimate (3.27) into (3.15) we have

$$\begin{aligned}
& \mu \int_{\Omega} (\partial_y^2 D^\alpha u \cdot \langle y \rangle^{2l+2k} D^\alpha u) dx dy \\
& \leq -\frac{\mu}{2} \|D^\alpha \partial_y u(t)\|_{L^2_{l+k}(\Omega)}^2 + \delta_1 \|(\partial_y u, \partial_y h)(t)\|_{\mathcal{H}_0^m}^2 + C\delta_1^{-1} \|(u, h)(t)\|_{\mathcal{H}_0^m}^2 (1 + \|(u, h)(t)\|_{\mathcal{H}_0^m}^2) \\
& \quad + C \sum_{|\beta| \leq m-1} \|\partial_\tau^\beta P_x(t)\|_{L^2(\mathbb{T}_x)}^2.
\end{aligned} \tag{3.28}$$

On the other hand, one can get the similar estimation on the term  $\kappa \int_{\Omega} (\partial_y^2 D^\alpha h \cdot \langle y \rangle^{2l+2k} D^\alpha h) dx dy$ :

$$\begin{aligned}
\kappa \int_{\Omega} (\partial_y^2 D^\alpha h \cdot \langle y \rangle^{2l+2k} D^\alpha h) dx dy & \leq -\frac{\kappa}{2} \|D^\alpha \partial_y h(t)\|_{L^2_{l+k}(\Omega)}^2 + \delta_1 \|(\partial_y u, \partial_y h)(t)\|_{\mathcal{H}_0^m}^2 \\
& \quad + C\delta_1^{-1} \|(u, h)(t)\|_{\mathcal{H}_0^m}^2 (1 + \|(u, h)(t)\|_{\mathcal{H}_0^m}^2).
\end{aligned} \tag{3.29}$$

Thus, we prove (3.10) by combining (3.28) with (3.29).

**Proof of (3.11).** From the definition (3.8) of  $I_1$  and  $I_2$ , we have

$$\begin{aligned}
I_1 & = [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]D^\alpha u - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]D^\alpha h \\
& \quad + [D^\alpha, (u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u - [D^\alpha, (h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h \\
& \quad + D^\alpha [U_x\phi'u + U\phi''v - H_x\phi'h - H\phi''g] \\
& \triangleq I_1^1 + I_1^2 + I_1^3,
\end{aligned}$$

and

$$\begin{aligned}
I_2 & = [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]D^\alpha h - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]D^\alpha u \\
& \quad + [D^\alpha, (u + U\phi')\partial_x + (v - U_x\phi)\partial_y]h - [D^\alpha, (h + H\phi')\partial_x + (g - H_x\phi)\partial_y]u \\
& \quad + D^\alpha [H_x\phi'u + H\phi''v - U_x\phi'h - U\phi''g] \\
& \triangleq I_2^1 + I_2^2 + I_2^3.
\end{aligned}$$

Thus, we divide the term  $-\int_{\Omega} (I_1 \cdot \langle y \rangle^{2l+2k} D^\alpha u + I_2 \cdot \langle y \rangle^{2l+2k} D^\alpha h) dx dy$  into three parts:

$$\begin{aligned}
& -\int_{\Omega} (I_1 \cdot \langle y \rangle^{2l+2k} D^\alpha u + I_2 \cdot \langle y \rangle^{2l+2k} D^\alpha h) dx dy \\
& = -\sum_{i=1}^3 \int_{\Omega} (I_1^i \cdot \langle y \rangle^{2l+2k} D^\alpha u + I_2^i \cdot \langle y \rangle^{2l+2k} D^\alpha h) dx dy \\
& := G_1 + G_2 + G_3,
\end{aligned} \tag{3.30}$$

and estimate each  $G_i, i = 1, 2, 3$  in the following. Firstly, note that

$$\phi(y) \equiv y, \quad \phi'(y) \equiv 1, \quad \phi^{(i)}(y) \equiv 0, \quad \text{for } y \geq 2R_0, \quad i \geq 2,$$

and then, there exists some positive constant  $C$  such that

$$\|\langle y \rangle^{i-1} \phi^{(i)}(y)\|_{L^\infty(\mathbb{R}_+)}, \|\langle y \rangle^\lambda \phi^{(j)}(y)\|_{L^\infty(\mathbb{R}_+)} \leq C, \quad \text{for } i = 0, 1, j \geq 2, \lambda \in \mathbb{R}, \tag{3.31}$$

**Estimate for  $G_1$ :** Note that

$$\partial_x(u + U\phi') + \partial_y(v - U_x\phi) = 0, \quad \partial_x(h + H\phi') + \partial_y(g - H_x\phi) = 0,$$

and the boundary conditions  $(v - U_x\phi)|_{y=0} = (g - H_x\phi)|_{y=0} = 0$ , we obtain that by integration by parts,

$$\begin{aligned}
G_1 & = -\frac{1}{2} \int_{\Omega} \left\{ \langle y \rangle^{2l+2k} [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y] (|D^\alpha u|^2 + |D^\alpha h|^2) \right\} dx dy \\
& \quad + \int_{\Omega} \left\{ \langle y \rangle^{2l+2k} [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y] (D^\alpha u \cdot D^\alpha h) \right\} dx dy \\
& = (l+k) \int_{\Omega} \left\{ \langle y \rangle^{2l+2k-1} (v - U_x\phi) \cdot (|D^\alpha u|^2 + |D^\alpha h|^2) \right\} dx dy
\end{aligned}$$

$$-2(l+k) \int_{\Omega} \left\{ \langle y \rangle^{2l+2k-1} (g - H_x \phi) \cdot (D^\alpha u \cdot D^\alpha h) \right\} dx dy.$$

Then, by using that  $v = -\partial_y^{-1} \partial_x u$ ,  $g = -\partial_y^{-1} \partial_x h$  and (3.31) for  $i = 0$ , we get that by virtue of (2.5) and Sobolev embedding inequality,

$$\begin{aligned} G_1 &\leq (l+k) \left( \left\| \frac{v - U_x \phi}{1+y} \right\|_{L^\infty(\Omega)} + \left\| \frac{g - H_x \phi}{1+y} \right\|_{L^\infty(\Omega)} \right) \cdot \|\langle y \rangle^{l+k} D^\alpha(u, h)(t)\|_{L^2(\Omega)}^2 \\ &\leq C(\|u_x(t)\|_{L^\infty(\Omega)} + \|h_x(t)\|_{L^\infty(\Omega)} + \|(U_x, H_x)(t)\|_{L^\infty(\mathbb{T}_x)}) \cdot \|\langle y \rangle^{l+k} D^\alpha(u, h)(t)\|_{L^2(\Omega)}^2 \\ &\leq C(\|(u, h)(t)\|_{\mathcal{H}_0^3} + \|(U_x, H_x)(t)\|_{L^\infty(\mathbb{T}_x)}) \|(u, h)(t)\|_{\mathcal{H}_1^m}^2. \end{aligned} \quad (3.32)$$

**Estimate for  $G_2$ :** For  $G_2$ , note that

$$G_2 \leq \|I_1^2(t)\|_{L_{l+k}^2(\Omega)} \|D^\alpha u(t)\|_{L_{l+k}^2(\Omega)} + \|I_2^2(t)\|_{L_{l+k}^2(\Omega)} \|D^\alpha h(t)\|_{L_{l+k}^2(\Omega)}. \quad (3.33)$$

Thus, we need to obtain  $\|I_1^2(t)\|_{L_{l+k}^2(\Omega)}$  and  $\|I_2^2(t)\|_{L_{l+k}^2(\Omega)}$ . To this end, we are going to estimate only the  $L_{l+k}^2$  of  $I_1^2$ , because the  $L_{l+k}^2$ -estimate on  $I_2^2$  can be obtained similarly.

Rewrite the quantity  $I_1^2$  as:

$$\begin{aligned} I_1^2 &= [D^\alpha, u \partial_x + v \partial_y] u - [D^\alpha, h \partial_x + g \partial_y] h \\ &\quad + [D^\alpha, U \phi' \partial_x - U_x \phi \partial_y] u - [D^\alpha, H \phi' \partial_x - H_x \phi \partial_y] h \\ &:= I_{1,1}^2 + I_{1,2}^2. \end{aligned} \quad (3.34)$$

In the following, we will estimate  $\|I_{1,1}^2\|_{L_{l+k}^2(\Omega)}$  and  $\|I_{1,2}^2\|_{L_{l+k}^2(\Omega)}$  respectively.

$L_{l+k}^2$ -estimate on  $I_{1,1}^2$ : The quantity  $I_{1,1}^2$  can be expressed as:

$$I_{1,1}^2 = \sum_{0 < \tilde{\alpha} \leq \alpha} \binom{\alpha}{\tilde{\alpha}} \left\{ \left( D^{\tilde{\alpha}} u \partial_x + D^{\tilde{\alpha}} v \partial_y \right) (D^{\alpha - \tilde{\alpha}} u) - \left( D^{\tilde{\alpha}} h \partial_x + D^{\tilde{\alpha}} g \partial_y \right) (D^{\alpha - \tilde{\alpha}} h) \right\}. \quad (3.35)$$

Let  $\tilde{\alpha} \triangleq (\tilde{\beta}, \tilde{k})$ , then we will study the terms in (3.35) through the following two cases corresponding to  $\tilde{k} = 0$  and  $\tilde{k} \geq 1$  respectively.

*Case 1:*  $\tilde{k} = 0$ . Firstly,  $D^{\tilde{\alpha}} = \partial_\tau^{\tilde{\beta}}$  and  $\tilde{\beta} \geq e_i, i = 1$  or  $2$  since  $|\tilde{\alpha}| > 0$ . Then, we obtain that by (2.4),

$$\begin{aligned} \|D^{\tilde{\alpha}} u \cdot \partial_x D^{\alpha - \tilde{\alpha}} u\|_{L_{l+k}^2(\Omega)} &= \|\partial_\tau^{\tilde{\beta} - e_i} (\partial_\tau^{e_i} u) \cdot D^{\alpha - \tilde{\alpha}} (\partial_x u)\|_{L_{l+k}^2(\Omega)} \\ &\leq C \|\partial_\tau^{e_i} u(t)\|_{\mathcal{H}_0^{m-1}} \|\partial_x u(t)\|_{\mathcal{H}_1^{m-1}} \leq C \|u(t)\|_{\mathcal{H}_1^m}^2, \end{aligned}$$

provided that  $m - 1 \geq 3$ . Similarly, it also holds

$$\|D^{\tilde{\alpha}} h \cdot \partial_x D^{\alpha - \tilde{\alpha}} h\|_{L_{l+k}^2(\Omega)} \leq C \|h(t)\|_{\mathcal{H}_1^m}^2.$$

On the other hand, by using  $v = -\partial_y^{-1} \partial_x u$ , we have

$$D^{\tilde{\alpha}} v \cdot \partial_y D^{\alpha - \tilde{\alpha}} u = -\partial_\tau^{\tilde{\beta}} \partial_y^{-1} (\partial_x u) \cdot \partial_\tau^{\beta - \tilde{\beta}} \partial_y^{k+1} u.$$

Then, when  $|\alpha| = |\beta| + k \leq m - 1$ , applying (2.7) to the right-hand side of the above equality yields

$$\begin{aligned} \|D^{\tilde{\alpha}} v \cdot \partial_y D^{\alpha - \tilde{\alpha}} u\|_{L_{l+k}^2(\Omega)} &= \|\partial_\tau^{\tilde{\beta}} \partial_y^{-1} (\partial_x u) \cdot \partial_\tau^{\beta - \tilde{\beta}} \partial_y^k (\partial_y u)\|_{L_{l+k}^2(\Omega)} \\ &\leq C \|\partial_x u(t)\|_{\mathcal{H}_0^{m-1}} \|\partial_y u(t)\|_{\mathcal{H}_{l+1}^{m-1}} \leq C \|u(t)\|_{\mathcal{H}_1^m}^2, \end{aligned}$$

provided that  $m - 1 \geq 3$ . When  $|\alpha| = |\beta| + k = m$ , it implies that  $k \geq 1$  since  $|\beta| \leq m - 1$ , and consequently, we get that by (2.7),

$$\begin{aligned} \|D^{\tilde{\alpha}} v \cdot \partial_y D^{\alpha - \tilde{\alpha}} u\|_{L_{l+k}^2(\Omega)} &= \|\partial_\tau^{\tilde{\beta} - e_i} \partial_y^{-1} (\partial_\tau^{e_i + e_2} u) \cdot \partial_\tau^{\beta - \tilde{\beta}} \partial_y^{k-1} (\partial_y^2 u)\|_{L_{l+1+(k-1)}^2(\Omega)} \\ &\leq C \|\partial_\tau^{e_i + e_2} u(t)\|_{\mathcal{H}_0^{m-2}} \|\partial_y^2 u(t)\|_{\mathcal{H}_{l+2}^{m-2}} \leq C \|u(t)\|_{\mathcal{H}_1^m}^2, \end{aligned}$$

provided that  $m - 2 \geq 3$ . Therefore, it holds that for  $|\alpha| = |\beta| + k \leq m, |\beta| \leq m - 1$ ,

$$\|D^{\tilde{\alpha}} v \cdot \partial_y D^{\alpha - \tilde{\alpha}} u\|_{L_{l+k}^2(\Omega)} \leq C \|u(t)\|_{\mathcal{H}_1^m}^2.$$

Similarly, one can obtain

$$\|D^{\tilde{\alpha}}g \cdot \partial_y D^{\alpha-\tilde{\alpha}}h\|_{L^2_{l+k}(\Omega)} \leq C\|h(t)\|_{\mathcal{H}_l^m}^2.$$

Thus, we conclude that for  $\tilde{k} = 0$  with  $\tilde{\alpha} = (\tilde{\beta}, \tilde{k})$ ,

$$\|(D^{\tilde{\alpha}}u \partial_x + D^{\tilde{\alpha}}v \partial_y)(D^{\alpha-\tilde{\alpha}}u) - (D^{\tilde{\alpha}}h \partial_x + D^{\tilde{\alpha}}g \partial_y)(D^{\alpha-\tilde{\alpha}}h)\|_{L^2_{l+k}(\Omega)} \leq C\|(u, h)(t)\|_{\mathcal{H}_l^m}^2. \quad (3.36)$$

*Case 2:*  $\tilde{k} \geq 1$ . It follows that  $\tilde{\alpha} \geq E_3$ , and then, the right-hand side of (3.35) becomes:

$$\begin{aligned} & (D^{\tilde{\alpha}}u \partial_x + D^{\tilde{\alpha}}v \partial_y)(D^{\alpha-\tilde{\alpha}}u) - (D^{\tilde{\alpha}}h \partial_x + D^{\tilde{\alpha}}g \partial_y)(D^{\alpha-\tilde{\alpha}}h) \\ &= (D^{\tilde{\alpha}}u \partial_x - D^{\tilde{\alpha}-E_3}(\partial_x u) \partial_y)(D^{\alpha-\tilde{\alpha}}u) - (D^{\tilde{\alpha}}h \partial_x - D^{\tilde{\alpha}-E_3}(\partial_x h) \partial_y)(D^{\alpha-\tilde{\alpha}}h). \end{aligned}$$

By applying (2.4) to the terms on the right-hand side of the above equality, we get

$$\begin{aligned} \|D^{\tilde{\alpha}}u \cdot \partial_x D^{\alpha-\tilde{\alpha}}u\|_{L^2_{l+k}(\Omega)} &= \|D^{\tilde{\alpha}-E_3}(\partial_y u) \cdot D^{\alpha-\tilde{\alpha}}(\partial_x u)\|_{L^2_{l+1+(k-1)}(\Omega)} \\ &\leq C\|\partial_y u(t)\|_{\mathcal{H}_{l+1}^{m-1}} \|\partial_x u(t)\|_{\mathcal{H}_0^{m-1}} \leq C\|u(t)\|_{\mathcal{H}_l^m}^2, \\ \|D^{\tilde{\alpha}-E_3}(\partial_x u) \cdot \partial_y D^{\alpha-\tilde{\alpha}}u\|_{L^2_{l+k}(\Omega)} &= \|D^{\tilde{\alpha}-E_3}(\partial_x u) \cdot D^{\alpha-\tilde{\alpha}}(\partial_y u)\|_{L^2_{l+1+(k-1)}(\Omega)} \\ &\leq C\|\partial_x u(t)\|_{\mathcal{H}_0^{m-1}} \|\partial_y u(t)\|_{\mathcal{H}_{l+1}^{m-1}} \leq C\|u(t)\|_{\mathcal{H}_l^m}^2, \\ \|D^{\tilde{\alpha}}h \cdot \partial_x D^{\alpha-\tilde{\alpha}}h\|_{L^2_{l+k}(\Omega)} &= \|D^{\tilde{\alpha}-E_3}(\partial_y h) \cdot D^{\alpha-\tilde{\alpha}}(\partial_x h)\|_{L^2_{l+1+(k-1)}(\Omega)} \\ &\leq C\|\partial_y h(t)\|_{\mathcal{H}_{l+1}^{m-1}} \|\partial_x h(t)\|_{\mathcal{H}_0^{m-1}} \leq C\|h(t)\|_{\mathcal{H}_l^m}^2, \\ \|D^{\tilde{\alpha}-E_3}(\partial_x h) \cdot \partial_y D^{\alpha-\tilde{\alpha}}h\|_{L^2_{l+k}(\Omega)} &= \|D^{\tilde{\alpha}-E_3}(\partial_x h) \cdot D^{\alpha-\tilde{\alpha}}(\partial_y h)\|_{L^2_{l+1+(k-1)}(\Omega)} \\ &\leq C\|\partial_x h(t)\|_{\mathcal{H}_0^{m-1}} \|\partial_y h(t)\|_{\mathcal{H}_{l+1}^{m-1}} \leq C\|h(t)\|_{\mathcal{H}_l^m}^2. \end{aligned}$$

Consequently, we actually conclude that for  $\tilde{k} \geq 1$  with  $\tilde{\alpha} = (\tilde{\beta}, \tilde{k})$ ,

$$\|(D^{\tilde{\alpha}}u \partial_x + D^{\tilde{\alpha}}v \partial_y)(D^{\alpha-\tilde{\alpha}}u) - (D^{\tilde{\alpha}}h \partial_x + D^{\tilde{\alpha}}g \partial_y)(D^{\alpha-\tilde{\alpha}}h)\|_{L^2_{l+k}(\Omega)} \leq C\|(u, h)(t)\|_{\mathcal{H}_l^m}^2. \quad (3.37)$$

Finally, based on the results obtained in the above two cases, it holds that by using (3.36) and (3.37) in (3.35),

$$\|I_{1,1}^2(t)\|_{L^2_{l+k}(\Omega)} \leq C\|(u, h)(t)\|_{\mathcal{H}_l^m}^2. \quad (3.38)$$

$L^2_{l+k}$ -estimate on  $I_{1,2}^2$ : Write

$$I_{1,2}^2 = \sum_{0 < \tilde{\alpha} \leq \alpha} \binom{\alpha}{\tilde{\alpha}} \left\{ \left( D^{\tilde{\alpha}}(U\phi') \partial_x - D^{\tilde{\alpha}}(U_x\phi) \partial_y \right) (D^{\alpha-\tilde{\alpha}}u) - \left( D^{\tilde{\alpha}}(H\phi') \partial_x - D^{\tilde{\alpha}}(H_x\phi) \partial_y \right) (D^{\alpha-\tilde{\alpha}}h) \right\}.$$

Let  $\tilde{\alpha} \triangleq (\tilde{\beta}, \tilde{k})$  and note that  $|\alpha - \tilde{\alpha}| \leq |\alpha| - 1 \leq m - 1$ . By using (3.31), we estimate each term on the right hand side of the above equality as follows:

$$\begin{aligned} \|D^{\tilde{\alpha}}(U\phi') \cdot \partial_x D^{\alpha-\tilde{\alpha}}u\|_{L^2_{l+k}(\Omega)} &\leq \|\langle y \rangle^{\tilde{k}} D^{\tilde{\alpha}}(U\phi')(t)\|_{L^\infty(\Omega)} \|\langle y \rangle^{l+k-\tilde{k}} \partial_x D^{\alpha-\tilde{\alpha}}u(t)\|_{L^2(\Omega)} \\ &\leq C\|\partial_\tau^{\tilde{\beta}}U(t)\|_{L^\infty(\mathbb{T}_x)} \|u(t)\|_{\mathcal{H}_l^m}, \\ \|D^{\tilde{\alpha}}(U_x\phi) \cdot \partial_y D^{\alpha-\tilde{\alpha}}u\|_{L^2_{l+k}(\Omega)} &\leq \|\langle y \rangle^{\tilde{k}-1} D^{\tilde{\alpha}}(U_x\phi)(t)\|_{L^\infty(\Omega)} \|\langle y \rangle^{l+k-\tilde{k}+1} \partial_y D^{\alpha-\tilde{\alpha}}u(t)\|_{L^2(\Omega)} \\ &\leq C\|\partial_\tau^{\tilde{\beta}}U_x(t)\|_{L^\infty(\mathbb{T}_x)} \|u(t)\|_{\mathcal{H}_l^m}, \end{aligned}$$

and similarly,

$$\begin{aligned} \|D^{\tilde{\alpha}}(H\phi') \cdot \partial_x D^{\alpha-\tilde{\alpha}}h\|_{L^2_{l+k}(\Omega)} &\leq C\|\partial_\tau^{\tilde{\beta}}H(t)\|_{L^\infty(\mathbb{T}_x)} \|h(t)\|_{\mathcal{H}_l^m}, \\ \|D^{\tilde{\alpha}}(H_x\phi) \cdot \partial_y D^{\alpha-\tilde{\alpha}}h\|_{L^2_{l+k}(\Omega)} &\leq C\|\partial_\tau^{\tilde{\beta}}H_x(t)\|_{L^\infty(\mathbb{T}_x)} \|h(t)\|_{\mathcal{H}_l^m}. \end{aligned}$$

Therefore, it follows

$$\|I_{1,2}^2(t)\|_{L_{i+k}^2(\Omega)} \leq C\|(u, h)(t)\|_{\mathcal{H}_t^m} \cdot \left( \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta(U, H)(t)\|_{L^\infty(\mathbb{T}_x)} \right). \quad (3.39)$$

Now, we can obtain the estimate of  $\|I_1^2\|_{L_{i+k}^2(\Omega)}$ . Indeed, plugging (3.38) and (3.39) into (3.34) yields

$$\|I_1^2(t)\|_{L_{i+k}^2(\Omega)} \leq C \left( \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta(U, H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_t^m} \right) \|(u, h)(t)\|_{\mathcal{H}_t^m}. \quad (3.40)$$

Similarly, one can also get

$$\|I_2^2(t)\|_{L_{i+k}^2(\Omega)} \leq C \left( \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta(U, H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_t^m} \right) \|(u, h)(t)\|_{\mathcal{H}_t^m}, \quad (3.41)$$

then, substituting (3.40) and (3.41) into (3.33) gives

$$\begin{aligned} G_2 &\leq C \left( \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta(U, H)(x)\|_{L^\infty(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_t^m} \right) \|(u, h)(t)\|_{\mathcal{H}_t^m} \|D^\alpha(u, h)(t)\|_{L_{i+k}^2(\Omega)} \\ &\leq C \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_t^m} \right) \|(u, h)(t)\|_{\mathcal{H}_t^m}^2. \end{aligned} \quad (3.42)$$

**Estimate on  $G_3$ :** For  $G_3$ , the Cauchy-Schwarz inequality implies

$$G_3 \leq \|I_1^3(t)\|_{L_{i+k}^2(\Omega)} \|D^\alpha u(t)\|_{L_{i+k}^2(\Omega)} + \|I_2^3(t)\|_{L_{i+k}^2(\Omega)} \|D^\alpha h(t)\|_{L_{i+k}^2(\Omega)}. \quad (3.43)$$

Then, it remains to estimate  $\|I_1^3(t)\|_{L_{i+k}^2(\Omega)}$  and  $\|I_2^3(t)\|_{L_{i+k}^2(\Omega)}$ . In the following, we are going to establish the weighted estimate on  $I_1^3$ , for example, and the weighed estimate on  $I_2^3$  can be obtained in a similar way.

Recall that  $D^\alpha = \partial_\tau^\beta \partial_y^k$ , we have

$$I_1^3 = \sum_{\tilde{\alpha} \leq \alpha} \binom{\alpha}{\tilde{\alpha}} \left[ D^{\tilde{\alpha}} u \cdot D^{\alpha-\tilde{\alpha}}(U_x \phi') + D^{\tilde{\alpha}} v \cdot D^{\alpha-\tilde{\alpha}}(U \phi'') - D^{\tilde{\alpha}} h \cdot D^{\alpha-\tilde{\alpha}}(H_x \phi') - D^{\tilde{\alpha}} g \cdot D^{\alpha-\tilde{\alpha}}(H \phi'') \right]. \quad (3.44)$$

Then, let  $\tilde{\alpha} \triangleq (\tilde{\beta}, \tilde{k})$ , and we estimate each term in (3.44) as follows. Firstly, by using (3.31) we have

$$\begin{aligned} \|D^{\tilde{\alpha}} u \cdot D^{\alpha-\tilde{\alpha}}(U_x \phi')\|_{L_{i+k}^2(\Omega)} &\leq \|\langle y \rangle^{l+\tilde{k}} D^{\tilde{\alpha}} u(t)\|_{L^2(\Omega)} \|\langle y \rangle^{k-\tilde{k}} D^{\alpha-\tilde{\alpha}}(U_x \phi')(t)\|_{L^\infty(\Omega)} \\ &\leq C \|u(t)\|_{\mathcal{H}_t^m} \|\partial_\tau^{\beta-\tilde{\beta}} U_x(t)\|_{L^\infty(\mathbb{T}_x)}, \end{aligned}$$

and similarly,

$$\|D^{\tilde{\alpha}} h \cdot D^{\alpha-\tilde{\alpha}}(H_x \phi')\|_{L_{i+k}^2(\Omega)} \leq C \|h(t)\|_{\mathcal{H}_t^m} \|\partial_\tau^{\beta-\tilde{\beta}} H_x(t)\|_{L^\infty(\mathbb{T}_x)}.$$

Secondly, as  $v = -\partial_y^{-1} \partial_x u$ , it reads

$$D^{\tilde{\alpha}} v \cdot D^{\alpha-\tilde{\alpha}}(U \phi'') = -D^{\tilde{\alpha}+E_2} \partial_y^{-1} u \cdot D^{\alpha-\tilde{\alpha}}(U \phi'').$$

Therefore, if  $\tilde{k} \geq 1$ , it follows that by (3.31),

$$\begin{aligned} \|D^{\tilde{\alpha}} v \cdot D^{\alpha-\tilde{\alpha}}(U \phi'')\|_{L_{i+k}^2(\Omega)} &= \|\partial_\tau^{\tilde{\beta}+e_2} \partial_y^{\tilde{k}-1} u \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{k-\tilde{k}}(U \phi'')\|_{L_{i+k}^2(\Omega)} \\ &\leq \|\langle y \rangle^{\tilde{k}-1} \partial_\tau^{\tilde{\beta}+e_2} \partial_y^{\tilde{k}-1} u(t)\|_{L^2(\Omega)} \|\langle y \rangle^{l+k-\tilde{k}+1} \partial_\tau^{\beta-\tilde{\beta}} \partial_y^{k-\tilde{k}}(U \phi'')(t)\|_{L^\infty(\Omega)} \\ &\leq C \|u(t)\|_{\mathcal{H}_0^m} \|\partial_\tau^{\beta-\tilde{\beta}} U(t)\|_{L^\infty(\mathbb{T}_x)}; \end{aligned}$$

if  $\tilde{k} = 0$ , we obtain that by (2.7) and (3.31),

$$\begin{aligned} \|D^{\tilde{\alpha}} v \cdot D^{\alpha-\tilde{\alpha}}(U \phi'')\|_{L_{i+k}^2(\Omega)} &= \left\| \frac{\partial_\tau^{\tilde{\beta}+e_2} \partial_y^{-1} u}{1+y} \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y^k(U \phi'') \right\|_{L_{i+k+1}^2(\Omega)} \\ &\leq \left\| \frac{\partial_\tau^{\tilde{\beta}+e_2} \partial_y^{-1} u(t)}{1+y} \right\|_{L^2(\Omega)} \|\langle y \rangle^{l+k+1} \partial_\tau^{\beta-\tilde{\beta}} \partial_y^k(U \phi'')(t)\|_{L^\infty(\Omega)} \\ &\leq C \|\partial_\tau^{\tilde{\beta}+e_2} u(t)\|_{L^2(\Omega)} \|\partial_\tau^{\beta-\tilde{\beta}} U(t)\|_{L^\infty(\mathbb{T}_x)} \end{aligned}$$

$$\leq C \|u(t)\|_{\mathcal{H}_0^m} \|\partial_\tau^{\beta-\tilde{\beta}} U(t)\|_{L^\infty(\mathbb{T}_x)},$$

provided that  $|\tilde{\beta}| \leq |\beta| \leq m-1$ . Combining the above two inequalities yields that

$$\|D^{\tilde{\alpha}} v \cdot D^{\alpha-\tilde{\alpha}}(U\phi'')\|_{L^2_{l+k}(\Omega)} \leq C \|u(t)\|_{\mathcal{H}_0^m} \cdot \left( \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta(U, H)(t)\|_{L^\infty(\mathbb{T}_x)} \right).$$

Similarly, we have

$$\|D^{\tilde{\alpha}} g \cdot D^{\alpha-\tilde{\alpha}}(H\phi'')\|_{L^2_{l+k}(\Omega)} \leq C \|h(t)\|_{\mathcal{H}_0^m} \cdot \left( \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta(U, H)(t)\|_{L^\infty(\mathbb{T}_x)} \right).$$

We take into account the above arguments, to conclude that

$$\|I_1^3(t)\|_{L^2_{l+k}(\Omega)} \leq C \|(u, h)(t)\|_{\mathcal{H}_l^m} \cdot \left( \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta(U, H)(t)\|_{L^\infty(\mathbb{T}_x)} \right). \quad (3.45)$$

Then, one can obtain a similar estimate of  $I_2^3$ :

$$\|I_2^3(t)\|_{L^2_{l+k}(\Omega)} \leq C \|(u, h)(t)\|_{\mathcal{H}_l^m} \cdot \left( \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta(U, H)(t)\|_{L^\infty(\mathbb{T}_x)} \right), \quad (3.46)$$

which implies that by plugging (3.45) and (3.46) into (3.43),

$$\begin{aligned} G_3 &\leq C \|D^\alpha(u, h)(t)\|_{L^2_{l+k}(\Omega)} \|(u, h)(t)\|_{\mathcal{H}_l^m} \cdot \left( \sum_{|\beta| \leq m+1} \|\partial_\tau^\beta(U, H)(x)\|_{L^\infty(\mathbb{T}_x)} \right) \\ &\leq C \|(u, h)(t)\|_{\mathcal{H}_l^m}^2 \cdot \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H)(t)\|_{L^2(\mathbb{T}_x)} \right). \end{aligned} \quad (3.47)$$

Now, as we have completed the estimates on  $G_i$ ,  $i = 1, 2, 3$  given by (3.32), (3.42) and (3.47) respectively, from (3.30) the conclusion of this step follows immediately:

$$\begin{aligned} & - \int_{\Omega} \left( I_1 \cdot \langle y \rangle^{2l+2k} D^\alpha u + I_2 \cdot \langle y \rangle^{2l+2k} D^\alpha h \right) dx dy \\ & \leq C \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_l^m} \right) \|(u, h)(t)\|_{\mathcal{H}_l^m}^2, \end{aligned}$$

and we complete the proof of (3.11).  $\square$

### 3.2. Weighted $H_l^m$ -estimates only in tangential variables.

Similar to the classical Prandtl equations, an essential difficulty for solving the problem (2.12) is the loss of one derivative in the tangential variable  $x$  in the terms  $v\partial_y u - g\partial_y h$  and  $v\partial_y h - g\partial_y u$ . In other words,  $v = -\partial_y^{-1}\partial_x u$  and  $g = -\partial_y^{-1}\partial_x h$ , by the divergence free conditions, create a loss of  $x$ -derivative that prevents us to apply the standard energy estimates. Precisely, consider the following equations of  $\partial_\tau^\beta(u, h)$  with  $|\beta| = m$ , by taking the  $m$ -th order tangential derivatives on the first two equations of (2.12)

$$\begin{cases} \partial_t \partial_\tau^\beta u + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y] \partial_\tau^\beta u - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y] \partial_\tau^\beta h - \mu \partial_y^2 \partial_\tau^\beta u \\ \quad + (\partial_y u + U\phi'') \partial_\tau^\beta v - (\partial_y h + H\phi'') \partial_\tau^\beta g = \partial_\tau^\beta r_1 + R_u^\beta, \\ \partial_t \partial_\tau^\beta h + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y] \partial_\tau^\beta h - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y] \partial_\tau^\beta u - \kappa \partial_y^2 \partial_\tau^\beta h \\ \quad + (\partial_y h + H\phi'') \partial_\tau^\beta v - (\partial_y u + U\phi'') \partial_\tau^\beta g = \partial_\tau^\beta r_2 + R_h^\beta, \end{cases} \quad (3.48)$$

where

$$\begin{cases} R_u^\beta = \partial_\tau^\beta (-U_x\phi'u + H_x\phi'h) - [\partial_\tau^\beta, U\phi'']v + [\partial_\tau^\beta, H\phi'']g - [\partial_\tau^\beta, (u + U\phi')\partial_x - U_x\phi\partial_y]u \\ \quad + [\partial_\tau^\beta, (h + H\phi')\partial_x - H_x\phi\partial_y]h - \sum_{0 < \tilde{\beta} < \beta} \binom{\beta}{\tilde{\beta}} \left( \partial_\tau^{\tilde{\beta}} v \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y u - \partial_\tau^{\tilde{\beta}} g \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y h \right), \\ R_h^\beta = \partial_\tau^\beta (-H_x\phi'u + U_x\phi'h) - [\partial_\tau^\beta, H\phi'']v + [\partial_\tau^\beta, U\phi'']g - [\partial_\tau^\beta, (u + U\phi')\partial_x - U_x\phi\partial_y]h \\ \quad + [\partial_\tau^\beta, (h + H\phi')\partial_x - H_x\phi\partial_y]u - \sum_{0 < \tilde{\beta} < \beta} \binom{\beta}{\tilde{\beta}} \left( \partial_\tau^{\tilde{\beta}} v \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y h - \partial_\tau^{\tilde{\beta}} g \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y u \right). \end{cases} \quad (3.49)$$



From the expression (3.49) and by using the inequalities (2.4)-(2.7), we can control the  $L_t^2(\Omega)$ -estimates of each term given in (3.49), and then obtain the estimates of  $\|R_u^\beta(t)\|_{L_t^2(\Omega)}$  and  $\|R_h^\beta(t)\|_{L_t^2(\Omega)}$ . For example, for  $\tilde{\beta} > 0$ , which implies that  $\tilde{\beta} \geq e_i, i = 1$  or  $2$ , by virtue of (2.4),

$$\begin{aligned} & \left\| \left[ \partial_\tau^{\tilde{\beta}}(u + U\phi')\partial_x - \partial_\tau^{\tilde{\beta}}(U_x\phi)\partial_y \right] (\partial_\tau^{\beta-\tilde{\beta}}u) \right\|_{L_t^2(\Omega)} \\ & \leq \left\| \left[ \partial_\tau^{\tilde{\beta}-e_i}(\partial_\tau^{e_i}u) \cdot \partial_\tau^{\beta-\tilde{\beta}}(\partial_x u) \right] \right\|_{L_t^2(\Omega)} + \left\| \partial_\tau^{\tilde{\beta}}(U\phi')(t) \right\|_{L^\infty(\Omega)} \left\| \partial_x \partial_\tau^{\beta-\tilde{\beta}}u(t) \right\|_{L_t^2(\Omega)} \\ & \quad + \left\| \frac{\partial_\tau^{\tilde{\beta}}(U_x\phi)(t)}{1+y} \right\|_{L^\infty(\Omega)} \left\| \partial_y \partial_\tau^{\beta-\tilde{\beta}}u(t) \right\|_{L_{i+1}^2(\Omega)} \\ & \leq C \left\| \partial_\tau^{e_i}u(t) \right\|_{\mathcal{H}_0^{m-1}} \left\| \partial_x u(t) \right\|_{\mathcal{H}_i^{m-1}} + C \left\| \partial_\tau^{\tilde{\beta}}(U, U_x)(t) \right\|_{L^\infty(\mathbb{T}_x)} \left\| u(t) \right\|_{\mathcal{H}_i^m} \\ & \leq C \left( \left\| \partial_\tau^{\tilde{\beta}}(U, U_x)(t) \right\|_{L^\infty(\mathbb{T}_x)} + \left\| u(t) \right\|_{\mathcal{H}_i^m} \right) \left\| u(t) \right\|_{\mathcal{H}_i^m}, \end{aligned}$$

provided  $m-1 \geq 3$  and  $|\beta - \tilde{\beta}| \leq m-1$ ; (2.7) gives that for  $\tilde{\beta} < \beta$

$$\begin{aligned} \left\| \partial_\tau^{\tilde{\beta}}v \cdot \partial_\tau^{\beta-\tilde{\beta}}(U\phi'') \right\|_{L_t^2(\Omega)} & \leq \left\| \frac{\partial_\tau^{\tilde{\beta}+e_2}\partial_y^{-1}u(t)}{1+y} \right\|_{L^2(\Omega)} \left\| \langle y \rangle^{l+1} \partial_\tau^{\beta-\tilde{\beta}}(U\phi'')(t) \right\|_{L^\infty(\Omega)} \\ & \leq C \left\| \partial_\tau^{\beta-\tilde{\beta}}U(t) \right\|_{L^\infty(\mathbb{T}_x)} \left\| u(t) \right\|_{\mathcal{H}_0^m}; \end{aligned}$$

moreover, for  $0 < \tilde{\beta} < \beta$  which implies that  $\tilde{\beta} \geq e_i, \beta - \tilde{\beta} \geq e_j, i, j = 1$  or  $2$ , (2.7) yields that

$$\begin{aligned} \left\| \partial_\tau^{\tilde{\beta}}v \cdot \partial_\tau^{\beta-\tilde{\beta}}(\partial_y u) \right\|_{L_t^2(\Omega)} & = \left\| \partial_\tau^{\tilde{\beta}-e_i}\partial_y^{-1}(\partial_\tau^{e_i+e_2}u) \cdot \partial_\tau^{\beta-\tilde{\beta}-e_j}(\partial_\tau^{e_j}\partial_y u) \right\|_{L_t^2(\Omega)} \\ & \leq C \left\| \partial_\tau^{e_i+e_2}u(t) \right\|_{\mathcal{H}_0^{m-2}} \left\| \partial_y \partial_\tau^{e_j}u(t) \right\|_{\mathcal{H}_{i+1}^{m-2}} \leq C \left\| u(t) \right\|_{\mathcal{H}_i^m}^2 \end{aligned}$$

provided  $m-2 \geq 3$ . The other terms in  $R_u^\beta$  and  $R_h^\beta$  can be estimated similarly so that

$$\|R_u^\beta(t)\|_{L_t^2(\Omega)}, \|R_h^\beta(t)\|_{L_t^2(\Omega)} \leq C \left( \sum_{|\beta| \leq m+2} \left\| \partial_\tau^\beta(U, H)(t) \right\|_{L^2(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_i^m} \right) \|(u, h)(t)\|_{\mathcal{H}_i^m}. \quad (3.50)$$

On the other hand, consider the equations (3.48), the main difficulty comes from the terms

$$(\partial_y u + U\phi'')\partial_\tau^\beta v - (\partial_y h + H\phi'')\partial_\tau^\beta g = -(\partial_y u + U\phi'') \cdot (\partial_y^{-1}\partial_\tau^{\beta+e_2}u) + (\partial_y h + H\phi'') \cdot (\partial_y^{-1}\partial_\tau^{\beta+e_2}h),$$

and

$$(\partial_y h + H\phi'')\partial_\tau^\beta v - (\partial_y u + U\phi'')\partial_\tau^\beta g = -(\partial_y h + H\phi'') \cdot (\partial_y^{-1}\partial_\tau^{\beta+e_2}u) + (\partial_y u + U\phi'') \cdot (\partial_y^{-1}\partial_\tau^{\beta+e_2}h),$$

that contain the  $m+1$ -th order tangential derivatives which can not be controlled by the standard energy method. To overcome this difficulty, we rely on the following two key observations. One is that from the equation (2.14),  $\partial_y^{-1}h$  satisfies the following equation (see also the equation (3.56) for  $\psi$ )

$$\partial_t(\partial_y^{-1}h) + (v - U_x\phi)(h + H\phi') - (g - H_x\phi)(u + U\phi') - \kappa\partial_y h = -H_t\phi + \kappa H\phi'',$$

or

$$\partial_t(\partial_y^{-1}h) + (h + H\phi')v + (u + U\phi')\partial_x(\partial_y^{-1}h) - U_x\phi h + H_x\phi u - \kappa\partial_y h = H_t\phi(\phi' - 1) + \kappa H\phi'',$$

by using  $g = -\partial_x\partial_y^{-1}h$  and the second relation of (1.6). This inspires us in the case of  $h + H\phi' > 0$ , to introduce the following two quantities

$$u_\beta := \partial_\tau^\beta u - \frac{\partial_y u + U\phi''}{h + H\phi'} \partial_\tau^\beta \partial_y^{-1}h, \quad h_\beta := \partial_\tau^\beta h - \frac{\partial_y h + H\phi''}{h + H\phi'} \partial_\tau^\beta \partial_y^{-1}h, \quad (3.51)$$

to eliminate the terms involving  $\partial_\tau^\beta v$ , then to avoid the loss of  $x$ -derivative on  $v$ . Note that the new quantities  $(u_\beta, h_\beta)$  are almost equivalent to  $\partial_\tau^\beta(u, h)$  in  $L_t^2$ -norm, that is,

$$\left\| \partial_\tau^\beta(u, h) \right\|_{L_t^2(\Omega)} \lesssim \left\| (u_\beta, h_\beta) \right\|_{L_t^2(\Omega)} \lesssim \left\| \partial_\tau^\beta(u, h) \right\|_{L_t^2(\Omega)}, \quad (3.52)$$

that will be proved at the end of this subsection.

Another observation is that by using the above two new unknowns  $(u_\beta, h_\beta)$  in (3.51), the regularity loss generated by  $g = -\partial_y^{-1}\partial_x h$ , can be cancelled by using the convection terms  $-(h+H\phi')\partial_x h$  and  $-(h+H\phi')\partial_x u$ , more precisely,

$$\begin{aligned} & -(h+H\phi')\partial_x\partial_\tau^\beta h - (\partial_y h + H\phi'')\partial_\tau^\beta g \\ &= -(h+H\phi')\partial_x\left(h_\beta + \frac{\partial_y h + H\phi''}{h+H\phi'}\partial_\tau^\beta\partial_y^{-1}h\right) + (\partial_y h + H\phi'') \cdot (\partial_y^{-1}\partial_\tau^{\beta+e_2}h) \\ &= -(h+H\phi')\partial_x h_\beta - (h+H\phi')\partial_x\left(\frac{\partial_y h + H\phi''}{h+H\phi'}\right) \cdot \partial_\tau^\beta\partial_y^{-1}h, \end{aligned}$$

and

$$\begin{aligned} & -(h+H\phi')\partial_x\partial_\tau^\beta u - (\partial_y u + U\phi'')\partial_\tau^\beta g \\ &= -(h+H\phi')\partial_x\left(u_\beta + \frac{\partial_y u + U\phi''}{h+H\phi'}\partial_\tau^\beta\partial_y^{-1}h\right) + (\partial_y u + U\phi'') \cdot (\partial_y^{-1}\partial_\tau^{\beta+e_2}h) \\ &= -(h+H\phi')\partial_x u_\beta - (h+H\phi')\partial_x\left(\frac{\partial_y u + U\phi''}{h+H\phi'}\right) \cdot \partial_\tau^\beta\partial_y^{-1}h. \end{aligned}$$

This cancellation mechanism reveals the stabilizing effect of the magnetic field on the boundary layer. Note that in the above expressions, the convection terms can be handled by the symmetric structure of the system. Based on the above discussion, we will carry out the estimation as follows. First of all, we always assume that there exists a positive constant  $\delta_0 \leq 1$ , such that

$$h(t, x, y) + H(t, x)\phi'(y) \geq \delta_0, \quad \text{for } (t, x, y) \in [0, T] \times \Omega. \quad (3.53)$$

Firstly, from the divergence free condition  $\partial_x h + \partial_y g = 0$ , there exists a stream function  $\psi$ , such that

$$h = \partial_y \psi, \quad g = -\partial_x \psi, \quad \psi|_{y=0} = 0. \quad (3.54)$$

Then, the equation (2.14) for  $h$  reads

$$\partial_t \partial_y \psi + \partial_y [(v - U_x \phi)(\partial_y \psi + H\phi') + (\partial_x \psi + H_x \phi)(u + U\phi')] - \kappa \partial_y^3 \psi = -H_t \phi' + \kappa H \phi^{(3)}. \quad (3.55)$$

By virtue of the boundary conditions:

$$\partial_t \psi|_{y=0} = \partial_x \psi|_{y=0} = \partial_y^2 \psi|_{y=0} = v|_{y=0} = 0,$$

and  $\phi(y) \equiv 0$  for  $y \in [0, R_0]$ , we integrate the equation (3.55) with respect to the variable  $y$  over  $[0, y]$ , to obtain

$$\partial_t \psi + [(u + U\phi')\partial_x + (v - U_x \phi)\partial_y]\psi + H_x \phi u + H\phi' v - \kappa \partial_y^2 \psi = r_3, \quad (3.56)$$

with

$$r_3 = H_t \phi(\phi' - 1) + \kappa H \phi^{(3)}. \quad (3.57)$$

Next, applying the  $m$ -th order tangential derivatives operator on (3.56) and by virtue of  $\partial_y \psi = h$ , it yields that

$$\partial_t \partial_\tau^\beta \psi + [(u + U\phi')\partial_x + (v - U_x \phi)\partial_y]\partial_\tau^\beta \psi + (h + H\phi')\partial_\tau^\beta v - \kappa \partial_y^2 \partial_\tau^\beta \psi = \partial_\tau^\beta r_3 + R_\psi^\beta, \quad (3.58)$$

where  $R_\psi^\beta$  is defined as follows:

$$R_\psi^\beta = -\partial_\tau^\beta (H_x \phi u) - [\partial_\tau^\beta, H\phi']v - [\partial_\tau^\beta, (u + U\phi')\partial_x - U_x \phi \partial_y]\psi - \sum_{0 < \tilde{\beta} < \beta} \binom{\beta}{\tilde{\beta}} (\partial_\tau^{\tilde{\beta}} v \cdot \partial_\tau^{\beta-\tilde{\beta}} \partial_y \psi). \quad (3.59)$$

By  $\psi = \partial_y^{-1}h$  and  $v = -\partial_x \partial_y^{-1}u$ , it gives

$$\begin{aligned} R_\psi^\beta &= -\partial_\tau^\beta (H_x \phi u) + [\partial_\tau^\beta, H\phi']\partial_x \partial_y^{-1}u - [\partial_\tau^\beta, (u + U\phi')]\partial_x \partial_y^{-1}h + [\partial_\tau^\beta, U_x \phi]h \\ &\quad + \sum_{0 < \tilde{\beta} < \beta} \binom{\beta}{\tilde{\beta}} (\partial_\tau^{\tilde{\beta}+e_2} \partial_y^{-1}u \cdot \partial_\tau^{\beta-\tilde{\beta}} h) \\ &= -\sum_{\tilde{\beta} \leq \beta} \binom{\beta}{\tilde{\beta}} [\partial_\tau^{\tilde{\beta}} (H_x \phi) \cdot \partial_\tau^{\beta-\tilde{\beta}} u] + \sum_{0 < \tilde{\beta} \leq \beta} \binom{\beta}{\tilde{\beta}} [\partial_\tau^{\tilde{\beta}} (H\phi') \cdot \partial_\tau^{\beta-\tilde{\beta}+e_2} \partial_y^{-1}u \end{aligned}$$

$$- \partial_\tau^{\tilde{\beta}}(u + U\phi') \cdot \partial_\tau^{\beta - \tilde{\beta} + e_2} \partial_y^{-1} h + \partial_\tau^{\tilde{\beta}}(U_x \phi) \cdot \partial_\tau^{\beta - \tilde{\beta}} h \Big] + \sum_{0 < \tilde{\beta} < \beta} \binom{\beta}{\tilde{\beta}} (\partial_\tau^{\tilde{\beta} + e_2} \partial_y^{-1} u \cdot \partial_\tau^{\beta - \tilde{\beta}} h),$$

and then, we can estimate  $\left\| \frac{R_\psi^\beta(t)}{1+y} \right\|_{L^2(\Omega)}$  from the above expression term by term. For example, it is easy to get that

$$\left\| \frac{\partial_\tau^{\tilde{\beta}}(H_x \phi) \cdot \partial_\tau^{\beta - \tilde{\beta}} u}{1+y} \right\|_{L^2(\Omega)} \leq \left\| \frac{\partial_\tau^{\tilde{\beta}}(H_x \phi)(t)}{1+y} \right\|_{L^\infty(\Omega)} \left\| \partial_\tau^{\beta - \tilde{\beta}} u(t) \right\|_{L^2(\Omega)} \leq C \|\partial_\tau^{\tilde{\beta}} H_x(t)\|_{L^\infty(\mathbb{T}_x)} \|u(t)\|_{\mathcal{H}_0^m},$$

and (2.7) implies that

$$\left\| \frac{\partial_\tau^{\tilde{\beta}}(H\phi') \cdot \partial_\tau^{\beta - \tilde{\beta} + e_2} \partial_y^{-1} u}{1+y} \right\|_{L^2(\Omega)} \leq \left\| \partial_\tau^{\tilde{\beta}}(H\phi')(t) \right\|_{L^\infty(\Omega)} \left\| \frac{\partial_\tau^{\beta - \tilde{\beta} + e_2} \partial_y^{-1} u(t)}{1+y} \right\|_{L^2(\Omega)} \leq C \|\partial_\tau^{\tilde{\beta}} H(t)\|_{L^\infty(\mathbb{T}_x)} \|u(t)\|_{\mathcal{H}_0^m},$$

provided  $|\beta - \tilde{\beta}| \leq |\beta| - 1 = m - 1$ . Also, (2.7) allows us to get that for  $\tilde{\beta} \geq e_i, i = 1$  or  $2$ ,

$$\begin{aligned} \left\| \frac{\partial_\tau^{\tilde{\beta}} u \cdot \partial_\tau^{\beta - \tilde{\beta} + e_2} \partial_y^{-1} h}{1+y} \right\|_{L^2(\Omega)} &= \left\| \partial_\tau^{\tilde{\beta} - e_i} (\partial_\tau^{e_i} u) \cdot \partial_\tau^{\beta - \tilde{\beta}} \partial_y^{-1} (\partial_x h) \right\|_{L^2_{-1}(\Omega)} \\ &\leq C \|\partial_\tau^{e_i} u(t)\|_{\mathcal{H}_0^{m-1}} \|\partial_x h(t)\|_{\mathcal{H}_0^{m-1}} \leq C \|(u, h)(t)\|_{\mathcal{H}_0^m}^2. \end{aligned}$$

The other terms in  $R_\psi^\beta$  can be estimated similarly, and we have

$$\left\| \frac{R_\psi^\beta(t)}{1+y} \right\|_{L^2(\Omega)} \leq C \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)\|_{\mathcal{H}_0^m} \right) \|(u, h)\|_{\mathcal{H}_0^m}. \quad (3.60)$$

Now, combining (3.51) with (3.54), we define new functions:

$$u_\beta = \partial_\tau^\beta u - \frac{\partial_y u + U\phi''}{h + H\phi'} \partial_\tau^\beta \psi, \quad h_\beta = \partial_\tau^\beta h - \frac{\partial_y h + H\phi''}{h + H\phi'} \partial_\tau^\beta \psi, \quad (3.61)$$

and denote

$$\eta_1 \triangleq \frac{\partial_y u + U\phi''}{h + H\phi'}, \quad \eta_2 \triangleq \frac{\partial_y h + H\phi''}{h + H\phi'}. \quad (3.62)$$

Then, by noting that  $\partial_\tau^\beta g = -\partial_x \partial_\tau^\beta \psi$  from (3.54), we compute (3.48)<sub>1</sub> - (3.56)  $\times \eta_1$  and (3.48)<sub>2</sub> - (3.56)  $\times \eta_2$  respectively, to obtain that

$$\begin{cases} \partial_t u_\beta + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u_\beta - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h_\beta - \mu\partial_y^2 u_\beta + (\kappa - \mu)\eta_1\partial_y h_\beta &= R_1^\beta, \\ \partial_t h_\beta + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]h_\beta - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]u_\beta - \kappa\partial_y^2 h_\beta &= R_2^\beta, \end{cases} \quad (3.63)$$

where

$$\begin{cases} R_1^\beta &= \partial_\tau^\beta r_1 - \eta_1 \partial_\tau^\beta r_3 + R_u^\beta - \eta_1 R_\psi^\beta + [2\mu\partial_y \eta_1 + (g - H_x\phi)\eta_2 + (\mu - \kappa)\eta_1\eta_2] \partial_\tau^\beta h - \zeta_1 \partial_\tau^\beta \psi, \\ R_2^\beta &= \partial_\tau^\beta r_1 - \eta_2 \partial_\tau^\beta r_2 + R_h^\beta - \eta_2 R_\psi^\beta + [2\kappa\partial_y \eta_2 + (g - H_x\phi)\eta_1] \partial_\tau^\beta h - \zeta_2 \partial_\tau^\beta \psi, \end{cases} \quad (3.64)$$

with

$$\begin{aligned} \zeta_1 &= \partial_t \eta_1 + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\eta_1 - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]\eta_2 - \mu\partial_y^2 \eta_1 + (\kappa - \mu)\eta_1\partial_y \eta_2, \\ \zeta_2 &= \partial_t \eta_2 + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]\eta_2 - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]\eta_1 - \kappa\partial_y^2 \eta_2. \end{aligned} \quad (3.65)$$

Also, direct calculation gives the corresponding initial-boundary values as follows:

$$\begin{cases} u_\beta|_{t=0} = \partial_\tau^\beta u(0, x, y) - \frac{\partial_y u_0(x, y) + U(0, x)\phi''(y)}{h_0(x, y) + H(0, x)\phi'(y)} \int_0^y \partial_\tau^\beta h(0, x, z) dz \triangleq u_{\beta 0}(x, y), \\ h_\beta|_{t=0} = \partial_\tau^\beta h(0, x, y) - \frac{\partial_y h_0(x, y) + H(0, x)\phi''(y)}{h_0(x, y) + H(0, x)\phi'(y)} \int_0^y \partial_\tau^\beta h(0, x, z) dz \triangleq h_{\beta 0}(x, y), \\ u_\beta|_{y=0} = 0, \quad \partial_y h_\beta|_{y=0} = 0. \end{cases} \quad (3.66)$$

Finally, we obtain the initial-boundary value problem for  $(u_\beta, h_\beta)$ :

$$\begin{cases} \partial_t u_\beta + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]u_\beta - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]h_\beta - \mu\partial_y^2 u_\beta + (\kappa - \mu)\eta_1\partial_y h_\beta = R_1^\beta, \\ \partial_t h_\beta + [(u + U\phi')\partial_x + (v - U_x\phi)\partial_y]h_\beta - [(h + H\phi')\partial_x + (g - H_x\phi)\partial_y]u_\beta - \kappa\partial_y^2 h_\beta = R_2^\beta, \\ (u_\beta, \partial_y h_\beta)|_{y=0} = 0, \quad (u_\beta, h_\beta)|_{t=0} = (u_{\beta 0}, h_{\beta 0})(x, y), \end{cases} \quad (3.67)$$

with the initial data  $(u_{\beta 0}, h_{\beta 0})(x, y)$  given by (3.66). Moreover, by combining  $\psi = \partial_y^{-1}h$  with (2.7),

$$\|\langle y \rangle^{-1}\partial_\tau^\beta \psi(t)\|_{L^2(\Omega)} \leq 2\|\partial_\tau^\beta h(t)\|_{L^2(\Omega)}. \quad (3.68)$$

From the expression (3.62) of  $\eta_1$  and  $\eta_2$ , by (3.53) and Sobolev embedding inequality we have that for  $\lambda \in \mathbb{R}$  and  $i = 1, 2$ ,

$$\begin{aligned} \|\langle y \rangle^\lambda \eta_i\|_{L^\infty(\Omega)} &\leq C\delta_0^{-1}(\|(U, H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u, h)\|_{\mathcal{H}_{\lambda-1}^3}), \\ \|\langle y \rangle^\lambda \partial_y \eta_i\|_{L^\infty(\Omega)} &\leq C\delta_0^{-2}(\|(U, H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u, h)\|_{\mathcal{H}_{\lambda-1}^4})^2, \end{aligned} \quad (3.69)$$

and

$$\|\langle y \rangle^\lambda \zeta_i\|_{L^\infty(\Omega)} \leq C\delta_0^{-3} \left( \sum_{|\beta| \leq 1} \|\partial_\tau^\beta (U, H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u, h)\|_{\mathcal{H}_{\lambda-1}^5} \right)^3, \quad i = 1, 2. \quad (3.70)$$

Then, for the terms  $R_1^\beta$  and  $R_2^\beta$  given by (3.64), from the above inequalities (3.68)-(3.70), the estimates (3.50) and (3.60) we obtain that for  $|\beta| = m \geq 5, l \geq 0$ ,

$$\begin{aligned} \|R_1^\beta(t)\|_{L_t^2(\Omega)} &\leq \|\partial_\tau^\beta r_1 - \eta_1\partial_\tau^\beta r_3\|_{L_t^2(\Omega)} + \|R_u^\beta\|_{L_t^2(\Omega)} + \|\langle y \rangle^{l+1}\eta_1\|_{L^\infty(\Omega)}\|\langle y \rangle^{-1}R_\psi^\beta\|_{L^2(\Omega)} \\ &\quad + (\|2\mu\partial_y\eta_1 + (\mu - \kappa)\eta_1\eta_2\|_{L^\infty(\Omega)} + \|\langle y \rangle^{-1}(g - H_x\phi)\|_{L^\infty(\Omega)}\|\langle y \rangle\eta_2\|_{L^\infty(\Omega)})\|\partial_\tau^\beta h\|_{L_t^2(\Omega)} \\ &\quad + \|\langle y \rangle^{l+1}\zeta_1\|_{L^\infty(\Omega)}\|\langle y \rangle^{-1}\partial_\tau^\beta \psi\|_{L^2(\Omega)} \\ &\leq \|\partial_\tau^\beta r_1 - \eta_1\partial_\tau^\beta r_3\|_{L_t^2(\Omega)} + C\delta_0^{-3} \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta (U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)\|_{\mathcal{H}_l^m} \right)^3 \|(u, h)(t)\|_{\mathcal{H}_l^m}, \end{aligned} \quad (3.71)$$

and

$$\begin{aligned} \|R_2^\beta(t)\|_{L_t^2(\Omega)} &\leq \|\partial_\tau^\beta r_2 - \eta_2\partial_\tau^\beta r_3\|_{L_t^2(\Omega)} + \|R_h^\beta\|_{L_t^2(\Omega)} + \|\langle y \rangle^{l+1}\eta_2\|_{L^\infty(\Omega)}\|\langle y \rangle^{-1}R_\psi^\beta\|_{L^2(\Omega)} \\ &\quad + (\|2\kappa\partial_y\eta_2\|_{L^\infty(\Omega)} + \|\langle y \rangle^{-1}(g - H_x\phi)\|_{L^\infty(\Omega)}\|\langle y \rangle\eta_1\|_{L^\infty(\Omega)})\|\partial_\tau^\beta h\|_{L_t^2(\Omega)} \\ &\quad + \|\langle y \rangle^{l+1}\zeta_2\|_{L^\infty(\Omega)}\|\langle y \rangle^{-1}\partial_\tau^\beta \psi\|_{L^2(\Omega)} \\ &\leq \|\partial_\tau^\beta r_2 - \eta_2\partial_\tau^\beta r_3\|_{L_t^2(\Omega)} + C\delta_0^{-3} \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta (U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)\|_{\mathcal{H}_l^m} \right)^3 \|(u, h)(t)\|_{\mathcal{H}_l^m}. \end{aligned} \quad (3.72)$$

Now, we are going to derive the following  $L_t^2$ -norms of  $(u_\beta, h_\beta)$ .

**Proposition 3.3.** [ $L_t^2$ -estimate on  $(u_\beta, h_\beta)$ ]

Under the hypotheses of Proposition 3.1, we have that for any  $t \in [0, T]$  and the quantity  $(u_\beta, h_\beta)$  given in (3.61),

$$\begin{aligned} &\sum_{|\beta|=m} \left( \frac{d}{dt} \|(u_\beta, h_\beta)(t)\|_{L_t^2(\mathbb{R}_+^2)}^2 + \mu\|\partial_y u_\beta(t)\|_{L_t^2(\Omega)}^2 + \kappa\|\partial_y h_\beta(t)\|_{L_t^2(\Omega)}^2 \right) \\ &\leq \sum_{|\beta|=m} \left( \|\partial_\tau^\beta r_1 - \eta_1\partial_\tau^\beta r_3\|_{L_t^2(\Omega)}^2 + \|\partial_\tau^\beta r_2 - \eta_2\partial_\tau^\beta r_3\|_{L_t^2(\Omega)}^2 \right) \\ &\quad + C\delta_0^{-2} \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta (U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_l^m} \right)^2 \left( \sum_{|\beta|=m} \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 \right) \\ &\quad + C\delta_0^{-4} \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta (U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)\|_{\mathcal{H}_l^m} \right)^4 \|(u, h)(t)\|_{\mathcal{H}_l^m}^2. \end{aligned} \quad (3.73)$$

**Proof.** Multiplying (3.67)<sub>1</sub> and (3.67)<sub>2</sub> by  $\langle y \rangle^{2l} u_\beta$  and  $\langle y \rangle^{2l} h_\beta$  respectively, and integrating them over  $\Omega$  with  $t \in [0, T]$ , we obtain that by integration by parts,

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(u_\beta, h_\beta)(t)\|_{L_t^2(\mathbb{R}_+^2)}^2 + \mu \|\partial_y u_\beta\|_{L_t^2(\Omega)}^2 + \kappa \|\partial_y h_\beta\|_{L_t^2(\Omega)}^2 \\ &= 2l \int_\Omega \langle y \rangle^{2l-1} [(v - U_x \phi) \frac{u_\beta^2 + h_\beta^2}{2} - (g - H_x \phi) u_\beta h_\beta] dx dy + (\mu - \kappa) \int_\Omega \langle y \rangle^{2l} (\eta_1 \partial_y h_\beta \cdot u_\beta) dx dy \\ & \quad + \int_\Omega \langle y \rangle^{2l} (u_\beta R_1^\beta + h_\beta R_2^\beta) dx dy - 2l \int_\Omega \langle y \rangle^{2l-1} (\mu u_\beta \partial_y u_\beta + \kappa h_\beta \partial_y h_\beta) dx dy, \end{aligned} \quad (3.74)$$

where we have used the boundary conditions in (3.67) and  $(v, g)|_{y=0} = 0$ .

By (2.5), it gives that

$$\begin{aligned} & \left| 2l \int_\Omega \langle y \rangle^{2l-1} [(v - U_x \phi) \frac{u_\beta^2 + h_\beta^2}{2} - (g - H_x \phi) u_\beta h_\beta] dx dy \right| \\ & \leq 2l \left( \left\| \frac{v - U_x \phi}{1 + y} \right\|_{L^\infty(\Omega)} + \left\| \frac{g - H_x \phi}{1 + y} \right\|_{L^\infty(\Omega)} \right) \|(u_\beta, h_\beta)\|_{L_t^2(\Omega)}^2 \\ & \leq 2l (\|(U_x, H_x)(t)\|_{L^\infty(\mathbb{T}_x)} + \|u_x(t)\|_{L^\infty(\Omega)} + \|h_x(t)\|_{L^\infty(\Omega)}) \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 \\ & \leq C (\|(U_x, H_x)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_t^m}) \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2. \end{aligned} \quad (3.75)$$

By integration by parts and the boundary condition  $u_\beta|_{y=0} = 0$ , we obtain that

$$\begin{aligned} & (\mu - \kappa) \int_\Omega \langle y \rangle^{2l} (\eta_1 \partial_y h_\beta \cdot u_\beta) dx dy \\ &= -\mu \int_\Omega h_\beta \partial_y (\langle y \rangle^{2l} \eta_1 u_\beta) dx dy - \kappa \int_\Omega \langle y \rangle^{2l} (\eta_1 \partial_y h_\beta \cdot u_\beta) dx dy \\ & \leq \frac{\mu}{4} \|\partial_y u_\beta(t)\|_{L_t^2(\Omega)}^2 + \frac{\kappa}{4} \|\partial_y h_\beta(t)\|_{L_t^2(\Omega)}^2 + C(1 + \|\eta_1(t)\|_{L^\infty(\Omega)}^2 + \|\partial_y \eta_1(t)\|_{L^\infty(\Omega)}) \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 \\ & \leq \frac{\mu}{4} \|\partial_y u_\beta(t)\|_{L_t^2(\Omega)}^2 + \frac{\kappa}{4} \|\partial_y h_\beta(t)\|_{L_t^2(\Omega)}^2 + C\delta_0^{-2} (\|(U, H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_t^m})^2 \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2, \end{aligned} \quad (3.76)$$

where we have used (3.69) in the above second inequality.

Next, it is easy to get that by (3.71) and (3.72),

$$\begin{aligned} \int_\Omega \langle y \rangle^{2l} (u_\beta R_1^\beta + h_\beta R_2^\beta) dx dy & \leq \|u_\beta(t)\|_{L_t^2(\Omega)} \|R_1^\beta(t)\|_{L_t^2(\Omega)} + \|h_\beta(t)\|_{L_t^2(\Omega)} \|R_2^\beta(t)\|_{L_t^2(\Omega)} \\ & \leq \|\partial_\tau^\beta r_1 - \eta_1 \partial_\tau^\beta r_3\|_{L_t^2(\Omega)}^2 + \|\partial_\tau^\beta r_2 - \eta_2 \partial_\tau^\beta r_3\|_{L_t^2(\Omega)}^2 \\ & \quad + C\delta_0^{-2} \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta (U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_t^m} \right)^2 \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 \\ & \quad + C\delta_0^{-4} \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta (U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_t^m} \right)^4 \|(u, h)(t)\|_{\mathcal{H}_t^m}^2. \end{aligned} \quad (3.77)$$

Also,

$$\begin{aligned} & \left| 2l \int_\Omega \langle y \rangle^{2l-1} (\mu u_\beta \partial_y u_\beta + \kappa h_\beta \partial_y h_\beta) dx dy \right| \\ & \leq \frac{\mu}{4} \|\partial_y u_\beta(t)\|_{L_t^2(\Omega)}^2 + \frac{\kappa}{4} \|\partial_y h_\beta(t)\|_{L_t^2(\Omega)}^2 + C \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2. \end{aligned} \quad (3.78)$$

Substituting (3.75)-(3.78) into (3.74) yields that

$$\begin{aligned} & \frac{d}{dt} \|(u_\beta, h_\beta)(t)\|_{L_t^2(\mathbb{R}_+^2)}^2 + \mu \|\partial_y u_\beta\|_{L_t^2(\Omega)}^2 + \kappa \|\partial_y h_\beta\|_{L_t^2(\Omega)}^2 \\ & \leq \|\partial_\tau^\beta r_1 - \eta_1 \partial_\tau^\beta r_3\|_{L_t^2(\Omega)}^2 + \|\partial_\tau^\beta r_2 - \eta_2 \partial_\tau^\beta r_3\|_{L_t^2(\Omega)}^2 \\ & \quad + C\delta_0^{-2} \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta (U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_t^m} \right)^2 \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 \end{aligned}$$

$$+ C\delta_0^{-4} \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)\|_{\mathcal{H}_t^m} \right)^4 \|(u, h)(t)\|_{\mathcal{H}_t^m}^2, \quad (3.79)$$

thus we prove (3.73) by taking the summation over all  $|\beta| = m$  in (3.79).  $\square$

Finally, we give the following result, which shows the almost equivalence in  $L_t^2$ -norm between  $\partial_\tau^\beta(u, h)$  and the quantities  $(u_\beta, h_\beta)$  given by (3.61).

**Lemma 3.4.** *[Equivalence between  $\|\partial_\tau^\beta(u, h)\|_{L_t^2}$  and  $\|(u_\beta, h_\beta)\|_{L_t^2}$ ]*

*If the smooth function  $(u, h)$  satisfies the problem (2.12) in  $[0, T]$ , and (3.53) holds, then for any  $t \in [0, T]$ ,  $l \geq 0$ , an integer  $m \geq 3$  and the quantity  $(u_\beta, h_\beta)$  with  $|\beta| = m$  defined by (3.61), we have*

$$M(t)^{-1} \|\partial_\tau^\beta(u, h)(t)\|_{L_t^2(\Omega)} \leq \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)} \leq M(t) \|\partial_\tau^\beta(u, h)(t)\|_{L_t^2(\Omega)}, \quad (3.80)$$

and

$$\|\partial_y \partial_\tau^\beta(u, h)(t)\|_{L_t^2(\Omega)} \leq \|\partial_y(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)} + M(t) \|h_\beta(t)\|_{L_t^2(\Omega)}, \quad (3.81)$$

where

$$M(t) := 2\delta_0^{-1} \left( C\|(U, H)(t)\|_{L^\infty(\mathbb{T}_x)} + \|\langle y \rangle^{l+1} \partial_y(u, h)(t)\|_{L^\infty(\Omega)} + \|\langle y \rangle^{l+1} \partial_y^2(u, h)(t)\|_{L^\infty(\Omega)} \right). \quad (3.82)$$

**Proof.** Firstly, from the definitions of  $u_\beta$  and  $h_\beta$  in (3.61), we have by using (3.68),

$$\begin{aligned} \|u_\beta(t)\|_{L_t^2(\Omega)} &\leq \|\partial_\tau^\beta u(t)\|_{L_t^2(\Omega)} + \|\langle y \rangle^{l+1} \eta_1(t)\|_{L^\infty(\Omega)} \|\langle y \rangle^{-1} \partial_\tau^\beta \psi(t)\|_{L^2(\Omega)} \\ &\leq \|\partial_\tau^\beta u(t)\|_{L_t^2(\Omega)} + 2\delta_0^{-1} (C\|U(t)\|_{L^\infty(\mathbb{T}_x)} + \|\langle y \rangle^{l+1} \partial_y u(t)\|_{L^\infty(\Omega)}) \|\partial_\tau^\beta h(t)\|_{L^2(\Omega)}, \end{aligned}$$

and

$$\begin{aligned} \|h_\beta(t)\|_{L_t^2(\Omega)} &\leq \|\partial_\tau^\beta h(t)\|_{L_t^2(\Omega)} + \|\langle y \rangle^{l+1} \eta_2(t)\|_{L^\infty(\Omega)} \|\langle y \rangle^{-1} \partial_\tau^\beta \psi(t)\|_{L^2(\Omega)} \\ &\leq 2\delta_0^{-1} (C\|H(t)\|_{L^\infty(\mathbb{T}_x)} + \|\langle y \rangle^{l+1} \partial_y h(t)\|_{L^\infty(\Omega)}) \|\partial_\tau^\beta h(t)\|_{L_t^2(\Omega)}. \end{aligned}$$

Thus, we have that by (3.82),

$$\|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)} \leq M(t) \|\partial_\tau^\beta(u, h)(t)\|_{L_t^2(\Omega)}. \quad (3.83)$$

On other hand, note that from  $\partial_y \psi = h$  and the expression of  $h_\beta$  in (3.61),

$$h_\beta = \partial_\tau^\beta h - \frac{\partial_y h + H\phi''}{h + H\phi'} \partial_\tau^\beta \psi = (h + H\phi') \cdot \partial_y \left( \frac{\partial_\tau^\beta \psi}{h + H\phi'} \right),$$

which implies that by  $\partial_\tau^\beta \psi|_{y=0} = 0$ ,

$$\partial_\tau^\beta \psi(t, x, y) = (h(t, x, y) + H(t, x)\phi'(y)) \cdot \int_0^y \frac{h_\beta(t, x, z)}{h(t, x, z) + H(t, x)\phi'(z)} dz. \quad (3.84)$$

Therefore, combining the definition (3.61) for  $(u_\beta, h_\beta)$  with (3.84), we have

$$\begin{cases} \partial_\tau^\beta u(t, x, y) = u_\beta(t, x, y) + (\partial_y u(t, x, y) + U(t, x)\phi''(y)) \cdot \int_0^y \frac{h_\beta(t, x, z)}{h(t, x, z) + H(t, x)\phi'(z)} dz, \\ \partial_\tau^\beta h(t, x, y) = h_\beta(t, x, y) + (\partial_y h(t, x, y) + H(t, x)\phi''(y)) \cdot \int_0^y \frac{h_\beta(t, x, z)}{h(t, x, z) + H(t, x)\phi'(z)} dz. \end{cases} \quad (3.85)$$

Then, by using (2.7),

$$\begin{aligned} \|\partial_\tau^\beta u(t)\|_{L_t^2(\Omega)} &\leq \|u_\beta(t)\|_{L_t^2(\Omega)} + \|\langle y \rangle^{l+1} (\partial_y u + U\phi'')(t)\|_{L^\infty(\Omega)} \left\| \frac{1}{1+y} \int_0^y \frac{h_\beta(t, x, z)}{h(t, x, z) + H(t, x)\phi'(z)} dz \right\|_{L^2(\Omega)} \\ &\leq \|u_\beta(t)\|_{L_t^2(\Omega)} + 2(C\|U(t)\|_{L^\infty(\mathbb{T}_x)} + \|\langle y \rangle^{l+1} \partial_y u(t)\|_{L^\infty(\Omega)}) \left\| \frac{h_\beta}{h + H\phi'} \right\|_{L^2(\Omega)} \\ &\leq \|u_\beta(t)\|_{L_t^2(\Omega)} + 2\delta_0^{-1} (C\|U(t)\|_{L^\infty(\mathbb{T}_x)} + \|\langle y \rangle^{l+1} \partial_y u(t)\|_{L^\infty(\Omega)}) \|h_\beta(t)\|_{L^2(\Omega)}, \end{aligned}$$

and similarly,

$$\|\partial_\tau^\beta h(t)\|_{L_t^2(\Omega)} \leq 2\delta_0^{-1} (C\|H(t)\|_{L^\infty(\mathbb{T}_x)} + \|\langle y \rangle^{l+1} \partial_y h(t)\|_{L^\infty(\Omega)}) \|h_\beta(t)\|_{L_t^2(\Omega)},$$

which implies that,

$$\|\partial_\tau^\beta(u, h)(t)\|_{L_t^2(\Omega)} \leq M(t) \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}, \quad (3.86)$$

provided that  $M(t)$  is given in (3.82). Thus, combining (3.83) with (3.86) yields (3.80).

Furthermore, by taking the derivation of (3.85) in  $y$ , we get the following forms of  $\partial_y \partial_\tau^\beta u$  and  $\partial_y \partial_\tau^\beta h$ :

$$\begin{cases} \partial_y \partial_\tau^\beta u(t, x, y) = \partial_y u_\beta(t, x, y) + \eta_1(t, x, y) h_\beta(t, x, y) + (\partial_y^2 u(t, x, y) + U(t, x) \phi^{(3)}(y)) \cdot \int_0^y \frac{h_\beta(t, x, z)}{h(t, x, z) + H(t, x) \phi'(z)} dz, \\ \partial_y \partial_\tau^\beta h(t, x, y) = \partial_y h_\beta(t, x, y) + \eta_2(t, x, y) h_\beta(t, x, y) + (\partial_y^2 h(t, x, y) + H(t, x) \phi^{(3)}(y)) \cdot \int_0^y \frac{h_\beta(t, x, z)}{h(t, x, z) + H(t, x) \phi'(z)} dz. \end{cases}$$

Then, it follows that by (2.7) and (3.69),

$$\begin{aligned} \|\partial_y \partial_\tau^\beta u(t)\|_{L_t^2(\Omega)} &\leq \|\partial_y u_\beta(t)\|_{L_t^2(\Omega)} + \|\eta_1(t)\|_{L^\infty(\Omega)} \|h_\beta(t)\|_{L_t^2(\Omega)} \\ &\quad + \|\langle y \rangle^{l+1} (\partial_y^2 u + U \phi^{(3)})(t)\|_{L^\infty(\Omega)} \left\| \frac{1}{1+y} \int_0^y \frac{h_\beta(t, x, z)}{h(t, x, z) + H(t, x) \phi'(z)} dz \right\|_{L^2(\Omega)} \\ &\leq \|\partial_y u_\beta(t)\|_{L_t^2(\Omega)} + \delta_0^{-1} (C \|U(t)\|_{L^\infty(\mathbb{T}_x)} + \|\partial_y u(t)\|_{L^\infty(\Omega)}) \|h_\beta(t)\|_{L_t^2(\Omega)} \\ &\quad + 2(C \|U(t)\|_{L^\infty(\mathbb{T}_x)} + \|\langle y \rangle^{l+1} \partial_y^2 u(t)\|_{L^\infty(\Omega)}) \left\| \frac{h_\beta}{h + H \phi'} \right\|_{L^2(\Omega)} \\ &\leq \|\partial_y u_\beta(t)\|_{L_t^2(\Omega)} + M(t) \|h_\beta(t)\|_{L_t^2(\Omega)}, \end{aligned}$$

and similarly,

$$\|\partial_y \partial_\tau^\beta h(t)\|_{L_t^2(\Omega)} \leq \|\partial_y h_\beta(t)\|_{L_t^2(\Omega)} + M(t) \|h_\beta(t)\|_{L_t^2(\Omega)}.$$

Combining the above two inequalities yields that by (3.82),

$$\|\partial_y \partial_\tau^\beta (u, h)(t)\|_{L_t^2(\Omega)} \leq \|\partial_y (u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)} + M(t) \|h_\beta(t)\|_{L_t^2(\Omega)}.$$

Thus we obtain (3.81) and this completes the proof.  $\square$

### 3.3. Closeness of the a priori estimates.

In this subsection, we will prove Proposition 3.1. Before that, we need some preliminaries. First of all, as we know that from (3.1),

$$\|\langle y \rangle^{l+1} \partial_y^i (u, h)(t)\|_{L^\infty(\Omega)} \leq \delta_0^{-1}, \quad \text{for } i = 1, 2, \quad t \in [0, T],$$

combining with the definitions (3.62) for  $\eta_i, i = 1, 2$ , and (3.82) for  $M(t)$ , it implies that for  $\delta_0$  sufficiently small,

$$\|\langle y \rangle^{l+1} \eta_i(t)\|_{L^\infty(\Omega)} \leq 2\delta_0^{-2}, \quad M(t) \leq 2\delta_0^{-1} (C \|U, H(t)\|_{L^\infty(\mathbb{T}_x)} + 2\delta_0^{-1}) \leq 5\delta_0^{-2}, \quad i = 1, 2. \quad (3.87)$$

Then, recall that  $D^\alpha = \partial_\tau^\beta \partial_y^k$ , we obtain that by (3.80) and (3.81) given in Lemma 3.4,

$$\begin{aligned} \|(u, h)(t)\|_{\mathcal{H}_t^m}^2 &= \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha (u, h)(t)\|_{L_t^2(\Omega)}^2 + \sum_{|\beta|=m} \|\partial_\tau^\beta (u, h)(t)\|_{L_t^2(\Omega)}^2 \\ &\leq \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha (u, h)(t)\|_{L_t^2(\Omega)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2, \end{aligned}$$

and

$$\begin{aligned} \|\partial_y (u, h)(t)\|_{\mathcal{H}_t^m}^2 &= \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha \partial_y (u, h)(t)\|_{L_t^2(\Omega)}^2 + \sum_{|\beta|=m} \|\partial_y \partial_\tau^\beta (u, h)(t)\|_{L_t^2(\Omega)}^2 \\ &\leq \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha \partial_y (u, h)(t)\|_{L_t^2(\Omega)}^2 + 2 \sum_{|\beta|=m} \|\partial_y (u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 + 50\delta_0^{-4} \sum_{|\beta|=m} \|h_\beta(t)\|_{L_t^2(\Omega)}^2. \end{aligned}$$

Consequently, we have the following

**Corollary 3.5.** *Under the assumptions of Proposition 3.1, for any  $t \in [0, T]$  and the quantity  $(u_\beta, h_\beta), |\beta| = m$  given by (3.61), it holds that*

$$\|(u, h)(t)\|_{\mathcal{H}_t^m}^2 \leq \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha (u, h)(t)\|_{L_t^2(\Omega)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2, \quad (3.88)$$

and

$$\|\partial_y(u, h)(t)\|_{\mathcal{H}_t^m}^2 \leq \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha \partial_y(u, h)(t)\|_{L_t^2(\Omega)}^2 + 2 \sum_{|\beta|=m} \|\partial_y(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 + 50\delta_0^{-4} \sum_{|\beta|=m} \|h_\beta(t)\|_{L_t^2(\Omega)}^2. \quad (3.89)$$

Now, we can derive the desired a priori estimates of  $(u, h)$  for the problem (2.12). From Proposition 3.2 and 3.3, it follows that for  $m \geq 5$  and any  $t \in [0, T]$ ,

$$\begin{aligned} & \frac{d}{dt} \left( \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(u, h)(t)\|_{L_t^2(\mathbb{R}_+^2)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 \right) \\ & + \mu \left( \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha \partial_y u(t)\|_{L_t^2(\mathbb{R}_+^2)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \|\partial_y u_\beta(t)\|_{L_t^2(\Omega)}^2 \right) \\ & + \kappa \left( \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha \partial_y h(t)\|_{L_t^2(\mathbb{R}_+^2)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \|\partial_y h_\beta(t)\|_{L_t^2(\Omega)}^2 \right) \\ & \leq \delta_1 C \|\partial_y(u, h)(t)\|_{\mathcal{H}_0^m}^2 + C\delta_1^{-1} \|(u, h)(t)\|_{\mathcal{H}_t^m}^2 (1 + \|(u, h)(t)\|_{\mathcal{H}_t^m}^2) + \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(r_1, r_2)(t)\|_{L_{t+k}^2(\Omega)}^2 \\ & + C \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H, P)(t)\|_{L^2(\mathbb{T}_x)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \left( \|\partial_\tau^\beta r_1 - \eta_1 \partial_\tau^\beta r_3\|_{L_t^2(\Omega)}^2 + \|\partial_\tau^\beta r_2 - \eta_2 \partial_\tau^\beta r_3\|_{L_t^2(\Omega)}^2 \right) \\ & + C\delta_0^{-6} \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)(t)\|_{\mathcal{H}_t^m} \right)^2 \left( \sum_{|\beta|=m} \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 \right) \\ & + C\delta_0^{-8} \left( \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H)(t)\|_{L^2(\mathbb{T}_x)} + \|(u, h)\|_{\mathcal{H}_t^m} \right)^4 \|(u, h)(t)\|_{\mathcal{H}_t^m}^2. \end{aligned} \quad (3.90)$$

Plugging the inequalities (3.88) and (3.89) given in Corollary 3.5 into (3.90), and choosing  $\delta_1$  small enough, we get

$$\begin{aligned} & \frac{d}{dt} \left( \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(u, h)(t)\|_{L_t^2(\mathbb{R}_+^2)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 \right) \\ & + \left( \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha \partial_y(u, h)(t)\|_{L_t^2(\mathbb{R}_+^2)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \|\partial_y(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 \right) \\ & \leq C \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(r_1, r_2)(t)\|_{L_{t+k}^2(\Omega)}^2 + C\delta_0^{-4} \sum_{|\beta|=m} \left( \|\partial_\tau^\beta(r_1, r_2)(t)\|_{L_t^2(\Omega)}^2 + 4\delta_0^{-4} \|\partial_\tau^\beta r_3\|_{L_{2-1}^2(\Omega)}^2 \right) \\ & + C\delta_0^{-8} \left( 1 + \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H, P)(t)\|_{L^2(\mathbb{T}_x)}^2 \right)^3 \\ & + C\delta_0^{-8} \left( \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(u, h)(t)\|_{L_t^2(\Omega)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \|(u_\beta, h_\beta)(t)\|_{L_t^2(\Omega)}^2 \right)^3, \end{aligned} \quad (3.91)$$

where we have used the fact that

$$\|\eta_i \partial_\tau^\beta r_3\|_{L_t^2(\Omega)} \leq \|\langle y \rangle^{i+1} \eta_i(t)\|_{L^\infty(\Omega)} \|\langle y \rangle^{-1} \partial_\tau^\beta r_3\|_{L^2(\Omega)} \leq 2\delta_0^{-2} \|\partial_\tau^\beta r_3\|_{L_{2-1}^2(\Omega)}, \quad i = 1, 2.$$

Denote by

$$F_0 := \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(u, h)(0)\|_{L_t^2(\Omega)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \|(u_{\beta 0}, h_{\beta 0})\|_{L_t^2(\Omega)}^2, \quad (3.92)$$



and

$$\begin{aligned}
F(t) := & C \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(r_1, r_2)(t)\|_{L^2_{1+k}(\Omega)}^2 + C\delta_0^{-4} \sum_{|\beta|=m} \left( \|\partial_\tau^\beta(r_1, r_2)(t)\|_{L^2_l(\Omega)}^2 + 4\delta_0^{-4} \|\partial_\tau^\beta r_3\|_{L^2_{-1}(\Omega)}^2 \right) \\
& + C\delta_0^{-8} \left( 1 + \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H, P)(t)\|_{L^2(\mathbb{T}_x)}^2 \right)^3. \tag{3.93}
\end{aligned}$$

By the comparison principle of ordinary differential equations in (3.91), it yields that

$$\begin{aligned}
& \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(u, h)(t)\|_{L^2_l(\Omega)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \|(u_\beta, h_\beta)(t)\|_{L^2_l(\Omega)}^2 \\
& + \int_0^t \left( \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha \partial_y(u, h)(s)\|_{L^2_l(\Omega)}^2 + 25\delta_0^{-4} \sum_{|\beta|=m} \|\partial_y(u_\beta, h_\beta)(s)\|_{L^2_l(\Omega)}^2 \right) ds \\
& \leq (F_0 + \int_0^t F(s) ds) \cdot \left\{ 1 - 2C\delta_0^{-8} (F_0 + \int_0^t F(s) ds)^2 t \right\}^{-\frac{1}{2}}. \tag{3.94}
\end{aligned}$$

Then, it implies that by combining (3.88) with (3.94),

$$\sup_{0 \leq s \leq t} \|(u, h)(s)\|_{\mathcal{H}_l^m} \leq (F_0 + \int_0^t F(s) ds)^{\frac{1}{2}} \cdot \left\{ 1 - 2C\delta_0^{-8} (F_0 + \int_0^t F(s) ds)^2 t \right\}^{-\frac{1}{4}}. \tag{3.95}$$

As we know

$$\langle y \rangle^{l+1} \partial_y^i(u, h)(t, x, y) = \langle y \rangle^{l+1} \partial_y^i(u_0, h_0)(x, y) + \int_0^t \langle y \rangle^{l+1} \partial_t \partial_y^i(u, h)(s, x, y) ds, \quad i = 1, 2,$$

and

$$h(t, x, y) = h_0(x, y) + \int_0^t \partial_t h(s, x, y) ds.$$

Then, by the Sobolev embedding inequality and (3.95) we have that for  $i = 1, 2$ ,

$$\begin{aligned}
& \|\langle y \rangle^{l+1} \partial_y^i(u, h)(t)\|_{L^\infty(\Omega)} \\
& \leq \|\langle y \rangle^{l+1} \partial_y^i(u_0, h_0)\|_{L^\infty(\Omega)} + \int_0^t \|\langle y \rangle^{l+1} \partial_t \partial_y^i(u, h)(s)\|_{L^\infty(\Omega)} ds \\
& \leq \|\langle y \rangle^{l+1} \partial_y^i(u_0, h_0)\|_{L^\infty(\Omega)} + C \left( \sup_{0 \leq s \leq t} \|(u, h)(s)\|_{\mathcal{H}_l^5} \right) \cdot t \\
& \leq \|\langle y \rangle^{l+1} \partial_y^i(u_0, h_0)\|_{L^\infty(\Omega)} + Ct \cdot (F_0 + \int_0^t F(s) ds)^{\frac{1}{2}} \left\{ 1 - 2C\delta_0^{-8} (F_0 + \int_0^t F(s) ds)^2 t \right\}^{-\frac{1}{4}}. \tag{3.96}
\end{aligned}$$

Similarly, one can obtain that

$$\begin{aligned}
h(t, x, y) & \geq h_0(x, y) - \int_0^t \|\partial_t h(s)\|_{L^\infty(\Omega)} ds \geq h_0(x, y) - C \left( \sup_{0 \leq s \leq t} \|h(s)\|_{\mathcal{H}_0^3} \right) \cdot t \\
& \geq h_0(x, y) - Ct \cdot (F_0 + \int_0^t F(s) ds)^{\frac{1}{2}} \left\{ 1 - 2C\delta_0^{-8} (F_0 + \int_0^t F(s) ds)^2 t \right\}^{-\frac{1}{4}}. \tag{3.97}
\end{aligned}$$

Therefore, we obtain the following

**Proposition 3.6.** *Under the assumptions of Proposition 3.1, there exists a constant  $C > 0$ , depending only on  $m, M_0$  and  $\phi$ , such that*

$$\sup_{0 \leq s \leq t} \|(u, h)(s)\|_{\mathcal{H}_l^m} \leq (F_0 + \int_0^t F(s) ds)^{\frac{1}{2}} \cdot \left\{ 1 - 2C\delta_0^{-8} (F_0 + \int_0^t F(s) ds)^2 t \right\}^{-\frac{1}{4}}, \tag{3.98}$$

for small time, where the quantities  $F_0$  and  $F(t)$  are defined by (3.92) and (3.93) respectively. Also, we have that for  $i = 1, 2$ ,

$$\|\langle y \rangle^{l+1} \partial_y^i(u, h)(t)\|_{L^\infty(\Omega)}$$

$$\begin{aligned}
&\leq \|\langle y \rangle^{l+1} \partial_y^i(u_0, h_0)\|_{L^\infty(\Omega)} + Ct \cdot \left( \sup_{0 \leq s \leq t} \|(u, h)(s)\|_{\mathcal{H}_t^i} \right) \\
&\leq \|\langle y \rangle^{l+1} \partial_y^i(u_0, h_0)\|_{L^\infty(\Omega)} + Ct \cdot (F_0 + \int_0^t F(s) ds)^{\frac{1}{2}} \left\{ 1 - 2C\delta_0^{-8} (F_0 + \int_0^t F(s) ds)^2 t \right\}^{-\frac{1}{4}}, \tag{3.99}
\end{aligned}$$

and

$$\begin{aligned}
h(t, x, y) &\geq h_0(x, y) - C \left( \sup_{0 \leq s \leq t} \|h(s)\|_{\mathcal{H}_0^3} \right) \cdot t \\
&\geq h_0(x, y) - Ct \cdot (F_0 + \int_0^t F(s) ds)^{\frac{1}{2}} \left\{ 1 - 2C\delta_0^{-8} (F_0 + \int_0^t F(s) ds)^2 t \right\}^{-\frac{1}{4}}. \tag{3.100}
\end{aligned}$$

From the above Proposition 3.6, we are ready to prove Proposition 3.1. Indeed, by using (1.10), (2.16) and the fact  $\|\partial_\tau^\beta r_3\|_{L_{-1}^2(\Omega)} \leq CM_0$  from the expression (3.57), it follows that from the definition (3.93) for  $F(t)$ ,

$$F(t) \leq C\delta_0^{-8} M_0^6. \tag{3.101}$$

Next, by direct calculation we know that  $D^\alpha(u, h)(0, x, y)$ ,  $|\alpha| \leq m$  can be expressed by the spatial derivatives of initial data  $(u_0, h_0)$  up to order  $2m$ . Then, combining with (3.66) we get that  $F_0$ , given by (3.92), is a polynomial of  $\|(u_0, h_0)\|_{H_T^{2m}(\Omega)}$ , and consequently

$$F_0 \leq \delta_0^{-8} \mathcal{P}(M_0 + \|(u_0, h_0)\|_{H_T^{2m}(\Omega)}). \tag{3.102}$$

Plugging (3.101) and (3.102) into (3.98)-(3.100), we derive the estimates (3.2)-(3.4), and then obtain the proof of Proposition 3.1.

#### 4. LOCAL-IN-TIME EXISTENCE AND UNIQUENESS

In this section, we will establish the local-in-time existence and uniqueness of solutions to the nonlinear problem (2.12).

##### 4.1. Existence.

For this, we consider a parabolic regularized system for problem (2.12), from which we can obtain the local (in time) existence of solution by using classical energy estimates. Precisely, for a small parameter  $0 < \epsilon < 1$ , we investigate the following problem:

$$\begin{cases} \partial_t u^\epsilon + [(u^\epsilon + U\phi')\partial_x + (v^\epsilon - U_x\phi)\partial_y]u^\epsilon - [(h^\epsilon + H\phi')\partial_x + (g^\epsilon - H_x\phi)\partial_y]h^\epsilon + U_x\phi'u^\epsilon + U\phi''v^\epsilon \\ \quad - H_x\phi'h^\epsilon - H\phi''g^\epsilon = \epsilon\partial_x^2 u^\epsilon + \mu\partial_y^2 u^\epsilon + r_1^\epsilon, \\ \partial_t h^\epsilon + [(u^\epsilon + U\phi')\partial_x + (v^\epsilon - U_x\phi)\partial_y]h^\epsilon - [(h^\epsilon + H\phi')\partial_x + (g^\epsilon - H_x\phi)\partial_y]u^\epsilon + H_x\phi'u^\epsilon + H\phi''v^\epsilon \\ \quad - U_x\phi'h^\epsilon - U\phi''g^\epsilon = \epsilon\partial_x^2 h^\epsilon + \kappa\partial_y^2 h^\epsilon + r_2^\epsilon, \\ \partial_x u^\epsilon + \partial_y v^\epsilon = 0, \quad \partial_x h^\epsilon + \partial_y g^\epsilon = 0, \\ (u^\epsilon, h^\epsilon)|_{t=0} = (u_0, h_0)(x, y), \quad (u^\epsilon, v^\epsilon, \partial_y h^\epsilon, g^\epsilon)|_{y=0} = 0, \end{cases} \tag{4.1}$$

where the source term

$$(r_1^\epsilon, r_2^\epsilon)(t, x, y) = (r_1, r_2) + \epsilon(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon)(t, x, y). \tag{4.2}$$

Here,  $(r_1, r_2)$  is the source term of the original problem (2.12), and  $(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon)$  is constructed to ensure that the initial data  $(u_0, h_0)$  also satisfies the compatibility conditions of (4.1) up to the order of  $m$ . Actually, we can use the given functions  $\partial_t^i(u, h)(0, x, y)$ ,  $0 \leq i \leq m$ , which can be derived from the equations and initial data of (2.12) by induction with respect to  $i$ , and it follows that  $\partial_t^i(u, h)(0, x, y)$  can be expressed as polynomials of the spatial derivatives, up to order  $2i$ , of the initial data  $(u_0, h_0)$ . Then, we may choose the corrector  $(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon)$  in the following form:

$$(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon)(t, x, y) := - \sum_{i=0}^m \left( \frac{t^i}{i!} \partial_x^2 \partial_t^i(u, h)(0, x, y) \right), \tag{4.3}$$

which yields that by direct calculation,

$$\partial_t^i(u^\epsilon, h^\epsilon)(0, x, y) = \partial_t^i(u, h)(0, x, y), \quad 0 \leq i \leq m.$$

Likewise, we can derive that  $\psi^\epsilon := \partial_y^{-1} h^\epsilon$  satisfies

$$\partial_t \psi^\epsilon + [(u^\epsilon + U\phi')\partial_x + (v^\epsilon - U_x\phi)\partial_y]\psi^\epsilon + H_x\phi u^\epsilon + H\phi'v^\epsilon - \kappa\partial_y^2\psi^\epsilon = r_3^\epsilon,$$

where

$$r_3^\epsilon = r_3 - \epsilon \sum_{i=0}^m \left( \frac{t^i}{i!} \int_0^y \partial_x^2 \partial_t^i h(0, x, z) dz \right) := r_3 + \epsilon \tilde{r}_3 \quad (4.4)$$

with  $r_3$  given by (3.57). Moreover, we have for  $\alpha = (\beta, k) = (\beta_1, \beta_2, k)$  with  $|\alpha| \leq m$ ,

$$\|D^\alpha(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon)(t)\|_{L_{t+k}^2(\Omega)}, \|\partial_\tau^\beta \tilde{r}_3^\epsilon(t)\|_{L_{-1}^2(\Omega)} \leq \sum_{\beta_1 \leq i \leq m} t^{i-\beta_1} \mathcal{P} \left( M_0 + \|(u_0, h_0)\|_{H_t^{2i+2+\beta_2+k}} \right). \quad (4.5)$$

Based on the a priori energy estimates established in Proposition 3.6, we can obtain

**Proposition 4.1.** *Under the hypotheses of Theorem 2.2, there exist a time  $0 < T_* \leq T$ , independent of  $\epsilon$ , and a solution  $(u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)$  to the initial boundary value problem (4.1) with  $(u^\epsilon, h^\epsilon) \in L^\infty(0, T_*; \mathcal{H}_t^m)$ , which satisfies the following uniform estimates in  $\epsilon$ :*

$$\sup_{0 \leq t \leq T_*} \|(u^\epsilon, h^\epsilon)(t)\|_{\mathcal{H}_t^m} \leq 2F_0^{\frac{1}{2}}, \quad (4.6)$$

where  $F_0$  is given by (3.92). Moreover, for  $t \in [0, T_*]$ ,  $(x, y) \in \Omega$ ,

$$\|\langle y \rangle^{l+1} \partial_y^i (u^\epsilon, h^\epsilon)(t)\|_{L^\infty(\Omega)} \leq \delta_0^{-1}, \quad h^\epsilon(t, x, y) + H(t, x)\phi'(y) \geq \delta_0, \quad i = 1, 2. \quad (4.7)$$

**Proof.** Since the problem (4.1) is a parabolic system, it is standard to show that (4.1) admits a solution in a time interval  $[0, T_\epsilon]$  ( $T_\epsilon$  may depend on  $\epsilon$ ) satisfying the estimates (4.7). Indeed, one can establish a priori estimates for (4.1), and then obtain the local existence of solution by the standard iteration and weak convergence methods.

On the other hand, we can derive the similar a priori estimates as in Proposition 3.6 for (4.1), so by the standard continuity argument we can obtain the existence of solution in a time interval  $[0, T_*]$ ,  $T_* > 0$  independent of  $\epsilon$ . Therefore, we only determine the uniform lifespan  $T_*$ , and verify the estimates (4.6) and (4.7).

According to Proposition 3.6, we can obtain the estimates for  $(u^\epsilon, h^\epsilon)$  similar as (3.98):

$$\sup_{0 \leq s \leq t} \|(u^\epsilon, h^\epsilon)(s)\|_{\mathcal{H}_t^m} \leq (F_0 + \int_0^t F^\epsilon(s) ds)^{\frac{1}{2}} \cdot \left\{ 1 - 2C\delta_0^{-8} (F_0 + \int_0^t F^\epsilon(s) ds)^2 t \right\}^{-\frac{1}{4}}, \quad (4.8)$$

as long as the quantity in  $\{\cdot\}$  on the right-hand side of (3.2) is positive, where the quantity  $F_0$  is given by (3.92), and  $F^\epsilon(t)$  is defined as follows (similar as (3.93)):

$$\begin{aligned} F^\epsilon(t) := & C \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(r_1^\epsilon, r_2^\epsilon)(t)\|_{L_{t+k}^2(\Omega)}^2 + C\delta_0^{-4} \sum_{|\beta|=m} \left( \|\partial_\tau^\beta(r_1^\epsilon, r_2^\epsilon)(t)\|_{L_t^2(\Omega)}^2 + 4\delta_0^{-4} \|\partial_\tau^\beta r_3^\epsilon\|_{L_{-1}^2(\Omega)}^2 \right) \\ & + C\delta_0^{-8} \left( 1 + \sum_{|\beta| \leq m+2} \|\partial_\tau^\beta(U, H, P)(t)\|_{L^2(\mathbb{T}_x)}^2 \right)^3. \end{aligned} \quad (4.9)$$

Substituting (4.2)-(4.4) into (4.9) and recalling  $F(t)$  defined by (3.93), it yields that

$$F^\epsilon(t) = F(t) + C\epsilon^2 \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m-1}} \|D^\alpha(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon)(t)\|_{L_{t+k}^2(\Omega)}^2 + C\epsilon^2 \delta_0^{-4} \sum_{|\beta|=m} \left( \|\partial_\tau^\beta(\tilde{r}_1^\epsilon, \tilde{r}_2^\epsilon)(t)\|_{L_t^2(\Omega)}^2 + 4\delta_0^{-4} \|\partial_\tau^\beta \tilde{r}_3^\epsilon\|_{L_{-1}^2(\Omega)}^2 \right),$$

which implies that from (3.101) and (4.5),

$$F^\epsilon(t) \leq C\delta_0^{-8} M_0^6 + \epsilon^2 \delta_0^{-8} \mathcal{P} (M_0 + \|(u_0, h_0)\|_{H_t^{3m+2}}) \leq \delta_0^{-8} \mathcal{P} (M_0 + \|(u_0, h_0)\|_{H_t^{3m+2}}).$$

Therefore, by choosing

$$T_1 := \min \left\{ \frac{\delta_0^8 F_0}{\mathcal{P} (M_0 + \|(u_0, h_0)\|_{H_t^{3m+2}})}, \frac{3\delta_0^8}{32CF_0^2} \right\}$$

in (4.8), we obtain (4.6) for  $T_* \leq T_1$ .

On the other hand, similar as the estimates (3.99) and (3.100) given in Proposition 3.6, we have the following bounds for  $\langle y \rangle^{l+1} \partial_y^i(u^\epsilon, h^\epsilon)$ ,  $i = 1, 2$  and  $h^\epsilon$ :

$$\|\langle y \rangle^{l+1} \partial_y^i(u^\epsilon, h^\epsilon)(t)\|_{L^\infty(\Omega)} \leq \|\langle y \rangle^{l+1} \partial_y^i(u_0, h_0)\|_{L^\infty(\Omega)} + Ct \cdot \left( \sup_{0 \leq s \leq t} \|(u^\epsilon, h^\epsilon)(s)\|_{\mathcal{H}_1^5} \right), \quad i = 1, 2, \quad (4.10)$$

and

$$h^\epsilon(t, x, y) \geq h_0(x, y) - Ct \cdot \left( \sup_{0 \leq s \leq t} \|(u^\epsilon, h^\epsilon)(s)\|_{\mathcal{H}_0^3} \right). \quad (4.11)$$

Then, from the assumptions (2.18) for the initial data  $(u_0, h_0)$ , and the chosen of  $T_1$  above, we obtain that by (4.8),

$$\begin{aligned} \|\langle y \rangle^{l+1} \partial_y^i(u^\epsilon, h^\epsilon)(t)\|_{L^\infty(\Omega)} &\leq (2\delta_0)^{-1} + 2CF_0^{\frac{1}{2}}t, \quad i = 1, 2, \\ h^\epsilon(t, x, y) + H(t, x)\phi'(y) &\geq 2\delta_0 + (H(t, x) - H(0, x))\phi'(y) - 2CF_0^{\frac{1}{2}}t \geq 2\delta_0 - C(M_0 + 2F_0^{\frac{1}{2}})t. \end{aligned}$$

So, let us choose

$$T_2 := \min \left\{ T_1, \frac{1}{4C\delta_0 F_0^{\frac{1}{2}}}, \frac{\delta_0}{C(M_0 + 2F_0^{\frac{1}{2}})} \right\},$$

then, (4.7) holds for  $T_* = T_2$ . Therefore, we find the lifespan  $T_* = T_2$  and establish the estimates (4.6) and (4.7), and consequently complete the proof of this proposition.  $\square$

From the above Proposition 4.1, we obtain the local existence of solutions  $(u^\epsilon, v^\epsilon, h^\epsilon, g^\epsilon)$  to the problem (4.1) and their uniform estimates in  $\epsilon$ . Now, by letting  $\epsilon \rightarrow 0$  we will obtain the solution to the original problem (2.12) through some compactness arguments. Indeed, from the uniform estimate (4.6), by the Lions-Aubin lemma and the compact embedding of  $H_l^m(\Omega)$  in  $H_{loc}^{m'}$  for  $m' < m$  (see [30, Lemma 6.2]), we know that there exists  $(u, h) \in L^\infty(0, T_*; \mathcal{H}_l^m) \cap \left( \bigcap_{m' < m-1} C^1([0, T_*]; H_{loc}^{m'}(\Omega)) \right)$ , such that, up to a subsequence,

$$\begin{aligned} \partial_t^i(u^\epsilon, h^\epsilon) &\xrightarrow{*} \partial_t^i(u, h), \quad \text{in } L^\infty(0, T_*; H_l^{m-i}(\Omega)), \quad 0 \leq i \leq m, \\ (u^\epsilon, h^\epsilon) &\rightarrow (u, h), \quad \text{in } C^1([0, T_*]; H_{loc}^{m'}(\Omega)). \end{aligned}$$

Then, by using the uniform convergence of  $(\partial_x u^\epsilon, \partial_x h^\epsilon)$  because of  $(\partial_x u^\epsilon, \partial_x h^\epsilon) \in Lip(\Omega_{T_*})$ , we get the pointwise convergence for  $(v^\epsilon, g^\epsilon)$ , i.e.,

$$(v^\epsilon, g^\epsilon) = \left( - \int_0^y \partial_x u^\epsilon dz, - \int_0^y \partial_x h^\epsilon dz \right) \rightarrow \left( - \int_0^y \partial_x u dz, - \int_0^y \partial_x h dz \right) := (v, g). \quad (4.12)$$

Now, we can pass the limit  $\epsilon \rightarrow 0$  in the problem (4.1), and obtain that  $(u, v, h, g)$ ,  $v$  and  $g$  given by (4.12), solves the original problem (2.12). As  $(u, h) \in L^\infty(0, T_*; \mathcal{H}_l^m)$  it is easy to get that  $(u, h) \in \bigcap_{i=0}^m W^{i, \infty}(0, T; H_l^{m-i}(\Omega))$ , and consequently (2.19) is proven. Moreover, the relation (2.20), respectively (2.21), follows immediately by combining the divergence free conditions  $v = -\partial_y^{-1} \partial_x u$ ,  $g = -\partial_y^{-1} \partial_x h$  with (2.7), respectively (2.8). Thus, we prove the local existence result of Theorem 2.2.

#### 4.2. Uniqueness.

We will show the uniqueness of the obtained solution to (2.12). Let  $(u^1, v^1, h^1, g^1)$  and  $(u^2, v^2, h^2, g^2)$  be two solutions in  $[0, T_*]$ , constructed in the previous subsection, with respect to the initial data  $(u_0^1, h_0^1)$  and  $(u_0^2, h_0^2)$  respectively. Set

$$(\tilde{u}, \tilde{v}, \tilde{h}, \tilde{g}) = (u^1 - u^2, v^1 - v^2, h^1 - h^2, g^1 - g^2),$$

then we have

$$\begin{cases} \partial_t \tilde{u} + [(u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y]\tilde{u} - [(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y]\tilde{h} - \mu \partial_y^2 \tilde{u} \\ \quad + (\partial_x u^2 + U_x\phi')\tilde{u} + (\partial_y u^2 + U\phi'')\tilde{v} - (\partial_x h^2 + H_x\phi')\tilde{h} - (\partial_y h^2 + H\phi'')\tilde{g} = 0, \\ \partial_t \tilde{h} + [(u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y]\tilde{h} - [(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y]\tilde{u} - \kappa \partial_y^2 \tilde{h} \\ \quad + (\partial_x h^2 + H_x\phi')\tilde{u} + (\partial_y h^2 + H\phi'')\tilde{v} - (\partial_x u^2 + U_x\phi')\tilde{h} - (\partial_y u^2 + U\phi'')\tilde{g} = 0, \\ \partial_x \tilde{u} + \partial_y \tilde{v} = 0, \quad \partial_x \tilde{h} + \partial_y \tilde{g} = 0, \\ (\tilde{u}, \tilde{h})|_{t=0} = (u_0^1 - u_0^2, h_0^1 - h_0^2), \quad (\tilde{u}, \tilde{v}, \partial_y \tilde{h}, \tilde{g})|_{y=0} = \mathbf{0}. \end{cases} \quad (4.13)$$

Denote by  $\tilde{\psi} := \partial_y^{-1}\tilde{h} = \partial_y^{-1}(h^1 - h^2)$ , then from the second equation (4.13)<sub>2</sub> of  $\tilde{h}$  and the divergence free conditions, we know that  $\tilde{\psi}$  satisfies the following equation:

$$\partial_t \tilde{\psi} + [(u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y]\tilde{\psi} - (g^2 - H_x\phi)\tilde{u} + (h^2 + H\phi')\tilde{v} - \kappa\partial_y^2\tilde{\psi} = 0. \quad (4.14)$$

Similar as (3.51), we introduce the new quantities:

$$\bar{u} := \tilde{u} - \frac{\partial_y u^2 + U\phi''}{h^2 + H\phi'}\tilde{\psi}, \quad \bar{h} := \tilde{h} - \frac{\partial_y h^2 + H\phi''}{h^2 + H\phi'}\tilde{\psi}, \quad (4.15)$$

and then,

$$\bar{u} := u^1 - u^2 - \eta_1^2 \partial_y^{-1}(h^1 - h^2), \quad \bar{h} := h^1 - h^2 - \eta_2^2 \partial_y^{-1}(h^1 - h^2), \quad (4.16)$$

where we denote

$$\eta_1^2 := \frac{\partial_y u^2 + U\phi''}{h^2 + H\phi'}, \quad \eta_2^2 := \frac{\partial_y h^2 + H\phi''}{h^2 + H\phi'}.$$

Next, we can obtain that through direct calculation,  $(\bar{u}, \bar{h})$  admits the following initial-boundary value problem:

$$\begin{cases} \partial_t \bar{u} + [(u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y]\bar{u} - [(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y]\bar{h} - \mu\partial_y^2\bar{u} + (\kappa - \mu)\eta_1^2\partial_y\bar{h} \\ \quad + a_1\bar{u} + b_1\bar{h} + c_1\tilde{\psi} = 0, \\ \partial_t \bar{h} + [(u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y]\bar{h} - [(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y]\bar{u} - \kappa\partial_y^2\bar{h} \\ \quad + a_2\bar{u} + b_2\bar{h} + c_2\tilde{\psi} = 0, \\ (\bar{u}, \partial_y\bar{h})|_{y=0} = 0, \quad (\bar{u}, \bar{h})|_{t=0} = (u_0^1 - u_0^2 - \eta_{10}^2 \partial_y^{-1}(h_0^1 - h_0^2), h_0^1 - h_0^2 - \eta_{20}^2 \partial_y^{-1}(h_0^1 - h_0^2)), \end{cases} \quad (4.17)$$

where

$$\begin{aligned} a_1 &= \partial_x u^2 + U_x\phi' + (g^2 - H_x\phi)\eta_1^2, \quad b_1 = (\kappa - \mu)\eta_1^2\eta_2^2 - 2\mu\partial_y\eta_1^2 - (\partial_x h^2 + H_x\phi') - (g^2 - H_x\phi)\eta_2^2, \\ c_1 &= [\partial_t + (u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y - \mu\partial_y^2]\eta_1^2 - [(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y]\eta_2^2 - 2\mu\eta_2^2\partial_y\eta_1^2 \\ &\quad + (\kappa - \mu)\eta_1^2[(\eta_2^2)^2 + \partial_y\eta_2^2] + (g^2 - H_x\phi)[(\eta_1^2)^2 - (\eta_2^2)^2] + (\partial_x u^2 + U_x\phi')\eta_1^2 - (\partial_x h^2 + H_x\phi')\eta_2^2, \\ a_2 &= \partial_x h^2 + H_x\phi' + (g^2 - H_x\phi)\eta_2^2, \quad b_2 = -2\kappa\partial_y\eta_2^2 - (\partial_x u^2 + U_x\phi') - (g^2 - H_x\phi)\eta_1^2, \\ c_2 &= [\partial_t + (u^1 + U\phi')\partial_x + (v^1 - U_x\phi)\partial_y - \kappa\partial_y^2]\eta_2^2 - [(h^1 + H\phi')\partial_x + (g^1 - H_x\phi)\partial_y]\eta_1^2 - 2\kappa\eta_2^2\partial_y\eta_2^2 \\ &\quad + (\partial_x h^2 + H_x\phi')\eta_1^2 - (\partial_x u^2 + U_x\phi')\eta_2^2, \end{aligned} \quad (4.18)$$

and

$$\eta_{10}^2(x, y) := \frac{\partial_y u_0^2 + U(0, x)\phi''(y)}{h_0^2 + H(0, x)\phi'(y)}, \quad \eta_{20}^2(x, y) := \frac{\partial_y h_0^2 + H(0, x)\phi''(y)}{h_0^2 + H(0, x)\phi'(y)}.$$

Combining (4.15) with the fact  $\tilde{\psi} = \partial_y^{-1}\tilde{h}$ , we get that

$$\bar{h} = (h^2 + H\phi') \cdot \partial_y \left( \frac{\tilde{\psi}}{h^2 + H\phi'} \right),$$

and then, by  $\tilde{\psi}|_{y=0} = 0$ ,

$$\tilde{\psi}(t, x, y) = (h^2(t, x, y) + H(t, x)\phi'(y)) \cdot \int_0^y \frac{\bar{h}(t, x, z)}{h^2(t, x, z) + H(t, x)\phi'(z)} dz. \quad (4.19)$$

Since  $h^2 + H\phi' \geq \delta_0$ , applying (2.7) in (4.19) gives

$$\left\| \frac{\tilde{\psi}(t)}{1+y} \right\|_{L^2(\Omega)} \leq 2\delta_0^{-1} \|h^2 + H\phi'\|_{L^\infty([0, T_*] \times \Omega)} \|\bar{h}(t)\|_{L^2(\Omega)}. \quad (4.20)$$

Moreover, through a similar process of getting the estimates (3.70), we can obtain that there exists a constant

$$C = C(T_*, \delta_0, \phi, U, H, \|(u^1, h^1)\|_{\mathcal{H}_i^5}, \|(u^2, h^2)\|_{\mathcal{H}_i^5}) > 0,$$

such that

$$\|a_i\|_{L^\infty([0, T_*] \times \Omega)}, \|b_i\|_{L^\infty([0, T_*] \times \Omega)}, \|(1+y)c_i\|_{L^\infty([0, T_*] \times \Omega)} \leq C, \quad i = 1, 2. \quad (4.21)$$

Thus, we have from (4.20) and (4.21),

$$\|(c_i \tilde{\psi})(t)\|_{L^2(\Omega)} \leq C \|\bar{h}(t)\|_{L^2(\Omega)}, \quad i = 1, 2. \quad (4.22)$$

**Proposition 4.2.** *Let  $(u^1, v^1, h^1, g^1)$  and  $(u^2, v^2, h^2, g^2)$  be two solutions of problem (2.12) with respect to the initial data  $(u_0^1, h_0^1)$  and  $(u_0^2, h_0^2)$  respectively, satisfying that  $(u^j, h^j) \in \bigcap_{i=0}^m W^{i,\infty}(0, T; H_i^{m-i}(\Omega))$  for  $m \geq 5$ ,  $j = 1, 2$ . Then, there exists a positive constant*

$$C = C\left(T_*, \delta_0, \phi, U, H, \|(u^1, h^1)\|_{\mathcal{H}_1^5}, \|(u^2, h^2)\|_{\mathcal{H}_1^5}\right) > 0,$$

such that for the quantities  $(\bar{u}, \bar{h})$  given by (4.16),

$$\frac{d}{dt} \|(\bar{u}, \bar{h})(t)\|_{L^2(\Omega)}^2 + \|(\partial_y \bar{u}, \partial_y \bar{h})(t)\|_{L^2(\Omega)}^2 \leq C \|(\bar{u}, \bar{h})\|_{L^2(\Omega)}^2. \quad (4.23)$$

The above Proposition 4.2 can be proved by the standard energy method and the estimates (4.21), (4.22), here we omit the proof for brevity of presentation. Then, by virtue of Proposition 4.2 we can prove the uniqueness of solutions to (2.12) as follows.

Firstly, if the initial data satisfying  $(u^1, h^1)|_{t=0} = (u^2, h^2)|_{t=0}$ , then we know that from (4.17),  $(\bar{u}, \bar{h})$  admits the zero initial data, which implies that  $(\bar{u}, \bar{h}) \equiv 0$  by applying Gronwall's lemma to (4.23). Secondly, it yields that  $\tilde{\psi} \equiv 0$  by plugging  $\bar{h} \equiv 0$  into (4.19). Then, from (4.16) we have  $(u^1, h^1) \equiv (u^2, h^2)$  immediately through the following calculation:

$$(u^1, h^1) - (u^2, h^2) = (\tilde{u}, \tilde{h}) = (\bar{u}, \bar{h}) + (\eta_1^2, \eta_2^2) \tilde{\psi} \equiv 0.$$

Finally, we obtain  $(v^1, g^1) \equiv (v^2, g^2)$  since  $v^i = -\partial_y^{-1} \partial_x u^i$  and  $g^i = -\partial_y^{-1} \partial_x h^i$  for  $i = 1, 2$ , and show the uniqueness of solutions.

*Remark 4.1.* We mention that in the independent recent preprint [13], the authors give a systematic derivation of MHD boundary layer models, and consider the linearization for the similar system as (1.8) around some shear flow. By using the analogous transformation to (3.61), they obtain the linear stability for the system in the Sobolev framework.

## 5. A COORDINATE TRANSFORMATION

In this section, we will introduce another method to study the initial-boundary value problem considered in this paper:

$$\begin{cases} \partial_t u_1 + u_1 \partial_x u_1 + u_2 \partial_y u_1 = h_1 \partial_x h_1 + h_2 \partial_y h_1 + \mu \partial_y^2 u_1, \\ \partial_t h_1 + \partial_y (u_2 h_1 - u_1 h_2) = \kappa \partial_y^2 h_1, \\ \partial_x u_1 + \partial_y u_2 = 0, \quad \partial_x h_1 + \partial_y h_2 = 0, \\ (u_1, u_2, \partial_y h_1, h_2)|_{y=0} = 0, \quad \lim_{y \rightarrow +\infty} (u_1, h_1) = (U, H). \end{cases} \quad (5.1)$$

As we mentioned in Subsectin 2.3, by the divergence free condition,

$$\partial_x h_1 + \partial_y h_2 = 0,$$

there exists a stream function  $\psi$ , such that

$$h_1 = \partial_y \psi, \quad h_2 = -\partial_x \psi, \quad \psi|_{y=0} = 0, \quad (5.2)$$

moreover,  $\psi$  satisfies

$$\partial_t \psi + u_1 \partial_x \psi + u_2 \partial_y \psi = \kappa \partial_y^2 \psi. \quad (5.3)$$

Under the assumptions that

$$h_1(t, x, y) > 0, \quad \text{or} \quad \partial_y \psi(t, x, y) > 0, \quad (5.4)$$

we can introduce the following transformation

$$\tau = t, \quad \xi = x, \quad \eta = \psi(t, x, y), \quad (5.5)$$

and then, (5.1) can be written in the new coordinates as follows:

$$\begin{cases} \partial_\tau u_1 + u_1 \partial_\xi u_1 - h_1 \partial_\xi h_1 + (\kappa - \mu) h_1 \partial_\eta h_1 \partial_\eta u_1 = \mu h_1^2 \partial_\eta^2 u_1, \\ \partial_\tau h_1 - h_1 \partial_\xi u_1 + u_1 \partial_\xi h_1 = \kappa h_1^2 \partial_\eta^2 h_1, \\ (u_1, h_1 \partial_\eta h_1)|_{y=0} = 0, \quad \lim_{\eta \rightarrow +\infty} (u_1, h_1) = (U, H). \end{cases} \quad (5.6)$$

*Remark 5.1.* The equations (5.6) are quasi-linear equations, and there is no loss of regularity term in (5.6), then we can use the classical Picard iteration scheme to establish the local existence. However, in order to guarantee the coordinates transformation to be valid, one needs to assume that  $h_1(t, x, y) > 0$ . Moreover, one can obtain the stability of solutions to (5.6) in the new coordinates  $(\tau, \xi, \eta)$ . It is necessary to transfer the well-posedness of solutions to the original equations (5.1). And then, there will be some loss of regularity.

*Remark 5.2.* Based on the well-posedness result for MHD boundary layer in the Sobolev framework given in this paper, we will show the validity of the vanishing limit of the viscous MHD equations (1.1) as  $\epsilon \rightarrow 0$  in a future work [26], that is, to show the solution to (1.1) converges to a solution of ideal MHD equations, corresponding to  $\epsilon = 0$  in (1.1), outside the boundary layer, and to a boundary layer profile studied in this paper inside the boundary layer.

#### APPENDIX A. SOME INEQUALITIES

In this appendix, we will prove the inequalities given in Lemma (2.1). Such inequalities can be found in [30] and [40], here we give a proof for readers' convenience.

**Proof of Lemma 2.1.** i) From  $\lim_{y \rightarrow +\infty} (fg)(x, y) = 0$ , it yields

$$\begin{aligned} \left| \int_{\mathbb{T}_x} (fg)|_{y=0} dx \right| &= \left| \int_{\Omega} \partial_y (fg) dx dy \right| \leq \int_{\Omega} |\partial_y f \cdot g| dx dy + \int_{\Omega} |f \cdot \partial_y g| dx dy \\ &\leq \|\partial_y f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \|\partial_y g\|_{L^2(\Omega)}, \end{aligned}$$

and we get (2.2). (2.3) follows immediately by letting  $g = f$  in (2.2).

ii) From  $m \geq 3$  and  $|\alpha| + |\tilde{\alpha}| \leq m$ , we know that there must be  $|\alpha| \leq m - 2$  or  $|\tilde{\alpha}| \leq m - 2$ . Without loss of generality, we assume that  $|\alpha| \leq m - 2$ , then for any  $l_1, l_2 \geq 0$  with  $l_1 + l_2 = l$ , we have that by using Sobolev embedding inequality,

$$\begin{aligned} \|(D^\alpha f \cdot D^{\tilde{\alpha}} g)(t, \cdot)\|_{L^2_{l_1+k+\tilde{k}}(\Omega)} &\leq \|\langle y \rangle^{l_1+k} D^\alpha f(t, \cdot)\|_{L^\infty(\Omega)} \cdot \|\langle y \rangle^{l_2+\tilde{k}} D^{\tilde{\alpha}} g(t, \cdot)\|_{L^2(\Omega)} \\ &\leq C \|\langle y \rangle^{l_1+k} D^\alpha f(t, \cdot)\|_{H^2(\Omega)} \|g(t)\|_{\mathcal{H}_{l_2}^{|\tilde{\alpha}|}} \\ &\leq C \|f(t)\|_{\mathcal{H}_{l_1}^{|\alpha|+2}(\Omega)} \|g(t)\|_{\mathcal{H}_{l_2}^m}, \end{aligned}$$

which implies (2.4) because of  $|\alpha| + 2 \leq m$ .

iii) For  $\lambda > \frac{1}{2}$ , it follows that by integration by parts,

$$\begin{aligned} \|\langle y \rangle^{-\lambda} (\partial_y^{-1} f)(y)\|_{L^2_y(\mathbb{R}_+)}^2 &= \int_0^{+\infty} \frac{[(\partial_y^{-1} f)(y)]^2}{1-2\lambda} d(1+y)^{1-2\lambda} = \frac{2}{2\lambda-1} \int_0^{+\infty} (1+y)^{1-2\lambda} f(y) \cdot (\partial_y^{-1} f)(y) dy \\ &\leq \frac{2}{2\lambda-1} \|\langle y \rangle^{-\lambda} (\partial_y^{-1} f)(y)\|_{L^2_y(\mathbb{R}_+)} \cdot \|\langle y \rangle^{1-\lambda} f(y)\|_{L^2_y(\mathbb{R}_+)}, \end{aligned}$$

which implies the first inequality of (2.5).

On the other hand, note that for  $\tilde{\lambda} > 0$ ,

$$\begin{aligned} |(\partial_y^{-1} f)(y)| &\leq \int_0^y |f(z)| dz \leq \|(1+z)^{1-\tilde{\lambda}} f(z)\|_{L^\infty(0,y)} \cdot \int_0^y (1+z)^{\tilde{\lambda}-1} dz \\ &\leq \frac{(1+y)^{\tilde{\lambda}} - 1}{\tilde{\lambda}} \|(1+y)^{1-\tilde{\lambda}} f(y)\|_{L^\infty_y(\mathbb{R}_+)}, \end{aligned}$$

which implies the second inequality of (2.5) immediately.

Next, as  $m \geq 3$  and  $|\alpha| + |\tilde{\beta}| \leq m$ , we also get  $|\alpha| \leq m-2$  or  $|\tilde{\beta}| \leq m-2$ . If  $|\alpha| \leq m-2$ , by using Sobolev embedding inequality and the first inequality of (2.5), we have for any  $\lambda > \frac{1}{2}$ ,

$$\begin{aligned} \|(D^\alpha g \cdot \partial_\tau^{\tilde{\beta}} \partial_y^{-1} h)(t, \cdot)\|_{L_{t+k}^2(\Omega)} &\leq \|\langle y \rangle^{l+\lambda+k} D^\alpha g(t, \cdot)\|_{L^\infty(\Omega)} \cdot \|\langle y \rangle^{-\lambda} \partial_\tau^{\tilde{\beta}} \partial_y^{-1} h(t, \cdot)\|_{L^2(\Omega)} \\ &\leq C \|\langle y \rangle^{l+\lambda+k} D^\alpha g(t, \cdot)\|_{H^2(\Omega)} \cdot \|\langle y \rangle^{1-\lambda} \partial_\tau^{\tilde{\beta}} h(t, \cdot)\|_{L^2(\Omega)} \\ &\leq C \|g(t)\|_{\mathcal{H}_{l+\lambda}^{|\alpha|+2}} \|h(t)\|_{\mathcal{H}_{1-\lambda}^{|\tilde{\beta}|}}. \end{aligned}$$

If  $|\tilde{\beta}| \leq m-2$ , by Sobolev embedding inequality and the second inequality of (2.5),

$$\begin{aligned} \|(D^\alpha g \cdot \partial_\tau^{\tilde{\beta}} \partial_y^{-1} h)(t, \cdot)\|_{L_{t+k}^2(\Omega)} &\leq \|\langle y \rangle^{l+\lambda+k} D^\alpha g(t, \cdot)\|_{L^2(\Omega)} \cdot \|\langle y \rangle^{-\lambda} \partial_\tau^{\tilde{\beta}} \partial_y^{-1} h(t, \cdot)\|_{L^\infty(\Omega)} \\ &\leq C \|g(t)\|_{\mathcal{H}_{l+\lambda}^{|\alpha|}} \cdot \|\langle y \rangle^{1-\lambda} \partial_\tau^{\tilde{\beta}} h(t, \cdot)\|_{H^2(\Omega)} \\ &\leq C \|g(t)\|_{\mathcal{H}_{l+\lambda}^{|\alpha|}} \|h(t)\|_{\mathcal{H}_{1-\lambda}^{|\tilde{\beta}|+2}}. \end{aligned}$$

Thus, we get the proof of (2.6), and then, (2.7) follows by letting  $\lambda = 1$  in (2.6).

iv) For any  $\lambda > \frac{1}{2}$ ,

$$|(\partial_y^{-1} f)(y)| \leq \|f(y)\|_{L_y^1(\mathbb{R}_+^2)} \leq \|\langle y \rangle^{-\lambda}\|_{L_y^2(\mathbb{R}_+)} \|\langle y \rangle^\lambda f\|_{L_y^2(\mathbb{R}_+)} \leq C \|\langle y \rangle^\lambda f\|_{L_y^2(\mathbb{R}_+)},$$

and we get (2.8).

For  $m \geq 2$  and  $|\alpha| + |\tilde{\beta}| \leq m$ , we get that  $|\alpha| \leq m-1$  or  $|\tilde{\beta}| \leq m-1$ . If  $|\alpha| \leq m-1$ , by using Sobolev embedding inequality and (2.8), we have for any  $\lambda > \frac{1}{2}$ ,

$$\begin{aligned} \|(D^\alpha f \cdot \partial_\tau^{\tilde{\beta}} \partial_y^{-1} g)(t, \cdot)\|_{L_{t+k}^2(\Omega)} &\leq \|\langle y \rangle^{l+k} D^\alpha f(t, \cdot)\|_{L_x^\infty L_y^2(\Omega)} \cdot \|\partial_\tau^{\tilde{\beta}} \partial_y^{-1} g(t, \cdot)\|_{L_x^2 L_y^\infty(\Omega)} \\ &\leq C \|\langle y \rangle^{l+k} D^\alpha f(t, \cdot)\|_{H^1(\Omega)} \cdot \|\langle y \rangle^\lambda \partial_\tau^{\tilde{\beta}} g(t, \cdot)\|_{L^2(\Omega)} \\ &\leq C \|f(t)\|_{\mathcal{H}_l^{|\alpha|+1}} \|g(t)\|_{\mathcal{H}_\lambda^{|\tilde{\beta}|}}. \end{aligned}$$

If  $|\tilde{\beta}| \leq m-1$ , by Sobolev embedding inequality and (2.8),

$$\begin{aligned} \|(D^\alpha f \cdot \partial_\tau^{\tilde{\beta}} \partial_y^{-1} g)(t, \cdot)\|_{L_{t+k}^2(\Omega)} &\leq \|\langle y \rangle^{l+k} D^\alpha f(t, \cdot)\|_{L^2(\Omega)} \cdot \|\partial_\tau^{\tilde{\beta}} \partial_y^{-1} g(t, \cdot)\|_{L^\infty(\Omega)} \\ &\leq C \|f(t)\|_{\mathcal{H}_l^{|\alpha|}} \cdot \|\langle y \rangle^\lambda \partial_\tau^{\tilde{\beta}} g(t, \cdot)\|_{H_x^1 L_y^2(\Omega)} \\ &\leq C \|f(t)\|_{\mathcal{H}_l^{|\alpha|}} \|g(t)\|_{\mathcal{H}_\lambda^{|\tilde{\beta}|+1}}. \end{aligned}$$

Thus, we get (2.9), and then complete the proof of this lemma.  $\square$

**Acknowledgements:** The second author is partially supported by NSFC (Grant No.11171213, No.11571231). The third author is supported by the General Research Fund of Hong Kong, CityU No. 11320016, and he would like to thank Pierre Degond for the initial discussion on this problem at Imperial College.

## REFERENCES

- [1] Alexander R., Wang Y.-G., Xu C.-J., Yang T., Well posedness of the Prandtl equation in Sobolev spaces. *J. Amer. Math. Soc.* **28** (2015), 3, 745-784.
- [2] Alfvén H., Existence of electromagnetic-hydrodynamic waves. *Nature* **150** (1942), 405-406.
- [3] Arkhipov V.N., Influence of magnetic field on boundary layer stability, *Soviet Physics (Doklady) Trans.* **4**, 1199-1201.
- [4] Cowling T. G., Magnetohydrodynamic. Interscience tracts on physics and astronomy, New-York, 1957.
- [5] Davidson P. A., An introduction to Magnetohydrodynamics. Cambridge University Press, Cambridge, 2001.
- [6] Drasin P., Stability of parallel flow in a parallel magnetic field at small magnetic Reynolds number. *J. Fluid Mech.* **8** (1960), 130-142.
- [7] Duvaut G., Lions J.-L., Inéquations en thermoélasticité et magnétohydrodynamique. *Arch. Rational Mech. Anal.* **46** (1972), 241-279.
- [8] E W.-N., Engquist B., Blowup of solutions of the unsteady Prandtl's equation. *Comm. Pure Appl. Math.* **50** (1997), 12, 1287-1293.



- [9] Gérard-Varet D., Dormy E., On the ill-posedness of the Prandtl equations. *J. Amer. Math. Soc.* **23** (2010), 591-609.
- [10] Gérard-Varet D., Maekawa Y., Masmoudi N., Gevrey stability of Prandtl expansions for 2D Navier-Stokes flows. Preprint, 2016. arXiv: 1607.06434.
- [11] Gérard-Varet D., Masmoudi N., Well-posedness for the Prandtl system without analyticity or monotonicity. *Ann. Sci. Éc. Norm. Supér.* **48** (2015), 6, 1273-1325.
- [12] Gérard-Varet D., Nguyen T., Remarks on the ill-posedness of the Prandtl equation. *Asymptot. Anal.* **77** (2012), 1-2, 71-88.
- [13] Gérard-Varet D., Prestipino M., Formal Derivation and Stability Analysis of Boundary Layer Models in MHD. Preprint, 2016. arXiv:1612.02641.
- [14] Grenier E., On the nonlinear instability of Euler and Prandtl equations. *Comm. Pure Appl. Math.* **53** (2000), 9, 1067-1091.
- [15] Guo Y., Nguyen T., A note on the Prandtl boundary layers. *Comm. Pure Appl. Math.* **64** (2011), 1416-1438.
- [16] Hartmann J., Theory of the laminar flow of an electrically conductive liquid in a homogeneous magnetic field. *K. Dan. Vidensk. Selsk. Mat. Fys. Medd.* **15**(1937), 6, 1-28.
- [17] Hartmann J., Lazarus F., Experimental investigations on the flow of mercury in a homogeneous magnetic field. *K. Dan. Vidensk. Selsk. Mat. Fys. Medd.* **15**(1937), 7, 1-45.
- [18] Ignatova M., Vicol V., Almost global existence for the Prandtl boundary layer equations. *Arch. Ration. Mech. Anal.* **220** (2016), 2, 809-848.
- [19] Kukavica I., Vicol V., On the local existence of analytic solutions to the Prandtl boundary layer equations. *Commun. Math. Sci.* **11** (2013), 1, 269-292.
- [20] Kukavica I., Masmoudi N., Vicol V., Wong T.-K., On the local well-posedness of the Prandtl and hydrostatic Euler equations with multiple monotonicity regions. *SIAM J. Math. Anal.* **46** (2014), 6, 3865-3890.
- [21] Li W.-X., Wu D., Xu C.-J., Gevrey class smoothing effect for the Prandtl equation. *SIAM J. Math. Anal.* **48** (2016), 3, 1672-1726.
- [22] Li W.-X., Yang T., Well-posedness in Gevrey space for the Prandtl equations with non-degenerate points. Preprint, 2016. arXiv: 1609.08430.
- [23] Liu C.-J., Wang Y.-G., Yang T., A well-posedness theory for the Prandtl equations in three space variables. To appear in *Adv. Math.*, arXiv:1405.5308.
- [24] Liu C.-J., Wang Y.-G., Yang T., A global existence of weak solutions to the Prandtl equations in three space variables. *Discrete Contin. Dyn. Syst. Ser. S* **9**(2016), 6, 2011-2029.
- [25] Liu C.-J., Wang Y.-G., Yang T., On the ill-posedness of the Prandtl equations in three space dimensions. *Arch. Ration. Mech. Anal.* **220**(2016), 1, 83-108.
- [26] Liu C.-J., Xie F., Yang T., MHD boundary layers in Sobolev spaces without monotonicity. II. convergence theory. Preprint, 2017.
- [27] Liu C.-J., Yang T., Ill-posedness of the Prandtl equations in Sobolev spaces around a shear flow with general decay. Preprint, 2016. arXiv: 1605.00102.
- [28] Lombardo M. C., Cannone M., Sammartino M., Well-posedness of the boundary layer equations. *SIAM J. Math. Anal.* **35**(2003), 4, 987-1004.
- [29] Maekawa Y., On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half plane. *Comm. Pure Appl. Math.*, **67** (2014), 7, 1045-1128.
- [30] Masmoudi N., Wong T.-K., Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods. *Comm. Pure Appl. Math.* **68** (2015), 10, 1683-1741.
- [31] Oleinik O. A., The Prandtl system of equations in boundary layer theory. *Soviet Math Dokl.* **4** (1963), 583-586.
- [32] Oleinik O. A., Samokhin V. N., Mathematical Models in Boundary Layers Theory. Chapman and Hall/CRC, 1999.
- [33] Prandtl L., Über flüssigkeits-bewegung bei sehr kleiner reibung. Verhandlungen des III. Internationalen Mathematiker Kongresses, Heidelberg. Teubner, Leipzig, (1904), 484-491.
- [34] Rossow V.J., Boundary layer stability diagrams for electrically conducting fluids in the presence of a magnetic field. NACA Technical Note **4282** (1958), NACA(Washington).
- [35] Sammartino M., Caffisch R.-E., Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations. *Comm. Math. Phys.* **192** (1998), 2, 433-461.
- [36] Sammartino M., Caffisch R.-E., Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution. *Comm. Math. Phys.* **192** (1998), 2, 463-491.
- [37] Sermange M., Temam R., Some mathematical questions related to the MHD equations. *Comm. Pure Appl. Math.* **36** (1983), 5, 635-664.

- [38] Xiao Y.-L., Xin Z.-P., Wu J.-H., Vanishing viscosity limit for the 3D magnetohydrodynamic system with a slip boundary condition. *J. Funct. Anal.* **257** (2009), 11, 3375-3394.
- [39] Xin Z.-P., Zhang L., On the global existence of solutions to the Prandtl system. *Adv. Math.* **181** (2004), 88-133.
- [40] Xu C.-J., Zhang X., Long time well-posedness of the Prandtl equations in Sobolev space. Preprint, 2016. arXiv:1511.04850.
- [41] Zhang P., Zhang Z.-F., Long time well-posedness of Prandtl system with small and analytic initial data. *J. Funct. Anal.* **270** (2016), 7, 2591-2615.

CHENG-JIE LIU

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF HONG KONG, TAT CHEE AVENUE, KOWLOON, HONG KONG  
*E-mail address:* cjliusjtu@gmail.com

FENG XIE

SCHOOL OF MATHEMATICAL SCIENCES, AND LSC-MOE, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI 200240, P. R. CHINA  
*E-mail address:* tzxief@sjtu.edu.cn

TONG YANG

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF HONG KONG, TAT CHEE AVENUE, KOWLOON, HONG KONG  
DEPARTMENT OF MATHEMATICS, JINAN UNIVERSITY, GUANGZHOU 510632, P. R. CHINA  
*E-mail address:* matyang@cityu.edu.hk