A Combination of Energy Method and Spectral Analysis for the Study of Equations of Gas Motion

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Abstract

There have been extensive studies on the large time behavior of solutions to systems on gas motions, such as the Navier-Stokes equations and the Boltzmann equation. Recently, an approach is introduced by combining the energy method and the spectral analysis to the study on the optimal rates of convergence to the asymptotic profiles. In this paper, we will first illustrate this method by using some simple model and then we will present some recent results on the Navier-Stokes equations and the Boltzmann equation. Precisely, we proved the stability of the non-trivial steady state for the Navier-Stokes equations with potential forces and also obtained the optimal rate of convergence of solutions toward the steady state. The same issue was also studied for the Boltzmann equation in the presence of the general time-space dependent forces. It is expected that this approach can also be applied to other dissipative systems in fluid dynamics and kinetic models such as the model system of radiating gas and the Vlasov-Poisson-Boltzmann system.

Contents

1 Introduction

Let X be a Banach space. Consider an ordinary differential equation on the functionals on X,

$$\frac{dy}{dt} = F(t, y), \quad t \in \mathbb{R}, \tag{1.1}$$

where $y : \mathbb{R} \to X$ is an unknown functional while $F : \mathbb{R} \times X \to X$ is a given mapping. We are interested in the rate of convergence of the solution y to a time asymptotic state $y_* \in X$. Since in this paper $y(t)$ is viewed as a small perturbation of the time asymptotic profile, without loss of generality, we set $y_* = 0$ for simplicity. Moreover, we assume that F has the form

$$F(t, y) = \mathbf{L}(t)y + G(t, y) + h(t), \quad \forall\, (t, y) \in \mathbb{R} \times X. \tag{1.2}$$

Here, for any $t \in \mathbb{R}$, $\mathbf{L}(t) : D \subset X \to X$ is a linear mapping, $G(t, \cdot) : X \to Y \subset X$ is a nonlinear mapping with $G(t, 0) = 0$, and $h(t) \in Y$ is a source term, where D, being the domain of definition of $\mathbf{L}(t)$, is a dense subset of X and Y is some subspace of X. Notice that we do not assume that the mappings $\mathbf{L}(t)$ and $G(t, \cdot)$ are t-independent.

As usual, the analysis on the linearization problem is useful to deal with a nonlinear problem with small perturbation. Thus, we first consider the linearized equation

$$\frac{dy}{dt} = \mathbf{L}(t)y, \quad t \in \mathbb{R}. \tag{1.3}$$

We call

$$U(\cdot, \cdot) : \{(t, s) \in \mathbb{R} \times \mathbb{R}; t \geq s\} \to \mathscr{L}(X, X)$$

the linear solution operator corresponding to (1.3) if, for any $s \in \mathbb{R}$ and $y_0 \in X$, $U(t, s)y_0$ is the solution to the Cauchy problem

$$\begin{cases} \dfrac{dy}{dt} = \mathbf{L}(t)y, \quad t > s, \\ y|_{t=s} = y_0. \end{cases} \tag{1.4}$$

Then by the Duhamel's principle, the nonlinear equation (1.1) with (1.2) can be written into the mild form

$$y(t) = U(t, s)y(s) + \int_s^t U(t, \tau)[G(\tau, y(\tau)) + h(\tau)]d\tau,$$

for any $t \geq s$. Based on this mild form, in general, a variety of problems to the nonlinear equation (1.1) including the Cauchy problem, the time-periodic problem, the stationary problem, can be solved by the contraction mapping theorem if $U(t,s)$ enjoys some good decay properties.

In both mathematics and physics, it is important to understand the mechanism on the time asymptotic behavior of the solution operator $U(t,s)$. In general, the time decay estimate for $U(t,s)$ may has the form

$$\|U(t,s)y_0\|_X \le C_Y(1+t-s)^{-\sigma}\|y_0\|_Y,$$

for any $t \ge s$ and any $y_0 \in Y$, where C_Y is a constant depending only on Y, and $\sigma > 0$ is called the index of decay rate.

The goal here is to obtain the optimal rate index σ for a given choice of spaces X and Y. It is well-known that the spectral analysis is an efficient approach to deal with this issue. In particular, by using the classical spectral analysis only, the time decay estimates for some typical linearized equations and systems coming from the fluid dynamics or kinetic theory such as the compressible Navier-Stokes equations and the Boltzmann equation, have been well established. However, these classical spectral analysis in general may fail to be successful if the linearized equation or system has variable coefficients which arise from other physical effects, such as the external forcing. For this, instead, one can further turn to the energy method. Indeed, in some cases, it turns out that under some smallness assumptions on the coefficients, the optimal time decay rates can be obtained by combining the energy method and the spectral analysis. Here the optimal decay rate for the nonlinear system means that it is the same as the one for the corresponding linearized system.

To explain this approach, in the next section, we use a simple model to illustrate the main idea. And then we apply it to the study of the compressible Navier-Stokes equations and the Boltzmann equation in the presence of the stationary potential forcing in Section 3 and Section 4, respectively. It should be pointed out that this approach can also be applied to other physical models and also to the study of existence of time periodic solutions.

Throughout this paper, C denotes a generic large positive constant and λ denotes a generic positive small constant. Further, $\|\cdot\|$ always denotes the L^2 norm for different space dimensions.

2 Analysis on a simple model

Consider a modified linear homogenous heat equation in n-dimensional space $\mathbb{R}^n$:

$$\begin{aligned} &\frac{\partial u}{\partial t} = \mathbf{L}^{\mathrm{h}}(t)u, \quad t\in\mathbb{R}, x\in\mathbb{R}^n,\\ &\mathbf{L}^{\mathrm{h}}(t)u := (1+a(t,x))\Delta_x u, \end{aligned} \tag{2.1}$$

where $u = u(t,x)$ is an unknown function and $\mathbf{L}^{\mathrm{h}}(t)$ is a Laplacian operator with a small given perturbation $a(t,x)$. Obviously, (2.1) is parabolic if $a(t,x)$ is sufficiently small and smooth. The solution to the Cauchy problem for the equation (2.1) with given initial data at time $s \in \mathbb{R}$,

$$u|_{t=s} = u_0,$$

is denoted by

$$u(t) = U_a^{\mathrm{h}}(t,s)u_0, \quad -\infty < s \leq t < \infty,$$

where $U_a^{\mathrm{h}}(t,s)$ is the corresponding linear solution operator. We now want to obtain the time decay estimates on $U_a^{\mathrm{h}}(t,s)$ in some Sobolev space by combining the energy method and the spectral analysis.

It is well known that for the trivial case when $a(t,x) \equiv 0$, i.e. for the classical heat equation, the solution operator $U_a^{\mathrm{h}}(t,s)$ reduces to

$$U_0^{\mathrm{h}}(t,s) = e^{(t-s)\Delta_x}, \quad -\infty < s \leq t < \infty,$$

where $\{e^{t\Delta_x}\}_{t\geq 0}$ denotes the semigroup generated by the heat equation

$$u_t = \Delta_x u, \quad x \in \mathbb{R}^n.$$

By the Fourier analysis, the following result concerned with the time decay estimates on $U_0^{\mathrm{h}}(t,s)$ is classical.

Proposition 2.1. *Let $n \geq 1, k \geq 0$ be integers and $1 \leq q \leq 2$. It holds that*

$$\|\partial_x^k U_0^{\mathrm{h}}(t,s)u_0\| \leq C(1+t-s)^{\sigma_n(q,k)}\|u_0\|_{H_x^k \cap L_x^q}, \tag{2.2}$$

for any $t \geq s$ and $u_0 \in H_x^k \cap L_x^q$, where C is some constant and $\sigma_n(q,k)$ is the rate index defined by

$$\sigma_n(q,k) = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{k}{2}. \tag{2.3}$$

However, in the case with the non-trivial coefficient $a(t,x)$, the direct Fourier analysis could fail to obtain the estimates similar to (2.2) because in the frequency space, the equation (2.1) contains a non-local term

$$\mathcal{F}a(t,\cdot) * i|\eta|^2 \mathcal{F}u,$$

where $\mathcal{F}$ is the Fourier operator and $*$ denotes the convolution. For our case, a way to overcome this difficulty is a combination of the energy method and the spectral estimate in Proposition 2.1. Actually, it leads to the optimal decay estimates. Firstly, we state the main estimates as follows.

Theorem 2.1. *Let $n \geq 3, N \geq 3$ be integers. Suppose that*

$$\delta_a := \sum_{|\alpha|\leq N-1} \|\partial_x^\alpha a\|_{L_{t,x}^\infty} + \sum_{1\leq|\alpha|\leq N} \||x|\partial_x^\alpha a\|_{L_{t,x}^\infty} + \|a\|_{L_t^\infty(L_x^2)}$$

is sufficiently small. Then it holds that

$$\|U_a^{\mathrm{h}}(t,s)u_0\| \leq C(1+t-s)^{-\sigma_n(1,0)}\|u_0\|_{H_x^N \cap L_x^1}, \tag{2.4}$$

$$\|\nabla_x U_a^{\mathrm{h}}(t,s)u_0\|_{H_x^{N-1}} \leq C(1+t-s)^{-\sigma_n(1,1)}\|u_0\|_{H_x^N \cap L_x^1}, \tag{2.5}$$

for any $t \geq s$ and $u_0 \in H_x^N \cap L_x^1$, where $\sigma_n(\cdot,\cdot)$ is defined by (2.3).

Remark 2.1. *Here, we consider the case when $u_0 \in H_x^N \cap L_x^1$ for simplicity. In fact, the general case when $u_0 \in H_x^N \cap L_x^q$ for $1 \le q \le 2$ can also be considered in a similar way based on the Proposition 2.1. For the consistency of the presentation, in the next two sections, we shall apply the analysis on the simple model considered in this section to the study on some complicated nonlinear physical models only when the initial perturbation is bounded in L^1. The investigation on the more general case can be found in those relevant papers.*

Proof of Theorem 2.1: Without loss of generality, suppose $s = 0$ and set $u = U_a^{\rm h}(t,0)u_0$. For any $t \ge 0$, u solves the Cauchy problem

$$\frac{\partial u}{\partial t} - \Delta_x u = a(t,x)\Delta_x u, \quad t > 0, x \in \mathbb{R}^n, \tag{2.6}$$

$$u|_{t=0} = u_0. \tag{2.7}$$

We will obtain the following two types of estimates whose combination will then give the desired time decay estimates.

Estimates of Type I (Based on the energy method). Fix α with $1 \le |\alpha| \le N$. By applying ∂_x^α to (2.6), multiplying it by $\partial_x^\alpha u$ and taking integration over $\mathbb{R}^n$, one has

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha u\|^2 + \|\nabla_x \partial_x^\alpha u\|^2 &= \langle \partial_x^\alpha [a(t,x)\Delta_x u], \partial_x^\alpha u\rangle \\ &= \langle a(t,x)\Delta_x \partial_x^\alpha u, \partial_x^\alpha u\rangle + \sum_{\beta<\alpha} C_\beta^\alpha \langle \partial_x^{\alpha-\beta} a(t,x)\Delta_x \partial_x^\beta u, \partial_x^\alpha u\rangle \\ &= I_1 + I_2.\end{aligned} \tag{2.8}$$

I_1 and I_2 can be estimated as follows. For I_1, it holds that

$$\begin{aligned} I_1 &= -\langle a(t,x)\nabla_x \partial_x^\alpha u, \nabla_x \partial_x^\alpha u\rangle - \langle \nabla_x a(t,x)\nabla_x \partial_x^\alpha u, \partial_x^\alpha u\rangle \\ &\le \|a\|_{L^\infty_{t,x}} \|\nabla_x \partial_x^\alpha u\|^2 + \||x|\nabla_x a\|_{L^\infty_{t,x}} \|\nabla_x \partial_x^\alpha u\| \left\|\frac{\partial_x^\alpha u}{|x|}\right\| \\ &\le C\delta_a \|\nabla_x \partial_x^\alpha u\|^2, \end{aligned}$$

where the Hardy inequality [16] has been used. For I_2, similarly, one has

$$\begin{aligned} I_2 &\le C\sum_{\beta<\alpha} \||x|\partial_x^{\alpha-\beta} a(t,x)\|_{L^\infty_{t,x}} \|\Delta_x \partial_x^\beta u\| \left\|\frac{\partial_x^\alpha u}{|x|}\right\| \\ &\le C\delta_a \|\nabla_x^2 u\|_{H_x^{N-1}} \|\nabla_x \partial_x^\alpha u\| \\ &\le C\delta_a \|\nabla_x^2 u\|_{H_x^{N-1}}^2. \end{aligned}$$

Putting the above two estimates into (2.8) and taking summation over $1 \le |\alpha| \le N$ yield

$$\frac{1}{2}\frac{d}{dt}\|\nabla_x u\|_{H_x^{N-1}}^2 + \lambda \|\nabla_x^2 u\|_{H_x^{N-1}}^2 \le 0,$$

where the smallness of δ_a has been used. Define the energy functional $\mathcal{E}^{\rm h}(u(t))$ on the higher order derivatives by

$$\mathcal{E}^{\rm h}(u(t))=\|\nabla_x u\|^2_{H_x^{N-1}}.$$

Then it is standard to show that the Lyapunov type inequality

$$\frac{d}{dt}\mathcal{E}^{\rm h}(u(t))+\lambda\mathcal{E}^{\rm h}(u(t))\leq C\|\nabla_x u\|^2 \tag{2.9}$$

holds good, which is the desired estimate of Type I. This inequality is not a closed inequality for $\mathcal{E}^{\rm h}(u(t))$ because the constant C on the right hand side is not small, but a remarkable feature is that the left hand side exhibits an exponential decay property in the sense that

$$\mathcal{E}^{\rm h}(u(t))\leq \mathcal{E}^{\rm h}(u_0)e^{-\lambda t}+C\int_0^t e^{-\lambda(t-s)}\|\nabla_x u\|^2 ds.$$

Thus, the energy inequality can be closed if the decay estimates on the lowest order derivative are known. Such estimates are available from the spectral estimates given in Proposition 2.1.

Estimates of Type II (Based on the spectral analysis). First, in terms of the emigroup $U_0^{\rm h}(t,0)$, one can write the Cauchy problem (2.6)-(2.7) in the mild form

$$u(t)=U_0^{\rm h}(t,0)u_0+\int_0^t U_0^{\rm h}(t,s)\{a(s,\cdot)\Delta_x u(s)\}ds, \tag{2.10}$$

for any $t\geq 0$. By using Proposition 2.1, we have

$$\begin{aligned}\|\nabla_x u\|\leq{}& C\|u_0\|_{H_x^1\cap L_x^1}(1+t)^{-\sigma_n(1,1)}\\ &+C\int_0^t(1+t-s)^{-\sigma_n(1,1)}\|a(s,\cdot)\Delta_x u(s)\|_{H_x^1\cap L_x^1}ds.\end{aligned}$$

By noticing

$$\begin{aligned}&\|a\Delta_x u\|_{L_x^1}\leq\|a\|\cdot\|\Delta_x u\|,\\ &\|a\Delta_x u\|_{H_x^1}\leq C(\|a\|_{L_x^\infty}+\|\nabla_x a\|_{L_x^\infty})\|\nabla_x^2 u\|_{H_x^1},\end{aligned}$$

it follows that

$$\begin{aligned}\|\nabla_x u\|\leq{}& C\|u_0\|_{H_x^1\cap L_x^1}(1+t)^{-\sigma_n(1,1)}\\ &+C\delta_a\int_0^t(1+t-s)^{-\sigma_n(1,1)}\|\nabla_x^2 u\|_{H_x^1}ds,\end{aligned} \tag{2.11}$$

which is the desired estimate of Type II.

Combinations of estimates of Type I and Type II. In order to get the time decay rate of the energy functional $\mathcal{E}^{\rm h}(u(\cdot))$, set

$$\mathcal{E}^{\rm h}_\infty(t)=\sup_{0\leq s\leq t}(1+s)^{2\sigma_n(1,1)}\mathcal{E}^{\rm h}(u(s)). \tag{2.12}$$

Notice that by the above definition, $\mathcal{E}^{\rm h}_\infty(t)$ is a non-decreasing function over $t\geq 0$. From the estimate (2.11) of Type II, it follows that

$$\|\nabla_x u\|\leq C(1+t)^{-\sigma_n(1,1)}\left(\|u_0\|_{H^1_x\cap L^1_x}+\delta_a\sqrt{\mathcal{E}^{\rm h}_\infty(t)}\right), \tag{2.13}$$

where we have used the fact that $n\geq 3$, $N\geq 3$ so that

$$\sigma_n(1,1)=\frac{n}{2}\left(1-\frac{1}{2}\right)+\frac{1}{2}=\frac{n}{4}+\frac{1}{2}>1,$$

and

$$\|\nabla^2_x u\|_{H^1_x}\leq\sqrt{\mathcal{E}^{\rm h}(u(t))}.$$

Hence, by the Gronwall's inequality, the estimate (2.9) of Type I together with (2.13) give that for any $t\geq 0$,

$$\begin{aligned}\mathcal{E}^{\rm h}(u(t))&\leq \mathcal{E}^{\rm h}(u_0)e^{-\lambda t}+C\int_0^t e^{-\lambda(t-s)}\|\nabla_x u\|^2ds\\&\leq \mathcal{E}^{\rm h}(u_0)e^{-\lambda t}+\int_0^t e^{-\lambda(t-s)}(1+s)^{-2\sigma_n(1,1)}ds\left(\|u_0\|^2_{H^1_x\cap L^1_x}+\delta_a^2\mathcal{E}^{\rm h}_\infty(t)\right)\\&\leq C(1+t)^{-2\sigma_n(1,1)}\left(\mathcal{E}^{\rm h}(u_0)+\|u_0\|^2_{H^1_x\cap L^1_x}+\delta_a^2\mathcal{E}^{\rm h}_\infty(t)\right),\end{aligned}$$

which implies that

$$\mathcal{E}^{\rm h}_\infty(t)\leq C\left(\mathcal{E}^{\rm h}(u_0)+\|u_0\|^2_{H^1_x\cap L^1_x}+\delta_a^2\mathcal{E}^{\rm h}_\infty(t)\right).$$

Since δ_a is small enough, it holds that

$$\mathcal{E}^{\rm h}_\infty(t)\leq C\left(\mathcal{E}^{\rm h}(u_0)+\|u_0\|^2_{H^1_x\cap L^1_x}\right)\leq C\|u_0\|^2_{H^N_x\cap L^1_x},$$

which from the definition (2.12) gives the time decay estimate (2.5).

Finally, by the mild form (2.10), the time decay estimate (2.4) on the solution itself follows from

$$\begin{aligned}\|u\|&\leq C\|u_0\|_{L^2_x\cap L^1_x}(1+t)^{-\sigma_n(1,0)}\\&\quad+C\int_0^t(1+t-s)^{-\sigma_n(1,0)}\|a(s,\cdot)\Delta_x u(s)\|_{L^2_x\cap L^1_x}ds\\&\leq C\|u_0\|_{L^2_x\cap L^1_x}(1+t)^{-\sigma_n(1,0)}\\&\quad+C\delta_a\int_0^t(1+t-s)^{-\sigma_n(1,0)}(1+s)^{-\sigma_n(1,1)}ds\sqrt{\mathcal{E}^{\rm h}_\infty(t)}\\&\leq C(1+t)^{-\sigma_n(1,0)}\|u_0\|_{H^N_x\cap L^1_x},\end{aligned}$$

where again Proposition 2.1 from the spectral analysis has been used together with $\sigma_n(1,1)>1$. This completes the proof of Theorem 2.1.

We conclude this section by a remark. From the proof of Theorem 2.1, one may consider a more general linear partial differential equations

$$\frac{\partial u}{\partial t} = P_a(\partial_x)u, \quad t \in \mathbb{R}, x \in \mathbb{R}^n. \tag{2.14}$$

Here, $u = u(t,x)$ is an unknown function, and $P_a(\partial_x)$ is a polynomial of differential operators in the form of

$$P_a(\partial_x) = \sum_{|\alpha|\le m} (c_\alpha + a_\alpha(t,x))\partial_x^\alpha,$$

where for each α, c_α is a constant and $a_\alpha(t,x)$ depending only on t and x is small in suitable norms. Consider the case when $a_\alpha(t,x) \equiv 0$ for any α, that is, an equation with constant coefficients

$$\frac{\partial u}{\partial t} = P_0(\partial_x)u, \quad t \in \mathbb{R}, x \in \mathbb{R}^n. \tag{2.15}$$

Under some situation, one can obtain the time decay estimates of the solutions to (2.15) by the help of Fourier transform. Then the above method can be used to obtain the time decay estimates on (2.14) under some smallness assumption on a_α.

3 Compressible Navier-Stokes Equations

In this section, we will apply the method illustrated in the last section to the system of the compressible Navier-Stokes equations with an external force. The following result is based on the work of [6]. Similar results were given in [7, 25, 32]. The case without any external forcing has been extensively studied, cf. [1, 2, 12, 14, 15, 17, 21, 24] and references therein.

Consider the initial value problem of the compressible Navier-Stokes equations with a potential external force in the whole space:

$$\begin{cases} \rho_t + \nabla_x \cdot (\rho v) = 0, \\ v_t + (v\cdot\nabla_x)v + \dfrac{\nabla_x P(\rho)}{\rho} = \dfrac{\mu}{\rho}\Delta_x v + \dfrac{\mu+\mu'}{\rho}\nabla_x(\nabla_x\cdot v) - \nabla_x\phi(x), \\ (\rho, v)(0,x) = (\rho_0, v_0)(x) \to (\rho_\infty, 0), \quad \text{as} \quad |x|\to\infty. \end{cases} \tag{3.1}$$

Here, $t > 0$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and the spatial dimension $n \ge 3$. $\rho = \rho(t,x) > 0$ and $u = u(t,x) = (u_1(t,x), \cdots, u_n(t,x))$ are the unknown functions representing the density and velocity respectively. $P = P(\rho)$ is the pressure function, $-\nabla_x\phi(x)$ is the time independent potential force, μ, μ' are the viscosity coefficients, and $(\rho_\infty, 0)$ is the state of initial data at infinity. In the following discussion, it is assumed that μ and μ' satisfy the usual physical conditions $\mu > 0$ and $\mu' + \frac{2}{n}\mu \ge 0$, while ρ_∞ is a positive constant and $P(\rho)$ is smooth in a neighborhood of ρ_∞ with $P'(\rho_\infty) > 0$.

For the Navier-Stokes equations $(3.1)_1$-$(3.1)_2$ with a potential external force, the stationary solution (ρ_*, u_*) is given by $(\rho_*(x), 0)$, where $\rho_*(x)$ satisfies, cf. [21],

$$\int_{\rho_\infty}^{\rho_*(x)} \frac{P'(s)}{s}ds + \phi(x) = 0. \tag{3.2}$$

To consider the time asymptotic stability and convergence rate of the stationary solutions, let's define the perturbations $\sigma(t,x)$ and $w(t,x)$ by

$$\sigma(t,x) = \rho(t,x) - \rho_*(x), \quad w(t,x) = \frac{\rho_\infty}{\sqrt{P'(\rho_\infty)}} v(t,x),$$

and introduce some new parameters μ_1, μ_2, γ defined by

$$\mu_1 = \frac{\mu}{\rho_\infty}, \quad \mu_2 = \frac{\mu + \mu'}{\rho_\infty}, \quad \gamma = \sqrt{P'(\rho_\infty)}.$$

Also, denote

$$\bar{\rho}(x) = \rho_*(x) - \rho_\infty.$$

The initial value problem (3.1) is reformulated into

$$\begin{cases} \sigma_t + \gamma \nabla_x \cdot w = S_1, \\ w_t - \mu_1 \Delta_x w - \mu_2 \nabla_x \nabla_x \cdot w + \gamma \nabla_x \sigma = S_2, \\ (\sigma, w)(0,x) = (\sigma_0, w_0)(x), \end{cases} \tag{3.3}$$

where

$$S_1 = -\frac{\mu_1 \gamma}{\mu} \nabla_x \cdot [(\sigma + \bar{\rho}) w], \tag{3.4}$$

$$\begin{aligned} S_2 = & -\frac{\mu_1^2 \gamma^2}{\mu^2} (w \cdot \nabla_x) w - \mu_1 \frac{\sigma + \bar{\rho}}{\sigma + \rho_*} \Delta_x w - \mu_2 \frac{\sigma + \bar{\rho}}{\sigma + \rho_*} \nabla_x \nabla_x \cdot w \\ & - \left[\frac{P'(\sigma + \rho_*)}{\sigma + \rho_*} - \frac{P'(\rho_*)}{\rho_*} \right] \nabla_x \bar{\rho} - \left[\frac{P'(\sigma + \rho_*)}{\sigma + \rho_*} - \frac{P'(\rho_\infty)}{\rho_\infty} \right] \nabla_x \sigma, \end{aligned} \tag{3.5}$$

and

$$(\sigma_0, w_0)(x) = \left(\rho_0 - \rho_*, \frac{\rho_\infty}{\sqrt{P'(\rho_\infty)}} v_0 \right)(x) \to (0,0) \quad \text{as} \quad |x| \to \infty.$$

The global existence of solutions to (3.3) was proved by Matsumura-Nishida [21], which is stated in the following proposition.

Proposition 3.1. *Let $n \geq 3$, $N \geq [\frac{n}{2}] + 2$ be integers. Suppose that*

$$\|(\sigma_0, w_0)\|_{H_x^N} + \|\phi\|_{W_x^{N+1,\infty}}$$

is sufficiently small. Then the initial value problem (3.3) has a unique global solution (σ, w) which satisfies

$$\|(\sigma, w)(t)\|_{H_x^N}^2 + \int_0^t (\|\nabla_x(\sigma, w)(s)\|_{H_x^{N-1}}^2 + \|\nabla_x w(s)\|_{H_x^N}^2)\, ds \leq C \|(\sigma_0, w_0)\|_{H_x^N}^2,$$

for any $t \geq 0$ and some constant C.

By using the method introduced in Section 2, one can obtain the optimal time convergence rates for solutions obtained in the above proposition. The result is stated as follows.

Theorem 3.1. *Suppose that the conditions in Proposition 3.1 hold. Moreover, assume that* $\|(\sigma_0, w_0)\|_{L^1_x}$ *is bounded and*

$$\|\phi\|_{L^2_x\cap L^\infty_x} + \sum_{|\alpha|\leq N} \|(1+|x|)\partial_x^\alpha \nabla_x \phi\|_{L^2_x\cap L^\infty} \tag{3.6}$$

is sufficiently small. Then, the perturbation (σ, w) *in Proposition 3.1 satisfies the following time decay estimates:*

$$\|(\sigma, w)(t)\| \leq C(1+t)^{-\sigma_n(1,0)}\|(\sigma_0, w_0)\|_{H^N_x\cap L^1_x}, \tag{3.7}$$

$$\|\nabla_x(\sigma, w)(t)\|_{H^{N-1}_x} \leq C(1+t)^{-\sigma_n(1,1)}\|(\sigma_0, w_0)\|_{H^N_x\cap L^1_x}, \tag{3.8}$$

for any $t\geq 0$*, where the rate index function* $\sigma_n(\cdot,\cdot)$ *is given by* (2.3).

We now try to sketch the proof of the above theorem. For later use, write

$$\delta_0 = \|(\sigma_0, w_0)\|_{H^N_x}, \quad K_0 = \|(\sigma_0, w_0)\|_{L^1_x},$$

and

$$\delta_\phi = \|\phi\|_{L^2_x\cap L^\infty_x} + \sum_{|\alpha|\leq N} \|(1+|x|)\partial_x^N \nabla_x \phi\|_{L^2_x\cap L^\infty}.$$

Notice that $\delta_0 > 0$ and $\delta_\phi > 0$ can be sufficiently small, while K_0 is kept as a finite number. Firstly, as in [7, 32], based on Proposition 3.1 on the existence of the global solution, we derive a Lyapunov-type inequality for the energy functional for higher order derivatives by using the energy method.

Lemma 3.1 (Estimate of Type I)**.** *Under the assumptions of Theorem 3.1, let* $u := (\sigma, w)$ *be the solution to the initial value problem (3.3). Then for any* $t \geq 0$*, it holds that*

$$\frac{d}{dt}\mathcal{E}^{\rm NS}(u(t)) + \lambda\mathcal{E}^{\rm NS}(u(t)) \leq C\|\nabla_x u(t)\|^2, \tag{3.9}$$

where the energy functional $\mathcal{E}^{\rm NS}(u(t))$ *is equivalent to* $\|\nabla_x u(t)\|^2_{H^{N-1}_x}$*, that is, there exists a positive constant* $C>0$ *such that*

$$\frac{1}{C}\|\nabla_x u(t)\|^2_{H^{N-1}_x} \leq \mathcal{E}^{\rm NS}(u(t)) \leq C\|\nabla_x u(t)\|^2_{H^{N-1}_x}, \quad t\geq 0. \tag{3.10}$$

Proof. For each multi-index α with $1\leq|\alpha|\leq N$, by applying ∂_x^α to (3.3), multiplying by $\partial_x^\alpha\sigma$, $\partial_x^\alpha w$ respectively, and then integrating over $\mathbb{R}^n$, we have from the sum of $(3.3)_1$-$(3.3)_2$ that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha u(t)\|^2 + \mu_1\|\nabla_x\partial_x^\alpha w(t)\|^2 + \mu_2\|\nabla_x\cdot\partial_x^\alpha w(t)\|^2\\
&= \langle\partial_x^\alpha\sigma(t), \partial_x^\alpha S_1(t)\rangle + \langle\partial_x^\alpha w(t), \partial_x^\alpha S_2(t)\rangle.
\end{aligned} \tag{3.11}$$

Before estimating the two terms on the right hand side of (3.11), we notice from (3.4) and (3.5) that the source term (S_1, S_2) has the following properties

$$S_1 \sim \partial_i\sigma w_i + \sigma\partial_i w_i + \partial_i\bar\rho w_i + \bar\rho\partial_i w_i, \tag{3.12}$$

$$\begin{aligned}
S_2 \sim\ & w_i\partial_i w_j + \sigma\partial_i\partial_i w_j + \sigma\partial_j\partial_i w_i + \sigma\partial_j\sigma\\
&+\bar\rho\partial_i\partial_i w_j + \bar\rho\partial_j\partial_i w_i + \partial_j\bar\rho\sigma + \bar\rho\partial_j\sigma.
\end{aligned} \tag{3.13}$$

Then for the first term on the right hand side of (3.11), it follows from (3.12) that

$$\begin{aligned}\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha S_1(t)\rangle \le\ & C|\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha \partial_i \sigma(t) w_i(t)\rangle| \\ & + C\sum_{|\beta|\le|\alpha|-1} C_\beta^\alpha |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \partial_i \sigma(t) \partial_x^{\alpha-\beta} w_i(t)\rangle| \\ & + C\sum_{|\beta|\le|\alpha|} C_\beta^\alpha |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \sigma(t) \partial_x^{\alpha-\beta} \partial_i w_i(t)\rangle| \\ & + C\sum_{|\beta|\le|\alpha|} C_\beta^\alpha |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \partial_i \bar\rho \partial_x^{\alpha-\beta} w_i(t)\rangle| \\ & + C\sum_{|\beta|\le|\alpha|} C_\beta^\alpha |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \bar\rho \partial_x^{\alpha-\beta} \partial_i w_i(t)\rangle| = \sum_{i=1}^5 I_i. \qquad (3.14)\end{aligned}$$

Next, we estimate each term I_i $(i = 1, \cdots, 5)$ on the right hand side of (3.14). As in [7, 32], by Proposition 3.1, we have

$$\sum_{j=1}^3 I_j \le C\delta_0 \sum_{1\le|\alpha|\le N} \|\partial_x^\alpha \sigma(t)\|^2 + C\delta_0 \sum_{1\le|\alpha|\le N+1} \|\partial_x^\alpha w(t)\|^2. \qquad (3.15)$$

For I_4 and I_5, firstly notice that (3.2) together with (3.6) imply

$$\|\bar\rho\|_{L_x^2\cap L_x^\infty} + \sum_{|\alpha|\le N} \|(1+|x|)\partial_x^\alpha \nabla_x \bar\rho\|_{L_x^2\cap L_x^\infty} \le C\delta_\phi. \qquad (3.16)$$

Hence, it follows that

$$\begin{aligned} I_4 =\ & C\left\{\sum_{|\beta|=|\alpha|} + \sum_{0\le|\beta|\le|\alpha|-1}\right\} |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \partial_i \bar\rho \partial_x^{\alpha-\beta} w_i(t)\rangle| \\ \le\ & \delta_\phi \|\partial_x^\alpha \sigma(t)\|^2 + \frac{C}{\delta_\phi}\|\partial_x^\alpha \partial_i \bar\rho w_i(t)\|^2 + \frac{C}{\delta_\phi}\sum_{0\le|\beta|\le|\alpha|-1} \|\partial_x^\beta \partial_i \bar\rho \partial_x^{\alpha-\beta} w_i(t)\|^2 \\ \le\ & \delta_\phi \|\partial_x^\alpha \sigma(t)\|^2 + \frac{C}{\delta_\phi}\|(1+|x|)\partial_x^\alpha \partial_i \bar\rho\|_{L^\infty}^2 \left\|\frac{w_i(t)}{1+|x|}\right\|^2 \\ & + \frac{C}{\delta_\phi}\sum_{0\le|\beta|\le|\alpha|-1} \|\partial_x^\beta \partial_i \bar\rho\|_{L^\infty}^2 \|\partial_x^{\alpha-\beta} \partial_i w_i(t)\|^2 \\ \le\ & \delta_\phi \sum_{1\le|\alpha|\le N} \|\partial_x^\alpha \sigma(t)\|^2 + C\delta_\phi \sum_{1\le|\alpha|\le N+1} \|\partial_x^\alpha w(t)\|^2, \qquad (3.17)\end{aligned}$$

and

$$
\begin{aligned}
I_5 &\leq C \sum_{0\leq|\beta|\leq|\alpha|} |\langle \partial_x^\alpha \sigma(t), \partial_x^\beta \bar{\rho} \partial_x^{\alpha-\beta} \partial_i w_i(t)\rangle| \\
&\leq \delta_\phi \|\partial_x^\alpha \sigma(t)\|^2 + \frac{C}{\delta_\phi} \sum_{0\leq|\beta|\leq|\alpha|} \|\partial_x^\beta \bar{\rho} \partial_x^{\alpha-\beta} \partial_i w_i(t)\|^2 \\
&\leq \delta_\phi \|\partial_x^\alpha \sigma(t)\|^2 + \frac{C}{\delta_\phi} \sum_{0\leq|\beta|\leq|\alpha|} \|\partial_x^\beta \bar{\rho}\|_{L^\infty}^2 \|\partial_x^{\alpha-\beta} \partial_i w_i(t)\|^2 \\
&\leq \delta_\phi \sum_{1\leq|\alpha|\leq N} \|\partial_x^\alpha \sigma(t)\|^2 + C\delta_\phi \sum_{1\leq|\alpha|\leq N+1} \|\partial_x^\alpha w(t)\|^2, \qquad (3.18)
\end{aligned}
$$

where the Hardy inequality has been used. Thus, (3.15), (3.17) and (3.18) give

$$
\langle \partial_x^\alpha \sigma(t), \partial_x^\alpha S_1(t)\rangle \leq (\delta_0+\delta_\phi) \sum_{1\leq|\alpha|\leq N} \|\partial_x^\alpha \sigma(t)\|^2 + C(\delta_0+\delta_\phi) \sum_{1\leq|\alpha|\leq N+1} \|\partial_x^\alpha w(t)\|^2. \quad (3.19)
$$

From (3.13), similar argument gives

$$
\langle \partial_x^\alpha w(t), \partial_x^\alpha S_2(t)\rangle \leq (\delta_0+\delta_\phi) \sum_{1\leq|\alpha|\leq N} \|\partial_x^\alpha \sigma(t)\|^2 + C(\delta_0+\delta_\phi) \sum_{1\leq|\alpha|\leq N+1} \|\partial_x^\alpha w(t)\|^2. \quad (3.20)
$$

Hence, (3.11) together with (3.19)-(3.20) yield

$$
\begin{aligned}
&\frac{d}{dt} \sum_{1\leq|\alpha|\leq N} \|\partial_x^\alpha u(t)\|^2 + \lambda \sum_{1\leq|\alpha|\leq N} \|\nabla_x \partial_x^\alpha w(t)\|^2 \\
&\leq C(\delta_0+\delta_\phi) \sum_{1\leq|\alpha|\leq N} \|\partial_x^\alpha \sigma(t)\|^2 + C(\delta_0+\delta_\phi) \sum_{1\leq|\alpha|\leq N+1} \|\partial_x^\alpha w(t)\|^2. \qquad (3.21)
\end{aligned}
$$

Furthermore, to include the estimation on $\|\nabla_x \partial_x^\alpha \sigma(t)\|^2$ when $1 \leq |\alpha| \leq N-1$, from $(3.3)_2$, we have

$$
\gamma \nabla_x \sigma = -w_t + \mu_1 \Delta_x w + \mu_2 \nabla_x(\nabla_x \cdot w) + S_2.
$$

After applying ∂_x^α with $1 \leq |\alpha| \leq N-1$ to the above equation, combining $(3.3)_1$ and performing the computations similar to (3.17) and (3.18) lead to

$$
\begin{aligned}
&\frac{\gamma}{2} \sum_{1\leq|\alpha|\leq N-1} \|\nabla_x \partial_x^\alpha \sigma(t)\|^2 + \frac{d}{dt} \sum_{1\leq|\alpha|\leq N-1} \langle \partial_x^\alpha w(t), \nabla_x \partial_x^\alpha \sigma(t)\rangle \\
&\leq C \sum_{1\leq|\alpha|\leq N-1} \|\partial_x^\alpha \nabla_x w(t)\|_{H_x^1}^2 + C(\delta_0+\delta_\phi) \sum_{1\leq|\alpha|\leq N} \|\partial_x^\alpha \sigma(t)\|^2 \\
&\qquad + C(\delta_0+\delta_\phi) \sum_{1\leq|\alpha|\leq N+1} \|\partial_x^\alpha w(t)\|^2. \qquad (3.22)
\end{aligned}
$$

Define

$$
\mathcal{E}^{\rm NS}(u(t)) = M \sum_{1\leq|\alpha|\leq N} \|\partial^\alpha u(t)\|^2 + \sum_{1\leq|\alpha|\leq N-1} \langle \partial_x^\alpha w(t), \nabla \partial_x^\alpha \sigma(t)\rangle,
$$

for some constant $M>0$. By choosing the constant $M>0$ sufficiently large and noticing that δ_0 and δ_ϕ can be sufficiently small, the linear combination of (3.21) and (3.22) leads to

$$\frac{d}{dt}\mathcal{E}^{\rm NS}(u(t))+\lambda\left(\|\nabla_x^2\sigma(t)\|_1^2+\|\nabla_x^2 w(t)\|_2^2\right)\leq C(\delta_0+\delta_\phi)\|\nabla_x u(t)\|^2,$$

which implies

$$\frac{d}{dt}\mathcal{E}^{\rm NS}(u(t))+c\|\nabla_x^2 u(t)\|_1^2\leq C(\delta_0+\delta_\phi)\|\nabla_x u(t)\|^2. \tag{3.23}$$

Adding the first-order derivatives $\|\nabla_x u(t)\|^2$ to both sides of (3.23) yields (3.9). This completes the proof of the lemma. □

Next, we will use the result obtained by the spectral analysis to estimate $\|\nabla_x u(t)\|$ with $u=(\sigma,w)$. For this purpose, we first recall the time decay properties of the solution semigroup for the linear isentropic compressible Navier-Stokes equations. Actually, the solution of (3.3) can be written in the mild form as

$$u(t)=U_0^{\rm NS}(t,0)u_0+\int_0^t U_0^{\rm NS}(t,s)S(u(s))ds, \tag{3.24}$$

where we have used the notations

$$u=(\sigma,w),\quad u_0=(\sigma_0,w_0),\quad S(u)=(S_1,S_2). \tag{3.25}$$

Here, $U_0^{\rm NS}(t,s)$ is the solution semigroup defined by

$$U_0^{\rm NS}(t,s)=e^{(t-s)\mathbb{A}},\quad t\geq s,$$

with $\mathbb{A}$ being a matrix-valued differential operator given by

$$\mathbb{A}=\begin{pmatrix}0 & -\gamma{\rm div}\\ -\gamma\nabla_x & \mu_1\Delta_x+\mu_2\nabla_x{\rm div}\end{pmatrix}.$$

The semigroup $e^{t\mathbb{A}}$ has the following time decay properties, cf. [14, 15].

Proposition 3.2. *Let $n\geq 3$, $k\geq 0$ be integers, and $1\leq q\leq 2$. It holds that*

$$\|\partial_x^k e^{t\mathbb{A}}u_0\|\leq C(1+t)^{-\sigma_n(q,k)}\|u_0\|_{H_x^k\cap L_x^q},$$

for any $t\geq 0$ and $u_0\in H_x^k\cap L_x^q$, where $\sigma_n(q,k)$ is defined by (2.3).

Now, we can use the above proposition to obtain the time decay estimate on $\|\nabla_x u(t)\|$ similar to the argument for the simple model studied in Section 2.

Lemma 3.2 (Estimate of Type II)**.** *Under the assumptions of Theorem 3.1, let $u=(\sigma,w)$ be the solution to the initial value problem* (3.3)*. Then one has*

$$\begin{aligned}\|\nabla_x u(t)\|\leq{}& C(\delta_0+K_0)(1+t)^{-\sigma_n(1,1)}\\ &+C(\delta_0+\delta_\phi)\int_0^t(1+t-s)^{-\sigma_n(1,1)}\|\nabla_x u(s)\|_{H_x^{N-1}}ds,\end{aligned} \tag{3.26}$$

where $K_0=\|U_0\|_{H_x^1\cap L_x^1}$ is finite.

Proof. From the integral formula (3.24) and Proposition 3.2, we have

$$\begin{aligned}\|\nabla_x u(t)\| \leq & C(\delta_0+K_0)(1+t)^{-\sigma_n(1,1)}\\ &+C\int_0^t (1+t-s)^{-\sigma_n(1,1)}\|S(u(s))\|_{H^1_x\cap L^1_x}ds,\end{aligned} \tag{3.27}$$

where $S(u)$ is given in (3.25) as well as (3.4)-(3.5). To derive (3.26), we need to control $\|S(u(t))\|_{H^1_x\cap L^1_x}$ by the L^2-norm of the derivatives of at least one order.

Firstly, we estimate those terms including $\bar{\rho}$. By (3.16), it follows that

$$\|\nabla_x\bar{\rho}\cdot w\|_{L^1_x} \leq \|(1+|x|)\nabla_x\bar{\rho}\|\cdot\left\|\frac{w}{1+|x|}\right\| \leq C\delta_\phi\|\nabla_x w\|,$$

$$\|\bar{\rho}\nabla_x\cdot w\|_{L^1_x} \leq \|\bar{\rho}\|\cdot\|\nabla_x\cdot w\| \leq C\delta_\phi\|\nabla_x w\|,$$

$$\|\nabla_x\bar{\rho}\cdot w\| \leq \|(1+|x|)\nabla_x\bar{\rho}\|_{L^\infty_x}\left\|\frac{w}{1+|x|}\right\| \leq C\delta_\phi\|\nabla_x w\|,$$

and

$$\|\bar{\rho}\nabla_x\cdot w\| \leq \|\bar{\rho}\|_{L^\infty_x}\|\nabla_x\cdot w\| \leq C\delta_\phi\|\nabla_x w\|.$$

Thus, the above inequalities give

$$\|\nabla_x\cdot(\bar{\rho}w)\|_{L^1_x},\ \|\nabla_x\cdot(\bar{\rho}w)\| \leq C\delta_\phi\|\nabla_x w\|.$$

Similarly, it holds that

$$\|\nabla_x\cdot(\bar{\rho}\sigma)\|_{L^1_x},\ \|\nabla_x\cdot(\bar{\rho}\sigma)\| \leq C\delta_\phi\|\nabla_x \sigma\|,$$

$$\|\nabla_x\cdot(\bar{\rho}w)\|_{H^1_x} \leq C\delta_\phi\|\nabla_x w\|_{H^1_x},\ \|\nabla_x\cdot(\bar{\rho}\sigma)\|_{H^1_x} \leq C\delta_\phi\|\nabla_x \sigma\|_{H^1_x},$$

$$\|\nabla_x^2(\bar{\rho}w)\|_{L^1_x} \leq C\delta_\phi\|\nabla_x w\|_{H^1_x},\ \|\nabla_x^2(\bar{\rho}w)\|_{H^1_x} \leq C\delta_\phi\|\nabla_x w\|_{H^2_x}.$$

The estimation on the other terms in $S(u(t))$ is straightforward so that we omit it for brevity. Hence, one has

$$\begin{aligned}\|S(u(t))\|_{L^1_x} \leq & C\|u(t)\|\|\nabla_x u(t)\|_{H^1_x}\\ &+C(\|\nabla_x\cdot(\bar{\rho}w)\|_{L^1_x}+\|\nabla_x\cdot(\bar{\rho}\sigma)\|_{L^1_x}+\|\nabla_x^2(\bar{\rho}w)\|_{L^1_x})\\ \leq & C(\delta_0+\delta_\phi)\|\nabla_x u(t)\|_{H^1_x},\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}\|S(u(t))\|_{H^1_x} \leq & C\|u(t)\|_{W^{1,\infty}_x}\|\nabla_x u(t)\|_{H^2_x}\\ &+C(\|\nabla_x\cdot(\bar{\rho}w)\|_{H^1_x}+\|\nabla_x\cdot(\bar{\rho}\sigma)\|_{H^1_x}+\|\nabla_x^2(\bar{\rho}w)\|_{H^1_x})\\ \leq & C(\delta_0+\delta_\phi)\|\nabla_x u(t)\|_{H^2_x}.\end{aligned} \tag{3.29}$$

Putting (3.28) and (3.29) into (3.27) gives the desired inequality (3.26). This completes the proof of the lemma. □

Proof of Theorem 3.1: Based on the estimates of Type I and Type II, the proof of Theorem 3.1 is almost the same as the one given in Section 2 for the simple model. For completeness, we sketch it as follows. Set

$$\mathcal{E}_\infty^{\rm NS}(t) = \sup_{0\le s\le t}(1+s)^{2\sigma_n(1,1)}\mathcal{E}^{\rm NS}(u(s)), \tag{3.30}$$

where $\mathcal{E}^{\rm NS}(u(t))$ is defined in Lemma 3.1. Notice that $\mathcal{E}_\infty^{\rm NS}(t)$ is non-decreasing over $t\ge 0$, and

$$\|\nabla_x u(s)\|_{H_x^2} \le C\sqrt{\mathcal{E}^{\rm NS}(u(s))} \le C(1+s)^{-\sigma_n(1,1)}\sqrt{\mathcal{E}_\infty^{\rm NS}(t)}, \quad 0\le s\le t.$$

Then it follows from (3.26) that

$$\begin{aligned}\|\nabla_x u(t)\| &\le C(\delta_0+K_0)(1+t)^{-\sigma_n(1,1)}\\ &\quad +C(\delta_0+\delta_\phi)\int_0^t (1+t-s)^{-\sigma_n(1,1)}(1+s)^{-\sigma_n(1,1)}ds\sqrt{\mathcal{E}_\infty^{\rm NS}(t)}\\ &\le C(1+t)^{-\sigma_n(1,1)}\left(\delta_0+K_0+(\delta_0+\delta_\phi)\sqrt{\mathcal{E}_\infty^{\rm NS}(t)}\right), \end{aligned}\tag{3.31}$$

because $\sigma_n(1,1)>1$ when $n\ge 3$. Hence, by the Gronwall's inequality, (3.9) and (3.31) give

$$\begin{aligned}\mathcal{E}^{\rm NS}(u(t)) &\le \mathcal{E}^{\rm NS}(u_0)e^{-\lambda t} + C\int_0^t e^{-\lambda(t-s)}\|\nabla_x u(s)\|^2 ds\\ &\le \mathcal{E}^{\rm NS}(u_0)e^{-\lambda t}\\ &\quad +C\int_0^t e^{-\lambda(t-s)}(1+s)^{-2\sigma_n(1,1)}ds\left[(\delta_0+K_0)^2+(\delta_0+\delta_\phi)^2\mathcal{E}_\infty^{\rm NS}(t)\right]\\ &\le C(1+t)^{-2\sigma_n(1,1)}\left[\mathcal{E}^{\rm NS}(u_0)+(\delta_0+K_0)^2+(\delta_0+\delta_\phi)^2\mathcal{E}_\infty^{\rm NS}(t)\right]. \end{aligned}\tag{3.32}$$

In terms of $\mathcal{E}_\infty^{\rm NS}(t)$, it follows from (3.32) that

$$\mathcal{E}_\infty^{\rm NS}(t) \le C\left[\mathcal{E}^{\rm NS}(u_0)+(\delta_0+K_0)^2+(\delta_0+\delta_\phi)^2\mathcal{E}_\infty^{\rm NS}(t)\right],$$

which implies

$$\mathcal{E}_\infty^{\rm NS}(t) \le C\mathcal{E}^{\rm NS}(u_0)+C(\delta_0+K_0)^2 \le C(\delta_0^2+K_0^2),$$

where we have used the assumption that δ_0 and δ_ϕ are sufficiently small. Hence, this gives (3.8) by noticing (3.10) and (3.30).

Next, by Proposition 3.2, (3.28) and (3.29), it follows from the integral formula (3.24) that

$$\begin{aligned}\|u(t)\| &\le C(\delta_0+K_0)(1+t)^{-\sigma_n(1,0)} + C\int_0^t (1+t-s)^{-\sigma_n(1,0)}\|S(u(s))\|_{L_x^1\cap L_x^2}ds\\ &\le C(\delta_0+K_0)(1+t)^{-\sigma_n(1,0)} + C(\delta_0+\delta_\phi)\int_0^t (1+t-s)^{-\sigma_n(1,0)}\|\nabla_x u(s)\|_{H_x^2}ds\\ &\le C(\delta_0+K_0)(1+t)^{-\sigma_n(1,0)}\\ &\quad +C(\delta_0+K_0)\int_0^t (1+t-s)^{-\sigma_n(1,0)}(1+s)^{-\sigma_n(1,1)}ds\\ &\le C(\delta_0+K_0)(1+t)^{-\sigma_n(1,0)}, \end{aligned}$$

where we again have used $\sigma_n(1,1) > 1$. Therefore, (3.7) is proved. This completes the proof of Theorem 3.1.

4 Boltzmann Equation

In this section, we will continue to use the method introduced in Section 2 to consider the optimal time decay estimates on the Boltzmann equation with an external force. The results of this section come from the recent series of works [4, 8, 33]. Some related results can also be found in [3, 10, 11, 18, 19, 34, 35] about the energy method for the Boltzmann equation, and [9, 26, 23, 27, 28, 29, 31] about the convergence rates of solutions by using different methods.

In the presence of a potential force, the Boltzmann equation for the hard-sphere gas in n-dimensional space $\mathbb{R}^n$ takes the form

$$\partial_t f + \xi \cdot \nabla_x f + \nabla_x \phi(x) \cdot \nabla_\xi f = Q(f, f). \tag{4.1}$$

Here, the unknown function $f = f(t, x, \xi)$ is non-negative standing for the number density of gas particles which have position $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and velocity $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$ at time $t \in \mathbb{R}$. The spatial dimension $n \geq 3$. $\phi(x)$ depending only on x is the potential of the external force. Q is the bilinear collision operator defined by

$$\begin{gathered} Q(f,g) = \frac{1}{2}\int_{\mathbb{R}^n \times S^{n-1}} (f' g'_* + f'_* g' - f g_* - f_* g)|(\xi - \xi_*)\cdot\omega| d\omega d\xi_*, \\ f = f(t,x,\xi), \quad f' = f(t,x,\xi'), \quad f_* = f(t,x,\xi_*), \quad f'_* = f(t,x,\xi'_*), \\ \textit{likewise for } g, \\ \xi' = \xi - [(\xi - \xi_*)\cdot\omega]\omega, \quad \xi'_* = \xi_* + [(\xi - \xi_*)\cdot\omega]\omega, \quad \omega \in S^{n-1}. \end{gathered}$$

The local Maxwellian $e^{\phi(x)}\mathbf{M}$ is a stationary solution to the Boltzmann equation (4.1), where

$$\mathbf{M} = \frac{1}{(2\pi)^{n/2}} \exp\left(-|\xi|^2/2\right),$$

is a global Maxwellian which has been normalized to have zero bulk velocity and unit density and temperature. Set the perturbation $u = u(t, x, \xi)$ by

$$f = e^{\phi(x)}\mathbf{M} + \sqrt{\mathbf{M}}u. \tag{4.2}$$

Then the Boltzmann equation (4.1) can be reformulated into

$$\partial_t u + \xi \cdot \nabla_x u + \nabla_x \phi(x) \cdot \nabla_\xi u - \frac{1}{2}\xi \cdot \nabla_x \phi(x) u = e^{\phi(x)}\mathbf{L}u + \Gamma(u,u), \tag{4.3}$$

where $\mathbf{L}$ is the linearized collision operator and Γ is the corresponding nonlinear collision operator, given by

$$\begin{aligned} \mathbf{L}u &= \frac{1}{\sqrt{\mathbf{M}}}\left[Q(\mathbf{M}, \sqrt{\mathbf{M}}u) + Q(\sqrt{\mathbf{M}}u, \mathbf{M})\right], \\ \Gamma(u,u) &= \frac{1}{\sqrt{\mathbf{M}}} Q(\sqrt{\mathbf{M}}u, \sqrt{\mathbf{M}}u). \end{aligned}$$

We shall consider the Cauchy problem of (4.3) with given initial data

$$u(0,x,\xi)=u_0(x,\xi),\quad (x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n. \tag{4.4}$$

Firstly, we recall some basic properties of the linearized operator $\mathbf{L}$. $\mathbf{L}$ can be decomposed into

$$\mathbf{L}=-\nu(\xi)+K,$$

where $\nu(\xi)$ is a multiplier operator called the collision frequency. For the case of the hard sphere gas, there exists a positive constant $\nu_0>0$ such that for any ξ,

$$\frac{1}{\nu_0}(1+|\xi|)\le\nu(\xi)\le\nu_0(1+|\xi|). \tag{4.5}$$

K is a self-adjoint compact operator on $L^2(\mathbb{R}^n)$ with a real symmetric integral kernel. The null-space of the operator $\mathbf{L}$ is the $(n+1)$-dimensional space of collision invariants

$$\mathcal{N}=Ker\mathbf{L}=\text{span}\left\{\sqrt{\mathbf{M}};\ \xi_i\sqrt{\mathbf{M}},i=1,\cdots,n;\ |\xi|^2\sqrt{\mathbf{M}}\right\}. \tag{4.6}$$

Following from the Boltzmann's H-theorem, the linearized collision operator $\mathbf{L}$ is non-positively definite and furthermore, $-\mathbf{L}$ is locally coercive in the sense that there is a constant $\lambda>0$ such that

$$-\int_{\mathbb{R}^n}u\mathbf{L}u\,d\xi\ge\lambda\int_{\mathbb{R}^n}\nu(\xi)\left(\{\mathbf{I}-\mathbf{P}\}u\right)^2d\xi,\quad\forall\,u\in D(\mathbf{L}), \tag{4.7}$$

for any fixed (t,x). Here, $\mathbf{P}$ denotes the projection operator from $L^2(\mathbb{R}^n)$ to $\mathcal{N}$ and $D(\mathbf{L})$ is the domain of $\mathbf{L}$ given by

$$D(\mathbf{L})=\left\{u\in L^2(\mathbb{R}^n)\ \Big|\ \nu(\xi)u\in L^2(\mathbb{R}^n)\right\}.$$

Let's also introduce some notations for the later presentation. We use $\langle\cdot,\cdot\rangle$ to denote the inner product in the Hilbert space $L^2(\mathbb{R}^n\times\mathbb{R}^n)$ or $L^2(\mathbb{R}^n)$ in the following without any ambiguity, and use $\|\cdot\|$ to denote the corresponding L^2 norm. Set

$$\langle u,v\rangle_\nu\equiv\langle\nu(\xi)u,v\rangle,$$

for any functions $u=u(x,\xi)$ and $v=v(x,\xi)$ to be the weighted inner product in $L^2(\mathbb{R}^n\times\mathbb{R}^n)$, and use $\|\cdot\|_\nu$ for the corresponding weighted L^2 norm.

To state the well-posedness of the Cauchy problem, we define the energy functional

$$[[u(t)]]^2\equiv\sum_{|\alpha|+|\beta|\le N}\|\partial_x^\alpha\partial_\xi^\beta u(t)\|^2, \tag{4.8}$$

and the dissipation rate

$$[[u(t)]]_\nu^2\equiv\sum_{|\alpha|+|\beta|\le N}\|\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u(t)\|_\nu^2+\sum_{0<|\alpha|\le N}\|\partial_x^\alpha\mathbf{P}u(t)\|_\nu^2, \tag{4.9}$$

where $N\ge[\frac{n}{2}]+3$ is an integer. Then one has

Proposition 4.1 (Well-posedness). *Suppose that* $\phi \in L^\infty_x$ *and*

$$\big\|(1+|x|)^2\nabla_x\phi\big\|_{L^\infty_x}+\sum_{2\leq|\alpha|\leq N}\|(1+|x|)\partial^\alpha_x\phi\|_{L^\infty_x}$$

is sufficiently small. If $[[u_0]]$ *is also sufficiently small, then the Cauchy problem admits a unique solution* $u(t,x,\xi)$ *which satisfies*

$$[[u(t)]]^2+\lambda\int_0^t[[u(s)]]^2_\nu ds\leq C[[u(0)]]^2,\ \forall\ t\geq 0.$$

For the convenience of the readers, we shall sketch the proof of the above proposition as follows. On the other hand, the goal of this section is to apply the method introduced in Section 2 to obtain the optimal time decay estimates for the Boltzmann equation stated as follows.

Theorem 4.1 (Optimal convergence rates). *Suppose that the conditions in Proposition 4.1 hold. Furthermore, assume that* $\|u_0\|_{L^2_\xi(L^1_x)}$ *is bounded and*

$$\||x|\nabla_x\phi\|_{L^2_x}$$

is sufficienlty small. Then the solution $u(t,x,\xi)$ *obtained in Proposition 4.1 satisfies*

$$[[u(t)]]\leq C(1+t)^{-\frac{n}{4}}\left([[u_0]]+\|u_0\|_{L^2_\xi(L^1_x)}\right), \tag{4.10}$$

for any $t\geq 0$.

The reason why we call the above decay estimate optimal is that it is the same as the case without external forcing. That is, when $\phi\equiv 0$, the solution semigroup $\{e^{\mathbf{B}t}\}_{t\geq 0}$, with

$$\mathbf{B}=-\xi\cdot\nabla_x+\mathbf{L},$$

has the decay estimate

$$\|\nabla^k_x e^{\mathbf{B}t}g\|_{L^2_{x,\xi}}\leq C(1+t)^{-\sigma_n(q,k)}(\|g\|_{Z_q}+\|\nabla^k_x g\|_{L^2_{x,\xi}}), \tag{4.11}$$

for the integer $k\geq 0$, $1\leq q\leq 2$ and for any function $g=g(x,\xi)$, where the spatial dimension $n\geq 3$, and $Z_q=L^2_\xi(L^q_x)$. This decay estimate (4.11) was proved by the spectral analysis, cf. Ukai [29, 30] and Nishida-Imai [23]. Note that the rate given in (4.10) is the same as that of $e^{\mathbf{B}t}$ without differentiation because $\sigma_n(1,0)=n/4$.

Sketch of the proof of Proposition 4.1: The proof is based on the energy method by combining the local existence and the closure of the a prior estimate. Here, we only illustrate the proof of the a prior estimate. That is, under the a prior assumption that $[[u(t)]]$ is sufficiently small, we want to show that there exists an energy functional $\mathcal{E}(u(t))$ and a dissipation rate $\mathcal{D}(u(t))$ which are equivalent to $[[u(t)]]$ and $[[u(t)]]_\nu$ respectively, such that

$$\frac{d}{dt}\mathcal{E}(u(t))+\lambda\mathcal{D}(u(t))\leq 0. \tag{4.12}$$

For this, introduce the macro-micro decomposition, cf. [11]:

$$\begin{cases} u(t,x,\xi) = u_1 + u_2, \\ u_1 \equiv \mathbf{P}u \in \mathcal{N}, \\ \mathbf{P}u = \left\{a(t,x) + \sum_{i=1}^n b_i(t,x)\xi_i + c(t,x)|\xi|^2\right\}\sqrt{\mathbf{M}}, \\ u_2 \equiv \{\mathbf{I}-\mathbf{P}\}u \in \mathcal{N}^\perp, \end{cases} \tag{4.13}$$

where u_1 denotes the macroscopic component of $u(t,x,\xi)$ with coefficients (a,b,c), and u_2 denotes the microscopic component of $u(t,x,\xi)$. In (4.9), the dissipation rate related to the microscopic part $\{\mathbf{I}-\mathbf{P}\}u$ can be obtained directly from the analysis on the perturbed equation (4.3) because of the dissipative property (4.7) of $\mathbf{L}$. However, it is more delicate to obtain the dissipation rate related to the macroscopic part $\mathbf{P}u$. It turns out that the coefficients a,b,c of $\mathbf{P}u$ satisfy a system which shares some similar properties of the linearized compressible Navier-Stokes equations with viscosity and heat conductivity. Thus the analytical techniques coming from the fluid dynamics as those used in Section 3 can be applied to obtain the dissipation rate on the macroscopic component. Precisely, we now give the equations satisfied by $u_1 = \mathbf{P}u$ or a,b,c as follows, cf. [11]. Firstly, the equation (4.3) can be rewritten as an equation of u_1:

$$\partial_t u_1 + \xi\cdot\nabla_x u_1 + \nabla_x\phi\cdot\nabla_\xi u_1 - \frac{1}{2}\xi\cdot\nabla_x\phi u_1 = r + \ell + h \tag{4.14}$$

with

$$\begin{aligned} &r = -\partial_t u_2, \\ &\ell = -\xi\cdot\nabla_x u_2 - \nabla_x\phi\cdot\nabla_\xi u_2 + \frac{1}{2}\xi\cdot\nabla_x\phi u_2 + e^\phi\mathbf{L}u_2, \\ &h = \Gamma(u,u). \end{aligned}$$

Furthermore, one can also obtain the evolution equations for (a,b,c) which defines u_1. In fact, by putting the expansion $(4.13)_3$ into (4.14) and collecting the coefficients with respect to the basis $\{e_k\}$ consisting of

$$\sqrt{\mathbf{M}},\ \left(\xi_i\sqrt{\mathbf{M}}\right)_{1\le i\le n},\ \left(|\xi_i|^2\sqrt{\mathbf{M}}\right)_{1\le i\le n},\ \left(\xi_i\xi_j\sqrt{\mathbf{M}}\right)_{1\le i<j\le n},\ \left(|\xi|^2\xi_i\sqrt{\mathbf{M}}\right)_{1\le i\le n}, \tag{4.15}$$

then one has the following macroscopic equations on the coefficients (a,b,c) of u_1:

$$\partial_t a + b\cdot\nabla_x\phi = \gamma^{(0)}, \tag{4.16}$$

$$\partial_t b_i + \partial_i a - (a\partial_i\phi - 2c\partial_i\phi) = \gamma_i^{(1)}, \tag{4.17}$$

$$\partial_t c + \partial_i b_i - b_i\partial_i\phi = \gamma_i^{(2)}, \tag{4.18}$$

$$\partial_i b_j + \partial_j b_i - (b_j\partial_i\phi + b_i\partial_j\phi) = \gamma_{ij}^{(2)},\quad i\neq j, \tag{4.19}$$

$$\partial_i c - c\partial_i\phi = \gamma_i^{(3)}, \tag{4.20}$$

where all terms on the right hand side are the coefficients of $r+\ell+h$ with respect to the corresponding elements in the basis (4.15) given by:

$$
\begin{aligned}
\gamma^{(0)} &\equiv -\partial_t\tilde{r}^{(0)}+\ell^{(0)}+h^{(0)},\\
\gamma_i^{(1)} &\equiv -\partial_t\tilde{r}_i^{(1)}+\ell_i^{(1)}+h_i^{(1)},\\
\gamma_i^{(2)} &\equiv -\partial_t\tilde{r}_i^{(2)}+\ell_i^{(2)}+h_i^{(2)},\\
\gamma_{ij}^{(2)} &\equiv -\partial_t\tilde{r}_{ij}^{(2)}+\ell_{ij}^{(2)}+h_{ij}^{(2)},\quad i\neq j,\\
\gamma_i^{(3)} &\equiv -\partial_t\tilde{r}_i^{(3)}+\ell_i^{(3)}+h_i^{(3)},
\end{aligned}
$$

where $r=-\partial_t\tilde{r}$. On the other hand, a,b,c also satisfy the local macroscopic balance laws. In fact, multiplying the equation (4.1) by the collision invariants in (4.6) and integrating the products over $\mathbb{R}^n$, one has

$$
\begin{aligned}
\partial_t\int_{\mathbb{R}^n} fd\xi+\nabla_x\cdot\int_{\mathbb{R}^n}\xi fd\xi=0,\\
\partial_t\int_{\mathbb{R}^n}\xi fd\xi+\nabla_x\cdot\int_{\mathbb{R}^n}\xi\otimes\xi fd\xi=\nabla_x\phi\int_{\mathbb{R}^n} fd\xi,\\
\partial_t\int_{\mathbb{R}^n}\frac{1}{2}|\xi|^2 fd\xi+\nabla_x\cdot\int_{\mathbb{R}^n}\frac{1}{2}|\xi|^2\xi fd\xi=\nabla_x\phi\cdot\int\xi fd\xi.
\end{aligned}
$$

By using the perturbation (4.2) and the decomposition (4.13), direct computation gives the macroscopic balance laws on the coefficients (a,b,c) of u_1:

$$
\partial_t a-\frac{1}{2}\nabla_x\cdot\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\rangle=-\frac{1}{2}b\cdot\nabla_x\phi, \tag{4.21}
$$

$$
\partial_t b_i+\partial_i[a+(n+2)c]+\nabla_x\cdot\langle\xi\xi_i\sqrt{\mathbf{M}},u_2\rangle=(a+nc)\partial_i\phi, \tag{4.22}
$$

$$
\partial_t c+\frac{1}{n}\nabla_x\cdot b+\frac{1}{2n}\nabla_x\cdot\langle|\xi|^2\xi\sqrt{\mathbf{M}},u_2\rangle=\frac{1}{2n}b\cdot\nabla_x\phi. \tag{4.23}
$$

An important observation from [11] is that for each j, b_j satisfies an elliptic-type equation:

$$
\begin{aligned}
-\Delta_x b_j-\partial_j\partial_j b_j=&\sum_{i\neq j}\partial_j\left[\gamma_i^{(2)}+b_i\partial_i\phi\right]-\sum_{i\neq j}\partial_i\left[\gamma_{ij}^{(2)}+b_j\partial_i\phi+b_i\partial_j\phi\right]\\
&-2\partial_j\left[\gamma_j^{(2)}+b_j\partial_j\phi\right].
\end{aligned}
$$

Moreover, the proper linear combination of the equations (4.16)-(4.20) and (4.21)-(4.23) gives

$$
\begin{aligned}
&\partial_t(a+nc)+\nabla_x\cdot b=0,\\
&\partial_t b+\nabla_x(a+nc)+2\nabla_x c-\Delta_x b-\frac{1}{n}\nabla_x\nabla_x\cdot b=(a+nc)\nabla_x\phi+R^b,\\
&\partial_t c+\frac{1}{n}\nabla_x\cdot b-\Delta_x c=\frac{1}{2n}b\cdot\nabla_x\phi+R^c,
\end{aligned}
$$

where $R^b = (R^b_1, \cdots, R^b_n)$ and R^c are defined by

$$R^b_j = -\nabla_x \cdot \langle \xi\xi_j \sqrt{\mathbf{M}}, u_2\rangle - \frac{1}{n}\partial_j \nabla_x \cdot \langle |\xi|^2 \xi \sqrt{\mathbf{M}}, u_2\rangle$$
$$-\sum_{i\neq j} \partial_i \left[\gamma^{(2)}_{ij} + b_j \partial_i \phi + b_i \partial_j \phi\right] - 2\partial_j \left[\gamma^{(2)}_j + b_j \partial_j \phi\right],$$
$$R^c = -\frac{1}{2n}\nabla_x \cdot \langle |\xi|^2 \xi \sqrt{\mathbf{M}}, u_2\rangle - \sum_i \partial_i \left[\gamma^{(3)}_i + c\partial_i \phi\right].$$

We now turn to the dissipation rate of the microscopic component:

$$\sum_{|\alpha|+|\beta|\leq N} \|\partial^\alpha_x \partial^\beta_\xi u_2\|^2_\nu = \|u_2\|^2_\nu + \sum_{1\leq|\alpha|\leq N} \|\partial^\alpha_x u_2\|^2_\nu + \sum_{k=1}^N \sum_{\substack{|\beta|=k\\ |\alpha|+|\beta|\leq N}} \|\partial^\alpha_x \partial^\beta_\xi u_2\|^2_\nu.$$

For this, set

$$\left\|(1+|x|)^2 \nabla_x \phi\right\|_{L^\infty_x} + \sum_{2\leq|\alpha|\leq N} \|(1+|x|)\partial^\alpha_x \phi\|_{L^\infty_x} \leq \delta,$$

for small $\delta > 0$. Then, one has the following estimates, cf. [5].

(i) Estimates on the zero-th order:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \lambda\|u_2\|^2_\nu \leq C[[u(t)]][[u(t)]]^2_\nu + C\delta\|\nabla_x u_1\|^2. \tag{4.24}$$

(ii) Estimates on only the spatial derivatives:

$$\frac{1}{2}\frac{d}{dt}\sum_{1\leq|\alpha|\leq N} \|\partial^\alpha_x u\|^2 + \lambda \sum_{1\leq|\alpha|\leq N} \|\partial^\alpha_x u_2\|^2_\nu \leq C[[u(t)]][[u(t)]]^2_\nu$$
$$+C\delta \sum_{1\leq|\alpha|\leq N} \|\partial^\alpha_x u_1\|^2 + C\delta \sum_{1\leq|\alpha|\leq N-1} \|\partial^\alpha_x \nabla_\xi u_2\|^2. \tag{4.25}$$

(iii) Estimates on the mixed derivatives:

$$\frac{1}{2}\frac{d}{dt}\sum_{\substack{|\beta|=k\\ |\alpha|+|\beta|\leq N}} \|\partial^\alpha_x \partial^\beta_\xi u_2\|^2 + \lambda \sum_{\substack{|\beta|=k\\ |\alpha|+|\beta|\leq N}} \|\partial^\alpha_x \partial^\beta_\xi u_2\|^2_\nu$$
$$\leq C[[u(t)]][[u(t)]]^2_\nu + C\sum_{|\alpha|\leq N-k+1} \|\partial^\alpha_x u_2\|^2_\nu + C\chi_{\{k\geq 2\}} \sum_{\substack{1\leq|\beta|\leq k-1\\ |\alpha|+|\beta|\leq N}} \|\partial^\alpha_x \partial^\beta_\xi u_2\|^2_\nu$$
$$+C\sum_{|\alpha|\leq N-k} \|\partial^\alpha_x \nabla_x (a,b,c)\|^2, \tag{4.26}$$

where the integer $k \geq 1$ and $\chi_{\{k\geq 2\}}$ denotes the characteristic function of the set $\{k \geq 2\}$.

Now, the dissipation rate of the microscopic component can be obtained by a suitable linear combination of (4.24), (4.25) and (4.26). In fact, the sum of (4.24) and (4.25) gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\leq N}\|\partial_x^\alpha u\|^2+\lambda\sum_{|\alpha|\leq N}\|\partial_x^\alpha u_2\|^2\leq C[[u(t)]][[u(t)]]_\nu^2\\ +C\delta\sum_{1\leq|\alpha|\leq N}\|\partial_x^\alpha u_1\|^2+C\delta\sum_{1\leq|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_\xi u_2\|^2.\end{aligned}\tag{4.27}$$

The linear combination of (4.26) with k taking values from 1 to N gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{\substack{|\alpha|+|\beta|\leq N\\|\beta|\geq1}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|^2+\lambda\sum_{\substack{|\alpha|+|\beta|\leq N\\|\beta|\geq1}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2\leq C[[u(t)]][[u(t)]]_\nu^2\\ +C\sum_{|\alpha|\leq N}\|\partial_x^\alpha u_2\|_\nu^2+C\sum_{|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_x(a,b,c)\|^2,\end{aligned}\tag{4.28}$$

where the energy functional is actually in the form of

$$\sum_{k=1}^N C_{N,k}\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\leq N}}\|\partial_x^\alpha\partial_\xi^\beta u_2\|^2,$$

for some properly chosen positive constants $C_{N,k}$. However, for simplicity, we shall ignore this detailed differences in the coefficients of the energy functional. Therefore, the linear combination of (4.27) and (4.28) gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\sum_{|\alpha|\leq N}\|\partial_x^\alpha u\|^2+\sum_{|\alpha|+|\beta|\leq N,|\beta|\geq1}\|\partial_x^\alpha\partial_\xi^\beta u_2\|^2\right)+\lambda\sum_{|\alpha|+|\beta|\leq N}\|\partial_x^\alpha\partial_\xi^\beta u_2\|_\nu^2\\ \leq C[[u(t)]][[u(t)]]_\nu^2+C\sum_{|\alpha|\leq N-1}\|\partial_x^\alpha\nabla_x(a,b,c)\|^2.\end{aligned}\tag{4.29}$$

Based on some analysis of the macroscopic equations (4.16), (4.17), (4.18), (4.19), (4.20) and the macroscopic balance laws (4.21), (4.22), (4.23), one can deduce

$$\begin{aligned}2\frac{d}{dt}\mathcal{I}(u(t))+\sum_{|\alpha|\leq N-1}\|\nabla_x\partial_x^\alpha(a,b,c)\|^2\\ \leq C\left\{\sum_{|\alpha|\leq N}\|\partial_x^\alpha u_2\|^2+[[u(t)]]^2[[u(t)]]_\nu^2\right\},\end{aligned}\tag{4.30}$$

where $\mathcal{I}(u(t))$ is the total interactive energy functional defined by

$$\mathcal{I}(u(t))=\sum_{|\alpha|\leq N-1}\sum_{i=1}^3\left[\mathcal{I}_{\alpha,i}^a(u(t))+\mathcal{I}_{\alpha,i}^b(u(t))+\mathcal{I}_{\alpha,i}^c(u(t))+\mathcal{I}_{\alpha,i}^{ab}(u(t))\right]$$

with $\mathcal{I}^a_{\alpha,i}(u(t))$, $\mathcal{I}^b_{\alpha,i}(u(t))$, $\mathcal{I}^c_{\alpha,i}(u(t))$ and $\mathcal{I}^{ab}_{\alpha,i}(u(t))$ being the individual interactive energy functionals defined by

$$\begin{aligned}
\mathcal{I}^a_{\alpha,i}(u(t)) &= \left\langle \partial^\alpha_x \tilde{r}^{(1)}_i, \partial_i \partial^\alpha_x a \right\rangle, \\
\mathcal{I}^b_{\alpha,i}(u(t)) &= -\sum_{j\neq i} \left\langle \partial^\alpha_x \tilde{r}^{(2)}_j, \partial_i \partial^\alpha_x b_i \right\rangle + \sum_{j\neq i} \left\langle \partial^\alpha_x \tilde{r}^{(2)}_{ji}, \partial_j \partial^\alpha_x b_i \right\rangle \\
&\quad +2 \left\langle \partial^\alpha_x \tilde{r}^{(2)}_i, \partial_i \partial^\alpha_x b_i \right\rangle, \\
\mathcal{I}^c_{\alpha,i}(u(t)) &= \left\langle \partial^\alpha_x \tilde{r}^{(3)}_i, \partial_i \partial^\alpha_x c \right\rangle, \\
\mathcal{I}^{ab}_{\alpha,i}(u(t)) &= \langle \partial_i \partial^\alpha_x a, \partial^\alpha_x b_i \rangle .
\end{aligned}$$

Notice that the proof of (4.30) can be found in [3, 5] by using an improved energy method.

Now one can obtain the full dissipation rate $[[u(t)]]^2_\nu$ as follows. The linear combination of (4.27) and (4.30) gives

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{M}{2}\sum_{|\alpha|\leq N} \|\partial^\alpha_x u\|^2 + 2\mathcal{I}(u(t))\right] \\
&\qquad +\lambda\left(\sum_{|\alpha|\leq N} \|\partial^\alpha_x u_2\|^2_\nu + \sum_{|\alpha|\leq N-1} \|\nabla_x \partial^\alpha_x (a,b,c)\|^2\right) \\
&\leq C[[u(t)]][[u(t)]]^2_\nu + C\delta \sum_{1\leq|\alpha|\leq N-1} \|\partial^\alpha_x \nabla_\xi u_2\|^2,
\end{aligned} \tag{4.31}$$

where $M > 0$ is large enough so that the energy functional in (4.31) is equivalent to its first part

$$\sum_{|\alpha|\leq N} \|\partial^\alpha_x u\|^2.$$

Recall the definitions of the norms $[[u(t)]]$ and $[[u(t)]]_\nu$ given in (4.8) and (4.9), respectively. Thus the linear combination of (4.29) and (4.31) gives the Lyapunov inequality

$$\frac{d}{dt}\mathcal{E}(u(t)) + \lambda \mathcal{D}(u(t)) \leq C[\sqrt{\mathcal{E}(u(t))} + \mathcal{E}(u(t)]\mathcal{D}(u(t)),$$

where the energy functional is in the form

$$\begin{aligned}
\mathcal{E}(u(t)) &\sim \sum_{|\alpha|\leq N} \|\partial^\alpha_x u\|^2 + \sum_{|\alpha|+|\beta|\leq N\ |\beta|\geq 1} \|\partial^\alpha_x \partial^\beta_\xi u_2\|^2 \\
&\quad + \frac{M}{2}\sum_{|\alpha|\leq N} \|\partial^\alpha_x u\|^2 + 2\mathcal{I}(u(t)) \\
&\sim [[u(t)]]^2,
\end{aligned}$$

and the dissipation rate is in the form

$$\begin{aligned}
\mathcal{D}(u(t)) &\sim \sum_{|\alpha|+|\beta|\leq N} \|\partial^\alpha_x \partial^\beta_\xi u_2\|^2_\nu + \sum_{|\alpha|\leq N-1} \|\nabla_x \partial^\alpha_x (a,b,c)\|^2 \\
&\sim [[u(t)]]^2_\nu.
\end{aligned}$$

Thus the uniform a prior estimate (4.12) holds under the assumption that $[[u(t)]]$ is small enough. This completes the proof of Proposition 4.1.

Proof of Theorem 4.1: The proof is divided into five steps.

Step 1. Let $u(t,x,\xi)$ be the solution to the Cauchy problem (4.3) and (4.4) which is obtained in Proposition 4.1. Denote δ_0, K_0, ϵ, respectively, by

$$\delta_0 = [[u_0]],\ K_0 = \|u_0\|_{Z_1},\ \delta_\phi = \sum_{|\alpha|\leq N} \|(1+|x|)\partial_x^\alpha \phi\|_{L_x^\infty} + \||x|\nabla_x\phi\|_{L_x^2}.$$

Notice that δ_0, δ_ϕ can be chosen to be sufficiently small while K_0 is kept to be just finite.

Step 2. From the energy estimates of higher order derivatives as in the proof of (4.12), one can obtain

$$\frac{d}{dt}\mathcal{E}^{\mathrm{BE}}(u(t)) + \lambda[[u(t)]]_\nu^2 \leq C\|\nabla_x u\|^2,$$

where $\mathcal{E}^{\mathrm{BE}}(u(t))$ is equivalent to the energy functional for higher order derivatives given by

$$\mathcal{E}^{\mathrm{BE}}(u(t)) \sim \sum_{|\alpha|+|\beta|\leq N} \|\partial_x^\alpha\partial_\xi^\beta\{\mathbf{I}-\mathbf{P}\}u(t)\|^2 + \sum_{0<|\alpha|\leq N}\|\partial_x^\alpha u(t)\|^2.$$

With the observation that

$$\mathcal{E}^{\mathrm{BE}}(u(t)) \leq C[[u(t)]]_\nu,$$

one has

$$\frac{d}{dt}\mathcal{E}^{\mathrm{BE}}(u(t)) + \lambda\mathcal{E}^{\mathrm{BE}}(u(t)) \leq C\|\nabla_x u\|^2, \tag{4.32}$$

which is the estimate of Type I.

Step 3. The solution u to the Cauchy problem (4.3) and (4.4) can be written as

$$u(t) = e^{\mathbf{B}t}u_0 + \int_0^t e^{\mathbf{B}(t-s)}\mathbf{S}(u(s))ds,$$

where $\mathbf{S}(u)$ is the source term given by

$$\mathbf{S}(u) = -\nabla_x\phi\cdot\nabla_\xi u + \frac{1}{2}\nabla_x\phi\cdot\xi u + (e^\phi - 1)\mathbf{L}u + \Gamma(u,u).$$

Now let's recall the following decomposition of the semigroup $e^{\mathbf{B}t}$ and the corresponding decay estimates.

Lemma 4.1 ([31])**.** *The semigroup $e^{\mathbf{B}t}$ can be decomposed into*

$$e^{\mathbf{B}t} = \mathbf{E}_0(t) + \mathbf{E}_1(t) + \mathbf{E}_2(t).$$

Here, $\mathbf{E}_0(t)$ is the linear transport operator with the collision frequency $\nu(\xi)$ defined by

$$\mathbf{E}_0(t)u \equiv e^{-\nu(\xi)t}u(x-\xi t,\xi).$$

$\mathbf{E}_1(t)\nu$ *has an algebraic decay while* $\mathbf{E}_2(t)\nu$ *has an exponential decay. Precisely, one has*

$$\|\nabla_x^m \mathbf{E}_1(t)\nu u\|_{Z_2} \leq C(1+t)^{-\sigma_n(q,m)}\|u\|_{Z_q},$$
$$\|\nabla_x^m \mathbf{E}_1(t)\{\mathbf{I}-\mathbf{P}\}\nu u\|_{Z_2} \leq C(1+t)^{-\sigma_n(q,m+1)}\|u\|_{Z_q},$$

and

$$\|\nabla_x^m \mathbf{E}_2(t)\nu u\|_{Z_2} \leq Ce^{-\lambda t}\|\nabla_x^m u\|_{Z_2},$$

for any non-negative integer m *and* $1 \leq q \leq 2$.

Thus, one can rewrite the solution u as a summation of two parts:

$$\begin{aligned}
u(t) &= I_1[u](t) + I_2[u](t),\\
I_1[u](t) &= e^{\mathbf{B}t}u_0 + \int_0^t \{\mathbf{E}_1(t-s) + \mathbf{E}_2(t-s)\}\mathbf{S}(u(s))ds,\\
I_2[u](t) &= \int_0^t \mathbf{E}_0(t-s)\mathbf{S}(u(s))ds.
\end{aligned}$$

Define

$$\mathcal{E}_\infty^{\mathrm{BE}}(t) = \sup_{0\leq s\leq t}(1+s)^{2\sigma_n(1,1)}\mathcal{E}^{\mathrm{BE}}(u(s)).$$

Since $\mathcal{E}_\infty^{\mathrm{BE}}(t)$ is a non-decreasing function, from Lemma 4.1, it is straightforward to show that, cf. [8],

$$\|\nabla_x I_1[u](t)\|^2 \leq C(1+t)^{-2\sigma_n(1,1)}[\delta_0^2 + K_0^2 + (\delta_0^2+\delta_\phi^2)\mathcal{E}_\infty^{\mathrm{BE}}(t)]. \tag{4.33}$$

Step 4. We need to consider carefully the estimate on $\|\nabla_x I_2[u](t)\|$ because the decay estimate as in (4.33) does not hold for $I_2[u]$. In fact, it follows from (4.32) and (4.33) that

$$\begin{aligned}
\mathcal{E}^{\mathrm{BE}}(u(t)) &\leq e^{-\lambda t}\mathcal{E}^{\mathrm{BE}}(u_0) + \int_0^t e^{-\lambda(t-s)}\|\nabla_x u(s)\|_{L^2_{x,\xi}}^2 ds\\
&\leq C(1+t)^{-2\sigma_n(1,1)}[\delta_0^2 + K_0^2 + (\delta_0^2+\delta_\phi^2)\mathcal{E}_\infty^{\mathrm{BE}}(t)]\\
&\quad + \int_0^t e^{-\lambda(t-s)}\|\nabla_x I_2[u](s)\|_{L^2_{x,\xi}}^2 ds.
\end{aligned}$$

To deal with the term related to $I_2[u]$, we need a technical lemma about the integral estimats on the solution to an inhomogeneous transport equation with damping.

Lemma 4.2 ([5])**.** *Define* $\mathcal{T}h(t,x,\xi)$ *to be the solution to the following Cauchy problem*

$$\begin{cases} \partial_t u + \xi\cdot\nabla_x u + \nu(\xi)u = \nu(\xi)h(t,x,\xi),\\ u|_{t=0} = 0. \end{cases}$$

Then, for any $\lambda \in (0,\nu_0)$ *where* ν_0 *is defined in* (4.5)*, there exists a constant* C *such that*

$$\int_0^t e^{-\lambda(t-s)}\|\mathcal{T}h(s)\|_{L^2_{x,\xi}}^2 ds \leq C\int_0^t e^{-\lambda(t-s)}\|h(s)\|_{L^2_{x,\xi}}^2 ds.$$

With this, one can write

$$\nabla_x I_2[u](t) = \int_0^t \mathbf{E}_0(t-s)\nabla_x \mathbf{S}(u(s))ds = \mathcal{T}(\nabla_x \mathbf{S}(u)/\nu).$$

The direct calculation yields

$$\|\nabla_x \mathbf{S}(u)/\nu\|^2_{L^2_{x,\xi}} \leq (\delta_0^2+\delta_\phi^2)\mathcal{E}^{\rm BE}(u(t)).$$

Thus, by using Lemma 4.2, one has

$$\begin{aligned}\mathcal{E}^{\rm BE}(u(t)) &\leq C(1+t)^{-2\sigma_n(1,1)}[\delta_0^2+K_0^2+(\delta_0^2+\delta_\phi^2)\mathcal{E}^{\rm BE}_\infty(t)]\\ &\quad+\int_0^t e^{-\lambda(t-s)}\|\mathcal{T}(\nabla_x\mathbf{S}(u)/\nu)\|^2_{L^2_{x,\xi}}ds\\ &\leq C(1+t)^{-2\sigma_n(1,1)}[\delta_0^2+K_0^2+(\delta_0^2+\delta_\phi^2)\mathcal{E}^{\rm BE}_\infty(t)].\end{aligned}$$

Since δ_0 and δ_ϕ can be sufficiently small, $\mathcal{E}^{\rm BE}_\infty(t)$ is bounded uniformly in t so that the following time decay estimates on the energy of higher order derivatives hold:

$$\mathcal{E}^{\rm BE}(u(t)) \leq C(\delta_0^2+K_0^2)(1+t)^{-2\sigma_n(1,1)}. \tag{4.34}$$

Step 5. By using (4.34), one can obtain the decay estimate on $\|u(t)\|_{L^2_{x,\xi}}$ in the same way as that in the last two steps. In fact, adding $\|\mathbf{P}u(t)\|^2_{L^2_{x,\xi}}$ to both sides of the uniform energy estimate (4.12) gives

$$\frac{d}{dt}\mathcal{E}(u(t))+\lambda\mathcal{E}(u(t)) \leq C\|u(t)\|^2_{L^2_{x,\xi}}.$$

The Gronwall's inequality then yields

$$\mathcal{E}(u(t)) \leq e^{-\lambda t}\mathcal{E}(u_0)+C\int_0^t e^{-\lambda(t-s)}\|u(s)\|^2_{L^2_{x,\xi}}ds.$$

With (4.34), the same procedures as those in Step 2 and Step 3 can be repeated to obtain

$$\mathcal{E}(u(t)) \leq C(\delta_0^2+K_0^2)(1+t)^{-2\sigma_n(1,0)}.$$

Since $\mathcal{E}(u(t))$ is an energy functional equivalent to $[[u(t)]]$ and $\sigma_n(1,0)=n/4$, (4.10) follows. This completes the proof of Theorem 4.1.

Remark 4.1. *From the proof presented above for the optimal decay rates in the case of the Boltzmann equation, it is impossible to obtain a similar estimate of Type II in the form* (2.11) *and* (3.26). *Instead, the estimate of Type II is replaced by a weaker integral estimate as stated in Lemma 4.2.*

5 Conclusions

In this paper we introduced an approach of how to obtain the optimal rates of convergence of solutions toward the possibly existing non-constant steady state for some dissipative evolution equations. The main motivation is inspired by the studies of the time-decay estimates on the linearized equations around the non-constant steady state. In fact, whenever there exists any non-constant steady state for some nonlinear evolution equations, then the corresponding linearized equations contain the variable coefficients in general, which leads that the usual method of the spectral analysis in the Fourier space is difficult to be applied to obtain the time-decay properties of the linearized solution operator. The same difficulties happen to the nonlinear case when we are trying to control the nonlinear terms in terms of the dissipations generated by the linearized operator. The current approach provides one of the ways to overcome these difficulties. The main observation is that at the level of linearization, the linear term with variable coefficients can be controlled by the dissipation of equations provided that the strength of perturbations around the constant steady state is small. Therefore, by using the energy estimates, one can obtain the dissipations of all the high-order derivatives on the basis of the lowest-order derivative, and on the other hand, the time-decay of the lowest-order derivative can be given in terms of the high-order derivatives by using the time-decay properties of the linearized solution operator with constant coefficients, the combination of which implies the optimal decay rates of solutions.

At this moment, it is expected that the approach introduced in this paper can also be applied to other dissipative evolution systems in fluid dynamics and kinetic models such as the model system of radiating gas and the Vlasov-Poisson-Boltzmann system. The common characteristic for these systems is that there also exist the non-constant steady states when in the presence of sources or the non-constant background density, respectively. Thus, it is interesting to apply the current method to study their asymptotical stability with the possible optimal convergence rates. The corresponding results could be reported in the future.

Finally, we remark that the current approach fails for the case when the strength of variable coefficients or perturbations around the constant states is large in the presence of sources or forces. In this case, the source or force should be regarded as an essential part of the equations under the consideration, which is generally a challenging research topic outside of the scope of this paper. The interested readers may refer to [20, 22] and references therein.

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