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MOMENT CLASSIFICATION OF INFINITE ENERGY SOLUTIONS TO THE HOMOGENEOUS BOLTZMANN EQUATION

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Abstract. In this paper, we will introduce a precise classification of characteristic functions in the Fourier space according to the moment constraint in the physical space of any order. Based on this, we construct measure valued solutions to the homogeneous Boltzmann equation with the exact moment condition as the initial data.

1. Introduction

Consider the spatially homogeneous Boltzmann equation,

$$\partial_t f(t,v) = Q(f,f)(t,v), \tag{1.1}$$

where $f(t,v)$ is the density distribution of particles with velocity $v \in \mathbb{R}^3$ at time t. The most interesting and important part of this equation is the collision operator given on the right hand side that captures the change rate of the density distribution through the elastic binary collisions:

$$Q(g,f)(v) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\left(v-v_*,\sigma\right)\left\{g(v'_*)f(v') - g(v_*)f(v)\right\} d\sigma dv_*\,,$$

where for $\sigma \in \mathbb{S}^2$

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma,\;\; v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma,$$

that follow from the conservation of momentum and energy,

$$v'+v'_* = v+v_*,\;\; |v'|^2+|v'_*|^2 = |v|^2+|v_*|^2.$$

The natural space of the unknown function $f(t,v)$ to the Boltzmann equation is the space of the probability distribution with suitable moment constraint that may reflect the boundedness of the momentum or energy.

The purpose of this paper is first to give a precise definition of the space of a probability distribution with α-order moment after Fourier transform, and then construct the measure valued solution to the Boltzmann equation in such space with the same parameter α for the initial data and the solution in the setting of Maxwellian type cross-sections.

More precisely, consider (1.1) with initial datum

$$f(0,v) = dF_0 \geq 0, \tag{1.2}$$

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where F_0 is a probability measure.

Motivated by the inverse power law, assume that the non-negative cross section B takes the form of

$$B(|v-v_*|, \cos\theta) = \Phi(|v-v_*|)b(\cos\theta), \quad \cos\theta = \frac{v-v_*}{|v-v_*|} \cdot \sigma\,, \;\; 0 \le \theta \le \frac{\pi}{2},$$

where

$$\Phi(|z|) = \Phi_\gamma(|z|) = |z|^\gamma, \;\text{ for some } \gamma > -3,$$
$$b(\cos\theta)\theta^{2+2s} \;\to K \;\text{ when }\; \theta \to 0+, \;\text{ for } 0 < s < 1 \text{ and } K > 0. \tag{1.3}$$

Throughout this paper, we will only consider the case when

$$\Phi(|v-v_*|) = 1,$$

that is called the Maxwellian molecule type cross section. In this case, the analysis relies on the good structure of the equation after taking Fourier transform in v by the Bobylev formula. And the other cases will be pursued by the authors in the future.

As usual, the range of θ can be restricted to $[0, \pi/2]$, by replacing $b(\cos\theta)$ by its "symmetrized" version

$$[b(\cos\theta) + b(\cos(\pi-\theta))]\mathbf{1}_{0\le\theta\le\pi/2}.$$

We can work on the problem with the following slightly more general assumption on the cross section

$$\exists \alpha_0 \in (0,2] \;\text{ such that }\; (\sin\theta/2)^{\alpha_0} b(\cos\theta)\sin\theta \in L^1((0,\pi/2]), \tag{1.4}$$

which is fulfilled for the function b in (1.3) if $2s < \alpha_0$.

Denote by $P_\alpha(\mathbb{R}^3)$, $\alpha \in [0,2]$ the set of probability measure F on $\mathbb{R}^3$, such that

$$\int_{\mathbb{R}^3} |v|^\alpha dF(v) < \infty,$$

and moreover when $1 < \alpha \le 2$, it requires that

$$\int_{\mathbb{R}^3} v_j dF(v) = 0, \;\; j = 1,2,3\,. \tag{1.5}$$

Following Jacob [5] and Cannone-Karch [2], call the Fourier transform of a probability measure $F \in P_0(\mathbb{R}^3)$, that is,

$$\varphi(\xi) = \hat{f}(\xi) = \mathcal{F}(F)(\xi) = \int_{\mathbb{R}^3} e^{-iv\cdot\xi} dF(v),$$

a characteristic function.

Put $\mathcal{K} = \mathcal{F}(P_0(\mathbb{R}^3))$. Inspired by a series of works by Toscani and his co-authors [3, 4, 11], Cannone-Karch defined a subspace $\mathcal{K}^\alpha$ for $\alpha \ge 0$ as follows:

$$\mathcal{K}^\alpha = \{\varphi \in \mathcal{K}\,;\, \|\varphi - 1\|_\alpha < \infty\}\,, \tag{1.6}$$

where

$$\|\varphi - 1\|_\alpha = \sup_{\xi\in\mathbb{R}^3} \frac{|\varphi(\xi) - 1|}{|\xi|^\alpha}. \tag{1.7}$$

The space $\mathcal{K}^\alpha$ endowed with the distance

$$\|\varphi - \tilde{\varphi}\|_\alpha = \sup_{\xi\in\mathbb{R}^3} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha} \tag{1.8}$$

is a complete metric space (see Proposition 3.10 of [2]). It follows that $\mathcal{K}^\alpha = \{1\}$ for all $\alpha > 2$ and the following embeddings (Lemma 3.12 of [2]) hold

$$\{1\} \subset \mathcal{K}^\alpha \subset \mathcal{K}^\beta \subset \mathcal{K}^0 = \mathcal{K} \quad \text{for all } 2 \geq \alpha \geq \beta \geq 0.$$

With this classification on the characteristic functions, the global existence of solution in $\mathcal{K}^\alpha$ was studied in [2](see also [7]). However, even though the inclusion $\mathcal{F}(P_\alpha(\mathbb{R}^3)) \subset \mathcal{K}^\alpha$ holds (see Lemma 3.15 of [2]), the space $\mathcal{K}^\alpha$ is strictly larger than $\mathcal{F}(P_\alpha(\mathbb{R}^3))$ for $\alpha \in (0,2)$, in other word, $\mathcal{F}^{-1}(\mathcal{K}^\alpha) \supsetneq P_\alpha(\mathbb{R}^3)$. Indeed, it is shown (see Remark 3.16 of [2]) that the function $\varphi_\alpha(\xi) = e^{-|\xi|^\alpha}$, with $\alpha \in (0,2)$, belongs to $\mathcal{K}^\alpha$, but $p_\alpha(v) = \mathcal{F}^{-1}(\varphi_\alpha)(v)$ that is the density of α-stable symmetric Lévy process, is not contained in $P_\alpha(\mathbb{R}^3)$.

On the other hand, we remark that $\mathcal{F}(P_2(\mathbb{R}^3)) = \mathcal{K}^2$. Indeed, this can be proved by contradiction. If there exists a $\varphi(\xi) \in \mathcal{K}^2$ such that $F = \mathcal{F}^{-1}(\varphi) \notin P_2$, then we may assume there exist $\omega_0 \in \mathbb{S}^{d-1}$and $A > 0$ such that

$$\int_{\{|\frac{v}{|v|} - \omega_0| < 10^{-10}\} \cap \{|v| \leq A\}} |v|^2 dF(v) \geq 100 \|1 - \varphi\|_2 \,,$$

from which we have a contradiction because

$$\begin{aligned}
&\|1-\varphi\|_2 \geq \sup_\xi \frac{\mathrm{Re}(1-\varphi(\xi))}{|\xi|^2} \\
&\geq 2 \int_{\{|\frac{v}{|v|} - \omega_0| < 10^{-10}\} \cap \{|v| \leq A\}} \frac{\sin^2\left\{\frac{|v||\xi|}{2}\left(\frac{v}{|v|}\cdot\frac{\xi}{|\xi|}\right)\right\}}{|v|^2|\xi|^2} |v|^2 dF(v) \ \text{ for } \frac{\xi}{|\xi|} = \omega_0,\ |\xi| = \frac{\pi}{A} \\
&\geq \frac{2}{\pi^2} \int_{\{|\frac{v}{|v|} - \omega_0| < 10^{-10}\} \cap \{|v| \leq A\}} \left(\frac{v}{|v|}\cdot\omega_0\right)^2 |v|^2 dF(v) > 50 \|1-\varphi\|_2,
\end{aligned}$$

by using

$$\sin z \geq \frac{2z}{\pi} \ \text{ when } \ 0 \leq z \leq \frac{\pi}{2}\,.$$

In order to capture the precise moment constraint in the Fourier space, another classification on the characteristic functions was introduced in [10] as follows:

$$\mathcal{M}^\alpha = \{\varphi \in \mathcal{K}\,;\, \|\varphi - 1\|_{\mathcal{M}^\alpha} < \infty\}\,,\ \ \alpha \in (0,2)\,, \tag{1.9}$$

where

$$\|\varphi - 1\|_{\mathcal{M}^\alpha} = \int_{\mathbb{R}^3} \frac{|\varphi(\xi) - 1|}{|\xi|^{3+\alpha}} d\xi\,. \tag{1.10}$$

It was shown in [10] that if $\alpha \in (0,1) \cap (1,2)$, then $\mathcal{M}^\alpha = \mathcal{F}(P_\alpha)$. However, for the case $\alpha = 1$, $\mathcal{M}^1 \subsetneq \mathcal{F}(P_1)$.

To give a more precise description of the characterization of P_α of any order, in this paper, we first introduce

$$\widetilde{\mathcal{M}}^\alpha = \{\varphi \in \mathcal{K}\,;\, \|Re\varphi - 1\|_{\mathcal{M}^\alpha} + ||\varphi - 1||_\alpha < \infty\}\,,\ \ \alpha \in (0,2)\,, \tag{1.11}$$

where $Re\varphi$ stands for the real part of $\varphi(\xi)$. Accordingly, the imaginary part of $\varphi(\xi)$ is denoted by $Im\varphi$.

For $\varphi, \tilde\varphi \in \widetilde{\mathcal{M}}^\alpha$, put

$$\|\varphi - \tilde\varphi\|_{\widetilde{\mathcal{M}}^\alpha} = \int_{\mathbb{R}^3} \frac{|Re\varphi(\xi) - Re\tilde\varphi(\xi)|}{|\xi|^{3+\alpha}} d\xi,$$

and, for any $0<\beta<\alpha<2, 0<\epsilon<1$, we introduce the distance in $\widetilde{\mathcal{M}}^\alpha$ as

$$dis_{\alpha,\beta,\epsilon}(\varphi,\tilde\varphi)=\|\varphi-\tilde\varphi\|_{\widetilde{\mathcal{M}}^\alpha}+\|\varphi-\tilde\varphi\|_\beta+\|\varphi-\tilde\varphi\|_\beta^\epsilon. \tag{1.12}$$

With the above preparation, the first main result in this paper can be stated as follows.

Theorem 1.1. *If $0<\beta<\alpha<\gamma\le 2, 0<\epsilon<1$, then the space $\widetilde{\mathcal{M}}^\alpha$ is a complete metric space endowed with the distance $dis_{\alpha,\beta,\epsilon}(\cdot,\cdot)$. Moreover, we have*

$$\mathcal{K}^\gamma\subset\widetilde{\mathcal{M}}^\alpha\subset\mathcal{K}^\alpha\subset\mathcal{K}^\beta, \tag{1.13}$$

$$\widetilde{\mathcal{M}}^\alpha=\mathcal{F}(P_\alpha(\mathbb{R}^3)). \tag{1.14}$$

Furthermore, $\lim_{n\to\infty} dis_{\alpha,\beta,\epsilon}(\varphi_n,\varphi)=0$, for $\varphi_n,\varphi\in\widetilde{\mathcal{M}}^\alpha$, implies

$$\lim_{n\to\infty}\int\psi(v)dF_n(v)=\int\psi(v)dF(v)\ \text{for any}\ \psi\in C(\mathbb{R}^3) \tag{1.15}$$
$$\text{satisfying the growth condition}\ |\psi(v)|\lesssim\langle v\rangle^\alpha,$$

where $F_n=\mathcal{F}^{-1}(\varphi_n), F=\mathcal{F}^{-1}(\varphi)\in P_\alpha(\mathbb{R}^3)$.

Remark 1.2. *If $\alpha\in(0,1)\cup(1,2)$, then $\widetilde{\mathcal{M}}^\alpha=\mathcal{M}^\alpha$. If $\alpha=1$, $\widetilde{\mathcal{M}}^\alpha\supsetneq\mathcal{M}^\alpha$.*

Proof. Firstly, for $F\in P_\alpha,\alpha\in(0,2)$, denote the Fourier transform of F by φ, then $1-Re\varphi=\int_v(1-\cos(\xi\cdot v))dF$. Hence

$$\int\frac{|1-Re\varphi|}{|\xi|^{3+\alpha}}d\xi=\iint\frac{1-\cos(\xi\cdot v)}{|\xi|^{3+\alpha}}dFd\xi=2\int\frac{\sin^2(\zeta\cdot\sigma/2)}{|\zeta|^{3+\alpha}}d\zeta\int|v|^\alpha dF. \tag{1.16}$$

It is proved in [2] that $\mathcal{F}(P_\alpha)\subset\mathcal{K}^\alpha$. This inclusion and (1.16) show $\widetilde{\mathcal{M}}^\alpha\supset\mathcal{F}(P^\alpha)$. To prove $\widetilde{\mathcal{M}}^\alpha\subset\mathcal{F}(P^\alpha)$, it suffices to show:

$$\text{for any } \varphi\in\widetilde{\mathcal{M}}^\alpha,\text{ if } \alpha>1\text{ , then } F=\mathcal{F}^{-1}(\varphi)\text{ satisfies (1.5).}$$

Assume there exists $\varphi\in\widetilde{\mathcal{M}}^\alpha\subset\mathcal{K}^\alpha$, such that $a=\int vdF\neq 0$, $F=\mathcal{F}^{-1}(\varphi)$. Since $F(\cdot+a)\in P_\alpha$, we know $e^{i\xi\cdot a}\varphi(\xi)=\varphi_a(\xi)=\mathcal{F}(F(\cdot+a))\in\mathcal{K}^\alpha$. Therefore, we have

$$\begin{aligned}\sup_\xi\frac{|e^{-i\xi\cdot a}-1|}{|\xi|^\alpha}&=\sup_\xi\frac{|1-e^{i\xi\cdot a}|}{|\xi|^\alpha}\\&\le\sup_\xi\frac{|1-\varphi|}{|\xi|^\alpha}+\sup_\xi\frac{|\varphi-e^{i\xi\cdot a}|}{|\xi|^\alpha}\\&=\sup_\xi\frac{|1-\varphi|}{|\xi|^\alpha}+\sup_\xi\frac{|\varphi_a-1|}{|\xi|^\alpha}<\infty.\end{aligned}$$

This gives a contradiction to the fact that if $\alpha>1, a\neq 0$, then $e^{-i\xi\cdot a}\notin\mathcal{K}^\alpha$. The inclusion (1.13) follows from

$$\int\frac{|1-Re\varphi|}{|\xi|^{3+\alpha}}d\xi\le||1-\varphi||_\gamma\int_{|\xi|<1}\frac{1}{|\xi|^{3+\alpha-\gamma}}d\xi+2\int_{|\xi|\ge1}\frac{1}{|\xi|^{3+\alpha}}d\xi.$$

Secondly, let $\{\varphi_n\}_{n=1}^\infty$ be a Cauchy sequence in $\{\widetilde{\mathcal{M}}^\alpha, dis_{\alpha,\beta,\epsilon}(\cdot,\cdot)\}$. Then, there exists $N>0$, such that

$$dis_{\alpha,\beta,\epsilon}(\varphi_N,\varphi_n)<1,\ \text{for any}\ n>N.$$

As $\{\varphi_n\}_{n=1}^{\infty}$ is also a Cauchy sequence in the complete space $\{\mathcal{K}^\beta, ||\cdot||_\beta\}$, there exists $\varphi \in \mathcal{K}^\beta$ such that

$$||\varphi_n - \varphi||_\beta = \sup_\xi \frac{|\varphi_n - \varphi|}{|\xi|^\beta} \to 0, \text{ as } n \to \infty.$$

Then, for any $\delta \in (0,1)$,

$$\int_{\delta<|\xi|<\delta^{-1}} \frac{|Re\varphi - Re\varphi_n|}{|\xi|^{3+\alpha}} d\xi \leq ||\varphi - \varphi_n||_\beta \int_{\delta<|\xi|<\delta^{-1}} \frac{1}{|\xi|^{3+\alpha-\beta}} d\xi \to 0, \text{ as } n \to \infty.$$

Notice that

$$\begin{aligned}&\int_{\delta<|\xi|<\delta^{-1}} \frac{|Re\varphi - 1|}{|\xi|^{3+\alpha}} d\xi \\ &\leq \int_{\delta<|\xi|<\delta^{-1}} \frac{|Re\varphi - Re\varphi_n|}{|\xi|^{3+\alpha}} d\xi + \int_{\delta<|\xi|<\delta^{-1}} \frac{|Re\varphi_n - Re\varphi_N|}{|\xi|^{3+\alpha}} d\xi \\ &\quad + \int_{\delta<|\xi|<\delta^{-1}} \frac{|Re\varphi_N - 1|}{|\xi|^{3+\alpha}} d\xi.\end{aligned}$$

Then by letting $n \to \infty$, we have

$$\int_{\delta<|\xi|<\delta^{-1}} \frac{|Re\varphi - 1|}{|\xi|^{3+\alpha}} d\xi \leq 1 + \int_{\delta<|\xi|<\delta^{-1}} \frac{|Re\varphi_N - 1|}{|\xi|^{3+\alpha}} d\xi \leq 1 + \int \frac{|Re\varphi_N - 1|}{|\xi|^{3+\alpha}} d\xi.$$

Since $\delta \in (0,1)$ is arbitrary, we obtain $\varphi \in \widetilde{\mathcal{M}}^\alpha$.

Finally, suppose that for $F_n, F \in P_\alpha(\mathbb{R}^3)$, we have

$$\varphi_n = \mathcal{F}(F_n), \varphi = \mathcal{F}(F) \in \widetilde{\mathcal{M}}^\alpha, \text{ and } \lim_{n\to\infty} dis_{\alpha,\beta,\epsilon}(\varphi_n, \varphi) = 0\,.$$

Note that for $R > 1$

$$\int_{\{|\xi|\leq 1/R\}} \frac{|1 - Re\varphi_n(\xi)|}{|\xi|^{3+\alpha}} d\xi \leq \int_{\{|\xi|\leq 1/R\}} \frac{|1 - Re\varphi(\xi)|}{|\xi|^{3+\alpha}} d\xi + \|\varphi_n - \varphi\|_{\widetilde{\mathcal{M}}^\alpha}\,,$$

then it follows from the proof of Proposition 2.2 in [10] that for any $\varepsilon_1 > 0$ there exist $R > 1$ and $N \in \mathbb{N}$ such that

$$\int_{\{|v|\geq R\}} |v|^\alpha dF_n(v) + \int_{\{|v|\geq R\}} |v|^\alpha dF(v) < \varepsilon_1 \text{ if } n \geq N.$$

This shows (1.15) because $\varphi_n \to \varphi$ in $\mathcal{S}'(\mathbb{R}^3)$, and hence, $F_n \to F$ in $\mathcal{S}'(\mathbb{R}^3)$. □

Hence, $\widetilde{\mathcal{M}}^\alpha$ represents precisely P_α in the Fourier space for $\alpha \in (0,2)$. It then leads to a question about how to describe the subspace in $\mathcal{K}$ corresponding to P_α with $\alpha > 2$. For this, we introduce the following spaces.

For each $n \geq 1 (n \in \mathbb{N}), \alpha \in (0,2]$, denote

$$\widetilde{P}_{2n+\alpha}(\mathbb{R}^3) = \{F \in P_0;\ \frac{(1+|v|^2)^n F}{\int (1+|v|^2)^n dF} \in P_\alpha(\mathbb{R}^3)\}.$$

It should be noted that if $\alpha > 1$ then the above definition requires

$$\int_{\mathbb{R}^3} v_j (1+|v|^2)^n dF = 0, j = 1,2,3\,. \tag{1.17}$$

We can then characterize $\widetilde{P}_{2n+\alpha}$ exactly by using the space $\widetilde{\mathcal{M}}^\alpha$ obtained above.

Corollary 1.3. *Let $n \in \mathbb{N}, n \geq 1$, then we have the following characterization: if $\alpha \in (0,2)$*

$$\mathcal{F}(\widetilde{P}_{2n+\alpha}) = (1-\Delta)^{-n}\widetilde{\mathcal{M}}^{\alpha};$$

If $\alpha = 2$,

$$\mathcal{F}(\widetilde{P}_{2n+2}) = (1-\Delta)^{-n}\mathcal{K}^{2},$$

where Δ is the Laplace operator and the space $(1-\Delta)^{-n}S$ is defined as:

$$\varphi(\xi) \in (1-\Delta)^{-n}S, \text{ if there exists } \psi(\xi) \in C_b(\mathbb{R}^3)$$
$$\text{such that } \psi = (1-\Delta)^{n}\varphi \text{ and } \frac{\psi(\xi)}{\psi(0)} \in S.$$

Proof. For any $F \in \widetilde{P}_{2n+\alpha}$, since $\frac{\left(1+|v|^2\right)^n dF}{\int \left(1+|v|^2\right)^n dF} \in P_\alpha(\mathbb{R}^3)$, we have

$$\mathcal{F}\left(\frac{(1+|v|^2)^n dF}{\int (1+|v|^2)^n dF}\right) = \frac{(1-\Delta)^n \mathcal{F}(dF)}{\int (1+|v|^2)^n dF} \in \widetilde{\mathcal{M}}^{\alpha}.$$

Hence, $\mathcal{F}(dF) \in (1-\Delta)^{-n}\widetilde{\mathcal{M}}^{\alpha}$. Inversely, if $\varphi \in (1-\Delta)^{-n}\widetilde{\mathcal{M}}^{\alpha}$, by the definition, there exists $\psi \in C_b(\mathbb{R}^3)$ such that $\psi = (1-\Delta)^n\varphi$ and $\psi/\psi(0) \in \widetilde{\mathcal{M}}^{\alpha}$. Then,

$$P_\alpha \ni \frac{\mathcal{F}^{-1}(\psi)}{\psi(0)} = \frac{\mathcal{F}^{-1}((1-\Delta)^n\varphi)}{\psi(0)} = \frac{\left(1+|v|^2\right)^n \mathcal{F}^{-1}(\varphi)}{\int \left(1+|v|^2\right)^n d\mathcal{F}^{-1}(\varphi)(v)}.$$

Moreover, if $\alpha > 1$,

$$\int_{\mathbb{R}^3} v_j d\mathcal{F}^{-1}(\psi)(v) = \int_{\mathbb{R}^3} v_j \left(1+|v|^2\right)^n d\mathcal{F}^{-1}(\varphi)(v) = 0, \;\; j = 1,2,3.$$

This shows $\mathcal{F}^{-1}(\varphi) \in \widetilde{P}_{2n+\alpha}$. $\square$

Remark 1.4. *It is worth to remark that $\mathcal{K}^2 \subsetneq (1-\Delta)^{-1}\mathcal{K}^0$, because the zero moment condition* (1.17) *is not assumed for the space $\mathcal{K}^0$.*

Thanks to the new characterization of P_α for any $\alpha \in (0,2)$ by its exact Fourier image $\widetilde{\mathcal{M}}^{\alpha}$, we can improve the previous results, given in [2, 7, 9, 10], concerning the existence and the smoothing effect of measure valued solutions to the Cauchy problem for the spatially homogeneous Boltzmann equation with the Maxwellian molecule type cross section without angular cutoff. The results will be stated in the following theorems.

Theorem 1.5. *Assume that b satisfies* (1.4) *for some $\alpha_0 \in (0,2)$ and let $\alpha \in (\alpha_0, 2)$. If $F_0 \in P_\alpha(\mathbb{R}^3)$, then there exists a unique measure valued solution $F_t \in C([0,\infty), P_\alpha(\mathbb{R}^3))$ to the Cauchy problem* (1.1)-(1.2)*, where the continuity with respect to t is in the topology defined in* (1.15)*.*

Remark 1.6. *Assume the initial data $F_0 \in \widetilde{P}_{2n+\alpha}$ with $\alpha \in (0,2]$. Since F_0 may belong to P_α up to the translation when $\alpha > 1$, by Theorem 1.5, we can obtain the corresponding solution $F(t) \in P_\alpha(\mathbb{R}^3)$. However, since $v_j\langle v\rangle^{2n}$ with $n \geq 1$ is not a collision invariant, we can not expect the condition* (1.17) *on the initial data can propagate in time. Hence, the solution in general does not belong to $\widetilde{P}_{2n+\alpha}$ for $1 < \alpha \leq 2$.*

Corollary 1.7. *Assume that b satisfies* (1.4)*, $n \geq 1, n \in \mathbb{N}, \alpha \in (0,2]$. Let F_t be the measure valued solution to the Cauchy problem* (1.1)-(1.2) *with respect to the initial data $F_0 \in P_0(\mathbb{R}^3)$ satisfying*

$$\int |v|^{2n+\alpha} dF_0 < \infty.$$

Then, for any $T > 0$, there exists $C > 0$ such that

$$\int |v|^{2n+\alpha} dF_t \leq Ce^{CT} \int |v|^{2n+\alpha} dF_0, \tag{1.18}$$

for any $t \in [0,T]$.

Proof. We first consider the case with angular cutoff cross section and use a modified weighted moment as follows. For $\delta > 0$, put

$$W_\delta(v) = \frac{\langle v \rangle^{2n+\alpha}}{1 + \delta \langle v \rangle^{2n+\alpha}} .$$

Since the function $x/(1+\delta x)$ is increasing in $[1, \infty]$ and $|v'| \leq |v| + |v_*|$, we have

$$\begin{aligned} W_\delta(v') &\lesssim \frac{\langle v \rangle^{2n+\alpha} + \langle v_* \rangle^{2n+\alpha}}{1 + \delta(\langle v \rangle^{2n+\alpha} + \langle v_* \rangle^{2n+\alpha})} \\ &= \frac{\langle v \rangle^{2n+\alpha}}{1 + \delta(\langle v \rangle^{2n+\alpha} + \langle v_* \rangle^{2n+\alpha})} + \frac{\langle v_* \rangle^{2n+\alpha}}{1 + \delta(\langle v \rangle^{2n+\alpha} + \langle v_* \rangle^{2n+\alpha})} \\ &\leq \frac{\langle v \rangle^{2n+\alpha}}{1 + \delta \langle v \rangle^{2n+\alpha}} + \frac{\langle v_* \rangle^{2n+\alpha}}{1 + \delta \langle v_* \rangle^{2n+\alpha}} = W_\delta(v) + W_\delta(v_*) . \end{aligned}$$

Therefore, $|W_\delta(v') - W_\delta(v)| \lesssim W_\delta(v) + W_\delta(v_*)$. Set $b_m(\cos\theta) = \min\{b(\cos\theta), m\}$ and let $f^m(t,v)$ be the unique solution of the corresponding Cauchy problem. For the simplicity of the notations, we consider the case where F_0 and F_t^m have density functions $f_0(v)$ and $f^m(t,v)$ respectively. The general case can be considered similarly. Then we have

$$\frac{d}{dt} \int f^m(t,v) W_\delta(v) dv \lesssim \Big(\int b_m d\sigma \Big) \Big(\int f^m(t,v) W_\delta(v) dv \Big) \Big(\int f^m(t,v_*) dv_* \Big) ,$$

which yields

$$\int f^m(t,v) W_\delta(v) dv \leq C_m e^{C_m t} \int f_0(v) W_\delta(v) dv.$$

Taking the limit $\delta \to +0$, we have $f^m(t,v) \in L^1_{2n+\alpha}$.

In order to overcome the angular singularity, we need a precise formula used in the Povzner inequality, cf. [6]. To be self-contained, we derive it as follows. Since $\sigma \in \mathbb{S}^2$, it can be written as

$$\sigma = \mathbf{k} \cos\theta + \sin\theta (\mathbf{h} \cos\varphi + \mathbf{i} \sin\varphi), \;\; \theta \in [0,\pi), \varphi \in [-\pi, \pi) ,$$

by an orthogonal basis in $\mathbb{R}^3$,

$$\mathbf{k} = \frac{v - v_*}{|v - v_*|}, \;\; \mathbf{i} = \frac{v \times v_*}{|v \times v_*|}, \;\; \mathbf{h} = \mathbf{i} \times \mathbf{k} = \frac{\big((v - v_*) \cdot v\big) v_* - \big((v - v_*) \cdot v_*\big) v}{|v - v_*| |v \times v_*|}.$$

It follows from $(v+v_*)\perp \mathbf{i}$ and the definition of $\mathbf{h}$ that

$$
\begin{aligned}
|v'|^2 =& |\frac{v+v_*}{2}|^2 + |\frac{v-v_*}{2}|^2 + \frac{|v-v_*|}{2}(v+v_*)\cdot\sigma \\
=& \frac{1}{4}(2|v|^2+2|v_*|^2) + \frac{|v-v_*|}{2}\Big((v+v_*)\cdot(\cos\theta\mathbf{k}+\sin\theta\cos\varphi\mathbf{h})\Big) \\
=& \frac{1}{2}(|v|^2+|v_*|^2) + \frac{\cos\theta}{2}(|v|^2-|v_*|^2) \\
&+ \frac{\sin\theta\cos\varphi}{2|v\times v_*|}\Big\{(v+v_*)\cdot\Big(\big((v-v_*)\cdot v\big)v_* - \big((v-v_*)\cdot v_*\big)v\Big)\Big\} \\
=& \frac{|v|^2(1+\cos\theta)}{2} + \frac{|v_*|^2(1-\cos\theta)}{2} + |v||v_*|\sin\alpha\sin\theta\cos\varphi\,,
\end{aligned}
$$

where α is the angle between v and v_*. Therefore, we have

$$
\begin{aligned}
|v'|^2 &= |v|^2\cos^2\frac{\theta}{2} + |v_*|^2\sin^2\frac{\theta}{2} + |v\times v_*|\sin\theta\cos\varphi \\
&= Y(\theta) + Z(\theta)\cos\varphi\,.
\end{aligned}
\tag{1.19}
$$

Similarly, we have

$$
\begin{aligned}
|v_*'|^2 &= |v_*|^2\cos^2\frac{\theta}{2} + |v|^2\sin^2\frac{\theta}{2} - |v\times v_*|\sin\theta\cos\varphi \\
&= Y(\pi-\theta) - Z(\theta)\cos\varphi\,.
\end{aligned}
\tag{1.20}
$$

If $\Psi(x) = \Psi_{2n+\alpha}(x) = (1+x)^{n+\alpha/2}$, then it follows from the change of variables that

$$
\frac{d}{dt}\int f^m(t,v)\langle v\rangle^{2n+\alpha}dv = \frac{1}{2}\iint f^m(t,v)f^m(t,v_*)K(v,v_*)dvdv_*,
$$

where

$$
\begin{aligned}
K(v,v_*) &= \int_{\mathbb{S}^2} b_m\big\{\Psi(|v'|^2)+\Psi(|v_*'|^2)-\Psi(|v|^2)-\Psi(|v_*|^2)\big\}d\sigma \\
&= 2\int_0^\pi\int_0^\pi b_m(\cos\theta)\big\{\Psi(|v'|^2)+\Psi(|v_*'|^2)-\Psi(|v|^2)-\Psi(|v_*|^2)\big\}\sin\theta d\theta d\varphi\,.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_0^\pi \Psi(Y(\theta)+Z(\theta)\cos\varphi)\;d\varphi \\
&= (\int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^\pi)\;\Psi(Y(\theta)+Z(\theta)\cos\varphi)\;d\varphi \\
&= \int_0^{\frac{\pi}{2}}\{\Psi(Y(\theta)+Z(\theta)\cos\varphi)+\Psi(Y(\theta)-Z(\theta)\cos\varphi)-2\Psi(Y(\theta))\}\;d\varphi + \pi\Psi(Y(\theta)),
\end{aligned}
$$

and by using integration by parts twice, we have

$$\begin{aligned}
&\int_0^\pi \Psi(|v'|^2)d\varphi = \int_0^\pi \Psi(Y(\theta)+Z(\theta)\cos\varphi)\ d\varphi\\
&= \pi\Psi(Y) + [\varphi\{\Psi(Y+Z\cos\varphi)+\Psi(Y-Z\cos\varphi)-2\Psi(Y)\}]_0^{\frac{\pi}{2}}\\
&\quad - \int_0^{\frac{\pi}{2}} \varphi\{\Psi'(Y+Z\cos\varphi)-\Psi'(Y-Z\cos\varphi)\}(-Z\sin\varphi)\ d\varphi\\
&= \pi\Psi(Y) + \int_0^{\frac{\pi}{2}} Z\varphi\sin\varphi(\Psi'(Y+Z\cos\varphi)-\Psi'(Y-Z\cos\varphi))\ d\varphi\\
&= \pi\Psi(Y) + Z[(\sin\varphi-\varphi\cos\varphi)\{\Psi'(Y+Z\cos\varphi)-\Psi'(Y-Z\cos\varphi)\}]_0^{\frac{\pi}{2}}\\
&\quad + Z^2\int_0^{\frac{\pi}{2}}(\sin\varphi-\varphi\cos\varphi)\{\Psi''(Y+Z\cos\varphi)+\Psi''(Y-Z\cos\varphi)\}\sin\varphi\ d\varphi\\
&= \pi\Psi(Y(\theta)) + Z^2\int_0^{\frac{\pi}{2}}(\sin\varphi-\varphi\cos\varphi)\sin\varphi\\
&\qquad \times\{(\Psi''(Y(\theta)+Z\cos\varphi)+\Psi''(Y(\theta)-Z\cos\varphi)\}\ d\varphi.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
\int_0^\pi \Psi(|v_*'|^2)d\varphi &= \pi\Psi(Y(\pi-\theta)) + Z^2\int_0^{\frac{\pi}{2}}(\sin\varphi-\varphi\cos\varphi)\sin\varphi\\
&\quad \times\{(\Psi''(Y(\pi-\theta)+Z\cos\varphi)+\Psi''(Y(\pi-\theta)-Z\cos\varphi)\}\ d\varphi.
\end{aligned}$$

In view of these formula, we consider $K(v,v_*)$ by dividing it into two parts as follows:

$$K(v,v_*) = -H(v,v_*) + G(v,v_*).$$

For the first part, we have

$$\begin{aligned}
&-H(v,v_*) = 2\pi\int_0^\pi b_m(\cos\theta)\\
&\qquad\times\Big\{\Psi(Y(\theta))+\Psi(Y(\pi-\theta)) - (\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2})\big(\Psi(|v|^2)+\Psi(|v_*|^2)\big)\Big\}d\theta\\
&= 2\pi\int_0^\pi b_m(\cos\theta)\Big[\{\Psi(|v|^2\cos^2\frac{\theta}{2}+|v_*|^2\sin^2\frac{\theta}{2}) - \cos^2\frac{\theta}{2}\Psi(|v|^2)-\sin^2\frac{\theta}{2}\Psi(|v_*|^2)\}\\
&\qquad + \{\Psi(|v_*|^2\cos^2\frac{\theta}{2}+|v|^2\sin^2\frac{\theta}{2}) - \cos^2\frac{\theta}{2}\Psi(|v_*|^2)-\sin^2\frac{\theta}{2}\Psi(|v|^2)\}\Big]d\theta \le 0,
\end{aligned}$$

where we have used the fact that Ψ is concave. On the other hand, if $Z_0 = Z(\theta)/(1+Y(\theta)) \in [0,1]$, then

$$\begin{aligned}
&Z^2\int_0^{\frac{\pi}{2}}(\sin\varphi-\varphi\cos\varphi)\sin\varphi\{\Psi''(Y+Z\cos\varphi)+\Psi''(Y-Z\cos\varphi)\}\ d\varphi\\
&\lesssim Z^2(1+Y)^{n-2+\alpha/2}\int_0^{\pi/2}\varphi^3\{(1+Z_0\cos\varphi)^{n-2+\alpha/2}+(1-Z_0\cos\varphi)^{n-2+\alpha/2}\}d\varphi\\
&\begin{cases} \lesssim Z^2 \lesssim |v|^2|v_*|^2\theta^2, & \text{if } n=1;\\ \lesssim Z^2(1+Y)^{n-2+\alpha/2} \lesssim (1+|v|^2+|v_*|^2)^{n+\alpha/2}\theta^2, & \text{if } n\ge 2\,. \end{cases}
\end{aligned}$$

Therefore, if $n=1$ then $G(v,v_*)\lesssim |v|^2|v_*|^2$, which implies

$$\frac{d}{dt}\int f^m(t,v)\langle v\rangle^{2n+\alpha}dv \le C_0\Big(\int |v|^2 f^m(t,v)dv\Big)^2 = C_0\Big(\int |v|^2 f_0(v)dv\Big)^2,$$

where the constant C_0 is independent of m. When $n\ge 2$, there exists another constant $C_1>0$ independent of m such that

$$G(v,v_*)\le C_1\big(\langle v\rangle^{2n+\alpha}+\langle v_*\rangle^{2n+\alpha}\big)\,,$$

which implies

$$\frac{d}{dt}\int f^m(t,v)\langle v\rangle^{2n+\alpha}dv \le 4C_1\int f^m(t,v)\langle v\rangle^{2n+\alpha}dv\,.$$

Finally, take a cutoff function $\chi(v)\in C_0^\infty(\mathbb{R}^3)$ satisfying $0\le\chi(v)\le 1$ and $\chi=1$ on $\{|v|\le 1\}$. Then, for any $R>0$, we have

$$\int f^m(t,v)\langle v\rangle^{2n+\alpha}\chi\left(\frac{v}{R}\right)dv \le Ce^{Ct}\int f_0(v)\langle v\rangle^{2n+\alpha}dv.$$

Since $f^m(t,v)\to f(t,v)$ in $\mathcal{S}'(\mathbb{R}^3_v)$ and $\langle v\rangle^{2n+\alpha}\chi\left(\frac{v}{R}\right)\in\mathcal{S}$, we get

$$\int f(t,v)\langle v\rangle^{2n+\alpha}\chi\left(\frac{v}{R}\right)dv \le Ce^{Ct}\int f_0(v)\langle v\rangle^{2n+\alpha}dv.$$

Letting $R\to\infty$ shows for any $T>0$, $F_t\in L^\infty([0,T],P_{2n+\alpha})$. And this completes the proof of Corollary 1.7. $\square$

The proof of the Theorem 1.5 will be given in the Fourier space. In fact, by letting $\varphi(t,\xi)=\mathcal{F}(F_t)$ and $\varphi_0=\mathcal{F}(F_0)$, it follows from the Bobylev formula that the Cauchy problem (1.1)-(1.2) is reduced to

$$\begin{cases} \partial_t\varphi(t,\xi) = \displaystyle\int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\Big(\varphi(t,\xi^+)\varphi(t,\xi^-)-\varphi(t,\xi)\varphi(t,0)\Big)d\sigma, \\ \\ \varphi(0,\xi)=\varphi_0(\xi), \ \text{ where } \xi^\pm = \dfrac{\xi}{2}\pm\dfrac{|\xi|}{2}\sigma\,. \end{cases} \tag{1.21}$$

By Theorem 1.1, to prove Theorem 1.5 it suffices to show

Theorem 1.8. *Assume that b satisfies* (1.4) *for some $\alpha_0\in(0,2)$. Let $2>\alpha>\beta>\max\{\alpha_0,\alpha/2\}$ and $\epsilon\in(0,1-\frac{\alpha_0}{\beta}]$. If the initial datum φ_0 belongs to $\widetilde{\mathcal{M}}^\alpha$, then there exists a unique classical solution $\varphi(t,\xi)\in C([0,\infty),\widetilde{\mathcal{M}}^\alpha)$ to the Cauchy problem* (1.21) *satisfying that, for all $t,s\in[0,T]$,*

$$||\varphi(t)-\varphi(s)||_\beta \lesssim e^{\lambda_\beta\max\{s,t\}}|t-s|, \tag{1.22}$$

$$||\varphi(t)-\varphi(s)||_{\widetilde{\mathcal{M}}^\alpha} \lesssim C(t,s)|t-s|, \tag{1.23}$$

where $C(t,s)=e^{\lambda_\alpha\max\{s,t\}}||\varphi_0-1||_{\widetilde{\mathcal{M}}^\alpha}+(e^{\lambda_\beta\max\{s,t\}}||1-\varphi_0||_\beta+1)^2$, and

$$\lambda_i = 2\pi\int_0^{\pi/2} b(\cos\theta)\left(\cos^i\frac{\theta}{2}+\sin^i\frac{\theta}{2}-1\right)\sin\theta d\theta>0\,, i=\alpha,\beta. \tag{1.24}$$

Furthermore, if $\varphi(t,\xi),\tilde\varphi(t,\xi)\in C([0,\infty),\widetilde{\mathcal{M}}^\alpha)$ are two solutions to the Cauchy problem (1.21) *with initial data $\varphi_0,\tilde\varphi_0\in\widetilde{\mathcal{M}}^\alpha$, respectively, then for any $t>0$, the following two stability estimates hold*

$$||\varphi(t)-\tilde\varphi(t)||_\alpha \le e^{\lambda_\alpha t}||\varphi_0-\tilde\varphi_0||_\alpha\,, \tag{1.25}$$

$$||\varphi(t)-\tilde{\varphi}(t)||_{\widetilde{\mathcal{M}}^\alpha} \lesssim e^{\lambda_\alpha t}||\varphi_0-\tilde{\varphi}_0||_{\widetilde{\mathcal{M}}^\alpha} + \frac{e^{2\lambda_\beta t}-e^{\lambda_\alpha t}}{2\lambda_\beta-\lambda_\alpha}A + \frac{e^{\lambda_\beta t}-e^{\lambda_\alpha t}}{\lambda_\beta-\lambda_\alpha}B\,, \tag{1.26}$$

where

$$A = \max\{||1-\varphi_0||_\beta, ||1-\tilde{\varphi}_0||_\beta\}\cdot||\varphi_0-\tilde{\varphi}_0||_\beta\,,$$
$$B = ||\varphi_0-\tilde{\varphi}_0||_\beta + ||1-\tilde{\varphi}_0||_\beta^{1-\varepsilon}||\varphi_0-\tilde{\varphi}_0||_\beta^{\varepsilon}.$$

Remark 1.9. *Since $\varphi_0, \tilde{\varphi}_0 \in \widetilde{\mathcal{M}}^\alpha \subset \mathcal{K}^\alpha$, the stability estimate* (1.25) *is nothing but (13) of* [7].

Finally, we give the following corollary about the regularity of the solutions.

Corollary 1.10. *Let $b(\cos\theta)$ satisfy* (1.4) *and let $\alpha \in (\alpha_0, 2]$. If $F_0 \in P_\alpha(\mathbb{R}^3)$ is not a single Dirac mass and $f(t,v)$ is the unique solution in $C([0,\infty), P_\alpha(\mathbb{R}^3))$ to the Cauchy problem* (1.1)-(1.2)*, then $f(t,\cdot)$ belongs to $L^1_\alpha(\mathbb{R}^3)\cap H^\infty(\mathbb{R}^3)$ for any $t>0$.*

Remark 1.11. *The case except $\alpha \neq 1$ in the above corollary was already proved in Theorem 1.8 in* [10]. *The newly defined space $\widetilde{\mathcal{M}}_\alpha$ in this paper fills the gap in the case when $\alpha = 1$.*

This ends the introduction and the proofs of Theorems 1.5 and 1.8 will be given in the next section.

2. Proof of Theorem 1.8

This section concerns with the existence of measure valued solutions in the new classification of the characteristic functions. We only need to prove Theorem 1.8 because Theorem 1.5 will then follow by using Theorem 1.1.

Let $b(\cdot)$ satisfy (1.4) . As usual, the existence for non-cutoff cross section is based on the cutoff approximations $b_n(\cdot) = \min\{b(\cdot), n\}$. Following the previous works [2, 7, 9, 10], define the following constants for $\alpha \in [\alpha_0, 2)$:

$$\gamma_\alpha^n = \int_{\mathbb{S}^2} b_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\sin^\alpha\frac{\theta}{2}+\cos^\alpha\frac{\theta}{2}\right)d\sigma > 0,$$
$$\lambda_\alpha^n = \int_{\mathbb{S}^2} b_n\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\sin^\alpha\frac{\theta}{2}+\cos^\alpha\frac{\theta}{2}-1\right)d\sigma = \gamma_\alpha^n - \gamma_2^n > 0,$$
$$\lambda_\alpha = \int_{\mathbb{S}^2} b\left(\frac{\xi\cdot\sigma}{|\xi|}\right)\left(\sin^\alpha\frac{\theta}{2}+\cos^\alpha\frac{\theta}{2}-1\right)d\sigma > 0. \tag{2.27}$$

Note that λ_α is finite and independent of ξ; furthermore, $\{\lambda_\alpha^n\}_{n=1}^\infty$ converges monotonically to λ_α.

Following Subsection 4.2 of [2], consider the nonlinear operator,

$$\mathcal{G}_n(\varphi)(\xi) \equiv \int_{\mathbb{S}^2} b_n\Big(\frac{\xi\cdot\sigma}{|\xi|}\Big)\varphi(\xi^+)\varphi(\xi^-)d\sigma.$$

Then, problem (1.21) can be formulated by

$$\varphi(\xi,t) = \varphi_0(\xi)e^{-\gamma_2^n t} + \int_0^t e^{-\gamma_2^n(t-\tau)}\mathcal{G}_n(\varphi(\cdot,\tau))(\xi)d\tau. \tag{2.28}$$

For the nonlinear operator $\mathcal{G}_n(\cdot)$, we have the following estimate.

Lemma 2.1. *Assume $b(\cdot)$ satisfies (1.4). Let $b_n = \min\{b, n\}$. Let $\max\{\alpha_0, \alpha/2\} < \beta < \alpha < 2$ and $\epsilon \in (0, 1 - \frac{\alpha_0}{\beta}]$. Then there exists $C > 0$ independent of n such that, for all $\varphi, \tilde{\varphi} \in \widetilde{\mathcal{M}}^\alpha$, we have*

$$||\mathcal{G}_n(\varphi) - \mathcal{G}_n(\tilde{\varphi})||_\beta \leq \gamma_\beta^n ||\varphi - \tilde{\varphi}||_\beta, \tag{2.29}$$

$$\begin{aligned} ||\mathcal{G}_n(\varphi) - \mathcal{G}_n(\tilde{\varphi})||_{\widetilde{\mathcal{M}}_\alpha} \leq& \gamma_\alpha^n ||\varphi - \tilde{\varphi}||_{\widetilde{\mathcal{M}}_\alpha} + C \max\{||1-\varphi||_\beta, ||1-\tilde{\varphi}||_\beta\} \cdot ||\varphi - \tilde{\varphi}||_\beta \\ &+ C||\varphi - \tilde{\varphi}||_\beta + C||1-\tilde{\varphi}||_\beta^{1-\varepsilon} ||\varphi - \tilde{\varphi}||_\beta^\varepsilon, \end{aligned} \tag{2.30}$$

for any $n \geq 1 (n \in \mathbb{N})$. In particular, if $\tilde{\varphi} \equiv 1$,

$$\int_{\mathbb{R}^3} \frac{|Re\mathcal{G}_n(\varphi) - \gamma_2^n|}{|\xi|^{3+\alpha}} d\xi \leq \gamma_\alpha^n ||\varphi - 1||_{\widetilde{\mathcal{M}}_\alpha} + C||\varphi - 1||_\beta^2 + C||\varphi - 1||_\beta.$$

Proof. The proof of (2.29) can be found in [2]. We only need to prove other estimates in the lemma. Firstly, note that

$$\begin{aligned} ||\mathcal{G}_n(\varphi) - \mathcal{G}_n(\tilde{\varphi})||_{\widetilde{\mathcal{M}}_\alpha} &= \int_{\mathbb{R}^3} \frac{|Re\mathcal{G}_n(\varphi) - Re\mathcal{G}_n(\tilde{\varphi})|}{|\xi|^{3+\alpha}} d\xi \\ &\leq \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b_n(\cdot) \Bigg(\frac{|Re\varphi^+ Re\varphi^- - Re\tilde{\varphi}^+ R\tilde{\varphi}^-|}{|\xi|^{3+\alpha}} \\ &\qquad + \frac{|Im\varphi^+ Im\varphi^- - Im\tilde{\varphi}^+ Im\tilde{\varphi}^-|}{|\xi|^{3+\alpha}} \Bigg) d\sigma d\xi \\ &= I_1 + I_2. \end{aligned}$$

For I_1, we can apply the Lemma 3.2 in [10]. That is,

$$I_1 = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b_n(\cdot) \frac{|Re\varphi^+ Re\varphi^- - Re\tilde{\varphi}^+ Re\tilde{\varphi}^-|}{|\xi|^{3+\alpha}} d\sigma d\xi \leq \gamma_\alpha^n \int_{\mathbb{R}^3} \frac{|Re\varphi - Re\tilde{\varphi}|}{|\xi|^{3+\alpha}} d\xi.$$

For I_2, we divide it into two parts: $I_{2,1}$ for $|\xi| < 1$ and $I_{2,2}$ for $|\xi| > 1$. Then,

$$\begin{aligned} I_{2,1} &= \int_{|\xi|<1} \int_{\mathbb{S}^2} b_n(\cdot) \frac{|Im\varphi^+ \cdot Im(\varphi^- - \tilde{\varphi}^-) + Im\tilde{\varphi}^- \cdot Im(\varphi^+ - \tilde{\varphi}^+)|}{|\xi|^{3+\alpha}} d\sigma d\xi \\ &\leq 2 \int_{|\xi|<1} \frac{d\xi}{|\xi|^{3+\alpha-2\beta}} \int_{\mathbb{S}^2} b(\cdot) \sin^\beta \frac{\theta}{2} \cos^\beta \frac{\theta}{2} d\sigma \cdot ||\varphi - \tilde{\varphi}||_\beta \\ &\qquad \cdot \max\{||1-\varphi||_\beta, ||1-\tilde{\varphi}||_\beta\} \\ &= C||\varphi - \tilde{\varphi}||_\beta \cdot \max\{||1-\varphi||_\beta, ||1-\tilde{\varphi}||_\beta\}. \end{aligned}$$

By using

$$\begin{aligned} \int_{|\xi|>1} \frac{|Im\varphi^+||Im(\varphi^- - \tilde{\varphi}^-)|}{|\xi|^{3+\alpha}} d\xi &\leq \int_{|\xi|>1} \frac{\sin^\beta \frac{\theta}{2}}{|\xi|^{3+\alpha-\beta}} \frac{|Im(\varphi^- - \tilde{\varphi}^-)|}{|\xi^-|^\beta} d\xi \\ &\leq C||\varphi - \tilde{\varphi}||_\beta \cdot \sin^\beta \frac{\theta}{2}, \end{aligned}$$

and

$$
\begin{aligned}
&\int_{|\xi|>1} \frac{|Im\tilde{\varphi}^-||Im(\varphi^+ - \tilde{\varphi}^+)|}{|\xi|^{3+\alpha}} d\xi \\
&\leq \left[\int_{|\xi|>1} \frac{|Im\tilde{\varphi}^-||Im(\varphi^+ - \tilde{\varphi}^+)|}{|\xi|^{3+\alpha}} d\xi\right]^{1-\varepsilon} \left[\int_{|\xi|>1} \frac{|Im\tilde{\varphi}^-||Im(\varphi^+ - \tilde{\varphi}^+)|}{|\xi|^{3+\alpha}} d\xi\right]^{\varepsilon} \\
&\leq \left[\int_{|\xi|>1} \frac{2\sin^\beta \frac{\theta}{2}}{|\xi|^{3+\alpha-\beta}} \cdot \frac{|Im\tilde{\varphi}^-|}{|\xi^-|^\beta} d\xi\right]^{1-\varepsilon} \left[\int_{|\xi|>1} \frac{\cos^\beta \frac{\theta}{2}}{|\xi|^{3+\alpha-\beta}} \cdot \frac{|Im(\varphi^+ - \tilde{\varphi}^+)|}{|\xi^+|^\beta} d\xi\right]^{\varepsilon} \\
&\leq 2\sin^{\beta(1-\varepsilon)} \frac{\theta}{2} \cos^{\beta\varepsilon} \frac{\theta}{2} \int_{|\xi|>1} \frac{1}{|\xi|^{3+\alpha-\beta}} d\xi ||1-\tilde{\varphi}||_\beta^{1-\varepsilon} ||\varphi - \tilde{\varphi}||_\beta^\varepsilon,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
I_{2,2} \leq & C \int b(\cdot) \sin^\beta \frac{\theta}{2} d\sigma ||\varphi - \tilde{\varphi}||_\beta \\
& + 2C \int b(\cdot) \sin^{\beta(1-\varepsilon)} \frac{\theta}{2} \cos^{\beta\varepsilon} \frac{\theta}{2} d\sigma ||1-\tilde{\varphi}||_\beta^{1-\varepsilon} ||\varphi - \tilde{\varphi}||_\beta^\varepsilon.
\end{aligned} \tag{2.31}
$$

Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}^3} \frac{|Re\mathcal{G}_n(\varphi) - Re\mathcal{G}_n(\tilde{\varphi})|}{|\xi|^{3+\alpha}} d\xi \leq & \gamma_\alpha^n \int_{\mathbb{R}^3} \frac{|Re\varphi - Re\tilde{\varphi}|}{|\xi|^{3+\alpha}} d\xi \\
& + C \max\{||1-\varphi||_\beta, ||1-\tilde{\varphi}||_\beta\} \cdot ||\varphi - \tilde{\varphi}||_\beta \\
& + C||\varphi - \tilde{\varphi}||_\beta + C||1-\tilde{\varphi}||_\beta^{1-\varepsilon} ||\varphi - \tilde{\varphi}||_\beta^\varepsilon.
\end{aligned}
$$

And this completes the proof of the lemma. □

2.1. **Existence under the cutoff assumption.** We are now ready to prove the existence of the solution under the cutoff assumption. The solution to (1.21) with b replaced by b_n can be obtained as a fixed point of (2.28) to the nonlinear operator

$$
\mathcal{F}_n(\varphi)(t,\xi) \equiv \varphi_0(\xi) e^{-\gamma_2^n t} + \int_0^t e^{-\gamma_2^n (t-\tau)} \mathcal{G}_n(\varphi(\tau))(\xi) d\tau,
$$

for a fixed $\varphi_0 \in \widetilde{\mathcal{M}}^\alpha$. For a fixed $T > 0$ to be determined later, denote

$$
\begin{aligned}
X_n = \{ & \varphi(t,\xi) \in C([0,T], \widetilde{\mathcal{M}}^\alpha) : \varphi(0,\xi) = \varphi_0(\xi), \\
& \forall t \in [0,T], ||1-\varphi(t,\cdot)||_\beta \leq e^{\lambda_\beta^n t} ||1-\varphi(0,\cdot)||_\beta \}
\end{aligned}
$$

supplemented with the metric

$$
||\varphi - \tilde{\varphi}||_{X_n} = \sup_{t\in[0,T]} dis_{\alpha,\beta,\epsilon}(\varphi(t), \tilde{\varphi}(t)),
$$

for $\varphi, \tilde{\varphi} \in X_n$.

Firstly, the following lemma gives a local in time existence of solution in X_n.

Lemma 2.2. *There exists $T > 0$, such that $\mathcal{F}_n : X_n \to X_n$ is a contraction mapping.*

Proof. Firstly, we prove that $\mathcal{F}_n$ maps X_n into itself. By Lemma 2.1, for any $\varphi \in X_n$, we have

$$\begin{aligned}
&\sup_{t\in[0,T]} ||\mathcal{F}_n(\varphi(t,\cdot)) - 1||_{\widetilde{\mathcal{M}}^\alpha} \\
&\le ||\varphi_0 - 1||_{\widetilde{\mathcal{M}}^\alpha} + \sup_{t\in[0,T]} \int_0^t ||\mathcal{G}_n(\varphi(\tau,\cdot)) - \gamma_2^n||_{\widetilde{\mathcal{M}}_\alpha} d\tau \\
&\le ||\varphi_0 - 1||_{\widetilde{\mathcal{M}}^\alpha} + \int_0^T \gamma_\alpha^n ||\varphi(\tau) - 1||_{\widetilde{\mathcal{M}}^\alpha} + C||\varphi(\tau) - 1||_\beta^2 + C||\varphi(\tau) - 1||_\beta d\tau \\
&\le ||\varphi_0 - 1||_{\widetilde{\mathcal{M}}^\alpha} + \gamma_\alpha^n T \sup_{t\in[0,T]} ||\varphi(t) - 1||_{\widetilde{\mathcal{M}}^\alpha} + CT(||\varphi_0 - 1||_\beta e^{\lambda_\beta^n T} + 1)^2 \\
&< \infty.
\end{aligned}$$

To show $\mathcal{F}_n(\varphi) \in X_n$, we also need to check that

$$\begin{aligned}
||1 - \mathcal{F}_n(\varphi)||_\beta &\le ||1 - \varphi_0||_\beta e^{-\gamma_2^n t} + \int_0^t e^{-\gamma_2^n (t-\tau)} ||\mathcal{G}(\varphi(\cdot,\tau)) - \mathcal{G}(1)||_\beta d\tau \\
&\le ||1 - \varphi_0||_\beta e^{-\gamma_2^n t} + \gamma_\beta^n \int_0^t e^{-\gamma_2^n (t-\tau)} ||\varphi(\cdot,\tau) - 1||_\beta d\tau \\
&\le ||1 - \varphi_0||_\beta e^{-\gamma_2^n t} + \gamma_\beta^n \int_0^t e^{-\gamma_2^n (t-\tau)} e^{\lambda_\beta^n \tau} ||\varphi_0 - 1||_\beta d\tau \\
&= e^{\lambda_\beta^n t} ||1 - \varphi_0||_\beta.
\end{aligned}$$

Moreover, assume that $\varphi(t,\xi) \in C([0,T]; \widetilde{\mathcal{M}}^\alpha)$. Since it follows from the definition of $\mathcal{F}_n$ that

$$\begin{aligned}
\mathcal{F}_n(\varphi)(t,\xi) - \mathcal{F}_n(\varphi)(s,\xi) &= \Big(\varphi_0(\xi) - 1\Big)\Big(e^{-\gamma_2^n t} - e^{-\gamma_2^n s}\Big) \\
&\quad + \int_s^t e^{-\gamma_2^n (t-\tau)} \Big(\mathcal{G}_n(\varphi(\tau))(\xi) - \gamma_2^n\Big) d\tau \\
&= I(t,s,\xi) + II(t,s,\xi),
\end{aligned}$$

we have $\mathcal{F}_n(\varphi)(t,\xi) \in C([0,T]; \widetilde{\mathcal{M}}^\alpha)$. In fact, as for $\mathcal{K}^\alpha$-norm, we have

$$\Big(\sup_\xi \frac{|I|}{|\xi|^\alpha}\Big)(t,s) \le \Big|e^{-\gamma_2^n t} - e^{-\gamma_2^n s}\Big| \sup_\xi \frac{|\varphi_0(\xi) - 1|}{|\xi|^\alpha} \to 0 \text{ as } |t-s| \to 0,$$

and

$$\Big(\sup_\xi \frac{|II|}{|\xi|^\alpha}\Big)(t,s) \le \int_s^t \frac{\big|\mathcal{G}_n(\varphi(\tau))(\xi) - \gamma_2^n\big|}{|\xi|^\alpha} d\tau \le |t-s| \gamma_\alpha^n \sup_{\tau\in[0,T]} \|1 - \varphi(\tau)\|_\alpha .$$

The proof for $\widetilde{\mathcal{M}}_\alpha$-norm is similar so that we omit the detail.

Secondly, to prove $\mathcal{F}_n$ is a contraction mapping, we apply Lemma 2.1 again to have that for any $\varphi, \tilde{\varphi} \in X_n$,

$$\begin{aligned}
&\sup_{t\in[0,T]} ||\mathcal{F}_n(\varphi(t,\cdot)) - \mathcal{F}_n(\tilde{\varphi}(t,\cdot))||_{\widetilde{\mathcal{M}}^\alpha} \\
&\le \sup_{t\in[0,T]} \int_0^t ||\mathcal{G}_n(\varphi(\tau,\cdot)) - \mathcal{G}_n(\varphi(\tau,\cdot))||_{\widetilde{\mathcal{M}}_\alpha} d\tau \\
&\le \int_0^T \gamma_\alpha^n ||\varphi(\tau) - \tilde{\varphi}(\tau)||_{\widetilde{\mathcal{M}}_\alpha} + C\max\{||1-\varphi(\tau)||_\beta, ||1-\tilde{\varphi}(\tau)||_\beta\} \cdot ||\varphi(\tau) - \tilde{\varphi}(\tau)||_\beta \\
&\quad + C||\varphi(\tau) - \tilde{\varphi}(\tau)||_\beta + C||1-\tilde{\varphi}(\tau)||_\beta^{1-\varepsilon} ||\varphi(\tau) - \tilde{\varphi}(\tau)||_\beta^\varepsilon d\tau \\
&\le \gamma_\alpha^n T \sup_{t\in[0,T]} ||\varphi - \tilde{\varphi}||_{\widetilde{\mathcal{M}}_\alpha} + C\int_0^T (e^{\lambda_\beta^n \tau}||1-\varphi_0||_\beta + 1) d\tau \cdot \sup_{t\in[0,T]} ||\varphi - \tilde{\varphi}||_\beta \\
&\quad + CT \sup_{t\in[0,T]} ||\varphi - \tilde{\varphi}||_\beta + C||1-\varphi_0||_\beta^{1-\varepsilon} \int_0^T e^{\lambda_\beta^n (1-\epsilon)\tau} d\tau \sup_{t\in[0,T]} ||\varphi - \tilde{\varphi}||_\beta^\varepsilon \\
&\le \gamma_\alpha^n T \sup_{t\in[0,T]} ||\varphi - \tilde{\varphi}||_{\widetilde{\mathcal{M}}_\alpha} + CT(e^{\lambda_\beta^n T}||1-\varphi_0||_\beta + 1) \cdot \sup_{t\in[0,T]} ||\varphi - \tilde{\varphi}||_\beta \\
&\quad + CT \sup_{t\in[0,T]} ||\varphi - \tilde{\varphi}||_\beta + CT||1-\varphi_0||_\beta^{1-\varepsilon} e^{\lambda_\beta^n (1-\epsilon)T} \sup_{t\in[0,T]} ||\varphi - \tilde{\varphi}||_\beta^\varepsilon \\
&\le (\gamma_\alpha^n + C)(3 + ||1-\varphi_0||_\beta + ||1-\varphi_0||_\beta^{1-\varepsilon}) e^{\lambda_\beta^n T} T \sup_{t\in[0,T]} dis_{\alpha,\beta,\epsilon}(\varphi, \tilde{\varphi}).
\end{aligned}$$

This together with the known estimate

$$\sup_{t\in[0,T]} ||\mathcal{F}_n(\varphi(t,\cdot)) - \mathcal{F}_n(\tilde{\varphi}(t,\cdot))||_\beta \le \gamma_\beta^n T \sup_{t\in[0,T]} ||\varphi(t,\cdot) - \tilde{\varphi}(t,\cdot)||_\beta,$$

we have

$$\begin{aligned}
&\sup_{t\in[0,T]} dis_{\alpha,\beta,\epsilon}(\mathcal{F}_n(\varphi) - \mathcal{F}_n(\tilde{\varphi})) \\
&\qquad \le [C_n(0)e^{\lambda_\beta^n T} T + \gamma_\beta^n T + (\gamma_\beta^n T)^\epsilon] \sup_{t\in[0,T]} dis_{\alpha,\beta,\epsilon}(\varphi, \tilde{\varphi}),
\end{aligned} \tag{2.32}$$

where $C_n(0) = (\gamma_\alpha^n + C)(3 + ||1-\varphi_0||_\beta + ||1-\varphi_0||_\beta^{1-\varepsilon})$. Therefore, $\mathcal{F}_n : X_n \to X_n$ is a contraction mapping if we select $T > 0$ small enough such that

$$C_n(0)e^{\lambda_\beta^n T} T + \gamma_\beta^n T + (\gamma_\beta^n T)^\epsilon < 1. \tag{2.33}$$

And then it completes the proof of the lemma. □

Lemma 2.2 shows that there exists a unique solution $\varphi(t,\xi) \in C([0,T_1], \widetilde{\mathcal{M}}_\alpha)$ to the problem (1.21) under the cut-off assumption, for some $T_1 > 0$ depending on the initial datum φ_0. To extend the solution to be global in time, we can apply the above argument for the initial data $\varphi(T_1,\xi)$, then one sufficient condition for $\mathcal{F}_n$ to be a contraction mapping in the next time interval $t \in [T_1, T_2]$ is

$$C_n(0)e^{\lambda_\beta^n (T_1+T_2)} T_2 + \gamma_\beta^n T_2 + (\gamma_\beta^n T_2)^\epsilon < 1,$$

where we have used the fact that $C_n(T_1) \le C_n(0)e^{\lambda_\beta^n T_1}$. This process can be continued. Assume the length of m-th extension of time interval is T_m, we can select the sequence $\{T_m\}_{m=1}^\infty$ as follows:

$$C_n(0)e^{\lambda_\beta^n (T_1+T_2+\cdots+T_m)} T_m + \gamma_\beta^n T_m + (\gamma_\beta^n T_m)^\epsilon = \frac{1}{2}, \quad \forall m \in \mathbb{N}.$$

It is straightforward to check that $\sum_{m=1}^{\infty} T_m = \infty$.

Remark 2.3. *Although we only obtain the solution in $X_n \subsetneq C([0,T],\widetilde{\mathcal{M}}^\alpha)$, the fixed point is unique in $C([0,T],\widetilde{\mathcal{M}}^\alpha)$. Assume $\psi \in C([0,T],\widetilde{\mathcal{M}}^\alpha)$ is a fixed point to the operator $\mathcal{F}_n$. Then*

$$\psi(t,\xi) = \varphi_0(\xi)e^{-\gamma_2^n t} + \int_0^t e^{-\gamma_2^n(t-\tau)}\mathcal{G}_n(\psi(\tau))(\xi)d\tau.$$

Let $t = 0$, we have $\psi(0,\xi) = \varphi_0$. Moreover,

$$\begin{aligned} e^{\gamma_2^n t}||\psi(t,\cdot) - 1||_\beta &\le ||\psi_0 - 1||_\beta + \int_0^t e^{\gamma_2^n \tau}||\mathcal{G}_n(\psi(\tau)) - \mathcal{G}_n(1)||_\beta d\tau \\ &\le ||\psi_0 - 1||_\beta + \gamma_\beta^n \int_0^t e^{\gamma_2^n \tau}||\psi(\tau) - 1||_\beta d\tau. \end{aligned}$$

Applying the Gronwall inequality yields

$$||\psi(t,\cdot) - 1||_\beta \le e^{\lambda_\beta^n t}||\psi_0 - 1||_\beta,$$

that is , $\psi \in X_n$.

2.2. **Stability under cutoff assumption.** Let $\varphi(t,\xi), \tilde{\varphi}(t,\xi) \in C([0,T],\widetilde{\mathcal{M}}_\alpha)$ be two solutions to the equation with cut-off cross section and with initial data $\varphi_0, \tilde{\varphi}_0$, respectively.

Let $H(t,\xi) = \varphi(t,\xi) - \tilde{\varphi}(t,\xi)$, then it is known from the previous works that

$$||\varphi(t,\xi) - \tilde{\varphi}(t,\xi)||_\beta \le e^{\lambda_\beta^n t}||\varphi_0 - \tilde{\varphi}_0||_\beta, \quad \text{for all } t > 0. \tag{2.34}$$

Starting from the equation, we can obtain

$$\begin{aligned} e^{\gamma_2^n t}\int \frac{|ReH(t,\xi)|}{|\xi|^{3+\alpha}}d\xi \le& \int \frac{|ReH(0,\xi)|}{|\xi|^{3+\alpha}}d\xi \\ &+ \int_0^t e^{\gamma_2^n \tau}\int \frac{|Re\mathcal{G}(\varphi(\tau)) - Re\mathcal{G}(\tilde{\varphi}(\tau))|}{|\xi|^{3+\alpha}}d\xi d\tau. \end{aligned} \tag{2.35}$$

By Lemma 2.1 and (2.34), we have

$$\begin{aligned} &\int \frac{|Re\mathcal{G}(\varphi(\tau)) - Re\mathcal{G}(\tilde{\varphi}(\tau))|}{|\xi|^{3+\alpha}}d\xi \\ &\quad \le \gamma_\alpha^n \int \frac{|ReH(\tau)|}{|\xi|^{3+\alpha}}d\xi + Ce^{2\lambda_\beta^n \tau}\max\{||1-\varphi_0||_\beta, ||1-\tilde{\varphi}_0||_\beta\}\cdot||\varphi_0 - \tilde{\varphi}_0||_\beta \\ &\quad\quad + Ce^{\lambda_\beta^n \tau}||\varphi_0 - \tilde{\varphi}_0||_\beta + Ce^{\lambda_\beta^n \tau}||1-\tilde{\varphi}_0||_\beta^{1-\varepsilon}||\varphi_0 - \tilde{\varphi}_0||_\beta^\varepsilon. \end{aligned}$$

To simplify the calculation, define the functions

$$f(t) = e^{\gamma_2^n t}\int \frac{|ReH(t,\xi)|}{|\xi|^{3+\alpha}}d\xi,$$

$$g(t) = A\int_0^t e^{(\gamma_\beta^n + \lambda_\beta^n)\tau}d\tau + B\int_0^t e^{\gamma_\beta^n \tau}d\tau,$$

where

$$\begin{aligned} A &= C\max\{||1-\varphi_0||_\beta, ||1-\tilde{\varphi}_0||_\beta\}\cdot||\varphi_0 - \tilde{\varphi}_0||_\beta, \\ B &= C(||\varphi_0 - \tilde{\varphi}_0||_\beta + ||1-\tilde{\varphi}_0||_\beta^{1-\varepsilon}||\varphi_0 - \tilde{\varphi}_0||_\beta^\varepsilon). \end{aligned}$$

Then, (2.35) becomes

$$f(t) \le f(0) + \gamma_\alpha^n \int_0^t f(\tau)d\tau + g(t).$$

Applying the Gronwall inequality gives

(2.36)

$$\int \frac{|ReH(t,\xi)|}{|\xi|^{3+\alpha}}d\xi \le e^{\lambda_\alpha^n t}\int \frac{|ReH(0,\xi)|}{|\xi|^{3+\alpha}}d\xi + \frac{e^{2\lambda_\beta^n t}-e^{\lambda_\alpha^n t}}{2\lambda_\beta^n-\lambda_\alpha^n}A + \frac{e^{\lambda_\beta^n t}-e^{\lambda_\alpha^n t}}{\lambda_\beta^n-\lambda_\alpha^n}B.$$

In particular, if $\tilde\varphi = 1$, we have

$$||\varphi(t,\xi)-1||_{\widetilde{\mathcal{M}}^\alpha} \le e^{\lambda_\alpha^n t}||\varphi_0-1||_{\widetilde{\mathcal{M}}^\alpha} + \frac{e^{2\lambda_\beta^n t}-e^{\lambda_\alpha^n t}}{2\lambda_\beta^n-\lambda_\alpha^n}A_0 + \frac{e^{\lambda_\beta^n t}-e^{\lambda_\alpha^n t}}{\lambda_\beta^n-\lambda_\alpha^n}B_0,$$

where $A_0 = C||\varphi_0-1||_\beta^2, B_0 = C||\varphi_0-1||_\beta$.

2.3. **Existence and stability without cut-off assumption.**

Assume $b(\cdot)$ satisfies (1.4). Let $b_n = \min\{b,n\}$. For any $\varphi_0 \in \widetilde{\mathcal{M}}^\alpha \subset \mathcal{K}^\beta, \alpha \in (\alpha_0, 2)$, we have a unique solution $\varphi_n(t,\xi) \in C([0,\infty);\widetilde{\mathcal{M}}^\alpha)$ to the Cauchy problem (1.21) with b replaced by b_n. From the $\mathcal{K}^\beta$-theory ([2, 7]), we know

$$||\varphi_n(t,\cdot)-1||_\beta \le e^{\lambda_\beta^n t}||\varphi_0-1||_\beta \le e^{\lambda_\beta t}||\varphi_0-1||_\beta, \text{ for all } \beta\in(\alpha_0,\alpha],$$

and it is proved that the sequence $\{\varphi_n(t,\xi)\}_{n=1}^\infty$ is bounded and equicontinuous. Therefore, by Ascoli-Arzelà theorem, there exists a subsequence of solutions, denoted by $\{\varphi_n\}$ again, which converges uniformly in every compact set of $[0,\infty)\times\mathbb{R}^3$. Moreover, the limit function

$$\varphi(t,\xi) = \lim_{n\to\infty}\varphi_n(t,\xi)$$

is a characteristic function in $\mathcal{K}^\alpha$ and it is a solution to the problem (1.21) with the initial data $\varphi(0,\xi)=\varphi_0$.

To prove $\varphi(t,\xi)\in\widetilde{\mathcal{M}}^\alpha$, by letting $\tilde\varphi = 1$, we obtain from the stability estimate that for any $0<\delta<1$

$$\begin{aligned}\int_{\delta<|\xi|<\delta^{-1}} \frac{|Re\varphi-1|}{|\xi|^{3+\alpha}}d\xi &= \lim_{n\to\infty}\int_{\delta<|\xi|<\delta^{-1}} \frac{|Re\varphi_n-1|}{|\xi|^{3+\alpha}}d\xi \\ &\le \lim_{n\to\infty} e^{\lambda_\alpha^n t}||\varphi_0-1||_{\widetilde{\mathcal{M}}^\alpha} + \frac{e^{2\lambda_\beta^n t}-e^{\lambda_\alpha^n t}}{2\lambda_\beta^n-\lambda_\alpha^n}A_0 + \frac{e^{\lambda_\beta^n t}-e^{\lambda_\alpha^n t}}{\lambda_\beta^n-\lambda_\alpha^n}B_0 \\ &\le e^{\lambda_\alpha t}||\varphi_0-1||_{\widetilde{\mathcal{M}}^\alpha} + te^{2\lambda_\beta t}A_0 + te^{\lambda_\beta t}B_0.\end{aligned}$$

Letting $\delta\to 0$, we obtain $\varphi(t,\xi)\in\widetilde{\mathcal{M}}^\alpha$ for each $t>0$. Similarly, the stability estimate (1.26) follows from (2.36) by letting $n\to\infty$.

To complete the proof of Theorem 1.8, we will show $\varphi(t,\xi) \in C([0,\infty), \widetilde{\mathcal{M}}^\alpha)$. For any $t, s > 0$, we have

$$\begin{aligned} &Re\varphi(t,\xi) - Re\varphi(s,\xi) \\ &= \int_s^t \int_{\mathbb{S}^2} b(\cdot)\{(Re\varphi(\tau,\xi^+)Re\varphi(\tau,\xi^-) - Re\varphi(\tau,\xi))d\sigma d\tau \\ &\quad - \int_s^t \int_{\mathbb{S}^2} b(\cdot) Im\varphi(\tau,\xi^+)Im\varphi(\tau,\xi^-)\} d\sigma d\tau \\ &= I - II. \end{aligned}$$

For I, by Proposition 3.3 in [10], we obtain

$$\int_{\mathbb{R}^3} \frac{|I|}{|\xi|^{3+\alpha}} d\xi \lesssim |t-s| e^{\lambda_\alpha \max\{s,t\}} ||\varphi_0 - 1||_{\widetilde{\mathcal{M}}^\alpha}.$$

For II,

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|II|}{|\xi|^{3+\alpha}} d\xi \le \int_s^t \int_{\mathbb{S}^2} b(\cdot) \Bigg\{ &\int_{|\xi|\ge 1} \frac{||1-\varphi(\tau,\xi^-)||_\beta}{|\xi|^{3+\alpha-\beta}} \sin^\beta \frac{\theta}{2} \\ &+ \int_{|\xi|<1} \frac{||1-\varphi(\tau,\xi^-)||_\beta^2}{|\xi|^{3+\alpha-2\beta}} \sin^\beta \frac{\theta}{2} \cos^\beta \frac{\theta}{2} \Bigg\} d\sigma d\tau \\ \lesssim (e^{\lambda_\beta \max\{s,t\}} &||1-\varphi_0||_\beta + 1)^2 |t-s|. \end{aligned}$$

Therefore,

$$||\varphi(t) - \varphi(s)||_{\widetilde{\mathcal{M}}^\alpha} \lesssim C(t,s)|t-s|, \text{ for all } t, s > 0,$$

where $C(t,s) = e^{\lambda_\beta \max\{s,t\}} ||\varphi_0 - 1||_{\widetilde{\mathcal{M}}^\alpha} + (e^{\lambda_\beta \max\{s,t\}} ||1-\varphi_0||_\beta + 1)^2$. And then it completes the proof of Theorem 1.8.

2.4. **Proof of Theorem 1.5.** We are now ready to complete the proof of Theorem 1.5. Assume b satisfies (1.4) for some $\alpha_0 \in (0,2)$ and let $\alpha \in [\alpha_0, 2)$. If $F_0 \in P_\alpha(\mathbb{R}^3)$, then $\varphi_0 = \mathcal{F}(F_0) \in \widetilde{\mathcal{M}}^\alpha$. By Theorem 1.8 and (1.14), there exists a unique measure valued solution $F_t \in C([0,\infty), P_\alpha(\mathbb{R}^3))$ to the problem (1.1)-(1.2). The continuity with respect to t is in the following sense:

$$\lim_{t\to t_0} \int \psi(v) dF(t,v) = \int \psi(v) dF(t_0, v), \text{ for any } \psi \in C(\mathbb{R}^3) \tag{2.37}$$
$$\text{satisfying the growth condition } |\psi(v)| \lesssim \langle v \rangle^\alpha.$$

This is true because from (1.22) and (1.23) , we have

$$dis_{\alpha,\beta,\epsilon}(\varphi(t), \varphi(t_0)) \to 0, \text{ as } t \to t_0.$$

Then (2.37) follows from Theorem 1.1, so that Theorem 1.5 holds.

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