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The Rate of Asymptotic Convergence of Strong Detonations for a Model Problem

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We prove that the strong detonation travelling waves for a viscous combustion model are nonlinearly stable, that is, for a given perturbation of a travelling wave, the solution converges to a shifted travelling wave asymptotically as $t \to \infty$. The rate of convergence is obtained. This work is a continuation of a previous work by done T.P. Liu and the first author in [7].

Key words: combustion, detonation wave, travelling wave, asymptotic stability

1. Introduction

We consider a viscous combustion model

$$(u + qz)_t + f(u)_x = \beta u_{xx}, \tag{1.1}$$

$$z_t = -K\phi(u)z, \tag{1.2}$$

where u represents the lumped fluid variable, $z \in [0, 1]$ represents the concentration of the reactant, and the positive constants β, q, and K represent the viscosity, heat release, and the reaction rate respectively. The flux function f is smooth and satisfies

$$f'(u) > 0, \tag{1.3}$$

$$f''(u) > 0. \tag{1.4}$$

The reaction rate function ϕ is smooth and satisfies

$$\phi(u) = \begin{cases} 1, & \text{for} \quad u > d, \\ 0, & \text{for} \quad u < 0, \end{cases} \tag{1.5}$$

$$\phi'(u) > 0, \qquad \text{for } \ 0 < u < d. \tag{1.6}$$

The travelling wave solutions to (1.1) and (1.2) have been studied in [8]. The initial value problem of the system has been studied in [13]. Let $\beta \to +0$ and

$K \to +\infty$, then the limits of solutions depend on βK. In this respect the Riemann problem for the case of $\beta K = 0$ has been studied in [14], where (1.3) is not necessary satisfied. A related problem is the nonlinear stability of travelling waves as $t \to +\infty$. The nonlinear stability of strong detonation waves was proved in [7]. For the other results of this system, see [2, 3, 6, 12].

A weighted energy method has been proved to be an effective approach to study stability problems (see [1, 5, 9, 11]). The purpose of this paper is to apply the weighted energy method to study the nonlinear stability problem of (1.1) (1.2). The result of this paper is a continuation of [7]. We will show that if q is sufficiently small, then a perturbation of a travelling strong detonation wave leads to a solution which tends to a shifted travelling strong detonation wave as $t \to +\infty$, and the rate is determined by the rate of initial perturbation as $|x| \to \infty$. T. Li has obtained a series of interesting results on another combustion model. In [4] the rate of asymptotic convergence of strong detonations was also proved. The previous works indicate that these two models share some common properties, which may hint some general results on more general models.

We make a brief statement of the results here. Let u_* and u^* be two constants with $u_* < u^*$, then it is known that there exist travelling shock waves $s_0(\xi)$, $\xi = x - \sigma t$, to a single conservation law,

$$s_t + f(s)_x = \beta s_{xx},$$

such that

$$f(u^*) - f(u_*) = \sigma(u^* - u_*). \tag{1.7}$$

$$\lim_{\xi \to +\infty} s_0(\xi) = u_*, \quad \lim_{\xi \to -\infty} s_0(\xi) = u^*.$$

Here (1.7) is the Rankine-Hugoniot condition. For a fixed value of $s_0(0) \in (u_*, u^*)$, $s_0(\xi)$ is unique. If $u_* > d$, where d is the positive constant in (1.5), then under some restrictions on the parameters q, β, K there exist travelling strong detonation waves $s(\xi)$, $\zeta(\xi)$, $\xi = x - \sigma t$, such that

$$f(u^*) - f(u_+) = \sigma(u^* - u_+ - q), \tag{1.8}$$

$$\lim_{\xi \to +\infty} (s(\xi), \zeta(\xi)) = (u_+, 1), \quad \lim_{\xi \to -\infty} (s(\xi), \zeta(\xi)) = (u^*, 0).$$

And under some other restrictions on q, β, K there exist travelling weak detonation waves $s(\xi)$, $\zeta(\xi)$, $\xi = x - \sigma t$ such that

$$f(u_*) - f(u_+) = \sigma(u_* - u_+ - q), \tag{1.9}$$

$$\lim_{\xi \to +\infty} (s(\xi), \zeta(\xi)) = (u_+, 1), \quad \lim_{\xi \to -\infty} (s(\xi), \zeta(\xi)) = (u_*, 0).$$

Since we consider strong detonation waves only in this paper, for simplicity we assume that $u_* < 0$ and $u^* > d$. It is easy to prove that strong detonation waves always exist for any positive numbers q, β, and K. From (1.8) (1.7) we can get $u_+ < u_*$.

Let the perturbation of $s(\xi)$ and $\zeta(\xi)$ be $u_0(\xi)$ and $z_0(\xi)$. We assume that $u_0(\xi)$ and $z_0(\xi)$ are sufficiently smooth and

$$\lim_{|\xi|\to\infty}(u_0(\xi) - s(\xi), z_0(\xi) - \zeta(\xi)) = 0.$$

Without loss of generality we assume that

$$\int_{-\infty}^{+\infty}(u_0(\xi) + qz_0(\xi) - s(\xi) - q\zeta(\xi))\,d\xi = 0, \tag{1.10}$$

otherwise a translation of (s,ζ) leads to (1.10). (1.10) implies that for all t

$$\int_{-\infty}^{+\infty}(u + qz - s - q\zeta)\,dx = 0, \tag{1.11}$$

where (u,z) is the solution to (1.1) (1.2) with initial condition

$$(u,z)|_{t=0} = (u_0, z_0), \tag{1.12}$$

and $s = s(x-\sigma t)$, $\zeta = \zeta(x - \sigma t)$. (1.11) enables us to consider the antiderivatives

$$v(x,t) = \int_{-\infty}^{x} U(\xi,t)\,d\xi,$$

where

$$U = u_1 + q(z-\zeta), \qquad u_1 = u - s. \tag{1.13}$$

Our main result is the following:

THEOREM. *Let ρ be a positive real number. We assume that*

$$\int_{-\infty}^{+\infty}(1+|x|)^\rho(v^2(x,0) + u_1^2(x,0) + u_{1x}^2(x,0))\,dx$$

is bounded, and

$$|z_0(x) - \zeta(x)|, |(z_0(x) - \zeta(x))_x| = O(|x|^{-\rho-2}), \quad as \ \ |x|\to\infty,$$

besides we assume that

$$\int_{-\infty}^{+\infty}(v^2(x,0) + u_1^2(x,0))\,dx$$

and q are sufficiently small, then

$$|u_1(x,t)| + |z(\xi+\sigma t, t) - \zeta(\xi)| \le O(1)(t^{-\rho/2}), \quad as \ \ t \to \infty.$$

Same decay rate for scalar viscous conservation laws was obtained in [9] [10]. Applying the weighted energy estimate, we will prove the theorem in the next section. In the proof we will always denote any generic constant by C.

2. Stability Analysis

The local solvability of the problem (1.1) (1.2) and (1.12) can be demostrated by iteration. Applying the Poisson formula to the heat equation we have

$$\begin{aligned}
u =& \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi\beta t}} e^{-(x-\xi)^2/(4\beta t)} u_0(\xi)\, d\xi \\
&- \int_0^t \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi\beta(t-\tau)}} e^{-(x-\xi)^2/(4\beta(t-\tau))} (qz_\tau + f(u)_\xi)\, d\xi d\tau \\
=& \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi\beta t}} e^{-(x-\xi)^2/(4\beta t)} u_0(\xi)\, d\xi \\
&+ \int_0^t \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi\beta(t-\tau)}} e^{-(x-\xi)^2/(4\beta(t-\tau))} qK\phi(u(\xi,\tau)) z_0(\xi) \\
&\qquad\qquad \cdot e^{-K\int_0^\tau \phi(u(\xi,\eta))\, d\eta}\, d\xi d\tau \\
&+ \int_0^t \int_{-\infty}^{+\infty} \frac{x-\xi}{4\sqrt{\pi}(\beta(t-\tau))^{3/2}} e^{-(x-\xi)^2/(4\beta(t-\tau))} f(u(\xi,\tau))\, d\xi d\tau.
\end{aligned}$$

■

The existence of u follows from Picard's method of contraction mappings, then z is obtained by solving the ordinary differential equation (1.2) with initial condition (1.12). The global existance of (u, z) is a consequence of the following energy estimate.

By (1.7) there is a constant $c \in (u_*, u^*)$ such that $f'(c) = \sigma$. A shift of the origin, if necessary, leads to $s(0) = c$. So one gets $f'(s(0)) = \sigma$.

The functions $s(x - \sigma t)$ and $\zeta(x - \sigma t)$ satisfy the equations (1.1) and (1.2), that is

$$(s + q\zeta)_t + f(s)_x = \beta s_{xx}, \tag{2.1}$$

$$\zeta_t = -K\phi(s)\zeta, \tag{2.2}$$

By integrating the difference of (1.1) and (2.1) we have

$$v_t + f(u) - f(s) = \beta u_{1x}. \tag{2.3}$$

We define a weight with $\alpha \geq 0$, $\delta > 0$, $\gamma \geq 0$,

$$(t+1)^\alpha W(\xi) = (t+1)^\alpha (1+\delta\xi^2)^{\gamma/2}, \quad \xi = x - \sigma t. \tag{2.4}$$

Multiplying the equation (2.3) by (2.4) and v and taking integration we get

$$\begin{aligned}\int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W v v_t \, dxdt + \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W v (f(u) - f(s)) \, dxdt \\ = \beta \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W v u_{1\,x} \, dxdt.\end{aligned} \tag{2.5}$$

The first term of (2.5) is equal to

$$\begin{aligned}&\frac{1}{2}\int_{-\infty}^{+\infty} (t+1)\alpha W v^2 \, dx - \frac{1}{2}\int_{-\infty}^{+\infty} W(x) v^2(x,0) \, dx \\ &\quad + \frac{1}{2}\int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha \sigma W' v^2 \, dxdt - \int_0^t \int_{-\infty}^{+\infty} \frac{\alpha}{2}(t+1)^{\alpha-1} W v^2 \, dxdt.\end{aligned}$$

The second term of (2.5) is equal to

$$\begin{aligned}&\int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W v (f'(s) u_1 + O(1) u_1^2) \, dxdt \\ &= \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W v \{ f'(s)(v_x - q(z-\zeta)) + O(1) u_1^2 \} \, dxdt \\ &= \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha \Big\{ - W f''(s) s_x \frac{v^2}{2} - W' f'(s) \frac{v^2}{2} - qWv(z-\zeta) f'(s) \\ &\qquad\qquad + O(1) W v u_1^2 \Big\} \, dxdt,\end{aligned}$$

where we have used integration by parts. Analogously the right hand side of (2.5) is equal to

$$\begin{aligned}&-\beta \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha (W U u_1 + W' v u_1) \, dxdt \\ &\quad = -\beta \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha (W u_1^2 + qWu_1(z-\zeta) + W' v u_1) \, dxdt.\end{aligned}$$

We define

$$h(\xi) = -W(\xi) f''(s(\xi)) s'(\xi) + W'(\xi)(\sigma - f'(s(\xi))),$$

then (2.5) is reduced to

$$\frac{1}{2}\int_{-\infty}^{+\infty} (t+1)^\alpha W v^2 \, dx + \frac{1}{2}\int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha h v^2 \, dxdt$$

$$+\beta\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_1^2\,dxdt$$
$$\le \frac{1}{2}\int_{-\infty}^{+\infty}W(x)v^2(x,0)\,dx+\frac{\alpha}{2}\int_0^t\int_{-\infty}^{+\infty}(t+1)^{\alpha-1}Wv^2\,dxdt$$
$$-\beta\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W' v u_1\,dxdt$$
$$-q\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W(\beta u_1-f'(s)v)(z-\zeta)\,dxdt$$
$$+C\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W v u_1^2\,dxdt. \tag{2.6}$$

To get an energy estimate from (2.6), we need to study the property of the function h, which is summerized in the following lemma:

LEMMA 1. *If q is small enough and $\gamma\neq 0$, then $h>0$ and $h(\xi)\sim|\xi|^{\gamma-1}$ as $|\xi|\to\infty$.*

Proof. Let $y=q\zeta$, then the integration of the equation (2.1) yields

$$s'=\beta^{-1}(-\sigma(s+y)+f(s)+c), \tag{2.7}$$

where

$$c=-f(u^*)+\sigma u^*.$$

The critical point A divides the curve $\{(s(\xi),y(\xi))|-\infty<\xi<+\infty\}$ into two parts: $s'(\xi)<0$ for $\xi>\xi_0$ and $s'(\xi)>0$ for $\xi<\xi_0$, cf. Fig. 1.

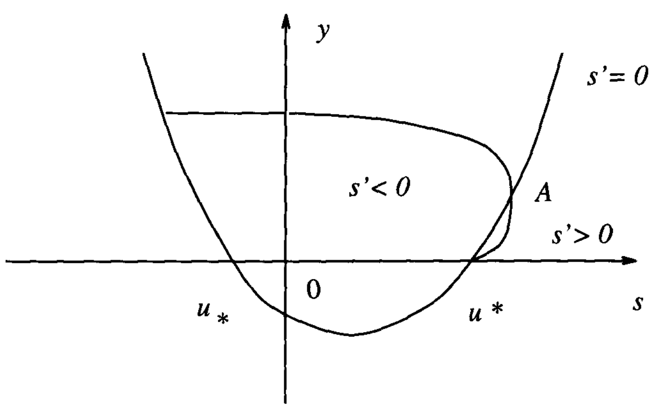

Fig. 1.

We have known that $u_*<s(0)<u^*$, and it is shown in Fig. 1 that $s(\xi_0)>u^*$, so $f'(s(\xi_0))>\sigma$, $\xi_0<0$.

If $\xi > 0$ then $W' > 0$, $f'(s) < \sigma$, and if $\xi_0 < \xi < 0$, then $W' < 0$, $f'(s) > \sigma$, hence $h > 0$ for the case of $\xi > \xi_0$. We turn now to consider the case of $\xi \le \xi_0$. Since $s'(\xi_0) = 0$, we get from the equation (2.7) that

$$(f' - \sigma)(s - u^*) = \sigma y$$

at the point ξ_0, where the argument of f' is a mean value., therefore

$$s(\xi_0) = u^* + \frac{\sigma y}{f' - \sigma},$$

but $y \in [0, q]$, so

$$s(\xi_0) \in \left[u^*, u^* + \frac{\sigma q}{f' - \sigma} \right]. \tag{2.8}$$

From $s \ge u^* > d$ we have $\phi \equiv 1$. The integration of the equation (2.2) yields

$$y = y(\xi_0) e^{K(\xi - \xi_0)/\sigma}. \tag{2.9}$$

We rewrite (2.7) as

$$s' = \beta^{-1}(f' - \sigma)(s - u^*) - \beta^{-1}\sigma y. \tag{2.10}$$

Let $p = \beta^{-1}(\sigma - f')$, the solution of (2.10) is

$$s = u^* + e^{-\int_{\xi_0}^{\xi} p\, d\xi'} \left\{ (s(\xi_0) - u^*) - \int_{\xi_0}^{\xi} \beta^{-1}\sigma y(\xi_0) e^{K(\xi - \xi_0)/\sigma} e^{\int_{\xi_0}^{\xi} p\, d\xi'}\, d\xi \right\},$$

then noting (2.8) we get

$$s - u^* \le Cq e^{b(\xi - \xi_0)}, \tag{2.11}$$

where $b > 0$. Applying (2.9) and (2.11) to (2.10) we have

$$|s'| \le Cq e^{\min(b, K/\sigma)(\xi - \xi_0)}.$$

The second term of $h(\xi)$ is positive, so if q is small enough, $h(\xi)$ is positive for $\xi \le \xi_0$.

Finally the definition (2.4) gives the degree of $h(\xi)$ as $|\xi| \to \infty$. □

Let us turn now to the estimate of (2.6). For the third term on the right hand side we have

$$\left| \beta \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W' v u_1\, dx dt \right| \le \frac{\beta}{4} \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W u_1^2\, dx dt + \beta \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha \frac{W'^2}{W} v^2\, dx dt. \tag{2.12}$$

If δ is small enough, then Lemma 1 implies

$$\beta \frac{W'^2}{W} \leq \frac{1}{8} h.$$

The right hand side of (2.12) is majorized by the left hand side of (2.6). To estimate the fourth and fifth terms of (2.6), we need the following *a priori* assumptions:

$$C|v| \leq \frac{\beta}{4}, \tag{2.13}$$

$$|u_1| < \min\left(\frac{|u_*|}{2}, \frac{u^* - d}{2}\right), \tag{2.14}$$

where d is given in (1.5). Using (2.13) the fifth term can be easily estimated. It remains to estimate the fourth term. Let $x_1, x_2 \in \mathbb{R}$ such that $s(x_1) = \frac{u_*}{2}$ and $s(x_2) = \frac{u^*+d}{2}$. Notice that $x_1 - x_2$ has an upper bound which is independent of q. We divide the domain $\Omega = \mathbb{R} \times [0, t)$ into four parts:

$$\Omega_1 = \Omega \cap \{x > x_1 + \sigma t\}, \qquad \Omega_2 = \Omega \cap \{x_2 + \sigma t < x < x_1 + \sigma t\},$$

$$\Omega_3 = \Omega \cap \{x_2 < x < x_2 + \sigma t\}, \qquad \Omega_4 = \Omega \cap \{x < x_2\}.$$

From (2.14) we see that $\phi(s) = \phi(u) \equiv 0$ in Ω_1 and $\phi(s) = \phi(u) \equiv 1$ in $\Omega_3 \cup \Omega_4$. Let all assumptions of the theorem be satisfied. We set

$$F(x,t) = q(t+1)^\alpha W(\beta u_1 - f'(s)v)(z - \zeta)$$

and estimate $\int\int_\Omega F(x,t)\, dxdt$. In the domain Ω_1

$$z - \zeta = z_0(x) - \zeta(x). \tag{2.15}$$

The Cauchy-Schwarz inequality gives

$$\left|\int\int_{\Omega_1} F(x,t)\, dxdt\right|$$
$$\leq Cq \int\int_{\Omega_1} (t+1)^\alpha W \left(|z_0(x) - \zeta(x)|^{1/2}(u_1^2 + v^2) + |z_0(x) - \zeta(x)|^{3/2}\right) dxdt$$

For small q it can be verified that

$$CqW|z_0(x) - \zeta(x)|^{1/2} \leq \frac{1}{8} h(x - \sigma t).$$

It follows that

$$\left|\iint\limits_{\Omega_1} F(x,t)\,dxdt\right| \le \iint\limits_{\Omega_1} \left\{\frac{1}{8}(t+1)^\alpha h(x-\sigma t)v^2 + Cq(t+1)^\alpha W u_1^2\right\} dxdt + Cq. \tag{2.16}$$

In the domain Ω_2 we define

$$t_1(x) = \begin{cases} 0, & \text{for} \quad x \le x_1, \\ \dfrac{x-x_1}{\sigma}, & \text{for} \quad x > x_1, \end{cases}$$

$$t_2(x) = \begin{cases} \dfrac{x-x_2}{\sigma}, & \text{for} \quad x < \sigma t + x_2, \\ t, & \text{for} \quad x \ge \sigma t + x_2, \end{cases}$$

then $\Omega_2 = \{(x,t); t_1(x) < t < t_2(x)\}$ and

$$\begin{aligned} z - \zeta &= z_0(x)e^{-K\int_0^t \phi(u)\,d\tau} - \zeta(x)e^{-K\int_0^t \phi(s)\,d\tau} \\ &= z_0(x)e^{-K\int_{t_1(x)}^t \phi(u)\,d\tau} - \zeta(x)e^{-K\int_{t_1(x)}^t \phi(s)\,d\tau} \\ &= z_0(x)\Big(e^{-K\int_{t_1(x)}^t \phi(u)\,d\tau} - e^{-K\int_{t_1(x)}^t \phi(s)\,d\tau}\Big) \\ &\qquad + (z_0(x)-\zeta(x))e^{-K\int_{t_1(x)}^t \phi(s)\,d\tau}, \end{aligned}$$

therefore

$$|z-\zeta| \le C\int_{t_1(x)}^t |u_1|\,d\tau + |z_0(x)-\zeta(x)|. \tag{2.17}$$

It follows that

$$\begin{aligned} \left|\iint\limits_{\Omega_2} F(x,t)\,dxdt\right| &\le Cq\iint\limits_{\Omega_2} (t+1)^\alpha W \\ &\quad \times \left\{u_1^2 + v^2 + \left(\int_{t_1(x)}^t |u_1|\,d\tau\right)^2 + (z_0(x)-\zeta(x))^2\right\} dxdt. \end{aligned} \tag{2.18}$$

Due to the Cauchy-Schwarz inequality we have

$$\iint\limits_{\Omega_2} (t+1)^\alpha W\Big(\int_{t_1(x)}^t |u_1|\,d\tau\Big)^2\,dxdt$$

$$\leq C \iint_{\Omega_2} (t+1)^\alpha W \Big(\int_{t_1(x)}^t u_1^2 \, d\tau \Big) \, dxdt$$

$$= C \int dx \int_{t_1(x)}^{t_2(x)} (t+1)^\alpha W \, dt \int_{t_1(x)}^t u_1^2 \, d\tau \equiv B.$$

But $(t+1)^\alpha/(\tau+1)^\alpha$ is bounded in Ω_2, so

$$\begin{aligned} B &\leq C \int dx \int_{t_1(x)}^{t_2(x)} dt \int_{t_1(x)}^t (\tau+1)^\alpha W u_1^2 \, d\tau \\ &\leq C \int dx \int_{t_1(x)}^{t_2(x)} (\tau+1)^\alpha W u_1^2 \, d\tau \\ &= C \iint_{\Omega_2} (t+1)^\alpha W u_1^2 \, dxdt. \end{aligned} \tag{2.19}$$

We substitute (2.19) into (2.18) and notice that W and h are in fact bounded functions in Ω_2, then we get

$$\left| \iint_{\Omega_2} F(x,t) \, dxdt \right| \leq Cq \iint_{\Omega_2} (t+1)^\alpha (W u_1^2 + h v^2) \, dxdt + Cq. \tag{2.20}$$

We have the following estimate in Ω_3:

$$|z - \zeta| \leq \Big(C \int_{t_1(x)}^{t_2(x)} |u_1| \, d\tau + |z_0(x) - \zeta(x)| \Big) e^{-K(t - t_2(x))}, \tag{2.21}$$

therefore

$$\left| \iint_{\Omega_3} F(x,t) \, dxdt \right| \leq Cq \iint_{\Omega_3} (t+1)^\alpha W e^{-K(t-t_2(x))} \times \Big\{ u_1^2 + v^2 + \int_{t_1(x)}^{t_2(x)} u_1^2 \, d\tau + (z_0(x) - \zeta(x))^2 \Big\} \, dxdt.$$

For small q it can be verified that

$$CqWe^{-K(t-t_2(x))} \leq \frac{1}{8} h(x - \sigma t).$$

And we have

$$\int_{t_2(x)}^t (t+1)^\alpha W e^{-K(t-t_2(x))} \, dt \leq C(t_2(x)+1)^\alpha \leq C(\tau+1)^\alpha,$$

and

$$\iint_{\Omega_3} (t+1)^\alpha W e^{-K(t-t_2(x))} (z_0(x) - \zeta(x))^2 \, dxdt \leq C,$$

thus

$$\left|\iint_{\Omega_3} F(x,t)\,dxdt\right| \le Cq\Big(\iint_{\Omega_3}(t+1)^\alpha W u_1^2\,dxdt + \iint_{\Omega_2}(t+1)^\alpha u_1^2\,dxdt + 1\Big) + \frac{1}{8}\iint_{\Omega_3}(t+1)^\alpha h v^2\,dxdt. \tag{2.22}$$

Since $\phi \equiv 1$ in Ω_4, we have

$$z - \zeta = (z_0(x) - \zeta(x))e^{-Kt},$$

therefore

$$\begin{aligned}&\left|\iint_{\Omega_4} F(x,t)\,dxdt\right| \\ &\le Cq\iint_{\Omega_4}(t+1)^\alpha\big(Wu_1^2 + W|z_0(x)-\zeta(x)|e^{-Kt}(v^2+1)\big)\,dxdt \\ &\le Cq\iint_{\Omega_4}(t+1)^\alpha W u_1^2\,dxdt + \frac{1}{8}\iint_{\Omega_4}(t+1)^\alpha h v^2\,dxdt + Cq. \end{aligned}\tag{2.23}$$

Let q be sufficiently small, then we use (2.12) (2.13) (2.16) (2.20) (2.22) (2.23) to estimate the right hand side of (2.6) and obtain

$$\begin{aligned}&\frac{1}{2}\int_{-\infty}^{+\infty}(t+1)^\alpha W v^2\,dx + \frac{1}{4}\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha h v^2\,dxdt \\ &\quad + \frac{\beta}{4}\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_1^2\,dxdt \\ &\le \frac{1}{2}\int_{-\infty}^{+\infty} W(x)v^2(x,0)\,dx + \frac{\alpha}{2}\int_0^t\int_{-\infty}^{+\infty}(t+1)^{\alpha-1}W v^2\,dxdt + Cq.\end{aligned}\tag{2.24}$$

We now turn to estimate the derivatives. We derive from (1.1) and (2.1) that

$$U_t + (f(u) - f(s))_x = \beta u_{1\,xx}. \tag{2.25}$$

We multiply both sides of (2.25) by the weight $(t+1)^\alpha W$ and u_1 and then take integration. For the first term we have

$$\begin{aligned}&\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_1 U_t\,dxdt \\ &= \frac{1}{2}\int_{-\infty}^{+\infty}(t+1)^\alpha W u_1^2\,dx - \frac{1}{2}\int_{-\infty}^{+\infty} W(x)u_1^2(x,0)\,dx\end{aligned}$$

$$+\frac{1}{2}\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha\sigma W' u_1^2\,dxdt-\int_0^t\int_{-\infty}^{+\infty}\alpha(t+1)^{\alpha-1}W\frac{u_1^2}{2}\,dxdt$$
$$+q\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_1(-K\phi(u)z+K\phi(s)\zeta)\,dxdt. \tag{2.26}$$

We notice that

$$-K\phi(u)z+K\phi(s)\zeta=-K\phi(z-\zeta)-K\phi' u_1\zeta.$$

The same procedure in estimating $F(x,t)$ is applied here, then the last term of (2.26) is majorized as the following:

$$\left|q\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_1(-K\phi(u)z+K\phi(s)\zeta)\,dxdt\right|$$
$$\le Cq\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_1^2\,dxdt+Cq.$$

For the second term of (2.25) we have

$$\begin{aligned}&\left|\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_1(f(u)-f(s))_x\,dxdt\right|\\&=\left|\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha(W u_{1\,x}+W' u_1)(f(u)-f(s))\,dxdt\right|\\&=\left|\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha(W u_1 u_{1\,x}+W' u_1^2)f'\,dxdt\right|\\&\le\frac{\beta}{4}\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_{1\,x}^2\,dxdt+C\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha(W+|W'|)u_1^2\,dxdt.\end{aligned}$$

For the right hand side of (2.25) the same procedure in deriving (2.12) is applied. We get the following estimation from (2.25) that

$$\begin{aligned}&\frac{1}{2}\int_{-\infty}^{+\infty}(t+1)^\alpha W u_1^2\,dx+\frac{\beta}{2}\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_{1\,x}^2\,dxdt\\&\quad+\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha h u_1^2\,dxdt\\&\le\frac{1}{2}\int_{-\infty}^{+\infty}W(x)u_1^2(x,0)\,dx+C\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_1^2\,dxdt\\&\quad+\frac{\alpha}{2}\int_0^t\int_{-\infty}^{+\infty}(t+1)^{\alpha-1}W u_1^2\,dxdt+Cq.\end{aligned}\tag{2.27}$$

We set $U_2=U_x$ and $u_2=u_{1\,x}$, then from (2.25) we have the equation

$$U_{2\,t}+(f(u)-f(s))_{xx}=\beta u_{2\,xx}.$$

Mutiply this equation by $(t+1)^\alpha W u_2$ and integrate over $[0,t]\times R$. Since

$$U_{2t} = u_{2t} + q(z-\zeta)_{xt},$$

we have

$$\begin{aligned}\int_0^t\int_{-\infty}^{+\infty} &(t+1)^\alpha W u_2(u_{2t}+q(z-\zeta)_{xt})\,dxdt\\ &+\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_2(f(u)-f(s))_{xx}\,dxdt\\ &=\beta\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_2 u_{2xx}\,dxdt. \end{aligned}\tag{2.28}$$

We now estimate (2.28) term by term. For the first term, we have

$$\begin{aligned}\int_0^t\int_{-\infty}^{+\infty}&(t+1)^\alpha W u_2(u_{2t}+q(z-\zeta)_{xt})\,dxdt\\ &=\frac{1}{2}\int_{-\infty}^{+\infty}(t+1)^\alpha W u_2^2\,dx-\frac{1}{2}\int_{-\infty}^{+\infty}W(x)u_2^2(x,0)\,dx\\ &-\frac{\alpha}{2}\int_0^t\int_{-\infty}^{+\infty}(t+1)^{\alpha-1}W u_2^2\,dxdt\\ &+\frac{\sigma}{2}\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W' u_2^2\,dxdt\\ &-qK\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W' u_2(\phi(z-\zeta)-\phi' u_1\zeta)\,dxdt\\ &-q\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_{2x}(\phi(z-\zeta)-\phi' u_1\zeta)\,dxdt.\end{aligned}$$

The second and the third term can be written as follows

$$\begin{aligned}\int_0^t\int_{-\infty}^{+\infty}&(t+1)^\alpha W u_2(f(u)-f(s))_{xx}\,dxdt\\ &=-\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W' u_2(f(u)-f(s))_x\,dxdt\\ &\quad-\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_{2x}(f'(u)u_x+f'(s)s_x)\,dxdt\\ &=-\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W' u_2(f' u_2-f' u_1 s_x)\,dxdt\\ &\quad-\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_{2x}(f' u_2-f' u_1 s_x)\,dxdt,\end{aligned}$$

and

$$\beta\int_0^t\int_{-\infty}^{+\infty}(t+1)^\alpha W u_2 u_{2xx}\,dxdt$$

$$= -\beta \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W u_{2x}^2 \, dxdt + \frac{\beta}{2} \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W'' u_2^2 \, dxdt.$$

Combining all the above estimation for (2.28) and using Cauchy inequality and q being sufficiently small, we have

$$\begin{aligned} &\int_{-\infty}^{+\infty} (t+1)^\alpha W u_2^2 \, dx + \beta \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W u_{2x}^2 \, dxdt \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha h u_2^2 \, dxdt \\ &\le \int_{-\infty}^{+\infty} W(x) u_2^2(x,0) \, dx + \alpha \int_0^t \int_{-\infty}^{+\infty} (t+1)^{\alpha-1} W u_2^2 \, dxdt \\ &\quad + C \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W (u_1^2 + u_2^2) \, dxdt + Cq. \end{aligned} \tag{2.29}$$

Combining (2.4), (2.27) and (2.29), we have for $\gamma \neq 0$

$$\begin{aligned} &\int_{-\infty}^{+\infty} (t+1)^\alpha W (v^2 + u_1^2 + u_2^2) \, dx + \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha h (v^2 + u_1^2 + u_2^2) \, dxdt \\ &\quad + \beta \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha W (u_1^2 + u_2^2 + u_{2x}^2) \, dxdt \\ &\le C \int_0^t W(x) (v^2(x,0) + u_1^2(x,0) + u_2^2(x,0)) \, dx \\ &\quad + C\alpha \int_0^t \int_{-\infty}^{+\infty} (t+1)^{\alpha-1} W (v^2 + u_1^2 + u_2^2) \, dxdt + Cq. \end{aligned} \tag{2.30}$$

We need one more estimate of

$$\begin{aligned} &\int_{-\infty}^{+\infty} (t+1)^\alpha W \{ (z-\zeta)^2 + ((z-\zeta)_x)^2 \} \, dx \\ &= \int_{x_1+\sigma t}^{\infty} + \int_{x_2+\sigma t}^{x_1+\sigma t} + \int_{x_2}^{x_2+\sigma t} + \int_{-\infty}^{x_2} \\ &\equiv I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We notice (2.15) and $t \le \frac{x-x_1}{\sigma}$ then get

$$|I_1| \le C.$$

We notice (2.17) and $(t+1)^\alpha/(\tau+1)^\alpha \le C$, then get

$$|I_2| \le C + C \iint_{\Omega_2} (t+1)^\alpha (u_1^2 + u_2^2) \, dxdt.$$

We notice (2.21) then get

$$|I_3| \le C\int_{x_2}^{x_2+\sigma t}(t+1)^\alpha W\Big\{\int_{t_1(x)}^{t_2(x)}(u_1^2+u_2^2)\,d\tau+(z_0(x)-\zeta(x))^2 \\ +((z_0(x)-\zeta(x))_x)^2\Big\}e^{-2K(t-t_2(x))}\,dx.$$

To estimate the first term of it we use the inequality

$$\frac{(t+1)^\alpha}{(\tau+1)^\alpha}e^{-K(t-t_2(x))}=\frac{(t+1)^\alpha e^{-Kt}}{(t_2(x)+1)^\alpha e^{-Kt_2(x)}}\cdot\frac{(t_2(x)+1)^\alpha}{(\tau+1)^\alpha}\le C,$$

and to estimate the second term we use

$$(t+1)^\alpha e^{-K(t-t_2(x))}\le(t+1)^\alpha e^{-Kt/2}\le C,\quad \text{for } t_2(x)\le\frac{t}{2},$$

and

$$(t+1)^\alpha(|z_0(x)-\zeta(x)|+|(z_0(x)-\zeta(x))_x|) \\ \le\left(\frac{2(x-x_2)}{\sigma}+1\right)^\alpha(|z_0(x)-\zeta(x)|+|(z_0(x)-\zeta(x))_x|)\le C, \\ \text{for } t_2(x)>\frac{t}{2},$$

then we get

$$|I_3|\le C+C\iint_{\Omega_2}(t+1)^\alpha(u_1^2+u_2^2)\,dxdt.$$

Finally it is easy to see that $|I_4|\le C$. It follows that

$$\int_{-\infty}^{+\infty}(t+1)^\alpha W\left\{(z-\zeta)^2+((z-\zeta)_x)^2\right\}dx \\ \le C+C\iint_{\Omega_2}(t+1)^\alpha(u_1^2+u_2^2)\,dxdt. \tag{2.31}$$

Based on the above estimates, we are going to prove the theorem as follows.

Proof of Theorem. Under the *a priori* estimates (2.13) and (2.14), all the estimates given above hold.

When $\rho=N>0$ is an integer, we inductively let $\alpha=0$ and $\gamma=N$ in (2.30), the left hand side of (2.30) can be bounded only by the initial data and q. Then we let $\alpha=1$ and $\gamma=N-1$ in (2.30), using Lemma 1, we get a similar estimate. This procedure can be continued up to $\alpha=N-1$ and $\gamma=1$.

For $\gamma=0$, we need the estimate for $\int_{-\infty}^{+\infty}(t+1)^\alpha v^2\,dx$ because the function h is no longer positive everywhere.

By the equation (4.4) in [7], we have

$$\int_{-\infty}^{+\infty} v^2(x,T)\,dx + \int_0^T \int_{-\infty}^{+\infty} v_x^2\,dxdt + \int_0^T \int_{-\infty}^{+\infty} |f''(s)s_x|v^2\,dxdt \\ \le \int_{-\infty}^{+\infty} v^2(x,0)\,dx + C\int_0^T \int_{-\infty}^{+\infty} (q^2(z-\zeta)^2 + q|v(z-\zeta)|)\,dxdt.$$

By the definition of v, v_x in the second term can replaced by u_1. From the deduction it is easy to see $(0,T)$ can be replaced by any interval (t_1,t_2).

Let $\Delta t = \frac{t}{M}$, $t_i = i\Delta t$. Then taking summation yields

$$\sum_{i=1}^{M}(t_i+1)^\alpha \Big\{ \int_{-\infty}^{+\infty} v^2(x,t_i)\,dx + \int_{t_{i-1}}^{t_i}\int_{-\infty}^{+\infty} u_1^2\,dxdt \\ + \int_{t_{i-1}}^{t_i}\int_{-\infty}^{+\infty} |f''(s)s_x|v^2\,dxdt \Big\} \\ \le \sum_{i=1}^{M}(t_i+1)^\alpha \Big\{ \int_{-\infty}^{+\infty} v^2(x,t_{i-1})\,dx \\ + C\int_{t_{i-1}}^{t_i}\int_{-\infty}^{+\infty} (q^2(z-\zeta)^2 + q|v(z-\zeta)|)\,dxdt \Big\}.$$

Let $M \to +\infty$, we have

$$\int_{-\infty}^{+\infty} (t+1)^\alpha v^2\,dx + \int_0^t\int_{-\infty}^{+\infty} (t+1)^\alpha u_1^2\,dxdt \\ + \int_0^t\int_{-\infty}^{+\infty} |f''(s)s_x|v^2\,dxdt \\ \le \int_{-\infty}^{+\infty} v^2(x,0)\,dx + \alpha\int_0^t\int_{-\infty}^{+\infty} (t+1)^{\alpha-1}v^2\,dxdt \\ + C\int_0^t\int_{-\infty}^{+\infty} (t+1)^\alpha(q^2(z-\zeta)^2 + q|v(z-\zeta)|)\,dxdt.$$

We now estimate the last term as follows.

$$\int_0^t\int_{-\infty}^{+\infty} (t+1)^\alpha |v(z-\zeta)|\,dxdt \\ \le \frac{1}{2}\iint_{\Omega_1\cup\Omega_4} (t+1)^{\alpha-1}v^2\,dxdt + \frac{1}{2}\iint_{\Omega_1\cup\Omega_4} (t+1)^{\alpha+1}(z-\zeta)^2\,dxdt \\ + \frac{1}{2}\iint_{\Omega_2} (t+1)^\alpha v^2\,dxdt + \frac{1}{2}\iint_{\Omega_2} (t+1)^\alpha (z-\zeta)^2\,dxdt \\ + \frac{1}{2}\iint_{\Omega_3} (t+1)^\alpha e^{-K(t-t_2(x))}v^2\,dxdt$$

$$+\frac{1}{2}\iint\limits_{\Omega_3}(t+1)^{\alpha}e^{K(t-t_2(x))}(z-\zeta)^2\,dxdt.$$

By (2.15), we have

$$\iint\limits_{\Omega_1}(t+1)^{\alpha+1}(z-\zeta)^2\,dxdt$$
$$\leq C\iint\limits_{\Omega_1}(t+1)^{\alpha+1}(|x|+1)^{-2N-4}\,dxdt\leq C.$$

and

$$\iint\limits_{\Omega_4}(t+1)^{\alpha+1}(z-\zeta)^2\,dxdt$$
$$=\iint\limits_{\Omega_4}(t+1)^{\alpha+1}(z_0(x)-\zeta(x))^2e^{-2Kt}\,dxdt\leq C.$$

Using (2.17) and (2.19) yields

$$\iint\limits_{\Omega_2}(t+1)^{\alpha}(z-\zeta)^2\,dxdt\leq C\iint\limits_{\Omega_2}(t+1)^{\alpha}u_1^2\,dxdt+C.$$

By (2.21), we have

$$\iint\limits_{\Omega_3}(t+1)^{\alpha}e^{K(t-t_2(x))}(z-\zeta)^2\,dxdt\leq C\iint\limits_{\Omega_2}(t+1)^{\alpha}u_1^2\,dxdt+C.$$

Since $|f''(s)s_x|\geq C>0$ in Ω_2, then

$$\iint\limits_{\Omega_2}(t+1)^{\alpha}v^2\,dxdt\leq C\iint\limits_{\Omega_2}|f''(s)s_x|v^2\,dxdt.$$

Let $\xi\in(x_2+\sigma t,x_1+\sigma t)$, we have

$$\iint\limits_{\Omega_3}(t+1)^{\alpha}e^{-K(t-t_2(x))}v^2(x,t)\,dxdt$$
$$=\iint\limits_{\Omega_3}(t+1)^{\alpha}e^{-K(t-t_2(x))}\left(v(\xi,t)+\int_{\xi}^{x}v_{\eta}(\eta,t)\,d\eta\right)^2dxdt$$
$$=\iint\limits_{\Omega_3}\frac{(t+1)^{\alpha}}{x_1-x_2}e^{-K(t-t_2(x))}\int_{x_2+\sigma t}^{x_1+\sigma t}\left(v(\xi,t)+\int_{\xi}^{x}v_{\eta}(\eta,t)\,d\eta\right)^2d\xi dxdt$$

$$
\begin{aligned}
&\le C \iint_{\Omega_3} (t+1)^\alpha e^{-K(t-t_2(x))} \int_{x_2+\sigma t}^{x_1+\sigma t} v^2(\xi,t)\, d\xi dx dt \\
&\quad + C \iint_{\Omega_3} (t+1)^\alpha e^{-K(t-t_2(x))} \int_{x_2+\sigma t}^{x_1+\sigma t} (\xi - x) \int_x^\xi v_\eta^2(\eta,t)\, d\eta d\xi dx dt \\
&= I_1 + I_2,
\end{aligned}
$$

where I_1 and I_2 are estimated as follows.

$$
I_1 \le C \iint_{\Omega_2} (t+1)^\alpha v^2(\xi,t)\, d\xi dt,
$$

and

$$
\begin{aligned}
I_2 &\le C \int_0^t (t+1)^\alpha\, dt \int_{x_2+\sigma t}^{x_1+\sigma t} d\xi \int_{x_2}^{\xi} v_\eta^2(\eta,t)\, d\eta \int_{x_2}^{x_2+\sigma t} e^{-K(t-t_2(x))}(\xi - x)\, dx \\
&\le C \iint_{\Omega_2\cup\Omega_3} (t+1)^\alpha v_\eta^2(\eta,t)\, d\eta dt \\
&\le C \iint_{\Omega_2\cup\Omega_3} (t+1)^\alpha u_1^2\, dx dt + Cq^2 \iint_{\Omega_2\cup\Omega_3} (t+1)^\alpha (z-\zeta)^2\, dx dt.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha |v(z-\zeta)|\, dx dt \\
&\qquad \le \iint_{\Omega_1\cup\Omega_4} (t+1)^{\alpha-1} v^2\, dx dt + C \iint_{\Omega_2} |f''(s)s_x| v^2\, dx dt \\
&\qquad\qquad + C \iint_{\Omega_2\cup\Omega_3} (t+1)^\alpha u_1^2\, dx dt + C.
\end{aligned}
$$

Finally

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} (t+1)^\alpha v^2\, dx + (1 - Cq) \int_0^t \int_{-\infty}^{+\infty} (t+1)^\alpha u_1^2\, dx dt \\
&\qquad + (1 - Cq) \int_0^t \int_{-\infty}^{+\infty} |f''(s)s_x| v^2\, dx dt \\
&\qquad \le (\alpha + Cq) \int_0^t \int_{-\infty}^{+\infty} (t+1)^{\alpha-1} v^2\, dx dt \\
&\qquad\qquad + \int_{-\infty}^{+\infty} v^2(x,0)\, dx + Cq. \qquad (2.32)
\end{aligned}
$$

Combining (2.32) and the above estimates yields

$$\int_{-\infty}^{+\infty}(t+1)^{\alpha}W(v^2+u_1^2+u_2^2)\,dx + \beta\int_0^t\int_{-\infty}^{+\infty}(t+1)^{\alpha}W(u_1^2+u_2^2+u_{2x}^2)\,dxdt \le C\int_{-\infty}^{+\infty}W^{(N)}(v^2(x,0)+u_1^2(x,0)+u_2^2(x,0))\,dx+Cq, \tag{2.33}$$

for $0\le\alpha\le N$ and $\gamma+\alpha=N$. Here $W^{(\theta)}$ is the weight defined in (2.4) with $\gamma=\theta$. (2.31) gives

$$\int_{-\infty}^{+\infty}(t+1)^{\alpha}W((z-\zeta)^2+((z-\zeta)_x)^2)\,dx \le C\int_{-\infty}^{+\infty}W^{(N)}(x)(v^2(x,0)+u_1^2(x,0)+u_2^2(x,0))\,dx+C. \tag{2.34}$$

Now it remains to verify the *a priori* estimates (2.13) and (2.14). By (4.4) in [7] and the discussion above, it is easy to show that when q and $\int_{-\infty}^{+\infty}(v^2(x,0)+u_1^2(x,0))\,dx$ are sufficiently small,

$$\int_{-\infty}^{+\infty}v^2\,dx,\ \int_{-\infty}^{+\infty}u_1^2\,dx,\ \int_{-\infty}^{+\infty}U^2\,dx$$

are sufficiently small. Since $\int_{-\infty}^{+\infty}u_2^2\,dx$ is bounded, using the Sobolev inequality

$$g^2(x)\le 2\left(\int_{-\infty}^{x}g^2\,dx\right)^{1/2}\left(\int_{-\infty}^{x}\left(\frac{dg}{dx}\right)^2dx\right)^{1/2},$$

and a simple continuity argument, we get (2.13) and (2.14). Therefore, by choosing $\alpha=N$ in (2.33) and (2.34) and using the Sobolev inequality again, we get the decay rate in the Theorem.

When $\rho>0$ is not an integer. As in [9] [10], using the above inductive procedure and the property of h, we have

$$(t+1)^{[\rho]}\int_{-\infty}^{+\infty}W^{(\rho-[\rho])}(v^2+u_1^2+u_2^2)\,dx + \int_0^t\int_{-\infty}^{+\infty}(t+1)^{[\rho]}W^{(\rho-[\rho]-1)}(v^2+u_1^2+u_2^2)\,dxdt + \beta\int_0^t\int_{-\infty}^{+\infty}(t+1)^{[\rho]}W^{(\rho-[\rho])}(u_1^2+u_2^2+u_{2x}^2)\,dxdt \le C\int_{-\infty}^{+\infty}W^{(\rho)}(v^2(x,0)+u_1^2(x,0)+u_2^2(x,0))\,dx+Cq. \tag{2.35}$$

For $\rho < \alpha < [\rho]+1$ and $\gamma = 0$, (2.32) (2.27) and (2.29) give

$$\begin{aligned}(t+1)^{\alpha}\int_{-\infty}^{+\infty}(v^2+u_1^2+u_2^2)\,dx+\beta\int_0^t\int_{-\infty}^{+\infty}(t+1)^{\alpha}(u_1^2+u_2^2+u_{2x}^2)\,dxdt \\ \le C\int_{-\infty}^{+\infty}(v^2(x,0)+u_1^2(x,0)+u_2^2(x,0))\,dx+Cq \\ +C\int_0^t\int_{-\infty}^{+\infty}(t+1)^{\alpha-1}(v^2+u_1^2+u_2^2)\,dxdt. \qquad (2.36)\end{aligned}$$

For simplicity, we denote $|\phi(\tau)|_\theta^2 = \int_{-\infty}^{+\infty} W^{(\theta)}(v^2+u_1^2+u_2^2)\,dx$. Now we only need to estimate the last term of (2.36). As in [9] [10], this can be done by using the Hölder inequality:

$$\begin{aligned}&\int_0^t(\tau+1)^{\alpha-1}|\phi(\tau)|_0^2\,d\tau \\ &= \int_0^t(\tau+1)^{\alpha-1}\int_{-\infty}^{+\infty}W^{(\rho-[\rho])([\rho]+1-\rho)-(\rho-[\rho])([\rho]+1-\rho)} \\ &\qquad\qquad \cdot(\phi^2)^{([\rho]+1-\rho)+(\rho-[\rho])}\,d\xi d\tau \\ &\le \int_0^t(\tau+1)^{\alpha-1}\left(\int_{-\infty}^{+\infty}W^{\rho-[\rho]}\phi^2\,d\xi\right)^{[\rho]+1-\rho} \\ &\qquad\qquad \cdot\left(\int_{-\infty}^{+\infty}W^{-([\rho]+1-\rho)}\phi^2\,d\xi\right)^{\rho-[\rho]}d\tau \\ &= \int_0^t(\tau+1)^{-([\rho]+1-\alpha)}\left((\tau+1)^{[\rho]}|\phi|_{\rho-[\rho]}^2\right)^{[\rho]+1-\rho} \\ &\qquad\qquad \cdot\left((\tau+1)^{[\rho]}|\phi|_{\rho-[\rho]-1}^2\right)^{\rho-[\rho]}d\tau \\ &\le C\left(|\phi(0)|_\rho^{2([\rho]+1-\rho)}+q\right)\int_0^t(\tau+1)^{-([\rho]+1-\alpha)}\left((\tau+1)^{[\rho]}|\phi|_{\rho-[\rho]-1}^2\right)^{\rho-[\rho]}d\tau \\ &\le C\left(|\phi(0)|_\rho^{2([\rho]+1-\rho)}+q\right)\left(\int_0^t(\tau+1)^{-([\rho]+1-\alpha)/([\rho]+1-\rho)}\,d\tau\right)^{[\rho]+1-\rho} \\ &\qquad\qquad \cdot\left(\int_0^t(\tau+1)^{[\rho]}|\phi|_{\rho-[\rho]-1}^2\,d\tau\right)^{\rho-[\rho]} \\ &\le C\left(|\phi(0)|_\rho^2+q\right)\left(\int_0^t(\tau+1)^{-([\rho]+1-\alpha)/([\rho]+1-\rho)}\,d\tau\right)^{[\rho]+1-\rho}.\end{aligned}$$

We have

$$\begin{aligned}&(t+1)^{\alpha}\int_{-\infty}^{+\infty}(v^2+u_1^2+u_2^2)\,dx \\ &\le C\int_{-\infty}^{+\infty}(v^2(x,0)+u_1^2(x,0)+u_2^2(x,0))\,dx+Cq+C(t+1)^{\alpha-\rho}(|\phi(0)|_\rho^2+q).\end{aligned}$$

Combining this estimate and the same argument for the case when ρ is an integer, we get the decay rate in the Theorem. This completes the proof of the Theorem.

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