Global Structure and Asymptotic Behavior of Weak Solutions to Flood Wave Equations

Tao Luo
Department of Mathematics, Georgetown University
Washington, DC 20057, USA
Fax: 1-202-687-6067. email:tl48@georgetown.edu

Tong Yang
Department of Mathematics, City University of Hong Kong
Kowloon, Hong Kong
Fax: 852-2788-8561. email:matyang@math.cityu.edu.hk

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Abstract

The present paper concerns with the global structure and asymptotic behavior of the discontinuous solutions to flood wave equations. By solving a free boundary problem, we first obtain the global structure and large time behavior of the weak solutions containing two shock waves. For the Cauchy problem with a class of initial data, we use Glimm scheme to obtain a uniform BV estimate both with respect to time and the relaxation parameter. This yields the global existence of BV solution and convergence to the equilibrium equation as the relaxation parameter tends to 0.

1 Introduction

The motion of flood wave can be described by the following system of hyperbolic conservation laws with a relaxation term, in Eulerian coordinates,

$$\begin{cases} h_\tau + (hu)_\xi = 0, \\ (hu)_\tau + (hu^2 + \frac{1}{2}g'h^2)_\xi = \frac{g'hS - C_f u^2}{\epsilon}, \end{cases} \tag{1.1}$$

where $g' = gcos\alpha$, $S = tan\alpha$, with $0 < \alpha < \pi/2$, g is the gravitational acceleration, α is a constant representing the inclination angle of the river, $C_f > 0$ is the constant frictional coefficient, $h > 0$ and $u > 0$ are the depth and velocity of the water respectively, $\epsilon > 0$ is the small relaxation parameter, and τ and ξ are the time and space variables respectively. The detailed physical background of (1.1) can be found in [24].

Since it is more convenient to consider the system (1.1) in the Lagrangian coordinates, we use the following usual transformation $x = \int_{\xi(\tau)}^{\xi} h(y,\tau)dy$ and $t = \tau$, here $\xi(\tau)$ is an arbitrary particle path satisfying $\dot{\xi}(\tau) = u(\xi(\tau),\tau)$. Under this transformation, the system (1.1) becomes

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \frac{g'S - C_f u^2 v}{\epsilon}. \end{cases} \tag{1.2}$$

where $v = 1/h$, and $p(v) = \frac{1}{2}g'v^{-2}$. This system is strictly hyperbolic when $0 < v < \infty$ with two distinct characteristic speeds $\lambda_1(v) = -\sqrt{-p'(v)} = -\sqrt{g'}v^{-3/2}$, $\lambda_2(v) = \sqrt{-p'(v)} = \sqrt{g'}v^{-3/2}$, and two Riemann invariants

$$w(u,v) = u + m(v), \qquad z(u,v) = u - m(v), \tag{1.3}$$

with $m(v) = -2\sqrt{g'/v}$ satisfying $m'(v) = \lambda_2(v)$. When the relaxation term $g'S - C_f u^2 v$ vanishes, the system is in equilibrium and the equilibrium equation corresponding to (1.2) is given by

$$v_t - f(v)_x = 0, \tag{1.4}$$

where $f(v) = \pm\sqrt{\frac{g'S}{C_f v}}$ satisfying $g'S - C_f(f(v))^2 v = 0$. In the following, we consider the case when (v,u) is in a small neighborhood of a point on the equilibrium curve $u = \sqrt{\frac{g'S}{C_f v}}$, i.e., $f(v) = \sqrt{\frac{g'S}{C_f v}}$. It is expected that system (1.2), as $t \to \infty$ or $\epsilon \to 0$, is well approximated by equilibrium equation (1.4) provided the subcharacteristic condition $|f'(v)| < \sqrt{-p'(v)}$ holds. This subcharacteristic condition serves as the stability condition (see [24] and [16]), and it turns out to be very simple in the present situation, i.e.,

$$\tan\alpha = S < 4C_f, \tag{1.5}$$

which means the inclination angel is less than a critical value. The previous works on the systems with relaxation are mainly on the smooth solutions with small derivatives (cf. [16]). In general, if the derivatives of initial data are not small, discontinuities will develop in a finite time. Therefore, it is quite natural to study the discontinuous solutions. Actually, for the flood wave, the discontinuities satisfying Rankine-Hugoniout condition represent the turbulent bores, or " hydraulic jumps" in water wave theory (cf. [24]).

The purpose of present paper is to investigate the global structure and large time behavior of the discontinuous solutions with Riemann data, and the relaxation limit behavior (as $\epsilon \to 0$) of the BV solutions for a class of initial data containing only interactions of shock wave and rarefaction wave. Our study shows that the first signals and wavefronts travel with the characteristic or shock speeds of (1.2). But as the time becomes large, the main information travel with the characteristic speeds or shock speeds of the reduced equilibrium equation (1.4).

For the Riemman problem of a class of Riemann data, we will show that the solution has the piecewise smooth structure separating by shock discontinuities. Then the $(x,\ t)$ plane can be divided into the different regions. The qualitative information is obtained on the solutions in each region which gives a global picture of the solution. For the Cauchy problem with the initial data which has the structure of $R_1S_2R_1S_2\cdots$, here R_i and S_j denote the i-rarefaction wave and j-shock respectively, we will show that this structure will maintain for all time if the subcharacterisitc condition holds. This kind of phenomena was found in [29] for the isentropic gas dynamics, and generalized in [12] to the general 2×2 homogeneous hyperbolic systems of conservation laws. Here we show that this is still true in the presence of the relaxation due to the subcharacterisitc condition. This enables us to get a uniform BV estimate for the solutions of (1.2), and show that the limit (as $\epsilon\to 0$) of the solutions is indeed governed by equilibrium equation (1.4).

We now present the main results in the paper. Before that, let us define the shock waves for (1.2) as follows.

A discontinuity along $x=x_1(t)$ is called a 1-shock if the Rankine- Hugoniout condition

$$\begin{cases} \frac{dx_1(t)}{dt}=-\sqrt{-\frac{p(v_+)-p(v_-)}{v_+-v_-}}, \\ u_+-u_-=-\dot{x}_1(t)(v_+-v_-), \\ p(v_+)-p(v_-)=\dot{x}_1(t)(u_+-u_-), \end{cases} \tag{1.6}$$

and entropy condition

$$v_+(t)<v_-(t),$$

hold, where

$$(v_-(t),u_-(t))=(v,u)(x_1(t)-0,t),(v_+(t),u_+(t))=(v,u)(x_1(t)+0,t).$$

The 2-shock can be defined similarly , which is a discontinuity $x=x_2(t)$ satisfying the Rankine- Hugoniout condition

$$\begin{cases} \frac{dx_2(t)}{dt}=\sqrt{-\frac{p(v_+)-p(v_-)}{v_+-v_-}} \\ u_+-u_-=-\dot{x}_2(t)(v_+-v_-), \\ p(v_+)-p(v_-)=\dot{x}_2(t)(u_+-u_-) \end{cases} \tag{1.7}$$

and entropy condition

$$v_+(t)>v_-(t),$$

where

$$(v_-(t),u_-(t))=(v,u)(x_2(t)-0,t),(v_+(t),u_+(t))=(v,u)(x_2(t)+0,t).$$

First, let us consider system (1.2) with the Riemann data

$$(v(x,0),u(x,0)) = \begin{cases} (v_l,u_l), & -\infty < x < 0, \\ (v_r,u_r) & 0 < x < +\infty, \end{cases} \tag{1.8}$$

and assume these two states (v_l, u_l) and (v_r, u_r) to be connected by $S_1 S_2$ in the (v, u) phase plane (see [22]). That is, there exists an intermediate state (v_m, u_m) such that (v_m, u_m) is connected to (v_l, u_l) by a 1-shock wave, i.e.

$$u_m - u_l = -\sqrt{-(p(v_m) - p(v_l))(v_m - v_l)}, v_l > v_m. \tag{1.9}$$

and (v_r, u_r) is connected to (v_m, u_m) by a 2-shock wave ,

$$u_r - u_m = \sqrt{-(p(v_m) - p(v_r))(v_m - v_r)}, \qquad v_m < v_r. \tag{1.10}$$

When (v_r, u_r) and (v_l, u_l) are close enough, it is easy to check $v_m > 0, u_m > 0$.

The structure of the solutions for Riemann problem to the homogeneous system corresponding to (1.2) is well known(see [22]) where the solutions can be resolved into elementary waves with self-similar structure. Compared to the homogeneous system, the structure of solutions for the Riemann problem of (1.2) is more complicated since there is no self-similar solution in the form of $(v(x/t), u(x/t))$ due to the inhomogeneity. Nevertheless, we will show that the Riemann problem of (1.2) can also be resolved into elementary waves. In fact, the Riemann problem of (1.2) and (1.8) for $0 < t \leq T$ can be formulated as the following free boundary problem.

FBP: 1-shock discontinuity $x = x_1(t)$ issuing form (0,0) satisfying the Rankine-Hugoniot condition, entropy condition

$$v(x_1(t)-, t) > v(x_1(t)+, t)$$

and the initial condition

$$\lim_{t\to 0}(v, u))(x_1(t)-, t) = (v_l, u_l),\ \lim_{t\to 0}(v, u))(x_1(t)+, t) = (v_m, u_m);$$

while a 2-shock $x = x_2(t)$ issuing from $(0, 0)$ satisfying the Rankine-Hugoniout condition, entropy condition

$$v(x_2(t)-, t) < v(x_2(t)+, t)$$

and the initial condition

$$lim_{t\to 0}(v, u))(x_2(t)-, t) = (v_m, u_m),\ lim_{t\to 0}(v, u))(x_2(t)+, t) = (v_r, u_r).$$

The solution (v, u) is smooth in the region

$$S(T) = \{(x, t)|0 < t \leq T, x_1(t) \leq x \leq x_2(t)\}.$$

In the outer region $O_1(T) = \{(x,t)|0 \le t \le T, -\infty < x < x_1(t)\}$, the solution is completely determined by the initial left state (v_l, u_l) because of the entropy condition. Similarly, the solution in $O_2(T) = \{(x,t)|0 \le t \le T, x_2(t) < x < \infty\}$ is completely determined by (v_r, u_r). In the following, for simplicity, we set $g' = 1$. Therefore,

$$f(v) = \sqrt{\frac{S}{C_f v}}.$$

It is easy to check that the solution in $O_1(T)$ is given by

$$(v,u)(x,t) = (v_l, u^l(x,t)) = (v_l, \sqrt{\frac{S}{C_f v_l}} \frac{1+y^l}{1-y^l}), \; x < x_1(t), \tag{1.11}$$

where

$$y^l = \frac{u_l\sqrt{C_f v_l} - \sqrt{S}}{u_l\sqrt{C_f v_l} + \sqrt{S}} \exp\left(-2\frac{\sqrt{SC_f v_l}}{\epsilon}t\right). \tag{1.12}$$

Actually, the solution (v,u) in $O_1(T)$ can be obtained by solving the following initial value problem of the system of ODEs.

$$v_t = 0, u_t = \frac{1}{\epsilon}(S - C_f u^2 v),$$

$$(v,u)|_{t=0} = (v_l, u_l).$$

Similarly, the solution in $O_2(T)$ is given by

$$(v,u)(x,t) = (v_r, u^r(x,t)) = (v_r, \sqrt{\frac{S}{C_f v_l}} \frac{1+y^r}{1-y^r}), \; x > x_2(t), \tag{1.13}$$

where

$$y^r = \frac{u_r\sqrt{C_f v_r} - \sqrt{S}}{u_r\sqrt{C_f v_r} + \sqrt{S}} \exp\left(-2\frac{\sqrt{SC_f v_r}}{\epsilon}t\right). \tag{1.14}$$

It follows from (1.11) and (1.13) that

$$|u^l(x,t) - f(v_l)| \le O(1)|u_l - f(v_l)| \exp\left(-\frac{\sqrt{SC_f v_l}}{\epsilon}t\right), \; x < x_1(t) \tag{1.15}$$

and

$$|u^r(x,t) - f(v_r)| \le O(1)|u_r - f(v_r)| \exp\left(-\frac{\sqrt{SC_f v_r}}{\epsilon}t\right), \; x > x_2(t). \tag{1.16}$$

Here and in the following, we use $O(1)$ to denote a generic positive bounded quantity independent of ϵ and t. (1.15) and (1.16) indicate that, in the outer region $O_i(T)$, $i = 1, 2$, the soltion (v,u) approaches to the equilibrium state $v = f(u)$ exponentially fast in $\frac{t}{\epsilon}$.

The local existence of the above free boundary problem is a simple corollary of Li and Yu's general theorem on quasilinear hyperbolic systems ([13]). In order to extend the local solution for all time, we need to establish a uniform C^1-estimate in the regions $S(T)$ defined above for $T > 0$. This will be carried out in Sections 2 and 3 by the observation that the subcharacteristic condition forces the discontinuity of the solution and its derivatives decay exponentially with respect to time. Thus, as $t \to \infty$, the solution of the free boundary problem will approach to a continuous function. Notice that the large time asymptotic state depends on the relationship between v_l and v_r. If $v_l > v_r$, the Riemann solution of equilibrium equation (1.4) with the Riemann data (v_l, v_r) is a rarefaction wave, which is expected to be the asymptotic state of the solution to the Riemann problem (1.2) and (1.8). We will not discuss this case here and leave it for the future investigation. In this paper, we only consider the case when

$$v_l < v_r. \tag{1.17}$$

In this case, the Riemann solution to the equilibrium equation (1.4) with the Riemann data (v_l, v_r) is a shock wave, and there is a shock profile which is the travelling wave solution of (1.2) in the form $(V, U)(y)$ with $y = (x - \sigma t)$ and $\sigma = -(f(v_r) - f(v_l)/(v_r - v_l)$ satisfying

$$\begin{cases} -\sigma V_y - U_y = 0, \\ -\sigma U_y + p(V)_y = \frac{g'S - C_f U^2 V}{\epsilon}, \end{cases} \tag{1.18}$$

$$(V, U)(-\infty) = (v_l, u_l), \qquad (V, U)(\infty) = (v_r, u_r). \tag{1.19}$$

This shock profile is shown to be the asymptotic state of the solution to the Riemann problem (1.2) and (1.8) under the condition (1.17). In fact, the subcharacteristic condition gives the existence and uniqueness up to a shift of the travelling wave solution ([16]). For the general 2×2 hyperbolic system with relaxation, it was shown ([16]) that the travelling wave is nonlinearly stable provided the subcharacteristic condition holds and the initial data are the small and smooth perturbation of the shock profile. In our present situation, although the initial data are discontinuous, the exponential decay of discontinuities leads to the convergence of the solution to the same smooth profile. Precisely, we will show that, as $t \to \infty$, the solution to the Riemann problem (1.2) and (1.8) will approach to the travelling wave with the shift x_0 which is determined by

$$\int (v(x, 0) - V(x + x_0))dx = 0. \tag{1.20}$$

Here and in the following, the integral is over the whole real line if not specified. It turns out that

$$x_0 = \int (v_0(x) - V(x))dx/(v_r - v_l). \tag{1.21}$$

The conservative form of $(1.2)_1$ implies

$$\int (v(x, t) - V(x + x_0 - \sigma t))dx = 0, \tag{1.22}$$

for all $t > 0$, here $v(x,t)$ is the solution to the Riemann problem (1.2) and (1.8).

Now we can state the Theorem 1.1 in the following, in which (V, U) is the travelling wave solution of (1.18) and (1.19), $[\ell](x(t))$ denotes the jump of the function ℓ along a curve $x = x(t)$, i.e., $[\ell](x(t) = \ell(x(t)+, t) - \ell(x(t)-, t)$.

Theorem 1.1. *(Structure and asymptotic behavior of the solutions to the Riemann problem) If $|v_r - v_l| + |u_r - f(v_r)|$ is small enough ($v_l \le v_r$), and the subcharacteristic condition (1.5) holds, then there exists a global smooth solution to the above (FBP) for any $T > 0$ in $S(T)$. Moreover, we have the following estimates:*
Along the shocks $x = x_i(t)(i = 1, 2)$,

$$|[u](x_i(t)| + |[v](x_i(t)| + |[u_x](x_i(t)| + |[v_x](x_i(t)| \le O(1)|v_r - v_l| \exp(-\alpha t), \tag{1.23}$$

for some $\alpha > 0$. Furthermore,

$$lim_{t\to\infty} sup_{x_1(t)\le x\le x_2(t)}(|v(x,t) - V(x + x_0 - \sigma t)| + |u(x,t) - U(x + x_0 - \sigma t)|) = 0. \tag{1.24}$$

We make the following remarks concerning Theorem 1.1 and its proof.

Remark 1. The decay estimate (1.23) of the shock strength plays a crucial role in our analysis because it provides the information of solution to our FBP, and serves a sort of boundary condition. The complicated structure of the source term makes it hard to get this decay estimate for (1.2), compared with that for the Euler equations with damping, as did by Hsiao and Tang in [6], where this decay property is easy to see from the Rakine-Hugoniot condition. To get this decay estimate for our system, we derive an ODE for the jumps along the shock curve in (x, t) plane. Moreover, to obtain the large time behavior, the decay estimate of the jump of derivatives of solutions is needed, this is a new estimate, compared with the estimate for the damping case in [6] where the estimate in derivatives is not necessary.

Remark 2. 2. Unlike the previous works for the smooth solution to the system with relaxation (cf. [11] and [20]) or the solution to the Riemann problem of the system with damping (cf. [6]) , where solely a characteristic or a energy method is used, a combination of these two methods is used in our case to get the uniform estimate of the solution. Actually, it seems to us neither a pure energy method nor a pure characteristic alone is enough to close our argument. The byproduct of this combination method is that the global existence and large time behavior of solutions are obtained at the same time. A key step in the estimate via the characteristic method is to decouple the equations governing the derivatives of two Riemann invariants. For this, we derive and solve a system of linear partial differential equations in (u, v) phase plane (see (2.38) and (2.39)) with the given data on the equilibrium curve $u = f(v)$.

As a corollary of Theorem 1.1, we have the following Theorem.

Theorem 1.2. *Let* $(v,\ u)$ *be the solutions of Riemann problem (1.2) and (1.8) as stated in Theorem 1.1,* (V, U) *be the travelling wave solution of (1.18) and (1.19). Then we have*

$$lim_{t\to\infty} sup_{-\infty<x<+\infty}(|v(x,t) - V(x + x_0 - \sigma t)| + |u(x,t) - U(x + x_0 - \sigma t)|) = 0. \quad (1.25)$$

As the second main result in the paper, we consider the Cauchy problem of (1.2) with the initial data

$$(v, u)(x, 0) = (v_0, u_0)(x), \quad (1.26)$$

which contains the states to be connected by a rarefaction wave and a shock wave in the sense stated later. For the general Cauchy problem, the existence of the BV solution via Glimm's method to the hyperbolic system with dissipation is always a hard problem, due to the fact that the local behavior and the large time asymptotic behavior are in general different. The structure of system (1.2) is only partially dissipative, compared with that considered by Dafermos and Hsiao in [6], where the dissipation is complete. In an interesting paper by Dafermos ([5]), a global existence of the BV solution to the system with damping is proved for the case when the two end states at $x = \pm\infty$ are the same. In the present paper, we consider a class of initial data which contain the interaction of shock and rarefaction waves, and the two end states at $x = \pm\infty$ are different. The global existence of BV solutions via a modified Glimm scheme for the p-system with relaxation or the wave-front tracking method was obtained in [18] and [1] respectively, for the pressure function $p(v) = 1/v$. For this pressure function, the geometry of shock curves in the phase plane of Riemann invariants is very special, i.e., the shock curves are parallel (cf. [21]). System (1.2) does not have this property. Another type of inhomogeneous hyperbolic system with source term was studied in [1], [14] and [15] by using the wave tracking method or Glimm scheme. For this type of system, the source term is required in L^1. System (1.2) can not enter in this framework.

The solution of Cauchy problem (1.2) and (1.26) is constructed by a modified Glimm's scheme introduced in [6]. At first, we select a space mesh-length r and a time mesh-length s satisfying the CFL condition

$$r/s > \max_{x\in R^1, t\geq 0} \lambda_2. \quad (1.27)$$

Let $\alpha_0,\ \alpha_1,\ \cdots, \alpha_n, \cdots$ be an equidistributed sequence of random number in $(-1, 1)$. After partitioning the upper half of the (x, t) plane into strips $T_n = \{(x, t) : -\infty < x < +\infty, ns \leq t < (n + 1)s\}$, $n = 0,\ 1,\ 2,\ \cdots$, we initiate the construction of the approximate solutions (u^s, v^s) by letting

$$(u^s(x, 0-),\ v^s(x, 0-)) = (u_0(x),\ v_0(x)). \quad (1.28)$$

Assuming that $(u^s,\ v^s)$ has already been determined on $\cup_{j=0}^{n-1} T_j$, we extend (u^s, v^s) to T_n as the admissible solution of the Cauchy problem

$$\begin{aligned} v_t - u_x &= 0, \\ u_t + p(v)_x &= 0, \end{aligned} \tag{1.29}$$

with the initial data at $t = ns$ as

$$\begin{aligned} u^s(x, ns) = u^r + \frac{s(g'S - C_f(u^s)^2 v^s)}{\epsilon}\left((m+\alpha_n)r, ns-\right), \\ v^s(x, ns) = v^s\left((m+\alpha_n)r, ns-\right), \end{aligned} \tag{1.30}$$

for $(m-1)r < x < (m+1)r$, $m+n$ odd. The Riemann problem to (1.29) can be resolved into shock waves and rarefaction waves. In (w, z) phase plane, the 1-rarefaction wave and 2-rarefaction wave curves starting from the state $(\bar{w}, \bar{z})$ are the sets of all the states satisfying

$$R_1 : \ z = \bar{z}, w \geq \bar{w}, \tag{1.31}$$

and

$$R_2 : \ w = \bar{w}, z \geq \bar{z}. \tag{1.32}$$

And the 1-shock wave and 2-shock wave curves starting from the state $(\bar{w}, \bar{z})$ are the sets of all the states satisfying

$$S_1 : \ z - \bar{z} = g_1(\bar{v}, \ w - \bar{w}), w \leq \bar{w} \tag{1.33}$$

$$S_2 : \ w - \bar{w} = g_2(\bar{v}, \ z - \bar{z}), z \leq \bar{z}, \tag{1.34}$$

where $\bar{v} = (4/(\bar{z} - \bar{w}))^2$ from (1.3). $Y = g_1(\bar{v}, X)$ is a function defined for $X \leq 0$ parameterized by α:

$$\begin{aligned} X &= -\frac{1}{2\bar{v}}\left(\sqrt{(\alpha-1)(\alpha^2-1)/\alpha} + \sqrt{2}(\alpha-1)\right) \\ Y &= -\frac{1}{2\bar{v}}\left(\sqrt{(\alpha-1)(\alpha^2-1)/\alpha} - \sqrt{2}(\alpha-1)\right), \end{aligned} \tag{1.35}$$

for $\alpha \geq 1$. While $Y = g_2(\bar{v}, X)$ is a function defined for $X \leq 0$ parameterized by α:

$$\begin{aligned} X &= -\frac{1}{2\bar{v}}\left(\sqrt{(1-\alpha)(1-\alpha^2)/\alpha} + \sqrt{2}(1-\alpha)\right) \\ Y &= -\frac{1}{2\bar{v}}\left(\sqrt{(1-\alpha)(1-\alpha^2)/\alpha} - \sqrt{2}(1-\alpha)\right), \end{aligned} \tag{1.36}$$

for $0 < \alpha \leq 1$. The functions g_1 and g_2 have the following properties:

$$0 \leq \frac{\partial g_i(\bar{v}, X)}{\partial X} < 1, \quad \frac{\partial^2 g_i(\bar{v}, X)}{\partial X^2} \leq 0, \quad g_i(\bar{v}, 0) = \frac{\partial g_i(\bar{v}, X)}{\partial X}|_{X=0} = 0 \tag{1.37}$$

for $\bar{v} > 0$, $X \leq 0$ and $i = 1, 2$, cf. [22].

We make the following three assumptions on the initial data $(v_0, u_0)(x)$, where $(w_0, z_0)(x)$ is the pair of the corresponding Riemann invariants.
A1: There exist positive constants M_1 and M_2 such that

$$M_1 < v_0(x) \leq M_2, \qquad |u_0(x)| \leq M_2, \tag{1.38}$$

for $-\infty < x < +\infty$, the total variation of u_0 is bounded;
A2: For any $x_1 < x_2$,

$$w_0(x_2) \geq w_0(x_1) + g_2\left(v_0(x_1), z_0(x_2) - z_0(x_1)\right), z_0(x_1) \geq z_0(x_2). \tag{1.39}$$

The above assumptions imply (cf. [22])

$$v_0(x_1) \leq v_0(x_2), \text{ for any } x_1 < x_2. \tag{1.40}$$

Let

$$\underline{v} = \lim_{x \to -\infty} v_0(x), \ \tilde{v} = \lim_{x \to +\infty} v_0(x)$$

and

$$\underline{u} = \lim_{x \to -\infty} u_0(x), \ \tilde{u} = \lim_{x \to +\infty} u_0(x)$$

The third assumption is
A3:

$$\underline{u} - f(\underline{v}) = 0. \tag{1.41}$$

Based on the above assumptions on the initial data, the Riemann problems in each building block can be resolved into R_1 and S_2. That is, in any T_n, the Riemann solutions have the same structure as in T_0. This enables us to get a uniform total variational estimate on the approximation sequence $\{u^s, v^s\}$, and to have a subsequence of $\{u^s, v^s\}$ converging almost everywhere to a function, denoted by (u^ϵ, v^ϵ), which is an entropy solution to the Cauchy problem (1.2) with the initial data (1.26). These results are stated in the following two theorems, in which (w^s, z^s) and (w^ϵ, z^ϵ) are the Riemann invariants corresponding to (u^s, v^s) in the scheme and (u^ϵ, v^ϵ) for (1.2) respectively, and $T.V.$ denotes the total variation.

Theorem 1.3. *Suppose the initial data (u_0, v_0) satisfy the assumptions A1, A2 and A3. There exist positive numbers δ and λ, such that if*

$$|\tilde{v} - \underline{v}| + T.V.(u_0) \leq \delta, \text{ and } 0 < s < \lambda\epsilon, \tag{1.42}$$

then the approximate solutions $(u^s, v^s)(x, t)$ can be constructed for all $t \geq 0$. For any $t \geq 0$, and $x_1 < x_2$ the two states $(u^s, v^s)(x_1, t)$ and $(u^s, v^s)(x_2, t)$ can be connected by a 1-rarefaction wave R_1 and a 2-shock wave S_2, that is,

$$w^s(x_2, t) \geq w^s(x_1, t) + g_2\left(v^s(x_1, t), z^s(x_2, t) - z(x_1, t)\right), \quad z^s(x_1, t) \geq z^s(x_2, t). \tag{1.43}$$

Moreover

$$\sup_{x\in R^1} |u^s - f(v^s)|(x,t) \le C\{|\tilde{v} - \underline{v}| + T.V.(u_0)\}, \tag{1.44}$$

and

$$\underline{v} \le v^s(x_1,t) \le v^s(x_2,t) \le \tilde{v}, \ \text{for } x_1 < x_2, \tag{1.45}$$

$$T.V.(u^s)(\cdot,t) \le O(1)\delta, \tag{1.46}$$

here C is a positive constant independent of s, t and ϵ.

Remark 3. The CFL condition in (1.27) takes the form

$$r/s > \lambda_2(\underline{v}).$$

And $|u_0 - f(v_0)|$ is small due to (1.41) and (1.42).

Based on this theorem, we can choose a subsequence of $\{u^s, v^s\}(x,t)$ converging almost everywhere to a function, denoted by $(u^\epsilon, v^\epsilon)(x,t)$. The limiting function $(u^\epsilon, v^\epsilon)(x,t)$ is indeed an entropy solution to the Cauchy problem (1.2) with the initial data (1.26).

Theorem 1.4. *$(u^\epsilon, v^\epsilon)(x,t)$ is a weak solution to the Cauchy problem (1.2) with the initial data (1.26). It satisfies the following entropy condition*

$$\partial_t \eta(u^\epsilon, v^\epsilon) + \partial_x q(u^\epsilon, v^\epsilon) - \frac{1}{\epsilon}\eta_{u^\epsilon}(u^\epsilon, v^\epsilon)(g'S - C_f(u^\epsilon)^2 v^\epsilon) \le 0, \tag{1.47}$$

in the sense of distribution, for any convex entropy-entropy flux pair $(\eta(u,v), q(u,v))$ of (1.29) satisfying $q_v = \eta_u p'(v), \ q_u = -\eta_v$. Moreover, (v^ϵ, u^ϵ) satisfies the following estimates:

$$\underline{v} \le v^\epsilon(x_1,t) \le v^\epsilon(x_2,t) \le \bar{v}, \ \text{for any } x_1 \le x_2, \ t \ge 0, \tag{1.48}$$

$$\sup_{x\in R^1} |u^\epsilon(x,t)| + T.V.(u^\epsilon) \le C_1 \ \text{for any } t \ge 0, \tag{1.49}$$

for some constant C_1 independent of t and ϵ.

The estimates (1.48) and (1.49) imply that a subsequence of $\{(u^\epsilon, v^\epsilon)(x,t)\}$ (still denoted by $\{(u^\epsilon, v^\epsilon)(x,t)\}$) can be chosen to converge almost everywhere to a function $(v,u)(x,t)$. An argument as in [18] leads to the following theorem.

Theorem 1.5. *$v(x,t)$ is the weak solution of the equilibrium equation (1.4) with the initial data $v(x,0) = v_0(x)$ and satisfies the entropy condition*

$$\Phi(v)_t + \Psi(v)_x \le 0, \tag{1.50}$$

in the sense of distribution for any convex entropy-entropy pairs (Φ, Ψ) with $\Psi'(v) = -\Phi'(v)f'(v)$ and $\Phi''(v) \ge 0$. Moreover,

$$u(x,t) = f(v)(x,t), \quad \text{as } t > 0, \ a.e.$$

Notice that all of the above results are based on the stability condition (1.5). If this condition is violated, which means the inclination angle exceeds or equal to the critical value, it is pointed out in [24] that the resulting flow is not necessarily completely chaotic or without structure. In favorable circumstances, it takes the form of "roll wave" with a periodic structure of discontinuous bores separated by smooth profiles. This case is considered in([10]) for system(1.2) with artificial viscosity and the weakly nonlinear limit is verified, where the underlying relaxation system is reduced to the Burgers equation with a source term, cf [10]. Such a limit is justified in ([10]) by using the energy method.

Relaxation problem attracts much attention in the recent years. A semilinear model proposed by Jin and Xin ([11]) has been extensively studied (cf. [11], [19], [27]). The Riemann problem of a modified Broadwell model with self-similar structure was investigated in ([7]). An interesting quasilinear model of gas dynamics was investigated in [28]. For the boundary layer problem, the readers can refer to [23], [25] and [20]. The general setting can be found in [3].

The rest of the paper is organized as follows. In Section 2, we use the characteristic method to get some estimates based on *a priori* assumption that v has positive lower and upper bounds. This assumption will be verified by the energy method in Section 3. These estimates enable us to obtain the large time behavior of the solution. Finally, Section 4 is devoted to the study of the Cauchy problem.

2 Estimate via Characteristic Method

In this and the next sections , we study the problem with the fixed ϵ, thus, we may let $\epsilon = 1$ in these two sections. In the following, we always use w and z to denote the Riemann invariants defined in (1.3). The free boundary problem has the boundaries 1-shock $x = x_1(t)$ and 2-shock $x = x_2(t)$. For our purpose, a careful analysis of the behavior of solutions on the boundaries is needed and it depends on the decay estimates on the solutions along the shock curves $x = x_i(t)$, $i = 1, 2$.

First, the Rankine - Hugoniot condition (1.7) gives us the following relation between $w_x(x_2(t)-)$ and $z_x(x_2(t)-)$ along the shock curve $x = x_2(t)$.

Lemma 2.1. *Along the 2-shock curve $x = x_2(t)$, it holds that*

$$(\dot{x}_2(t) + \lambda_2^-)^3[w_x] - (\dot{x}_2(t) - \lambda_2^-)^3[z_x] = -4\dot{x}_2(t)\lambda_2^- C_f[u^2v], \tag{2.1}$$

where $\lambda_2^{\pm} = \lambda_2(x_2(t)\pm, t)$, $w_x^{\pm} = w_x(x_2(t)\pm, t)$ and $z_x^{\pm} = z_x(x_2(t)\pm, t)$. In this lemma and its proof, we use $[\cdot]$ to denote the jump of a function along $x = x_2(t)$, e.g. $[\ell] = \ell(x_2(t)+, t) - \ell(x_2(t)-, t)$.

Proof. Along 2- shock $x = x_2(t)$, we have

$$-[u]^2 = [p(v)][v], \qquad \dot{x}_2(t)[v] = -[u]. \tag{2.2}$$

Differentiating the first equation in (2.2) with respect to t along the shock, we have

$$
\begin{aligned}
&3\dot{x}_2(t)\big([w_x]+[z_x]\big)+3\dot{x}_2(t)\big([\lambda_2 w_x]-[\lambda_2 z_x]\big)\\
&+\dot{x}_2(t)^3[w_x/\lambda_2]-\dot{x}_2(t)^3[z_x/\lambda_2]+[\lambda_2^2 w_x]+[\lambda_2^2 z_x]\\
&=-4\dot{x}_2(t)C_f[u^2v],
\end{aligned}
\tag{2.3}
$$

where we use $u_x=(w_x+z_x)/2$ and $v_x=(w_x-z_x)/(2m'(v))=(w_x-z_x)/(2\lambda_2)$. Notice that $w_x(x_2(t)+,t)=z_x(x_2(t)+,t)=0$ in view of (1.13), (2.3) implies (2.1) by some rearrangements. □

With the same proof of the above lemma 2.1, we have

Lemma 2.2. *Along the 1-shock curve $x=x_1(t)$, it holds that*

$$
\begin{aligned}
&(\dot{x}_1(t)+\lambda_2(x_1(t)+,t))^3[w_x]|_{x_1(t)}-(\dot{x}_1(t)-\lambda_2(x_1(t)+,t))^3[z_x]|_{x_1(t)}\\
&=-4\dot{x}_1(t)\lambda_2(x_1(t)+,t)C_f[u^2v]_{x=x_1(t)},
\end{aligned}
\tag{2.4}
$$

here $[\cdot]_{x_1(t)}$ is the jump of a function along $x=x_1(t)$, e.g., $[\ell]_{x_1(t)}=\ell(x_1(t)+,t)-\ell(x_1(t)-,t)$.

Remark 4. Since $w_x(x_2(t)+,t)=z_x(x_2(t)+,t)=0$ and $w_x(x_1(t)-,t)=z_x(x_1(t)-,t)=0$ (cf. (1.11) and (1.13)) (2.1) and (2.4) imply

$$
\begin{aligned}
&-(\dot{x}_2(t)+\lambda_2(x_2(t)-,t))^3w_x(x_2(t)-,t)+(\dot{x}_2(t)-\lambda_2(x_2(t)-,t))^3z_x(x_2(t)-,t)\\
&=-4\dot{x}_2(t)\lambda_2(x_2(t)-,t)C_f[u^2v]_{x_2(t)},
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
&(\dot{x}_1(t)+\lambda_2(x_2(t)+,t))^3w_x(x_1(t)+,t)-(\dot{x}_1(t)-\lambda_2(x_1(t)+,t))^3z_x(x_2(t)+,t)\\
&=-4\dot{x}_2(t)\lambda_2(x_2(t)+,t)C_f[u^2v]_{x_1(t)}.
\end{aligned}
\tag{2.6}
$$

The following lemma gives the decay estimates along the shock curve $x=x_2(t)$.

Lemma 2.3. *Along the 2-shock $x=x_2(t)$, if $|[v]|_{x_2(t)}|$, $|w_x^-(t)|+|z_x^-(t)|=:|w_x(x_2(t)-,t)|+|z_x(x_2(t)-,t)|$ and $|u_r-f(v_r)|$ are small enough, then it holds that*
(i) there exists a function $k_1(t)$ with the uniform positive lower bound α i.e. $k_1(t)\geq\alpha>0$ such that

$$
[u](t)=:u(x_2(t)+,t)-u(x_2(t)-,t)=(u_r-u_m)\exp(-\int_0^t k_1(s)ds), \tag{2.7}
$$

$$
[v](t)=:v(x_2(t)+,t)-v(x_2(t)-,t)=(v_r-v_m)\frac{\dot{x}_2(0)}{\dot{x}_2(t)}\exp(-\int_0^t k_1(s)ds). \tag{2.8}
$$

(ii)

$$
\begin{aligned}
&|\frac{du^-(t)}{dt}|+|\frac{dv^-(t)}{dt}|\\
&\leq O(1)(|v_r-v_m|+|u_r-f(v_r)|)e^{-\gamma t}
\end{aligned}
\tag{2.9}
$$

for some $\gamma>0$, where $u^-(t)=u(x_2(t)-,t),\qquad v^-(t)=v(x_2(t)-,t)$.

Remark 5. It follows from (2.8) that

$$0 < v_r - v(x_2(t)-, t) \le (v_r - v_m)\exp(-\alpha t), \tag{2.10}$$

for $0 \le t \le T$. This means the discontinuity will not disappear in any finite time, but decay exponentially.

Proof of Lemma 2.3. We use $[\cdot]$ to denote the jump along the 2-shock curve $x = x_2(t)$. Differentiating $(1.7)_3$ with respect to t, we obtain

$$\frac{d^2x_2(t)}{dt^2}[u] + \dot{x}_2(t)\frac{d[u]}{dt} = \frac{d[p(v)]}{dt}. \tag{2.11}$$

On the other hand, by virtue of (1.2) and (1.7), we have

$$\begin{aligned}
&\frac{d[p(v)]}{dt}\\
&= [p(v)]_t + \dot{x}_2(t)[p(v)]_x\\
&= [p(v)]_t - \dot{x}_2(t)(C_f[u^2v] + [u]_t)\\
&= [p(v)]_t - \dot{x}_2(t)(C_f[u^2v] + \frac{d[u]}{dt} - \dot{x}_2(t)[u]_x)\\
&= [p'(v)]u_x^- + (p'(v_r) + \dot{x}_2^2(t))[u_x] - \dot{x}_2(t)C_f[u^2v] - \dot{x}_2(t)\frac{d[u]}{dt}.
\end{aligned} \tag{2.12}$$

(1.13) and (1.16) imply

$$\begin{aligned}
&[u^2v] = [u^2]v^+ + u_-^2[v]\\
&= \{(u^+ + u^-)v^+ - \frac{(u^-)^2}{\dot{x}_2(t)}\}[u]\\
&= \{(2u^r v_r - \frac{(u^r)^2}{\sqrt{-p'(v_r)}}\}[u] + a_1(v^+, v^-)[u]\\
&= \{(2f(v_r)v_r - \frac{(f(v_r))^2}{\sqrt{-p'(v_r)}}\}[u] + a_1(v^+, v^-)[u]\\
&+ k_2(u_r - f(v_r))exp(-\sqrt{SC - fv_r}t)[u]\\
&= \frac{2S}{C_f u_r}(1 - \sqrt{\frac{S}{4C_f}})[u] + a_1(v^+, v^-)[u] + a_2(u_r - f(v_r))exp(-\sqrt{SC - fv_r}t)[u],
\end{aligned} \tag{2.13}$$

where $|a_1(v^+, v^-)| \le O(1)|v^+ - v^-|$ and $|a_2(u_r - f(v_r))| \le O(1)|u_r - f(v_r)|$. Thus, we arrive at

$$2\dot{x}_2(t)\frac{d([u])}{dt} = -\dot{x}_2(t)\frac{2S}{u_r}(1 - \sqrt{\frac{S}{4C_f}})[u] - \frac{d^2x_2(t)}{dt^2}[u] + I, \tag{2.14}$$

where $|I| \le O(1)(|u_x^-| + |u_r - f(v_r)|)[u]$. We estimate $\frac{d^2x_2(t)}{dt^2}$ as follows. Differentiate $(1.7)_1$ with respect t along $x = x_2(t)$ we get

$$-\frac{d^2x_2(t)}{dt^2}[v] - \dot{x}_2(t)(u_x^- + \dot{x}_2(t)v_x^-) = \frac{d[u]}{dt}. \tag{2.15}$$

By virtue of (2.11), (2.12) and (2.15) using the fact $\dot{x}_2(t)[v] = -[u]$, we get

$$\begin{aligned} & 3\frac{d^2x_2(t)}{dt^2}[u] \\ & = [p'(v)]u_x^- + (p'(v_r) + \dot{x}_2^2(t))[u_x] \\ & - \dot{x}_2(t)C_f[u^2v] + 2(\dot{x}_2(t))^2(u_x^- + \dot{x}_2(t)v_x^-). \end{aligned} \tag{2.16}$$

Obviously,

$$\begin{aligned} & (u_x^- + \dot{x}_2(t)v_x^-) \\ & = \frac{1}{2}(1 + \frac{\dot{x}_2(t)}{\lambda_2(v^-)})w_x^- + \frac{1}{2}(1 - \frac{\dot{x}_2(t)}{\lambda_2(v^-)})z_x^- \end{aligned} \tag{2.17}$$

and

$$|1 - \frac{\dot{x}_2(t)}{\lambda_2(v^-)}| \le O(1)|v_r - v_m|. \tag{2.18}$$

Moreover, it follows form (2.5) that

$$\begin{aligned} & |w_x^- - \frac{4\dot{x}_2(t)\lambda_2(v^-)}{(\dot{x}_2(t) + \lambda_2(v^-))^3}C_f[u^2v]| \\ & \le O(1)|z_x^-||[v]|^3 \le O(1)|z_x^-||[u]|^3. \end{aligned} \tag{2.19}$$

Here we have used the fact that $|\dot{x}_2(t) - \lambda_2(v^-)| \le O(1)|[v]|$. Use this fact again, (2.19) becomes

$$|w_x^- - \frac{C_f}{2\dot{x}_2(t)}[u^2v]| \le O(1)|z_x^-||[u]|^3 + O(1)[u]^2. \tag{2.20}$$

(2.17) and (2.20) imply

$$\frac{dv^-(t)}{dt} = u_x^- + \dot{x}_2(t)v_x^- = \frac{C_f}{2\dot{x}_2(t)}[u^2v] + II, \tag{2.21}$$

where

$$|II| \le O(1)(|z_x^-||[u]|^3 + [u]^2 + |w_x^-||[u]|).$$

It follows from (2.16) and (2.21) that

$$\frac{d^2x_2(t)}{dt^2}[u] = [p'(v)]u_x^- + (p'(v_r) + \dot{x}_2^2(t))[u_x] + II, \tag{2.22}$$

where II is the same as that in (2.21). Since $|[p'(v)]| \le O(1)|[u]|$, $|p'(v_r)+\dot{x}_2^2(t)| \le O(1)|[v]| = O(1)|[u]|$, we obtain, from (2.22) that,

$$|\frac{d^2x_2(t)}{dt^2}| \le O(1)(|u_x(x_2(t)-,t)| + |v_x(x_1(t)-,t)| + |[u]|). \tag{2.23}$$

Combining the above estimates together, by virtue of the fact $1 - \sqrt{\frac{S}{4C_f}} > 0$ due to the subcharacteristic condition, (2.7) follows then. (2.8) is obtained by the Rankine-Hugoniout condition. (2.7), (2.8) and (2.14) imply (2.9).

By the same proof of the above lemma, we have the following decay estimate along 1-shock $x = x_1(t)$.

Lemma 2.4. *Along the 1-shock $x = x_1(t)$, if $|[v]|_{x_1(t)}|$, $|w_x(x_1(t)+,t)| + |z_x(x_1(t)+,t)|$ and $|u_l - f(v_l)|$ are small enough, then there exists a positive number γ such that*

$$\begin{aligned} &|u(x_1(t)+,t) - u(x_1(t)-,t)| + |v(x_1(t)+,t) - v(x_1(t)-,t)| \\ &\le O(1)(|v_m - v_l| + |u_l - f(v_l)) \exp(-\gamma t), \end{aligned} \tag{2.24}$$

$$\begin{aligned} &|\frac{du^+(t)}{dt}| + |\frac{dv^+(t)}{dt}| \\ &\le O(1)(|v_m - v_l| + |u_l - f(v_l)|)e^{-\gamma t} \end{aligned} \tag{2.25}$$

for some $\gamma > 0$, where $u^+(t) = u(x_1(t)+,t), \qquad v^+(t) = v(x_1(t)+,t)$.

(2.5), (2.6) and lemmas 2.3, 2.4 give the following estimates immediately

$$|w_x(x_2(t)-,t)| \le O(1)|v_r - v_m|e^{-\gamma t}(|z_x(x_2(t)-,t)| + 1), \tag{2.26}$$

$$|z_x(x_1(t)+,t)| \le O(1)|v_m - v_l|e^{-\gamma t}(|w_x(x_1(t)+,t)| + 1). \tag{2.27}$$

We turn to the estimates of the solutions in the region where the solution is smooth. System (1.2) can be written as

$$\begin{cases} w_t + \lambda_1 w_x = F(u,v), \\ z_t + \lambda_2 z_x = F(u,v), \end{cases} \tag{2.28}$$

wherever the solution is smooth , here and in the following $F(u,v) = S - C_f u^2 v$.
Using the notation

$$\frac{d^-}{dt} = \frac{\partial}{\partial t} + \lambda_1 \frac{\partial}{\partial x}, \qquad \frac{d^+}{dt} = \frac{\partial}{\partial t} + \lambda_2 \frac{\partial}{\partial x},$$

the above equation becomes

$$\begin{cases} \frac{d^- w}{dt} = F(u,v), \\ \frac{d^+ z}{dt} = F(u,v), \end{cases}$$

It is easy to check

$$\frac{d^- v}{dt} = z_x, \frac{d^+ v}{dt} = w_x. \tag{2.29}$$

Also

$$\begin{aligned} \frac{d^- u}{dt} &= \frac{1}{2}(w_t + \lambda_1 w_x) + \frac{1}{2}(z_t + \lambda_1 z_x) \\ &= \frac{1}{2}(w_t + \lambda_1 w_x) + \frac{1}{2}(z_t + \lambda_2 z_x) + \frac{1}{2}(\lambda_1 - \lambda_2) z_x \\ &= F(u,v) + \lambda_1 z_x. \end{aligned} \tag{2.30}$$

Similarly

$$\frac{d^+ u}{dt} = F(u,v) + \lambda_2 w_x. \tag{2.31}$$

Therefore, for any smooth function $q(u,v)$, it holds that

$$\frac{d^- q(u,v)}{dt} = (F(u,v) + \lambda_1 z_x) q_u + q_v z_x, \tag{2.32}$$

and

$$\frac{d^+ q(u,v)}{dt} = (F(u,v) + \lambda_2 w_x) q_u + q_v w_x \tag{2.33}$$

By virtue of (2.29), we have

$$\begin{aligned}&\frac{d^-(\lambda_2^{1/2} w_x)}{dt}\\&= -G_+(u,v)\lambda_2^{1/2} w_x - G_-(u,v)\lambda_2^{1/2} z_x\\&+ \frac{p''(v)}{4p'(v)} \cdot (-p'(v))^{-1/4} \lambda_2^{1/2} {w_x}^2,\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}&\frac{d^+\lambda_2^{1/2} z_x}{dt}\\&= -G_-(u,v)\lambda_2^{1/2} z_x - G_+(u,v)\lambda_2^{1/2} w_x\\&+ \frac{p''(v)}{4p'(v)} \cdot (-p'(v))^{-1/4} \lambda_2^{1/2} {z_x}^2,\end{aligned} \tag{2.35}$$

where and in the following

$$G_\pm(u,v) = -\frac{1}{2}(F_u \pm F_v(-p'(v))^{-1/2}).$$

Therefore, for any smooth function $\alpha(u,v)$ and $\beta(u,v)$,

$$\begin{aligned}&\frac{d^-(\lambda_2^{1/2} w_x + \alpha(u,v))}{dt}\\&= -G_+(u,v)\lambda_2^{1/2} w_x\\&- G_-(u,v)\lambda_2^{1/2} z_x + F(u,v)\alpha_u + (\lambda_1\alpha_u + \alpha_v)(-p'(v))^{-1/4}\lambda_2^{1/2} z_x,\end{aligned} \tag{2.36}$$

and

$$\begin{aligned}&\frac{d^+(\lambda_2^{1/2} z_x + \beta(u,v))}{dt}\\&= -G_+(u,v)\lambda_2^{1/2} w_x\\&- G_-(u,v)\lambda_2^{1/2} z_x + F(u,v)\beta_u + (\lambda_2\beta_u + \beta_v)(-p'(v))^{-1/4}\lambda_2^{1/2} w_x.\end{aligned} \tag{2.37}$$

We choose α and β such that

$$\lambda_1\alpha_u + \alpha_v = G_-(u,v)(-p'(v))^{1/4}, \tag{2.38}$$

$$\lambda_2\beta_u + \beta_v = G_+(u,v)(-p'(v))^{1/4}. \tag{2.39}$$

Then one gets

$$\frac{d^-(\lambda_2^{1/2}w_x+\alpha(u,v))}{dt}$$
$$=-G_+(u,v)\lambda_2^{1/2}w_x+F(u,v)\alpha_u+\frac{p''(v)}{4p'(v)}\cdot(-p'(v))^{-1/4}\lambda_2^{1/2}{w_x}^2, \tag{2.40}$$

and

$$\frac{d^+(\lambda_2^{1/2}z_x+\beta(u,v))}{dt}$$
$$=-G_-(u,v)\lambda_2^{1/2}z_x+F(u,v)\beta_u+\frac{p''(v)}{4p'(v)}\cdot(-p'(v))^{-1/4}\lambda_2^{1/2}{z_x}^2. \tag{2.41}$$

Simple calculation yields

$$\begin{aligned}G_\pm &= C_f u^2 v^{3/2}(\frac{2v^{-1/2}}{u}\pm 1)\\ &= C_f uv(2-\sqrt{S/C_f})+C_f uv^{3/2}(f(v)\pm u).\end{aligned} \tag{2.42}$$

Differentiating (2.38) and (2.39) to u respectively, we obtain

$$\lambda_1(\alpha_u)_u+(\alpha_u)_v=(G_-(u,v)(-p'(v))^{1/4})_u=2C_f(v^{1/4}-uv^{3/4}), \tag{2.43}$$

and

$$\lambda_2(\beta_u)_u+(\beta_u)_v=(G_+(u,v)(-p'(v))^{1/4})_u=2C_f(v^{1/4}+uv^{3/4}). \tag{2.44}$$

We require, along the equilibrium curve $u=f(v)$,

$$\alpha_u(f(v),v)=0, \tag{2.45}$$

and

$$\beta_u(f(v),v)=0. \tag{2.46}$$

The Cauchy problem (2.43) with the data (2.45) given on the curve $u=f(v)$ can be solved explicitly at least locally because the curve $u=f(v)$ is not the characteristic curve of (2.43) in view of the subcharacteristic condition (1.5). In fact, we have

$$\alpha_u(u,v)=\int_{\bar v}^{v}2C_f\{s^{1/4}-s^{3/4}(2\sqrt{s^{-1}}+u-2\sqrt{v^{-1}}\}ds, \tag{2.47}$$

where $\bar v$ is determined by $\sqrt{\bar v^{-1}}=\frac{2\sqrt{v^{-1}}-u}{2-\sqrt{S/C_f}}$. Thus, $\bar v$ is determined uniquely when $2\sqrt{v^{-1}}-u>0$, this is guaranteed when $|u-f(v)|$ is small due to the subcharacteristic condition (1.5). Similarly, we have

$$\beta_u(u,v)=\int_{\hat v}^{v}2C_f\{s^{1/4}+s^{3/4}(-2\sqrt{s^{-1}}+u+2\sqrt{v^{-1}}\}ds, \tag{2.48}$$

where $\hat{v}$ is determined by $\sqrt{\hat{v}^{-1}} = \frac{2\sqrt{v^{-1}}+u}{2+\sqrt{S/C_f}}$. Now that we have determined $\alpha_u(u,v)$ and $\beta_u(u,v)$, we can choose $\alpha(u,v)$ and $\beta(u,v)$ as follows. At first, we choose a constant $M > 0$ and set $\alpha(f(v),v) = \beta(f(v),v) = M$. Since $M > 0$, we can claim that $\alpha(u,v) \geq 0$ and $\beta(u,v) \geq 0$ when $|u - f(v)|$ is small.

Lemma 2.5. *If $|u - f(v)|$ is small, and (u,v) is bounded in the quarter $\{(u,v)|u > 0, v > 0\}$, then*

$$|\alpha_u|(u,v) \leq O(1)|u - f(v)|, \qquad |\beta_u|(u,v) \leq O(1)|u - f(v)|, \tag{2.49}$$

$$|alpha(u,v) - M| \leq O(1)|u - f(v)|^2, \tag{2.50}$$

$$|\beta(u,v) - M| \leq O(1)|u - f(v)|^2. \tag{2.51}$$

Let $W = \lambda_2^{1/2} w_x + \alpha(u,v)$ and $Z = \lambda_2^{1/2} z_x + \beta(u,v)$, then

$$\frac{d^- W}{dt} = -a_1(x,t)W + a_1(x,t)\alpha + F(u,v)\alpha_u, \tag{2.52}$$

where

$$a_1(x,t) = G_+(u,v) + \frac{p''(v)}{4p'(v)} \cdot (-p'(v))^{-1/4} \lambda_2^{1/2} w_x. \tag{2.53}$$

While

$$\frac{d^+ Z}{dt} = -a_2(x,t)Z + a_2(x,t)\beta + F(u,v)\beta_u, \tag{2.54}$$

where

$$a_2(x,t) = G_-(u,v) + \frac{p''(v)}{4p'(v)} \cdot (-p'(v))^{-1/4} \lambda_2^{1/2} z_x. \tag{2.55}$$

We are coming up with the following ODE

$$\frac{dY}{dt} + a(t)Y(t) = a(t)B(t) + C(t), \tag{2.56}$$

for which, it is obvious

$$Y(t)\exp(\int_{t_0}^t a(s)ds) = Y(t_0) + \int_{t_0}^t B(\tau)a(\tau)\exp(\int_{t_0}^\tau a(s)ds)d\tau + \int_{t_0}^t C(\tau)\exp(\int_{t_0}^\tau a(s)ds)d\tau.$$

Therefore,

Lemma 2.6.

$$\begin{aligned}
&Y(t_0)\exp(-\int_{t_0}^t a(s)ds) + (1 - \exp(\int_{t_0}^t -a(s)ds) \min_{t_0 \leq \tau \leq t} B(\tau) - \max_{t_0 \leq \tau \leq t} |C(\tau)| \\
&\leq Y(t) \\
&\leq Y(t_0)\exp(-\int_{t_0}^t a(s)ds) + (1 - \exp(\int_{t_0}^t -a(s)ds) \max_{t_0 \leq \tau \leq t} B(\tau) + \max_{t_0 \leq \tau \leq t} |C(\tau)|,
\end{aligned} \tag{2.57}$$

whenever $B(\tau) \geq 0$ and $a(\tau) \geq 0$ for $t_0 \leq \tau \leq t$.

With this preparation, we are able to give the estimate of solution in the interior of $S(T)$. At first, we have the following lemma.

Lemma 2.7. *Suppose there exist two positive constants* v_1 *and* v_2 *independent of* t *such that* $v_1 \le v(x,t) \le v_2$ *as* $(x,t) \in S(T)$. *Then we have*

$$|w_x|(x,t) \le O(1)|v_r - v_m| + O(1)max_{S(T)}|u - f(v)|^2 \tag{2.58}$$

and

$$|z_x|(x,t) \le O(1)|v_m - v_l| + O(1)max_{S(T)}|u - f(v)|^2 \tag{2.59}$$

as $(x,t) \in S(T)$, *provided* $|v_r - v_l| + |u_r - f(v_r)|$, $w_x(x,t)$ *and* $z_x(x,t)$ *are small.*

Proof. For any $(\bar{x},\bar{t}) \in S(T)$, we draw a 1-characteristic $x = \tilde{x}_1(t)$ and a 2-characteristic $x = \tilde{x}_2(t)$ through $(\bar{x},\bar{t})$, which intersect the 2-shock $x = x_2(t)$ at $(x_2(t_0), t_0)$ and 1-shock $x = x_1(t)$ at $(x_1(t_1), t_1)$. Set $W = \lambda_2^{1/2} w_x + \alpha(u,v)$ and $Z = \lambda_2^{1/2} z_x + \beta(u,v)$. Then we obtain from (2.52) and Lemma 2. 6 that (here we should notice that $a_i(x,t) > 0 (i = 1,2)$ as $(x,t) \in S(T)$ when $|w_x| + |z_x|$ are small),

$$\begin{aligned}
&(\lambda_2^{1/2} w_x + \alpha(u,v))(\bar{x},\bar{t}) \\
&\le \lambda_2^{1/2} w_x((x_2(t_0)-, t_0) + max_{t_0 \le s \le \bar{t}} \alpha(u,v)(\tilde{x}_1(s), s) \\
&+ O(1)max_{S(T)}|u - f(v)|^2.
\end{aligned} \tag{2.60}$$

Therefore,

$$\begin{aligned}
&\lambda_2^{1/2} w_x(\bar{x},\bar{t}) \\
&\le \lambda_2^{1/2} w_x((x_2(t_0)-, t_0) + max_{t_0 \le s \le \bar{t}} \alpha(u,v)(\tilde{x}_1(s), s) - \alpha(u,v)(\bar{x},\bar{t}) \\
&+ O(1)max_{S(T)}|u - f(v)|^2.
\end{aligned} \tag{2.61}$$

Similarly, one obtains,

$$\begin{aligned}
&\lambda_2^{1/2} w_x \\
&\ge \lambda_2^{1/2} w_x((x_2(t_0)-, t_0) + min_{t_0 \le s \le \bar{t}} \alpha(u,v)(\tilde{x}_1(s), s) - \alpha(u,v)(\bar{x},\bar{t}) \\
&- O(1)max_{S(T)}|u - f(v)|^2.
\end{aligned} \tag{2.62}$$

Since $\alpha(u,v) = M + O(1)|u - f(v)|^2$, (2.61) and (2.62) and (2.50) imply (2.58). (2.59) can be obtained by the same argument by virtue of (2.54) and (2.51). □

In view of (2.32) and (2.33), we get

$$\frac{d^-(u - f(v))}{dt} = F(u,v) + (\lambda_1 - f'(v))z_x, \tag{2.63}$$

and

$$\frac{d^+(u - f(v))}{dt} = F(u,v) + (\lambda_2 - f'(v))w_x. \tag{2.64}$$

Notice that $F(u,v) = -(\sqrt{C_f S v} + C_f uv)(u - f(v))$. Therefore,

$$\begin{aligned}
&\frac{d^-|u - f(v)|}{dt} \\
&= -(\sqrt{C_f S v} + C_f uv)|u - f(v)| + (\lambda_2 - f'(v))z_x \cdot sign(u - f(v)) \\
&\leq -\gamma|u - f(v)| + O(1)|z_x|,
\end{aligned} \tag{2.65}$$

as long as $u > 0$. Here and in the following throughout this paper, we use a generic positive constant γ in the exponentially decay term $e^{-\gamma t}$. (2.65) leads to the following lemma, in view of (2.7), (2.8) and (1.16)

Lemma 2.8.

$$|u - f(v)|(x,t)| \leq e^{-\gamma t}|u_r - f(v_r)| + O(1) \sup_{(x,t)\in S(T)} |z_x(x,t)|, \tag{2.66}$$

for $(x,t) \in S(T)$ *if* $u > 0$*, and thus*

$$\begin{aligned}
&f(v_r) - e^{-\gamma t}|u_r - f(v_r)| - O(1)sup_{(x,t)\in S(T)}|z_x(x,t)| \\
&\leq u(x,t) \\
&\leq f(v_l) + e^{-\sqrt{S}t}|u_r - f(v_r)| + O(1)sup_{(x,t)\in S(T)}|z_x(x,t)|.
\end{aligned} \tag{2.67}$$

Combining this lemma with lemma 2.7, we have

Lemma 2.9. *Suppose there exist two positive constants* v_1 *and* v_2 *independent of* t *such that* $v_1 \leq v(x,t) \leq v_2$ *as* $(x,t) \in S(T)$*. Then we have*

$$|w_x|(x,t) \leq O(1)|v_r - v_l| + |u_r - f(v_r)|, \tag{2.68}$$

$$|z_x|(x,t) \leq O(1)|v_r - v_l| + |u_r - f(v_r)|, \tag{2.69}$$

$$|u - f(v)|(x,t) \leq O(1)(|v_r - v_l| + |u_r - f(v_r)|), \tag{2.70}$$

and as $(x,t) \in S(T)$*, provided* $|v_r - v_l| + |u_r - f(v_r)|$ *, is small.*

Proof. This lemma can be proved by a standard continuation argument. For this purpose, we observe that $lim_{t\to 0+}(w_x, z_x)(x_1(t)+, t) = lim_{t\to 0+}(w_x, z_x)(x_2(t)-, t)$. This, together with (2.5) and (2.6), gives

$$|lim_{t\to 0+}(w_x, z_x)(x_1(t)+, t)| \leq O(1)(|v_r - v_l| + |u_r - f(v_r)|) \tag{2.71}$$

and

$$|lim_{t\to 0+}(w_x, z_x)(x_2(t)-, t)| \leq O(1)(|v_r - v_l| + |u_r - f(v_r)|). \tag{2.72}$$

By virtue of these two inequalities , (2.26), (2.27), Lemma 2.7 and 2.8, a standard continuation leads to (2.68), (2.69) and (2.70). □

We turn to the decay estimate of derivatives along the shocks.

Lemma 2.10. *If $|v_r - v_l| + |u_r - f(v_r)$ is small, then we have, for some positive constant γ*

$$|z_x(x_2(t)-,t)| \leq O(1)|v_r - v_l|e^{-\gamma t}, \tag{2.73}$$

provided $z_{xx}(x_2(t)-,t)$ is bounded. Similarly

$$|w_x(x_1(t)+,t)| \leq O(1)|v_r - v_l|e^{-\gamma t}, \tag{2.74}$$

provided $w_{xx}(x_1(t)+,t)$ is bounded.

Proof. At first, along the 2-shock $x = x_2(t)$, we have (all the terms in the following formula are evaluated at $x = x_2(t)-$),

$$\begin{aligned}
&\frac{dz_x(x_2(t)-,t)}{dt} = (z_x)_t + \dot{x}_2(t)z_{xx} \\
&= (z_x)_t + \lambda_2 z_{xx} + (\dot{x}_2(t) - \lambda_2)z_{xx} \\
&= (z_x)_t + (\lambda_2 z_x)x - (\lambda_2)_x z_x + (\dot{x}_2(t) - \lambda_2)z_{xx} \\
&= \{-C_f(u^2v)_x - (\lambda_2)_x z_x + (\dot{x}_2(t) - \lambda_2)z_{xx}\}(x_2(t)-,t).
\end{aligned} \tag{2.75}$$

On the other hand, a straightforward calculation yields,

$$\begin{aligned}
&- C_f(u^2v)_x \\
&= -C_f(uv - \frac{u^2}{2\lambda_2})z_x - C_f(uv + \frac{u^2}{2\lambda_2})w_x
\end{aligned} \tag{2.76}$$

and

$$\begin{aligned}
&(uv - \frac{u^2}{2\lambda_2})(x_2(t)-,t) \\
&\geq u_r(v_r - \frac{f(v_r)}{2\lambda_2(v_r)} - O(1)(|u_r - f(v_r) + |v_r - v_m|) \\
&= u_r v_r(1 - \sqrt{\frac{S}{4C_f}}) - O(1)(|u_r - f(v_r) + |v_r - v_m|) > 0
\end{aligned} \tag{2.77}$$

in view of the subcharacteristic condition (1.5). Since

$$|\dot{x}_2(t) - \lambda_2(x_2(t)-,t)| \leq O(1)|v_r - v_m|e^{-\gamma t}$$

for some $\gamma > 0$. (2.73) follows from (2.75) and (2.76) immediately , with the help of the following estimate

$$\begin{aligned}
&|\lambda_{2x}z_x|(x_2(t)-,t) \leq O(1)(z_x^2 + w_x^2)(x_2(t)-,t) \\
&\leq O(1)z_x^2(x_2(t)-,t) + O(1)e^{-\gamma t}(|z_x| + 1)^2
\end{aligned} \tag{2.78}$$

here (2.26) is used.

(2.74) can be obtained similarly. □

In lemma 2.10, we need the bounds of $|w_{xx}|$ and $|z_{xx}|$, which are given in the following lemmas.

Lemma 2.11. *Suppose there exist two positive constants v_1 and v_2 independent of t such that $v_1 \le v(x,t) \le v_2$ as $(x,t) \in S(T)$. Then we have , along the shocks $x = x_2(t)$ and $x = x_1(t)$*

$$\begin{aligned}&|w_{xx}(x_2(t)-,t)|\\&\le O(1)(|v_r - v_l|e^{-\gamma t}|z_{xx}((x_2(t)-,t)|\\&+ O(1)(|v_r - v_l| + |u_r - f(v_r)|),\end{aligned} \tag{2.79}$$

and

$$\begin{aligned}&|z_{xx}(x_1(t)+,t)|\\&\le O(1)(|v_r - v_l|e^{-\gamma t}|w_{xx}((x_1(t)+,t)|\\&+ O(1)(|v_r - v_l| + |u_r - f(v_r)|),\end{aligned} \tag{2.80}$$

for some $\gamma > 0$, provided $|v_r - v_l| + |u_r - f(v_r)|$ is small enough.

Proof. Differentiate (2.5) along $x = x_2(t)$, by virtue of Lemma 2.3, we have,

$$\begin{aligned}&|(w_{xx}\dot{x}_2(t) + w_{xt}(x_2(t)-,t)|\\&\le O(1)(w_x^2 + z_x^2 + |v_r - v_m|(|w_x| + z_x|)\\&+ O(1)(|v_r - v_m|e^{-\gamma t}|(z_{xx}\dot{x}_2(t) + z_{xt}|) + O(1)|\frac{d[u^2v](x_2(t),t)}{dt},\end{aligned} \tag{2.81}$$

in this inequality, each term is evaluated at $x = x_2(t)-$.
On the other hand, since

$$w_{xt} = -\lambda_2 w_{xx} - C_f(u^2v)_x - \lambda_{1x}w_x,$$

we have

$$\begin{aligned}&|(w_{xx}\dot{x}_2(t) + w_{xt}(x_2(t)-,t)|\\&\ge (c|w_{xx}| - O(1)(|w_x| + |z_x|))(x_2(t)-,t).\end{aligned} \tag{2.82}$$

for some $c > 0$, if $v(x,t)$ has positive lower and upper bounds. Moreover, since

$$\begin{aligned}&z_{xt} + \dot{x}_2(t)z_{xx}\\&= -C_f(u^2v)_x + (\dot{x}_2(t) - \lambda_2)z_{xx},\end{aligned}$$

we have

$$\begin{aligned}&|z_{xt} + \dot{x}_2(t)z_{xx}|(x_2(t)-,t)\\&\le O(1)|v_r - v_l|e^{-\gamma t}|z_x x(x_2(t)-,t)|\\&+ (|v_r - v_l| + |u_r - f(v_r)|).\end{aligned} \tag{2.83}$$

(2.79) follows form the above estimates. (2.80) can be obtained similarly. □

With the estimates (2.79) and (2.80), we can derive the estimates for w_{xx} and z_{xx} in $S(T)$ by a similar approach as that used for the estimates of w_x and z_x, by virtue of the estimates we obtained so far. The idea is to get the differential equations for $\lambda_2^{1/2} w_{xx}$ and $\lambda_2^{1/2} z_{xx}$ along 1-characteristic and 2-characteristic respectively. This can be carried out by respectively differentiating (2.40) and (2.41) with respect to x, and apply Lemma 2.6 to the resultant equations. Precisely, we have the following lemma, the proof of which is sketched as above and thus omitted.

Lemma 2.12. *Suppose there exist two positive constants v_1 and v_2 independent of t such that $v_1 \leq v(x,t) \leq v_2$ as $(x,t) \in S(T)$. Then we have*

$$|w_{xx}|(x,t) \leq O(1)|v_r - v_l| + |u_r - f(v_r)|, \tag{2.84}$$

$$|z_{xx}|(x,t) \leq O(1)|v_r - v_l| + |u_r - f(v_r)|, \tag{2.85}$$

as $(x,t) \in S(T)$, provided $|v_r - v_l| + |u_r - f(v_r)|$ is small enough.

Combining lemma 2.12 and 2.10 together, we get the decay estimates for w_x and z_x along 1-shock $x_1(t)$ and 2-shock $x_2(t)$. We end this section by the putting all the estimates of the derivatives we have obtained together as the following lemma.

Lemma 2.13. *Suppose there exist two positive constants v_1 and v_2 independent of t such that $v_1 \leq v(x,t) \leq v_2$ as $(x,t) \in S(T)$. Then we have*

$$\begin{aligned} &sup_{(x,t)\in S(T)}(|w_x| + |z_x| + |w_{xx}|(x,t) + |z_{xx}|)(x,t) \\ &\leq O(1)|v_r - v_l| + |u_r - f(v_r)|, \end{aligned} \tag{2.86}$$

$$\begin{aligned} &(|w_x| + |z_x|)(x_1(t)+,t) + (|w_x| + |z_x|)(x_2(t)-,t) \\ &\leq O(1)(|v_r - v_l| + |u_r - f(v_r)|)e^{-\gamma t}, \end{aligned} \tag{2.87}$$

for some $\gamma > 0$, provided $|v_r - v_l| + |u_r - f(v_r)|$ is small enough.

3 Energy Estimate

With the estimates obtained in Section 2, we can use energy method to get the desired estimates as follows.

Let

$$\phi(x,t) = \int_{x_1(t)}^{x} (v(y,t) - V(y + x_0 - \sigma t)dy,$$

$$\psi(x,t) = u(x,t) - U(x + x_0 - \sigma t),$$

where x_0 is determined by (1.21). The following equations hold true as $x_1(t) < x < x_2(t)$,

$$\phi_t(x,t) = \psi(x,t) - m(t),$$

and

$$\psi_t + (p(V+\phi_x) - p(V))_x = -C_f(v(u+U)\psi + U^2\phi_x),$$

where

$$m(t) = (v(x_1(t),t) - V(x_1(t)+x_0-\sigma t))\dot{x}_1(t) + (u(x_1(t),t) - U(x_1(t)+x_0-\sigma t)).$$

Therefore, when $x_1(t) < x < x_2(t)$,

$$\begin{aligned}&\phi_{tt} + (P(V+\phi_x) - P(V))_x + 2C_f UV\phi_t + C_f U^2\phi_x \\ &+ 2C_f(U\phi_x\phi_t + UVm(t) + \phi_x m(t)) + C_f(V+\phi_x)(\phi_t + m(t))^2 \\ &= -m'(t).\end{aligned} \tag{3.1}$$

We will work on the equation (3.1) by using the energy method in $S(T)$. For this purpose, we establish the following estimates of ϕ and its derivatives and the source term $m(t)$ on the boundaries of $S(T)$.

Lemma 3.1. *Suppose there exist two positive constants v_1 and v_2 independent of t such that $v_1 \le v(x,t) \le v_2$ as $(x,t) \in S(T)$. Then*

$$|\phi(x_2(t)-,t)| \le O(1)(|v_r - v_l| + |u_r - f(v_r))|e^{-\gamma t}, \tag{3.2}$$

$$\begin{aligned}&|\phi_x(x_2(t)-,t)| + |\phi_t(x_2(t)-,t)| + |\phi_{xx}(x_2(t)-,t)|\phi_{xt}(x_2(t)-,t)| \\ &\le O(1)(|v_r - v_l| + |u_r - f(v_r))e^{-\gamma t},\end{aligned} \tag{3.3}$$

$$\begin{aligned}&|\phi_x(x_1(t)+,t)| + |\phi_t(x_1(t)+,t)| + |\phi_{xx}(x_1(t)+,t) + |\phi_{xt}(x_1(t)+,t)| \\ &\le O(1)(|v_r - v_l| + |u_r - f(v_r)|)e^{-\gamma t},\end{aligned} \tag{3.4}$$

$$|m(t)| + |m'(t)| \le O(1)(|v_r - v_l| + |u_r - f(v_r))e^{-\gamma t}, \tag{3.5}$$

where $\gamma > 0$ is a positive constant, provided $|v_r - v_l| + |u_r - f(v_r)|$ is small.

Proof. From the conservation law $(1.2)_1$, we have

$$d\{\int_{-\infty}^{+\infty} (v(y,t) - V(y+x_0-\sigma t))dy\}/dt = 0.$$

Since x_0 is chosen such that

$$\int_{-\infty}^{+\infty} (v_0(y) - V(y+x_0))dy = 0,$$

therefore

$$\int_{-\infty}^{+\infty} (v(y,t) - V(y+x_0-\sigma t))dy = 0. \tag{3.6}$$

Since $x_2(t) - \sigma t \geq ct$ for some positive constant c due to the subcharacterisitc condition $\dot{x}_2(t) > \lambda_2(v_r) > \sigma$, and $|v(y) - v_r| \leq O(1)|v_r - v_l|e^{-\gamma y}$, we have the following estimate

$$\begin{aligned}
&\int_{x_2(t)}^{\infty} |v(y,t) - V(y + x_0 - \sigma t)|dy \\
&\leq \int_{x_2(t)}^{\infty} |v_r - V(y + x_0 - \sigma t)|dy \\
&\leq O(1)|v_r - v_l|e^{-\gamma t}.
\end{aligned} \tag{3.7}$$

Similarly,

$$\begin{aligned}
&\int_{-\infty}^{x_1(t)} |v(y,t) - V(y + x_0 - \sigma t)|dy \\
&\leq O(1)|v_r - v_l|e^{-\gamma t}.
\end{aligned} \tag{3.8}$$

(3.2) follows then from (3.6), (3.7) and (3.8). Since

$$|v(x_1(t)+,t) - v_l| + |u(x_1(t),t) - u_l| \leq O(1)|v_r - v_l|e^{-\gamma t}, \tag{3.9}$$

and

$$|V(x_1(t) + x_0 - \sigma t) - v_l| + |U(x_1(t) + x_0 - \sigma t) - u_l| \leq O(1)|v_r - v_l|e^{-\gamma t}, \tag{3.10}$$

we have

$$|m(t)| \leq O(1)|v_r - v_l|e^{-\gamma t}, \tag{3.11}$$

for some $\gamma > 0$. On the other hand,

$$\begin{aligned}
m'(t) &= \big(d(v(x_1(t)+,t))/dt - V'(x_1(t) + x_0 - \sigma t)\big)(\dot{x}_1(t) - \sigma) \\
&+ \big(v(x_1(t)+,t) - V(x_1(t) + x_0 - \sigma t)\big)\frac{d^2x_1(t)}{dt^2 t} \\
&+ \big(d(u(x_1(t)+,t))/dt - U'(x_1(t) + x_0 - \sigma t)\big)(\dot{x}_1(t) - \sigma) \\
&+ \big(u(x_1(t)+,t) - U(x_1(t) + x_0 - \sigma t)\big)\frac{d^2x_1(t)}{dt^2}
\end{aligned} \tag{3.12}$$

We estimate each term in(3.12) as follows. Since $x_1(t) \leq -O(1)$ and $\sigma > 0$, we have

$$\begin{aligned}
&|V'(x_1(t) + x_0 - \sigma t)| + |V(x_1(t) + x_0 - \sigma t) - v_l| \\
&\leq O(1)|v_r - v_l|e^{-\gamma t},
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&|U'(x_1(t) + x_0 - \sigma t)| + |U(x_1(t) + x_0 - \sigma t) - f(v_l)| \\
&\leq O(1)|v_r - v_l|e^{-\gamma t}.
\end{aligned} \tag{3.14}$$

Here we have used the property of the shock profile, i.e

$$|V'(y)| + |U'(y)| + |V(y) - v_l| + |U(y) - f(v_l)|$$
$$\leq O(1)|v_r - v_l|e^{-\gamma|y|}, \tag{3.15}$$

for $y < 0$. On the other hand, (2.24), (2.25), (1.11) and (1.15) imply

$$|d(v(x_1(t)+,t))/dt| + |v(x_1(t)+,t) - v_l|$$
$$\leq O(1)(|v_r - v_l| + |u_r - f(v_r)|)e^{-\gamma t}, \tag{3.16}$$

and

$$|d(u(x_1(t)+,t))/dt| + |u(x_1(t)+,t) - f(v_l)|$$
$$\leq O(1)(|v_r - v_l| + |u_r - f(v_r)|e^{-\gamma t}. \tag{3.17}$$

(3.12) and (3.13)-(3.17) imply

$$|m'(t)| \leq O(1)(|v_r - v_l| + |u_r - f(v_r)|)e^{-\gamma t}. \tag{3.18}$$

The other estimates are the direct consequences of Lemma 2.3, lemma 2.4 and (2.87) . □

We give the follow estimate on the shock profile V.

Lemma 3.2.

$$0 < V'(y) \leq O(1)|v_r - v_l|e^{-\gamma|y|}, v_l < V(y) < v_r$$

as $y \in (-\infty, \infty)$.

The proof of this lemma can be found in [16]. Let us define $\delta(T) = sup_{(x,t)\in R(T)}\{|\phi(x,t)| + |\phi_x(x,t)| + |\phi_t(x,t)|\}$. We have, from lemma 2.13 that, $|\phi_{xx}| + |\phi_{xt}|$ is small if $\delta(T)$ and $|v_r - v_l| + |u_r - f(v_r)|$ are small. This is useful in the energy estimate.

Lemma 3.3. *Suppose ϕ is smooth in $S(T)$, if $\delta(T)$ and $|v_r - v_l| + |u_r - f(v_r)|$ are suitably small, then we have the following estimate*

$$\int_{x_1(t)}^{x_2(t)} (\phi^2 + \phi_x^2 + \phi_x^2)(x,t)dx$$
$$+ \int_0^t \int_{x_1(s)}^{x_2(s)} (V'\phi^2 + \phi_x^2 + \phi_x^2)(x,s)dxds$$
$$\leq O(1)|v_r - v_l|, \tag{3.19}$$

for $0 \leq t \leq T$.

Proof. We multiply the equation (3.1) by ϕ, and integrate the resultant equation over $(x_1(t), x_2(t))$. We estimate each term as follows.

$$\int_{x_1(t)}^{x_2(t)} \phi_{tt}\phi dx = \int_{x_1(t)}^{x_2(t)} \{(\phi_t\phi)_t - \phi_t^2\}dx$$
$$= d(\int_{x_1(t)}^{x_2(t)} \phi_t\phi dx)/dt - \int_{x_1(t)}^{x_2(t)} \phi_t^2 dx - (\phi_t\phi)(x_2(t)-, t)\dot{x}_2(t) + (\phi_t\phi)(x_1(t)+, t)\dot{x}_1(t). \quad (3.20)$$

From lemma 3.1, we have the estimate

$$|(\phi_t\phi)(x_2(t)-, t)\dot{x}_2(t)| \leq O(1)|v_r - v_l|e^{-\gamma t},$$

and

$$(\phi_t\phi)(x_1(t)+, t) = 0.$$

Integration by parts gives

$$\int_{x_1(t)}^{x_2(t)} (p(V + \phi_x) - p(V))_x\phi dx$$
$$= (p(V + \phi_x) - p(V))\phi(x_2(t)-, t) + \int_{x_1(t)}^{x_2(t)} (p(V) - p(V + \phi_x))\phi_x dx. \quad (3.21)$$

Here we have used the fact $\phi(x_1(t), t) = 0$, which will be used many times in the following in this section without pointing out explicitly. As far as the first term on the right hand side of (3.21), the estimates in lemma 3.1 give

$$|(p(V + \phi_x) - p(V))\phi(x_2(t)-, t)| \leq O(1)|v_r - v_l|e^{-\gamma t}.$$

One can get the following estimates by virtue of the integration by parts and the estimates on the boundaries $x = x_1(t)$ and $x = x_2(t)$ in lemma 3.1.

$$\int_{x_1(t)}^{x_2(t)} 2C_f UV\phi_t\phi dx$$
$$= C_f d(\int_{x_1(t)}^{x_2(t)} C_f UV\phi^2 dx)/dt - \int_{x_1(t)}^{x_2(t)} C_f(UV)_t\phi^2 dx - C_f UV\phi^2(x_2(t) - t)\dot{x}_2(t), \quad (3.22)$$

$$|C_f UV\phi^2(x_2(t) - t)\dot{x}_2(t)| \leq O(1)|v_r - v_l|e^{-\gamma t};$$

$$\int_{x_1(t)}^{x_2(t)} C_f U^2\phi_x\phi dx$$
$$= -\int_{x_1(t)}^{x_2(t)} C_f(UU_x\phi^2 dx + \frac{1}{2}C_f U^2\phi^2(x_2(t)-, t), \quad (3.23)$$

and

$$|\frac{1}{2}C_f U^2\phi^2(x_2(t)-, t)| \leq O(1)|v_r - v_l|e^{-\gamma t};$$

$$|\int_{x_1(t)}^{x_2(t)} C_f\{2U\phi_x\phi_t + 2U\phi_x m(t) + (V+\phi_x)(\phi_t + m(t))^2\}\phi dx|$$
$$\leq O(1)(\delta(T) + |v_r - v_l|)\int_{x_1(t)}^{x_2(t)} (\phi_t^2 + \phi_x^2)dx + O(1)|v_r - v_l|e^{-\gamma t}; \tag{3.24}$$

$$|\int_{x_1(t)}^{x_2(t)} m'(t)\phi dx| \leq O(1)|v_r - v_l|(1+t)e^{-\gamma t}. \tag{3.25}$$

Gathering the above estimates , we obtain

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} (\phi_t\phi + C_f UV\phi^2)dx)$$
$$+ \int_{x_1(t)}^{x_2(t)} \{(p(V) - p(V+\phi_x))\phi_x - C_f((UV)_t + UU_x)\phi^2 - \phi_t^2\}$$
$$\leq O(1)|v_r - v_l|e^{-\gamma t}$$
$$+ O(1)(\delta(T) + |v_r - v_l|)\int_{x_1(t)}^{x_2(t)} (\phi_t^2 + \phi_x^2)dx. \tag{3.26}$$

Multiplying (3.1) by ϕ_t, integrating the resulting equation over $(x_1(t), x_2(t))$, one obtains, by virtue of the integration by parts and the estimates in lemma 3.1, that

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} (\frac{1}{2}\phi_t^2 + (p(V) - p(V+\phi_x)\phi_x + \frac{p'(V+\phi_x)}{2}\phi_x^2)dx)$$
$$+ \int_{x_1(t)}^{x_2(t)} \{2C_f UV\phi_t^2 + C_f U^2\phi_x\phi_t\}$$
$$\leq O(1)|v_r - v_l|e^{-\gamma t}$$
$$+ O(1)(\delta(T) + |v_r - v_l|)\int_{x_1(t)}^{x_2(t)} (\phi_t^2 + \phi_x^2)dx. \tag{3.27}$$

Actually, the method used for (3.27) is very similar to that for (3.26). (3.26) + (3.27) $\times\, k$ gives

$$\frac{d}{dt}\int_{x_1(t)}^{x_2(t)} \{\phi_t\phi + C_f UV\phi^2 + \frac{k}{2}\phi_t^2 + k(p(V) - p(V+\phi_x))\phi_x + k\frac{p'(V+\phi_x)}{2}\phi_x^2\}dx$$
$$+ \int_{x_1(t)}^{x_2(t)} \{(p(V) - p(V+\phi_x))\phi_x + (2C_f kUV - 1)\phi_t^2 + kC_f U^2\phi_x\phi_t\}dx$$
$$+ \int_{x_1(t)}^{x_2(t)} C_f\sigma^2 VV'\phi^2 dx$$
$$\leq O(1)|v_r - v_l|e^{-\gamma t}$$
$$+ O(1)(\delta(T) + |v_r - v_l|)\int_{x_1(t)}^{x_2(t)} (\phi_t^2 + \phi_x^2)dx, \tag{3.28}$$

where k is a positive number which will be determined as follows. Here we have used the fact

$$-C_f(UV)_t - C_f UU_x = C_f\sigma^2 VV'$$

and $V' > 0$. We choose k to guarantee that each term on the right hand side of (3.28) is positive. This will follow the following estimate. Since

$$U = \sigma(v_l - V) + f(v_l) \geq f(v_l) - O(1)|v_r - v_l|,$$

we have

$$\begin{aligned} &\phi_t\phi + C_f UV\phi^2 + \frac{k}{2}\phi_t^2 \\ &\geq C_f f(v_l)v_l\phi^2 + \phi_t\phi + \frac{k}{2}\phi_t^2 - O(1)|v_r - v_l|\phi^2. \end{aligned} \tag{3.29}$$

We require $1 - 4C_f f(v_l)v_l \cdot \frac{k}{2} = 1 - 2kC_f f(v_l)v_l < 0$, to guarantee that the quadratic form $C_f f(v_l)v_l\phi^2 + \phi_t\phi + \frac{k}{2}\phi_t^2$ is positively definite. Thus, when

$$k > (2kC_f f(v_l)v_l)^{-1} = (4C_f S v_l)^{-1/2} \tag{3.30}$$

and $|v_r - v_l|$ is small, we have

$$\begin{aligned} &\phi_t\phi + C_f UV\phi^2 + \frac{k}{2}\phi_t^2 \\ &\geq O(1)(\phi^2 + \phi_t^2). \end{aligned} \tag{3.31}$$

We estimate the second term in (3.28) as follows. At first

$$\begin{aligned} &(p(V) - p(V + \phi_x))\phi_x + (2C_f kUV - 1)\phi_t^2 + kC_f U^2\phi_x\phi_t \\ &\geq -p'(v_l)\phi_x^2 + (2kC_f f(v_l)v_l - 1)\phi_t^2 + kC_f(f(v_l))^2\phi_x\phi_t \\ &- O(1)|v_r - v_l|(\phi_x^2 + \phi_t^2) - O(1)|\phi_x|^3. \end{aligned} \tag{3.32}$$

We require

$$k^2C_f^2(f(v_l))^4 + 4p'(v_l)(2kC_f f(v_l)v_l - 1) < 0 \tag{3.33}$$

to guarantee that the quadratic form $-p'(v_l)\phi_x^2 + (2kC_f f(v_l)v_l - 1)\phi_t^2 + kC_f(f(v_l))^2\phi_x$ is positively definite. Since $p'(v) = -v^{-3}$ and $f(v) = \sqrt{S}(C_f v)^{-1/2}$, (3.33) is equivalent to

$$(4\sqrt{C_f}S^{-3/2} - 2S^{-3/2}\sqrt{4C_f - S})v_l^{-1/2} < k < (4\sqrt{C_f}S^{-3/2} + 2S^{-3/2}\sqrt{4C_f - S})v_l^{-1/2}. \tag{3.34}$$

Here the subcharacteristic condition is used. In view of (3.30) and (3.34), if we choose

$$\begin{aligned} &\max\{(4\sqrt{C_f}S^{-3/2} - 2S^{-3/2}\sqrt{4C_f - S})v_l^{-1/2}, (4C_f S)^{-1/2}v_l^{-1/2}\} \\ &< k \\ &< (4\sqrt{C_f}S^{-3/2} + 2S^{-3/2}\sqrt{4C_f - S})v_l^{-1/2}, \end{aligned} \tag{3.35}$$

then we can claim (3.31) and the following estimate

$$
\begin{aligned}
&(p(V) - p(V + \phi_x))\phi_x + (2C_f kUV - 1)\phi_t^2 + kC_f U^2 \phi_x \phi_t \\
&\geq O(1)(\phi_x^2 + \phi_t^2)
\end{aligned} \tag{3.36}
$$

provided $|v_r - v_l|$ and $\delta(T)$ are small. Since $S < 4C_f$,

$$
(4C_f S)^{-1/2} v_l^{1/2} < (4\sqrt{C_f} S^{-3/2} + 2S^{-3/2}\sqrt{4C_f - S}) v_l^{-1/2}.
$$

Thus the positive number k satisfying (3.35) can be actually chosen.

Integrating the equation (3.28) over $[0, t]$, by virtue of the above estimates, one obtains (3.19). □

Differentiate (3.1) with respect to x, we get

$$
\phi_{xtt} + (P(V + \phi_x) - P(V))_{xx} + 2C_f UV \phi_{xt} + C_f U^2 \phi_{xx} + I = 0, \tag{3.37}
$$

where

$$
\begin{aligned}
I =& 2C_f (UV)_x \phi_t + C_f (U^2)_x \phi_x \\
&+ C_f \partial_x \{2U\phi_x\phi_t + 2UVm(t) + 2U\phi_x m(t) + (V + \phi_x)(\phi_t + m(t))^2\}
\end{aligned}
$$

. Multiplying (3.37) by ϕ_x, integrating the resulting equation over the region $\{(x, s) : (x_1(s) < x < x_2(s),\ t_0 < s < t\}$, by virtue of the estimates on the boundaries (Lemma 3.1) and Lemma (3.3), we obtain

$$
\begin{aligned}
&\int_{x_1(t)}^{x_2(t)} (\phi_{xt}\phi_x(x, t) dx \\
&+ \int_{t_0}^{t} \int_{x_1(s)}^{x_2(s)} \{(-p'(V + \phi_x)\phi_{xx}^2 - \phi_{xt}^2\}(x, s) dx ds \\
&\leq \int_{x_1(t_0)}^{x_2(t_0)} (\phi_{xt}\phi_x(x, t) dx \\
&+ O(1)(\bar{\epsilon} + |v_r - v_l| + \delta(T)) \int_0^t \int_{x_1(s)}^{x_2(s)} \{\phi_{xx}^2 + \phi_{xt}^2\}(x, s) dx ds \\
&+ O(1)(\bar{\epsilon})^{-1} |v_r - v_l|,
\end{aligned} \tag{3.38}
$$

provided $\delta(T)$ and $|v_r - v_l| + |u_r - f(v_r)|$ are small. $\bar{\epsilon}$ in (3.38) is an arbitrary positive number, which arises when one uses the Cauchy-Schwartz inequality. Multiplying (3.37) by ϕ_{xt}, integrating the resulting equation over the region $\{(x, s) : (x_1(s) < x < x_2(s),\ t_0 < s <$

$t\}$, with the help of the estimates on the boundaries (Lemma 3.1) and Lemma (3.3), we get

$$\begin{aligned}
&\frac{1}{2}\int_{x_1(t)}^{x_2(t)} \{\phi_{xt}^2 + (-p'(V+\phi_x)\phi_{xx}^2\}(x,t)dx \\
&+ \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \{2C_f UV phi_{xt}^2 + C_f U^2 \phi_{xx}\phi_{xt}\}(x,s)dxds \\
&\leq \int_{x_1(t_0)}^{x_2(t_0)} \{\phi_{xt}^2 + (-p'(V+\phi_x)\phi_{xx}^2\}(x,t_0)dxdx \\
&+ O(1)(\bar{\delta} + |v_r - v_l| + \delta(T)) \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} \{\phi_{xx}^2 + phi_{xt}^2\}(x,s)dxds \\
&+ O(1)(\bar{\delta})^{-1}|v_r - v_l|, \qquad (3.39)
\end{aligned}$$

provided $\delta(T)$ and $|v_r - v_l|$ are small. $\bar{\delta}$ in (3.39) is an arbitrary positive number, which again arises when one uses the Cauchy-Schwartz inequality.

By a similar method to get (3.36), (3.38)+(3.39)$\times k_1$ with a suitably chosen positive number k_1 gives the following estimate

Lemma 3.4. *Suppose ϕ is smooth in $S(T)$, if $\delta(T)$ and $|v_r - v_l|$ are suitably small, then we have the following estimate*

$$\begin{aligned}
&\int_{x_1(t)}^{x_2(t)} (\phi_{xx}^2 + \phi_{xt}^2)(x,t)dx \\
&+ \int_{t_0}^{t}\int_{x_1(s)}^{x_2(s)} (\phi_{xx}^2 + \phi_{xt}^2)(x,s)dxds \\
&\leq O(1)|v_r - v_l| + \int_{x_1(t_0)}^{x_2(t_0)} (\phi_{xx}^2 + \phi_{xt}^2)(x,t_0)dx, \qquad (3.40)
\end{aligned}$$

for $0 < t_0 \leq t \leq T$.

In order to complete the proof of (ii) in Theorem 1.1, we choose a sufficiently small t_0, the local existence result ([13]) and a standard continuation argument yields the global existence in $S(T)$. (1.24) follows easily from the above estimates. Theorem 1.2 is a direct corollary of (1.24) because we have the estimates (1.15) and (1.16) for $x < x_1(t)$ and $x > x_2(t)$, and the shock profile $(V,U)(x-\sigma t)$ tends to $(v_l, f(v_l))$ and $(v_r, f(v_r))$ exponentially fast as $x - \sigma t \to \pm\infty$ respectively.

4 Cauchy Problem

In this section, we will prove Theorems 1.3-1.5. This will be carried out by several lemmas. The notation used in this section can be found in section 1.

Lemma 4.1. *Assume that the subcharacteristic condition (1.5) is true. Then there exist positive numbers δ and λ such that if the two states (v_l, u_l) and (v_r, u_r) are connected by R_1 and S_2 $(v_l v_r > 0)$, then $(v_l, u_l + \frac{s(S-C_f(u_l)^2 v_l}{\epsilon})$ and $(v_r, u_r + \frac{s(S-C_f(u_r)^2 v_r}{\epsilon})$ are connected by R_1 and S_2 too provided $0 < s < \lambda\epsilon$, and $|u_l - f(v_l)| + |u_r - f(v_r)| \leq \delta$.*

Proof. At first, let us assume $0 < s < \lambda\epsilon$ and $|u_l - f(v_l)| + |u_r - f(v_r)| \leq \delta$, then we will show later that we can suitably choose δ and λ as desired. The two states (v_l, u_l) and (v_r, u_r) are connected by R_1 and S_2 is equivalent to the following condition (cf. [22]),

$$w_r \geq w_l + g_2 (v_l, z_r - z_l), z_l \geq z_r, \tag{4.1}$$

where (w_l, z_l) and (w_r, z_r) are the Riemann invariants corresponding to (v_l, u_l) and (v_r, u_r) respectively, g_2 is the function in (1.34).

Let $(\bar{w}_l, \bar{z}_l)$ and $(\bar{w}_r, \bar{z}_r)$ be the Riemann invariants corresponding to $(v_l, u_l + \frac{s(S-C_f(u_l)^2 v_l}{\epsilon})$ and $(v_r, u_r + \frac{s(S-C_f(u_r)^2 v_r}{\epsilon})$ respectively. Therefore

$$\bar{w}_l = w_l + \frac{s}{\epsilon}(S - C_f(u_l)^2 v_l) \tag{4.2}$$

Similarly,

$$\bar{z}_l = z_l + \frac{s}{\epsilon}(S - C_f(u_l)^2 v_l) \tag{4.3}$$

$$\bar{w}_r = w_r + \frac{s}{\epsilon}(S - C_f(u_r)^2 v_r), \tag{4.4}$$

and

$$\bar{z}_r = z_r + \frac{s}{\epsilon}(S - C_f(u_r)^2 v_r). \tag{4.5}$$

For the simplicity of the notation, we use ΔA to denote $A_r - A_l$ for a quantity A, for example, $\Delta w = w_r - w_l$. By virtue of this setting, (4.2) and (4.3) imply

$$\Delta\bar{z} = \bar{z}_r - \bar{z}_l = \Delta z + \frac{s}{\epsilon}\Delta(S - C_f(u)^2 v). \tag{4.6}$$

We estimate $\Delta(S - C_f(u)^2 v)$ as follows. At first

$$\begin{aligned}
&\Delta(S - C_f(u)^2 v) \\
&= \Delta\{(\sqrt{SC_f v} + C_f uv)(f(v) - u)\} \\
&= (f(v_r) - u_r)\Delta(\sqrt{SC_f v} + C_f uv) \\
&+ (\sqrt{SC_f v_l} + C_f u_l v_l) \left(\sqrt{\frac{S}{C_f}}\frac{\Delta z - \Delta w}{4} - \frac{\Delta z + \Delta w}{2}\right) \\
&= (f(v_r) - u_r)\Delta(\sqrt{SC_f v} + C_f uv) \\
&+ \frac{1}{2}(\sqrt{SC_f v_l} + C_f u_l v_l) \left((-1 + \sqrt{\frac{S}{4C_f}})\Delta z - (1 + \sqrt{\frac{S}{4C_f}})\Delta w\right).
\end{aligned} \tag{4.7}$$

On the other hand,

$$\Delta(\sqrt{SC_f v} + C_f uv) = a_1 \Delta w + a_2 \Delta z, \tag{4.8}$$

where a_1 and a_2 are two functions of (u_l, v_l) and (u_r, v_r), which are bounded whenever (u_l, v_l) and (u_r, v_r) are bounded from above and away from 0.
(4.7) and (4.8) imply

$$\begin{aligned}
&\Delta(S - C_f(u)^2 v) \\
&= -\{\frac{1}{2}(\sqrt{SC_f v_l} + C_f u_l v_l)(1 - \sqrt{\frac{S}{4C_f}}) - a_1(f(v_r) - u_r)\}\Delta z \\
&- \{\frac{1}{2}(\sqrt{SC_f v_l} + C_f u_l v_l)(1 + \sqrt{\frac{S}{4C_f}}) - a_2(f(v_r) - u_r)\}\Delta w.
\end{aligned} \tag{4.9}$$

In view of subcharacteristic condition (1.5), we have

$$1 - \sqrt{\frac{S}{4C_f}} > 0. \tag{4.10}$$

Thus, we can choose $\delta > 0$ such that

$$\frac{1}{2}(\sqrt{SC_f v_l} + C_f u_l v_l)(1 - \sqrt{\frac{S}{4C_f}}) - a_1(f(v_r) - u_r) > 0 \tag{4.11}$$

and

$$\frac{1}{2}(\sqrt{SC_f v_l} + C_f u_l v_l)(1 + \sqrt{\frac{S}{4C_f}}) - a_1(f(v_r) - u_r) > 0, \tag{4.12}$$

provided $|f(v_r) - u_r| < \delta$. On the hand, by virtue of (4.1) and (1.37), we have

$$\Delta w \geq g_2(v_l, \Delta z) \geq \Delta z. \tag{4.13}$$

This together with (4.9) gives

$$\begin{aligned}
&\Delta(S - C_f(u)^2 v) \\
&\leq -\{\frac{1}{2}(\sqrt{SC_f v_l} + C_f u_l v_l)(1 - \sqrt{\frac{S}{4C_f}}) - a_1(f(v_r) - u_r)\}\Delta z \\
&- \{\frac{1}{2}(\sqrt{SC_f v_l} + C_f u_l v_l)(1 + \sqrt{\frac{S}{4C_f}}) - a_2(f(v_r) - u_r)\}\Delta z \\
&\leq -(\sqrt{SC_f v_l} + C_f u_l v_l - O(1)\delta)\Delta z.
\end{aligned} \tag{4.14}$$

(4.6) and (4.14) imply

$$\Delta \bar{z} \leq \{1 - \frac{s}{\epsilon}(\sqrt{SC_f v_l} + C_f u_l v_l - O(1)\delta)\}\Delta z. \tag{4.15}$$

We can choose a positive number such that

$$1 - \frac{s}{\epsilon}(\sqrt{SC_f v_l} + C_f u_l v_l - O(1)\delta) > 0$$

provided $0 < s < \lambda\epsilon$.
By virtue of (4.15) and (4.16), one has

$$\bar{z}_r - \bar{z}_l \leq 0. \tag{4.16}$$

The next step is to show

$$\Delta\bar{w} \geq g_2(v_l, \Delta\bar{z}). \tag{4.17}$$

We show this by discussing the two cases.
Case 1. $\Delta(S - C_f u^2 v) \geq 0$.
In this case, we have

$$\begin{aligned} & g_2(v_l, \Delta\bar{z}) - g_2(v_l, \Delta z) \\ & = \frac{\partial g_2(v_l, X)}{\partial X}|_{X=\theta}(\Delta\bar{z} - \Delta z) \\ & = \frac{\partial g_2(v_l, X)}{\partial X}|_{X=\theta}\frac{s}{\epsilon}\Delta(S - C_f u^2 v), \end{aligned} \tag{4.18}$$

here θ is between Δz and $\Delta\bar{z}$. Since $0 \leq \frac{\partial g_2(v_l,X)}{\partial X}|_{X=\theta} < 1$ (cf. (1.37)), we have

$$g_2(v_l, \Delta z) \geq g_2(v_l, \Delta\bar{z}) - \frac{s}{\epsilon}\Delta(S - C_f u^2 v). \tag{4.19}$$

By virtue of (4.1) and (4.20), we obtain

$$\begin{aligned} & \Delta\bar{w} \\ & = \Delta w + \frac{s}{\epsilon}\Delta(S - C_f u^2 v) \\ & \geq g_2(v_l, \Delta z) + \frac{s}{\epsilon}\Delta(S - C_f u^2 v) \\ & \geq g_2(v_l, \Delta\bar{z}). \end{aligned} \tag{4.20}$$

(4.18) follows then in this case.
Case 2. $\Delta(S - C_f u^2 v) < 0$.
We have, in view of (4.9), that

$$\Delta w \geq c(-\Delta z) \geq 0, \tag{4.21}$$

for some $c > 0$, if we choose δ small. Using (4.8) again, by virtue of (4.11) and (4.21), we have, for some $c_1 > 0$ and $c_2 > 0$,

$$\begin{aligned}
&\Delta\bar{w} \\
&= \Delta w + \frac{s}{\epsilon}\Delta(S - C_f u^2 v) \\
&\geq \Delta w + \frac{s}{\epsilon}(c_1(-\Delta z) - c_2\Delta w) \\
&\geq \Delta w + \frac{s}{\epsilon}(c_1(-\Delta w)/c - c_2\Delta w) \\
&= \left(1 - \frac{s}{\epsilon}((c_1/c) + c_2)\right)\Delta w.
\end{aligned} \tag{4.22}$$

We can choose $\lambda > 0$ such that $1 - \frac{s}{\epsilon}((c_1/c) + c_2) > 0$ provided $0 < s < \lambda\epsilon$. Since $\delta w \geq 0$ (cf. (4.21)), we have $\delta\bar{w} \geq 0$. On the other hand, we have form (1.37) and (4.17) that $g_2(v_l, \Delta\bar{z}) \leq 0$, (4.18) follows then in this case, which completes the proof of this lemma. □

With Lemma 4.1, we can prove Theorem 4.1 by the following argument. Suppose $|u^s - f(v^s)|(x,t)$ is sufficiently small for $x \in R^1$ and $t \geq 0$. Then by Lemma 4.1, the Riemann solutions in each time step $T_n = \{(x,t) :\ x \in R^1,\ ns < t < (n+1)s\}$ $(n \geq 0$ are all R_1 and S_2. Thus, $v^s(x,t)$ is nondecreasing in x (cf. [22]). Moreover, since v is a parameter for the wave curves of (1.29), we have

$$T.V.(u^s(\cdot,t) \leq O(1)(T.V.(u_0) + T.V.(v_0)). \tag{4.23}$$

The smallness of $|u^s - f(v^s)|(x,t)$ can be verified as follows. Let $\underline{u}(t) = \lim_{x\to-\infty} u^s(x,t)$ and $\underline{v}(t) = \lim_{x\to-\infty} v^s(x,t)$. By (1.41) and our scheme, we have

$$\underline{u}(t) - f(\underline{v}(t)) = 0. \tag{4.24}$$

The smallness of $|u^s - f(v^s)|(x,t)$ thus follows from (4.23) and (4.24) if the total variation of initial data is small. Theorem 1.3 is then proved.
Once we have Theorem 1.3, the proof of theorems 1.4 and 1.5 becomes standard (cf. [18])

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