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# Compactness Framework of $L^p$ Approximate Solutions for Scalar Conservation Laws

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In this paper, we study the strong convergence of a sequence of uniform $L^p_{\text{loc}}(\mathbf{R} \times \mathbf{R}^+)$ bounded approximate solutions $\{u^\epsilon(x,t)\}$ to the following scalar conservation laws

$$u_t + f(x,t,u)_x + g(x,t,u) = 0, \qquad x \in \mathbf{R}, \mathbf{t} > \mathbf{0},$$

with initial data

$$u(x,0) = u_0(x) \in L^p(\mathbf{R}) \cap L^2(\mathbf{R}), \qquad x \in \mathbf{R}, 1 < p \leq +\infty.$$

Without the convexity assumption and growth condition at infinity for $f(x,t,u)$, we prove strong convergence of a subsequence of $\{u^\varepsilon(x,t)\}$. Under a more general growth condition than those in the previous work, we prove the existence of weak solution for the equation. The result obtained here generalizes those in earlier work. Some applications of the results are also given at the end of this paper.
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## 1. INTRODUCTION

We study the strong convergence of a sequence of uniform $L^p_{\text{loc}}(\mathbf{R}\times\mathbf{R}^+)$ bounded approximate solutions $\{u^\varepsilon(x,t)\}$ for the following scalar conservation laws

$$u_t + f(x,t,u)_x + g(x,t,u) = 0, \qquad x \in \mathbf{R}, \mathbf{t} > \mathbf{0}, \tag{1.1}$$

with initial data

$$u(x,0) = u_0(x) \in L^p(\mathbf{R}) \cap L^2(\mathbf{R}), \qquad x \in \mathbf{R}, 1 < p \leq +\infty. \tag{1.2}$$

When $f(x,t,u) = F(u)$, $g(x,t,u) \equiv 0$, (1.1) *and* (1.2) *were studied in* [11]. *The result obtained there is as follows.*

THEOREM 1.1. *Let* $\Omega$ *be an open bounded set in* $\mathbf{R}\times\mathbf{R}^+$, *let* $F(u) \in C^2(\mathbf{R})$ *satisfy the following conditions,*

$$(\mathrm{C}_1) \qquad F''(u) > 0 \text{ for } u \in \mathbf{R};$$

$$(\mathrm{C}_2) \qquad F(u) = o(|u|^p) \text{ as } |u| \to \infty, \tag{1.3}$$

*and* $|F'(u)| \leq C(1 + |u|^{p-1})$ *for some* $p > 1$.

*Suppose* $\{u^\varepsilon(x,t)\}$ *is a sequence of the approximate solutions of* (1.1), (1.2) *which are uniformly bounded in* $L^p(\Omega)$ *such that*

$$\frac{\partial}{\partial t}\eta(u^\varepsilon(x,t)) + \frac{\partial}{\partial x}q(u^\varepsilon(x,t)) \in H^{-1}_{\text{loc}}(\Omega) \tag{1.4}$$

*for every function pair* $(\eta(u), q(u)) \in C^2(\mathbf{R}, \mathbf{R}^2)$ *which satisfies*

$$q'(u) = \eta'(u)F'(u) \tag{1.5}$$

*with* $\eta(u)$ *of compact support. Then there exists a subsequence* (*still labeled by* $\{u^\varepsilon(x,t)\}$) *such that for all* $1 \leq r \leq p$

$$u^\varepsilon(x,t) \to u(x,t) \qquad \text{strongly in } L^r(\Omega).$$

The above result is generalized in [8] to the case when the flux function $F(u)$ does not satisfy the convexity condition $(\mathrm{C}_1)$, but a more strict growth condition. Precisely, the result in [8] is the following:

THEOREM 1.2. *Let* $\Omega \subset \mathbf{R}\times\mathbf{R}^+$ *be an open bounded set and let* $F(u) \in C^2(\mathbf{R})$ *satisfy the following conditions*:

$(\mathrm{C}_3)$ $\quad \operatorname{meas}\{u : F''(u) = 0\} = 0$;

$(\mathrm{C}_4)$ $\quad F(u) = o(|u|^{s+1}), F'(u) = o(|u|^s)$ *as* $|u| \to \infty$ *for some* $s \geq 0$.

*Furthermore, suppose that* $\{u^\varepsilon(x,t)\}$ *is uniformly bounded in* $L^{2(s+1)}(\Omega)$ *and*

$$\frac{\partial}{\partial t} I_n(u^\varepsilon(x,t)) + \frac{\partial}{\partial x} \Phi_n(u^\varepsilon(x,t)),$$

*and*

$$\frac{\partial}{\partial t} \Phi_n(u^\varepsilon(x,t)) + \frac{\partial}{\partial x} \Psi_n(u^\varepsilon(x,t)), \tag{1.6}$$

*lie in a compact set of* $H^{-1}_{\mathrm{loc}}(\Omega)$. *Then there exists a subsequence* (*still labeled by* $\{u^\varepsilon(x,t)\}$) *such that for all* $1 \le r \le 2(s+1)$

$$u^\varepsilon(x,t) \to u(x,t) \qquad \textit{strongly in } L^r(\Omega).$$

Here $I_n(u), \Phi_n(u), \Psi_n(u)$ are defined as

$$I_n(u) = \begin{cases} u, & \text{for } |u| \le n, \\ 0, & \text{for } |u| \ge 2n, \end{cases}$$

$$I_n(u) \in C^2(\mathbf{R}), \qquad |I_n(u)| \le |u|, \qquad |I_n'(u)| \le 2,$$

$$\Phi_n(u) = \int_0^u I_n'(s) F'(s)\, ds, \qquad \Psi_n(u) = \int_0^u \Phi_n'(s) F'(s)\, ds.$$

In this work, to have the strong convergence of the solutions sequence from the sequence of uniform $L^p_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+)$ bounded approximate solutions $\{u^\varepsilon(x,t)\}$, the flux function $F(u)$ is assumed to satisfy the growth condition

$$\lim_{|u| \to \infty} \frac{F(u)}{|u|^p} = 0.$$

Therefore, according to the above strong convergence results, one cannot deduce whether there exists a subsequence of $\{u^\varepsilon(x,t)\}$ for the viscous Burger's equations

$$\begin{cases} u_t^\varepsilon + \frac{1}{2}\left((u^\varepsilon)^2\right)_x = \varepsilon u_{xx}^\varepsilon, & x \in \mathbf{R}, t > 0, \\ u^\varepsilon(x,0) = u_0(x) \in L^2(\mathbf{R}), & x \in \mathbf{R}, \end{cases} \tag{1.7}$$

which strongly converges to an admissible solution $u(x,t)$ of the non-viscous Burger's equation

$$\begin{cases} u_t + \frac{1}{2}(u^2)_x = 0, & x \in \mathbf{R}, t > 0, \\ u(x,0) = u_0(x) \in L^2(\mathbf{R}), & x \in \mathbf{R}. \end{cases} \tag{1.8}$$

In this paper, we will consider the system (1.1) and (1.2). By constructing the entropy-entropy flux pairs (3.4), (3.6) of Lax type with $\eta_{\pm m}(u)$ which are compactly supported, we succeed in reducing the generalized Young measures $\nu_{x,t} \in \mathrm{Prob}(\mathbf{R}^2)$, which determine the weak limit of the sequence of uniform $L^p_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+)$ bounded approximate solutions $\{u^\varepsilon(x,t)\}$ of (1.1), (1.2), to Dirac measures. By applying the theory of compensated compactness, a strong convergence theorem on the sequence is established. In our analysis, no convexity condition and growth conditions at infinity are imposed on the flux function $f(x,t,u)$.

The rest of the paper is arranged as follows: In Section 2, we give an alternative proof of the representation of Young measures which enables us to establish a general compactness framework for the application of compensated compactness theory. In Section 3, we construct several families of entropy-entropy flux of Lax's type. In Section 4, we prove that the family of Young measures $\nu_{x,t}$ is indeed a family of Dirac masses using the compactness framework. In the last section, we apply our compactness result to some models. We prove the strong convergence of the sequence of uniform $L^p_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+)$ $(1 < p < \infty)$ bounded viscosity solutions $\{u^\varepsilon(x,t)\}$ for the Cauchy problem (1.1), (1.2); and the existence of global entropy solutions to a combustion model with $L^p_{\mathrm{loc}}$ bounded initial data when the flux function satisfies a general growth rate.

Throughout this paper, we will use $C(\theta)$ to denote a generic constant depending only on $\theta$.

## 2. PRELIMINARIES

For the paper to be self-contained, we first give several fundamental results for later use.

From now on, $\mathbf{R}^{\mathbf{N}}$ denotes $N$-dimensional real Euclidean space; $\overline{\mathbf{R}}^N = \mathbf{R}^N \cup \{\pm\infty\}$; $\mathbf{R}^1 = \mathbf{R}$; $\mathbf{R}^+ = \{a \geq 0,\ a \in \mathbf{R}\}$; $C(\mathbf{R}^N)$ is the space of continuous functions; while $C_0(\mathbf{R}^N)$ is the space of continuous functions with compact support; $M(\mathbf{R}^N)$ is the dual space of $C_0(\mathbf{R}^N)$.

THEOREM 2.1. *Let $\Omega \subset \mathbf{R}^n$ be measurable, and $u^\varepsilon(x): \Omega \to \overline{\mathbf{R}}^s$ be a sequence of measurable functions. Then there exists a subsequence (still labeled by $\{u^\varepsilon(x)\}$) and a family of positive measures $\mu_x \in M(\mathbf{R}^s)$, depending* *measurably on* $x \in \Omega$, *such that for any* $f(x, y) \in C_0(\Omega \times \overline{\mathbf{R}}_s)$

$$f(x, u^\varepsilon(x)) \overset{*}{\rightharpoonup} \langle \mu_x, f(x, \lambda) \rangle = \int_{\mathbf{R}^s} f(x, \lambda)\, d\mu_x, \qquad \text{in } L^\infty(\Omega).$$

We refer the readers to Hormander [4, Theorem 5.2.5] for a proof.

The following is a corollary of Theorem 2.1.

COROLLARY 2.2. *Suppose that* $\{u^\varepsilon(x)\}$ *is uniformly bounded in* $L^p_{\mathrm{loc}}(\mathbf{R}^n)$ $(1 < p \le \infty)$. *Then there exists a subsequence (still labeled by* $\{u^\varepsilon(x)\}$*) and a family of positive measures* $\mu_x \in M(\mathbf{R}^s)$, $x \in \mathbf{R}^n$, *such that for any bounded measurable set* $A \subset \mathbf{R}^n$

$$f(x, u^\varepsilon(x)) \rightharpoonup \langle \mu_x, f(x, \lambda) \rangle = \int_{\mathbf{R}^s} f(x, \lambda)\, d\mu_x, \qquad \text{in } L^1(A),$$

*whenever* $f(x, y) \in C(\mathbf{R}^n \times \mathbf{R}^s)$ *satisfies*

$$\lim_{|\lambda| \to \infty} \frac{f(x, \lambda)}{|\lambda|^p} = 0, \qquad \text{uniformly for all } x \in A.$$

*Proof.* The proof is similar to that of Corollary 2.2 in [7]. To be self-contained, we give the proof as follows.

Without loss of generality, let $f(x, u) \ge 0$. We introduce $f^m(x, u) \in C_0(\mathbf{R}^n \times \mathbf{R}^s)$ by defining $f^m(x, u) = \theta^m(x, u) f(x, u)$, where $\theta^m(x, u) \in C_0(\mathbf{R}^n \times \mathbf{R}^s)$ is given by

$$\theta^m(x, u) = \theta_1^m(x)\theta_2^m(u),$$

$$\theta_1^m(x) = m^n \int_{\mathbf{R}^n} \chi_A(y) \prod_{i=1}^n \rho(m(x_i - y_i))\, dy,$$

$$\theta_2^m(u) = \begin{cases} 1, & \text{for } |u| \le m, \\ 1 + m - |u|, & \text{for } m < |u| \le m + 1, \\ 0, & \text{for } |u| > m + 1, \end{cases}$$

$$\chi_A(x) = \begin{cases} 1, & \text{for } x \in A, \\ 0, & \text{for } x \notin A, \end{cases}$$

where $\rho(x)$ is a mollifier; that is, $0 \le \rho(x) \in C_0^\infty(\mathbf{R})$, $\operatorname{supp} \rho(x) \subset [-1, 1]$, and $\int_{\mathbf{R}} \rho(x)\, dx = 1$.

It's easy to see that $f^m(x, u) \in C_0(A_{1/m} \times B_{m+1})$ with

$$A_{1/m} = \left\{ x \in \mathbf{R}^n, \operatorname{dist}(x, A) \le \frac{1}{m} \right\},$$

$$B_{m+1} = \{ y \in \mathbf{R}^S, |y| \le m + 1 \}.$$

We claim that for each $\phi(x) \in L^\infty(A)$

$$\lim_{m\to\infty}\int_{A_{1/m}} \bar{\phi} f^m(x, u^\varepsilon(x))\,dx = \lim_{m\to\infty}\int_A \phi f^m(x, u^\varepsilon(x))\,dx$$

$$= \int_A \phi f(x, u^\varepsilon(x))\,dx \qquad (2.1)$$

uniformly in $\varepsilon$, where

$$\bar{\phi}(x) = \begin{cases} \phi(x), & \text{for } x \in A, \\ 0, & \text{for } x \notin A. \end{cases}$$

Indeed

$$\left|\int_A \phi\{f^m(x, u^\varepsilon(x)) - f(x, u^\varepsilon(x))\}\,dx\right|$$

$$\leq \left|\int_A \phi\theta_2^m(u^\varepsilon(x)) f(x, u^\varepsilon(x))(\theta_1^m(x) - \chi_A(x))\,dx\right|$$

$$+ \left|\int_A \phi\{\theta_2^m(u^\varepsilon(x)) - 1\} f(x, u^\varepsilon(x))\,dx\right|$$

$$\leq C\|\phi\|_{L^\infty(A)}\|\theta_1^m(x) - \chi_A(x)\|_{L^\infty(A)}\int_A |u^\varepsilon(s)|^p\,dx$$

$$+ \|\phi\|_{L^\infty(A)}\int_{\{x\in A,\,|u^\varepsilon(x)|\geq m\}} f(x, u^\varepsilon(x))\,dx$$

$$\leq C\|\phi\|_{L^\infty(A)}\|u^\varepsilon(x)\|_{L^p(A)}^p\left\{\|\theta_1^m(x) - \chi_A(x)\|_{L^\infty(A)} + \sup_{\substack{|\lambda|\geq m\\ x\in A}}\frac{f(x,\lambda)}{|\lambda|^p}\right\},$$

which tends to zero uniformly in $\varepsilon$ as $m \to \infty$.

On the other hand, since $f^m(x, u) \in C_0(A_{1/m} \times B_{m+1})$, then by Theorem 2.1, there exists a subsequence (still labeled by $\{u^\varepsilon(x)\}$) and a family of positive measures $\mu_x \in M(\mathbf{R}^s)$ such that for each $m$ and any $\bar{\phi} \in L^\infty(A_{1/m})$

$$\lim_{\varepsilon\to 0}\int_{A_{1/m}} \bar{\phi} f^m(x, u^\varepsilon(x))\,dx = \int_{A_{1/m}} \bar{\phi}\langle \mu_x, f^m(x,\lambda)\rangle\,dx$$

$$= \int_A \phi\langle \mu_x, f^m(x,\lambda)\rangle\,dx. \qquad (2.2)$$

Using (2.1) and (2.2), if we can prove for any $\phi \in L^\infty(A)$

$$\lim_{m\to\infty} \int_A \phi \langle \mu_x, f^m(x,\lambda)\rangle\, dx = \int_A \phi \langle \mu_x, f(x,\lambda)\rangle\, dx, \tag{2.3}$$

then by Lebesgue Theorem, Corollary 2.2 follows.

In fact, from the definition of $f^m(x,u)$, we have

$$\begin{aligned} &\left| \int_A \phi \langle \mu_x, f^m(x,\lambda) - f(x,\lambda)\rangle\, dx \right| \\ &\quad \le \left| \int_A \phi \{\theta_1^m(x) - \chi_A(x)\} \langle \mu_x, \theta_2^m(\lambda) f(x,\lambda)\rangle\, dx \right| \\ &\qquad + \left| \int_A \phi \langle \mu_x, (\theta_2^m(\lambda) - 1) f(x,\lambda)\rangle\, dx \right|. \end{aligned} \tag{2.4}$$

Since

$$\begin{aligned} &\left| \int_A \phi \{\theta_1^m(x) - \chi_A(x)\} \langle \mu_x, \theta_2^m(\lambda) f(x,\lambda)\rangle\, dx \right| \\ &\quad \le C \|\phi\|_{L^\infty(A)} \|\theta_1^m(x) - \chi_A(x)\|_{L^\infty(A)} \left| \int_A \langle \mu_x, f(x,\lambda)\rangle\, dx \right| \\ &\quad \to 0, \qquad \text{as } m \to \infty, \end{aligned} \tag{2.5}$$

and from the monotone convergence theorem

$$\lim_{m\to\infty} \left| \int_A \phi \langle \mu_x, (\theta_2^m(\lambda) - 1) f(x,\lambda)\rangle\, dx \right| = 0. \tag{2.6}$$

From (2.4), (2.5), and (2.6), we can easily deduce that (2.3) holds. This completes the proof of Corollary 2.2. Q.E.D.

Now we state the following compactness result without a proof for brevity of the paper. The interested readers can refer to [3, 7, 11, 13].

THEOREM 2.3. *Let $\{u^\varepsilon(x)\}$ be uniformly bounded in $L^p_{\mathrm{loc}}(\mathbf{R}^n)$ $(1 < p \le \infty)$. If the corresponding generalized Young measures $\mu_x$ are Dirac masses, i.e.,*

$\mu_x = \delta_{u(x)}$, *then there exists a subsequence* (*still labeled by* $\{u^\varepsilon(x)\}$) *such that*

$$u^\varepsilon(x) \to u(x), \qquad \textit{strongly in } L^r_{\text{loc}}(\mathbf{R}^n)$$

*for all r satisfying* $1 \le r \le p$.

## 3. ESTIMATES OF ENTROPY-ENTROPY FLUX

In this section, we study the entropy-entropy flux pairs for the system (1.1). We will construct the entropy-entropy flux pairs of Lax type with compact support, and then derive some estimates for these pairs which will be used later.

A pair of real-valued functions $(\eta(u), q(x,t,u))$ is called an entropy-entropy flux pair of (1.1), if for all smooth solutions of (1.1), we have

$$\frac{\partial}{\partial u} q(x,t,u) = \frac{\partial}{\partial u} f(x,t,u)\eta'(u). \tag{3.1}$$

In the following discussion, we will use the $C^2$ entropy-entropy flux pair $(\eta(u), q(x,t,u))$ with $\eta(u)$ of compact support. These entropy-entropy flux pairs are shown to satisfy the following property. For convenience, we introduce some notations first.

DEFINITION 3.1. $\phi(x,t,u) \in B_k(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R})$ if for every $U > 0$, there exists $C(U)$ independent of $x, t$, such that

$$\left| \frac{\partial^j}{\partial u^j} \phi(x,t,u) \right| \le C(U), \qquad |u| \le U, 0 \le j \le i.$$

PROPOSITION 3.2. *Let* $f(x,t,u) \in C^{1,1,2}(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}) \cap B_2(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R})$. *Suppose* $\eta(u)$ *is any* $C^2$ *function of compact support. Then there exists an entropy-entropy flux* $(\eta(u), q(x,t,u))$ *such that* $q(x,t,u)$ *satisfies*

$$q(x,t,u) \in C^{1,1,2}(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}) \cap B_0(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}).$$

*Proof.* Since $\eta(u) \in C_0^2(\mathbf{R})$ and $f(x,t,u) \in C^{1,1,2}(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R})$, then $q(x,t,u) \in C^{1,1,2}(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R})$ follows immediately. We now prove there exists $q(x,t,u) \in B_0(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R})$ corresponding to $\eta(u)$. Let $\operatorname{supp} \eta(u) \subset [-N, N]$. Then integrating (3.1) gives

$$q(x,t,u) = q(x,t,-N) + \frac{\partial}{\partial u} f(x,t,u)\eta(u) - \int_{-N}^{u} \frac{\partial^2 f(x,t,s)}{\partial s^2} \eta(s)\, ds.$$

By choosing $q(x, t, -N) = Q_0$, where $Q_0$ is dependent of $x, t$, we have

$$q(x,t,u) = \begin{cases} Q_0, & u < -N, \\ Q_0 + \dfrac{\partial}{\partial u} f(x,t,u)\eta(u) + \displaystyle\int_{-N}^{u} \dfrac{\partial^2 f(x,t,s)}{\partial s^2} \eta(s)\, ds, & \\ & |u| \le N, \\ Q_0 - \displaystyle\int_{-N}^{N} \dfrac{\partial^2 f(x,t,s)}{\partial s^2} \eta(s)\, ds, & u > N. \end{cases}$$

Hence $q(x, t, u) \in B_0(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R})$. Q.E.D.

In the rest of the section, we will estimate the entropy flux $q(x, t, u)$ in detail. Let

$$E = \Big\{(\eta, q) : \eta(u) \in C_0^2(\mathbf{R}),$$
$$q(x,t,u) \in C^{1,1,2}(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}) \cap B_0(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R})\Big\},$$

and

$$E' = \Big\{(\eta, q) : \eta(u) \in C^2(\mathbf{R}) \cap B_0(\mathbf{R}),$$
$$q(x,t,u) \in C^{1,1,2}(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}) \cap B_0(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R})\Big\}.$$

We now construct the following entropy-entropy flux pairs $(\eta_{\pm m}(u), q_{\pm m}(x, t, u))$ of Lax type with $(\eta_{\pm m}(u), q_{\pm m}(x, t, u)) \in E$, for $m \in \mathbf{N}$. Let

$$\eta_{\pm m}(u) = e^{\pm A(m)u} \phi_m(u),$$

where $\phi_m(u) \in C_0^2(\mathbf{R})$ with supp $\phi_m(u) \subset [-m, m]$, $\|\phi_m(u)\|_{L^\infty} \le C_0$, and

$$A(m) = \left( \max\left\{ m, \max_{|u| \le m} \left| \frac{\partial^2 f(x,t,u)}{\partial u^2} \right| \right\} \right)^2.$$

By Proposition 3.2, we can find the corresponding entropy flux $q_{\pm m}(x, t, u)$ with respect to $\eta_{\pm m}(u)$, i.e.,

$$q_{\pm m}(x,t,u) = \begin{cases} C_{\pm m}^-, & u < -m, \\ e^{\pm A(m)u} \left( \phi_m(u) \dfrac{\partial f(x,t,u)}{\partial u} + F_{\pm m}(x,t,u) \right), & \\ & |u| \le m, \\ C_{\pm m}^+(x,t), & u > m, \end{cases}$$

where the $C^-_{\pm m}$ are constants and $C^+_{\pm m}(x,t)$ are $C^1$ bounded functions of $(x,t)$. Using (3.1) we have the following ordinary equation for $F_{\pm m}(x,t,u)$,

$$\frac{\partial F_{\pm m}(x,t,u)}{\partial u} \pm A(m) F_{\pm m}(x,t,u) + \phi_m \frac{\partial^2 f(x,t,u)}{\partial u^2} = 0, \qquad \text{for } |u| < m.$$

Especially, we can choose

$$F_m(x,t,u) = -e^{-A(m)u} \int_{-m}^{u} \phi_m(s) \frac{\partial^2 f(x,t,s)}{\partial s^2} e^{A(m)s}\, ds,$$

$$F_{-m}(x,t,u) = e^{A(m)u} \int_{u}^{m} \phi m(s) \frac{\partial^2 f(x,t,s)}{\partial s^2} e^{-A(m)s}\, ds.$$

Thus we have

$$|F_{\pm m}(x,t,u)| \le \frac{C_0}{\sqrt{A(m)}}, \qquad \text{for all } |u| < m. \tag{3.2}$$

The above entropy-entropy pairs $(\eta_{\pm m}, q_{\pm m})$ will be used to prove the strong convergence of the uniform $L^p_{\mathrm{loc}}$ bounded sequence$\{u^\varepsilon(x,t)\}$ in the following section.

## 4. COMPACTNESS FRAMEWORK

In this section, we will establish a framework for the study of the strong convergence of the approximate solutions $\{u^\varepsilon(x,t)\}$ or, equivalently, the corresponding family of Young measures $\nu_{x,t} \in \mathrm{Prob}(\mathbf{R}^2)$ to the Cauchy problem (1.1), (1.2).

THEOREM 4.1. *Let $\Omega \subset \mathbf{R} \times \mathbf{R}^+$ be any open bounded set, suppose that $f(x,t,u) \in C^{1,1,2}(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R}) \cap B_2(\mathbf{R} \times \mathbf{R}^+ \times \mathbf{R})$, $\mathrm{meas}\{\partial^2 f(x,t,u)/\partial u^2 = 0\} = 0$, and $\{u^\varepsilon(x,t)\}$ is a sequence of approximate solutions of the Cauchy problem* (1.1), (1.2) *which is uniformly bounded with respect to $\varepsilon$ in $L^p_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+)$ $(1 < p \le \infty)$. If we assume further*

$$\frac{\partial}{\partial t}\eta(u^\varepsilon(x,t)) + \frac{\partial}{\partial x} q(x,t,u^\varepsilon(x,t)) \in H^{-1}_{\mathrm{loc}}(\Omega) \tag{4.1}$$

*for all entropy-entropy flux pairs $(\eta(u), q(x,t,u)) \in E \cup E'$, then there exists a subsequence (still labeled by $\{u^\varepsilon(x,t)\}$) such that*

$$u^\varepsilon(x,t) \to u(x,t), \qquad \textit{strongly in } L^r_{\mathrm{loc}}(\mathbf{R} \times \mathbf{R}^+) \textit{ as } \varepsilon \to 0,$$

*where $1 \le r \le p$.*

*Furthermore, if the functions* $f(x,t,u)$ *and* $g(x,t,u)$ *satisfy the growth conditions*

$$\left|\frac{\partial f(x,t,u)}{\partial u}\right| \le C(1+|u|^{p-1}),$$
$$\left|\frac{\partial g(x,t,u)}{\partial u}\right| \le C(1+|u|^{p-1}), \qquad (x,t) \in \mathbf{R}\times\mathbf{R}^+,$$

*where C is a constant independent of* $(x,t,u)$, *then the limit function* $u(x,t)$ *is a global weak solution for the Cauchy problem* (1.1), (1.2); *that is,*

$$\iint_{t>0} (u\phi_t + f(x,t,u)\phi_x + g(x,t,u)\phi)\,dx\,dt + \int_{\mathbf{R}} u_0(x)\phi(x,0)\,dx = 0$$

*for any* $\phi \in C_0(\mathbf{R}\times\mathbf{R}^+)$.

*Proof.* From Theorem 2.3, we only need to show that the Young measures $\nu_{x,t}$, which are uniquely determined by the sequence of approximate solutions $\{u^\varepsilon(x,t)\}$ of (1.1), (1.2), are the Dirac masses.

We will prove this by contradiction. Suppose the Young measures $\nu_{x,t}$ are not Dirac masses. Then we first show that

$$\operatorname{supp}\nu_{x,t} = \{-\infty,\infty\} \qquad \text{or} \qquad \operatorname{supp}\nu_{x,t} = \{-\infty,u_0,\infty\}. \tag{4.2}$$

for some $u_0 \in (-\infty,\infty)$. To do this, we discuss the following four cases.

(a) there exists a sequence $\{u_N^+\}$ such that

$$\lim_{N\to\infty} u_N^+ = \infty, \qquad \text{and} \qquad \nu_{x,t}(N(u_N^+)) \ne 0,$$

where $N(\theta)$ is a neighborhood containing $\theta$.

(b) for any sequence $\{u_N^+\}$ with $\lim_{N\to\infty} u_N^+ = \infty$, $\nu_{x,t}(N(u_N^+)) = 0$. Denote the smallest $u$ such that $\nu_{x,t}(s>u) = 0$ by $u^+$;

(c) there exists a sequence $\{u_N^-\}$ such that

$$\lim_{N\to\infty} u_N^- = -\infty, \qquad \text{and} \qquad \nu_{x,t}(N(u_N^-)) \ne 0;$$

(c) for any sequence $\{u_N^-\}$ with $\lim_{N\to\infty} u_N^- = -\infty$, but $\nu_{x,t}(N(u_N^-)) = 0$, denote the largest $u$ such that $\nu_{x,t}(s<u) = 0$ by $u^-$.

In fact, for $\nu_{x,t}$, one of the following cases holds:

*Case* 1. Cases (a), (c) hold.

*Case* 2. Cases (a), (d) hold.

*Case* 3. Cases (b), (c) hold.

*Case* 4. Cases (b), (d) hold.

Let $Q$ denote the smallest characteristic rectangle

$$Q = \{u : u^{+} \leq u \leq u^{+}\}$$

which contains the support of $\nu_{x,t}$, where $u^{-}$ and $u^{+}$ may be $-\infty$ or $\infty$.

In Case 1, as in [3], we introduce the probability measures $\mu_{\pm m}$ on $Q$ defined by

$$\langle \mu_{\pm m}, h\rangle = \frac{\langle \nu_{x,t}, h\eta_{\pm m}\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle},$$

where $h = h(x,t,u)$ denotes an arbitrary function in $C(\mathbf{R} \times \mathbf{R}^{+} \times \mathbf{R}) \cap B_0(\mathbf{R} \times \mathbf{R}^{+} \times \mathbf{R})$, and $(\eta_{\pm m}, q_{\pm m})$ is the entropy-entropy flux defined in Section 3.

As a consequence of weak $*$ compactness, there exists probability measures $\delta^{\pm}$ on $\mathbf{R}$ such that

$$\langle \delta^{\pm}, h\rangle = \lim_{m\to\infty} \langle \mu_{\pm m}, h\rangle, \tag{4.3}$$

by selecting an appropriate subsequence. As in [3], we can deduce that the measures $\delta^{+}$ and $\delta^{-}$ are concentrated at $\infty$ and $-\infty$ respectively, i.e., $\operatorname{supp}\delta^{+} = \{\infty\}$ and $\operatorname{supp}\delta^{-} = \{-\infty\}$.

By the assumptions of Theorem 4.1, we have by applying the Tartar–Murat Lemma,

$$\begin{aligned}\langle \nu_{x,t}, \eta_1(\lambda)q_2(x,t,\lambda) - \eta_2(\lambda)q_1(x,t,\lambda)\rangle \\ = \langle \nu_{x,t}, \eta_1(\lambda)\rangle\langle \nu_{x,t}, q_2(x,t,\lambda)\rangle - \langle \nu_{x,t}, \eta_2(\lambda)\rangle\langle \nu_{x,t}, q_1(x,t,\lambda)\rangle,\end{aligned} \tag{4.4}$$

for any entropy-entropy flux pairs $(\eta_i(u), q_i(x,t,u)) \in E \cup E', i = 1, 2$.

Thus we have for sufficiently large $m$

$$\langle \nu_{x,t}, q\rangle - \langle \nu_{x,t}, \eta\rangle \frac{\langle \nu_{x,t} q_{\pm m}\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle} = \frac{\langle \nu_{x,t}, q\eta_{\pm m} - q_{\pm m}\eta\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle}, \tag{4.5}$$

whenever $(\eta, q) \in E \cup E'$.

Let $\operatorname{supp}\eta(u)\subset[-N,N]$. Then for sufficiently large $m$

$$\frac{\langle \nu_{x,t}, q\eta_{\pm m} - q_{\pm m}\eta\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle} = \frac{\langle \nu_{x,t}, q\eta_{\pm m}\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle} - \frac{(\langle \nu_{x,t}, q_{\pm m}\eta\rangle)|_{|u|\le N}}{\langle \nu_{x,t}, \eta_{\pm m}\rangle}$$
$$= \frac{\langle \nu_{x,t}, q\eta_{\pm m}\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle} - \frac{\langle \nu_{x,t}, f_u\eta\eta_{\pm m}\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle}$$
$$- \frac{\langle \nu_{x,t}, \eta F_{\pm m}\eta_{\pm m}\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle}. \tag{4.6}$$

From (3.9), we have

$$\left|\frac{\langle \nu_{x,t}, \eta F_{\pm m}\eta_{\pm m}\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle}\right| \le \frac{C}{\sqrt{A(m)}}, \tag{4.7}$$

where $C$ is a positive constant independent of $m$.

Thus from (4.3), (4.6), (4.7), we have

$$\lim_{m\to\infty} \frac{\langle \nu_{x,t}, q\eta_{\pm m} - q_{\pm m}\eta\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle} = \langle \delta^{\pm}, q - f_u\eta\rangle. \tag{4.8}$$

Equations (4.5) and (4.8) show that the following limit exists

$$\lim_{m\to\infty} \frac{\langle \nu_{x,t}, q_{\pm m}\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle}.$$

Let

$$\lambda^{\pm} = \lim_{m\to\infty} \frac{\langle \nu_{x,t}, q_{\pm m}\rangle}{\langle \nu_{x,t}, \eta_{\pm m}\rangle}. \tag{4.9}$$

Then letting $m\to\infty$ in (4.5), we have

$$\langle \delta^{\pm}, q - f_u\eta\rangle = \langle \nu_{x,t}, q\rangle - \lambda^{\pm}\langle \nu_{x,t}, \eta\rangle. \tag{4.10}$$

Next, we claim that

$$\lambda^{+} = \lambda^{-}. \tag{4.11}$$

Indeed, we have from (4.4)

$$\frac{\langle \nu_{x,t}, q_m\rangle}{\langle \nu_{x,t}, \eta_m\rangle} - \frac{\langle \nu_{x,t}, q_{-m}\rangle}{\nu_{x,t}, \eta_{-m}\rangle} = \frac{\langle \nu_{x,t}, q_m\eta_{-m} - q_{-m}\eta_m\rangle}{\langle \nu_{x,t}, \eta_m\rangle\langle \nu_{x,t}, \eta_{-m}\rangle}. \tag{4.12}$$

It follows for sufficiently large $m$ with $m > u_N^+, u_N^- < -m$ for some $N$. Then

$$\langle \nu_{x,t}, \eta_m \rangle \geq \langle \nu_{x,t}, \eta_m \rangle|_{u \geq u_N^+} \geq C_4 e^{A(m)u_N^+}$$

and

$$\langle \nu_{x,t}, \eta_{-m} \rangle \geq \langle \nu_{x,t}, \eta_{-m} \rangle|_{u \leq u_N^-} \geq C_5 e^{-A(m)u_N^-}.$$

Hence, for large $m$ the denominator on the right hand side of (4.12) approaches infinity. The numerator is in general $O(1/\sqrt{A(m)})$. Thus the right hand side approaches zero while the left approaches $\lambda^+ - \lambda^-$. This proves the assertion (4.11).

Equations (4.10) and (4.11) show that

$$\langle \delta^+, q - f_u \eta \rangle = \langle \delta^-, q - f_u \eta \rangle; \tag{4.13}$$

that is,

$$q(x,t,\infty) - \frac{\partial f(x,t,\infty)}{\partial u}\eta(\infty) = q(x,t,-\infty) - \frac{\partial f(x,t,-\infty)}{\partial u}\eta(-\infty) \tag{4.14}$$

for any entropy-entropy flux pair $(\eta(u), q(x,t,u)) \in E \cup E'$.

Integrating (3.1) over $(-\infty, +\infty)$, we have

$$\begin{aligned} q(x,t,\infty) = q(x,t,-\infty) &+ \frac{\partial f(x,t,\infty)}{\partial u}\eta(\infty) \\ &- \frac{\partial f(x,t,-\infty)}{\partial u}\eta(-\infty) - \int_{-\infty}^{\infty} \frac{\partial^2 f(x,t,s)}{\partial s^2}\eta(s)\, ds. \end{aligned} \tag{4.15}$$

Combining (4.14) with (4.15), we have

$$\int_{-\infty}^{\infty} \frac{\partial^2 f(x,t,s)}{\partial s^2}\eta(s)\, ds = 0, \qquad (x,t) \in \mathbf{R} \times \mathbf{R}^+, \tag{4.16}$$

for any $C^2$ compact supported entropy $\eta(u)$.

Let

$$S_1(x,t) = \left\{ u \in \mathbf{R} : \frac{\partial^2 f(x,t,u)}{\partial u^2} > 0 \right\},$$

$$S_2(x,t) = \left\{ u \in \mathbf{R} : \frac{\partial^2 f(x,t,u)}{\partial u^2} < 0 \right\},$$

and

$$S_3(x,t) = \left\{ u \in \mathbf{R} : \frac{\partial^2 f(x,t,u)}{\partial u^2} = 0 \right\}.$$

Then, for every $(x,t) \in \mathbf{R} \times \mathbf{R}^+$, $S_1(x,t)$ and $S_2(x,t)$ are an open subset of $\mathbf{R}$. Thus there exist two sequences of non-overlapping open intervals $\{(a^1_{i_k}(x,t), a^1_{i_k+1}(x,t))\}$ and $\{(a^2_{i_k}(x,t), a^2_{i_k+1}(x,t))\}$, at most countable, such that

$$S_1(x,t) = \bigcup_k \left( a^1_{i_k}(x,t), a^1_{i_k+1}(x,t) \right),$$
$$S_2(x,t) = \bigcup_k \left( a^2_{i_k}(x,t), a^2_{i_k+1}(x,t) \right)$$

and

$$\left( a^\alpha_{i_k}(x,t), a^\alpha_{i_k+1}(x,t) \right) \cap \left( a^\beta_{i_l}(x,t), a^\beta_{i_l+1}(x,t) \right) = \varnothing,$$

for all $\alpha, \beta \in \{1, 2\}, k, l$ with $(\alpha, k) \neq (\beta, l)$. Thus

$$S_3(x,t) \subset \bigcup_k \left\{ a^1_{i_k+1}(x,t), a^2_{i_k+1}(x,t) \right\}$$

and

$$\mathbf{R} \subset \bigcup_{n=1}^{2} \left( \bigcup_k \left( a^n_{i_k}(x,t), a^n_{i_k+1}(x,t) \right] \right)$$

for every $(x,t) \in \mathbf{R} \times \mathbf{R}^+$.

Choosing $\eta(u)$ of the following form in (4.16)

$$\eta(u) = \begin{cases} 1, & \text{for } u \in I_0, \\ 0, & \text{for } u \notin \left( a^1_{i_{k_0}}(x,t), a^1_{i_{k_0}+1}(x,t) \right), \end{cases}$$
$$0 \leq \eta(u) \in C^2(\mathbf{R}),$$

where $I_0 = (\frac{1}{4}(3a^1_{i_{k_0}}(x,t) + a^1_{i_{k_0}+1}(x,t)), \frac{1}{4}(a^1_{i_{k_0}}(x,t) + 3a^1_{i_{k_0}+1}(x,t)))$, we have

$$0 = \int_{-\infty}^{\infty} \frac{\partial^2 f(x,t,s)}{\partial s^2} \eta(s)\, ds \geq \int_{I_0} \frac{\partial^2 f(x,t,s)}{\partial s^2}\, ds > 0.$$

This contradiction shows that Case 1 is impossible.

For Case 2, we can prove that the measures $\delta^+$ and $\delta^-$ are concentrated at $\infty$ and $u^-$, respectively, i.e., $\delta^+ = \delta_\infty$ and $\delta^- = \delta_{u^-}$. Similar to Case 1, we deduce $u^- = \infty$. Thus

$$\operatorname{supp} \nu_{x,t} = \{-\infty, \infty\}. \tag{4.17}$$

Case 3 is similar to Case 2.

For Case 4, we can prove $u^- = u^-$ similar to Case 2. Thus

$$\operatorname{supp} \nu_{x,t} = \{-\infty, u_0, \infty\} \tag{4.18}$$

for some $u_0 \in (-\infty, \infty)$.

From (4.17) and (4.18), there exist $a_i$ $(i = 1,2,3)$ with $a_1 \geq 0, a_2 \geq 0, a_3 \geq 0, \sum_{i=1}^3 a_i = 1$ such that

$$\nu_{x,t} = a_1 \delta_{-\infty} + a_2 \delta_{u_0} + a_3 \delta_\infty. \tag{4.19}$$

From (4.4) and (4.19), we have

$$\begin{aligned} &a_1\big(\eta_1(-\infty) q_2(x,t,-\infty) - \eta_2(-\infty) q_1(x,t,-\infty)\big) \\ &\quad + a_2\big(\eta_1(u_0) q_2(x,t,u_0) - \eta_2(u_0) q_1(x,t,u_0)\big) \\ &\quad + a_3\big(\eta_1(\infty) q_2(x,t,\infty) - \eta_2(\infty) q_1(x,t,\infty)\big) \\ &\quad = \big(a_1\eta_1(-\infty) + a_2\eta_1(u_0) + a_3\eta_1(\infty)\big) \\ &\qquad \times \big(a_1 q_2(x,t,-\infty) + a_2 q_2(x,t,u_0) + a_3 q_2(x,t,\infty)\big) \\ &\qquad - \big(a_1\eta_2(-\infty) + a_2\eta_2(u_0) + a_3\eta_2(\infty)\big) \\ &\qquad\quad \times \big(a_1 q_1(x,t,-\infty) + a_2 q_1(x,t,u_0) + a_3 q_1(x,t,\infty)\big) \end{aligned} \tag{4.20}$$

for any entropy-entropy flux pairs $(\eta_i(u), q_i(x,t,u)) \in E \cup E'$, $i = 1,2$.

Let

$$\eta_1(u) = \begin{cases} 0, & \text{for } u \leq u_0, \\ 1, & \text{for } u > u_0 + 1, \end{cases}$$

$$\eta_2(u) = \begin{cases} 0, & \text{for } u < u_0 - 1, \\ 1, & \text{for } u \geq u_0, \end{cases}$$

$$q_1(x,t,u) = \begin{cases} \alpha_1, & \text{for } u \le u_0, \\ \alpha_1 + \int_{u_0}^{u} f_u(x,t,u) \frac{\partial}{\partial u} \eta_1(u)\, du, & \text{for } u_0 < u \le u_0 + 1, \\ \alpha_1 + \int_{u_0}^{u_0+1} f_u(x,t,u) \frac{\partial}{\partial u} \eta_1(u)\, du, & \text{for } u > u_0 + 1, \end{cases}$$

$$q_2(x,t,u) = \begin{cases} \alpha_2, & \text{for } u \le u_0 - 1, \\ \alpha_2 + \int_{u_0-1}^{u} f_u(x,t,u) \frac{\partial}{\partial u} \eta_2(u)\, du, & \text{for } u_0 - 1 < u \le u_0, \\ \alpha_2 + \int_{u_0-1}^{u_0} f_u(x,t,u) \frac{\partial}{\partial u} \eta_2(u)\, du, & \text{for } u > u_0, \end{cases}$$

with $(\eta_i(u), q_i(x,t,u)) \in E'$, where the $\alpha_i$ are arbitrary constants. Then (4.20) deduces

$$\begin{aligned} -\alpha_2\alpha_1 + \alpha_3\biggl(\alpha_2 - \alpha_1 + \int_{u_0-1}^{u_0} f_u(x,t,u) \frac{\partial}{\partial u} \eta_2(u)\, du \\ - \int_{u_0}^{u_0+1} f_u(x,t,u) \frac{\partial}{\partial u} \eta_1(u)\, du\biggr) \\ = a_3\biggl(a_1\alpha_2 + a_2\alpha_2 + a_3\alpha_2 + (a_2 + a_3)\int_{u_0-1}^{u_0} f_u(x,t,u) \frac{\partial}{\partial u} \eta_2(u)\, du\biggr) \\ -(a_2 + a_3)\biggl(a_1\alpha_1 + a_2\alpha_1 + a_3\alpha_1 \\ + a_3\int_{u_0}^{u_0+1} f_u(x,t,u) \frac{\partial}{\partial u} \eta_1(u)\, du\biggr). \end{aligned}$$

Let $\alpha_1 = \alpha_2 = 0$. We have

$$\begin{aligned} a_3\biggl(\int_{u_0-1}^{u_0} f_u(x,t,u) \frac{\partial}{\partial u} \eta_2(u)\, du - \int_{u_0}^{u_0+1} f_u(x,t,u) \frac{\partial}{\partial u} \eta_1(u)\, du\biggr) \\ = a_3(a_2 + a_3)\biggl(\int_{u_0-1}^{u_0} f_u(x,t,u) \frac{\partial}{\partial u} \eta_2(u)\, du \\ - \int_{u_0}^{u_0+1} f_u(x,t,u) \frac{\partial}{\partial u} \eta_1(u)\, du\biggr), \end{aligned}$$

i.e., $a_3 = (a_2 + a_3)a_3$, which implies $a_1 = 0$, or $a_3 = 0$.

By (4.20), it is easy to show that the measures $\nu_{x,t}$ are indeed Dirac masses.

From Theorem 2.3, the proof of the first half of Theorem 4.1 is completed.

Under the growth conditions, it is standard to show that the limit function $u(x,t)$ is a global weak solution for the Cauchy problem (1.1) and (1.2). This completes the proof of Theorem 4.1. Q.E.D.

## 5. APPLICATIONS

In this section, we apply our compactness framework to some equations such as the scalar conservation laws and a combustion model.

### 5.1. *The Scalar Conservation Laws*

In this subsection, we use our compactness framework to prove the global existence of a generalized admissible solution to the following scalar conservation laws

$$u_t + f(u)_x = 0, \qquad (x,t) \in \mathbf{R} \times \mathbf{R}^+, \tag{5.1}$$

with initial data

$$u(x,0) = u_0(x) \in L^p \cap L^2(\mathbf{R}), \qquad x \in \mathbf{R}, 2 \le p \le \infty. \tag{5.2}$$

Our results can be stated as follows:

THEOREM 5.1. *Let* $f(u) \in C^2(\mathbf{R})$, $\operatorname{meas}\{u : f''(u) = 0\} = 0$, *and let* $f(u)$ *satisfy the growth condition*

$$|f'(u)| \le C\left(1 + |u|^{p-1}\right). \tag{5.3}$$

*Then the Cauchy problem* (5.1), (5.2) *admits a global weak solution* $u(x,t)$.

*Proof.* We first consider the Cauchy problem for the related parabolic system of (5.1), (5.2)

$$u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, \qquad (x,t) \in \mathbf{R} \times \mathbf{R}^+, \tag{5.4}$$

with initial data

$$u^\varepsilon(x,0) = u_0^\varepsilon(x) = \frac{1}{\varepsilon}\int_{\mathbf{R}} u_0(y)\,\rho\left(\frac{x-y}{\varepsilon}\right) dy, \qquad x \in \mathbf{R}. \tag{5.5}$$

Here $\rho(x)$ is a mollifier; that is, $0 \le \rho(x) \in C_0^\infty(\mathbf{R})$, supp $\rho(x) \subset [-1,1]$, and $\int_{\mathbf{R}} \rho(x)\,dx = 1$.

From the classical Young inequality

$$\|f\|_{q'} \leq \|g\|_p \|h\|_r, \qquad \frac{1}{r} = \frac{1}{p'} + \frac{1}{q'}, \frac{1}{p} + \frac{1}{p'} = 1 \ (p, r \geq 1),$$

where

$$f(x) = \int_{\mathbf{R}} g(y) h(x-y)\, dy,$$

we get

$$\|u^\varepsilon(x,0)\|_{L^\infty} \leq \frac{1}{\varepsilon}\|u_0\|_{L^p}\|\rho\|_{L^r} \leq C(\varepsilon), \qquad r = \frac{p}{p-1}. \tag{5.6}$$

From (5.4), (5.6), it is easy to prove that, for any fixed $\varepsilon > 0$, there exists a unique global smooth solution $u^\varepsilon(x,t)$ of the Cauchy problem (5.4), (5.5) for $t > 0$.

To prove Theorem 5.1, by using our compactness framework, we only need to show that the following two assertions hold:

$$\text{(i)} \quad u^\varepsilon(x,t) \in L_b^p(\mathbf{R} \times \mathbf{R}^+), \tag{5.7}$$

(ii) For every bounded open set $\Omega \subset \mathbf{R} \times \mathbf{R}^+$ and every $C^2$ entropy-entropy flux pair $(\eta(u), q(u))$ with $\eta(u)$ of compact support,

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \in H^{-1}_{\mathrm{loc}}(\Omega). \tag{5.8}$$

Assertion (5.7) can be obtained easily. To prove (5.8), according to the well known Murat's Lemma (see, for example, [3, 4, 7, 10, 11, 13]), we only need to verify

$$\varepsilon \iint_\Omega (u_x^\varepsilon)^2\, dx\, dt \leq C. \tag{5.9}$$

This kind of estimate can be obtained by standard method when $u_0(x) \in L^2(\mathbf{R})$. This completes the proof of Theorem 5.1. Q.E.D.

### 5.2. *A Hyperbolic Model of Combustion*

In this subsection, we consider the following hyperbolic model of combustion (see [8, 9, 24]),

$$\begin{cases} (u + qz)_t + f(u)_x = 0, \\ z_t + k\phi(u) z = 0, \end{cases} \tag{5.10}$$

with initial data,

$$(u(x,0), z(x,0)) = (u_0(x), z_0(x)) \in L^p \cap L^2(\mathbf{R} \times \mathbf{R}^+),$$
$$2 \le p < \infty,\ x \in \mathbf{R}. \quad (5.11)$$

Here $z(x,t)$ denotes the density of unburned fraction in fluid, $k$ is the rate of chemical reaction, $q$ is the binding energy, both of them are positive constants, $f(u)$ is smooth on $\mathbf{R}$, and $\phi(u)$ is defined as

$$\phi(u) = \begin{cases} 1, & u > 0, \\ 0, & u < 0. \end{cases} \quad (5.12)$$

This model was first studied by Majda (see [9]) and has been widely investigated, cf. [8, 14]. For the Cauchy problem (5.10), (5.11), we first give the following definition of global weak solution:

DEFINITION 5.2. A pair of functions $(u(x,t), z(x,t)) \in L^p(\mathbf{R} \times \mathbf{R}^+) \times L^q(\mathbf{R} \times \mathbf{R}^+)$ $(1 \le p < \infty, 1 \le q < \infty)$ is called a weak solution of the Cauchy problem (5.10), (5.11) with initial data $(u_0(x), z_0(x)) \in L^p(\mathbf{R}) \times L^q(\mathbf{R})$ if the following equalities

$$\iint_{t \ge 0} \{(u + qz)\phi_t + f(u)\phi_x\}\,dx\,dt + \int_{\mathbf{R}} (u_0(x) + qz_0(x))\phi(x,0)\,dx = 0,$$

$$\iint_{t \ge 0} z\phi_t\,dx\,dt - \iint_G k\phi(u)z\phi\,dx\,dt + \int_{\mathbf{R}} z_0(x)\phi(x,0)\,dx = 0 \quad (5.13)$$

hold for any $\phi(x,t) \in C_0^1(\mathbf{R} \times \mathbf{R}^+)$, where $G = \mathbf{R} \times \mathbf{R}^+ - S_0$; $S_0 = \{(x,t) : u(x,t) = 0\}$.

We are going to prove the following theorem.

THEOREM 5.3. *Let* $f(u) \in C^2(\mathbf{R})$, $\operatorname{meas}\{u : f''(u) = 0\} = 0$, *and* $f(u)$ *satisfy the growth condition* (5.3). *The the Cauchy problem* (5.10), (5.11) *admits a global weak solution* $(u(x,t), z(x,t))$ *defined by* (5.13).

*Proof.* The proof of Theorem 5.3 is equivalent to proving the convergence of the sequence of solutions $\{u^\varepsilon(x,t), z^\varepsilon(x,t)\}$ of the Cauchy problem

$$\begin{cases} u_t^\varepsilon + f(u^\varepsilon)_x - kq\phi^\varepsilon(u^\varepsilon)z^\varepsilon = \varepsilon u_{xx}^\varepsilon, \\ z_t^\varepsilon = -k\phi^\varepsilon(u^\varepsilon)z^\varepsilon, \end{cases} \quad (5.14)$$

with initial data

$$(u^\varepsilon(x,0), z^\varepsilon(x,0)) = (u_0^\varepsilon(x), z_0^\varepsilon(x)), \quad (5.15)$$

where $\phi^\varepsilon(u) = (1/\varepsilon)\int_{\mathbf{R}} \phi(v)\rho((u-v)/\varepsilon)\,dv$, $u_0^\varepsilon(x) = (1/\varepsilon)\int_{\mathbf{R}} u_0(y)\rho((x-y)/\varepsilon)\,dy$, $z_0^\varepsilon(x) = (1/\varepsilon)\int_{\mathbf{R}} z_0(y)\rho((x-y)/\varepsilon)\,dy$, and show the limit function $(u(x,t), z(x,t))$ satisfies (5.13). For this purpose, as observed in [8], we only need to prove the strong convergence of $\{u^\varepsilon(x,t)\}$ as $\varepsilon \to 0$. Using our compactness framework, it is equivalent to verify

$$\text{(i)} \quad u^\varepsilon(x,t) \in L_b^p(\mathbf{R}), \tag{5.16}$$

(ii) For each bounded open set $\Omega \subset \mathbf{R} \times \mathbf{R}^+$ and any $C^2$ entropy-entropy flux pair $(\eta(u), q(u))$ with $\eta(u)$ of compact support

$$\eta(u^\varepsilon)_t + q(u^\varepsilon)_x \in H_{\mathrm{loc}}^{-1}(\Omega). \tag{5.17}$$

By Murat's Lemma again, (ii) can be rewritten equivalently as

$$\varepsilon \iint_\Omega (u_x^\varepsilon)^2\,dx\,dt \le C. \tag{5.18}$$

Now we turn to prove (5.16) and (5.18).

Using the second equations in (5.14) and $z_0(x) \in L^p \cap L^2(\mathbf{R})$, we have

$$z^\varepsilon(x,t) = z_0^\varepsilon(x)\exp\left(-\int_0^t k\phi^\varepsilon(u^\varepsilon(\tau,x))\,d\tau\right),$$

thus

$$\begin{cases} \|z^\varepsilon(x,t)\|_{L^p(\mathbf{R})} \le \|z_0(x)\|_{L^p(\mathbf{R})}, \\ \|z^\varepsilon(x,t)\|_{L^2(\mathbf{R})} \le \|z_0(x)\|_{L^2(\mathbf{R})}. \end{cases} \tag{5.19}$$

By (5.19), it is straightforward to prove (5.16), cf. [8].

To prove (5.18), multiplying $(5.14)_1$ by $u^\varepsilon$ and integrating over $[0,t]\times\mathbf{R}$, we have

$$\begin{aligned} &\tfrac{1}{2}\int_{\mathbf{R}} |u^\varepsilon|^2\,dx + \varepsilon\int_0^t\int_{\mathbf{R}} (u_x^\varepsilon)^2\,dx\,dt \\ &\quad = \tfrac{1}{2}\int_{\mathbf{R}} |u_0^\varepsilon|^2\,dx + kq\int_0^t\int_{\mathbf{R}} \phi^\varepsilon(u^\varepsilon)u^\varepsilon z^\varepsilon\,dx\,dt \\ &\quad \le \tfrac{1}{2}\int_{\mathbf{R}} |u_0|^2\,dx + kg\left\{\int_0^t\int_{\mathbf{R}} |u^\varepsilon|^2\,dx\,dt + \int_0^t\int_{\mathbf{R}} |z^\varepsilon|^2\,dx\,dt\right\}. \end{aligned} \tag{5.20}$$

Thus Gronwall's inequality gives

$$\varepsilon\int_0^t\int_{\mathbf{R}} |u_x^\varepsilon|^2\,dx\,dt \le C\big(\|u_0\|_{L^2(\mathbf{R})}, t, k, q, \|z_0\|_{L^2(\mathbf{R})}\big) < \infty.$$

This completes the proof of Theorem 5.3. Q.E.D.

*Remark.* In [8], to get the global existence result to the Cauchy problem (5.14), (5.15), the flux function $f(u)$ is assumed to satisfy a more strict growth condition. Our result obtained above generalizes the previous work for (5.14) and (5.15) to a more general function $f(u)$.

In fact, since $f(u)$ and $\phi^\varepsilon(u)$ are smooth functions and $(u_0^\varepsilon(x), z_0^\varepsilon(x)) \in C_b^\infty(\mathbf{R}, \mathbf{R}^2)$, then the Cauchy problem (5.14), (5.15) admits a unique smooth solution $(u^\varepsilon(x,t), z^\varepsilon(x,t))$ on $\Pi_{t_1} = \{(x,t): 0 \le t \le t_1,\ x \in \mathbf{R}\}$, where $t_1$ depends only on $\|u_0^\varepsilon(x)\|_{L^\infty(\mathbf{R})}$ and $\|z_0^\varepsilon(x)\|_{L^\infty(\mathbf{R})}$. Suppose that the solution $(u^\varepsilon(x,t), z^\varepsilon(x,t))$ has been extended up to time $t = T$. To prove $T = \infty$, we only need to prove the following estimates:

(i) $\|u^\varepsilon(x,t)\|_{L^\infty([0,T]\times\mathbf{R})} \le M(T, \|u_0^\varepsilon(x)\|_{L^\infty(\mathbf{R})}, \|z_0^\varepsilon(x)\|_{L^\infty(\mathbf{R})})$, and

(ii) $\|z^\varepsilon(x,t)\|_{L^\infty([0,T]\times\mathbf{R})} \le \|z_0^\varepsilon(x)\|_{L^\infty(\mathbf{R})}$.

Estimate (ii) follows from $z^\varepsilon(x,t) = z_0^\varepsilon(x)\exp(-\int_0^t k\phi^\varepsilon(u^\varepsilon(\tau,x))\,d\tau)$. As to (i), since $u^\varepsilon(x,t)$ satisfies

$$u_t^\varepsilon + f(u^\varepsilon)_x - kq\phi^\varepsilon(u^\varepsilon)z^\varepsilon = \varepsilon u_{xx}^\varepsilon,$$

and

$$|\phi^\varepsilon(u^\varepsilon)| \le 1, \qquad \|z^\varepsilon(x,t)\|_{L^\infty(\mathbf{R})} \le \|z_0^\varepsilon(x)\|_{L^\infty(\mathbf{R})},$$

estimate (i) holds due to the standard maximum principle.

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