Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2016,**36B**(4):1015–1029
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

TIME PERIODIC SOLUTION TO THE COMPRESSIBLE NAVIER-STOKES EQUATIONS IN A PERIODIC DOMAIN*

Chunhua JIN (金春花)
School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China
E-mail: *jinchhua@126.com*

Tong YANG (杨彤)†
Department of Mathematics, City University of Hong Kong, Hong Kong, China
E-mail: *matyang@cityu.edu.hk*

Dedicated to Professor Boling Guo on the occasion of his 80th birthday

Abstract This article is concerned with the time periodic solution to the isentropic compressible Navier-Stokes equations in a periodic domain. Using an approach of parabolic regularization, we first obtain the existence of the time periodic solution to a regularized problem under some smallness and symmetry assumptions on the external force. The result for the original compressible Navier-Stokes equations is then obtained by a limiting process. The uniqueness of the periodic solution is also given.

Key words Time periodic solution; compressible Navier-Stokes equation; topology degree; energy method

2010 MR Subject Classification 35Q30; 35B10; 35M10

1 Introduction

Consider the compressible Navier-Stokes equations for isentropic fluid

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ \rho(u_t + (u\cdot\nabla)u) + \nabla P(\rho) = \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u + \rho f(x,t), \end{cases} \tag{1.1}$$

where $x \in (-L, L)^3$, $\rho(x,t)$, $u(x,t) = (u_1, u_2, u_3)(x,t)$ are the density and velocity, $P(\rho)$ is the pressure, $f = (f_1, f_2, f_3)(x,t)$ is a given external force, which is assumed to be periodic in both

*Received April 16, 2015; revised January 19, 2016. C. Jin's research is supported by the Program for New Century Excellent Talents in University of the Ministry of Education (NCET-13-0804), NSFC (11471127), Guangdong Natural Science Funds for Distinguished Young Scholar (2015A030306029), The Excellent Young Teachers Program of Guangdong Province (HS2015007), Pearl River S&T Nova Program of Guangzhou (2013J2200064), and Special support program of Guangdong Province; T. Yang is supported by the General Research Fund of Hong Kong, CityU 104511.

†Corresponding author

space and time with period $2L$ and T respectively, and μ and λ are the viscosity coefficients satisfying

$$\mu > 0, \lambda + \frac{2\mu}{3} \geq 0.$$

The purpose of this article is to study the existence of periodic solution around a constant state $(\bar{\rho}, 0)$. Rewrite (1.1) in the perturbation form as

$$\begin{cases} \sigma_t + \bar{\rho}\nabla \cdot u = -\nabla \cdot (\sigma u), \\ u_t - (1 - h(\sigma))(\bar{\mu}\Delta u + (\bar{\mu} + \bar{\lambda})\nabla \mathrm{div} u) + \gamma\bar{\rho}\nabla\sigma = -(u \cdot \nabla)u - g(\sigma)\nabla\sigma + f(x,t), \end{cases} \tag{1.2}$$

where $\sigma = \rho - \bar{\rho}$, $\bar{\mu} = \mu/\bar{\rho}$, $\bar{\lambda} = \lambda/\bar{\rho}$, $\gamma = P'(\bar{\rho})/\bar{\rho}^2$, and

$$h(\sigma) = \frac{\sigma}{\sigma + \bar{\rho}} \quad \text{and} \quad g(\sigma) = \frac{P'(\sigma + \bar{\rho})}{\sigma + \bar{\rho}} - \frac{P'(\bar{\rho})}{\bar{\rho}}.$$

Before stating the main result, we give some notations, which will be used throughout this article,

$$\Omega = (-L, L)^3, \quad Q_T = \Omega \times (0, T),$$

and for multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$\partial_x^\alpha = \partial_x^{\alpha_1}\partial_x^{\alpha_2}\partial_x^{\alpha_3}, \quad |\alpha| = \sum_{i=1}^{3} \alpha_i,$$

$$|u(\cdot,t)|_K^2 = \sum_{|\alpha|=K} \|\partial^\alpha u(\cdot,t)\|_{L^2(\Omega)}^2, \quad \|u\|_{H^k} = \|u(\cdot,t)\|_{H^k(\Omega)},$$

$$\begin{aligned} \mathcal{X} = \{(\sigma, u) :& \sigma \in L^\infty((0,T); H^K(\Omega)), u \in L^2((0,T); H^{K+1}(\Omega)) \cap L^\infty((0,T); H^K(\Omega)); \\ & \sigma, u \text{ satisfies (a), (b), (c) of Theorem 1.1}\}. \end{aligned}$$

Theorem 1.1 Assume that $P(\rho)$ is a smooth function in a neighborhood of $\bar{\rho}$, $K \geq 4$ is an integer, and $f \in L^2((0,T); H^{K-1}(\Omega))$ with $f(-x,t) = -f(x,t)$. If $\int_0^T \|f(\cdot,t)\|_{H^{K-1}}^2 \mathrm{d}t$ is suitably small, then the problem (1.2) admits a solution $(\sigma, u) \in \mathcal{X} \cap G_K^R$ such that

(a) (σ, u) is periodic with the space period $2L$ and time period T;

(b) $\int_\Omega \sigma(x,t)\mathrm{d}x \equiv 0$, $\int_\Omega u(x,t)\mathrm{d}x \equiv 0$;

(c) $\sigma(x,t) = \sigma(-x,t)$, $u(x,t) = -u(-x,t)$.

Moreover, the solution (σ, u) is unique within this class, provided that $\sup\limits_{t \in (0,T)} \|(\sigma, u)(s)\|_{H^K}$ is sufficiently small.

Remark 1.2 Theorem 1.1 shows that there is a periodic solution with constant total mass, momentum. The condition on symmetry of the external force f is to ensure that the Poincáre inequality holds. In fact, when there is a given external force acting on the fluid, only the conservation of the total mass holds in general. It is worth noting that some structural condition on f seems to be necessary, because (1.1) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho u \mathrm{d}x = \int_\Omega \rho f \mathrm{d}x.$$

If $f \geq 0$ (or $f \leq 0$) for any $(x,t) \in Q_T$ with $f \not\equiv 0$, then we have $\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho u \mathrm{d}x > 0$ (or < 0) even when f is very small. This implies that the smallness on the external force itself cannot guarantee the existence of time periodic solution.

Let us review some previous related works. The time periodic solution to the incompressible Navier-Stokes equations has been extensively studied; see, for example [1–5]. Precisely, [1] is about the existence of time periodic solutions in a bounded domain Ω when the boundary moves periodically in time. In this article, Serrin introduced a method to solve the periodic problem, that is, for a time-periodic force f and any initial value, the solution of the corresponding initial-value problem will converge (as $t \to \infty$) to some state, which is considered as an initial value , and this yields a time-periodic solution. Another approach was introduced by Yudovich [2]. He considered the Poincaré map from an initial value $u(x, 0)$ to the state $u(x, T)$, where T is the period of the given external force f, and $u(x, t)$ is the solution corresponding to the initial data $u(x, 0)$. A time-periodic solution is then identified as a fixed point of this Poincaré map. This approach is also introduced in the same time by Prodi in another article [3]. In [4], the authors considered the reproductive property, which can be regarded as a generalization of time periodicity, that is, a solution with the same value as the initial data at some time T without the assumption of the periodicity on the external force f. If f is time periodic, then the reproductive solution is also time periodic. Recently, [5] extends these results to the inhomogeneous boundary conditions and gives the existence and uniqueness of small time periodic strong solutions under the assumption that the inhomogeneous boundary data and the external force f are small; moreover, using the reproductive property, [5] yields the existence of time periodic weak solution without any smallness assumption on boundary value and the external force f.

On the other hand, for the compressible Navier-Stokes equations, there are only a few works on time periodic solutions. Among them, in 1983, Valli [9] first studied the existence of periodic solutions in a three spatially bounded domain with non-slip boundary condition, and using Serrin's method, he showed the existence of a small strong periodic solution when the external force f is small enough. In [10], Matsumura and Nishida considered the periodic solution for a one dimensional bounded domain in the lagrangian mass coordinate with external force or a piston. In a recent work [11], Feireisl, Mucha et al considered a full Navier-Stokes-Fourier system confined to a smooth bounded domain with no-slip boundary conditions, and obtained the existence of periodic solutions. As the authors in [11] pointed out, such a condition is in fact necessary, as energetically closed fluid systems do not possess non-trivial (changing in time) periodic solutions due to the second law of thermodynamics.

In addition to these results in a bounded domain, there are also some results for unbounded domain problems; for example, in [6–8], the authors showed the existence of time periodic solutions for the incompressible Navier-Stokes equations in the whole three-dimensional space, the half space and exterior domains, respectively. It is worth noticing that most of these existence results depend closely on the decay rate estimate of the corresponding linearized equation, where the technique of $L^q - L^r$ estimate is used. However, this technique seems ineffective for the compressible Navier-Stokes equations, due to the fact that $\mathrm{div} u \neq 0$, and the appearance of some other nonlinear terms. Hence, for the compressible Navier-Stokes equations, by using the spectral analysis method for the optimal decay estimates, in the previous work, the existence of periodic solutions is obtained only when the space dimension $N \geq 5$ in $\mathbb{R}^N$. In a recent work [12, 13], Březina and Kagei considered the time-periodic parallel flow in an n-dimensional infinite layer $\mathbb{R}^{N-1} \times (0, l)$. In these article, the authors constructed a periodic solution using the solution of a one-dimensional heat type equation in a bounded domain $(0,l)$ for a special time-periodic external force of the form $f=(f_1(x_n,t),0,\cdots,0,f_n(x_n))$.

In this article, we focus on the compressible Navier-Stokes equations with periodic boundary. The main idea is based on an approach of parabolic regularization, and this method allows us to overcome difficult problems in proving the compactness of the operator. Thus, using the energy estimates and the topological degree method, we obtain the existence of periodic solutions for the regularized problem, then the existence of periodic solutions for (1.2) is obtained by a limiting process. On the basis of this, we also show the uniqueness of small periodic solutions.

2 Existence of Periodic Solutions

We will prove the existence of time periodic solutions in this section. For this, we first consider the following regularized problem

$$\begin{cases} \sigma_t-\varepsilon\Delta\sigma+\bar\rho\nabla\cdot u=-\nabla\cdot(\sigma u), \\ u_t-(1-h(\sigma))(\bar\mu\Delta u+(\bar\mu+\bar\lambda)\nabla\mathrm{div}u)+\gamma\bar\rho\nabla\sigma=-(u\cdot\nabla)u-g(\sigma)\nabla\sigma+f(x,t). \end{cases}\tag{2.1}$$

Set

$$\mathcal{G}_K=\left\{(\rho,\omega)\in L^2((0,T);H^K(\Omega))\cap L^\infty((0,T);H^{K-1}(\Omega));\rho,\omega \text{ satisfies (a), (b), (c)}\right\};$$

and

$$\mathcal{G}_K^R=\left\{(\rho,\omega)\in\mathcal{G}_K;\sup_{0<t<T}\|(\rho,\omega(\cdot,t)\|_{H^{K-1}}^2+\int_0^T\|(\rho,\omega)(\cdot,t)\|_{H^K}^2\mathrm{d}t<R^2\right\}.$$

Define an operator

$$F:\quad \mathcal{G}_K^R\times[0,1]\to\mathcal{G}_K,$$
$$((\rho,\omega),\tau)\to(\sigma,u),$$

with $K\geq 4$, where (σ,u) is the solution of the problem

$$\begin{cases} \sigma_t-\varepsilon\Delta\sigma+\bar\rho\nabla\cdot u=-\tau\nabla\cdot(\rho\omega), \\ u_t-(1-\tau h(\rho))(\bar\mu\Delta u+(\bar\mu+\bar\lambda)\nabla\mathrm{div}u)+\gamma\bar\rho\nabla\sigma=-\tau((\omega\cdot\nabla)\omega+g(\rho)\nabla\rho)+\tau f(x,t). \end{cases}\tag{2.2}$$

The following lemma shows that the operator F is well defined.

Lemma 2.1 Assume that R is suitably small. Then for any $(\rho,\omega)\in\mathcal{G}_K^R$, $\tau\in[0,1]$, the problem (2.2) admits a unique solution (σ,u) in $\mathcal{G}_K$.

Proof Noticing that $K\geq 4$, we have

$$\|\rho\|_{L^\infty}\leq\mu\sup_{0<t<T}\|\rho(\cdot,t)\|_{H^{K-1}}\leq\mu R.$$

Then, when R is suitably small, we have $|h(\rho)|\leq 1/2$. By the classical theory of parabolic equations, the existence of solutions can be obtained. To prove uniqueness, suppose to the contrary, there exists two solutions (σ_1,u_1), (σ_2,u_2) for some $(\rho,\omega)\in\mathcal{G}_K^R$, $\tau\in[0,1]$. Let $\sigma=\sigma_1-\sigma_2$, $u=u_1-u_2$. Then, we have

$$\begin{cases} \sigma_t-\varepsilon\Delta\sigma+\bar\rho\nabla\cdot u=0, \\ u_t-(1-\tau h(\rho))(\bar\mu\Delta u+(\bar\mu+\bar\lambda)\nabla\mathrm{div}u)+\gamma\bar\rho\nabla\sigma=0. \end{cases}$$

From the proof of (2.3) in the following Lemma 2.2, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^2\right)+\varepsilon\gamma|\sigma(\cdot,t)|_{K+1}^2+\frac{2}{3}\bar\mu|u(\cdot,t)|_{K+1}^2\le 0.$$

Then, the Poincáre inequality implies $(\sigma,u)=(0,0)$, and the uniqueness of solutions is proved. On the other hand, as both $(\sigma(x,t),u(x,t))$ and $(\sigma(-x,t),-u(-x,t))$ are the solution of (2.2), by uniqueness, $\sigma(x,t)=\sigma(-x,t)$, $u(x,t)=-u(-x,t)$. And this completes the proof of the lemma. □

Next, we show that F is a compact operator.

Lemma 2.2 If R is suitably small, then the operator F is compact.

Proof Assume that (σ,u) is the solution of problem (2.2). For each multi-index α with $|\alpha|=K$, by applying ∂_x^α to (2.2), multiplying the first and second equation by $\gamma\partial_x^\alpha\sigma$, $\partial_x^\alpha u$ respectively, and taking integration over Ω, we obtain

$$\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\partial^\alpha\sigma|^2\mathrm{d}x+\varepsilon\gamma\int_\Omega|\nabla\partial^\alpha\sigma|^2\mathrm{d}x-\gamma\bar\rho\int_\Omega\partial^\alpha u\partial^\alpha(\nabla\sigma)\mathrm{d}x=-\tau\gamma\int_\Omega\partial^\alpha(\nabla\cdot(\rho\omega))\partial^\alpha\sigma\mathrm{d}x$$

and

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\partial^\alpha u|^2\mathrm{d}x+(1-\tau h(\rho))\left(\bar\mu\int_\Omega|\nabla\partial^\alpha u|^2\mathrm{d}x+(\bar\mu+\bar\lambda)\int_\Omega|\partial^\alpha\mathrm{div}u|^2\mathrm{d}x\right)\\ =&-\gamma\bar\rho\int_\Omega\partial^\alpha u\partial^\alpha(\nabla\sigma)\mathrm{d}x+\tau\int_\Omega\left(\bar\mu h'(\rho)\partial^\alpha\nabla u\partial^\alpha u\nabla\rho+(\bar\mu+\bar\lambda)h'(\rho)\partial^\alpha u\partial^\alpha(\mathrm{div}u)\nabla\rho\right)\mathrm{d}x\\ &+\int_\Omega\sum_{1\le l\le\alpha}\binom{\alpha}{l}\partial^l(1-\tau h(\rho))\partial^{\alpha-l}(\bar\mu\Delta u+(\bar\mu+\bar\lambda)\nabla\mathrm{div}u)\partial^\alpha u\mathrm{d}x\\ &+\tau\int_\Omega\partial^{\alpha-1}((\omega\cdot\nabla)\omega+g(\rho)\nabla\rho)\partial^{\alpha+1}u\mathrm{d}x+\tau\int_\Omega\partial^{\alpha-1}f\partial^{\alpha+1}u\mathrm{d}x.\end{aligned}$$

Summing up the above equalities yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\gamma|\partial^\alpha\sigma|^2+|\partial^\alpha u|^2\right)\mathrm{d}x+\varepsilon\gamma\int_\Omega|\nabla\partial^\alpha\sigma|^2\mathrm{d}x\\ &+(1-\tau h(\rho))\left(\bar\mu\int_\Omega|\nabla\partial^\alpha u|^2\mathrm{d}x+(\bar\mu+\bar\lambda)\int_\Omega|\partial^\alpha\mathrm{div}u|^2\mathrm{d}x\right)\\ =&-\tau\gamma\int_\Omega\partial^\alpha(\nabla\cdot(\rho\omega))\partial^\alpha\sigma\mathrm{d}x\\ &+\tau\int_\Omega\left(\bar\mu h'(\rho)\partial^\alpha\nabla u\partial^\alpha u\nabla\rho+(\bar\mu+\bar\lambda)h'(\rho)\partial^\alpha u\partial^\alpha(\mathrm{div}u)\nabla\rho\right)\mathrm{d}x\\ &+\int_\Omega\sum_{1\le l\le\alpha}\binom{\alpha}{l}\partial^l(1-\tau h(\rho))\partial^{\alpha-l}(\bar\mu\Delta u+(\bar\mu+\bar\lambda)\nabla\mathrm{div}u)\partial^\alpha u\mathrm{d}x\\ &+\tau\int_\Omega\partial^{\alpha-1}((\omega\cdot\nabla)\omega+g(\rho)\nabla\rho)\partial^{\alpha+1}u\mathrm{d}x+\tau\int_\Omega\partial^{\alpha-1}f\partial^{\alpha+1}u\mathrm{d}x\\ =&I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

Firstly, note that

$$\|\rho\|_{L^\infty(Q_T)}\le C\sup_{0<t<T}\|\rho(\cdot,t)\|_{H^{K-1}}<CR.$$

Then for sufficiently small R, we have

$$\|\rho\|_{L^\infty(Q_T)}\le\frac{\bar\rho}{2},$$

which implies that

$$(1-\tau h(\rho))\geq \frac{2}{3}.$$

By the periodic boundary condition, we see that $\|\partial^\beta(\rho,u)\|_{L^2}\leq C\|\nabla\partial^\beta(\rho,u)\|_{L^2}$ for any multi-index β with $|\beta|\geq 0$. Note that $|\alpha|\geq 4$, then we have

$$\begin{aligned}|I_1|&\leq C\|\nabla\partial^\alpha\sigma\|_{L^2}(\|\partial^\alpha\rho\|_{L^2}\|\omega\|_{L^\infty}+\|\partial^\alpha\omega\|_{L^2}\|\rho\|_{L^\infty})\\&\leq \frac{\varepsilon\gamma}{2}\|\nabla\partial^\alpha\sigma\|_{L^2}^2+C_1(|\omega|_K^2\|\rho\|_{H^{K-1}}^2+|\rho|_K^2\|\omega\|_{H^{K-1}}^2),\end{aligned}$$

$$|I_2|\leq C\|\nabla\partial^\alpha u\|_{L^2}^2\|\nabla\rho\|_{L^\infty}\leq C_2R\|\nabla\partial^\alpha u\|_{L^2}^2,$$

$$\begin{aligned}|I_3|&\leq C\|\partial^\alpha u\|_{L^2}(\|\rho\|_{L^\infty}\|\nabla^2u\|_{H^{K-1}}+\|\nabla^2u\|_{L^\infty}\|\rho\|_{H^{K-1}})\\&\leq C\|\rho\|_{H^{K-1}}|u|_{K+1}^2\leq C_3R|u|_{K+1}^2,\end{aligned}$$

$$\begin{aligned}|I_4|&\leq C(|\omega|_K\|\omega\|_{L^\infty}+|\omega|_{K-1}\|\nabla\omega\|_{L^\infty}+|\rho|_K\|\rho\|_{L^\infty}+|\rho|_{K-1}\|\nabla\rho\|_{L^\infty})\|\partial^{\alpha+1}u\|_{L^2}\\&\leq C_4(|\omega|_K\|\omega\|_{H^{K-1}}+|\rho|_K\|\rho\|_{H^{K-1}})|u|_{K+1},\end{aligned}$$

$$|I_5|\leq |f|_{K-1}|u|_{K+1}.$$

Then for sufficiently small $R>0$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^2\right)+\varepsilon\gamma|\sigma(\cdot,t)|_{K+1}^2+\frac{2}{3}\bar{\mu}|u(\cdot,t)|_{K+1}^2\\&\leq C_5(|\omega(\cdot,t)|_K^2+|\rho(\cdot,t)|_K^2)(\|\omega(\cdot,t)\|_{H^{K-1}}^2+\|\rho(\cdot,t)\|_{H^{K-1}}^2)+C_6|f(\cdot,t)|_{K-1}^2.\end{aligned}\tag{2.3}$$

Integrating this inequality from 0 to T yields

$$\begin{aligned}&\varepsilon\gamma\int_0^T|\sigma(\cdot,t)|_{K+1}^2\mathrm{d}t+\frac{2}{3}\bar{\mu}\int_0^T|u(\cdot,t)|_{K+1}^2\mathrm{d}t\\&\leq C_5\sup_{0<t<T}(\|\omega(\cdot,t)\|_{H^{K-1}}^2+\|\rho(\cdot,t)\|_{H^{K-1}}^2)\int_0^T(|\omega(\cdot,t)|_K^2+|\rho(\cdot,t)|_K^2)\mathrm{d}t\\&\quad+C_6\int_0^T|f(\cdot,t)|_{K-1}^2\mathrm{d}t\\&=M^*.\end{aligned}\tag{2.4}$$

Then, there exists a time $t^*\in(0,T)$ such that

$$\varepsilon\gamma T|\sigma(\cdot,t^*)|_{K+1}^2+\frac{2}{3}\bar{\mu}T|u(\cdot,t^*)|_{K+1}^2\leq M^*.$$

By Poincáre inequality, we have $\gamma|\sigma(\cdot,t^*)|_K^2+|u(\cdot,t^*)|_K^2\leq CM^*$. Then, integrating (2.3) from t^* to t for any $t\in(t^*,T]$, we have

$$\gamma|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^2\leq CM^*.$$

By the time periodicity, we further have

$$\gamma|\sigma(\cdot,0)|_K^2+|u(\cdot,0)|_K^2\leq CM^*.$$

Repeating the above process yields

$$\sup_{0<t<T}\{\gamma\|\sigma(\cdot,t)\|_{H^K}^2+\|u(\cdot,t)\|_{H^K}^2\}\leq CM^*.\tag{2.5}$$

Integrating (2.3) from t to $t+h$, then integrating it from 0 to T over t, we have

$$\begin{aligned}&\int_0^T(\gamma|\sigma(\cdot,t+h)|_K^2+|u(\cdot,t+h)|_K^2)-(\gamma|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^2)\mathrm{d}t\\ \le\,&2C_5h\sup_{0<t<T}(\|\omega(\cdot,t)\|_{H^{K-1}}^2+\|\rho(\cdot,t)\|_{H^{K-1}}^2)\int_0^T(|\omega(\cdot,s)|_K^2+|\rho(\cdot,s)|_K^2)\mathrm{d}s\\ &+2C_6h\int_0^T|f(\cdot,s)|_{K-1}^2\mathrm{d}s. \end{aligned}\tag{2.6}$$

For each multi-index β with $|\beta|=K-1$, by applying ∂_x^β to (2.2), and multiplying the first and second equation by $(\partial_x^\beta\sigma)_t$, $(\partial_x^\beta u)_t$ respectively, then integrating over Q_T, we have

$$\begin{aligned}&\int_0^T(\|(\partial^\beta\sigma)_t\|_{L^2}^2+\|(\partial^\beta u)_t\|_{L^2}^2\mathrm{d}t\\ \le\,&C\int_0^T\big(|u(\cdot,t)|_K^2+|\rho(\cdot,t)|_{K-1}^2|\omega(\cdot,t)|_K^2+|\rho(\cdot,t)|_K^2|\omega(\cdot,t)|_{K-1}^2+|\rho(\cdot,t)|_{K-1}^2\\ &|u(\cdot,t)|_{K+1}^2+|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^4+|\rho(\cdot,t)|_{K-1}^2|\rho(\cdot,t)|_K^2+|f(\cdot,s)|_{K-1}^2\big)\,\mathrm{d}t\\ \le\,&C\sup_{0<t<T}|\rho(\cdot,t)|_{K-1}^2\int_0^T(1+|\omega(\cdot,t)|_K^2+|\rho(\cdot,t)|_K^2+|u(\cdot,t)|_{K+1}^2)\mathrm{d}t+C\int_0^T|f(\cdot,s)|_{K-1}^2\mathrm{d}t\\ &+C\sup_{0<t<T}|\omega(\cdot,t)|_{K-1}^2\int_0^T|\rho(\cdot,t)|_K^2\mathrm{d}t+CT\sup_{0<t<T}(|u(\cdot,t)|_K^2+|u(\cdot,t)|_K^4). \end{aligned}\tag{2.7}$$

From (2.4), (2.5), (2.6), and (2.7), we see that F is a compact operator and this completes the proof of the lemma. □

The next lemma is about the continuity of the operator F.

Lemma 2.3 When R is suitably small, the operator F is continuous.

Proof Assume that $(\rho_n,\omega_n)\subset\mathcal{G}_K^R$, $\tau_n\in[0,1]$, $(\rho,\omega)\subset\mathcal{G}_K^R$, $\tau\in[0,1]$, and

$$\lim_{n\to\infty}\sup_{t\in(0,T)}\|(\rho_n-\rho,\omega_n-\omega)(\cdot,t)\|_{H^{K-1}}^2+\int_0^T\|(\rho_n-\rho,\omega_n-\omega)(\cdot,t)\|_{H^K}^2\mathrm{d}t=0$$

as $\lim\limits_{n\to\infty}\tau_n=\tau$. Let $(\sigma_n,u_n)=F((\rho_n,\omega_n),\tau_n)$, $(\sigma,u)=F((\rho,\omega),\tau)$. Then, $(\sigma_n-\sigma,u_n-u)$ is a periodic solution of following equation

$$\begin{cases}\widetilde\sigma_t-\varepsilon\Delta\widetilde\sigma+\bar\rho\nabla\cdot\widetilde u=(\tau-\tau_n)\nabla\cdot(\rho\omega)-\tau_n\nabla\cdot((\rho_n-\rho)\omega+\rho_n(\omega_n-\omega)),\\ \quad\widetilde u_t-(1-\tau h(\rho_n))(\bar\mu\Delta\widetilde u+(\bar\mu+\bar\lambda)\nabla\mathrm{div}\widetilde u)+\gamma\bar\rho\nabla\widetilde\sigma\\ =(\tau-\tau_n)h(\rho)(\bar\mu\Delta u+(\bar\mu+\bar\lambda)\nabla\mathrm{div}u)\\ \quad+\tau_n(h(\rho)-h(\rho_n))(\bar\mu\Delta u+(\bar\mu+\bar\lambda)\nabla\mathrm{div}u)\\ \quad-(\tau_n-\tau)((\omega_n\cdot\nabla)\omega_n+g(\rho_n)\nabla\rho_n)-\tau\big((\omega_n-\omega)\nabla\omega_n+\omega\nabla(\omega_n-\omega)\big)\\ \quad-\tau\big((g(\rho_n)-g(\rho))\nabla\rho_n+g(\rho)\nabla(\rho_n-\rho)\big)+(\tau_n-\tau)f(x,t),\end{cases}\tag{2.8}$$

with periodic boundary condition. Similar to the proof for the compactness of the operator F, and noticing that

$$\int_0^T\|u(\cdot,t)\|_{H^{K+1}}^2\mathrm{d}t<C,$$

we obtain

$$\lim_{n\to\infty}\sup_{t\in(0,T)}\|(\sigma_n-\sigma,u_n-u)(\cdot,t)\|_{H^{K-1}}^2+\int_0^T\|(\sigma_n-\sigma,u_n-u)(\cdot,t)\|_{H^K}^2\mathrm{d}t=0,$$

which gives the continuity of the operator F. □

From Lemmas 2.2 and 2.3, we see that the operator F is completely continuous.

Proposition 2.4 Assume that

$$f(x,t)=-f(-x,t),\quad f\in L^2((0,T);H^{K-1})$$

and $\int_0^T\|f(\cdot,t)\|_{H^K}^2\mathrm{d}t$ is sufficiently small, then problem (2.1) admits a periodic solution $u\in\mathcal{G}_K^R$.

Proof Firstly, we see that solving problem (2.1) in $\mathcal{G}_K$ is equivalent to solving the equation

$$U-F(U,1)=0,U=(\sigma,u)\in\mathcal{G}_K.$$

In what follows, we apply the topological degree theory. We first choose $R>0$ such that

$$(I-F(\cdot,\tau))(\partial\hat{B}_R(0))\neq 0,\ \text{for any }\tau\in[0,1],\tag{2.9}$$

where $\hat{B}_R(0)$ is the ball of radius R centered at the origin in $\mathcal{G}_K$. If (2.9) holds, then to show the existence of solution, we only need to show that

$$\mathbf{deg}(I-F(\cdot,1),\hat{B}_R(0),0)\neq 0.\tag{2.10}$$

For this, we will show that there exists $R>0$ such that (2.9) holds. We prove it by contradiction. Let $((\sigma,u),\tau)$ be a solution of (2.9) for some small $R>0$ by replacing (ρ,ω) with (σ,u) such that $((\sigma,u),\tau)$ satisfies

$$\begin{cases}\sigma_t-\varepsilon\Delta\sigma+\bar{\rho}\nabla\cdot u=-\tau\nabla\cdot(\sigma u),\\ u_t-(1-\tau h(\sigma))(\bar{\mu}\Delta u+(\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}u)+\gamma\bar{\rho}\nabla\sigma=-\tau(u\cdot\nabla u+g(\sigma)\nabla\sigma)+\tau f(x,t).\end{cases}\tag{2.11}$$

For each multi-index α with $|\alpha|=K$, by applying ∂_x^α to (2.11), and multiplying the equations by $\gamma\partial_x^\alpha\sigma$, $\partial_x^\alpha u$ respectively, then the integrating over Ω, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\alpha|=K}\int_\Omega\left(\gamma|\partial^\alpha\sigma|^2+|\partial^\alpha u|^2\right)\mathrm{d}x+\varepsilon\gamma\sum_{|\alpha|=K}\int_\Omega|\nabla\partial^\alpha\sigma|^2\mathrm{d}x\\ &+(1-\tau h(\sigma))\sum_{|\alpha|=K}\left(\bar{\mu}\int_\Omega|\nabla\partial^\alpha u|^2\mathrm{d}x+(\bar{\mu}+\bar{\lambda})\int_\Omega|\partial^\alpha\mathrm{div}u|^2\mathrm{d}x\right)\\ =&-\tau\gamma\sum_{|\alpha|=K}\int_\Omega\partial^\alpha(\nabla\cdot(\sigma u))\partial^\alpha\sigma\mathrm{d}x\\ &+\tau\sum_{|\alpha|=K}\int_\Omega\left(\bar{\mu}h'(\sigma)\partial^\alpha\nabla u\partial^\alpha u\nabla\sigma+(\bar{\mu}+\bar{\lambda})h'(\sigma)\partial^\alpha u\partial^\alpha(\mathrm{div}u)\nabla\sigma\right)\mathrm{d}x\\ &+\sum_{|\alpha|=K}\int_\Omega\sum_{1\le l\le\alpha}\binom{\alpha}{l}\partial^l(1-\tau h(\sigma))\partial^{\alpha-l}(\bar{\mu}\Delta u+(\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}u)\partial^\alpha u\mathrm{d}x\\ &+\tau\sum_{|\alpha|=K}\int_\Omega\partial^{\alpha-1}((u\cdot\nabla)u+g(\sigma)\nabla\sigma)\partial^{\alpha+1}u\mathrm{d}x+\tau\sum_{|\alpha|=K}\int_\Omega\partial^{\alpha-1}f\partial^{\alpha+1}u\mathrm{d}x\\ =&J_1+J_2+J_3+J_4+J_5.\end{aligned}\tag{2.12}$$

Next, we estimate each term on the right hand side of (2.12). For J_1, we have

$$|J_1|\le\frac{\tau\gamma}{2}\sum_{|\alpha|=K}\|\nabla u\|_{L^\infty}\|\partial^\alpha\sigma\|_{L^2}^2$$

$$
\begin{aligned}
&+\tau\gamma\int_\Omega\left(\sum_{0\le l\le\alpha-1}\binom{\alpha}{l}\partial^l(\nabla\sigma)\partial^{\alpha-l}u\partial^\alpha\sigma+\partial^l\sigma\partial^{\alpha-l}(\nabla\cdot u)\partial^\alpha\sigma\right)\mathrm{d}x\\
\le&\sum_{|\alpha|=K}\frac{\tau\gamma}{2}\|\nabla u\|_{L^\infty}\|\partial^\alpha\sigma\|_{L^2}^2+\sum_{|\alpha|=K}\|\partial^\alpha\sigma\|_{L^2}\big(\|\nabla u\|_{L^\infty}\|\nabla\sigma\|_{H^{K-1}}\\
&+\|\nabla u\|_{H^{K-1}}\|\nabla\sigma\|_{L^\infty}+\|\sigma\|_{L^\infty}\|\nabla^2u\|_{H^{K-1}}+\|\nabla^2u\|_{L^\infty}\|\sigma\|_{H^{K-1}}\big)\\
\le&\tilde{C}_1\|u\|_{H^{K-1}}\|\sigma\|_{H^K}^2+\hat{C}_1|\sigma|_K^2\|\sigma\|_{H^{K-1}}^2+\eta|u|_{K+1}^2.
\end{aligned}
$$

For other terms, we have

$$
|J_2|\le C\sum_{|\alpha|=K}\|\nabla\sigma\|_{L^\infty}\|\partial^\alpha\nabla u\|_{L^2}\|\partial^\alpha u\|_{L^2}\le\tilde{C}_2|\sigma|_{K-1}^2|u|_K^2+\eta|u|_{K+1}^2,
$$

$$
\begin{aligned}
|J_3|&\le C\sum_{|\alpha|=K}\|\partial^\alpha u\|_{L^2}(\|\sigma\|_{L^\infty}\|\nabla^2u\|_{H^{K-1}}+\|\nabla^2u\|_{L^\infty}\|\sigma\|_{H^{K-1}})\\
&\le C|\sigma|_{K-1}|u|_{K+1}|u|_K\le\tilde{C}_3|\sigma|_{K-1}^2|u|_K^2+\eta|u|_{K+1}^2,
\end{aligned}
$$

$$
\begin{aligned}
|J_4|&\le C(|u|_K\|u\|_{L^\infty}+|u|_{K-1}\|\nabla u\|_{L^\infty}+|\sigma|_K\|\sigma\|_{L^\infty}+|\sigma|_{K-1}\|\nabla\sigma\|_{L^\infty})|u|_{K+1}\\
&\le\tilde{C}(|u|_K|u|_{K-1}+|\sigma|_K|\sigma|_{K-1})|u|_{K+1}\\
&\le\tilde{C}_4(|u|_K^2|u|_{K-1}^2+|\sigma|_K^2|\sigma|_{K-1}^2)+\eta|u|_{K+1}^2,
\end{aligned}
$$

$$
|J_5|\le|f|_{K-1}|u|_{K+1}\le\tilde{C}_5|f|_{K-1}^2+\eta|u|_{K+1}^2.
$$

Taking $5\eta<\frac{1}{3}\bar{\mu}$ to be sufficiently small, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^2\right)+\varepsilon\gamma|\sigma(\cdot,t)|_{K+1}^2+\frac{1}{3}\bar{\mu}|u(\cdot,t)|_{K+1}^2\\
\le&\widehat{C}\left(|u(\cdot,t)|_{K-1}^2+|\sigma(\cdot,t)|_{K-1}^2\right)\left(|u(\cdot,t)|_K^2+|\sigma(\cdot,t)|_K^2\right)+\delta|\sigma(\cdot,t)|_K^2+\tilde{C}_5|f(\cdot,t)|_{K-1}^2,
\end{aligned}\tag{2.13}
$$

where δ is a small constant to be determined later.

We now turn to estimate $|\sigma|_K$. For $|\beta|=K-1$, applying ∂_x^β to $(2.11)_2$, and multiplying the resulting equation by $\partial_x^\beta\nabla\sigma$, then integrating them over Ω, we obtain

$$
\begin{aligned}
&\gamma\bar{\rho}\int_\Omega|\nabla\partial^\beta\sigma|^2\mathrm{d}x+\int_\Omega\partial^\beta u_t\nabla\partial^\beta\sigma\mathrm{d}x\\
=&\int_\Omega\partial^\beta\{(1-\tau h(\sigma))(\bar{\mu}\Delta u+(\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}u)\}\nabla\partial^\beta\sigma\mathrm{d}x\\
&-\tau\int_\Omega\partial^\beta(u\cdot\nabla u+g(\sigma)\nabla\sigma)\nabla\partial^\beta\sigma\mathrm{d}x+\tau\int_\Omega\partial^\beta f(x,t)\nabla\partial^\beta\sigma\mathrm{d}x.
\end{aligned}
$$

On the other hand, applying $\nabla\partial_x^\beta$ to $(2.11)_1$, and multiplying the resulting identity by $\partial_x^\beta u$, and then integrating over Ω, it yields

$$
\int_\Omega\partial^\beta u\nabla\partial^\beta\sigma_t\mathrm{d}x-\varepsilon\int_\Omega\partial^\beta\nabla\sigma\partial^\beta\Delta u\mathrm{d}x-\tau\bar{\rho}\int_\Omega|\partial^\beta(\mathrm{div}u)|^2\mathrm{d}x=\tau\int_\Omega\partial^\beta(\mathrm{div}\sigma u)\partial^\beta(\mathrm{div}u)\mathrm{d}x.
$$

Summing up the above inequalities, we obtain

$$
\begin{aligned}
&\gamma\bar{\rho}|\sigma|_K^2+\sum_{|\beta|=K-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\partial^\beta u\nabla\partial^\beta\sigma\mathrm{d}x\\
\le&\varepsilon|\sigma|_K|u|_{K+1}+\tau\bar{\rho}|u|_K^2+|\sigma u|_K|u|_K+\frac{4}{3}|u|_{K+1}|\sigma|_K+|u|_{K-1}|u|_K|\sigma|_K
\end{aligned}
$$

$$
\begin{aligned}
&+|\sigma|_{K-1}|\sigma|_K^2+|f|_{K-1}|\sigma|_K \\
\le& \frac{\gamma\bar{\rho}}{4}|\sigma|_K^2+\widehat{C}_2|u|_{K+1}^2+\widehat{C}_3\left(|u(\cdot,t)|_{K-1}^2+|\sigma(\cdot,t)|_{K-1}^2\right)\left(|u(\cdot,t)|_K^2+|\sigma(\cdot,t)|_K^2\right)+C|f|_{K-1}^2.
\end{aligned}
$$

By multiplying inequality (2.13) by A with $\frac{1}{6}\bar{\mu}A>\widehat{C}_2$, and taking δ sufficiently small with $\delta A<\frac{\gamma\bar{\rho}}{4}$, then summing up the above inequalities, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{A}{2}(\gamma|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^2)+\sum_{|\beta|=K-1}\int_\Omega \partial^\beta u\nabla\partial^\beta\sigma\mathrm{d}x\right)+A\varepsilon\gamma|\sigma(\cdot,t)|_{K+1}^2 \\
&+\frac{1}{6}\bar{\mu}A|u(\cdot,t)|_{K+1}^2+\frac{\gamma\bar{\rho}}{4}|\sigma|_K^2 \\
\le& M_1\left(|u(\cdot,t)|_{K-1}^2+|\sigma(\cdot,t)|_{K-1}^2\right)\left(|u(\cdot,t)|_K^2+|\sigma(\cdot,t)|_K^2\right)+M_2|f|_{K-1}^2. \qquad (2.14)
\end{aligned}
$$

Let

$$
\xi(t)=\frac{A}{2}(\gamma|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^2)+\sum_{|\beta|=K-1}\int_\Omega \partial^\beta u\nabla\partial^\beta\sigma\mathrm{d}x.
$$

It is easy to see that

$$
\underline{C}(|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^2)\le\xi(t)\le\overline{C}(|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^2).
$$

Note that we also have

$$
\frac{1}{6}\bar{\mu}A|u(\cdot,t)|_{K+1}^2+\frac{\gamma\bar{\rho}}{4}|\sigma|_K^2\ge\underline{M}(|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^2),
$$

for some positive constant $\underline{M}$. Integrating (2.14) from 0 to T yields

$$
\begin{aligned}
&\underline{M}\int_0^T(|\sigma(\cdot,t)|_K^2+|u(\cdot,t)|_K^2)\mathrm{d}t \\
\le& M_1\sup_{0<t<T}\left(|u(\cdot,t)|_{K-1}^2+|\sigma(\cdot,t)|_{K-1}^2\right)\int_0^T\left(|u(\cdot,t)|_K^2+|\sigma(\cdot,t)|_K^2\right)\mathrm{d}t+M_2\int_0^T|f|_{K-1}^2\mathrm{d}t \\
\le& M_1R^2+M_2\int_0^T|f|_{K-1}^2\mathrm{d}t.
\end{aligned}
$$

Using the Mean Value Theorem, there exists $\tau\in(0,T)$ such that

$$
\underline{M}T(|\sigma(\cdot,\tau)|_K^2+|u(\cdot,\tau)|_K^2)\le M_1R^2+M_2\int_0^T|f|_{K-1}^2\mathrm{d}t.
$$

Integrating (2.14) from τ to t for any $t\in(\tau,T]$, we have

$$
\xi(t)\le\xi(\tau)+M_1R^2+M_2\int_0^T|f|_{K-1}^2\mathrm{d}t\le\widetilde{M}_1R^2+\widetilde{M}_2\int_0^T|f|_{K-1}^2\mathrm{d}t.
$$

Noticing that $\xi(0)=\xi(T)$ and integrating (2.14) from 0 to t for any $t\in[0,T]$, we have

$$
\begin{aligned}
\sup_{0<t<T}\xi(t)\le&\xi(0)+M_1R^2+M_2\int_0^T|f|_{K-1}^2\mathrm{d}t \\
\le&(M_1+\widetilde{M}_1)R^2+(M_2+\widetilde{M}_2)\int_0^T|f|_{K-1}^2\mathrm{d}t. \qquad (2.15)
\end{aligned}
$$

Then, we obtain

$$
\sup_{0<t<T}\|(\sigma,u)(\cdot,t)\|_{H^{K-1}}^2+\int_0^T\|(\sigma,u)(\cdot,t)\|_{H^K}^2\mathrm{d}t
$$

$$
\begin{aligned}
&\leq \sup_{0<t<T} \|(\sigma,u)(\cdot,t)\|^2_{H^K} + \int_0^T \|(\sigma,u)(\cdot,t)\|^2_{H^K}\mathrm{d}t \\
&\leq M^*R^2 + C^*\int_0^T |f|^2_{K-1}\mathrm{d}t,
\end{aligned}
$$

which implies

$$
R \leq M^*R^2 + C^*\int_0^T |f|^2_{K-1}\mathrm{d}t.
$$

Take $R < \frac{1}{2M^*}$, and let $C^*\int_0^T |f|^2_{K-1}\mathrm{d}t < \frac{1}{2}R$, then the above inequality is a contradiction. Hence, (2.9) holds. On the other hand, when $\tau = 0$, $(\sigma, u) = 0$, that is, $F(\cdot, 0) = 0$. Therefore, for the chosen $R > 0$, we have

$$
\mathbf{deg}(I - F(\cdot,1), \hat{B}_R(0), 0) = \mathbf{deg}(I - F(\cdot,0), \hat{B}_R(0), 0) = \mathbf{deg}(I, \hat{B}_R(0), 0) = 1.
$$

Thus, (2.10) holds. That is the problem (2.1) admits a time periodic solution $U \in \mathcal{G}^R_K$. The proof is completed. □

Proof of Theorem 1.1 (Existence) To avoid any confusion, we denote the solution of the regularized problem (2.1) by $(\sigma_\varepsilon, u_\varepsilon)$. From the proof of Proposition 2.4, we see that

$$
\sup_{0<t<T} \|(\sigma_\varepsilon,u_\varepsilon)(\cdot,t)\|^2_{H^K} + \int_0^T (\|\sigma_\varepsilon(\cdot,t)\|^2_{H^K} + \|u_\varepsilon(\cdot,t)\|^2_{H^{K+1}})\mathrm{d}t \leq CR,
$$

where R is independent of ε. On the other hand, integrating from t to $t+h$ of (2.13), then integrating it from 0 to T, we have

$$
\int_0^T \left(\gamma|\sigma(\cdot,t+h)|^2_K + |u(\cdot,t+h)|^2_K\right) - \left(\gamma|\sigma(\cdot,t)|^2_K + |u(\cdot,t)|^2_K\right)\mathrm{d}t \leq Ch. \tag{2.16}
$$

Thus, there exists a subsequence of $(\sigma_\varepsilon, u_\varepsilon)$, denoted by $(\sigma_{\varepsilon_n}, u_{\varepsilon_n})$, such that

$$
\begin{aligned}
&(\sigma_{\varepsilon_n}, u_{\varepsilon_n}) \overset{*}{\rightharpoonup} (\sigma,u) \text{ in } L^\infty((0,T);H^K(\Omega)); \\
&u_{\varepsilon_n} \rightharpoonup u \text{ in } L^2((0,T);H^{K+1}(\Omega)); \\
&\sigma_{\varepsilon_n} \to \sigma \text{ in } L^2((0,T);H^{K-1}(\Omega)); \\
&u_{\varepsilon_n} \to u \text{ in } L^2((0,T);H^K(\Omega)).
\end{aligned}
$$

In what follows, we show that $\sigma_\varepsilon \in C^{\alpha,\beta}(\Omega\times(0,T)))$. Clearly, we have $\sigma_\varepsilon(x,t) \in C^\alpha(\Omega)$ for $\alpha \in (0,1)$ for any t because $\sigma_\varepsilon \in L^\infty((0,T);H^K(\Omega))$ with $K \geq 4$. So, we only need to show that there exists a constant $\beta \in (0,1)$ such that

$$
|\sigma_\varepsilon(x,t_1) - \sigma_\varepsilon(x,t_2)| \leq C|t_1-t_2|^\beta \tag{2.17}
$$

holds for any $t_1, t_2 \in (0,T)$, $x \in \Omega$. Without loss of generality, assume that $t_2 \leq t_1$. Take a ball B_r of radius r centered at x, with $r = |t_1-t_2|^\eta$, $\eta = \frac{1}{2\alpha+3}$. Recalling (2.7) and using Poincáre inequality, we have

$$
\begin{aligned}
\int_{B_r} |\sigma_\varepsilon(y,t_1) - \sigma_\varepsilon(y,t_2)|\mathrm{d}y &= \int_{B_r}\left|\int_{t_2}^{t_1}\frac{\partial\sigma_\varepsilon(y,t)}{\partial t}\mathrm{d}t\right|\mathrm{d}y \\
&\leq C\left(\int_{t_2}^{t_1}\int_{B_r}\left|\frac{\partial\sigma_\varepsilon(y,t)}{\partial t}\right|^2\mathrm{d}y\mathrm{d}t\right)^{1/2}|t_1-t_2|^{1/2}r^{3/2} \\
&\leq C|t_1-t_2|^{1/2}r^{3/2}.
\end{aligned}
$$

By Mean Value Theorem, there exists $x^* \in B_r$ such that

$$|\sigma_\varepsilon(x^*,t_1)-\sigma_\varepsilon(x^*,t_2)| \le C|t_1-t_2|^{1/2}r^{-3/2} = C|t_1-t_2|^{(1-3\eta)/2}.$$

Then, we have

$$\begin{aligned}
&|\sigma_\varepsilon(x,t_1)-\sigma_\varepsilon(x,t_2)|\\
\le{}& |\sigma_\varepsilon(x,t_1)-\sigma_\varepsilon(x^*,t_1)| + |\sigma_\varepsilon(x^*,t_1)-\sigma_\varepsilon(x^*,t_2)| + |\sigma_\varepsilon(x^*,t_2)-\sigma_\varepsilon(x,t_2)|\\
\le{}& C(|t_1-t_2|^{\eta\alpha} + |t_1-t_2|^{(1-3\eta)/2})\\
\le{}& C|t_1-t_2|^{\alpha/(2\alpha+3)}.
\end{aligned}$$

Similarly, we also have

$$|u_\varepsilon(x_1,t_1)-u_\varepsilon(x_2,t_2)| \le C|x_1-x_2|^{\tilde\alpha} + |t_1-t_2|^{\tilde\beta}, \tag{2.18}$$

for $\tilde\alpha, \tilde\beta \in (0,1)$, where C is independent of ε. Thus, by Arzela-Ascoli Theorem, we have

$$(\sigma_{\varepsilon_n}, u_{\varepsilon_n}) \to (\sigma, u) \text{ uniformly.}$$

Thus, $(\sigma,u)\in\mathcal{X}$ is a time periodic solution of (2.1). And this completes the proof of Theorem 1.1. □

3 The Uniqueness of Periodic Solutions

In this section, we consider the uniqueness of periodic solutions. Let $(\sigma_1,u_1),(\sigma_2,u_2)\in \mathcal{G}_K^R\cap\mathcal{X}$ be the time periodic solutions of the problem. Set $\sigma=\sigma_1-\sigma_2$, $u=u_1-u_2$. Then, (σ,u) is a periodic solution of following equation

$$\sigma_t + \bar\rho\nabla\cdot u = -\nabla\cdot(\sigma u_1) - \nabla\cdot(\sigma_2 u), \tag{3.1}$$

$$\begin{aligned}
&u_t - (\bar\mu\Delta u + (\bar\mu+\bar\lambda)\nabla \mathrm{div} u) + \gamma\bar\rho\nabla\sigma\\
={}&(h(\sigma_1)-h(\sigma_2))(\bar\mu\Delta u_2 + (\bar\mu+\bar\lambda)\nabla\mathrm{div}u_2) - h(\sigma_1)(\bar\mu\Delta u + (\bar\mu+\bar\lambda)\nabla\mathrm{div}u)\\
&- u\nabla u_1 - u_2\nabla u - (g(\sigma_1)-g(\sigma_2))\nabla\sigma_1 - g(\sigma_2)\nabla\sigma,
\end{aligned}\tag{3.2}$$

with periodic boundary condition. We will show the uniqueness if $\sup\limits_{t\in(0,T)}\|(\sigma_i,u_i)(s)\|_{H^K}$ is sufficiently small.

Proof of Theorem 1.1 (Uniqueness) We assume that

$$\sup_{t\in(0,T)}\|(\sigma_i,u_i)(s)\|_{H^K} \le \delta$$

for some sufficiently small $\delta>0$. Note that $K\ge 4$. For each multi-index α with $|\alpha|=3\le K-1$, by applying ∂_x^α to (3.1), (3.2), and multiplying them by $\gamma\partial_x^\alpha\sigma$, $\partial_x^\alpha u$ respectively, then integrating over Ω, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\sum_{|\alpha|=3}(\gamma|\partial^\alpha\sigma|^2 + |\partial^\alpha u|^2)\mathrm{d}x + \bar\mu\sum_{|\alpha|=3}\int_\Omega|\nabla\partial^\alpha u|^2\mathrm{d}x + (\bar\mu+\bar\lambda)\sum_{|\alpha|=3}\int_\Omega|\partial^\alpha\mathrm{div}u|^2\mathrm{d}x\\
={}&-\gamma\sum_{|\alpha|=3}\int_\Omega\partial^\alpha(\nabla\cdot(\sigma u_1))\partial^\alpha\sigma\mathrm{d}x - \gamma\sum_{|\alpha|=3}\int_\Omega\partial^\alpha(\nabla\cdot(\sigma_2 u))\partial^\alpha\sigma\mathrm{d}x\\
&+\sum_{|\alpha|=3}\int_\Omega\partial^{\alpha-1}\big((h(\sigma_1)-h(\sigma_2))(\bar\mu\Delta u_2 + (\bar\mu+\bar\lambda)\nabla\mathrm{div}u_2)\big)\partial^{\alpha+1}u\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{|\alpha|=3}\int_\Omega \partial^{\alpha-1}\big(h(\sigma_1)(\bar{\mu}\Delta u+(\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}u)\big)\partial^{\alpha+1}u\mathrm{d}x-\sum_{|\alpha|=3}\int_\Omega \partial^\alpha(u\nabla u_1)\partial^\alpha u\mathrm{d}x\\
&-\sum_{|\alpha|=3}\int_\Omega \partial^\alpha(u_2\nabla u)\partial^\alpha u\mathrm{d}x-\sum_{|\alpha|=3}\int_\Omega \partial^\alpha((g(\sigma_1)-g(\sigma_2))\nabla\sigma_1)\partial^\alpha u\mathrm{d}x\\
&-\sum_{|\alpha|=3}\int_\Omega \partial^\alpha(g(\sigma_2)\nabla\sigma)\partial^\alpha u\mathrm{d}x\\
=&\,I_1+I_2+\cdots+I_8.
\end{aligned}
$$

Next, we estimate each term on the right hand side of the above equation. Firstly, for I_1, we have

$$
\begin{aligned}
|I_1|\le&\frac{\gamma}{2}\sum_{|\alpha|=3}\|\nabla u_1\|_{L^\infty}\|\partial^\alpha\sigma\|_{L^2}^2\\
&+\gamma\int_\Omega\left(\sum_{0\le l\le 2}\binom{\alpha}{l}\partial^l(\nabla\sigma)\partial^{\alpha-l}u_1\partial^\alpha\sigma+\partial^l\sigma\partial^{\alpha-l}(\nabla\cdot u_1)\partial^\alpha\sigma\right)\mathrm{d}x\\
\le&\sum_{|\alpha|=3}\frac{\tau\gamma}{2}\|\nabla u_1\|_{L^\infty}\|\partial^\alpha\sigma\|_{L^2}^2+\sum_{|\alpha|=3}\|\partial^\alpha\sigma\|_{L^2}\big(\|\nabla u_1\|_{L^\infty}\|\nabla\sigma\|_{H^2}\\
&+\|\nabla u_1\|_{H^2}\|\nabla\sigma\|_{L^\infty}+\|\sigma\|_{L^\infty}\|\nabla^2u_1\|_{H^2}+\|\nabla^2u_1\|_{L^\infty}\|\sigma\|_{H^2}\big)\\
\le&\,C\delta|\sigma|_3^2.
\end{aligned}
$$

For I_2, we have

$$|I_2|\le\gamma|\sigma|_3(\|\sigma_2\|_{L^\infty}|u|_4+\|u\|_{L^\infty}|\sigma_2|_4)\le C\delta(|\sigma|_3^2+|u|_4^2).$$

Moreover,

$$
\begin{aligned}
|I_3|&\le|u|_4|(h(\sigma_1)-h(\sigma_2))(\bar{\mu}\Delta u_2+(\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}u_2)|_2\\
&\le C|u|_4(\|\sigma\|_{L^\infty}|u_2|_4+|\sigma|_2\|\Delta u_2\|_{L^\infty})\\
&\le C\delta(|\sigma|_3^2+|u|_4^2),
\end{aligned}
$$

and

$$|I_4|\le C|u|_4^2|\sigma_2|_2\le C\delta|u|_4^2.$$

Furthermore,

$$|I_5|\le C|u|_3^2|u_1|_4\le C\delta|u|_3^2,$$

$$|I_6|\le C\|\nabla u_2\|_{L^\infty}|u|_3^2+|u|_3\|\nabla u\|_{L^\infty}|u_2|_3\le C\delta|u|_3^2,$$

$$|I_7|\le|u|_4|(g(\sigma_1)-g(\sigma_2))\nabla\sigma_1|_2\le C\delta(|u|_4^2+|\sigma|_3^2),$$

$$|I_8|\le|u|_4|g(\sigma_2)\nabla\sigma|_2\le C\delta(|u|_4^2+|\sigma|_3^2).$$

Here, we use the fact that

$$|g(\sigma_1)-g(\sigma_2)|_k\le C|\sigma|_{\max\{2,k\}},$$

for smooth function g. In fact, note that

$$g(\sigma_1)-g(\sigma_2)=\frac{g(\sigma_1)-g(\sigma_2)}{\sigma_1-\sigma_2}(\sigma_1-\sigma_2)=f(\sigma_1,\sigma_2)\sigma.$$

Then,

$$|g(\sigma_1)-g(\sigma_2)|_k \le |f|_k\|\sigma\|_{L^\infty} + |\sigma|_k|f|_{L^\infty} \le C|\sigma|_{\max\{2,k\}}.$$

Hence, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\gamma|\sigma(\cdot,t)|_3^2+|u(\cdot,t)|_3^2)\mathrm{d}x+2\bar{\mu}|u(\cdot,t)|_4^2 \le \widetilde{C}_1\delta(|u(\cdot,t)|_4^2+|\sigma(\cdot,t)|_3^2). \tag{3.3}$$

For each multi-index β with $|\beta|=2$, by applying ∂_x^β, $\nabla\partial_x^\beta$ to (3.2), (3.1), and multiplying the resulting equations by $\partial_x^\beta\nabla\sigma$, $\partial_x^\beta u$ respectively, summing up the two resulting equations, then integrating over Ω, we have

$$\begin{aligned}
&\gamma\bar{\rho}\sum_{|\beta|=2}\int_\Omega|\nabla\partial^\beta\sigma|^2\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\beta|=2}\int_\Omega\partial^\beta u\nabla\partial^\beta\sigma\mathrm{d}x\\
=&\sum_{|\beta|=2}\int_\Omega\partial^\beta(\bar{\mu}\Delta u+(\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}u)\nabla\partial^\beta\sigma\mathrm{d}x\\
&+\sum_{|\beta|=2}\int_\Omega\partial^\beta((h(\sigma_1)-h(\sigma_2))(\bar{\mu}\Delta u_2+(\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}u_2))\nabla\partial^\beta\sigma\mathrm{d}x\\
&-\sum_{|\beta|=2}\int_\Omega\partial^\beta(h(\sigma_2)(\bar{\mu}\Delta u+(\bar{\mu}+\bar{\lambda})\nabla\mathrm{div}u))\nabla\partial^\beta\sigma\mathrm{d}x\\
&-\sum_{|\beta|=2}\int_\Omega\partial^\beta(u\nabla u_1+u_2\nabla u)\nabla\partial^\beta\sigma\mathrm{d}x\\
&-\sum_{|\beta|=2}\int_\Omega\partial^\beta((g(\sigma_1)-g(\sigma_2))\nabla\sigma_1)\nabla\partial^\beta\sigma\mathrm{d}x-\sum_{|\beta|=2}\int_\Omega\partial^\beta(g(\sigma_2)\nabla\sigma)\nabla\partial^\beta\sigma\mathrm{d}x\\
&+\bar{\rho}\sum_{|\beta|=2}\int_\Omega|\partial^\beta\mathrm{div}u|^2\mathrm{d}x+\sum_{|\beta|=2}\int_\Omega\partial^\beta\mathrm{div}u\cdot\partial^\beta\mathrm{div}(\sigma u_1)\mathrm{d}x\\
&+\sum_{|\beta|=2}\int_\Omega\partial^\beta\mathrm{div}u\cdot\partial^\beta\mathrm{div}(\sigma_2 u)\mathrm{d}x\\
=&J_1+J_2+\cdots+J_9.
\end{aligned}$$

In what follows, we estimate each term on the right hand side of the above equation as follows:

$$\begin{gathered}
|J_1|\le|\sigma|_3|u|_4,\ |J_2|\le C|\sigma|_3^2|u_2|_4\le C\delta|\sigma|_3^2,\\
|J_3|\le C|\sigma|_3|u|_4|\sigma_2|_2\le C\delta(|\sigma|_3^2+|u|_4^2),\\
|J_4|\le C(|u|_2|u_1|_3+|u|_3|u_2|_2)|\sigma|_3\le C\delta(|\sigma|_3^2+|u|_4^2),\\
|J_5|\le C|\sigma|_2|\sigma_1|_3|\sigma|_3\le C\delta|\sigma|_3^2,\ |J_6|\le C\delta|\sigma|_3^2,\\
|J_7|\le\bar{\rho}|u|_3^2,\ |J_8|\le C|u|_3|\sigma|_3|u_1|_3\le C\delta(|\sigma|_3^2+|u|_4^2),\ |J_9|\le C\delta|u|_4^2.
\end{gathered}$$

Thus, we obtain

$$\gamma\bar{\rho}|\sigma(\cdot,t)|_3^2+2\frac{\mathrm{d}}{\mathrm{d}t}\sum_{|\beta|=2}\int_\Omega\partial^\beta u\nabla\partial^\beta\sigma\mathrm{d}x\le\widetilde{C}_2\delta(|\sigma|_3^2+|u|_4^2)+\widetilde{C}_3|u|_4^2. \tag{3.4}$$

Combining with (3.3), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(A\int_\Omega(\gamma|\sigma(\cdot,t)|_3^2+|u(\cdot,t)|_3^2)\mathrm{d}x+2\sum_{|\beta|=2}\int_\Omega\partial^\beta u\nabla\partial^\beta\sigma\mathrm{d}x\right)$$

$$+ (2\bar{\mu}A - \widetilde{C}_3)|u(\cdot,t)|_4^2 + \gamma\bar{\rho}|\sigma(\cdot,t)|_3^2$$
$$\le (\widetilde{C}_1 A + \widetilde{C}_2)\delta(|u(\cdot,t)|_4^2 + |\sigma(\cdot,t)|_3^2).$$

Taking $A = \frac{\widetilde{C}_3}{\bar{\mu}}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(A\int_\Omega (\gamma|\sigma(\cdot,t)|_3^2 + |u(\cdot,t)|_3^2)\mathrm{d}x + 2\sum_{|\beta|=2}\int_\Omega \partial^\beta u \nabla \partial^\beta \sigma \mathrm{d}x\right)$$
$$+ \left(\widetilde{C}_3 - (\frac{\widetilde{C}_1\widetilde{C}_3}{\bar{\mu}} + \widetilde{C}_2)\delta\right)|u(\cdot,t)|_4^2 + \left(\gamma\bar{\rho} - (\frac{\widetilde{C}_1\widetilde{C}_3}{\bar{\mu}} + \widetilde{C}_2)\delta\right)|\sigma(\cdot,t)|_3^2 \le 0.$$

Integrating the above inequality from 0 to T, then choosing δ suitably small, we have

$$\int_0^T |u(\cdot,t)|_4^2 \mathrm{d}t + \int_0^T |\sigma(\cdot,t)|_3^2 \mathrm{d}t \le 0,$$

which implies that $u = \sigma = 0$ a.e. in Q_T. The uniqueness is then proved. □

Remark 3.1 From the proof of uniqueness, it is not difficult to see that for the compressible Navier-Stokes equations without external force, that is (1.2) with $f \equiv 0$, there is no small non-trivial time periodic solution.

References

[1] Serrin J. A note on the existence of periodic solutions of the Navier-Stokes equations. Arch Rational Mech Anal, 1959, **3**: 120–122

[2] Yudovich V T. Periodic motions of a viscous incompressible fluid. Soviet Math, 1960, **1**: 1214–1217

[3] Prodi G. Qualche risultato riguardo alle equazioni di Navier-Stokes nel caso bidimensionale. Rend Sem Mat Univ Padova, 1960, **30**: 1–15

[4] Kaniel S, Shinbrot M. A Reproductive Property of the Navier-Stokes Equations. Arch Rational Mech Anal, 1967, **24**: 265–288

[5] Farwig R, Okabe T. Periodic solutions of the Navier-Stokes equations with inhomogeneous boundary conditions. Ann Univ Ferrara, 2010, **56**: 249–281

[6] Maremonti P. Existence and stability of time periodic solution of the Navier-Stokes equations in the whole space. Nonlinearity, 1991, **4**: 503–529

[7] Maremonti P. Some theorems of existence for solutions of the Navier-Stokes equations with slip boundary conditions in half-space. Rich Mat, 1991, **40**: 81–135

[8] Kozono H, Nakao M. Periodic solutions of the Navier-Stokes equations in unbounded domains. Tohoku Math J, 1996, **48**: 33–50

[9] Valli A. Periodic and stationary solutions for compressible Navier-Stokes equations via a stability method. Ann Scuola Norm Sup Pisa Cl Sci, 1983, **10**(4): 607–647

[10] Matsumura A, Nishida T. Periodic solutions of a viscous gas equation. Recent topics in nonlinear PDE IV (Kyoto, 1988). Lecture Notes in Num Appl Anal, 1989, **10**: 49–82

[11] Feireisl E, Mucha P B, Novotny A, Pokorny M. Time-Periodic Solutions to the Full Navier-Stokes-Fourier System. Arch Rational Mech Anal, 2012, **204**: 745–786

[12] Březina J, Kagei K. Decay properties of solutions to the linearized compressible navier-stokes equation around time-periodic parallel flow. Mathematical Models and Methods in Applied Sciences, 2012, **22**: 1–53

[13] Březina J, Kagei K. Spectral properties of the linearized compressible Navier-Stokes equation around time-periodic parallel flow. J Differential Equations, 2013, **255**: 1132–1195

[14] Ma H F, Ukai S, Yang T. Time periodic solutions of compressible Navier-Stokes equations. J Differential Equations, 2010, **248**: 2275–2293