THE BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF IN THE WHOLE SPACE: I, AN ESSENTIAL COERCIVITY ESTIMATE

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ABSTRACT. It is known that the singularity in the non-cutoff cross-section of the Boltzmann equation leads to the gain of regularity and gain of weight in the velocity variable. By defining and analyzing a non-isotropy norm which precisely captures the dissipation in the linearized collision operator, we give a new and precise coercivity estimate for the non-cutoff Boltzmann equation for general physical cross sections.

1. Introduction

This is the first part of a series of papers related to the inhomogeneous Boltzmann equation without angular cut-off, in the whole space and for general physical cross-sections. This global project is a natural continuation of our previous study [6] which was specialized to Maxwellian type cross sections.

The present part is concerned with an essential coercivity estimate of the linearized collision operator, in the framework of general cross sections. As shown in [6] for the special Maxwellian case, this estimate plays an important role for the related Cauchy problem.

Based on this and connected estimations, in the second and third papers, [7, 8], we will prove the global existence of classical non-negative solutions to the Boltzmann equation without angular cutoff, together with convergence rates to the equilibrium, for the soft and hard potentials respectively, so that we are able to cover a general physical setting. Finally, in the fourth paper, [9], we will prove the full regularization property of the solution for any positive time. On the whole, our series of works will establish a satisfactory theory on the well-posedness and full regularity of classical solutions.

To simplify the exposition, we shall work in velocity space dimension 3, which is the most important physical case, but our results hold true for any dimension $n \geq 2$.

Consider

$$f_t + v \cdot \nabla_x f = Q(f, f). \tag{1.1}$$

As usual, $f = f(t, x, v)$ is the density distribution function of particles with space location $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time t. Here, the right hand side of (1.1) is the Boltzmann bilinear collision operator, which is given in the classical

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$\sigma-$representation by

$$Q(g,f)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B\left(v-v_*,\sigma\right)\left\{g'_*f'-g_*f\right\}d\sigma dv_*\,,$$

where $f'_*=f(t,x,v'_*), f'=f(t,x,v'), f_*=f(t,x,v_*), f=f(t,x,v)$, and for $\sigma\in\mathbb{S}^2$,

$$v'=\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma,\ \ v'_*=\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma,$$

give the relations between the post and pre collisional velocities. Recall that these relations follow from the conservation of momentum and kinetic energy, that is, $v+v_*=v'+v'_*$ and $|v|^2+|v_*|^2=|v'|^2+|v'_*|^2$.

The kernel $B(v-v_*,\sigma)$ appearing in the collision operator is called the collision cross-section and varies according to different physical settings.

For the monoatomic gas, the non-negative cross-section $B(z,\sigma)$ depends only on $|z|$ and the scalar product $\frac{z}{|z|}\cdot\sigma$. In most cases, the kernel $B(z,\sigma)$ cannot be expressed explicitly, but to capture its main properties, one may assume that it takes the form

$$B(v-v_*,\cos\theta)=\Phi(|v-v_*|)b(\cos\theta),\quad \cos\theta=\frac{v-v_*}{|v-v_*|}\cdot\sigma,\ \ 0\le\theta\le\frac{\pi}{2}.$$

There are two classical and important physical models to be kept in mind. The first one is the hard sphere model where each particle is viewed as an identical ball. For this model, the cross-session is proportional to $|v-v_*|$ and has no singularity in θ. Another model is the inverse power law potential model in which the force between a pair of particles is proportional to $\nabla U(\rho)$ and $U(\rho)$ is proportional to ρ^{-r} with $r>1$. Here, ρ is the distance between the two particles. In the later case, the kinetic part in the cross-section behaves like

$$\Phi(|v-v_*|)=|v-v_*|^{\gamma},\qquad \gamma=1-\frac{4}{r}, \tag{1.2}$$

and a singular factor in the collision angle arises, such that

$$\lim_{\theta\to 0+}b(\cos\theta)\theta^{2+2s}=K, \tag{1.3}$$

for some constant $K>0$, and where $0<s=\frac{1}{r}<1$.

For the model of inverse power law potential, the cases with $1<r<4$, $r=4$ and $r>4$ correspond to the so-called soft, Maxwellian molecule and hard potentials respectively. Moreover, since the Boltzmann equation is well defined when $r>1$ for this model, one has $\gamma>-3$, $0<s<1$, and

$$-1<\gamma+2s=1-\frac{2}{r}=1-2s<1.$$

Herein, we will be concerned with the precise coercivity estimate of the linearized collisional operator, for cross-sections in this general setting.

That is, we only assume that the indices in the cross-section satisfy $\gamma>-3$ and $0<s<1$ in the kinetic and angular parts respectively. In particular, note that this assumption includes the inverse power law potential as an example.

For later use, we will need to compare the original cross-session with the situation when its kinetic part is mollified. That is, for the function $\Phi(z)$ appearing in the cross-section, we denote by $\tilde\Phi(z)$ its smoothed version defined by

$$\tilde\Phi(z)=(1+|z|^2)^{\frac{\gamma}{2}}\equiv<z>^{\gamma}\,. \tag{1.4}$$

To show the dependence of the estimates on the mollified or non-mollified kinetic factor in the cross-session, we will use the notations $Q^{\tilde\Phi}$ and Q^{Φ} to denote the Boltzmann collisional operator when the kinetic part is $\tilde\Phi$ and Φ respectively. In particular, $Q = Q^{\Phi}$. This upper-script will be also used for other operators as well.

As usual, and keeping the same notations as in [6], we consider the Boltzmann equation around a normalized global Maxwellian distribution

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Since μ is the global equilibrium state satisfying $Q(\mu, \mu) = 0$, by setting $f = \mu + \sqrt{\mu} g$, we have

$$Q(\mu + \sqrt{\mu} g,\, \mu + \sqrt{\mu}\, g) = Q(\mu,\, \sqrt{\mu}\, g) + Q(\sqrt{\mu}\, g,\, \mu) + Q(\sqrt{\mu}\, g,\, \sqrt{\mu}\, g).$$

Define the following standard nonlinear operator

$$\Gamma(g,\, h) = \mu^{-1/2} Q(\sqrt{\mu}\, g,\, \sqrt{\mu}\, h).$$

Then the linearized Boltzmann collision operator takes the form

$$\mathcal{L} g = \mathcal{L}_1\, g + \mathcal{L}_2\, g = -\Gamma(\sqrt{\mu}\,,\, g) - \Gamma(g,\, \sqrt{\mu}\,),$$

and the original equation (1.1) becomes the following equation on the perturbation g

$$g_t + v \cdot \nabla_x g + \mathcal{L} g = \Gamma(g,\, g), \quad t > 0\,. \tag{1.5}$$

This close to equilibrium framework is classical for the Boltzmann equation with angular cutoff, but much less is known for the Boltzmann equation without angular cutoff, though the spectrum of the linearized operator without angular cut-off was analyzed a long time ago by Pao in [18].

However, since the late 1990s, the regularizing effect on the solution, produced by the singularity of the cross-section, has become reachable by rigorous analysis. Let us mention the systematic work on the entropy dissipation method initiated by Alexandre [1] and developed firstly by Lions [15], and then by many others, cf [3, 20, 21] and references therein. Since then, various works have been done on deriving the coercivity estimates in different settings and in different norms for different purposes. In particular, this kind of coercivity estimates has displayed some non-isotropic property in the very loose sense that, on one hand one gets a gain of the regularity in Sobolev norm of fractional order; and on the other hand, one also get a gain the moment to some fractional power in the velocity variable, cf. [2, 3, 4, 5, 6, 10, 11, 12, 14, 16, 17, 19, 20, 21] and references therein. Furthermore, these coercivity estimates have been proven to be very useful in getting the global existence and gain of full regularity in all variables for the Boltzmann equation without angular cutoff, as shown in our previous work [6]. For details about the recent progress in some of the directions mentionned previously, readers are referred to the survey paper by Alexandre, [2].

Since the coercivity estimate plays an important role in the study on the angular non-cutoff Boltzmann equation, the precise statement of this estimate in terms of the indices in the cross-section, that is, γ and s, has been pursued by many people. The main purpose of this paper is to present a precise estimate that gives the essential properties of this singular behavior. This main result will be stated in the next theorem. Let us note that this result is proved in a general setting and it improves on previous results, such as those obtained in [4, 5, 6, 16, 17]. And this estimate will be used in our papers [7, 8, 9] on the global existence and full regularity of the Boltzmann equation without angular cutoff in the general setting.

To derive the desired coercivity estimate, we generalize the non-isotropic norm introduced in [6]. The introduction of this norm was motivated by the study on the Landau equation which can be viewed as the grazing limit of the Boltzmann equation. It is known that for the Landau equation, see for example [13], that the essential norm in order to capture the dissipation of the linearized Landau operator can be defined just as the Dirichlet form of the linearized operator. By doing so, a norm can be well defined without loss of any dissipative information in the operator and this can be done directly for the Landau equation mainly because the corresponding Landau operator is a differential operator. However, for the Boltzmann equation without angular cutoff, the collision operator is a singular integral operator so that a direct analog is not obvious or feasible. In order to extract the essential property in the linearized collision operator, in [6], a non-isotropic norm was introduced. Therefore, in the first part of this paper, we will show that this non-isotropic norm is in fact equivalent to the Dirichlet form of the linearized collision operator. Then by analyzing the properties of the non-isotropic norm, we obtain the precise coercivity estimate of the linearized collision operator.

The application of the non-isotropic norm and the coercivity estimate for the global well-posedness theory and the full regularity for the solution to the Cauchy problem will be given in our papers [7, 8, 9]. At this point, let us mention the different approach undertaken by Gressman and Strain [11, 12] for the global well-posedness issue.

To state our main theorem, we recall that the linearized operator $\mathcal{L}$ has the following null space, which is spanned by the set of collision invariants:

$$\mathcal{N} = \text{Span}\left\{\sqrt{\mu}\,,\, v_1\sqrt{\mu}\,,\, v_2\sqrt{\mu}\,,\, v_3\sqrt{\mu}\,,\, |v|^2\sqrt{\mu}\ \right\}, \tag{1.6}$$

that is, $\Big(\mathcal{L}g,\, g\Big)_{L^2(\mathbb{R}^3_v)} = 0$ if and only if $g \in \mathcal{N}$.

Herein, $(.,.)_{L^2} = (.,.)_{L^2(\mathbb{R}^3_v)}$ denotes the usual scalar product of functions in $L^2 = L^2(\mathbb{R}^3)$.

We shall use the standard weighted Sobolev space defined, for $k, l \in \mathbb{R}$, as

$$H^k_l = H^k_l(\mathbb{R}^3_v) = \{f \in \mathcal{S}'(\mathbb{R}^3_v);\ W^l f \in H^k(\mathbb{R}^3_v)\}$$

where $H^k = H^k(\mathbb{R}^3)$ is the usual Sobolev space, and $W^l(v) = <v>^l = (1+|v|^2)^{l/2}$ is the weight function. In particular, in the case $k = 0$, we also use the notation $L^2_l = H^0_l$.

Theorem 1.1. *Assume that the cross-section satisfies (1.2),* (1.3) *with* $0 < s < 1$ *and* $\gamma > -3$*. Then there exist two generic constants* $C_1, C_2 > 0$ *such that*

$$\begin{aligned} C_1 &\left\{\|(\mathbf{I}-\mathbf{P})g\|^2_{H^s_{\gamma/2}} + \|(\mathbf{I}-\mathbf{P})g\|^2_{L^2_{s+\gamma/2}}\right\} \\ &\le \Big(\mathcal{L}g,\, g\Big)_{L^2} \le C_2\, \|g\|^2_{H^s_{s+\gamma/2}}\,, \end{aligned} \tag{1.7}$$

where $\mathbf{P}$ *is the* L^2*-orthogonal projection onto the null space* $\mathcal{N}$.

This essential coercivity estimate of the linearized collisional operator will enable us in [7, 8] to prove the global existence of classical solutions to the Boltzmann equation. For this purpose, the analysis on the nonlinear operator is necessary. By using this essential coercivity estimate and the analytic techniques used in the proofs below, we will also give a clear upper bound estimate on the nonlinear operator when $\gamma > -3/2$ which is stated in the next section, the remaining values of γ being given in [7].

The rest of the paper is arranged as follows. In the next section, we extend the definition of the non-isotropic norm introduced in [6] and then state the main estimates in this paper. The proof of the upper and lower bound estimates of the non-isotropic norm will be given in Section 3. In Section 4, we will prove the equivalence of the Dirichlet form of the linearized collison operator and the square of the non-isotropic norm. The equivalence of the non-isotropic norms with different kinetic factors and different weights will be shown in Section 5. Finally, in the last section, we will prove an upper bound estimate on the nonlinear collision operator which is useful for the well-posedness theory for the Boltzmann equation.

Notations: Herein, letters f, g stand for various smooth functions, while C, c... stand for various numerical constants, independent from functions f, g.., and which may vary from line to line. Notation $A \lesssim B$ means that there exists a constant C such that $A \le CB$, and similarly for $A \gtrsim B$. While $A \sim B$ means that there exist two generic constants $C_1, C_2 > 0$ such that

$$C_1 A \le B \le C_2 A.$$

2. NON-ISOTROPIC NORM AND MAIN ESTIMATES

First of all, let us recall that

$$\Big(\mathcal{L}g,\, g\Big)_{L^2} = -\Big(\Gamma(\sqrt{\mu}\,, g) + \Gamma(g,\, \sqrt{\mu}\,),\, g\Big)_{L^2} \ge 0.$$

It is known that $\mathcal{L}$ is a non-isotropic operator, so, extending [6], we define a non-isotropic norm associated with the cross-section $\Phi(|v - v^*|)b(\cos\theta)$ by the following formula

$$\begin{aligned} |||g|||^2 &= \iiint \Phi(|v - v^*|)b(\cos\theta)\mu_* \left(g' - g\right)^2 \\ &\quad + \iiint \Phi(|v - v^*|)b(\cos\theta)g_*^2\left(\sqrt{\mu'} \, - \sqrt{\mu}\,\right)^2 \\ &= J_1 + J_2\,, \end{aligned} \tag{2.1}$$

where the integration is over $\mathbb{R}^3_v \times \mathbb{R}^3_{v_*} \times SS^2_\sigma$. Note that it is a norm with respect to the velocity variable $v \in \mathbb{R}^3$ only. As we will see later, the reason that this norm is called non-isotropic is due to the fact that it combines both differentiation and weight to some orders due to the singularity of cross-section $b(\cos\theta)$. In fact, compared to the previous works on the Landau case [13], this non-isotropic norm is much more involved because we bascially deal with a singular integral operator that behaves like a fractional order differential operator, though with different weight factors according to the direction of the frequency variable, a fact which is clearly well understood for the Landau case.

The following proposition gives the equivalence between the non-isotropic norm and the Dirichlet form of $\mathcal{L}$:

Proposition 2.1. *Assume that the cross-section satisfies (1.2),* (1.3) *with* $0 < s < 1$ *and* $\gamma > -3$. *Then there exist two generic constants* $C_1, C_2 > 0$ *such that*

$$C_1 |||(\mathbf{I} - \mathbf{P})g|||^2 \le \Big(\mathcal{L}g,\, g\Big)_{L^2} \le 2\Big(\mathcal{L}_1 g,\, g\Big)_{L^2} \le C_2 |||g|||^2. \tag{2.2}$$

Concerning the lower and upper bounds of the non-isotropic norm we have

Proposition 2.2. *Assume that the cross-section satisfies (1.2),* (1.3) *with* $0 < s < 1$ *and* $\gamma > -3$*. Then there exist two generic constants* $C_1, C_2 > 0$ *such that*

$$C_1 \left\{ \|g\|^2_{H^s_{\gamma/2}} + \|g\|^2_{L^2_{s+\gamma/2}} \right\} \leq |||g|||^2 \leq C_2 \, \|g\|^2_{H^s_{s+\gamma/2}} \, . \tag{2.3}$$

Our main result, that is, Theorem 1.1, is then a direct consequence of the above two propositions.

In the following, we will use the lower script Φ on the non-isotropic norm, and so use the notation $|||g|||_\Phi$ if we need to specify its dependence on the kinetic factor Φ. Notations J_1^Φ, J_2^Φ will be also used for the same purpose.

Part of the proof on the lower bound of the non-isotropic norm given in Proposition 2.2 is essentially due to the following equivalence relations between the non-isotropic norms with Φ and $\tilde{\Phi} = \langle v - v_* \rangle^\gamma$.

Proposition 2.3. *Assume that the cross-section satisfies (1.2),* (1.3) *with* $0 < s < 1$ *and* $\gamma > -3$*. Then we have*

$$|||g|||_\Phi \sim |||g|||_{\tilde{\Phi}} \, . \tag{2.4}$$

Concerning the dependence on the index γ in $\Phi_\gamma = |v - v_*|^\gamma$, we have

Proposition 2.4. *Assume that the cross-section satisfies (1.2),* (1.3) *with* $0 < s < 1$ *and* $\gamma > -3$*. Then for any* $\beta > -3$*, we have*

$$|||g|||_{\Phi_\gamma} \sim |||\langle v \rangle^{(\gamma-\beta)/2} g|||_{\Phi_\beta} \, . \tag{2.5}$$

To end this section, we give the following upper bound estimate on the non-linear term $\Gamma(\cdot,\cdot)$ defined in (1.5), which holds only for the restricted value $\gamma > -3/2$. The upper bound estimate for the general index $\gamma > -3$ will be given in our paper [7], since it is more involved.

Proposition 2.5. *Assume that the cross-section satisfies* (1.2), (1.3) *with* $0 < s < 1$ *and* $\gamma > -3/2$*. Then*

$$\begin{aligned} \Big|\Big(\Gamma(f,g), h\Big)\Big| \lesssim \Big\{ &\|f\|_{L^2_{s+\gamma/2}} |||g|||_{\Phi_\gamma} + \|g\|_{L^2_{s+\gamma/2}} |||f|||_{\Phi_\gamma} \\ &+ \min \Big(\|f\|_{L^2} \|g\|_{L^2_{s+\gamma/2}} \, , \, \|f\|_{L^2_{s+\gamma/2}} \|g\|_{L^2} \Big) \Big\} |||h|||_{\Phi_\gamma} \, . \end{aligned}$$

3. Bounds on the non-isotropic norm

This section is devoted to the proof of Proposition 2.2. Throughout this paper, we will often use the following elementary estimate stated in velocity dimension n, since it will be needed for both cases $n = 2$ and $n = 3$.

Lemma 3.1. *Let the velocity dimension be* n, $n \in \mathbb{N}$, $\rho > 0, \delta \in \mathbb{R}$ *and let* $\mu_{\rho,\delta}(u) = \langle u \rangle^\delta e^{-\rho |u|^2}$ *for* $u \in \mathbb{R}^n$*. If* $\alpha > -n$ *and* $\beta \in \mathbb{R}$*, then we have*

$$I_{\alpha,\beta}(u) = \int_{\mathbb{R}^n} |w|^\alpha \langle w \rangle^\beta \mu_{\rho,\delta}(w+u) dw \sim \langle u \rangle^{\alpha+\beta} \, . \tag{3.1}$$

Proof. Since we have

$$\langle u \rangle^\beta \langle u + w \rangle^{-|\beta|} \leq \langle w \rangle^\beta \leq \langle u \rangle^\beta \langle u + w \rangle^{|\beta|},$$

it suffices to show (3.1) with $\beta = 0$, by taking $\mu_{\rho,\delta\pm|\beta|}$ instead of $\mu_{\rho,\delta}$. Taking into account the fact that $\alpha > -n$, this estimate is obvious when $|u| \le 1$. If $|u| \ge 1$, then we have

$$I_{\alpha,0}(u) \ge 4^{-|\alpha|}\langle u\rangle^\alpha \int_{\{|u+w|\le 1/2\}} \mu_{\rho,\delta}(u+w)dw \gtrsim \langle u\rangle^\alpha,$$

because $|u+w| \le 1/2$ implies that $4^{-1}\langle u\rangle \le |w| \le 4\langle u\rangle$. Noticing that $2|w| \ge \langle w\rangle$ if $|w| \ge 1$, we have

$$\begin{aligned} I_{\alpha,0}(u) &\le \Big(\max_{|w|\le 1} \mu_{\rho,\delta}(u+w)\Big) \int_{\{|w|\le 1\}} |w|^\alpha dw + 2^{|\alpha|} \int_{\{|w|\ge 1\}} \langle w\rangle^\alpha \mu_{\rho,\delta}(u+w)dw \\ &\lesssim \Big(\langle u\rangle^{|\delta|} e^{-\rho|u|^2/2} + \langle u\rangle^\alpha \int_{\mathbb{R}^n} \langle u+w\rangle^\alpha \mu_{\rho,\delta}(u+w)dw\Big) \lesssim \langle u\rangle^\alpha . \end{aligned}$$

And this completes the proof of the lemma.

□

Recall from (2.1) that the non-isotropic norm contains two parts, denoted by J_1 and J_2 respectively. The estimation on each part will be given in the following subsections. We start with the estimation on J_2 because the analysis is easier.

J_2-estimate. Let us start with the following upper bound on J_2.

Lemma 3.2. *Under the same assumptions as in Theorem 1.1, we have*

$$J_2 := \iiint b(\cos\theta)\Phi(|v-v_*|)g_*^2\big(\sqrt{\mu'} - \sqrt{\mu}\,\big)^2 dvdv_*d\sigma \lesssim \|g\|^2_{L^2_{s+\gamma/2}} .$$

Proof. Note that

$$\begin{aligned} J_2 &\le 2\iiint b\Phi(|v-v_*|)g_*^2\Big(\mu'^{1/4} - \mu^{1/4}\Big)^2\Big(\mu'^{1/2} + \mu^{1/2}\Big)dvdv_*d\sigma \\ &\lesssim \iiint b\,|v'-v_*|^\gamma g_*^2\Big(\mu'^{1/4} - \mu^{1/4}\Big)^2 \mu'^{1/2} dvdv_*d\sigma \\ &\quad + \iiint b\,|v-v_*|^\gamma g_*^2\Big(\mu'^{1/4} - \mu^{1/4}\Big)^2 \mu^{1/2} dvdv_*d\sigma \\ &= F_1 + F_2 . \end{aligned}$$

By the regular change of variables $v \to v'$, we have

$$\begin{aligned} F_1 &\lesssim \iint |v'-v_*|^\gamma \Big(\int b(\cos\theta)\min\Big(|v'-v_*|^2\theta^2, 1\Big)d\sigma\Big) g_*^2 \mu'^{1/2} dv'dv_* \\ &\lesssim \int\Big(\int |v'-v_*|^{\gamma+2s}\mu' dv'\Big)g_*^2 dv_* \lesssim \|g\|^2_{L^2_{s+\gamma/2}} , \end{aligned}$$

where we have used Lemma 3.1 in the case $n = 3$ to get the last inequality. A direct estimation show thats the same bound holds true for F_2. And this completes the proof of the lemma. □

Remark 3.3. *Note that the above lemma holds even if Φ is replaced by $\tilde\Phi$ by using Lemma 3.1.*

We now turn to the lower bound for J_2.

Lemma 3.4. *Under the assumptions* (1.2) *and* (1.3), *with* $\gamma > -3$, *there exists a constant* $C > 0$ *such that*

$$J_2 := \iiint b(\cos\theta)\Phi(|v-v_*|)g_*^2\big(\sqrt{\mu'} - \sqrt{\mu}\,\big)^2 dv dv_* d\sigma \ge C\|g\|^2_{L^2_{s+\gamma/2}}.$$

Proof. We will apply the argument used in [20]. By shifting to the ω-representation,

$$v' = v - \big((v-v_*)\cdot\omega\big)\omega \quad v'_* = v + \big((v-v_*)\cdot\omega\big)\omega \ \ ,\omega \in SS^2,$$

in view of the change of variables $(v, v_*) \to (v_*, v)$, we get,

$$J_2 = 4\iiint b(\cos\theta)\sin(\theta/2)\Phi(|v-v_*|)g^2\big(\sqrt{\mu'_*} - \sqrt{\mu_*}\,\big)^2 dv dv_* d\omega\,,$$

because $d\sigma = 4\sin(\theta/2)d\omega$. Then, we use the Carleman representation. The idea of this representation is to replace the set of variables (v, v_*, ω) by the set (v, v', v'_*). Here, $v, v' \in \mathbb{R}^3$ and $v'_* \in E_{vv'}$, where $E_{vv'}$ is the hyperplane passing through v and orthogonal to $v - v'$. By using the formula

$$dv_* d\omega = \frac{dv'_* dv'}{|v-v'|^2},$$

cf. page 347 of [20], and by taking the change of variables

$$(v, v', v'_*) \to (v, v+h, v+y),$$

with $h \in \mathbb{R}^3$ and $y \in E_h$, where E_h is the hyperplane orthogonal to h passing through the origin in $\mathbb{R}^3$, we have

$$\begin{aligned} J_2 \sim \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_h}\int_{y\in E_h\cap\{|y|\ge|h|\}} &\frac{|y|^{1+2s+\gamma}}{|h|^{1+2s}} g(v)^2 \\ &\times\big(\sqrt{\mu(v+y)} - \sqrt{\mu(v+y+h)}\,\big)^2 dv \frac{dh dy}{|h|^2}\,, \end{aligned}$$

because

$$\begin{aligned} &|h| = |v'-v| = |v'_* - v|\tan\frac{\theta}{2} = |y|\tan\frac{\theta}{2}, \quad \theta\in[0,\pi/2]\,,\\ &b(\cos\theta)\sin(\theta/2)\Phi(|v-v_*|) \sim \frac{|v_*-v'|^{1+2s+\gamma}}{|v-v'|^{1+2s}}\mathbf{1}_{\{|v'_*-v|\ge|v'-v|\}}\,. \end{aligned}$$

We decompose $v = v_1 + v_2$, where v_2 is the orthogonal projection of v on E_h. Since μ is invariant by rotation, we may assume $v = (0, 0, |v|)$ without loss of generality. By introducing the polar coordinates

$$h = (\rho\sin\vartheta\cos\phi, \rho\sin\vartheta\sin\phi, \rho\cos\vartheta)\,, \vartheta\in[0,\pi],\ \phi\in[0,2\pi], \rho > 0,$$

we obtain $|v_1| = |v||\cos\vartheta|$, $|v_1 + h| = |\,|v|\cos\vartheta + \rho|$ and $|v_2| = |v|\sin\vartheta$. Note that if $0 < \vartheta \le \pi/2$, then

$$\begin{aligned} \big(\sqrt{\mu(v+y)} - \sqrt{\mu(v+y+h)}\,\big)^2 &= \mu(v_2+y)\big(\sqrt{\mu(v_1)} - \sqrt{\mu(v_1+h)}\,\big)^2\\ &\ge \mu(v_2+y)\mu(v_1)\big(1-e^{-\rho^2/4}\big)^2/(2\pi)^{3/2}. \end{aligned}$$

Therefore, we have for any $\delta > 0$

$$
\begin{aligned}
J_2 \geq C \int_{\mathbb{R}^3_v} g(v)^2 \Big\{ \int_{\mathbb{R}^3_h} & \frac{\big(\sqrt{\mu(v_1)} \; - \sqrt{\mu(v_1+h)}\; \big)^2}{|h|^{3+2s}} \\
& \times \Big(\int_{y \in E_h \cap \{|y| \geq |h|\}} |y|^{1+2s+\gamma} \mu(v_2+y) dy \Big) dh \Big\} dv \\
\geq C \int_{\mathbb{R}^3_v} g(v)^2 \Big\{ \int_{\pi/2 - 1/\langle v \rangle}^{\pi/2} & \mu(v_1) \left(\int_0^\delta \frac{\big(1 - e^{-\rho^2/4}\big)^2}{\rho^{1+2s}} \right. \\
& \times \Big(\int_{y \in E_h} |y|^{1+2s+\gamma} \mu(v_2+y) dy \\
& \left. - \int_{y \in E_h \cap \{|y| \leq \rho\}} |y|^{1+2s+\gamma} \mu(v_2+y) dy \Big) d\rho \right) \sin\vartheta d\vartheta \Big\} dv \,.
\end{aligned}
$$

Since we have

$$
\int_{y \in E_h \cap \{|y| \leq \rho\}} |y|^{1+2s+\gamma} \mu(v_2+y) dy \leq \delta^{2s} \int_{y \in E_h} |y|^{1+\gamma} \mu(v_2+y) dy, \quad \text{if } \rho \leq \delta \;,
$$

and it follows from Lemma 3.1 in the case $n = 2$, that

$$
\int_{y \in E_h} |y|^\beta \mu(v_2+y) dy \sim \langle v_2 \rangle^\beta \;\text{ if } \beta > -2 \;,
$$

there exist two constants $C_1, C_2 > 0$ such that if $\rho \leq \delta$, we have

$$
\begin{aligned}
\int_{y \in E_h} |y|^{1+2s+\gamma} \mu(v_2+y) dy - \int_{y \in E_h \cap \{|y| \leq \rho\}} & |y|^{1+2s+\gamma} \mu(v_2+y) dy \\
& \geq C_1 \langle v_2 \rangle^{1+2s+\gamma} - C_2 \delta^{2s} \langle v_2 \rangle^{1+\gamma} \,.
\end{aligned}
$$

Taking a sufficiently small $\delta > 0$ gives

$$
\begin{aligned}
J_2 \geq C \int_{\mathbb{R}^3_v} g(v)^2 \Big\{ & \int_{\pi/2 - 1/\langle v \rangle}^{\pi/2} \mu(v_1) \\
& \times \left(\int_0^\delta \frac{\big(1 - e^{-\rho^2/4}\big)^2}{\rho^{1+2s}} d\rho \right) \langle v_2 \rangle^{1+2s+\gamma} \sin\vartheta d\vartheta \Big\} dv \\
\geq C_\delta & \int_{\mathbb{R}^3_v} \langle v \rangle^{2s+\gamma} g(v)^2 \Big\{ \int_{\pi/2 - 1/\langle v \rangle}^{\pi/2} e^{-|v|^2 \cos^2\vartheta} \langle v \rangle d\vartheta \Big\} dv \\
\geq C_\delta & \|g\|^2_{s+\gamma/2} \,.
\end{aligned}
$$

The proof of the lemma is now completed. □

Remark 3.5. *In the above proof, the factor* $|y|^\gamma$ *can be replaced by* $\langle y \rangle^\gamma$*, so that Lemma 3.4 is valid even if* Φ *is replaced by* $\tilde{\Phi} = \langle v - v_* \rangle^\gamma$*. By the above lemma together with Lemma 3.2 and the Remark after it, we can conclude*

$$
J_2^\Phi \sim \|g\|^2_{L^2_{s+\gamma/2}} \sim J_2^{\tilde{\Phi}} \,. \tag{3.2}
$$

J_1**-estimate.** We now turn to the estimation of the first term of the non-isotropic norm, that is, J_1. We will firstly show that the singular behavior of the cross-section when $v = v^*$ can be smoothed out. This point is given by the following proposition.

Proposition 3.6. *Under the same assumtion as in Theorem 1.1, we have*

$$J_1^{\Phi} + \|g\|^2_{L^2_{s+\gamma/2}} \sim J_1^{\widetilde{\Phi}} + \|g\|^2_{L^2_{s+\gamma/2}} . \tag{3.3}$$

Remark 3.7. *This proposition is nothing but Proposition 2.3 by Remark 3.5.*

Proof. By using similar arguments as in the proof of Lemma 3.4, it follows from the Carleman representation that

$$\begin{aligned} J_1^{\Phi} &\sim \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_h}\int_{y\in E_h\cap\{|y|\geq|h|\}} \frac{|y|^{1+2s+\gamma}}{|h|^{1+2s}}\mu(v)\big(g(v+y) \ - g(v+y+h)\ \big)^2 dv\frac{dhdy}{|h|^2} \\ &= \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_h}\int_{y\in E_h\cap\{|y|\geq|h|\}} \frac{|y|^{1+2s+\gamma}}{|h|^{1+2s}}\mu(v+y)\big(g(v) \ - g(v+h)\ \big)^2 dv\frac{dhdy}{|h|^2}\,, \end{aligned}$$

where the last equality is a direct consequence of the change of variables $(v+y, y) \to (v, -y)$.

Similarly, we have

$$J_1^{\widetilde{\Phi}} \sim \int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_h}\int_{y\in E_h\cap\{|y|\geq|h|\}} \frac{|y|^{1+2s}\langle y\rangle^{\gamma}}{|h|^{1+2s}}\mu(v+y)\big(g(v+h) \ - g(v)\ \big)^2 dv\frac{dhdy}{|h|^2}\,.$$

We claim that

$$\begin{aligned} &\int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_h}\int_{y\in E_h\cap\{|y|\leq|h|\}} \frac{|y|^{1+2s+\gamma}}{|h|^{1+2s}}\mu(v+y)\big(g(v+h) \ - g(v)\ \big)^2 dv\frac{dhdy}{|h|^2} \\ &\qquad \lesssim \|g\|^2_{L^2_{s+\gamma/2}}\,, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_h}\int_{y\in E_h\cap\{|y|\leq|h|\}} \frac{|y|^{1+2s}\langle y\rangle^{\gamma}}{|h|^{1+2s}}\mu(v+y)\big(g(v+h) \ - g(v)\ \big)^2 dv\frac{dhdy}{|h|^2} \\ &\qquad \lesssim \|g\|^2_{L^2_{s+\gamma/2}}\,. \end{aligned} \tag{3.5}$$

Note carefully that the integration in these estimates is performed for "large" values of h.

Once we admit those estimates, to conclude the proof of the lemma, it suffices to show that

$$G(v,h) = \int_{y\in E_h} |y|^{1+2s+\gamma}\mu(v+y)dy \sim \int_{y\in E_h} |y|^{1+2s}\langle y\rangle^{\gamma}\mu(v+y)dy = \tilde{G}(v,h).$$

We decompose $v = v_1 + v_2$, where v_2 is the orthogonal projection of v on E_h. Then we have $\mu(v+y) = \mu(v_1)\mu(v_2+y)$, whence it follows from Lemma 3.1 together with $1 + 2s + \gamma > -2$ that

$$G(v,h) \sim \mu(v_1)\langle v_2\rangle^{1+2s+\gamma} \sim \tilde{G}(v,h)\,.$$

It remains to show (3.4) and (3.5). We write

$$\begin{aligned} &\int_{\mathbb{R}^3_v}\int_{\mathbb{R}^3_h}\int_{y\in E_h\cap\{|y|\leq|h|\}} \frac{|y|^{1+2s+\gamma}}{|h|^{1+2s}}\mu(v+y)\big(g(v+h) \ - g(v)\ \big)^2 dv\frac{dhdy}{|h|^2} \\ &= \int_{\mathbb{R}^3_v}\int_{\{|h|\leq 1\}}\int_{y\in E_h\cap\{|y|\leq|h|\}} \quad + \int_{\mathbb{R}^3_v}\int_{\{|h|\geq 1\}}\int_{y\in E_h\cap\{|y|\leq|h|\}} \quad = A_1 + A_2\,. \end{aligned}$$

Take a small $\delta > 0$ such that $\gamma - \delta > -3$. Then, in view of $1+\gamma-\delta > -2$, we have

$$
\begin{aligned}
A_1 &\le \int_{\mathbb{R}^3_v} \int_{\{|h|\le 1\}} \int_{y\in E_h} \frac{|y|^{1+\gamma-\delta}}{|h|^{1-\delta}} \mu(v+y)\big(g(v+h) \ - g(v)\ \big)^2 dv \frac{dhdy}{|h|^2} \\
&= \int_{\mathbb{R}^3_v} \mu(v_1) \int_{\{|h|\le 1\}} \Big(\int_{y\in E_h} |y|^{1+\gamma-\delta} \mu(v_2+y) dy \Big) \big(g(v+h) \ - g(v)\ \big)^2 \frac{dh}{|h|^{3-\delta}} dv \\
&\lesssim \int_{\mathbb{R}^3_v} \mu(v_1) \langle v_2\rangle^{1+\gamma-\delta} \int_{\{|h|\le 1\}} \big(g(v+h) \ - g(v)\ \big)^2 \frac{dh}{|h|^{3-\delta}} dv \\
&\lesssim \int_{\mathbb{R}^3_v} \int_{\{|h|\le 1\}} \Big(\mu(v_1-h) + \mu(v_1)\Big) \langle v_2\rangle^{1+\gamma-\delta} \big|g(v)\big|^2 \frac{dh}{|h|^{3-\delta}} dv \,,
\end{aligned}
$$

where we have used the change of variables $v+h \to v$ for the factor $g(v+h)$. As in the proof of Lemma 3.4, by assuming $v = (0,0,|v|)$, we introduce the polar coordinates

$$
h = (\rho \sin\vartheta \cos\phi, \rho\sin\vartheta\sin\phi, \rho\cos\vartheta)\,, \vartheta \in [0,\pi],\ \phi\in[0,2\pi], \rho > 0\,.
$$

Since $|v_1| = |v||\cos\vartheta|$, $|v_1 - h| = |\,|v|\cos\vartheta - \rho|$ and $|v_2| = |v|\sin\vartheta$, by using the change of varible $|v|\cos\vartheta = r$, we obtain

$$
\begin{aligned}
A_1 \lesssim & \int_{\mathbb{R}^3_v} \big|g(v)\big|^2 \int_0^1 \frac{1}{\rho^{1-\delta}} \\
& \times \left(\int_{-|v|}^{|v|} \frac{(1+|v|^2 - r^2)^{(1+\gamma-\delta)/2}}{|v|} \Big(e^{-|r-\rho|^2/2} + e^{-r^2/2} \Big) dr \right) d\rho dv \,.
\end{aligned}
$$

Similarly, if $1+2s-\delta > 1$, then we have

$$
\begin{aligned}
A_2 &\le \int_{\mathbb{R}^3_v} \int_{\{|h|\ge 1\}} \int_{y\in E_h} \frac{|y|^{1+\gamma+2s-\delta}}{|h|^{1+2s-\delta}} \mu(v+y)\big(g(v+h) \ - g(v)\ \big)^2 dv \frac{dhdy}{|h|^2} \\
&\lesssim \int_{\mathbb{R}^3_v} \big|g(v)\big|^2 \int_1^\infty \frac{1}{\rho^{1+2s-\delta}} \\
&\quad \times \left(\int_{-|v|}^{|v|} \frac{(1+|v|^2 - r^2)^{(1+\gamma+2s-\delta)/2}}{|v|} \Big(e^{-|r-\rho|^2/2} + e^{-r^2/2} \Big) dr \right) d\rho dv \,.
\end{aligned}
$$

If $1+\gamma+2s-\delta \ge 0$, then

$$
\begin{aligned}
K(v,\rho) &= \int_{-|v|}^{|v|} \frac{(1+|v|^2 - r^2)^{(1+\gamma+2s-\delta)/2}}{|v|} \Big(e^{-|r-\rho|^2/2} + e^{-r^2/2} \Big) dr \\
&\le \langle v\rangle^{(\gamma+2s-\delta)/2} \int_{-|v|}^{|v|} \Big(e^{-|r-\rho|^2/2} + e^{-r^2/2} \Big) dr \lesssim \langle v\rangle^{\gamma+2s},
\end{aligned}
$$

which shows

$$
A_2 \lesssim \int_{\mathbb{R}^3_v} \big|g(v)\big|^2 \int_1^\infty \frac{K(v,\rho)}{\rho^{1+2s-\delta}} d\rho dv \lesssim \int \langle v\rangle^{\gamma+2s} \big|g(v)\big|^2 dv \,. \tag{3.6}
$$

On the other hand, if $1+\gamma+2s-\delta<0$ and $|v|\geq 3$, then

$$\begin{aligned}K(v,\rho)&\lesssim \int_0^{|v|}\frac{(|v|^2-r^2)^{(1+\gamma+2s-\delta)/2}}{|v|}\Big(e^{-|r-\rho|^2/2}+3e^{-r^2/2}\Big)dr\\&\lesssim |v|^{(-1+\gamma+2s-\delta)/2}\int_0^{|v|}\Big(|v|-r\Big)^{(1+\gamma+2s-\delta)/2}\Big(e^{-|r-\rho|^2/2}+3e^{-r^2/2}\Big)dr\\&\lesssim \langle v\rangle^{\gamma+2s}+|v|^{(-1+\gamma+2s-\delta)/2}\int_{|v|/2}^{|v|}\Big(|v|-r\Big)^{(1+\gamma+2s-\delta)/2}3e^{-|r-\rho|^2/2}dr,\end{aligned}$$

because

$$\begin{aligned}\int_0^{|v|}\Big(|v|-r\Big)^{(1+\gamma+2s-\delta)/2}e^{-|r|^2/2}dr&\lesssim |v|^{(1+\gamma+2s-\delta)/2}\int_0^{|v|/2}e^{-|r|^2/2}dr\\&\quad+e^{-|v|^2/8}\int_{|v|/2}^{|v|}\Big(|v|-r\Big)^{(1+\gamma+2s-\delta)/2}dr,\end{aligned}$$

where we have used that $(1+\gamma+2s-\delta)/2>-1$ for small $\delta>0$ that follows from the assumption $\gamma>-3$. We now consider

$$\begin{aligned}&\int_1^\infty\frac{d\rho}{\rho^{1+2s-\delta}}\int_{|v|/2}^{|v|}\Big(|v|-r\Big)^{(1+\gamma+2s-\delta)/2}e^{-|r-\rho|^2/2}dr\\&\leq\int_{|v|/2}^{|v|}\Big(|v|-r\Big)^{(1+\gamma+2s-\delta)/2}\Big(\int_{\{|r-\rho|\leq\sqrt{2\log|v|}\}}(|v|/3)^{-(1+2s-\delta)}d\rho\Big)dr\\&+\int_{|v|/2}^{|v|}\Big(|v|-r\Big)^{(1+\gamma+2s-\delta)/2}\Big(\int_{\{|r-\rho|\geq\sqrt{2\log|v|}\}}\frac{|v|^{-1}d\rho}{\rho^{1+2s-\delta}}\Big)dr\\&\lesssim\Big(|v|^{(1+\gamma+2s-\delta)/2}\sqrt{2\log|v|}+|v|^{(1+\gamma+2s-\delta)/2}\Big)\lesssim\langle v\rangle^{(1+\gamma+2s)/2}.\end{aligned}$$

Therefore, in the case when $1+\gamma+2s-\delta<0$, we also have (3.6). Similarly, we have

$$A_1\lesssim\int_{\mathbb{R}^3_v}\big|g(v)\big|^2\int_0^1\frac{K(v,\rho)}{\rho^{1-\delta}}d\rho dv\lesssim\int\langle v\rangle^{\gamma+2s}\big|g(v)\big|^2dv\,,$$

which shows (3.4). The proof of (3.5) is similar, and thus the proof of the proposition is completed.

□

Lemma 3.8. *There exist constants $C_1,C_2>0$ such that*

$$J_1\geq C_1\|\langle v\rangle^{\gamma/2}g\|^2_{H^s}-C_2\|g\|^2_{L^2_{s+\gamma/2}}. \tag{3.7}$$

The same conclusion holds even if μ replaced by μ^ρ for any fixed $\rho>0$.

Proof. It follows from Proposition 3.6 that

$$
\begin{aligned}
C\Big(J_1^{\Phi} + \|g\|^2_{L^2_{s+\gamma/2}}\Big) &\geq 2\, J_1^{\widetilde{\Phi}} \\
&\geq \iiint b(\cos\theta)\frac{\mu_*}{\langle v_*\rangle^{|\gamma|}}\left(\langle v'\rangle^{\gamma/2}g' - \langle v\rangle^{\gamma/2}g\right)^2 d\sigma dv dv_* \\
&\quad - 2\iiint b(\cos\theta)\frac{\mu_*}{\langle v_*\rangle^{|\gamma|}}\Big(\langle v\rangle^{\gamma/2} - \langle v'\rangle^{\gamma/2}\Big)^2|g|^2 d\sigma dv dv_* \\
&= A_1 + A_2,
\end{aligned}
\tag{3.8}
$$

because $\widetilde{\Phi}(|v-v_*|) \sim \langle v'-v_*\rangle^{\gamma} \geq \dfrac{\langle v'\rangle^{\gamma}}{\langle v_*\rangle^{|\gamma|}}$ and $2(a+b)^2 \geq a^2 - 2b^2$. Setting $\tilde{g} = \langle v\rangle^{\gamma/2}g$ and noting $C_\gamma\mu(v)\langle v\rangle^{-|\gamma|} \geq \mu(2v)$ for a $C_\gamma > 0$, as in Proposition 1 of [3], we have

$$
\begin{aligned}
C_\gamma\, A_1 &\geq \int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\mu(2v_*)\Big(\tilde{g}(v) - \tilde{g}(v')\Big)^2 d\sigma dv_* dv \\
&= (4\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big\{\hat{\mu}(0)|\hat{\tilde{g}}(\xi)|^2 + \hat{\mu}(0)|\hat{\tilde{g}}(\xi^+)|^2 \\
&\qquad\qquad - 2Re\,\hat{\mu}(\xi^-/2)\hat{\tilde{g}}(\xi^+)\bar{\hat{\tilde{g}}}(\xi)\Big\}d\sigma d\xi \\
&\geq \frac{1}{2(4\pi)^3}\int_{\mathbb{R}^3}|\hat{\tilde{g}}(\xi)|^2\Big\{\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)(\hat{\mu}(0) - |\hat{\mu}(\xi^-/2)|)d\sigma\Big\}d\xi\,.
\end{aligned}
$$

Since we have $\hat{\mu}(0) - |\hat{\mu}(\xi^-/2)| = c(1-e^{-|\xi^-|^2/8}) \geq c'|\xi^-|^2$ if $|\xi^-| \leq 1$, in view of $|\xi^-|^2 = |\xi|^2\sin^2\theta/2 \geq |\xi|^2(\theta/\pi)^2$, we obtain for $|\xi| \geq 1$

$$
\begin{aligned}
\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)(\hat{\mu}(0) - |\hat{\mu}(\xi^-)|)d\sigma &\geq \int_{|\xi|(\theta/\pi)\leq 1}\sin\theta b(\cos\theta)|\xi|^2(\theta/\pi)^2 d\theta \\
&\geq c''K|\xi|^2\int_0^{1/|\xi|}\theta^{-1-2s}\theta^2 d\theta \\
&= c''K|\xi|^2|\xi|^{2s-2}/(2-2s).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
A_1 &\geq C_1\int_{|\xi|\geq 1}|\xi|^{2s}|\hat{\tilde{g}}(\xi)|^2 d\xi \geq C_1 2^{-2s}\int_{|\xi|\geq 1}(1+|\xi|^2)^s|\hat{\tilde{g}}(\xi)|^2 d\xi \\
&\geq C_1 2^{-2s}\|\langle v\rangle^{\gamma/2}g\|^2_{H^s(\mathbb{R}^3_v)} - C_1\|\langle v\rangle^{\gamma/2}g\|^2_{L^2(\mathbb{R}^3_v)}\,.
\end{aligned}
\tag{3.9}
$$

As for A_2, we note that if $v_\tau = v' + \tau(v-v')$ for $\tau \in [0,1]$, then

$$
\langle v\rangle \leq \langle v-v_*\rangle + \langle v_*\rangle \leq \sqrt{2}\langle v_\tau - v_*\rangle + \langle v_*\rangle \leq (1+\sqrt{2})\langle v_\tau\rangle\langle v_*\rangle,
$$

and $\langle v_\tau\rangle \leq (1+\sqrt{2})\langle v\rangle\langle v_*\rangle$, which show $\langle v_\tau\rangle^{\beta} \leq C_\beta\langle v\rangle^{\beta}\langle v_*\rangle^{|\beta|}$ for any $\beta \in \mathbb{R}$. It follows that

$$
\begin{aligned}
\Big|\langle v\rangle^{\gamma/2} - \langle v'\rangle^{\gamma/2}\Big| &\leq C_\gamma\int_0^1\langle v' + \tau(v-v')\rangle^{(\gamma/2-1)}d\tau|v-v_*|\theta \\
&\leq C'_\gamma\Big(\langle v\rangle^{(\gamma/2-1)}\langle v_*\rangle^{|\gamma/2-1|}\Big)\langle v-v_*\rangle\theta,
\end{aligned}
$$

and thus we have

$$A_2 \le C \iint \frac{\mu_*}{\langle v_*\rangle^{|\gamma|}} |g|^2 \Big\{ \Big(\langle v\rangle^{(\gamma-2)} \langle v_*\rangle^{|\gamma-2|} \Big)^2 \Big(\int_0^{\langle v-v_*\rangle^{-1}} \theta^{-1-2s} \Big(\langle v-v_*\rangle \theta \Big)^2 d\theta \Big)$$
$$+ \int_{\langle v-v_*\rangle^{-1}}^{\pi/2} \Big(\langle v\rangle^{\gamma} + \langle v\rangle^{\gamma} \langle v_*\rangle^{|\gamma|} \Big) \theta^{-1-2s} d\theta \Big\} dv dv_*$$
$$\le C \iint \Big(\langle v\rangle^{2s+\gamma} \langle v_*\rangle^{2s+\max(|\gamma-2|-|\gamma|,0)} \Big) \mu_* |g|^2 dv dv_* \le C \|g\|^2_{L^2_{s+\gamma/2}},$$

which together with (3.9) yields the desired estimate (3.7). The last estimate of the lemma is obvious by replacing μ by μ^ρ in each step of the above arguments, so that the proof of the lemma is completed. □

Lemma 3.4 together with Lemma 3.8 imply that we have the following lower bound on the non-isotropic norm,

$$|||g|||^2 \gtrsim \Big(\|g\|^2_{H^s_{\gamma/2}} + \|g\|^2_{L^2_{s+\gamma/2}} \Big). \tag{3.10}$$

Therefore, to complete the proof of Proposition 2.2, it remains to show

Lemma 3.9. *Let $\gamma > -3$. Then we have*

$$J_1 \lesssim \|g\|^2_{H^s_{s+\gamma/2}} + \|g\|^2_{L^2_{s+\gamma/2}}.$$

The same conclusion holds even if μ in J_1 is replaced by μ^ρ for any fixed $\rho > 0$.

Proof. As for Lemma 3.8, it follows from Proposition 3.6 that, for suitable constants $C_1, C_2 > 0$, we have

$$C_1 J_1^{\Phi} - C_2 \|g\|^2_{L^2_{s+\gamma/2}} \le J_1^{\widetilde{\Phi}}$$
$$\le 2 \iiint b(\cos\theta) \mu_* \langle v_*\rangle^{|\gamma|} \Big(\langle v'\rangle^{\gamma/2} g' - \langle v\rangle^{\gamma/2} g \Big)^2 d\sigma dv dv_* \tag{3.11}$$
$$+ 2 \iiint b(\cos\theta) \mu_* \langle v_*\rangle^{|\gamma|} \Big(\langle v\rangle^{\gamma/2} - \langle v'\rangle^{\gamma/2} \Big)^2 |g|^2 d\sigma dv dv_*$$
$$= B_1 + B_2,$$

because $\widetilde{\Phi}(|v-v_*|) \sim \langle v'-v_*\rangle^{\gamma} \le \langle v'\rangle^{\gamma} \langle v_*\rangle^{|\gamma|}$ and $(a+b)^2 \le 2(a^2+b^2)$.

By the same argument for A_2 in the proof of Lemma 3.8, we get $B_2 \lesssim \|g\|_{L^2_{s+\gamma/2}}$.

To estimate B_1, we apply Theorem 2.1 of [5] about the upper bound on the collision operator in the Maxwellian molecule case. It follows from (2.1.9) of [5] with $(m, \alpha) = (-s, -s)$ that

$$\Big| \Big(Q^{\Phi_0}(F, G), G \Big) \Big| \lesssim \|F\|_{L^1_{s+2s}} \|G\|^2_{H^s_s}.$$

Since $2a(b-a) = -(b-a)^2 + (a^2 - b^2)$, we get

$$\Big(Q^{\Phi_0}(F, G), G \Big) = \iiint b F_* G (G' - G)$$
$$= -\frac{1}{2} \iiint b F_* (G' - G)^2 + \frac{1}{2} \iint F_* \big(G'^2 - G^2 \big),$$

and therefore

$$\Big|\iiint bF_*(G'-G)^2\Big| \le 2\Big|\Big(Q^{\Phi_0}(F,G),G\Big)\Big| + \Big|\iint F_*\big({G'}^2-G^2\big)\Big| \\ \lesssim \|F\|_{L^1_{3s}}\|G\|^2_{H^s_s} + \|F\|_{L^1}\|G\|^2_{L^2}\,,$$

where we have used the cancellation lemma from [3] for the second term. Choosing $F=\mu\langle v\rangle^{|\gamma|}$ and $G=\langle v\rangle^{\gamma/2}g$, it follows that $B_1\lesssim \|g\|^2_{H^s_{s+\gamma/2}}$, completing the proof of the lemma. □

4. Equivalence to the linearized operator

We will now show that the Dirichlet form of the linearized collision operator is equivalent to the square of the non-isotropic norm, and therefore, the proof of Proposition 2.1 will be given. Let us note that for the bilinear operator $\Gamma(\,\cdot,\,\cdot\,)$, for suitable functions f,g, one has

$$\Big(\Gamma(f,\,g),\,h\Big)_{L^2} = \iiint b(\cos\theta)\Phi(|v-v^*|)\sqrt{\mu_*}\,\big(f'_*g'-f_*g\big)h \\ = \iiint b(\cos\theta)\Phi(|v-v^*|)\sqrt{\mu'_*}\,\big(f_*g-f'_*g'\big)h'\,,$$

and by adding these two lines, it follows that

$$\Big(\Gamma(f,\,g),\,h\Big)_{L^2} = \frac{1}{2}\iiint b(\cos\theta)\Phi(|v-v^*|)\Big(f'_*g'-f_*g\Big)\Big(\sqrt{\mu_*}\,h-\sqrt{\mu'_*}\,h'\Big)\,. \tag{4.1}$$

The following lemma shows that $\mathcal{L}_1$ dominates $\mathcal{L}$.

Lemma 4.1. *Under the conditions* (1.2),(1.3) *on the cross-section with* $0<s<1$ *and* $\gamma\in\mathbb{R}$*, we have*

$$\Big(\mathcal{L}_1 g,\,g\Big)_{L^2} \ge \frac{1}{2}\Big(\mathcal{L}g,\,g\Big)_{L^2}\,. \tag{4.2}$$

Proof. By standard changes of variables, the following computations hold true

$$\Big(\mathcal{L}_1 g,\,g\Big)_{L^2} = -\Big(\Gamma(\sqrt{\mu}\,,g),\,g\Big)_{L^2} \\ = \frac{1}{2}\iiint B(|v-v^*|,\cos\theta)\Big((\mu'_*)^{1/2}g'-(\mu_*)^{1/2}g\Big)^2 dv_*d\sigma dv \\ = \frac{1}{2}\iiint B(|v-v^*|,\cos\theta)\Big((\mu')^{1/2}g'_*-(\mu)^{1/2}g_*\Big)^2 dv_*d\sigma dv \\ = \frac{1}{4}\iiint B(|v-v^*|,\cos\theta) \\ \times\Big\{\Big((\mu'_*)^{1/2}g'-(\mu_*)^{1/2}g\Big)^2+\Big((\mu')^{1/2}g'_*-(\mu)^{1/2}g_*\Big)^2\Big\}\,,$$

and

$$\begin{aligned}
\Big(\mathcal{L}g,\, g\Big)_{L^2} &= -\Big(\Gamma(\sqrt{\mu}\,, g) + \Gamma(g,\, \sqrt{\mu}\,),\, g\Big)_{L^2(\mathbb{R}^3_v)} \\
&= \iiint B\left((\mu_*)^{1/2} g - (\mu'_*)^{1/2} g' + g_*(\mu)^{1/2} - g'_*(\mu')^{1/2}\right)(\mu_*)^{1/2}\, g \\
&= \iiint B\left((\mu'_*)^{1/2} g' - (\mu_*)^{1/2} g + g'_*(\mu')^{1/2} - g_*(\mu)^{1/2}\right)(\mu'_*)^{1/2}\, g' \\
&= \iiint B\left((\mu)^{1/2} g_* - (\mu')^{1/2} g'_* + g(\mu_*)^{1/2} - g'(\mu'_*)^{1/2}\right)(\mu)^{1/2}\, g_* \\
&= \iiint B\left((\mu')^{1/2} g'_* - (\mu)^{1/2} g_* + g'(\mu'_*)^{1/2} - g(\mu_*)^{1/2}\right)(\mu')^{1/2}\, g'_* \\
&= \frac{1}{4}\iiint B\left\{\left((\mu_*)^{1/2} g - (\mu'_*)^{1/2} g'\right) + \left((\mu)^{1/2} g_* - (\mu')^{1/2} g'_*\right)\right\}^2.
\end{aligned}$$

Therefore, (4.2) follows from $(\alpha+\beta)^2 \le 2(\alpha^2+\beta^2)$ and the proof is completed. □

Now for the term $\mathcal{L}_2$, we have

Lemma 4.2. *One has*

$$\left|\Big(\mathcal{L}_2 g,\, h\Big)_{L^2}\right| \lesssim \|\mu^{1/10^3} g\|_{L^2} \|\mu^{1/10^3} h\|_{L^2}\,.$$

Proof. It follows from (4.1) that

$$\begin{aligned}
\Big(\mathcal{L}_2(g),\, h\Big)_{L^2} &= -\frac{1}{2}\iiint B\Big(g'_*\sqrt{\mu'} - g_*\sqrt{\mu}\Big)\Big(\sqrt{\mu_*}\, h - \sqrt{\mu'_*}\, h'\Big) \\
&= \Big(g\,, \mathcal{L}_2(h)\Big)_{L^2}\,,
\end{aligned}$$

that is, $\mathcal{L}_2$ is symmetric. Hence it suffices to show the lemma in the case when $g = h$. Putting $G = \sqrt{\mu} g$, we have

$$\begin{aligned}
-\mathcal{L}_2 g =& \mu^{-1/2} Q(G,\, \mu) \\
=& \mu^{-1/2} \iint b(\cos\theta)\Phi(|v - v_*|) G'_*\Big(\mu' - \mu\Big) dv_* d\sigma \\
&+ \sqrt{\mu} \iint b(\cos\theta)\Phi(|v - v_*|)\Big(G'_* - G_*\Big) dv_* d\sigma \\
=& I_1(v) + I_2(v).
\end{aligned}$$

Thanks to the cancellation lemma, we have $I_2(v) = \sqrt{\mu(v)}\big(S * G\big)(v)$ with $S(v) \sim |v|^\gamma$, whence we have

$$\begin{aligned}
\left|(I_2, g)_{L^2}\right| &\lesssim \int_v \int_{v_*} |v - v_*|^\gamma \sqrt{\mu}\sqrt{\mu}_* |g||g_*| dv dv_* \\
&\lesssim \int_v \int_{v_*} |v - v_*|^\gamma \Big\{(\mu_*^{1/4}\mu^{1/4} g)^2 + (\mu^{1/4}\mu_*^{1/4} g_*)^2\Big\} dv dv_* \\
&\lesssim \|\langle v\rangle^\gamma \mu^{1/4} g\|^2_{L^2} \lesssim \|\mu^{1/8} g\|^2_{L^2}\,,
\end{aligned} \tag{4.3}$$

by means of Lemma 3.1.

Writing

$$\mu' - \mu \,=\, \sqrt{\mu'}\big(\sqrt{\mu'} - \sqrt{\mu}\big) + \sqrt{\mu}\big(\sqrt{\mu'} - \sqrt{\mu}\big)$$

and using $\sqrt{\mu'\mu'_*} = \sqrt{\mu\mu_*}$, we have

$$I_1(v) = \iint b(\cos\theta)\Phi(|v-v_*|)g'_*\Big(\sqrt{\mu_*}+\sqrt{\mu'_*}\Big)\ \Big(\sqrt{\mu'}-\sqrt{\mu}\Big)dv_*d\sigma.$$

Hence

$$\begin{aligned}\Big(I_1,\, g\Big)_{L^2} =& \iiint b(\cos\theta)\Phi(|v-v_*|)g'_*\Big(\sqrt{\mu_*}-\sqrt{\mu'_*}\Big)\ \Big(\sqrt{\mu'}-\sqrt{\mu}\Big)g dv dv_* d\sigma\\ &+2\iiint b(\cos\theta)\Phi(|v-v_*|)G_*\Big(\sqrt{\mu}-\sqrt{\mu'}\Big)\, g' dv dv_* d\sigma\\ =&A_1+A_2\,,\end{aligned}$$

where we have used the change of variables $(v,v_*)\to(v',v'_*)$ for the second term. We can write

$$\begin{aligned}A_1 = \iiint b(\cos\theta)\Phi(|v-v_*|)&\Big({\mu_*}^{1/4}-{\mu'_*}^{1/4}\Big)\ \Big({\mu'}^{1/4}-\mu^{1/4}\Big)g'_*g\\ &\times\Big({\mu_*}^{1/4}+{\mu'_*}^{1/4}\Big)\ \Big({\mu'}^{1/4}+\mu^{1/4}\Big)d\sigma dv dv_*\,.\end{aligned}$$

Since we have

$$\begin{aligned}|v'_*|^2 &\le (|v'_*-v'|+|v'|)^2 \le (\sqrt{2}|v_*-v'|+|v'|)^2\\ &\le (\sqrt2|v_*|+(\sqrt2+1)|v'|)^2 \le 4|v_*|^2+2(\sqrt2+1)^2|v'|^2,\end{aligned}$$

and in the same way, $|v|^2 \le 4|v'|^2+2(\sqrt2+1)^2|v_*|^2$, we get, by adding the two corresponding inequalities, that $\mu_*\mu' \le (\mu'_*\mu)^{1/(10+4\sqrt2)}$. Moreover, we have $\mu'_*\mu' = \mu_*\mu \le (\mu'_*\mu)^{1/5}$ because $|v'_*|^2 \le (|v'_*-v|+|v|)^2 \le (|v_*-v|+|v|)^2 \le 2|v_*|^2+8|v|^2$. Noticing that

$$\Big|\Big({\mu_*}^{1/4}-{\mu'_*}^{1/4}\Big)\ \Big({\mu'}^{1/4}-\mu^{1/4}\Big)\Big| \lesssim |v-v'_*|^2\theta^2,$$

we get

$$\begin{aligned}|A_1| &\lesssim \iint |v-v'_*|^{\gamma+2}\left\{\int_0^{\pi/2}\theta^{1-2s}d\theta\right\}(\mu'_*\mu)^{1/80}g'_*g dv dv'_*\\ &\lesssim \iint |v-v'_*|^{\gamma}(\mu'_*\mu)^{1/160}g'_*g dv dv'_* \lesssim \|\mu^{1/10^3}g\|^2_{L^2}\,,\end{aligned}\tag{4.4}$$

by an argument similar to the analysis of I_1.

For A_2, we use the regular change of variable $v\to v'$, and denote its inverse transformation by $v'\to\psi_\sigma(v')=v$. Then

$$\begin{aligned}A_2 = 2\iint \sqrt{\mu}_*g_*\Bigg\{\int_{SS^2} &b\Big(\frac{\psi_\sigma(v')-v_*}{|\psi_\sigma(v')-v_*|}\cdot\sigma\Big)\Phi(|\psi_\sigma(v')-v_*|)\\ &\times\Big(\sqrt{\mu(\psi_\sigma(v'))}-\sqrt{\mu(v')}\Big)\Big|\frac{\partial(\psi_\sigma(v'))}{\partial(v')}\Big|d\sigma\Bigg\}\, g(v')dv_*dv'\,.\end{aligned}$$

It follows from the Taylor expansion that

$$\begin{aligned}\sqrt{\mu(\psi_\sigma(v'))}-\sqrt{\mu(v')} &= \Big(\nabla\sqrt{\mu}\Big)(v')\cdot\Big(\psi_\sigma(v')-v'\Big)\\ &+\int_0^1(1-\tau)\Big(\nabla^2\sqrt{\mu}\Big)(v'+\tau(\psi_\sigma(v')-v'))\ \Big(\psi_\sigma(v')-v'\Big)^2d\tau.\end{aligned}$$

Note that the integral with respect to σ corresponding to the first order term vanishes, by means of the symmetry on SS^2. Putting $v'_\tau = v' + \tau(\psi_\sigma(v') - v')$, we have $|v'|^2 \leq (|v' - v_*| + |v_*|)^2 \leq ((|v'_\tau - v_*| + |v_*|)^2 \leq 2|v'_\tau|^2 + 8|v_*|^2$, so that

$$\left|\sqrt{\mu(v_*)}\Big(\nabla^2\sqrt{\mu}\Big)(v' + \tau(\psi_\sigma(v') - v'))\right| \lesssim (\mu(v_*)\,\mu(v'))^{1/12}\,.$$

Since $|\psi_\sigma(v') - v'| \lesssim |v' - v_*|\theta$, we have

$$|A_2| \lesssim \iint \left\{\int_0^{\pi/2} \theta^{1-2s} d\theta\right\} |v' - v_*|^{\gamma+2}(\mu_*\,\mu')^{1/12}|g_*|\,|g'|dv_*dv'$$
$$\lesssim \iint |v' - v_*|^{\gamma}(\mu_*\,\mu')^{1/24}|g_*|\,|g'|dv_*dv' \lesssim \|\mu^{1/10^3} g\|^2_{L^2}.$$

Together with (4.3) and (4.4), this yields the desired estimate and completes the proof of the lemma. □

Recalling

$$\begin{aligned} |||g|||^2 =& \iiint b(\cos\theta)\Phi(|v - v_*|)\mu_*\left(g' - g\right)^2 \\ &+ \iiint b(\cos\theta)\Phi(|v - v_*|)g_*^2\left(\sqrt{\mu'} - \sqrt{\mu}\,\right)^2 \\ =& J_1 + J_2\,, \end{aligned} \tag{4.5}$$

let us first note the following inequality between $\Big(\mathcal{L}_1 g,\, g\Big)_{L^2}$, corresponding to the first term of the linear operator, and the non-isotropic norm.

Proposition 4.3. *Let $\gamma > -3$. There exists a constant $C > 0$ such that*

$$|||g|||^2 \geq \Big(\mathcal{L}_1 g,\, g\Big)_{L^2} \geq \frac{1}{10}|||g|||^2 - C\|g\|^2_{L^2_{\gamma/2}}\,. \tag{4.6}$$

Proof. The equalities

$$\begin{aligned} &2\Big(\mathcal{L}_1 g,\, g\Big)_{L^2} = -2\Big(\Gamma(\sqrt{\mu},\, g),\, g\Big)_{L^2} \\ &= \iiint B\,\Big((\mu'_*)^{1/2} g' - (\mu_*)^{1/2}\, g\Big)^2\, dv dv_* d\sigma \\ &= \iiint B\,\Big((\mu'_*)^{1/2}(g' - g) + g((\mu'_*)^{1/2} - (\mu_*)^{1/2}\Big)^2\, dv dv_* d\sigma, \end{aligned} \tag{4.7}$$

together with the inequality

$$2(a^2 + b^2) \geq (a + b)^2 \geq \frac{1}{2}a^2 - b^2$$

yields

$$|||g|||^2 \geq \Big(\mathcal{L}_1 g,\, g\Big)_{L^2} \geq \frac{1}{4}J_1 - \frac{1}{2}J_2 \geq \frac{1}{4}|||g|||^2 - \frac{3}{4}J_2\,. \tag{4.8}$$

It follows from the equality $(a + b)^2 = a^2 + b^2 + 2ab$ that

$$2\Big(\mathcal{L}_1 g,\, g\Big)_{L^2} \geq J_2 - C\|g\|^2_{L^2_{\gamma/2}}\,, \tag{4.9}$$

which yields the desired estimate (4.6) together with (4.8).

Indeed, note that

$$
\begin{aligned}
&2\Big(\mathcal{L}_1 g,\, g\Big)_{L^2}\\
&= \iiint B\, \Big((\mu'_*)^{1/2}(g'-g) + g((\mu'_*)^{1/2} - (\mu_*)^{1/2})\Big)^2 dvdv_*d\sigma\\
&= J_1 + J_2 + 2\iiint B\,(g'-g)g(\mu'_*)^{1/2}\Big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\Big)dvdv_*d\sigma\,.
\end{aligned}
$$

Using the identity $2(\beta-\alpha)\alpha = \beta^2 - \alpha^2 - (\beta-\alpha)^2$, we have

$$
\begin{aligned}
&2(g'-g)g(\mu'_*)^{1/2}\Big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\Big)\\
&\quad= \frac{1}{2}\Big(g'^2 - g^2 - (g'-g)^2\Big)\Big(\mu'_* - \mu_* + \big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\big)^2\Big)\\
&\quad= -\frac{1}{2}\Big(g'-g\Big)^2\Big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\Big)^2 + \frac{1}{2}\big(g^2 - {g'}^2\big)\big(\mu_* - \mu'_*\big)\\
&\quad\quad + \frac{1}{2}\Big(g'-g\Big)^2\big(\mu_* - \mu'_*\big) + \frac{1}{2}\big({g'}^2 - g^2\big)\Big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\Big)^2\\
&\quad= I_1 + I_2 + I_3 + I_4\,.
\end{aligned}
$$

Using the change of variables $(v', v'_*) \to (v, v_*)$, we see that

$$
\Big|\iiint B\, I_2 dvdv_*d\sigma\Big| = \Big|\iiint B\,\mu_*\big(g^2 - g'^2\big)dvdv_*d\sigma\Big| \le C\|g\|^2_{L^2_{\gamma/2}}\,,
$$

by means of the cancellation lemma. Furthermore,

$$
\begin{aligned}
\iiint B\, I_1 dvdv_*d\sigma = &-\frac{1}{2}\iiint B(\mu_* + \mu'_*)\Big(g'-g\Big)^2 dvdv_*d\sigma\\
&+ \iiint B(\mu_*)^{1/2}(\mu'_*)^{1/2}\Big(g'-g\Big)^2 dvdv_*d\sigma \ge -J_1\,,
\end{aligned}
$$

where we have used the change of variables $(v', v'_*) \to (v, v_*)$. Thus, we obtain (4.9) because the integrals corresponding to the last two terms I_3 and I_4 vanish, ending the proof of the proposition. □

End of the proof of Proposition 2.1: It follows from (4.6) and (4.2) that

$$
|||g|||^2 \ge \Big(\mathcal{L}_1 g,\, g\Big)_{L^2} \ge \frac{1}{2}\Big(\mathcal{L}g,\, g\Big)_{L^2}\,. \tag{4.10}
$$

On the other hand, note that $\Big(\mathcal{L}g,\, g\Big)_{L^2} = \Big(\mathcal{L}(\mathbf{I}-\mathbf{P})g,\, (\mathbf{I}-\mathbf{P})g\Big)_{L^2}$, from the very definition of the projection operator $\mathbf{P}$.

Thus, from Proposition 4.3 and Lemma 4.2, we get

$$
\begin{aligned}
\Big(\mathcal{L}g,\, g\Big)_{L^2} &= \Big(\mathcal{L}_1(\mathbf{I}-\mathbf{P})g,\, (\mathbf{I}-\mathbf{P})g\Big)_{L^2} + \Big(\mathcal{L}_2(\mathbf{I}-\mathbf{P})g,\, (\mathbf{I}-\mathbf{P})g\Big)_{L^2}\\
&\ge \frac{1}{10}|||(\mathbf{I}-\mathbf{P})g|||^2 - C\|(\mathbf{I}-\mathbf{P})g\|^2_{L^2_{\gamma/2}}\,.
\end{aligned}
$$

Since it is known from [16] that we have

$$
\Big(\mathcal{L}g,\, g\Big)_{L^2} \ge C\|(\mathbf{I}-\mathbf{P})g\|^2_{L^2_{\gamma/2}}\,,
$$

we get on the whole

$$
|||(\mathbf{I}-\mathbf{P})g|||^2 \le C\Big(\mathcal{L}g,\, g\Big)_{L^2}\,. \tag{4.11}
$$

5. Non-isotropic norms with different kinetic factors

This section is devoted to the proof of Proposition 2.4. That is, we will show some equivalence relations between the non-isotropic norms with different kinetic factors and different weights.

For the proof, we introduce some further notations. Let $\rho > 0$, $\mu_\rho(v) = \mu(v)^\rho$, and set

$$J_{1,\rho}^{\Phi_\gamma}(g) = \iiint \Phi_\gamma(|v-v^*|) b(\cos\theta) \mu_{\rho,*} \left(g' - g\right)^2 dv dv_* d\sigma\,.$$

We simply write $J_1^{\Phi_\gamma}(g)$ if $\rho = 1$, and also introduce the notation $J_{2,\rho}^{\Phi_\gamma}(g)$ similarly with μ replaced by μ_ρ.

Then it follows from (3.2) and the change of variables $v \to v/\sqrt{\rho}$ that

$$J_{2,\rho}^{\Phi_\gamma}(g) \sim \|g\|^2_{L^2_{s+\gamma/2}} = \|\langle v\rangle^{(\gamma-\beta)/2} g\|^2_{L^2_{s+\beta/2}} \sim J_{2,\rho}^{\Phi_\beta}(\langle v\rangle^{(\gamma-\beta)/2} g)\,. \tag{5.1}$$

By the last assertions of Lemmas 3.8 and 3.9, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|g\|^2_{H^s_{\gamma/2}} \le J_{1,\rho}^{\Phi_\gamma}(g) + \|g\|^2_{L^2_{s+\gamma/2}} \le C_2 \|g\|^2_{H^s_{s+\gamma/2}}\,. \tag{5.2}$$

Furthermore, it follows from (3.8), (3.11) and the proofs of Lemmas 3.8 and 3.9 that

$$J_{1,2}^{\Phi_0}(\langle v\rangle^{\gamma/2} g) \lesssim J_1^{\Phi_\gamma}(g) \lesssim J_{1,1/2}^{\Phi_0}(\langle v\rangle^{\gamma/2} g), \qquad \text{modulo } \|g|^2_{L^2_{s+\gamma/2}}\,,$$

because we have $C_1\mu_2 \le \mu\langle v\rangle^{\pm|\gamma|} \le C_2 \mu_{1/2}$.

Therefore, to complete the proof of Proposition 2.3, it suffices to show that for any $\rho, \rho' > 0$

$$J_{1,\rho}^{\Phi_0}(g) \sim J_{1,\rho'}^{\Phi_0}(g), \qquad \text{modulo } \|g\|^2_{L^2_s}\,. \tag{5.3}$$

In fact, note that

$$J_1^{\Phi_\gamma}(g) \sim J_{1,\rho}^{\Phi_0}(\langle v\rangle^{\gamma/2} g) \sim J_1^{\Phi_\beta}(\langle v\rangle^{(\gamma-\beta)/2} g), \ \text{ modulo } \|g\|^2_{L^2_{s+\gamma/2}}.$$

This equivalence looks quite obvious, however, for completeness, we shall give a proof. In fact, (5.3) is a direct consequence of the following lemma, by taking $f = \mu_{\rho'}$.

Lemma 5.1. *Assume that* (1.3) *with* $0 < s < 1$. *Then there exists a constant* $C > 0$ *such that*

$$\iiint b\, f_*^2 (g'-g)^2 d\sigma dv dv_* \le C \|f\|^2_{L^2_s} \Big(J_{1,\rho}^{\Phi_0}(g) + \|g\|^2_{L^2_s} \Big)\,. \tag{5.4}$$

Once the equivalence (5.3) has been established, we have

Corollary 5.2. *Assume that* (1.3) *holds with* $0 < s < 1$. *Then there exists a constant* $C > 0$ *such that*

$$\iiint b\, f_*^2 (g'-g)^2 d\sigma dv dv_* \le C \|f\|^2_{L^2_s} |||g|||^2_{\Phi_0}\,. \tag{5.5}$$

Proof. It is enough to consider the case $\rho = 1$. As in the proof of Lemma 3.8, it follows from Proposition 2 of [3] that

$$
\begin{aligned}
J_1^{\Phi_0}(g) &= \iiint b(\cos\theta)\mu_*(g'-g)^2 dv_* d\sigma dv \\
&= \frac{1}{(2\pi)^3}\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\Big(\widehat{\mu}(0)|\widehat{g}(\xi)|^2 + |\widehat{g}(\xi^+)|^2 \\
&\qquad - 2Re\ \widehat{\mu}(\xi^-)\widehat{g}(\xi^+)\overline{\widehat{g}}(\xi)\Big)d\xi d\sigma \\
&= \frac{1}{(2\pi)^3}\iint b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big(\widehat{\mu}(0)|\widehat{g}(\xi) - \widehat{g}(\xi^+)|^2 \\
&\qquad + 2Re\Big(\widehat{\mu}(0)-\widehat{\mu}(\xi^-)\Big)\widehat{g}(\xi^+)\overline{\widehat{g}}(\xi)\Big)d\xi d\sigma, \qquad (5.6)
\end{aligned}
$$

and

$$
\begin{aligned}
A &= \iiint b(\cos\theta)f_*^2(g'-g)^2 dv_* d\sigma dv \\
&= \frac{1}{(2\pi)^3}\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(\widehat{f^2}(0)|\widehat{g}(\xi) - \widehat{g}(\xi^+)|^2\right. \\
&\qquad \left. + 2Re\left(\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\right)\widehat{g}(\xi^+)\overline{\widehat{g}}(\xi)\right)d\xi d\sigma\,.
\end{aligned}
$$

Since $\widehat{f^2}(0) = \|f\|^2_{L^2}$ and $\widehat{\mu}(0) = c_0 > 0$, we obtain

$$
\begin{aligned}
c_0 A &= c_0\iiint b(\cos\theta)f_*^2(g'-g)^2 dv_* d\sigma dv \\
&= \|f\|^2_{L^2} J_1^{\Phi_0}(g) \\
&\quad - \frac{2}{(2\pi)^3}\|f\|^2_{L^2}\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) Re\left(\widehat{\mu}(0) - \widehat{\mu}(\xi^-)\right)\widehat{g}(\xi^+)\overline{\widehat{g}}(\xi)\Big)d\xi d\sigma \\
&\quad + \frac{2c_0}{(2\pi)^3}\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) Re\left(\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\right)\widehat{g}(\xi^+)\overline{\widehat{g}}(\xi)\Big)d\xi d\sigma \\
&= \|f\|^2_{L^2} J_1^{\Phi_0}(g) + A_1 + A_2\,.
\end{aligned}
$$

Write

$$
\begin{aligned}
A_2 =& \frac{2c_0}{(2\pi)^3}\Big\{\int |\widehat{g}(\xi)|^2\Big(\int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) Re\left(\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\right)d\sigma\Big)d\xi \\
&+ \iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right) Re\left(\widehat{f^2}(0) - \widehat{f^2}(\xi^-)\right)\left(\widehat{g}(\xi^+) - \widehat{g}(\xi)\right)\overline{\widehat{g}}(\xi)d\xi d\sigma\Big\} \\
=& A_{2,1} + A_{2,2}\,.
\end{aligned}
$$

It follows from Cauchy-Schwarz's inequality that

$$
\begin{aligned}
|A_{2,2}| \le& C\Big(\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)|\widehat{f^2}(0) - \widehat{f^2}(\xi^-)|^2|\widehat{g}|^2(\xi)d\xi d\sigma\Big)^{1/2} \\
&\times\Big(\iint b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)|\widehat{g}(\xi^+) - \widehat{g}(\xi)|^2 d\xi d\sigma\Big)^{1/2} \\
=& B_1^{1/2}\times B_2^{1/2}\,.
\end{aligned}
$$

Since

$$
|\widehat{f^2}(0) - \widehat{f^2}(\xi^-)| \le \int f^2(v)|1 - e^{-iv\cdot\xi^-}|dv,
$$

we have

$$\begin{aligned}B_1 &\le C\iiint |\widehat{g}(\xi)|^2 f^2(v)f^2(w)\\ &\quad\times\Big(\int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)(|1-e^{-iv\cdot\xi^-}|^2+|1-e^{-iw\cdot\xi^-}|^2)d\sigma\Big)dvdwd\xi\\ &\le C\|g\|^2_{H^s}\|f\|^2_{L^2}\|f\|^2_{L^2_s},\end{aligned}$$

because

$$\begin{aligned}&\int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)|1-e^{-iv\cdot\xi^-}|^2 d\sigma\\ &\le C\Big(\int_0^{(\langle v\rangle\langle\xi\rangle)^{-1}}\theta^{-1-2s}(|v||\xi|)^2\theta^2 d\theta+\int_{(\langle v\rangle\langle\xi\rangle)^{-1}}^{\pi/2}\theta^{-1-2s}d\theta\Big)\\ &\le C\langle v\rangle^{2s}\langle\xi\rangle^{2s}.\end{aligned}$$

Then we have $|A_{2,1}|\le C\|g\|^2_{H^s}\|f\|^2_{L^2_s}$ because

$$\begin{aligned}&\int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)Re\left(\widehat{f^2}(0)-\widehat{f^2}(\xi^-)\right)d\sigma\\ &=\int f^2(v)\Big(\int b\left(\frac{\xi}{|\xi|}\cdot\sigma\right)\left(1-\cos(v\cdot\xi^-)\right)d\sigma\Big)dv\\ &\le C\langle\xi\rangle^{2s}\int f^2(v)\langle v\rangle^{2s}dv.\end{aligned}$$

Since $\widehat{\mu}(\xi)$ is real-valued, it follows that

$$Re\left(\widehat{\mu}(0)-\widehat{\mu}(\xi^-)\right)\widehat{g}(\xi^+)\overline{\widehat{g}}(\xi)=\Big(\int\left(1-\cos(v\cdot\xi^-)\right)\mu(v)dv\Big)\,Re\,\widehat{g}(\xi^+)\overline{\widehat{g}}(\xi).$$

Therefore, by using Cauchy-Schwarz's inequality and the change of variables $\xi\to\xi^+$ (see the proof of Lemma 2.8 in [6]), we obtain $|A_1|\le C\|f\|^2_{L^2}\|g\|^2_{H^s}$. Furthermore, it follows from (5.6) that

$$\begin{aligned}B_2&=\iint b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)|\widehat{g}(\xi)-\widehat{g}(\xi^+)|^2 d\xi d\sigma\\ &\le C\Big(J_1^{\Phi_0}(g)+\|g\|^2_{H^s}\Big),\end{aligned}$$

which yields $|A_{2,2}|\le C\|f\|_{L^2}\|f\|_{L^2_s}\|g\|_{H^s}\left(J_1^{\Phi_0}(g)+\|g\|^2_{H^s}\right)^{1/2}$. Hence

$$|A_2|\le C\|f\|^2_{L^2_s}\|g\|_{H^s}\left(J_1^{\Phi_0}(g)+\|g\|^2_{H^s}\right)^{1/2}.$$

Finally, we have

$$A\le C\|f\|^2_{L^2_s}\|g\|_{H^s}\left(J_1^{\Phi_0}(g)+\|g\|^2_{H^s}\right)^{1/2}\le C\|f\|^2_{L^2_s}\Big(J_1^{\Phi_0}(g)+\|g\|^2_{L^2_s}\Big),$$

by means of (5.2) with $\gamma=0$, completing the proof of the lemma. □

6. Estimation on the nonlinear operator

Based on the previous analysis, we will give an explicit and simple upper bound estimate on the Boltzmann nonlinear operator when $\gamma > -\frac{3}{2}$. In fact, an upper bound estimate is also available for the case when $-3 < \gamma \le -\frac{3}{2}$. However, since it is more complicated and not so simple, it will be given in [7].

In any case, our reason for including these non linear estimations here is to show the usefulness of the estimates on the linearized operator and the non-isotropic norm.

Firstly, we consider the case when $\gamma \ge 0$.

Lemma 6.1. *Let $\gamma \ge 0$. Assume that* (1.3) *with* $0 < s < 1$. *Then*

$$\iiint \Phi_\gamma(|v-v_*|) b\, f_*^2 (g'-g)^2 d\sigma dv dv_* \lesssim \|f\|^2_{L^2_{s+\gamma/2}} |||g|||^2_{\Phi_\gamma}\,. \tag{6.1}$$

Proof. Since $\Phi_\gamma(|v-v_*|) \lesssim \langle v'\rangle^\gamma + \langle v_*\rangle^\gamma$, we have

$$\begin{aligned}
&\iiint b(\cos\theta)\Phi(|v-v_*|) f_*^2 (g'-g)^2 d\sigma dv dv_* \\
&\quad\lesssim \iiint b(\cos\theta) f_*^2 \Big(\langle v'\rangle^{\gamma/2} g' - \langle v\rangle^{\gamma/2} g\Big)^2 d\sigma dv dv_* \\
&\qquad + \iiint b(\cos\theta)\Big(\langle v_*\rangle^{\gamma/2} f_*\Big)^2 (g'-g)^2 d\sigma dv dv_* \\
&\qquad + \iiint b(\cos\theta) f_*^2 \Big(\langle v\rangle^{\gamma/2} - \langle v'\rangle^{\gamma/2}\Big)^2 |g|^2 d\sigma dv dv_* \\
&\quad = A_1 + A_2 + A_3\,.
\end{aligned}$$

Noticing that

$$\begin{aligned}
\Big|\langle v\rangle^{\gamma/2} - \langle v'\rangle^{\gamma/2}\Big| &\le C_\gamma \int_0^1 \langle v' + \tau(v-v')\rangle^{(\gamma/2-1)^+} d\tau |v-v_*|\theta \\
&\le C'_\gamma \Big(\langle v\rangle^{(\gamma/2-1)^+} + \langle v_*\rangle^{(\gamma/2-1)^+}\Big)\langle v-v_*\rangle\theta,
\end{aligned}$$

we have

$$\begin{aligned}
A_3 &\lesssim \iint f_*^2 |g|^2 \Big\{\Big(\langle v\rangle^{(\gamma/2-1)^+} + \langle v_*\rangle^{(\gamma/2-1)^+}\Big)^2 \\
&\qquad \times \Big(\int_0^{\langle v-v_*\rangle^{-1}} \theta^{-1-2s}\Big(\langle v-v_*\rangle\theta\Big)^2 d\theta\Big) \\
&\qquad + \int_{\langle v-v_*\rangle^{-1}}^{\pi/2} \Big(\langle v\rangle^{\gamma/2} + \langle v_*\rangle^{\gamma/2}\Big)^2 \theta^{-1-2s} d\theta\Big\} dv dv_* \\
&\lesssim \iint \Big(\langle v\rangle^{2s+\gamma} + \langle v_*\rangle^{2s+\gamma}\Big) f_*^2 |g|^2 dv dv_* \\
&\lesssim \Big(\|f\|^2_{L^2_{s+\gamma/2}}\|g\|^2_{L^2} + \|f\|^2_{L^2}\|g\|^2_{L^2_{s+\gamma/2}}\Big).
\end{aligned}$$

Applying Corollary 5.2 to A_1 and A_2, it follows that

$$\begin{aligned}
A_1 + A_2 &\lesssim \|f\|^2_{L^2_s} |||\langle v\rangle^{\gamma/2} g|||^2_{\Phi_0} + \|\langle v\rangle^{\gamma/2} f\|^2_{L^2_s} |||g|||^2_{\Phi_0} \\
&\lesssim \|f\|^2_{L^2_{s+\gamma/2}} |||g|||^2_{\Phi_\gamma}\,,
\end{aligned}$$

where we have used Proposition 2.4 in the last inequality. ∎

Proof of Proposition 2.5 for the case when $\gamma \geq 0$

Note that

$$\begin{aligned}
(6.2)\qquad \Big(\Gamma(f,\,g),\,h\Big)_{L^2} &= \Big(\mu^{-1/2}Q(\mu^{1/2}f,\,\mu^{1/2}g),\,h\Big)_{L^2}\\
&= \iiint \Phi_\gamma b(\cos\theta)\mu_*^{1/2}\Big(f'_*g' - f_*g\Big)\,h\\
&= \frac{1}{2}\iiint \Phi_\gamma b(\cos\theta)\Big(f'_*g' - f_*g\Big)\Big(\mu_*^{1/2}h - \mu_*^{1/2'}h'\Big)\\
&\leq \frac{1}{2}\Big(\iiint \Phi_\gamma b(\cos\theta)\Big(f'_*g' - f_*g\Big)^2\Big)^{1/2}\\
&\quad\times\Big(\iiint \Phi_\gamma b(\cos\theta)\Big((\mu_*)^{1/2}h - (\mu'_*)^{1/2}h'\Big)^2\Big)^{1/2}\\
&\leq \frac{1}{2}A^{1/2}\times B^{1/2}.
\end{aligned}$$

For B, we have

$$\begin{aligned}
B &= \iiint \Phi_\gamma b(\cos\theta)\Big((\mu'_*)^{1/2}(h'-h) + h\big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\big)\Big)^2\\
&\leq 2\iiint \Phi_\gamma b(\cos\theta)\left\{\mu'_*(h'-h)^2 + h^2\Big((\mu'_*)^{1/2} - (\mu_*)^{1/2}\Big)^2\right\}\\
&\leq 2\iiint \Phi_\gamma b(\cos\theta)\mu_*(h'-h)^2 + 2\iiint \Phi_\gamma b(\cos\theta)h_*^2\Big((\mu')^{1/2} - \mu^{1/2}\Big)^2\\
&= 2|||h|||^2_{\Phi_\gamma},
\end{aligned}$$

where we have used the change of variables $(v, v_*) \to (v', v'_*)$ for the first term and $(v, v_*) \to (v_*, v)$ for the second term. Similarly,

$$\begin{aligned}
A &= \iiint \Phi_\gamma b(\cos\theta)\Big(f'_*(g'-g) + g(f'_* - f_*)\Big)^2\\
&\leq 2\iiint \Phi_\gamma b(\cos\theta)\left\{{f'_*}^2(g'-g)^2 + g^2(f'_* - f_*)^2\right\}\\
&\leq 2\iiint \Phi_\gamma b(\cos\theta){f_*}^2(g'-g)^2 + 2\iiint \Phi_\gamma b(\cos\theta){g_*}^2(f'-f)^2.
\end{aligned}$$

Then (6.1) implies that

$$A \lesssim ||f||^2_{L^2_{s+\gamma/2}}|||g|||^2_{\Phi_\gamma} + ||g||^2_{L^2_{s+\gamma/2}}|||f|||^2_{\Phi_\gamma},$$

which completes the proof in the case when $\gamma \geq 0$.

Proof of Proposition 2.5 in the case when $-3/2 < \gamma < 0$

As in Section 5, it is easy to check that for any fixed $\rho > 0$,

$$\begin{aligned}
(6.3)\qquad |||g|||^2_{\Phi_\gamma} &\sim J^{\Phi_\gamma}_{1,\rho}(g) + J^{\Phi_\gamma}_{2,\rho}(g) \sim J^{\Phi_\gamma}_{1,\rho}(g) + \|g\|^2_{L^2_{s+\gamma/2}}\\
&\sim \iiint \frac{\Phi_{2\gamma}}{\tilde{\Phi}_\gamma}b\mu_{\rho,*}(g'-g)^2 + \iiint \frac{\Phi_{2\gamma}}{\tilde{\Phi}_\gamma}bg_*^2\Big(\sqrt{\mu'_\rho} - \sqrt{\mu_\rho}\Big)^2,
\end{aligned}$$

where the assumption $2\gamma > -3$ is required for the existence of the above integral, and more precisely for

$$\int |v_*|^{2\gamma} \langle v_* \rangle^{2s-\gamma} \mu_\rho(v+v_*) dv_* \sim \langle v \rangle^{\gamma+2s} .$$

Instead of (6.2), we write

$$\begin{aligned}\Big(\Gamma(f,g),h\Big) &= \iiint b\Phi_\gamma \mu_*^{1/2}\Big(f'_* g' - f_* g\Big) h dv dv_* d\sigma \\ &= \frac{1}{2}\iiint \Big(b\tilde{\Phi}_\gamma\Big)^{1/2}\Big(f'_* g' - f_* g\Big)\Big(b\frac{\Phi_{2\gamma}}{\tilde{\Phi}_\gamma}\Big)^{1/2} {\mu'_*}^{1/4}\Big({\mu_*}^{1/4} h - {\mu'_*}^{1/4} h'\Big) \\ &\quad + \frac{1}{2}\iiint \Big(b\tilde{\Phi}_\gamma\Big)^{1/2}\Big(f'_* g' - f_* g\Big)\Big(b\frac{\Phi_{2\gamma}}{\tilde{\Phi}_\gamma}\Big)^{1/2} {\mu_*}^{1/4}\Big({\mu_*}^{1/4} - {\mu'_*}^{1/4}\Big) h .\end{aligned}$$

Noticing that

$${\mu_*}^{1/4} h - {\mu'_*}^{1/4} h' = {\mu'_*}^{1/4}\Big(h - h'\Big) + \Big({\mu_*}^{1/4} - {\mu'_*}^{1/4}\Big) h ,$$

by Cauchy-Schwarz's inequality and (6.3), we have

$$\begin{aligned}\Big|\Big(\Gamma(f,g),h\Big)\Big| &\lesssim \Big(\iiint b\tilde{\Phi}_\gamma \mu_*^{1/2}\Big(f'_* g' - f_* g\Big)^2 d\sigma dv dv_*\Big)^{1/2} |||h|||_{\Phi_\gamma} \\ &= A^{1/2} |||h|||_{\Phi_\gamma} .\end{aligned}$$

We estimate

$$\begin{aligned}A \leq 3\Big(&\iiint b\tilde{\Phi}_\gamma \mu_*^{1/4}\Big(\big(\mu^{1/8} f\big)'_* - \big(\mu^{1/8} f\big)_*\Big)^2 g^2 d\sigma dv dv_* \\ &+ \iiint b\tilde{\Phi}_\gamma \mu_*^{1/8}\Big(\big(\mu^{1/8} f\big)'_*\Big)^2\Big(g' - g\Big)^2 d\sigma dv dv_* \\ &+ \iiint b\tilde{\Phi}_\gamma \mu_*^{1/4}\Big(\mu_*^{1/8} - {\mu'_*}^{1/8}\Big)^2\Big(f'_* g'\Big)^2 d\sigma dv dv_*\Big) \\ = A_1 &+ A_2 + A_3 .\end{aligned}$$

Since $\tilde{\Phi}_\gamma(|v-v_*|)\mu_*^{1/4} \lesssim \langle v \rangle^\gamma$, we have by means of Corollary 5.2

$$\begin{aligned}A_1 &\lesssim \iiint b\big(\langle v \rangle^{\gamma/2} g\big)^2\Big(\big(\mu^{1/8} f\big)'_* - \big(\mu^{1/8} f\big)_*\Big)^2 d\sigma dv dv_* \\ &\lesssim \|\langle v \rangle^{\gamma/2} g\|^2_{L^2_s} |||\mu^{1/8} f|||^2_{\Phi_0} \lesssim \|g\|^2_{L^2_{s+\gamma/2}} |||f|||^2_{\Phi_\gamma} ,\end{aligned}$$

where we have used Propostions 2.4 and 2.2 in the last inequality. As for A_2, we decompose it as follows

$$\begin{aligned}A_2 &\lesssim \iiint b\Big(\big(\mu^{1/8} f\big)'_*\Big)^2\Big(\big(\langle v \rangle^{\gamma/2} g\big)' - \big(\langle v \rangle^{\gamma/2} g\big)\Big)^2 d\sigma dv dv_* \\ &\quad + \iiint b\Big(\langle v \rangle^{\gamma/2} - \langle v' \rangle^{\gamma/2}\Big)^2\Big(\big(\mu^{1/8} f\big)'_*\Big)^2 g'^2 d\sigma dv dv_* \\ &= A_{2,1} + A_{2,2} .\end{aligned}$$

Apply Corollary 5.2 again to $A_{2,1}$. Then

$$A_{2,1} \lesssim \|\mu^{1/8} f\|^2_{L^2_s} |||\langle v \rangle^{\gamma/2} g|||^2_{\Phi_0} \lesssim \|f\|^2_{L^2_{s+\gamma/2}} |||g|||^2_{\Phi_\gamma} .$$

The estimation for $A_{2,2}$ is the same as the one for A_2 in the proof of Lemma 3.8. By using the change of variables $(v', v'_*) \to (v, v_*)$, we obtain

$$A_{2,2} \lesssim \iint \Big(\langle v\rangle^{2s+\gamma}\langle v_*\rangle^{2s+2}\Big)\big(\mu^{1/8}f\big)_*^2|g|^2 dv dv_* \lesssim \|\mu^{1/10}f\|_{L^2}^2\|g\|_{L^2_{s+\gamma/2}}^2 .$$

Noticing that $\Big(\mu_*^{1/8} - {\mu'_*}^{1/8}\Big)^2 \lesssim \min(|v-v_*|^2\theta^2, 1)$, we have

$$\begin{aligned} A_3 &\lesssim \iint \tilde\Phi_\gamma\Big(\int_{SS^2} b(\cos\theta)\min(|v-v_*|^2\theta^2,1)d\sigma\Big)f_*^2 g^2 dv dv_* \\ &\lesssim \iint \langle v-v_*\rangle^{\gamma+2s} f_*^2 g^2 dv dv_* \\ &\lesssim \iint \langle v_*\rangle^{\gamma+2s} f_*^2 \langle v\rangle^{\gamma+2s} g^2 dv dv_* \lesssim \|f\|_{L^2_{s+\gamma/2}}^2\|g\|_{L^2_{s+\gamma/2}}^2 , \end{aligned}$$

if $\gamma+2s \ge 0$ because of $\langle v-v_*\rangle^{\gamma+2s} \le \langle v_*\rangle^{\gamma+2s}\langle v\rangle^{\gamma+2s}$.

To consider the case $\gamma+2s<0$, we divide $\mathbb{R}^3_v \times \mathbb{R}^3_{v_*}$ into three parts

$$\begin{aligned} &U_1 = \{|v-v_*| \le |v_*|/8\}\,,\ \ U_2 = \{|v-v_*| > |v_*|/8\}\cap\{|v_*|\le 1\}\,, \\ &U_3 = \{|v-v_*| > |v_*|/8\}\cap\{|v_*|>1\}\,. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{1}{3}A_3 &= \iiint b\tilde\Phi_\gamma {\mu'_*}^{1/4}\Big({\mu'_*}^{1/8} - \mu_*^{1/8}\Big)^2\Big(f_* g\Big)^2 d\sigma dv dv_* \\ &= \iint_{U_1}\int d\sigma dv dv_* + \iint_{U_2}\int d\sigma dv dv_* + \iint_{U_3}\int d\sigma dv dv_* \\ &= A_{3,1} + A_{3,2} + A_{3,3}\,. \end{aligned}$$

Since $|v'-v_*| \le |v-v_*| \le |v_*|/8$ implies $7|v_*|/8 \le |v'|, |v| \le 9|v_*|/8$ and $|v'_*|^2 = |v|^2 + |v_*|^2 - |v'|^2 \ge |v_*|^2/2$. Hence, we have ${\mu'_*}^{1/4} \le C\mu_*^{1/8} \le C(\mu_*\mu)^{1/20}$ on U_1, which leads to

$$A_{3,1} \lesssim \iint (\mu\mu_*)^{1/20}\langle v-v_*\rangle^{\gamma+2s} f_*^2 g^2 dv dv_* \le C\|f\|_{L^2_{s+\gamma/2}}^2\|g\|_{L^2_{s+\gamma/2}}^2 .$$

Furthermore, we have

$$A_{3,2} \lesssim \iint_{U_2} \langle v-v_*\rangle^{\gamma+2s} f_*^2 g^2 dv dv_* \lesssim \|f\|_{L^2_{s+\gamma/2}}^2\|g\|_{L^2_{s+\gamma/2}}^2,$$

because $\langle v-v_*\rangle^{-1} \le \langle v\rangle^{-1}\langle v_*\rangle^{-1}\langle v_*\rangle^2 \le 2\langle v\rangle^{-1}\langle v_*\rangle^{-1}$ on U_2. Since $\langle v-v_*\rangle^{-1} \le 8|v_*|^{-1} \le 16\langle v_*\rangle^{-1}$ on U_3, we get

$$A_{3,3} \lesssim \|f\|_{L^2_{s+\gamma/2}}^2\|g\|_{L^2}^2 .$$

Therefore, we have in the case when $\gamma+2s<0$

$$A_3 \lesssim \|f\|_{L^2_{s+\gamma/2}}^2\|g\|_{L^2}^2 .$$

If one considers another partition in R^6_{v,v_*} with v and v_* exchanged, then the estimate

$$A_3 \lesssim \|f\|_{L^2}^2\|g\|_{L^2_{s+\gamma/2}}^2$$

holds, because $|v'_*-v| \le |v_*-v| \le |v|/8$ implies $7|v|/8 \le |v'_*|, |v_*| \le 9|v|/8$.

As a conclusion, when $\gamma > -3/2$ and $\gamma + 2s \le 0$ we have

$$\left|\Big(\Gamma(f,g),h\Big)\right| \lesssim \Big\{\|f\|_{L^2_{s+\gamma/2}}|||g|||_{\Phi_\gamma} + \|g\|_{L^2_{s+\gamma/2}}|||f|||_{\Phi_\gamma} + \min\Big(\|f\|_{L^2}\|g\|_{L^2_{s+\gamma/2}}\,,\ \|f\|_{L^2_{s+\gamma/2}}\|g\|_{L^2}\Big)\Big\}|||h|||_{\Phi_\gamma}\,.$$

The proof of Proposition 2.5 is then completed.

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