

Convergence to the Self-similar Solutions to the Homogeneous Boltzmann Equation

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Abstract

The Boltzmann H-theorem implies that the solution to the Boltzmann equation tends to an equilibrium, that is, a Maxwellian when time tends to infinity. This has been proved in various settings when the initial energy is finite. However, when the initial energy is infinite, the time asymptotic state is no longer described by a Maxwellian, but a self-similar solution obtained by Bobylev-Cercignani. The purpose of this paper is to rigorously justify this for the spatially homogeneous problem with Maxwellian molecule type cross section without angular cutoff.

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1 Introduction

Consider the homogeneous Boltzmann equation

$$\partial_t f(t, v) = Q(f, f)(t, v), \quad v \in \mathbb{R}^3, \quad t \in \mathbb{R}^+ \quad (1.1)$$

with initial data

$$f(0, v) = f_0(v) \geq 0, \quad v \in \mathbb{R}^3, \quad (1.2)$$

where the non-negative unknown function $f(t, v)$ is the distribution density function of particles with velocity $v \in \mathbb{R}^3$ at time $t \in \mathbb{R}^+$. The right hand side of (1.1) is the Boltzmann bilinear collision operator corresponding to the Maxwellian molecule type cross section

$$Q(g, f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{B} \left(\frac{v - v_*}{|v - v_*|} \cdot \sigma \right) \left(f(v')g(v'_*) - f(v)g(v_*) \right) d\sigma dv_*. \quad (1.3)$$

Here for $\sigma \in \mathbb{S}^2$

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma,$$

from the conservation of momentum and energy,

$$v' + v'_* = v + v_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2.$$

The Maxwellian molecule type cross section $\mathcal{B}(\tau)$ in (1.3) is a non-negative function depending only on the deviation angle $\theta = \cos^{-1}\left(\frac{v-v_*}{|v-v_*|} \cdot \sigma\right)$. As usual, θ is restricted to $0 \leq \theta \leq \frac{\pi}{2}$ by replacing $\mathcal{B}(\cos \theta)$ by its ‘‘symmetrized’’ version $[\mathcal{B}(\cos \theta) + \mathcal{B}(\pi - \cos \theta)]\mathbf{1}_{0 \leq \theta \leq \pi/2}$. Moreover, motivated by inverse power laws, throughout this paper, we assume

$$\lim_{\theta \rightarrow 0_+} \mathcal{B}(\cos \theta) \theta^{2+2s} = b_0 \quad (1.4)$$

for positive constants $s \in (0, 1)$ and $b_0 > 0$.

As in [7, 8, 9, 11, 16], the Cauchy problem (1.1) and (1.2) is considered in the set of probability measures on \mathbb{R}^3 . For presentation, we first introduce some function spaces defined in the previous literatures. For $\alpha \in [0, 2]$, $\mathcal{P}^\alpha(\mathbb{R}^3)$ denotes the probability density function f on \mathbb{R}^3 such that

$$\int_{\mathbb{R}^3} |v|^\alpha f(v) dv < \infty,$$

and moreover when $\alpha \geq 1$, it requires that

$$\int_{\mathbb{R}^3} v_j f(v) dv = 0, \quad j = 1, 2, 3.$$

Following [7], a characteristic function $\varphi(t, \xi)$ is the Fourier transform of $f(t, v) \in \mathcal{P}^0(\mathbb{R}^3)$ with respect to v :

$$\varphi(t, \xi) = \hat{f}(t, \xi) = \mathcal{F}(f)(t, \xi) = \int_{\mathbb{R}^3} e^{-iv \cdot \xi} f(t, v) dv. \quad (1.5)$$

For each $\alpha \in [0, 2]$, set $\tilde{\mathcal{P}}^\alpha(\mathbb{R}^3) = \mathcal{F}^{-1}(\mathcal{K}^\alpha(\mathbb{R}^3))$ with $\mathcal{K}(\mathbb{R}^3) = \mathcal{F}(\mathcal{P}^0(\mathbb{R}^3))$ and

$$\mathcal{K}^\alpha(\mathbb{R}^3) = \left\{ \varphi \in \mathcal{K}(\mathbb{R}^3) : \|\varphi - 1\|_{\mathcal{D}^\alpha} < \infty \right\}.$$

Here the distance \mathcal{D}^α between two suitable functions $\varphi(\xi)$ and $\tilde{\varphi}(\xi)$ with $\alpha > 0$ is defined by

$$\|\varphi - \tilde{\varphi}\|_{\mathcal{D}^\alpha} \equiv \sup_{0 \neq \xi \in \mathbb{R}^3} \frac{|\varphi(\xi) - \tilde{\varphi}(\xi)|}{|\xi|^\alpha}.$$

Then the set $\mathcal{K}^\alpha(\mathbb{R}^3)$ endowed with the distance \mathcal{D}^α is a complete metric space. It follows from Lemma 3.12 of [7], that $\mathcal{K}^\alpha(\mathbb{R}^3) = \{1\}$ for all $\alpha > 2$ and the embeddings $\{1\} \subset \mathcal{K}^\alpha(\mathbb{R}^3) \subset \mathcal{K}^\beta(\mathbb{R}^3) \subset \mathcal{K}(\mathbb{R}^3)$ hold for $2 \geq \alpha \geq \beta \geq 0$.

The advantage of considering the Maxwellian molecule cross section is that the Bobylev formula is in a simple form. That is, by taking the Fourier transform (1.5) of the equation (1.1) leads to the following equation for the new unknown $\varphi = \varphi(t, \xi)$:

$$\partial_t \varphi(t, \xi) = \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right) (\varphi(t, \xi^+) \varphi(t, \xi^-) - \varphi(t, \xi)) d\sigma, \quad (1.6)$$

where we have used

$$\varphi(t, 0) = \int_{\mathbb{R}^3} f(t, v) dv = 1.$$

Here,

$$\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \xi^- = \frac{\xi - |\xi|\sigma}{2} \quad (1.7)$$

satisfying

$$\xi^+ + \xi^- = \xi, \quad |\xi^+|^2 + |\xi^-|^2 = |\xi|^2. \quad (1.8)$$

From now on, we consider the Cauchy problem for (1.6) with initial condition

$$\varphi(0, \xi) = \varphi_0(\xi). \quad (1.9)$$

For $\alpha \in (2s, 2]$, it is shown in [7, 11, 12] that this Cauchy problem admits a unique global solution $\varphi(t, \xi) \in C([0, \infty), \mathcal{K}^\alpha(\mathbb{R}^3))$ for every $\varphi_0(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$. Moreover, $f(t, \cdot) \in L^1(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3)$ for any $t > 0$ if $\mathcal{F}^{-1}(\varphi_0)(v)$ is not a single Dirac mass, cf. [12, 13].

To study the large time behavior of the solution, it depends on whether the initial energy is finite or not, and in the above setting, it depends on the parameter α , cf. [2, 7, 8, 9, 11, 12, 14, 15, 16] and the references cited therein:

- When $\alpha = 2$, the initial datum has finite energy so that the solution tends to the Maxwellian defined by the initial datum. This was indeed proved in the early work by Tanaka [15] using probability theory in the weak convergence in probability. And it was also proved later in [9, 14, 16] by using analytic methods about convergence in Toscani metrics. Moreover, if some moment higher than the second order is assumed to be bounded, the convergence in the $\mathcal{D}^{2+\delta}$ -distance with $\delta > 0$ is shown to be exponentially decay in time, cf. [9];
- When $2s < \alpha < 2$, the initial energy is infinite so that the solution will no longer tend to an equilibrium, but to a self-similar solution

$$f_{\alpha, K}(t, v) = e^{-3\mu_\alpha t} \Psi_{\alpha, K}(ve^{-\mu_\alpha t})$$

constructed in [5, 6], where

$$\mu_\alpha = \frac{\lambda_\alpha}{\alpha}, \quad \lambda_\alpha \equiv \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\xi \cdot \sigma}{|\xi|}\right) \left(\frac{|\xi^-|^\alpha + |\xi^+|^\alpha}{|\xi|^\alpha} - 1\right) d\sigma. \quad (1.10)$$

Here, $K > 0$ is any given constant and $\Psi_{\alpha, K}(v)$ is a radially symmetric non-negative function satisfying

$$\int_{\mathbb{R}^3} \Psi_{\alpha, K}(v) dv = 1, \quad \hat{\Psi}_{\alpha, K}(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3), \quad \lim_{|\eta| \rightarrow 0} \frac{1 - \hat{\Psi}_{\alpha, K}(\eta)}{|\eta|^\alpha} = K. \quad (1.11)$$

The regularity of the self-similar solution in $H^\infty(\mathbb{R}^3)$ was proved in [12, 13]. However, the convergence to the self-similar solution $f_{\alpha, K}(t, v)$ is not well understood even though there are some works, cf. [6, 7, 8] about pointwise convergence in radially symmetric setting or in weak topology with scaling. In fact, even how to show convergence in distribution sense has been a problem.

The main difficulties in studying the convergence to the self-similar solutions come from the fact that the self-similar solution has infinite energy and it decays to zero exponentially in time except in L_1 norm. The purpose of this paper is to show strong convergence holds when $\alpha \in (\max\{2s, 1\}, 2]$ under some conditions on the initial perturbation.

For this, we first consider the $\mathcal{D}^{2+\delta}$ distance between two solutions. For $f_0(v) \in \tilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ and $g_0(v) \in \tilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$, as in [9, 10], set

$$\begin{aligned} \tilde{P}(t, \xi) &= e^{-At} \tilde{P}(0, \xi), \\ \tilde{P}(0, \xi) &= \frac{1}{2} \sum_{j, l=1}^3 \xi_j \xi_l P_{jl}(0) X(\xi), \\ P_{jl}(0) &= \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3} |v|^2 \right) (f_0(v) - g_0(v)) dv, \end{aligned} \quad (1.12)$$

where

$$A = \frac{3}{4} \int_{\mathbb{S}^2} \mathcal{B}\left(\frac{\sigma \cdot \xi}{|\xi|}\right) \left(1 - \left(\frac{\sigma \cdot \xi}{|\xi|}\right)^2\right) d\sigma, \quad (1.13)$$

δ_{jl} is the Kronecker delta and $X(\xi) \equiv X(|\xi|)$ is a smooth radially symmetric function satisfying $0 \leq X(\xi) \leq 1$ and $X(\xi) = 1$ for $|\xi| \leq 1$ and $X(\xi) = 0$ for $|\xi| \geq 2$.

The first result in this paper on the $\mathcal{D}^{2+\delta}$ time asymptotic stability of the solutions is given by

Theorem 1.1. *Suppose $f_0(v), g_0(v) \in \tilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ for $\alpha \in (\max\{2s, 1\}, 2]$. Let $\hat{f}(t, \xi)$ and $\hat{g}(t, \xi)$ be the corresponding two global solutions of the Cauchy problem (1.6) with initial data $\hat{f}_0(\xi)$ and $\hat{g}_0(\xi)$ respectively. Assume for some $\delta \in (0, \alpha] \cap (0, \frac{A}{\mu_\alpha})$, the initial data satisfy*

$$\int_{\mathbb{R}^3} |v|^2 (f_0(v) - g_0(v)) dv = 0, \quad (1.14)$$

$$\begin{cases} \int_{\mathbb{R}^3} |v|^2 |f_0(v) - g_0(v)| dv < +\infty, \\ \left\| \hat{f}_0(\cdot) - \hat{g}_0(\cdot) - \tilde{P}(0, \cdot) \right\|_{\mathcal{D}^{2+\delta}} < +\infty. \end{cases} \quad (1.15)$$

Then there exists some positive constant $C_1 > 0$ independent of t and ξ such that

$$\left\| \hat{f}(t, \cdot) - \hat{g}(t, \cdot) - \tilde{P}(t, \cdot) \right\|_{\mathcal{D}^{2+\delta}} \leq C_1 e^{-\eta_0 t}, \quad t \geq 0. \quad (1.16)$$

Here, $\eta_0 = \min \{A - \delta\mu_\alpha, B\}$ and

$$B = \int_{\mathbb{S}^2} \mathcal{B} \left(\frac{\sigma \cdot \xi}{|\xi|} \right) \left(1 - \left| \cos \frac{\theta}{2} \right|^{2+\delta} - \left| \sin \frac{\theta}{2} \right|^{2+\delta} \right) d\sigma, \quad \cos \theta = \frac{\sigma \cdot \xi}{|\xi|}. \quad (1.17)$$

Note that for the $\mathcal{D}^{2+\delta}$ convergence to the self-similar solution, one can simply take $g_0 = \Psi_{\alpha, K}(v)$. Based on this, in order to obtain a convergence in strong topology, such as in the Sobolev norms, we will give a uniform in time estimate on the solution in H^N -norm that is given in

Theorem 1.2. *For $\max\{1, 2s\} < \alpha < 2$, assume that $f_0(v) \in \tilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ satisfies (1.14)-(1.15) and is not a single Dirac mass, $g_0(v) = \Psi_{\alpha, K}(v)$. Then for any given positive constant $t_1 > 0$ and any $N \in \mathbb{N}$, there exists a positive constant $C_2(t_1, N)$ independent of t such that*

$$\sup_{t \in [t_1, +\infty)} \left\{ \|f(t, \cdot)\|_{H^N} \right\} \leq C_2(t_1, N). \quad (1.18)$$

Consequently, there exists a positive constant $C_3(t_1, N)$ independent of t such that

$$\left\| f(t, \cdot) - f_{\alpha, K}(t, \cdot) \right\|_{H^N} = \left\| f(t, v) - e^{-3\mu_\alpha t} \Psi_{\alpha, K}(ve^{-\mu_\alpha t}) \right\|_{H^N} \leq C_3(t_1, N) e^{-\frac{\eta_0 t}{2}} \quad (1.19)$$

holds for any $t \geq t_1$. Since

$$e^{-\frac{3\mu_\alpha t}{2}} \left\| \Psi_{\alpha, K}(\cdot) \right\|_{L^2} \leq \left\| f_{\alpha, K}(t, \cdot) \right\|_{H^N} \leq e^{-\frac{3\mu_\alpha t}{2}} \left\| \Psi_{\alpha, K}(\cdot) \right\|_{H^N}, \quad (1.20)$$

(1.19) and (1.20) imply that when

$$\mu_\alpha < \frac{\eta_0}{3}, \quad (1.21)$$

the convergence rate given in (1.19) is faster than the decay rate of the self-similar solution itself. Hence in this case, the infinite energy solution $f(t, v)$ converges to the self-similar solution $f_{\alpha, K}(t, v)$ exponentially in time.

Remark 1.1. *Since $\mu_\alpha \rightarrow 0+$ as $\alpha \rightarrow 2$, the condition (1.21) holds when α is close to 2.*

For the case with finite energy, the above stability estimates give a better convergence description on the solution obtained in the previous literatures, which extends the exponential convergence result in the Toscani metrics $\mathcal{D}^{2+\delta}$ with $\delta > 0$, cf. [9], to the Sobolev space $H^N(\mathbb{R}^3)$ for any $N \in \mathbb{N}$. In fact, we have

Corollary 1.1. *Suppose that $f_0(v) \in \mathcal{P}^2(\mathbb{R}^3)$ is not a single Dirac mass and satisfies*

$$\int_{\mathbb{R}^3} |v|^2 f_0(v) dv = 3, \quad \left\| \hat{f}_0(\cdot) - \mu(\cdot) - \tilde{P}(0, \cdot) \right\|_{\mathcal{D}^{2+\delta}} < +\infty, \quad (1.22)$$

for some positive constant $\delta \in (0, 2]$ with $\mu = (2\pi)^{-\frac{3}{2}} e^{-|v|^2/2}$. Then for any $N \in \mathbb{N}$, there exist positive constants $C_4, C_5(t_1, N) > 0$ independent of t such that

$$\left\| \hat{f}(t, \cdot) - \mu(\cdot) - \tilde{P}(t, \cdot) \right\|_{\mathcal{D}^{2+\delta}} \leq C_4 e^{-\eta_1 t}, \quad t > 0, \quad (1.23)$$

and

$$\sup_{t \in [t_1, +\infty)} \left\{ \|f(t, \cdot)\|_{H^N} \right\} \leq C_5(t_1, N), \quad t \geq t_1. \quad (1.24)$$

Here $t_1 > 0$ is any given positive constant and $\eta_1 = \min\{A, B\}$.

A direct consequence of (1.23) and (1.24) gives

$$\left\| f(t, \cdot) - \mu(\cdot) \right\|_{H^N} \leq C_6(t_1, N) e^{-\frac{\eta_1 t}{2}}, \quad (1.25)$$

for some positive constant $C_6(t_1, N)$ depending only on t_1 and N .

Remark 1.2. *Two comments on the above two theorems:*

- By Lemma 2.6, sufficient conditions for the requirements (1.15) and (1.22) are

$$\int_{\mathbb{R}^3} |v|^{2+\delta} |f_0(v) - g_0(v)| dv < +\infty,$$

and

$$\int_{\mathbb{R}^3} |v|^{2+\delta} |f_0(v) - \mu(v)| dv < +\infty,$$

respectively.

- The convergence rate in Corollary 1.1 is faster than the corresponding rates in Theorem 1.1 and Theorem 1.2.

Before the end of the introduction, we list some notations used throughout the paper. Firstly, C, C_i with $i \in \mathbb{N}$, and $O(1)$ are used for some generic large positive constants and ε, κ stand for some generic small positive constants. When the dependence needs to be explicitly pointed out, then the notations like $C(\cdot, \cdot)$ are used. For multi-index $\beta = (\beta_1, \beta_2, \beta_3)$, $\partial_v^\beta = \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}$. And $A \lesssim B$ means that there is a constant $C > 0$ such that $A \leq CB$, and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

The rest of this paper will be organized as follows: Some known results concerning the global solvability, stability, and regularity of solutions to the Cauchy problem (1.6) and (1.9) in $\mathcal{K}^\alpha(\mathbb{R}^3)$ are recalled in Section 2. Moreover, some properties of the approximations of the initial data in $\mathcal{K}^\alpha(\mathbb{R}^3)$ will also be given in this section. And then the proofs of Theorem 1.1, Theorem 1.2, and Corollary 1.1 will be given in the next three sections respectively.

2 Preliminaries

In this section, we will first recall the global solvability, stability and regularity results on the Cauchy problem (1.6) and (1.9) obtained in [5, 6, 7, 8, 11, 12, 13]. And then we will study the properties of the approximation $f_{0R}(v)$ on the initial data $f_0(v)$ defined in (2.3) for later stability estimates.

For the Cauchy problem (1.6) and (1.9), the following estimates are proved in [7, 8, 11, 12, 13].

Lemma 2.1. For $\alpha \in (2s, 2]$, if $\varphi_0(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$, then the Cauchy problem (1.6) and (1.9) admits a unique global classical solution $\varphi(t, \xi) \equiv \hat{f}(t, \xi) \in C([0, \infty), \mathcal{K}^\alpha(\mathbb{R}^3))$ satisfying

$$\left\| \varphi(t, \cdot) - 1 \right\|_{\mathcal{D}^\alpha} \leq e^{\lambda_\alpha t} \left\| \varphi_0(\cdot) - 1 \right\|_{\mathcal{D}^\alpha}. \quad (2.1)$$

If $\psi(t, \xi) \in C([0, \infty), \mathcal{K}^\alpha(\mathbb{R}^3))$ is another solution with initial data $\psi_0(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$, then

$$\left\| \varphi(t, \cdot) - \psi(t, \cdot) \right\|_{\mathcal{D}^\alpha} \leq e^{\lambda_\alpha t} \left\| \varphi_0(\cdot) - \psi_0(\cdot) \right\|_{\mathcal{D}^\alpha}. \quad (2.2)$$

Furthermore, if $f_0(v) = \mathcal{F}^{-1}(\varphi_0)(v)$ is not a single Dirac mass, then $f(t, \cdot) \in L^1(\mathbb{R}^3) \cap \mathcal{P}^\beta(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3)$ for $t > 0$ and $0 < \beta < \alpha$.

And for self-similar solution $f_{\alpha, K}(t, v)$ constructed in [5, 6], by [12, 13], we have

Lemma 2.2. For $\alpha \in (2s, 2)$ and a constant $K > 0$, there exists a radially symmetric function $\hat{\Psi}_{\alpha, K}(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$ satisfying (1.11) such that

$$f_{\alpha, K}(t, v) = e^{-3\mu_\alpha t} \Psi_{\alpha, K}(ve^{-\mu_\alpha t})$$

is a solution of the Cauchy problem (1.1) with initial datum $\Psi_{\alpha, K}(v)$. Moreover, $\Psi_{\alpha, K}(t, \cdot) \in L^1(\mathbb{R}^3) \cap \mathcal{P}^\beta(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3)$ for $0 < \beta < \alpha$.

The relation between $\mathcal{P}^\alpha(\mathbb{R}^3)$ and $\mathcal{K}^\alpha(\mathbb{R}^3)$ was given in [7] and [12] and it can be stated as follows.

Lemma 2.3. It holds that

- (i). For $\alpha \in (0, 2]$, if $h(v) \in \mathcal{P}^\alpha(\mathbb{R}^3)$, then $\hat{h}(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$;
- (ii). For $\alpha \in (0, 2]$, if $\hat{h}(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$, then for any $0 < \beta < \alpha$, $h(v) \in \mathcal{P}^\beta(\mathbb{R}^3)$.

Since the energy of the initial data is infinite, for analysis, we will first approximate it by a cutoff on the large velocity so that the moment of any order is bounded. And then it remains to show that the solution with this kind of approximation has uniform bound independent of the cutoff parameter. On the other hand, the approximate solution can not be arbitrary because it has to be in the function space \mathcal{K}^α .

For $\alpha \in (2s, 2]$ and $f_0(v) \in \tilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$, let $X(v)$ be the smooth function defined in the construction of $\tilde{P}(t, \xi)$ and set $X_R(v) = X(v/R)$, define

$$f_{0R}(v) = \tilde{f}_{0R}(v + a_R^f), \quad \tilde{f}_{0R}(v) = \frac{f_0(v)X_R(v)}{\int_{\mathbb{R}^3} f_0(v)X_R(v)dv} \quad (2.3)$$

with

$$a_R^f = \int_{\mathbb{R}^3} v \tilde{f}_{0R}(v) dv = \frac{\int_{\mathbb{R}^3} v f_0(v) X_R(v) dv}{\int_{\mathbb{R}^3} f_0(v) X_R(v) dv}. \quad (2.4)$$

The properties of the approximation function are given in

Lemma 2.4. For $1 < \beta < \alpha \leq 2$, if we choose $R > 0$ sufficiently large, then

- (i). $\hat{f}_{0R}(\xi), \hat{g}_{0R}(\xi) \in \mathcal{K}^2(\mathbb{R}^3)$, and for sufficiently large $R > 0$ it holds

$$\left\| \hat{f}_{0R}(\cdot) - \hat{g}_{0R}(\cdot) \right\|_{\mathcal{D}^2} \leq C_7 \left(1 + \int_{\mathbb{R}^3} |v| (f_0(v) + g_0(v)) dv + \int_{\mathbb{R}^3} |v|^2 |f_0(v) - g_0(v)| dv \right). \quad (2.5)$$

Here the positive constant C_7 depends only on $\int_{\mathbb{R}^3} (1 + |v|^\beta)(f_0(v) + g_0(v)) dv$;

(ii). For $1 < \beta < \alpha \leq 2$ and sufficiently large $R > 0$, $f_{0R}(v) \in \mathcal{P}^\beta(\mathbb{R}^3)$ with $\mathcal{P}^\beta(\mathbb{R}^3)$ -norm being uniformly bounded, precisely,

$$\int_{\mathbb{R}^3} |v|^\beta f_{0R}(v) dv \lesssim \int_{\mathbb{R}^3} (1 + |v|^\beta) f_0(v) dv. \quad (2.6)$$

Thus

$$\left\| \hat{f}_{0R}(\cdot) - 1 \right\|_{\mathcal{D}^\beta} \lesssim 1, \quad \left\| \hat{f}_{0R}(\cdot) - \hat{f}_0(\cdot) \right\|_{\mathcal{D}^\beta} \lesssim 1, \quad (2.7)$$

and

$$\lim_{R \rightarrow +\infty} \left\| \hat{f}_{0R}(\cdot) - \hat{f}_0(\cdot) \right\|_{\mathcal{D}^\beta} = 0. \quad (2.8)$$

Proof. We first prove (2.6)-(2.8). Since it is straightforward to verify (2.6), (2.7) is a direct consequence of (2.6) and Lemma 2.3. We only prove (2.8) as follows: For this, note that

$$\lim_{R \rightarrow +\infty} \int_{\mathbb{R}^3} f_0(v) X_R(v) dv = 1.$$

Choose R sufficiently large, we have

$$\int_{\mathbb{R}^3} f_0(v) X_R(v) dv \geq \frac{1}{2}, \quad \int_{\mathbb{R}^3} g_0(v) X_R(v) dv \geq \frac{1}{2}. \quad (2.9)$$

Thus

$$\begin{aligned} |a_R^f| &\leq 2 \left| \int_{\mathbb{R}^3} v f_0(v) X_R(v) dv \right| \\ &= 2 \left| \int_{\mathbb{R}^3} v f_0(v) (1 - X_R(v)) dv \right| \leq 2R^{1-\beta} \int_{\mathbb{R}^3} |v|^\beta f_0(v) dv. \end{aligned} \quad (2.10)$$

Similarly,

$$|a_R^g| \leq 2R^{1-\beta} \int_{\mathbb{R}^3} |v|^\beta g_0(v) dv. \quad (2.11)$$

From (2.9), (2.10), and the fact that

$$\begin{aligned} \left| \hat{f}_{0R}(\xi) - \hat{f}_0(\xi) \right| &\leq \int_{\mathbb{R}^3} |f_{0R}(v) - f_0(v)| dv \\ &\leq \left(\int_{\mathbb{R}^3} f_0(v) X_R(v) dv \right)^{-1} \int_{\mathbb{R}^3} \left| f_0(v + a_R^f) - f_0(v) \right| dv \\ &\quad + \left(\int_{\mathbb{R}^3} f_0(v) X_R(v) dv \right)^{-1} \int_{\mathbb{R}^3} f_0(v) \left| X_R(v + a_R^f) - X_R(v) \right| dv, \end{aligned}$$

we obtain

$$\lim_{R \rightarrow +\infty} \sup_{\xi \in \mathbb{R}^3} \left| \hat{f}_{0R}(\xi) - \hat{f}_0(\xi) \right| = 0. \quad (2.12)$$

On the other hand, $\hat{f}_0(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3)$ implies that $\left\| 1 - \hat{f}_0 \right\|_{\mathcal{D}^\alpha} \lesssim 1$. Consequently, for $1 < \beta < \alpha \leq 2$, it holds that

$$\frac{|1 - \hat{f}_0(\eta)|}{|\eta|^\beta} \leq \left\| 1 - \hat{f}_0 \right\|_{\mathcal{D}^\alpha} |\eta|^{\alpha-\beta},$$

so that for each $\varepsilon > 0$, there exists a $\delta_1(\varepsilon) > 0$ such that

$$\frac{|1 - \hat{f}_0(\eta)|}{|\eta|^\beta} < \frac{\varepsilon}{2} \quad (2.13)$$

holds for any $|\eta| < \delta_1$.

Choose $\tilde{\beta} \in (\beta, \alpha)$ so that $\|1 - \hat{f}_{0R}\|_{\mathcal{D}^{\tilde{\beta}}}$ is bounded by a constant independent of R because of (2.7). Then

$$\frac{|1 - \hat{f}_{0R}(\eta)|}{|\eta|^\beta} \leq \|1 - \hat{f}_{0R}\|_{\mathcal{D}^{\tilde{\beta}}} |\eta|^{\tilde{\beta}-\beta} \lesssim |\eta|^{\tilde{\beta}-\beta} < \frac{\varepsilon}{2} \quad (2.14)$$

provided that $|\eta| < \delta_2(\varepsilon)$ for some sufficiently small $\delta_2 > 0$.

(2.13) together with (2.14) imply that for any $\varepsilon > 0$ and $|\eta| < \delta = \min\{\delta_1, \delta_2\}$, we have

$$\frac{|\hat{f}_{0R}(\eta) - \hat{f}_0(\eta)|}{|\eta|^\beta} \leq \frac{|1 - \hat{f}_{0R}(\eta)|}{|\eta|^\beta} + \frac{|1 - \hat{f}_0(\eta)|}{|\eta|^\beta} < \varepsilon. \quad (2.15)$$

And (2.8) follows directly from (2.12) and (2.15).

Now it remains to prove (2.5). Set

$$k(v, \xi) = \frac{e^{-iv \cdot \xi} + iv \cdot \xi - 1}{|\xi|^2}, \quad (2.16)$$

then

$$\begin{aligned} \frac{|\hat{f}_{0R}(\xi) - \hat{g}_{0R}(\xi)|}{|\xi|^2} &= \left| \int_{\mathbb{R}^3} \left(k(v - a_R^f, \xi) \frac{f_0(v)X_R(v)}{\int_{\mathbb{R}^3} f_0(v)X_R(v)dv} - k(v - a_R^g, \xi) \frac{g_0(v)X_R(v)}{\int_{\mathbb{R}^3} g_0(v)X_R(v)dv} \right) dv \right| \\ &\lesssim \underbrace{\left| \int_{\mathbb{R}^3} \left(k(v - a_R^f, \xi) - k(v - a_R^g, \xi) \right) \frac{f_0(v)X_R(v)}{\int_{\mathbb{R}^3} f_0(v)X_R(v)dv} dv \right|}_{I_1} \\ &\quad + \underbrace{\left| \int_{\mathbb{R}^3} k(v - a_R^g, \xi) \left(\frac{f_0(v)X_R(v)}{\int_{\mathbb{R}^3} f_0(v)X_R(v)dv} - \frac{g_0(v)X_R(v)}{\int_{\mathbb{R}^3} g_0(v)X_R(v)dv} \right) dv \right|}_{I_2}. \end{aligned} \quad (2.17)$$

Firstly, from (2.9), (2.11) and the fact

$$|k(v - a_R^g, \xi)| \lesssim |v - a_R^g|^2,$$

we have

$$\begin{aligned}
I_2 &\leq 2 \left| \int_{\mathbb{R}^3} k(v - a_R^g, \xi) (f_0(v) - g_0(v)) X_R(v) dv \right| \\
&\quad + 4 \left| \int_{\mathbb{R}^3} k(v - a_R^g, \xi) g_0(v) X_R(v) \left(\int_{\mathbb{R}^3} (f_0(v) - g_0(v)) X_R(v) dv \right) dv \right| \\
&\lesssim \int_{\mathbb{R}^3} (1 + |v|^2) |f_0(v) - g_0(v)| dv + \int_{\mathbb{R}^3} (1 + |v|^2) g_0(v) X_R(v) \left| \int_{\mathbb{R}^3} (f_0(v) - g_0(v)) (1 - X_R(v)) dv \right| dv \quad (2.18) \\
&\lesssim \int_{\mathbb{R}^3} (1 + |v|^2) |f_0(v) - g_0(v)| dv + \int_{\mathbb{R}^3} (1 + |v|^\beta) g_0(v) dv \cdot R^{2-\beta} \cdot \left| \int_{\mathbb{R}^3} (f_0(v) - g_0(v)) (1 - X_R(v)) dv \right| \\
&\lesssim \int_{\mathbb{R}^3} (1 + |v|^2) |f_0(v) - g_0(v)| dv + \int_{\mathbb{R}^3} (1 + |v|^\beta) g_0(v) dv \cdot \int_{\mathbb{R}^3} |v|^{2-\beta} |f_0(v) - g_0(v)| dv \\
&\lesssim \int_{\mathbb{R}^3} (1 + |v|^2) |f_0(v) - g_0(v)| dv \cdot \left(1 + \int_{\mathbb{R}^3} (1 + |v|^\beta) g_0(v) dv \right).
\end{aligned}$$

For I_1 , by noticing

$$\left| k(v - a_R^f, \xi) - k(v - a_R^g, \xi) \right| = |\xi|^{-2} \left| e^{-i(v - a_R^f) \cdot \xi} \left(e^{-i(a_R^g - a_R^f) \cdot \xi} - 1 \right) + i(a_R^g - a_R^f) \cdot \xi \right|, \quad (2.19)$$

we have for $|\xi| \geq 1$ that

$$\left| k(v - a_R^f, \xi) - k(v - a_R^g, \xi) \right| \lesssim |a_R^g - a_R^f| \lesssim R^{1-\beta}. \quad (2.20)$$

For $|\xi| \leq 1$, it holds that

$$\begin{aligned}
&\left| k(v - a_R^f, \xi) - k(v - a_R^g, \xi) \right| \\
&\lesssim |\xi|^{-2} \left| \left[(1 + O(1)|v - a_R^g| |\xi|) \left(-i(a_R^g - a_R^f) \cdot \xi + O(1) |a_R^g - a_R^f|^2 |\xi|^2 \right) + i(a_R^g - a_R^f) \cdot \xi \right] \right| \\
&\lesssim |a_R^g - a_R^f|^2 + |v - a_R^g| |a_R^g - a_R^f| + |v - a_R^g| |a_R^g - a_R^f|^2 |\xi| \\
&\lesssim (1 + |v|) R^{1-\beta}.
\end{aligned} \quad (2.21)$$

Thus, (2.19), (2.20) and (2.21) imply that

$$\left| k(v - a_R^f, \xi) - k(v - a_R^g, \xi) \right| \lesssim 1 + |v|, \quad (2.22)$$

and consequently

$$I_1 \lesssim \int_{\mathbb{R}^3} (1 + |v|) f_0(v) dv. \quad (2.23)$$

Inserting (2.18) and (2.23) into (2.17) yields (2.5) and this completes the proof of Lemma 2.4. \square

Now let

$$P_{jl}^R(0) \equiv \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3} |v|^2 \right) (f_{0R}(v) - g_{0R}(v)) dv \quad (2.24)$$

be the approximation of $P_{jl}(0)$ defined by (1.12)₃. The following lemma gives the convergence of $P_{jl}^R(0)$ to $P_{jl}(0)$ as $R \rightarrow +\infty$.

Lemma 2.5. *Assume*

$$\int_{\mathbb{R}^3} |v|^2 |f_0(v) - g_0(v)| dv < +\infty, \quad (2.25)$$

then

$$\lim_{R \rightarrow +\infty} P_{jl}^R(0) = P_{jl}(0). \quad (2.26)$$

Proof. Notice that

$$\begin{aligned} P_{jl}^R(0) &= \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3} |v|^2 \right) \frac{f_0(v + a_R^f) X_R(v + a_R^f) - f_0(v + a_R^g) X_R(v + a_R^g)}{\int_{\mathbb{R}^3} f_0(v) X_R(v) dv} dv \\ &\quad + \left(\int_{\mathbb{R}^3} f_0(v) X_R(v) dv \right)^{-1} \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3} |v|^2 \right) (f_0(v + a_R^g) - g_0(v + a_R^g)) X_R(v + a_R^g) dv \\ &\quad + \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3} |v|^2 \right) g_0(v + a_R^g) X_R(v + a_R^g) \left(\left(\int_{\mathbb{R}^3} f_0(v) X_R(v) dv \right)^{-1} - \left(\int_{\mathbb{R}^3} g_0(v) X_R(v) dv \right)^{-1} \right) dv \\ &= \underbrace{\int_{\mathbb{R}^3} \left[\left((v_j - a_{Rj}^f) (v_l - a_{Rl}^f) - \frac{\delta_{jl}}{3} |v - a_R^f|^2 \right) - \left((v_j - a_{Rj}^g) (v_l - a_{Rl}^g) - \frac{\delta_{jl}}{3} |v - a_R^g|^2 \right) \right] \frac{f_0(v) X_R(v)}{\int_{\mathbb{R}^3} f_0(v) X_R(v) dv} dv}_{I_3} \\ &\quad + \underbrace{\int_{\mathbb{R}^3} \left((v_j - a_{Rj}^g) (v_l - a_{Rl}^g) - \frac{\delta_{jl}}{3} |v - a_R^g|^2 \right) \frac{(f_0(v) - g_0(v)) X_R(v)}{\int_{\mathbb{R}^3} f_0(v) X_R(v) dv} dv}_{I_4} \\ &\quad + \underbrace{\int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3} |v|^2 \right) g_0(v + a_R^g) X_R(v + a_R^g) \frac{\int_{\mathbb{R}^3} (f_0(v) - g_0(v)) X_R(v) dv}{\int_{\mathbb{R}^3} f_0(v) X_R(v) dv \cdot \int_{\mathbb{R}^3} g_0(v) X_R(v) dv} dv}_{I_5}. \end{aligned} \quad (2.27)$$

We have from

$$\begin{aligned} &\left| \left((v_j - a_{Rj}^f) (v_l - a_{Rl}^f) - \frac{\delta_{jl}}{3} |v - a_R^f|^2 \right) - \left((v_j - a_{Rj}^g) (v_l - a_{Rl}^g) - \frac{\delta_{jl}}{3} |v - a_R^g|^2 \right) \right| \\ &\lesssim |a_R^f - a_R^g| |v| + |a_R^g|^2 + |a_R^f|^2, \end{aligned}$$

and (2.9) that for $R \geq R_1$

$$|I_3| \lesssim \left(|a_R^g| + |a_R^f| \right) \int_{\mathbb{R}^3} (1 + |v|) f_0(v) dv.$$

This together with (2.10)-(2.11) imply that

$$\lim_{R \rightarrow +\infty} I_3 = 0. \quad (2.28)$$

For I_5 , if R is sufficiently large, we have from (2.9), (2.11) and the assumption (2.25) that

$$\begin{aligned}
|I_5| &\lesssim \left| \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3} |v|^2 \right) g_0(v + a_R^g) X_R(v + a_R^g) \left(\int_{\mathbb{R}^3} (f_0(v) - g_0(v)) X_R(v) dv \right) dv \right| \\
&\lesssim \int_{\mathbb{R}^3} \left(1 + |v - a_R^g|^2 \right) g_0(v + a_R^g) X_R(v + a_R^g) \left| \int_{\mathbb{R}^3} (f_0(v) - g_0(v)) (1 - X_R(v)) dv \right| dv \\
&\lesssim \int_{\mathbb{R}^3} (1 + |v|^\beta) R^{2-\beta} g_0(v) X_R(v) \left| \int_{\mathbb{R}^3} (f_0(v) - g_0(v)) (1 - X_R(v)) dv \right| dv \\
&\lesssim R^{2(1-\beta)} \left(\int_{\mathbb{R}^3} (1 + |v|^\beta) g_0(v) dv \right) \left(\int_{\mathbb{R}^3} |v|^\beta |f_0(v) - g_0(v)| dv \right) \lesssim R^{2(1-\beta)}.
\end{aligned}$$

Thus

$$\lim_{R \rightarrow +\infty} I_5 = 0. \quad (2.29)$$

Finally for I_4 , from (2.10), (2.11), the assumption (2.25) and $\lim_{R \rightarrow +\infty} \int_{\mathbb{R}^3} f_0(v) X_R(v) dv = 1$, the dominated convergence theorem yields

$$\lim_{R \rightarrow +\infty} I_4 = \int_{\mathbb{R}^3} \left(v_j v_l - \frac{\delta_{jl}}{3} |v|^2 \right) (f_0(v) - g_0(v)) dv = P_{jl}(0). \quad (2.30)$$

Inserting (2.28), (2.29) and (2.30) into (2.28) gives (2.26). This completes the proof of the lemma. \square

In the last lemma of this section, a sufficient condition for (1.15) on $f_0(v) - g_0(v)$ used in Theorem 1.1 is given.

Lemma 2.6. *Let $0 < \delta \leq 1$, it holds that*

$$\left\| \hat{f}_0(\cdot) - \hat{g}_0(\cdot) - \tilde{P}(0, \cdot) \right\|_{\mathcal{D}^{2+\delta}} \lesssim \int_{\mathbb{R}^3} (1 + |v|^{2+\delta}) |f_0(v) - g_0(v)| dv. \quad (2.31)$$

Proof. In fact, by the assumption (1.14), we have

$$\begin{aligned}
\hat{f}_0(\xi) - \hat{g}_0(\xi) - \tilde{P}(0, \xi) &= \int_{\mathbb{R}^3} \left[e^{-iv \cdot \xi} - 1 + iv \cdot \xi - \frac{1}{2} \sum_{j,l=1}^3 \xi_j \xi_l X(\xi) \left(v_j v_l - \frac{\delta_{jl}}{3} |v|^2 \right) \right] (f_0(v) - g_0(v)) dv \\
&= \int_{\mathbb{R}^3} \left[e^{-iv \cdot \xi} - 1 + iv \cdot \xi - \frac{1}{2} \sum_{j,l=1}^3 \xi_j \xi_l X(\xi) v_j v_l \right] (f_0(v) - g_0(v)) dv.
\end{aligned}$$

The Taylor expansion of $e^{-iv \cdot \xi} - 1 + iv \cdot \xi$ to the second order implies that

$$\left| e^{-iv \cdot \xi} - 1 + iv \cdot \xi - \frac{1}{2} \sum_{j,l=1}^3 \xi_j \xi_l X(\xi) v_j v_l \right| \lesssim |v|^2 |\xi|^2,$$

and the Taylor expansion to $e^{-iv \cdot \xi} - 1 + iv \cdot \xi - \frac{1}{2} \sum_{j,l=1}^3 \xi_j \xi_l X(\xi) v_j v_l$ to the third order gives

$$\left| e^{-iv \cdot \xi} - 1 + iv \cdot \xi - \frac{1}{2} \sum_{j,l=1}^3 \xi_j \xi_l X(\xi) v_j v_l \right| \lesssim (1 + |v|^3) |\xi|^3.$$

Thus interpolation yields

$$\left| e^{-iv \cdot \xi} - 1 + iv \cdot \xi - \frac{1}{2} \sum_{j,l=1}^3 \xi_j \xi_l X(\xi) v_j v_l \right| \lesssim (1 + |v|^{2+\delta}) |\xi|^{2+\delta},$$

for $0 < \delta \leq 1$. With this, (2.31) follows. And this completes the proof of the lemma. \square

3 Proof of Theorem 1.1

To prove Theorem 1.1, as in [7], we first approximate the cross section by a sequence of bounded cross sections defined by

$$\mathcal{B}_n(s) = \min \{ \mathcal{B}(s), n \}, \quad n \in \mathbb{N}. \quad (3.1)$$

Then consider

$$\partial_t H_n + H_n = \frac{1}{\bar{\sigma}_n} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{(v - v_*) \cdot \sigma}{|v - v_*|} \right) H_n(v') H_n(v'_*) d\sigma dv_*, \quad (3.2)$$

$$H_n(0, v) = H_0(v). \quad (3.3)$$

Here

$$\bar{\sigma}_n = \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) d\sigma. \quad (3.4)$$

For $\alpha \in (\max\{2s, 1\}, 2]$ and $f_0(v), g_0(v) \in \tilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$, let $f_{0R}(v)$ and $g_{0R}(v)$ be the approximation of $f_0(v)$ and $g_0(v)$ constructed in the previous section. Since $f_{0R}(v), g_{0R}(v) \in \mathcal{P}^2(\mathbb{R}^3) \subset \mathcal{K}^2(\mathbb{R}^3)$, it follows from Lemma 2.1 that the Cauchy problem (3.2)-(3.3) with $H_0(v) = f_{0R}(v)$ ($H_0(v) = g_{0R}(v)$) admits a unique non-negative global solution $F_R^n(t, v)$ ($G_R^n(t, v)$) satisfying $\hat{F}_R^n(t, \xi) \in C([0, \infty), \mathcal{K}^2(\mathbb{R}^3))$ ($\hat{G}_R^n(t, \xi) \in C([0, \infty), \mathcal{K}^2(\mathbb{R}^3))$). Moreover, for $\max\{2s, 1\} < \beta < \alpha \leq 2$, (2.2), Lemma 2.3 and Lemma 2.4 imply that

$$\begin{cases} \left\| \hat{F}_R^n(t, \cdot) - \hat{F}_n(t, \cdot) \right\|_{\mathcal{D}^\beta} \leq e^{\lambda_\beta^n t} \left\| \hat{f}_{0R}(\cdot) - \hat{f}_0(\cdot) \right\|_{\mathcal{D}^\beta} \lesssim e^{\lambda_\beta^n t}, \\ \left\| \hat{G}_R^n(t, \cdot) - \hat{G}_n(t, \cdot) \right\|_{\mathcal{D}^\beta} \leq e^{\lambda_\beta^n t} \left\| \hat{g}_{0R}(\cdot) - \hat{g}_0(\cdot) \right\|_{\mathcal{D}^\beta} \lesssim e^{\lambda_\beta^n t}. \end{cases} \quad (3.5)$$

Here $F_n(t, v)$ and $G_n(t, v)$ denote the unique non-negative solutions of the Cauchy problem (3.2)-(3.3) with initial data $f_0(v) \in \tilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ and $g_0(v) \in \tilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ respectively, and

$$\lambda_\alpha^n = \frac{1}{\bar{\sigma}_n} \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(\frac{|\xi^+|^\alpha + |\xi^-|^\alpha}{|\xi|^\alpha} - 1 \right) d\sigma. \quad (3.6)$$

Furthermore,

$$\int_{\mathbb{R}^3} |v|^2 F_R^n(t, v) dv = \int_{\mathbb{R}^3} |v|^2 f_{0R}(v) dv < +\infty, \quad \int_{\mathbb{R}^3} |v|^2 G_R^n(t, v) dv = \int_{\mathbb{R}^3} |v|^2 g_{0R}(v) dv < +\infty.$$

Consequently, Lemma 2.1 yields

$$\left\| \hat{F}_R^n(t, \cdot) - \hat{G}_R^n(t, \cdot) \right\|_{\mathcal{D}^2} \leq \left\| \hat{f}_{0R}(\cdot) - \hat{g}_{0R}(\cdot) \right\|_{\mathcal{D}^2} \lesssim 1, \quad t > 0. \quad (3.7)$$

Noticing that

$$\left| \hat{F}_R^n(t, \xi) - \hat{F}_n(t, \xi) \right| \leq |\xi|^\beta \left\| \hat{F}_R^n(t, \cdot) - \hat{F}_n(t, \cdot) \right\|_{\mathcal{D}^\beta} \leq |\xi|^\beta e^{\lambda_\beta^n t} \left\| \hat{f}_{0R}(\cdot) - \hat{f}_0(\cdot) \right\|_{\mathcal{D}^\beta},$$

where (3.5) has been used, from (2.8), we have

Lemma 3.1. *The limit*

$$\lim_{R \rightarrow +\infty} \left(\hat{F}_R^n(t, \xi), \hat{G}_R^n(t, \xi) \right) = \left(\hat{F}_n(t, \xi), \hat{G}_n(t, \xi) \right)$$

holds uniformly, locally with respect to $t \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^3$.

Putting

$$\Phi_1^{nR}(t, \xi) = \hat{F}_R^n(t, \xi) - \hat{G}_R^n(t, \xi) - \tilde{P}_R^n(t, \xi) \quad (3.8)$$

with

$$\begin{aligned} \tilde{P}_R^n(t, \xi) &= \frac{1}{2} e^{-A_n t} \sum_{j,l=1}^3 P_{jl}^R(0) \xi_j \xi_l X(\xi), \\ A_n &= \frac{3}{4\bar{\sigma}_n} \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left[1 - \left(\frac{\xi \cdot \sigma}{|\xi|} \right)^2 \right] d\sigma, \end{aligned} \quad (3.9)$$

we now deduce the equation for $\Phi_1^{nR}(t, \xi)$. Set

$$\hat{Q}_n^+ \left(\hat{F}, \hat{G} \right) = \frac{1}{\bar{\sigma}_n} \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \hat{F}(t, \xi^+) \hat{G}(t, \xi^-) d\sigma. \quad (3.10)$$

Since $\hat{F}_R^n(t, \xi)$ and $\hat{G}_R^n(t, \xi)$ satisfy

$$\begin{cases} \partial_t \hat{F}_R^n + \hat{F}_R^n = \hat{Q}^+ \left(\hat{F}_R^n, \hat{F}_R^n \right), \\ \partial_t \hat{G}_R^n + \hat{G}_R^n = \hat{Q}^+ \left(\hat{G}_R^n, \hat{G}_R^n \right), \end{cases}$$

we have

$$\begin{aligned} \partial_t \Phi_1^{nR} + \Phi_1^{nR} &= - \left(\partial_t \tilde{P}_R^n + \tilde{P}_R^n \right) + \hat{Q}^+ \left(\Phi_1^{nR}, \hat{F}_R^n \right) + \hat{Q}^+ \left(\hat{G}_R^n, \Phi_1^{nR} \right) \\ &\quad + \hat{Q}^+ \left(\tilde{P}_R^n, \hat{F}_R^n \right) + \hat{Q}^+ \left(\hat{G}_R^n, \tilde{P}_R^n \right), \\ \Phi_1^{nR}(0, \xi) &= \hat{f}_{0R}(\xi) - \hat{g}_{0R}(\xi) - \tilde{P}_R^n(0, \xi). \end{aligned} \quad (3.11)$$

Let

$$\Phi_1^n(t, \xi) = \hat{F}^n(t, \xi) - \hat{G}^n(t, \xi) - \tilde{P}^n(t, \xi). \quad (3.12)$$

By taking $R \rightarrow +\infty$, we have from Lemma 3.1 and Lemma 2.5 that $\Phi_1^{nR}(t, \xi) \rightarrow \Phi_1^n(t, \xi)$ uniformly, locally with respect to $t \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^3$ as $R \rightarrow +\infty$. To derive the equation for $\Phi_1^n(t, \xi)$, we firstly study

$$E_R^n(t, \xi) = \hat{Q}^+ \left(\tilde{P}_R^n, \hat{F}_R^n \right) + \hat{Q}^+ \left(\hat{G}_R^n, \tilde{P}_R^n \right) - \left(\partial_t \tilde{P}_R^n(t, \xi) + \tilde{P}_R^n(t, \xi) \right). \quad (3.13)$$

In fact, for $E_R^n(t, \xi)$, we have

Lemma 3.2. *It holds that*

$$\lim_{R \rightarrow +\infty} E_R^n(t, \xi) = E^n(t, \xi), \quad (3.14)$$

uniformly, locally with respect to $t \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^3$. And $E^n(t, \xi)$ satisfies

$$|E^n(t, \xi)| \leq \begin{cases} O(1) |\xi|^{2+\delta} e^{-\left(A_n - \frac{\delta \lambda_n}{\alpha}\right)t}, & |\xi| \leq 1, \\ O(1) e^{-A_n t}, & |\xi| \geq 1 \end{cases} \quad (3.15)$$

for any $\delta \in (0, \alpha] \cap \left(0, \frac{\alpha A_n}{\lambda_n}\right)$ and some positive constant $O(1)$ independent of t, ξ, R and n .

Proof. Since

$$\begin{aligned} E_R^n(t, \xi) &= - \left(\partial_t \tilde{P}_R^n(t, \xi) + \tilde{P}_R^n(t, \xi) \right) \\ &\quad + \frac{1}{2\bar{\sigma}_n} \sum_{j,l=1}^3 e^{-A_n t} P_{jl}^R(0) \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left[\xi_j^+ \xi_l^+ X(\xi^+) + \xi_j^- \xi_l^- X(\xi^-) \right] d\sigma \\ &\quad + \frac{1}{2\bar{\sigma}_n} \sum_{j,l=1}^3 e^{-A_n t} P_{jl}^R(0) \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left[\xi_j^+ \xi_l^+ X(\xi^+) \left(\hat{F}_R^n(t, \xi^-) - 1 \right) + \xi_j^- \xi_l^- X(\xi^-) \left(\hat{G}_R^n(t, \xi^+) - 1 \right) \right] d\sigma, \end{aligned}$$

it follows from Lemma 3.1 and Lemma 2.5 that

$$\lim_{R \rightarrow +\infty} E_R^n(t, \xi) = E^n(t, \xi) \quad (3.16)$$

uniformly, locally with respect to $t \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^3$. Here

$$\begin{aligned} E^n(t, \xi) &= - \left(\partial_t \tilde{P}^n(t, \xi) + \tilde{P}^n(t, \xi) \right) + \underbrace{\frac{1}{2\bar{\sigma}_n} \sum_{j,l=1}^3 e^{-A_n t} P_{jl}(0) \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left[\xi_j^+ \xi_l^+ X(\xi^+) + \xi_j^- \xi_l^- X(\xi^-) \right] d\sigma}_{I_6} \quad (3.17) \\ &\quad + \underbrace{\frac{1}{2\bar{\sigma}_n} \sum_{j,l=1}^3 e^{-A_n t} P_{jl}(0) \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left[\xi_j^+ \xi_l^+ X(\xi^+) \left(\hat{F}^n(t, \xi^-) - 1 \right) + \xi_j^- \xi_l^- X(\xi^-) \left(\hat{G}^n(t, \xi^+) - 1 \right) \right] d\sigma}_{I_7}, \end{aligned}$$

and

$$\tilde{P}^n(t, \xi) = \frac{1}{2} e^{-A_n t} \sum_{j,l=1}^3 P_{jl}(0) \xi_j \xi_l X(\xi). \quad (3.18)$$

To estimate the bounds on I_6 and I_7 , firstly note that for $|\xi| \geq 1$,

$$\left| \hat{F}^n(t, \xi) \right| \leq 1, \quad \left| \hat{G}^n(t, \xi) \right| \leq 1$$

imply that

$$\left| E^n(t, \xi) \right| \leq O(1)(1 + A_n) e^{-A_n t} \leq O(1) e^{-A_n t}. \quad (3.19)$$

Here we have used the fact that A_n has a uniform upper bound for any $n \in \mathbb{N}$.

If $|\xi| \leq 1$, then $|\xi^\pm| \leq 1$ so that $X(\xi^\pm) \equiv 1$. Hence, as obtained in [9], we have

$$\xi_j^- \xi_l^- + \xi_j^+ \xi_l^+ = \frac{1}{2} (\xi_j \xi_l + |\xi|^2 \sigma_j \sigma_l),$$

and

$$\frac{1}{\bar{\sigma}_n} \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \sigma_j \sigma_l d\sigma = \frac{2A_n}{3} \delta_{jl} + (1 - 2A_n) \frac{\xi_j \xi_l}{|\xi|^2}.$$

Then

$$\begin{aligned} I_6 &= \frac{1}{2\bar{\sigma}_n} \sum_{j,l=1}^3 e^{-A_n t} P_{jl}(0) \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left[\xi_j^+ \xi_l^+ + \xi_j^- \xi_l^- \right] d\sigma \\ &= \frac{1}{4\bar{\sigma}_n} \sum_{j,l=1}^3 e^{-A_n t} P_{jl}(0) \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left[\xi_j \xi_l + |\xi|^2 \sigma_j \sigma_l \right] d\sigma \\ &= \frac{1}{4} \sum_{j,l=1}^3 e^{-A_n t} P_{jl}(0) \xi_j \xi_l + \frac{1}{4} \sum_{j,l=1}^3 e^{-A_n t} P_{jl}(0) \left[(1 - 2A_n) \xi_j \xi_l + \frac{2A_n}{3} \delta_{jl} |\xi|^2 \right] \\ &= (1 - A_n) \tilde{P}^n(t, \xi) = \partial_t \tilde{P}^n(t, \xi) + \tilde{P}^n(t, \xi). \end{aligned}$$

Thus for $|\xi| \leq 1$, it holds that

$$I_6 - \left(\partial_t \tilde{P}^n(t, \xi) + \tilde{P}^n(t, \xi) \right) = 0. \quad (3.20)$$

For I_7 when $|\xi| \leq 1$, we have from the assumption $f_0(v), g_0(v) \in \tilde{\mathcal{P}}^\alpha(\mathbb{R}^3)$ and Lemma 2.1 that

$$\begin{aligned} \left| \hat{F}_n(t, \xi) - 1 \right| &\leq |\xi|^\alpha \left\| \hat{F}_n(t, \cdot) - 1 \right\|_{\mathcal{D}^\alpha} \leq e^{\lambda_\alpha^n t} |\xi|^\alpha \left\| \hat{f}_0(\cdot) - 1 \right\|_{\mathcal{D}^\alpha}, \\ \left| \hat{G}_n(t, \xi) - 1 \right| &\leq |\xi|^\alpha \left\| \hat{G}_n(t, \cdot) - 1 \right\|_{\mathcal{D}^\alpha} \leq e^{\lambda_\alpha^n t} |\xi|^\alpha \left\| \hat{g}_0(\cdot) - 1 \right\|_{\mathcal{D}^\alpha}. \end{aligned}$$

The above estimates together with $\left| \hat{F}_n(t, \xi) \right| \leq 1$ and $\left| \hat{G}_n(t, \xi) \right| \leq 1$ imply that

$$\left| \hat{F}_n(t, \xi) - 1 \right| + \left| \hat{G}_n(t, \xi) - 1 \right| \leq O(1) |\xi|^{\varepsilon\alpha} e^{\varepsilon\lambda_\alpha^n t} \quad (3.21)$$

for any $\varepsilon \in (0, 1]$. Consequently, for $|\xi| \leq 1$,

$$|I_7| \leq O(1) |\xi|^{2+\varepsilon\alpha} e^{-(A_n - \varepsilon\lambda_\alpha^n)t}. \quad (3.22)$$

(3.20) together with (3.22) imply that

$$|E^n(t, \xi)| \leq O(1) |\xi|^{2+\varepsilon\alpha} e^{-(A_n - \varepsilon\lambda_\alpha^n)t}, \quad |\xi| \leq 1. \quad (3.23)$$

With (3.19) and (3.23), let $\delta = \varepsilon\alpha$, the estimate (3.15) follows immediately. This completes the proof of the lemma. \square

Now by letting $R \rightarrow +\infty$ in (3.11), we get from Lemma 2.5, Lemma 3.1 and Lemma 3.2 that $\Phi_1^n(t, \xi) = \hat{F}^n(t, \xi) - \hat{G}^n(t, \xi) - \tilde{P}^n(t, \xi)$ solves

$$\partial_t \Phi_1^n + \Phi_1^n = \hat{Q}^+ \left(\Phi_1^n, \hat{F}^n \right) + \hat{Q}^+ \left(\hat{G}^n, \Phi_1^n \right) + E^n(t, \xi), \quad (3.24)$$

$$\Phi_1^n(0, \xi) = \hat{f}_0(\xi) - \hat{g}_0(\xi) - \tilde{P}^n(0, \xi). \quad (3.25)$$

Here $E^n(t, \xi)$ satisfies (3.15). By Lemmas 2.5, 3.1 and 3.2, $\Phi_1^n(t, \xi)$, $\hat{F}^n(t, \xi)$, $\hat{G}^n(t, \xi)$ and $E^n(t, \xi)$ are continuous functions of $(t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3$ and satisfy (3.24) in the sense of distribution. Since $\Phi_1^n(t, \xi)$, $\hat{F}^n(t, \xi)$, $\hat{G}^n(t, \xi)$, and $E^n(t, \xi)$ are uniformly bounded, $\partial_t \Phi_1^n(t, \xi)$ is also uniformly bounded so that $\Phi_1^n(t, \xi)$ is globally Lipschitz continuous with respect to t . Hence (3.24) holds almost everywhere. Furthermore, by the continuity of $\Phi_1^n(t, \xi)$, $\hat{F}^n(t, \xi)$, $\hat{G}^n(t, \xi)$ and $E^n(t, \xi)$, we have that $\partial_t \Phi_1^n(t, \xi)$ is a continuous function of $(t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3$ and consequently $\Phi_1^n(t, \xi)$ satisfies (3.24) everywhere.

The next lemma is about the upper bound on $\|\Phi_1^n(t, \cdot)\|_{\mathcal{D}^{2+\delta}}$ for some $\delta \in (0, \alpha] \cap \left(0, \frac{\alpha A_n}{\lambda_n}\right)$.

Lemma 3.3. *If $\|\Phi_1^n(0, \cdot)\|_{\mathcal{D}^{2+\delta}} < +\infty$ with $\delta \in (0, \alpha] \cap \left(0, \frac{\alpha A_n}{\lambda_n}\right)$, then*

$$\|\Phi_1^n(t, \cdot)\|_{\mathcal{D}^{2+\delta}} \lesssim e^{-\eta_0^n t}. \quad (3.26)$$

Here $\eta_0^n = \min \left\{ B_n, A_n - \frac{\delta \lambda_n^n}{\alpha} \right\}$ with

$$B_n = \frac{1}{\sigma_n} \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\sigma \cdot \xi}{|\xi|} \right) \left(1 - \left| \cos \frac{\theta}{2} \right|^{2+\delta} - \left| \sin \frac{\theta}{2} \right|^{2+\delta} \right) d\sigma, \quad \cos \theta = \frac{\sigma \cdot \xi}{|\xi|}. \quad (3.27)$$

Proof. The proof is divided into two steps, the first step is to show that $\frac{\Phi_1^n(t, \xi)}{|\xi|^{2+\delta}} \in L^\infty(\mathbb{R}^3)$. Indeed, for $\kappa > 0$, we have from (3.24) that

$$\begin{aligned} & \left(\frac{\Phi_1^n(t, \xi)}{|\xi|^2 (|\xi|^\delta + \kappa)} \right)_t + \frac{\Phi_1^n(t, \xi)}{|\xi|^2 (|\xi|^\delta + \kappa)} \\ &= \frac{1}{\bar{\sigma}_n} \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\sigma \cdot \xi}{|\xi|} \right) \left[\frac{\Phi_1^n(t, \xi^+)}{|\xi^+|^2 (|\xi^+|^\delta + \kappa)} \frac{|\xi^+|^2 (|\xi^+|^\delta + \kappa)}{|\xi|^2 (|\xi|^\delta + \kappa)} \hat{F}^n(t, \xi^-) \right. \\ & \quad \left. + \hat{G}^n(t, \xi^+) \frac{\Phi_1^n(t, \xi^-)}{|\xi^-|^2 (|\xi^-|^\delta + \kappa)} \frac{|\xi^-|^2 (|\xi^-|^\delta + \kappa)}{|\xi|^2 (|\xi|^\delta + \kappa)} \right] d\sigma + \frac{E^n(t, \xi)}{|\xi|^2 (|\xi|^\delta + \kappa)}. \end{aligned} \quad (3.28)$$

On the other hand, by letting $R \rightarrow +\infty$ in (3.7), we have from Lemma 3.1 that for $t > 0$

$$\left\| \hat{F}^n(t, \cdot) - \hat{G}^n(t, \cdot) \right\|_{\mathcal{D}^2} \leq \left\| \hat{f}_0(\cdot) - \hat{g}_0(\cdot) \right\|_{\mathcal{D}^2} \lesssim 1. \quad (3.29)$$

(3.29) together with the definition of $\tilde{P}^n(t, \xi)$ imply that

$$\frac{\Phi_1^n(t, \xi)}{|\xi|^2 (|\xi|^\delta + \kappa)} \in L^\infty(\mathbb{R}^3), \quad t > 0,$$

for any $\kappa > 0$. Hence, by (3.28), Lemma 3.2 and the fact that $|\xi^\pm| \leq |\xi|$, $|\hat{F}^n(t, \xi)| \leq 1$, $|\hat{G}^n(t, \xi)| \leq 1$, we can deduce by using the Gronwall inequality that there exists a positive constant $C(T) > 0$ independent of κ, n and ξ such that

$$\sup_{0 \neq \xi \in \mathbb{R}^3} \left\{ \frac{|\Phi_1^n(t, \xi)|}{|\xi|^2 (|\xi|^\delta + \kappa)} \right\} \leq C(T), \quad (3.30)$$

holds for $0 \leq t \leq T$. Here $T > 0$ is any given positive constant.

Since the positive constant $C(T) > 0$ in (3.30) is independent of κ , we have from (3.30) by letting $\kappa \rightarrow 0_+$ that

$$\sup_{0 \neq \xi \in \mathbb{R}^3} \left\{ \frac{|\Phi_1^n(t, \xi)|}{|\xi|^{2+\delta}} \right\} \leq C(T), \quad 0 \leq t \leq T. \quad (3.31)$$

With (3.31), set

$$\Phi_2^n(t, \xi) = \frac{\Phi_1^n(t, \xi)}{|\xi|^{2+\delta}}, \quad (3.32)$$

we can get from (3.24) and the fact $|\xi^\pm|^2 = \frac{|\xi|^2 \pm \xi \cdot \sigma |\xi|}{2} = \frac{|\xi|^2 (1 \pm \cos \theta)}{2}$ that

$$\begin{aligned} \partial_t \Phi_2^n + \Phi_2^n &= \frac{1}{\bar{\sigma}_n} \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left[\Phi_2^n(t, \xi^+) \hat{F}^n(t, \xi^-) \left| \cos \frac{\theta}{2} \right|^{2+\delta} \right. \\ & \quad \left. + \hat{G}^n(t, \xi^+) \Phi_2^n(t, \xi^-) \left| \sin \frac{\theta}{2} \right|^{2+\delta} \right] d\sigma + \frac{E^n(t, \xi)}{|\xi|^{2+\delta}}. \end{aligned} \quad (3.33)$$

A direct consequence of (3.33) yields

$$|\partial_t \Phi_2^n + \Phi_2^n| \leq (1 - B_n) \|\Phi_2^n(t, \cdot)\|_{L^\infty} + O(1) e^{-(A_n - \delta \lambda_\alpha^n / \alpha) t}. \quad (3.34)$$

Since $0 < B_n < 1$, we can apply the argument used in [9] to have

$$|\Phi_2^n(t, \xi)| \leq O(1) e^{-\eta_0 t}, \quad (3.35)$$

so that (3.26) follows. This completes the proof of the lemma. \square

We now turn to prove Theorem 1.1. Let $F_n(t, v)$ and $G_n(t, v)$ be the unique solutions of the Cauchy problem (3.2)-(3.3) with initial data $f_0(v)$ and $g_0(v)$ respectively, then

$$\begin{cases} f_n(t, v) \equiv F_n(\bar{\sigma}_n t, v), \\ g_n(t, v) \equiv G_n(\bar{\sigma}_n t, v), \end{cases} \quad (3.36)$$

solve

$$\begin{cases} \partial_t f_n + \bar{\sigma}_n f_n = \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\sigma \cdot (v - v_*)}{|v - v_*|} \right) f_n(v') f_n(v'_*) dv_* d\sigma, \\ f_n(0, v) = f_0(v), \end{cases} \quad (3.37)$$

and

$$\begin{cases} \partial_t g_n + \bar{\sigma}_n g_n = \int_{\mathbb{S}^2} \mathcal{B}_n \left(\frac{\sigma \cdot (v - v_*)}{|v - v_*|} \right) g_n(v') g_n(v'_*) dv_* d\sigma, \\ g_n(0, v) = g_0(v), \end{cases} \quad (3.38)$$

respectively.

The estimate (3.26) in Lemma 3.3 gives

$$\left\| \hat{F}_n(t, \cdot) - \hat{G}_n(t, \cdot) - \tilde{P}^n(t, \cdot) \right\|_{\mathcal{D}^{2+\delta}} \leq O(1) e^{-\eta_0^n t}. \quad (3.39)$$

Putting (3.36) and (3.39) together yields

$$\left\| \hat{f}_n(t, \cdot) - \hat{g}_n(t, \cdot) - \tilde{P}^n(\bar{\sigma}_n t, \cdot) \right\|_{\mathcal{D}^{2+\delta}} \leq O(1) e^{-\bar{\sigma}_n \eta_0^n t}. \quad (3.40)$$

Noticing that

$$\lim_{n \rightarrow +\infty} \bar{\sigma}_n A_n = A, \quad \lim_{n \rightarrow +\infty} \bar{\sigma}_n B_n = B, \quad \lim_{n \rightarrow +\infty} \bar{\sigma}_n \lambda_\alpha^n = \lambda_\alpha,$$

we have

$$\lim_{n \rightarrow +\infty} \bar{\sigma}_n \eta_0^n = \eta_0, \quad \lim_{n \rightarrow +\infty} \tilde{P}^n(\bar{\sigma}_n t, \xi) = \tilde{P}(t, \xi). \quad (3.41)$$

On the other hand, it is shown in [7, 11] that $(\hat{f}_n(t, \xi), \hat{g}_n(t, \xi)) \rightarrow (\hat{f}(t, \xi), \hat{g}(t, \xi))$ uniformly as $n \rightarrow +\infty$, locally in with respect to $(t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3$. By (3.40), we obtain from (3.41) that

$$\left\| \hat{f}(t, \cdot) - \hat{g}(t, \cdot) - \tilde{P}(t, \cdot) \right\|_{\mathcal{D}^{2+\delta}} \lesssim O(1) e^{-\eta_0 t}. \quad (3.42)$$

(3.42) is exactly (1.16) and thus the proof of Theorem 1.1 is completed.

4 Proof of Theorem 1.2

To prove Theorem 1.2, compared with Theorem 1.1, we only need to obtain the uniform $H^N(\mathbb{R}^3)$ -estimate (1.18) on $f(t, v)$ and the key point is to deduce the following coercivity estimate.

Lemma 4.1. *There exists a sufficiently large positive constant $t_1 > 0$ such that*

$$\int_{\mathbb{S}^2} \mathcal{B} \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(1 - \left| \hat{f}(t, \xi^-) \right| \right) d\sigma \geq \kappa e^{2s\mu_\alpha t} |\xi|^{2s}, \quad \text{if } |\xi| \geq 2, \quad (4.1)$$

holds for any $t \geq t_1$ and some positive constant $\kappa > 0$ which depends only on t_1 .

Proof. Notice that

$$\hat{\Psi}_{\alpha,K}(\xi) \in \mathcal{K}^\alpha(\mathbb{R}^3), \quad \Psi_{\alpha,K}(v) \in L^1(\mathbb{R}^3) \cap H^\infty(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \Psi_{\alpha,K}(v) dv = 1,$$

we have from Theorem 1.1 of [12] that for $1 < \beta < \alpha < 2$ that $\Psi_{\alpha,K}(v) \in \mathcal{P}^\beta(\mathbb{R}^3)$, and consequently Lemma 3 of [1] shows that there exists a positive constant $\kappa_1 > 0$ independent of t and ξ such that

$$1 - \left| \hat{\Psi}_{\alpha,K}(\xi) \right| \geq \kappa_1 \min \{1, |\xi|^2\}.$$

Hence, we have

$$1 - \left| \hat{\Psi}_{\alpha,K}(e^{\mu_\alpha t} \xi) \right| \geq \kappa_1 \min \left\{1, |e^{\mu_\alpha t} \xi|^2\right\}, \quad \forall (t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3. \quad (4.2)$$

On the other hand, we have from the $\mathcal{D}^{2+\delta}$ -stability estimate given in Theorem 1.1 that

$$\begin{aligned} \left| \hat{f}(t, \xi) - \hat{\Psi}_{\alpha,K}(e^{\mu_\alpha t} \xi) \right| &\leq O(1) |\xi|^{2+\delta} e^{-\eta_0 t} + \left| \tilde{P}(t, \xi) \right| \\ &\leq \kappa_2 (|\xi|^{2+\delta} + |\xi|^2) e^{-\eta_0 t} \end{aligned} \quad (4.3)$$

holds for any $(t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3$ with a constant $\kappa_2 > 0$ independent of t and ξ .

A direct consequence of (4.2) and (4.3) is

$$\begin{aligned} 1 - \left| \hat{f}(t, \xi) \right| &\geq \left(1 - \left| \hat{\Psi}_{\alpha,K}(e^{\mu_\alpha t} \xi) \right| \right) - \left| \hat{f}(t, \xi) - \hat{\Psi}_{\alpha,K}(e^{\mu_\alpha t} \xi) \right| \\ &\geq \kappa_1 \min \left\{1, |e^{\mu_\alpha t} \xi|^2\right\} - \kappa_2 (|\xi|^{2+\delta} + |\xi|^2) e^{-\eta_0 t}, \quad \forall (t, \xi) \in \mathbb{R}^+ \times \mathbb{R}^3. \end{aligned} \quad (4.4)$$

Thus

$$1 - \left| \hat{f}(t, \xi) \right| \geq \max \left\{0, \kappa_1 \min \left\{1, |e^{\mu_\alpha t} \xi|^2\right\} - \kappa_2 (|\xi|^{2+\delta} + |\xi|^2) e^{-\eta_0 t}\right\}. \quad (4.5)$$

With (4.5), we now turn to prove (4.1). Firstly, note that $|\xi^-|^2 = |\xi|^2 \sin^2 \frac{\theta}{2}$. If we choose $t_1 > 0$ sufficiently large such that

$$\kappa_1 e^{2\mu_\alpha t} - 2\kappa_2 e^{-\eta_0 t} \geq \frac{\kappa_1}{2} e^{2\mu_\alpha t} + \frac{\kappa_1}{2} e^{2\mu_\alpha t_1} - 2\kappa_2 e^{-\eta_0 t_1} \geq \frac{\kappa_1}{2} e^{2\mu_\alpha t}, \quad \forall t \geq t_1, \quad (4.6)$$

then for $t \geq t_1$, $|\xi| \geq 2$ and θ sufficiently small such that

$$\theta \in \left[0, \frac{2}{e^{\mu_\alpha t} |\xi|}\right] \subset \left[0, \frac{\pi}{2}\right), \quad (4.7)$$

we have

$$\begin{aligned} &\kappa_1 \min \left\{1, |e^{\mu_\alpha t} \xi^-|^2\right\} - \kappa_2 (|\xi^-|^{2+\delta} + |\xi^-|^2) e^{-\eta_0 t} \\ &\geq \kappa_1 |e^{\mu_\alpha t} \xi^-|^2 - \kappa_2 (|\xi^-|^{2+\delta} + |\xi^-|^2) e^{-\eta_0 t} \\ &\geq (\kappa_1 e^{2\mu_\alpha t} - 2\kappa_2 e^{-\eta_0 t}) |\xi^-|^2 \geq \frac{\kappa_1}{2} e^{2\mu_\alpha t} |\xi^-|^2. \end{aligned} \quad (4.8)$$

Thus for the case when $t \geq t_1$, $|\xi| \geq 2$ and θ satisfies (4.7), one can deduce from the assumption (1.4), the estimates

(4.5), (4.6) and (4.8) that

$$\begin{aligned}
& \int_{\mathbb{S}^2} \mathcal{B} \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(1 - |\hat{f}(t, \xi^-)| \right) d\sigma \\
& \geq 2\pi \int_0^{\frac{2}{e^{\mu_\alpha t} |\xi|}} \mathcal{B}(\cos \theta) (\kappa_1 e^{2\mu_\alpha t} - 2\kappa_2 e^{-\eta_0 t}) |\xi^-|^2 \sin \theta d\theta \\
& \geq \pi (\kappa_1 e^{2\mu_\alpha t}) |\xi|^2 \int_0^{\frac{2}{e^{\mu_\alpha t} |\xi|}} \mathcal{B}(\cos \theta) \sin^2 \frac{\theta}{2} \sin \theta d\theta \\
& \geq \frac{2\kappa_1 e^{2\mu_\alpha t}}{\pi^2} |\xi|^2 \int_0^{\frac{2}{e^{\mu_\alpha t} |\xi|}} \mathcal{B}(\cos \theta) \theta^3 d\theta \\
& \geq \frac{b_0 \kappa_1 e^{2\mu_\alpha t}}{\pi^2} |\xi|^2 \int_0^{\frac{2}{e^{\mu_\alpha t} |\xi|}} \theta^{1-2s} d\theta = \frac{2^{1-2s} b_0 \kappa_1 e^{2s\mu_\alpha t}}{(1-s)\pi^2} |\xi|^{2s}.
\end{aligned}$$

Here we have used the fact that $\sin \theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$. This completes the proof of the lemma. \square

With Lemma 4.1, we now deduce the uniform estimate on $f(t, v)$. Let $\varphi(t, \xi)$ be the Fourier transform of $f(t, v)$ with respect to v . For any $N \in \mathbb{N}$, let $M(\xi) = \widetilde{M}(|\xi|^2) \left(1 - X \left(\frac{|\xi|^2}{4} \right) \right)$ with $\widetilde{M}(t) = t^N$ and $X(t)$ defined as in Theorem 1.1. Multiplying (1.6) by $2M^2(\xi)\overline{\varphi}(t, \xi)$ with $\overline{\varphi}(t, \xi)$ being the complex conjugate of $\varphi(t, \xi)$ gives

$$\begin{aligned}
& \frac{d}{dt} \left(\int_{\mathbb{R}^3} |M(\xi)\varphi(t, \xi)|^2 d\xi \right) \tag{4.9} \\
& = 2 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{B} \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \Re \left\{ (\varphi(t, \xi^+) \varphi(t, \xi^-) - \varphi(t, \xi)) M^2(\xi) \overline{\varphi}(t, \xi) \right\} d\sigma d\xi \\
& = \underbrace{- \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{B} \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(|M(\xi)\varphi(t, \xi)|^2 + |M(\xi^+)\varphi(t, \xi^+)|^2 - 2\Re \left\{ \varphi(t, \xi^-) (M(\xi^+)\varphi(t, \xi^+)) \overline{M(\xi)\varphi(t, \xi)} \right\} \right) d\sigma d\xi}_{J_1} \\
& \quad - \underbrace{\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{B} \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \left(|M(\xi)\varphi(t, \xi)|^2 - |M(\xi^+)\varphi(t, \xi^+)|^2 \right) d\sigma d\xi}_{J_2} \\
& \quad - 2 \underbrace{\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{B} \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \Re \left\{ \varphi(t, \xi^-) (M(\xi) - M(\xi^+)) \varphi(t, \xi^+) \overline{M(\xi)\varphi(t, \xi)} \right\} d\sigma d\xi}_{J_3}.
\end{aligned}$$

We estimate J_1 , J_2 and J_3 term by term as follows. Since $\text{Supp } M(\xi) \subset \{\xi \in \mathbb{R}^3, |\xi| \geq 2\}$, it follows firstly from Lemma 4.1 that

$$J_1 \lesssim -e^{2s\mu_\alpha t} \int_{\mathbb{R}^3} |\xi|^{2s} |M(\xi)\varphi(t, \xi)|^2 d\xi, \tag{4.10}$$

because

$$\begin{aligned} & |M(\xi)\varphi(t, \xi)|^2 + |M(\xi^+)\varphi(t, \xi^+)|^2 - 2\Re \left\{ \varphi(t, \xi^-) (M(\xi^+)\varphi(t, \xi^+)) \overline{M(\xi)\varphi(t, \xi)} \right\} \\ & \geq (1 - |\varphi(t, \xi^-)|) \left(|M(\xi)\varphi(t, \xi)|^2 + |M(\xi^+)\varphi(t, \xi^+)|^2 \right) \\ & \geq (1 - |\varphi(t, \xi^-)|) |M(\xi)\varphi(t, \xi)|^2. \end{aligned}$$

For J_2 , if we use the change of variable $\xi \rightarrow \xi^+$ for the term $M(\xi^+)\varphi(t, \xi^+)$, the cancelation lemma (Lemma 1 of [1]) implies that

$$\begin{aligned} |J_2| &= 2\pi \left| \int_{\mathbb{R}^3} |M(\xi)\varphi(t, \xi)|^2 \left(\int_0^{\frac{\pi}{2}} \mathcal{B}(\cos \theta) \sin \theta \left(1 - \cos^{-3} \left(\frac{\theta}{2} \right) \right) d\theta \right) d\xi \right| \\ &\lesssim \int_{\mathbb{R}^3} |M(\xi)\varphi(t, \xi)|^2 d\xi. \end{aligned} \quad (4.11)$$

For J_3 , note that

$$\begin{aligned} J_3 &= \underbrace{\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{B} \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \Re \left\{ \varphi(t, \xi^-) \left(\widetilde{M}(\xi) - \widetilde{M}(\xi^+) \right) \left\{ 1 - X \left(\frac{|\xi^+|^2}{4} \right) \right\} \varphi(t, \xi^+) \overline{M(\xi)\varphi(t, \xi)} \right\} d\sigma d\xi}_{J_3^1} \\ &\quad + \underbrace{\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \mathcal{B} \left(\frac{\xi \cdot \sigma}{|\xi|} \right) \Re \left\{ \varphi(t, \xi^-) \widetilde{M}(\xi) \left(X \left(\frac{|\xi^+|^2}{4} \right) - X \left(\frac{|\xi|^2}{4} \right) \right) \varphi(t, \xi^+) \overline{M(\xi)\varphi(t, \xi)} \right\} d\sigma d\xi}_{J_3^2}. \end{aligned}$$

We estimate J_3^1 and J_3^2 separately.

For J_3^1 , since $|\xi^+|^2 = |\xi|^2 \cos^2 \frac{\theta}{2} \sim |\xi|^2$ for $\theta \in [0, \frac{\pi}{2}]$ and $|\xi|^2 - |\xi^+|^2 = |\xi|^2 \sin^2 \frac{\theta}{2}$, we have

$$\left| \widetilde{M}(\xi) - \widetilde{M}(\xi^+) \right| \lesssim \sin^2 \left(\frac{\theta}{2} \right) \widetilde{M}(\xi^+),$$

and consequently

$$\begin{aligned} |J_3^1| &\lesssim \int_{\mathbb{R}^3} \left(\int_0^{\frac{\pi}{2}} \mathcal{B}(\cos \theta) \sin \theta \sin^2 \left(\frac{\theta}{2} \right) d\theta \right) |M(\xi)\varphi(t, \xi)| \cdot |M(\xi^+)\varphi(t, \xi^+)| d\xi \\ &\lesssim \int_{\mathbb{R}^3} |M(\xi)\varphi(t, \xi)|^2 d\xi. \end{aligned} \quad (4.12)$$

Here we have used the fact that $|\varphi(t, \xi^-)| \leq 1$.

For J_3^2 , since

$$\begin{aligned} X \left(\frac{|\xi|^2}{4} \right) - X \left(\frac{|\xi^+|^2}{4} \right) &= X' \left(\frac{\eta|\xi|^2 + (1-\eta)|\xi^+|^2}{4} \right) \frac{|\xi|^2 - |\xi^+|^2}{4} \\ &= \frac{|\xi|^2 \sin^2 \left(\frac{\theta}{2} \right)}{4} X' \left(\frac{\eta|\xi|^2 + (1-\eta)|\xi^+|^2}{4} \right), \quad \eta \in [0, 1], \end{aligned}$$

and

$$|\xi^+|^2 \leq |\xi|^2 \leq 2|\xi^+|^2, \quad \text{Supp} \left\{ X' \left(\frac{|\xi|^2}{4} \right) \right\} \subset \{ \xi \in \mathbb{R}^3 : 4 \leq |\xi|^2 \leq 8 \},$$

we obtain

$$\text{Supp} \left\{ \widetilde{M}(\xi) \left(X \left(\frac{|\xi|^2}{4} \right) - X \left(\frac{|\xi^+|^2}{4} \right) \right) \right\} \subset \{ \xi \in \mathbb{R}^3 : 4 \leq |\xi|^2 \leq 16 \}.$$

Hence, there exists a constant $C_N > 0$ depending on N such that

$$\begin{aligned} |J_3^2| &\leq 4^{2N} \int_{|\xi| \leq 4} \left(\int_0^{\frac{\pi}{2}} \mathcal{B}(\cos \theta) \sin \theta \sin^2 \left(\frac{\theta}{2} \right) d\theta \right) |\varphi(t, \xi)| \cdot |\varphi(t, \xi^+)| d\xi \\ &\leq C_N \end{aligned} \quad (4.13)$$

because of $|\varphi(t, \xi)| \leq 1$. (4.12) together with (4.13) shows that there exists a $C_1 > 0$ such that

$$|J_3| \leq C_1 \int_{\mathbb{R}^3} |\xi|^{2s} |M(\xi) \varphi(t, \xi)|^2 d\xi + C_N. \quad (4.14)$$

Inserting (4.10), (4.11) and (4.14) into (4.9), we have, for another $C'_N > 0$,

$$\frac{d}{dt} \left(\int_{\mathbb{R}^3} |M(\xi) \varphi(t, \xi)|^2 d\xi \right) + \int_{\mathbb{R}^3} |M(\xi) \varphi(t, \xi)|^2 d\xi \leq C'_N,$$

which gives

$$\int_{\mathbb{R}^3} |M(\xi) \varphi(t, \xi)|^2 d\xi \leq e^{-(t-t_1)} \int_{\mathbb{R}^3} |M(\xi) \varphi(t_1, \xi)|^2 d\xi + C'_N, \quad t \geq t_1. \quad (4.15)$$

Noting $|\varphi(\xi)| \leq 1$ again, by means of (4.15) we see that for any $N \in \mathbb{N}$ there exists a $C(t_1, N) > 0$ such that

$$\sup_{t \in [t_1, \infty)} \left\{ \|f(t)\|_{H^N} \right\} \leq C(t_1, N) < +\infty, \quad \forall N \in \mathbb{N}.$$

This and (1.20) give

$$\sup_{t \in [t_1, \infty)} \left\{ \|f(t, \cdot) - f_{\alpha, K}(t, \cdot)\|_{H^N} \right\} \leq C(t_1, N) < +\infty, \quad \forall N \in \mathbb{N}, \quad t \geq t_1. \quad (4.16)$$

Moreover,

$$\begin{aligned} &\left| \hat{f}(t, \xi) - \hat{\Psi}_{\alpha, K}(e^{\mu_{\alpha} t} \xi) \right|^2 \\ &\lesssim e^{-\eta_0 t} |\xi|^{2+\delta} \left| \hat{f}(t, \xi) - \hat{\Psi}_{\alpha, K}(e^{\mu_{\alpha} t} \xi) - \tilde{P}(t, \xi) \right| + \left| \tilde{P}(t, \xi) \right|^2 \\ &\lesssim e^{-\eta_0 t} |\xi|^{2+\delta} \left| \hat{f}(t, \xi) - \hat{\Psi}_{\alpha, K}(e^{\mu_{\alpha} t} \xi) \right| + e^{-At} |\xi|^4 X(\xi) (|\xi|^\delta e^{-\eta_0 t} + e^{-At}). \end{aligned} \quad (4.17)$$

(4.16) and (4.17) yield

$$\begin{aligned} &\|f(t, \cdot) - f_{\alpha, K}(t, \cdot)\|_{H^N}^2 \\ &= \int_{\mathbb{R}^3} (1 + |\xi|^2)^N \left| \hat{f}(t, \xi) - \hat{\Psi}_{\alpha, K}(e^{\mu_{\alpha} t} \xi) \right|^2 d\xi \\ &\lesssim e^{-\eta_0 t} \int_{\mathbb{R}^3} (1 + |\xi|^2)^N |\xi|^{2+\delta} \left| \hat{f}(t, \xi) - \hat{\Psi}_{\alpha, K}(e^{\mu_{\alpha} t} \xi) \right| d\xi \\ &\quad + e^{-At} \int_{\mathbb{R}^3} X(\xi) (e^{-\eta_0 t} + e^{-At}) d\xi \lesssim e^{-\eta_0 t}. \end{aligned} \quad (4.18)$$

Here we have used the fact that

$$\begin{aligned} & \int_{\mathbb{R}^3} (1 + |\xi|^2)^N |\xi|^{2+\delta} \left| \hat{f}(t, \xi) - \hat{\Psi}_{\alpha, K}(e^{\mu_\alpha t} \xi) \right| d\xi \\ & \leq \left(\int_{\mathbb{R}^3} (1 + |\xi|^2)^{2(N+1)} |\xi|^{2(2+\delta)} \left| \hat{f}(t, \xi) - \hat{\Psi}_{\alpha, K}(e^{\mu_\alpha t} \xi) \right|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} (1 + |\xi|^2)^{-2} d\xi \right)^{\frac{1}{2}} \\ & \leq C(t_1, N), \quad t \geq t_1. \end{aligned}$$

(4.18) is exactly (1.19) and the proof of Theorem 1.2 is completed.

5 Proof of Corollary 1.1

We prove Corollary 1.1 in this last section. Firstly of all, note that Theorem 1.1 and Theorem 1.2 hold for $\alpha = 2$. The purpose of Corollary 1.1 is to have a better convergence rate in the case of finite energy.

In fact, compared with Theorem 1.1 and Theorem 1.2, the main difference is that now the initial data $f_0(v)$ is of finite energy and consequently the corresponding global solution $F_n(t, v)$ of the Cauchy problem (3.2)-(3.3) with $H_0(v) = f_{0R}(v)$ also has finite energy, i.e.

$$\int_{\mathbb{R}^3} |v|^2 F_n^R(t, v) dv = \int_{\mathbb{R}^3} |v|^2 f_{0R}(v) dv \lesssim 1 + \int_{\mathbb{R}^3} |v|^2 f_0(v) dv < +\infty. \quad (5.1)$$

With (5.1), it is straightforward to show that

$$\left| \hat{F}_n(t, \xi) - 1 \right| \leq O(1)|\xi|^2, \quad |\mu(\xi) - 1| \leq O(1)|\xi|^2. \quad (5.2)$$

Consequently, the term I_7 in (3.17) can be estimated by

$$|I_7| \leq \begin{cases} O(1)|\xi|^4 e^{-A_n t}, & |\xi| \leq 1, \\ O(1)e^{-A_n t}, & |\xi| \geq 1, \quad t \in \mathbb{R}^+. \end{cases} \quad (5.3)$$

Here, $G_n(t, v) = \mu(v)$.

Having (5.3), the proof of Corollary 1.1 is the same as the ones for Theorem 1.1 and Theorem 1.2. Thus, we omit the detail for brevity.

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