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Waiting time for a non-Newtonian polytropic filtration equation with convection ☆

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ABSTRACT

In this paper, we study the waiting time phenomena for a class of non-Newtonian polytropic filtration equation with convection. According to the nonlinearity in the diffusion and the convection terms, some necessary and sufficient conditions on the initial data are given for the existence of waiting time.

Contents

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1. Introduction

Consider a class of non-Newtonian polytropic filtration equation with convection in the following form,

$$\frac{\partial u}{\partial t} = \left(\left|(u^m)_x\right|^{p-2}(u^m)_x\right)_x + \lambda\left(u^q\right)_x, \quad (x,t) \in Q, \tag{1}$$

with initial data given by

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{2}$$

Here, $Q = \mathbb{R} \times \mathbb{R}^+$, $p > 1$, $m(p-1) > 1$, $q > 0$, $\lambda \in \mathbb{R}$. In the following discussion, $u_0(x)$ is assumed to be a non-negative and continuous function with compact support.

Eq. (1) arises from the study of a variety of diffusive phenomena, such as soil physics, fluid dynamics, combustion theory, reaction chemistry, cf. [5,4,6,16] and the references therein. In fact, this equation can be used to describe the filtration of water in a homogeneous porous medium [18], the evolution of the thickness of a two-dimensional plane ice sheet [8], transporting of thermal energy in plasma, and the Kolmogorov backward equation [12] for a statistical description of Brownian motion of a particle in a fluid.

In this paper, we will study the waiting time phenomenon of solutions to the problem (1)–(2). For this purpose, set

$$\xi_-(t) = \inf\left\{x \in \mathbb{R};\ u(x,t) > 0\right\}, \qquad \xi_+(t) = \sup\left\{x \in \mathbb{R};\ u(x,t) > 0\right\}.$$

By the well-known property of finite speed of propagation of perturbation in this kind of equations, cf. for example [19], both $\xi_-(t)$ and $\xi_+(t)$ are finite provided that the initial data have compact support under some general conditions on the parameters p, m, q as stated later. Here, $\xi_-(t)$ and $\xi_+(t)$ are called the interfaces of solutions. The appearance of the interfaces can be found in many physical situations, for example, the wetting fronts separating wet and dry regions of the porous medium in the soil-moisture infiltration; the leading edge of the fluid flow in a thin viscous film with the unknown variable representing the thickness of the film, cf. [9].

When the initial data is appropriately smooth near the boundary of the support, these interfaces may be stationary for certain time, called the waiting time. This kind of phenomena have been investigated by many authors, see for example [7,13–15]. In particular, Knerr [13], Aronson, Caffarelli and Kamin [1] and Vazquez [17] considered the waiting time for the special case when $p = 2$, $\lambda = 0$ in Eq. (1), namely the porous media equation

$$\frac{\partial u}{\partial t} = \left(u^m\right)_{xx}, \tag{3}$$

where $m > 1$. Some upper and lower bound estimates of the waiting time were also given in these works. Recently, some works have been done on the waiting time property for the equations with convection with necessary and sufficient conditions on the initial data, see for example [3,9,2,10], where the typical case of Eq. (1) when $p = 2$ is studied. Notice that such phenomenon never appears for fast diffusive equations in general because of the infinite speed of propagation which corresponds to the case for Eq. (1) with $m(p-1) \leqslant 1$ and $\lambda = 0$.

One of the motivations of the study in this paper comes from the study on the time evolution on the vacuum boundary for compressible Euler equations with damping, cf. [14]. It was known that for large time, the solution to the Euler equations with damping can be described by a nonlinear diffusion equation by applying the Darcy's law, cf. [11]. Even though there have been extensive studies on the waiting time problem for scalar diffusion equations, there is almost no theoretical results for systems. In this paper, we give a detailed description on the waiting time phenomena for the non-Newtonian polytropic filtration equation with convection. Some new phenomena about the balance between diffusion and convection is observed.

To discuss the influence of the convection on the waiting time property, we notice that the direction of the convection plays an important role. On the other hand, even though the case when $\lambda > 0$ is different from the one when $\lambda < 0$, they are just related by replacing x by $-x$. Thus, without loss of generality, it suffices to consider the case $\lambda \geqslant 0$. Notice that when $\lambda > 0$, the convection direction is in the negative direction along the x-axis. We will show that convection can be classified into three classes, the strong, mild and weak convections, according to their influence on the waiting time. For this, there are two critical values

$$q_0 = 1, \qquad q_c = \frac{(m+1)(p-1)}{p},$$

so that the three classes correspond to the following decomposition on the interval for q:

$$(0, +\infty) = (0, 1] \cup (1, q_c) \cup [q_c, +\infty).$$

It is worth mentioning that the critical value q_c also appears in finding the self-similar solution to (1). In fact, if $u(x,t)$ is a solution of (1), then for any $r > 0$, $r^\alpha u(r^\beta x, rt)$ is also the solution of Eq. (1) with $\alpha = \frac{p-1}{pq-(m+1)(p-1)}$, $\beta = \frac{m(p-1)-q}{(m+1)(p-1)-pq}$ when $q \neq \frac{(m+1)(p-1)}{p}$.

The analysis in this paper shows that for the weak convection case when $q \in [q_c, +\infty)$, the waiting time property of solutions is completely analogous to the case without convection. In this case, the necessary and sufficient condition on the initial data $u_0(x)$ for the existence of waiting time is given. For the mild convection when $q \in (1, q_c)$, the influence of the convection cannot be ignored. And the behavior of the left and right interfaces are quite different because of the direction of the convection. For the right interface, the restriction on the initial data for the existence of waiting time is weaker and it depends on m, p and q. For the left interface, since the convection dominates the diffusion, the restriction on the initial data depends only on the exponent index of the convection term in the unknown function for the existence of the waiting time. Finally, for the strong convection case when $q \in (0, 1]$, the left interface exists only when $q = 1$. On the other hand, even though the right interface always exists, it travels to either the left or right in general and the existence of waiting time appears only in some particular situations.

The rest of this paper is organized as follows. In the next section, we will prove some preliminary lemmas and propositions on the solutions. Subsequently, the weak, mild and strong convections will be discussed in the remaining three sections separately.

2. Preliminaries

Since the initial data $u_0(x)$ is a function with compact support, and we are interested in the property of interfaces, throughout this paper, we assume that

$$\operatorname{supp} u_0 = [a_1, a_2].$$

For the left interface $\xi_-(t)$ of the solution to Eq. (1), since the convection and the diffusion are in the same direction, we shall see that it will not move to the right. However, the right interface $\xi_+(t)$ may move either to left or right because the direction of the convection is opposite to that of the diffusion. In fact, both shrinking and expanding of supports are possible even for the special case $p = 2$, see [3].

Due to the degeneracy, Eq. (1) may not have classical solutions in general, and hence we consider non-negative solutions in the following weak sense.

Definition 2.1. Let

$$E = \left\{ u \in C(\overline{Q}_T);\ u^m \in W^{1,p}_{\mathrm{loc}}(Q_T) \right\}, \quad Q_T = \mathbb{R} \times (0, T).$$

A function $u \in E$ is called a local weak solution of the problem (1)–(2) provided that for some $T > 0$ and any continuously differential function φ with $\varphi(x,t) = 0$ for large $|x|$ and $t = T$, it holds

$$\iint\limits_{Q_T} u\varphi_t\,dx\,dt + \int\limits_{\mathbb{R}} u_0(x)\varphi(x,0)\,dx = \iint\limits_{Q_T} \left|\left(u^m\right)_x\right|^{p-2}\left(u^m\right)_x\varphi_x\,dx\,dt + \iint\limits_{Q_T} \lambda u^q\varphi_x\,dx\,dt.$$

We will now give several technical lemmas for potential comparison, which will be used in the later waiting time analysis.

Lemma 2.1. *Let u_1 and u_2 be two weak upper and lower solutions of Eq.* (1) *with $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively, where $\lambda_1 \geqslant \lambda_2$, $u_i(x,0) = u_{i0}(x) \in L^1(\mathbb{R})$ $(i = 1,2)$, and*

$$\int\limits_{-\infty}^{x} u_{10}(s)\,ds \geqslant \int\limits_{-\infty}^{x} u_{20}(s)\,ds \quad \text{for any } x \in \mathbb{R}. \tag{4}$$

Then for any $t > 0$, we have

$$\int\limits_{-\infty}^{x} u_1(s,t)\,ds \geqslant \int\limits_{-\infty}^{x} u_2(s,t)\,ds \quad \text{for any } x \in \mathbb{R}. \tag{5}$$

Lemma 2.2. *Let u_1 and u_2 be two weak upper and lower solutions of Eq.* (1) *with $\lambda = \lambda_1$ and $\lambda = \lambda_2$ respectively, where $\lambda_1 \leqslant \lambda_2$, $u_i(x,0) = u_{i0}(x) \in L^1(\mathbb{R})$ $(i = 1,2)$, and*

$$\int\limits_{x}^{+\infty} u_{10}(s)\,ds \geqslant \int\limits_{x}^{+\infty} u_{20}(s)\,ds \quad \text{for any } x \in \mathbb{R}. \tag{6}$$

Then for any $t > 0$, we have

$$\int\limits_{x}^{+\infty} u_1(s,t)\,ds \geqslant \int\limits_{x}^{+\infty} u_2(s,t)\,ds \quad \text{for any } x \in \mathbb{R}. \tag{7}$$

We will only prove Lemma 2.1 because the proof of Lemma 2.2 is similar.

Proof of Lemma 2.1. Without loss of generality, we may assume that $u_i(x,t) > 0$, $i = 1,2$. Otherwise, we take sequences $\{u_{i0}^n\}$ to approximate u_{i0} with $u_{i0}(x) > 0$ satisfying (4), such that the corresponding solutions $u_i^n(x,t)$ converge to $u_i(x,t)$ at least in $C([0,T]; L^1(\mathbb{R}))$.

Let

$$\omega_i(x,t) = e^{-\sigma t}\int\limits_{-\infty}^{x} u_i(s,t)\,ds, \quad i = 1,2,$$

with $\sigma > 0$. We may also assume that ω_1, ω_2 are appropriately smooth because the following equation always admits classical solutions

$$\frac{\partial\omega}{\partial t} = m^{p-1}e^{(p-2)\sigma t}u^{(m-1)(p-1)}|\omega_{xx}|^{p-2}\omega_{xx} + \lambda e^{(q-1)\sigma t}\omega_x^q - \sigma\omega,$$

provided that $u(x,t)>0$. The straightforward calculation shows that

$$\int_{-\infty}^{+\infty} u_1(s,t)\,ds \geqslant \int_{-\infty}^{+\infty} u_{10}(s)\,ds$$

and

$$\int_{-\infty}^{+\infty} u_2(s,t)\,ds \leqslant \int_{-\infty}^{+\infty} u_{20}(s)\,ds.$$

And by (4), we have

$$\lim_{x\to+\infty} \omega_1(x,t)-\omega_2(x,t) \geqslant 0. \tag{8}$$

In what follows, we shall show that (5) holds by contradiction. Suppose that (5) fails. Then there exists a point $(x^*,t^*)\in\mathbb{R}\times\mathbb{R}^+$ such that $\omega_1(x^*,t^*)<\omega_2(x^*,t^*)$. Let

$$Q_{t^*}=\mathbb{R}\times\left[0,t^*\right].$$

Then $\omega_1(x,t)-\omega_2(x,t)$ admits a minimum at $(x_0,t_0)\in Q_{t^*}$. Hence, we have

(i) $\omega_{1x}(x_0,t_0)=\omega_{2x}(x_0,t_0)>0$;
(ii) $\frac{\partial(\omega_1-\omega_2)}{\partial t}(x_0,t_0)\leqslant 0$;
(iii) $((\omega_{1x})^m-(\omega_{2x})^m)_x=m(\omega_{1x})^{m-1}(\omega_{1xx}-\omega_{2xx})\geqslant 0$.

On the other hand, note that

$$\begin{aligned}\frac{\partial(\omega_1-\omega_2)}{\partial t} \geqslant{}& e^{(m(p-1)-1)\sigma t}\left(\left|\left((\omega_{1x})^m\right)_x\right|^{p-2}\left((\omega_{1x})^m\right)_x-\left|\left((\omega_{2x})^m\right)_x\right|^{p-2}\left((\omega_{2x})^m\right)_x\right)\\&+\lambda_1 e^{(q-1)\sigma t}\omega_{1x}^q-\lambda_2 e^{(q-1)\sigma t}\omega_{2x}^q-\sigma(\omega_1-\omega_2).\end{aligned}$$

Now since at the point (x_0,t_0), the right-hand side of the above inequality is larger than 0, while the left-hand side is less than or equal to 0. This is a contradiction which implies the conclusion in the lemma. □

Remark 2.1. By a similar argument, under the same conditions in Lemma 2.1, it can be verified that the above two lemmas remain true in the domains $(-\infty,\delta]\times[0,T]$ and $[\delta,\infty)\times[0,T]$ respectively if in addition, the solutions u_1 and u_2 satisfy

$$\int_{-\infty}^{\delta} u_1(s,t)\,ds \geqslant \int_{-\infty}^{\delta} u_2(s,t)\,ds, \quad \text{for any } t\in(0,T],$$

and

$$\int_{\delta}^{+\infty} u_1(s,t)\,ds \geqslant \int_{\delta}^{+\infty} u_2(s,t)\,ds, \quad \text{for any } t\in(0,T],$$

respectively.

The following comparison between the interfaces of the upper and lower solutions follows from a simple argument so that we omit its proof for brevity.

Lemma 2.3. *If u_1 and u_2 satisfy*

$$\int_{-\infty}^{x} u_1(s,t)\,ds \geqslant \int_{-\infty}^{x} u_2(s,t)\,ds,$$

then

$$\xi_{1-}(t) \leqslant \xi_{2-}(t).$$

On the other hand, if

$$\int_{x}^{+\infty} u_1(s,t)\,ds \geqslant \int_{x}^{+\infty} u_2(s,t)\,ds,$$

then

$$\xi_{1+}(t) \geqslant \xi_{2+}(t).$$

Denote the interfaces for the case when $\lambda = 0$ by $\xi^*_\pm(t)$, and those for the case when $\lambda > 0$ by $\xi_\pm(t)$. By the above lemmas, for the same initial value $u_0(x)$, we have

$$\xi^*_-(t) \geqslant \xi_-(t), \qquad \xi^*_+(t) \geqslant \xi_+(t). \tag{9}$$

However, from the classical result on the polytropic filtration equation, cf. for example [19], we know that $\xi^*_-(t)$ is non-increasing, which means $\xi_-(t)$ is also non-increasing. In fact, suppose on the contrary, there exist t_1, t_2 with $t_1 < t_2$ such that $\xi_-(t_1) < \xi_-(t_2)$. Denote the solution when $\lambda > 0$ by $u(x,t)$, and the solution when $\lambda = 0$ by $u^*(x,t)$ with the initial value $u(x,t_1)$, then we have $\xi_-(t_1) \equiv \xi^*_-(t_1) \geqslant \xi^*_-(t_2) \geqslant \xi_-(t_2)$, which leads to a contradiction. Thus we have the following proposition.

Proposition 2.1. *Let $u(x,t)$ be a weak solution of the problem* (1)–(2). *Then for any $q > 0$, $\xi_-(t)$ is non-increasing.*

The above proposition shows that no matter the convection is either weak or strong, it does not change the monotonicity of the left interface. However, due to the direction of the convection, only for weak or mild convection, the right interface $\xi_+(t)$ can remain non-decreasing. More precisely, we have the following proposition.

Proposition 2.2. *Let $q > 1$, and $u(x,t)$ be a weak solution of the problem* (1)–(2)*. Suppose $u(x_0,0) > 0$. Then it holds that*

$$u(x_0,t) > 0, \quad \textit{for all } t \geqslant 0.$$

Proof. Without loss of generality, we may choose $x_0 = 0$. Consider the function

$$\underline{u}(x,t) = \left(\rho^{\frac{p}{p-1}} - \rho^{-r}|x|^{\frac{p}{p-1}}\right)_+^{\alpha} e^{-\sigma t},$$

where $\alpha = \max\{\frac{p}{m(p-1)-1}, \frac{1}{q-1}\} > 0$, and ρ, σ, r are positive constants to be determined later. Firstly, it is easy to see that for sufficiently small $\rho > 0$, we have

$$\underline{u}(x,0) \leqslant u_0(x), \quad \text{for all } x \in (-\infty, +\infty).$$

Straightforward calculation gives

$$\begin{aligned}
&-\frac{\partial \underline{u}}{\partial t} + \left(\left|\left(\underline{u}^m\right)_x\right|^{p-2}\left(\left(\underline{u}^m\right)_x\right)\right)_x + \lambda\left(\underline{u}^q\right)_x \\
&\geqslant \underline{u}\Bigg(\sigma - \left(\frac{\alpha m p}{(p-1)\rho^r}\right)^{p-1}\frac{(1+p(\alpha m-1))}{e^{\sigma(m(p-1)-1)t}}\left(\rho^{\frac{p}{p-1}} - \rho^{-r}|x|^{\frac{p}{p-1}}\right)_+^{\alpha(m(p-1)-1)-(p-1)} \\
&\quad + p(\alpha m-1)\left(\frac{\alpha m p}{(p-1)\rho^r}\right)^{p-1}\rho^{\frac{p}{p-1}}e^{-\sigma(m(p-1)-1)t}\left(\rho^{\frac{p}{p-1}} - \rho^{-r}|x|^{\frac{p}{p-1}}\right)_+^{\alpha(m(p-1)-1)-p} \\
&\quad - \frac{\lambda\alpha q}{p-1}\rho^{-r}|x|^{1/(p-1)}e^{-\sigma(q-1)t}\left(\rho^{\frac{p}{p-1}} - \rho^{-r}|x|^{\frac{p}{p-1}}\right)_+^{\alpha(q-1)-1}\Bigg) \\
&\geqslant \underline{u}\Bigg(\sigma - \left(\frac{\alpha m p}{(p-1)\rho^r}\right)^{p-1}\rho^{\frac{p}{p-1}}e^{-\sigma(m(p-1)-1)t}\left(\rho^{\frac{p}{p-1}} - \rho^{-r}|x|^{\frac{p}{p-1}}\right)_+^{\alpha(m(p-1)-1)-p} \\
&\quad - \frac{\lambda\alpha q}{p-1}\rho^{\frac{1}{p-1}-r\frac{p-1}{p}}e^{-\sigma(q-1)t}\left(\rho^{\frac{p}{p-1}} - \rho^{-r}|x|^{\frac{p}{p-1}}\right)_+^{\alpha(q-1)-1}\Bigg).
\end{aligned}$$

Take $r = \frac{p}{(p-1)^2}$. Noticing that $\alpha(m(p-1)-1)-p, \alpha(q-1)-1 > 0$, then for any $T > 0$, when ρ is sufficiently small and σ is sufficiently large, we have

$$-\frac{\partial \underline{u}}{\partial t} + \left(\left|\left(\underline{u}^m\right)_x\right|^{p-2}\left(\left(\underline{u}^m\right)_x\right)\right)_x + \lambda\left(\underline{u}^q\right)_x \geqslant 0.$$

By comparison, we obtain that $u(x,t) \geqslant \underline{u}(x,t)$ for $t \in [0,T]$. The proof is completed. □

From the above propositions, we see that when $q > 1$, $\xi_+(t)$ is non-decreasing.

In the following three sections, we will study the waiting time property by using Lemmas 2.1–2.3. In addition, it is worth to point out that these lemmas can also be used for the fast diffusion equations. For example, they can be used to show that the generalized Burgers equation

$$\frac{\partial u}{\partial t} = \left(u^m\right)_{xx} - u u_x, \quad m \leqslant 1, \tag{10}$$

has no waiting time. In fact, the above equation has the same property as the fast diffusion porous media equation, that is, for any non-trivial initial data $u_0(x) \geqslant 0$, we have $u(x,t) > 0$ for any $x \in \mathbb{R}$ as long as $t > 0$. In fact, after replacing x by $-x$ in the above equation, the generalized Burgers equation becomes

$$\frac{\partial u}{\partial t} = \left(u^m\right)_{xx} + u u_x, \quad m \leqslant 1. \tag{11}$$

If the interfaces exist, from Lemma 2.1, the left interface of the above equation should be on the left of the left interface of the corresponding pure diffusion equation. Since it is well known that there is no interface for the fast diffusion equation, this implies that the Burgers equation has no right interface. As for the left interface, let us consider the following equation

$$\frac{\partial u}{\partial t} = \left(u^m\right)_{xx} + Mu_x, \quad m \leqslant 1. \tag{12}$$

Let $v(x,t) = u(x - Mt, t)$. Then we have

$$\frac{\partial v}{\partial t} = \left(v^m\right)_{xx}.$$

Thus, Eq. (12) has no interface. However, there always exists $M > 0$ such that $u(x,t) < M$ for $t \in (0, t^*)$ for some $t^* > 0$. Then, by comparing with Eq. (12), we know that the solution to Eq. (11) has no right interface, which implies that this also holds for Eq. (10).

3. Weak convection when $q \geqslant q_c$

In this section, we study the interfaces for the case when $q \geqslant q_c$. For this case, the convection is so weak that the waiting time property will not be influenced by convection, namely, the diffusion will dominate the convection. As mentioned in the introduction, we denote

$$q_c = \frac{(m+1)(p-1)}{p}.$$

Theorem 3.1. *Let $u(x,t)$ be a weak solution of the problem* (1)–(2). *Then there exists a waiting time $T > 0$ such that $\xi_-(t) = \xi_-(0)$ for all $t \in (0, T]$, if and only if*

$$\lim_{x \to \xi_-(0)^+} \left|x - \xi_-(0)\right|^{(m+1)(p-1)/(1-m(p-1))} \int_{-\infty}^{x} u_0(s)\, ds < \infty. \tag{13}$$

Proof. Without loss of generality, we may assume that $\xi_-(0) = 0$. We first show that the condition is sufficient. Denote

$$M_0 = \int_{-\infty}^{+\infty} u_0(s)\, ds, \qquad M_1 = \left(\lim_{x \to 0^+} |x|^{(m+1)(p-1)/(1-m(p-1))} \int_{-\infty}^{x} u_0(s)\, ds\right) + 1.$$

By (13), there exists a constant δ with $0 < \delta < 1$ such that

$$\int_{-\infty}^{x} u_0(s)\, ds \leqslant M_1 x_+^{(m+1)(p-1)/(m(p-1)-1)}, \quad \text{for } x \in (-\infty, \delta]. \tag{14}$$

In what follows, we shall complete the proof by constructing an upper solution. Let

$$\bar{u}(x,t) = \begin{cases} (\frac{T}{T-t})^{\beta} A x^{\alpha}, & (x,t) \in [0, \delta] \times [0, T], \\ 0, & (x,t) \in (-\infty, 0) \times [0, T], \end{cases}$$

where $\alpha = \frac{p}{m(p-1)-1}$, $\beta = \min\{\frac{1}{m(p-1)-1}, \frac{1}{q-1}\}$ and T, A are to be determined. Then

$$\frac{\partial \bar{u}}{\partial t} - \left(\left|\left(\bar{u}^m\right)_x\right|^{p-2}\left(\bar{u}^m\right)_x\right)_x - \lambda\left(\bar{u}^q\right)_x$$
$$\geqslant Ax^\alpha\left(\frac{T}{T-t}\right)^{\beta+1}\left(\beta T^{-1} - \frac{(m+1)(p-1)(mp)^{p-1}}{(m(p-1)-1)^p}A^{m(p-1)-1} - \frac{\lambda pqA^{q-1}}{m(p-1)-1}x^{\alpha(q-1)-1}\right).$$

Notice that $q \geqslant \frac{(m+1)(p-1)}{p}$ implies $\alpha(q-1)-1 \geqslant 0$. Let

$$T = \beta\left(\frac{(m+1)(p-1)(mp)^{p-1}}{(m(p-1)-1)^p}A^{m(p-1)-1} + \frac{\lambda pqA^{q-1}}{m(p-1)-1}\delta^{\alpha(q-1)-1}\right)^{-1}.$$

Then one has

$$\frac{\partial \bar{u}}{\partial t} - \left(\left|\left(\bar{u}^m\right)_x\right|^{p-2}\left(\bar{u}^m\right)_x\right)_x - \lambda\left(\bar{u}^q\right)_x \geqslant 0.$$

Furthermore, we see that

$$\int_{-\infty}^{x} \bar{u}(s,0)\,ds \geqslant \int_{-\infty}^{x} As^\alpha\,ds = \frac{Ax_+^{(m+1)(p-1)/(m(p-1)-1)}}{\alpha+1},$$
$$\int_{-\infty}^{\delta} \bar{u}(s,t)\,ds \geqslant \frac{A\delta^{(m+1)(p-1)/(m(p-1)-1)}}{\alpha+1}.$$

By taking

$$A = \max\left\{M_1(\alpha+1), M_0(\alpha+1)\delta^{-(m+1)(p-1)/(m(p-1)-1)}\right\},$$

we have

$$\int_{-\infty}^{x} \bar{u}(s,0)\,ds \geqslant \int_{-\infty}^{x} u_0(s)\,ds, \qquad \int_{-\infty}^{\delta} \bar{u}(s,t)\,ds \geqslant M_0 \geqslant \int_{-\infty}^{\delta} u(s,t)\,ds.$$

By Lemma 2.1 and Remark 2.1, we conclude that

$$\int_{-\infty}^{x} u(s,t)\,ds \leqslant \int_{-\infty}^{x} \bar{u}(s,t)\,ds, \quad \text{for any } (x,t) \in (-\infty,\delta] \times [0,T].$$

Thus, by Lemma 2.3, we arrive at $\xi_-(t) \geqslant 0$ for $t \in [0,T]$, which implies that $\xi_-(t) = 0$ for $t \in [0,T]$ because $\xi_-(t)$ is non-increasing.

We now turn to show that the condition is necessary. Assume that (13) does not hold, that is the limit goes to infinity. For any given $C > 0$, there exists $\delta(C) > 0$ such that

$$\int_{-\infty}^{x} u_0(s)\,ds > Cx_+^{(m+1)(p-1)/(m(p-1)-1)}, \quad \text{for } x \in (-\infty,\delta). \tag{15}$$

Let

$$\underline{u} = \big(K^p t + K(M_1 - t)(x - \hat{x})\big)_+^r,$$

where $r = (p-1)/(m(p-1)-1)$, $\hat{x} \in (0, \delta)$ is arbitrary, and K, M_1 are to be determined. We first note that

$$\int_{-\infty}^{x} \underline{u}(s,0)\,ds = \frac{1}{r+1} K^r M_1^r (x - \hat{x})_+^{r+1}.$$

Take $K = \frac{C^{1/r}(r+1)^{1/r}}{M_1} |\hat{x}|^{1/(p-1)}$. Then we have

$$\int_{-\infty}^{x} \underline{u}(s,0)\,ds \leqslant \int_{-\infty}^{x} u_0(s)\,ds, \quad \text{for } x \in (-\infty, \delta].$$

Furthermore, we choose $\hat{x}$ sufficiently small. Given τ with $0 < \tau < \min\{1, \frac{M_1}{2}\}$, then for any $t \in [0, \tau]$, we have

$$\int_{-\infty}^{\delta} \underline{u}(s,t)\,ds = \frac{1}{K(r+1)(M_1 - t)} \big[K^p t + K(M_1 - t)(\delta - \hat{x})\big]_+^{r+1} \leqslant \int_{-\infty}^{\delta} u(s,t)\,ds.$$

Direct calculation gives

$$\begin{aligned}\underline{u}_t - \big(\big|(\underline{u}^m)_x\big|^{p-2}(\underline{u}^m)_x\big)_x &= Kr\big(K^p t + K(M_1 - t)(x - \hat{x})\big)_+^{((1-m)(p-1)+1)/(m(p-1)-1)} \\ &\quad \times \big(K^{p-1} - (x - \hat{x}) - (mr)^{p-1} K^{p-1} |M_1 - t|^p\big).\end{aligned}$$

It suffices to consider the case $x > -\frac{2k^{p-1}}{M_1}$ because $\underline{u} \equiv 0$ when $x \leqslant -\frac{2k^{p-1}}{M_1}$. For simplicity, we denote

$$I_1 = K^{p-1} - (x - \hat{x}) - (mr)^{p-1} K^{p-1} |M_1 - t|^p.$$

Then we have

$$\begin{aligned} I_1 &\leqslant K^{p-1} + \frac{2}{M_1} K^{p-1} + \hat{x} - (rm)^{p-1} K^{p-1} (M_1/2)^p \\ &= \left(1 + \frac{2}{M_1} + \left(\frac{M_1}{(C(r+1))^{1/r}}\right)^{p-1} - (rm)^{p-1}(M_1/2)^p\right) K^{p-1}. \end{aligned}$$

By choosing an appropriately large M_1, we obtain $I_1 \leqslant 0$, namely

$$\underline{u}_t - \big(\big|(\underline{u}^m)_x\big|^{p-2}(\underline{u}^m)_x\big)_x \leqslant 0.$$

By the above estimates and by using Lemma 2.1, we obtain

$$\int_{-\infty}^{x} \underline{u}(s,t)\,ds \leqslant \int_{-\infty}^{x} u(s,t)\,ds, \quad \text{for } x \in (-\infty, \delta].$$

Lemma 2.3 implies that $\xi_-(t) \leqslant \underline{\xi}_-(t,\hat{x})$. Here

$$\underline{\xi}_-(t,\hat{x}) = \inf\{x; \underline{u}(x,t,K(\hat{x}),\hat{x}) > 0\}.$$

Then one has

$$\underline{\xi}_-(t,\hat{x}) = \hat{x} - \frac{K^{p-1}t}{M_1 - t} = \left(\left(\frac{M_1}{C^{1/r}(r+1)^{1/r}}\right)^{p-1} - \frac{t}{M_1 - t}\right)K^{p-1}.$$

For any small $t > 0$, choose C to be sufficiently large such that

$$\underline{\xi}_-(t,\hat{x}) < 0.$$

Therefore $\xi_-(t) < 0$ holds and this completes the proof of the theorem. □

Theorem 3.2. *Let $u(x,t)$ be a weak solution of the problem* (1)–(2)*. Then there exists a waiting time $t^* > 0$ such that $\xi_+(t) = \xi_+(0)$ for all $t \in [0,t^*]$ if and only if*

$$\lim_{x\to\xi_+(0)^-} \left|x - \xi_+(0)\right|^{(m+1)(p-1)/(1-m(p-1))} \int_x^{+\infty} u_0(s)\,ds < \infty. \tag{16}$$

Proof. By the result of [15], we see that there is a waiting time t^* for $\lambda = 0$, if (16) holds. While by Lemmas 2.2 and 2.3, we know that

$$\xi_+(t) \leqslant \xi_+^*(t),$$

which implies that $\xi_+(t) = 0$ for $t \in [0,t^*]$ because $\xi_+(t)$ is non-decreasing. In what follows, it suffices to show that the condition (16) is necessary. Without loss of generality, we still assume that $\xi_+(0) = 0$. Suppose that

$$\lim_{x\to\xi_+(0)^-} \left|x - \xi_+(0)\right|^{(m+1)(p-1)/(1-m(p-1))} \int_x^{+\infty} u_0(s)\,ds = \infty.$$

Then for any $C > 0$, there exists $\delta(C) > 0$ such that

$$\int_x^{\infty} u_0(x) > C|x|^{(m+1)(p-1)/(m(p-1)-1)}, \quad \text{for any } x \in [-\delta, 0].$$

For any $\hat{x} \in (-\delta, 0)$, let

$$\underline{u} = \left(K^p t - K(M_1 - M_2 t)(x - \hat{x})\right)_+^r, \quad (x,t) \in [-\delta, +\infty) \times [0,T],$$

where $T < \min\{1, \frac{M_1}{2M_2}\}$ and $r = \frac{p-1}{m(p-1)-1}$. Note that

$$\underline{u}_t - \left(\left|(\underline{u}^m)_x\right|^{p-2}(\underline{u}^m)_x\right)_x - \lambda\left(\underline{u}^q\right)_x \leqslant 0$$

is equivalent to

$$I_2 = K^{p-1} + M_2(x-\hat{x}) - (mr)^{p-1}K^{p-1}|M_1 - M_2t|^p$$
$$+ \lambda q\big(K^pt - K(M_1 - M_2t)(x-\hat{x})\big)_+^{(q-1)r}(M_1 - M_2t) \leqslant 0.$$

It is easy to verify that $(q-1)r \geqslant \frac{p-1}{p}$. Choose M_1 sufficiently large such that $(mr)^{p-1}(M_1/2)^p > 2$. When $x - \hat{x} \leqslant -K^{p-1}$, by noticing that $x \geqslant -\delta$, then $|x-\hat{x}| < 2\delta$, and we have

$$I_2 \leqslant -M_2|x-\hat{x}| + \lambda q M_1\big((M_1+t)|x-\hat{x}|^{p/(p-1)}\big)^{r(q-1)}$$
$$\leqslant |x-\hat{x}|\big(c\lambda q M_1(M_1+1)^{r(q-1)} - M_2\big),$$

where $c = (2\delta)^{\frac{rp(q-1)}{(p-1)}-1}$. Let $M_2 = c\lambda q M_1(M_1+1)^{r(q-1)}$. We have $I_2 \leqslant 0$.

When $x - \hat{x} > -K^{p-1}$, also note that $\underline{u} \equiv 0$ if $x - \hat{x} \geqslant \frac{2K^{p-1}}{M_1}$. Thus, we only need to consider the case when $x - \hat{x} < \frac{2K^{p-1}}{M_1}$. Recalling $(q-1)r \geqslant \frac{p-1}{p}$, for this, it is easy to see that if $K^p(M_1+1) < 1$,

$$I_2 \leqslant \left(1 + \frac{2M_2}{M_1} - (mr)^{p-1}(M_1/2)^p\right)K^{p-1} + \lambda q M_1\big(K^p(M_1+t)\big)^{r(q-1)}$$
$$\leqslant \left(1 + \frac{2M_2}{M_1} - (mr)^{p-1}(M_1/2)^p\right)K^{p-1} + \lambda q M_1 K^{p-1}(M_1+1)^{(p-1)/p}$$
$$\leqslant \left(1 + \frac{2M_2}{M_1} - (mr)^{p-1}(M_1/2)^p + \lambda q M_1(M_1+1)^{(p-1)/p}\right)K^{p-1}.$$

By taking M_1 sufficiently large, the above inequality implies $I_2 \leqslant 0$. Furthermore, we have

$$\int_x^{+\infty} \underline{u}_0(s)\,ds = \frac{1}{r+1}(KM_1)^r(\hat{x}-x)_+^{r+1}.$$

Take

$$K = \frac{1}{M_1}\big((r+1)C\big)^{1/r}|\hat{x}|^{1/(p-1)}.$$

Then when M_1 is sufficiently large, $K^p(M_1+1) < 1$. Namely $I_2 \leqslant 0$. On the other hand, for $x \in [-\delta, 0]$, we have

$$\int_x^{+\infty} \underline{u}_0(s)\,ds \leqslant C|x|^{(m+1)(p-1)/(m(p-1)-1)} \leqslant \int_x^{+\infty} u_0(s)\,ds.$$

Moreover, if $\hat{x}$ is small enough, we also have

$$\int_{-\delta}^{+\infty} \underline{u}(s,t)\,ds = \frac{1}{K(r+1)(M_1 - M_2t)}\big(K^pt + K(M_1 - M_2t)(\hat{x}+\delta)\big)^{r+1} \leqslant \int_{-\delta}^{+\infty} u(s,t)\,ds.$$

By Lemmas 2.2, 2.3 and Remark 2.1, we can conclude that

$$\xi_+(t) \geqslant \underline{\xi}_+(t,\hat{x}).$$

It is easy to check that

$$\underline{\xi}_+(t,\hat{x}) = \hat{x} + \frac{K^{p-1}t}{M_1 - M_2 t} = \left(\frac{((r+1)C)^{m(p-1)-1}t}{M_1^{p-1}(M_1 - M_2 t)} - 1\right)|\hat{x}|.$$

Then for small $t > 0$, there exists a sufficiently large $C > 0$, such that $\underline{\xi}_+(t,\hat{x}) > 0$. And this implies that

$$\xi_+(t) > 0,$$

which completes the proof of the theorem. □

Remark 3.1. Since in the weak convection case, the convection has no effect on the existence of waiting time, the proofs and the results in Theorems 3.1 and 3.2 hold when $\lambda = 0$, namely, for the equation

$$\frac{\partial u}{\partial t} = \left(\left|(u^m)_x\right|^{p-2}(u^m)_x\right)_x, \quad (x,t) \in Q.$$

In fact, the two theorems imply that the waiting time exists if and only if there exist some constants $b, \delta > 0$ such that

$$u_0(x) \leqslant b\left|\xi_\pm(0) - x\right|^\gamma \quad \text{for } \xi_+ - \delta \leqslant x \leqslant \xi_+, \text{ or } \xi_-(0) \leqslant x \leqslant \xi_- + \delta,$$

for $\gamma = \frac{p}{m(p-1)-1}$. And this agrees with Knerr's results for the porous medium equation, namely, the case $p = 2$, $\lambda = 0$ [13].

4. Mild convection when $q_0 < q < q_c$

In this section, we turn to consider the mild convection case when $1 < q < q_c$. We will show that the convection plays an important role on the interfaces. In particular, the influence of the convection on the left interface and the right interface is different because of the direction of the convection.

Firstly, for the left interface $\xi_-(t)$, the following theorem shows that the existence of waiting time depends only on the convection exponent q.

Theorem 4.1. *Let $u(x,t)$ be a weak solution of the problem* (1)–(2)*. Then there exists a waiting time $T > 0$ such that $\xi_-(t) = \xi_-(0)$ for all $t \in (0, T]$ if and only if*

$$\lim_{x \to \xi_-(0)^+} \left|x - \xi_-(0)\right|^{q/(1-q)} \int_{-\infty}^{x} u_0(s)\,ds < \infty. \tag{17}$$

Proof. Here, we still assume $\xi_-(0) = 0$. To show that the condition (17) is sufficient, let

$$\widehat{M} = \int_{-\infty}^{+\infty} u_0(s)\,ds,$$

$$M^* = 1 + \lim_{x \to \xi_-(0)^+} \left|x - \xi_-(0)\right|^{q/(1-q)} \int_{-\infty}^{x} u_0(s)\,ds.$$

If (17) holds, then there exists $\delta > 0$ such that

$$\int_{-\infty}^{x} u_0(s)\,ds \leqslant M^* x_+^{q/(q-1)}, \quad \text{for any } x \in (-\infty, \delta]. \tag{18}$$

Let

$$\bar{u}(x,t) = (M_1 - M_2 t)^{\frac{1}{1-m(p-1)}} x_+^{1/(q-1)}, \quad (x,t) \in (-\infty, \delta] \times [0, T],$$

where $T = \frac{M_1}{2M_2}$. Then

$$\frac{\partial \bar{u}}{\partial t} - \left(\left|(\bar{u}^m)_x\right|^{p-2}(\bar{u}^m)_x\right)_x - \lambda(\bar{u}^q)_x \geqslant 0$$

is guaranteed by

$$I_3 = \frac{M_2}{m(p-1)-1} - \frac{(m+1-q)(p-1)m^{p-1}}{(q-1)^p} x_+^{\frac{m(p-1)-1}{q-1} - p} - \frac{q\lambda}{q-1} M_1^{\frac{m(p-1)-q}{m(p-1)-1}} \geqslant 0.$$

By noticing that

$$\frac{m(p-1)-1}{q-1} - p \geqslant 0,$$

it follows that $I_3 \geqslant 0$ for fixed $M_1 > 0$ if we choose M_2 sufficiently large. In addition, by

$$\int_{-\infty}^{x} \bar{u}(s,0)\,ds = \frac{q-1}{q} M_1^{-1/(m(p-1)-1)} x_+^{q/(q-1)},$$

and by recalling (18), we see that if $M_1 \leqslant (\frac{qM^*}{q-1})^{1-m(p-1)}$, then for any $x \in (-\infty, \delta]$,

$$\int_{-\infty}^{x} \bar{u}(s,0)\,ds \geqslant M^* x_+^{q/(q-1)} \geqslant \int_{-\infty}^{x} u_0(s)\,ds. \tag{19}$$

Moreover, we have

$$\int_{-\infty}^{\delta} \bar{u}(s,t)\,ds \geqslant \frac{q-1}{q} M_1^{-1/(m(p-1)-1)} \delta^{q/(q-1)} \geqslant \widehat{M} \geqslant \int_{-\infty}^{\delta} u(s,t)\,ds, \tag{20}$$

for any $t \in (0, T]$, if $M_1 \leqslant (\frac{q-1}{\widehat{M}q}\delta^{q/(q-1)})^{m(p-1)-1}$. Take

$$M_1 = \min\left\{\left(\frac{qM^*}{q-1}\right)^{1-m(p-1)}, \left(\frac{q-1}{\widehat{M}q}\delta^{q/(q-1)}\right)^{m(p-1)-1}\right\}.$$

Then (19) and (20) hold. By combining this with Lemma 2.1 and Remark 2.1, we arrive at

$$\int_{-\infty}^{x} \bar{u}(s,t)\,ds \geqslant \int_{-\infty}^{x} u(s,t)\,ds, \quad \text{for any } (x,t) \in (-\infty,\delta] \times [0,T].$$

By Lemma 2.3, we then have

$$\xi_-(t) = 0, \quad \text{for all } t \in [0,T].$$

It remains to show the condition (17) is necessary. Suppose that $\frac{\int_{-\infty}^{x} u_0(s)\,ds}{x^{q/(q-1)}} \to +\infty$ as $x \to 0^+$. Then for any $C > 0$, there exists $\delta > 0$ such that for any $x \in [-\infty,\delta]$,

$$\int_{-\infty}^{x} u_0(s)\,ds \geqslant C x_+^{q/(q-1)}.$$

Here, we introduce a family of traveling wave solutions $v(x,t,K,\hat{x})$ defined by

$$v(x,t,K,\hat{x}) = v_k\big((x+Kt-\hat{x})_+\big) = v_k(y), \quad (x,t) \in \mathbb{R} \times [0,1],$$

where $\hat{x} \in (0,\delta)$. Here, v_k satisfies

$$K v_k(y) = \big|\big(v_k^m\big)'\big|^{p-2}\big(v_k^m\big)'(y) + \lambda v_k^q(y).$$

That is,

$$\big(v_k^m\big)'(y) = \big|v_k\big(K-\lambda v_k^{q-1}\big)\big|^{\hat{p}-2} v_k\big(K-\lambda v_k^{q-1}\big),$$

where $\hat{p}$ satisfies $\frac{1}{p} + \frac{1}{\hat{p}} = 1$. This implies that $v_k(y) \leqslant (\frac{K}{\lambda})^{1/(q-1)}$ if $v_k(0) \leqslant (\frac{K}{\lambda})^{1/(q-1)}$. Indeed, $v_k(y) > (\frac{K}{\lambda})^{1/(q-1)}$ is impossible. Otherwise, there must exist y^*, such that $v_k(y^*) = (\frac{K}{\lambda})^{1/(q-1)}$, and $v_k(y) > (\frac{K}{\lambda})^{1/(q-1)}$ for $y^* < y < y^* + \delta$ with $\delta > 0$ sufficiently small. Then there exists $\tilde{y} \in (y^*, y^*+\delta)$ such that $(v_k^m)'(\tilde{y}) \geqslant 0$, while by the equation v_k satisfied, we see that $(v_k^m)'(y) < 0$ for $y \in (y^*, y^*+\delta)$. And this is a contradiction. In addition, we have

$$\int_{0}^{v_k(y)} \frac{m s^{m-1}}{|Ks-\lambda s^q|^{\hat{p}-2}(Ks-\lambda s^q)}\,ds = y.$$

Thus,

$$\int_{-\infty}^{x} v(s,0,K,\hat{x})\,ds = \int_{\hat{x}}^{x} v_k\big((s-\hat{x})_+\big)\,ds \leqslant \left(\frac{K}{\lambda}\right)^{1/(q-1)} (x-\hat{x})_+.$$

Let $K = \lambda C^{q-1}|\hat{x}|$. Then for any $x \in (-\infty,\delta]$, we have

$$\int_{-\infty}^{x} v(s,0,K,\hat{x})\,ds \leqslant C|x|^{q/(q-1)} \leqslant \int_{-\infty}^{x} u_0(s)\,ds.$$

On the other hand, if $|\hat{x}|$ is small enough, we also have for any $t \in [0, 1]$,

$$\int_{-\infty}^{\delta} v\big(s,t,K,|\hat{x}|\big)\,ds \leqslant \int_{\hat{x}-Kt}^{\delta} \|v\|_\infty\,ds \leqslant (\delta + Kt - \hat{x})\left(\frac{K}{\lambda}\right)^{1/(q-1)}$$

$$\leqslant C(\delta + K)|\hat{x}|^{1/(q-1)} \leqslant \int_{-\infty}^{\delta} u(s,t)\,ds.$$

Then by Lemma 2.1 and Remark 2.1, we have

$$\int_{-\infty}^{x} v(s,t,K,\hat{x})\,ds \leqslant \int_{-\infty}^{x} u(s,t)\,ds, \quad (x,t) \in (-\infty,\delta] \times [0,1].$$

Hence when $t \in [0,1]$, $\xi_-(t) \leqslant \underline{\xi}_-(t,\hat{x})$ holds. Also note that

$$\underline{\xi}_-(t,\hat{x}) = \big(1 - \lambda C^{q-1}t\big)|\hat{x}|.$$

Then for any small $t > 0$, there exists large enough C such that $\underline{\xi}_-(t,\hat{x}) < 0$. Thus, $\xi_-(t) < 0$. And this completes the proof of the theorem. □

Next, we turn to consider the right interface $\xi_+(t)$. The following theorem shows that the convection has influence on the right interface.

Theorem 4.2. *Let $u(x,t)$ be a weak solution of the problem* (1)–(2)*. Denote*

$$\widehat{M}(q,u_0) \equiv \lim_{x\to\xi_+(0)^-} \big|x - \xi_+(0)\big|^{((m+1)(p-1)-q)/(q-m(p-1))} \int_{x}^{+\infty} u_0(s)\,ds. \tag{21}$$

Then there exists a critical value

$$\widehat{M}_c(q) = \left(\lambda\left(\frac{m(p-1)-q}{m(p-1)}\right)^{p-1}\right)^{\frac{1}{m(p-1)-q}} \frac{m(p-1)-q}{(m+1)(p-1)-q},$$

such that

(i) *if $\widehat{M}(q,u_0) < \widehat{M}_c(q)$, then there exists $t^* > 0$ such that $\xi_+(t) = \xi_+(0)$ for all $t \in [0,t^*]$;*
(ii) *if $\widehat{M}_c(q) < \widehat{M}(q,u_0) \leqslant +\infty$, then there exist $\widetilde{C}, \tilde{t} > 0$ such that*

$$\xi_+(t) \geqslant \xi_+(0) + \widetilde{C}t^{\frac{m(p-1)-q}{(m+1)(p-1)-qp}} \quad \textit{for any } t \in [0,\tilde{t}].$$

Proof. Without loss of generality, we still assume that $\xi_+(0) = 0$.

(i) Let $v(x,t) = C^*(-x)_+^{\alpha}$, where $\alpha = \frac{p-1}{m(p-1)-q}$, $C^* = (\lambda(\frac{m(p-1)-q}{m(p-1)})^{p-1})^{\frac{1}{m(p-1)-q}}$. v is a stationary solution of Eq. (1). By (21) and the continuity of u, there exist $\delta > 0$, $t^* > 0$ such that for any $x \in [-\delta, +\infty)$, $t \in [0,t^*]$

$$\int_x^{+\infty} u_0(s)\,ds < C^* \frac{m(p-1)-q}{(m+1)(p-1)-q}(-x)_+^{((m+1)(p-1)-q)/(m(p-1)-q)},$$

$$\int_{-\delta}^{+\infty} u(s,t)\,ds \leqslant C^* \frac{m(p-1)-q}{(m+1)(p-1)-q}\delta_+^{((m+1)(p-1)-q)/(m(p-1)-q)}.$$

Then we have

$$\int_x^{+\infty} v(s,0)\,ds = \frac{C^*}{\alpha+1}(-x)_+^{\alpha+1} \geqslant \int_x^{+\infty} u_0(s)\,ds$$

and

$$\int_{-\delta}^{+\infty} v(s,t)\,ds = \frac{C^*}{\alpha+1}\delta_+^{\alpha+1} \geqslant \int_{-\delta}^{+\infty} u(s,t)\,ds.$$

Thus we deduce

$$\int_x^{+\infty} v(s,t)\,ds \geqslant \int_x^{+\infty} u(s,t)\,ds, \quad \text{for any } (x,t) \in [-\delta,+\infty) \times \big[0,t^*\big],$$

which implies that

$$\xi_+(t) = 0, \quad \text{for any } t \in \big[0,t^*\big].$$

Note that $v(x,t) \equiv 0$ for any $x \geqslant 0$. Then (i) holds.

(ii) By the condition given in (ii), there exists $\delta > 0$ such that for any $x \in [-\delta,+\infty)$,

$$\int_x^{+\infty} u_0(s)\,ds \geqslant C^* \frac{m(p-1)-q}{(m+1)(p-1)-q}(-x)_+^{((m+1)(p-1)-q)/(m(p-1)-q)}.$$

In what follows, we construct another family of traveling wave solutions. For any $\hat{x} \in (-\delta,0)$, set

$$v(x,t,K,\hat{x}) = v_k\big(\big(Kt - K^\alpha(x-\hat{x})\big)_+\big),$$

where $0 < \alpha < 1/p$ is a constant. Note that

$$Kv_k = K^{\alpha p}\big|\big(v_k^m\big)'\big|^{p-2}\big(v_k^m\big)' - K^\alpha \lambda v_k^q,$$

that is,

$$\big(v_k^m\big)' = \big(K^{1-\alpha p}v_k + K^{-\alpha(p-1)}\lambda v_k^q\big)^{1/(p-1)}.$$

And notice that

$$y = \int_0^{v_k(y)} \frac{ms^{m-1}}{(K^{1-\alpha p}s + K^{-\alpha(p-1)}\lambda s^q)^{1/(p-1)}}\,ds. \tag{22}$$

Now we want to prove

$$\int_x^{+\infty} v(s,0,K,\hat{x})\,ds \leqslant \int_x^{+\infty} u_0(s)\,ds, \quad \text{for } x \in [-\delta, +\infty).$$

For this, it suffices to consider the case when $-\delta \leqslant x < \hat{x}$ because $v(s,0,K,\hat{x}) \equiv 0$ if $x \geqslant \hat{x}$. Notice that the above inequality holds if

$$(\hat{x} - x)_+ \leqslant \int_0^{C^*|x|^{\frac{p-1}{m(p-1)-q}}} \frac{ms^{m-1-\frac{1}{p-1}}}{(K^{1-\alpha} + \lambda s^{q-1})^{1/(p-1)}}\,ds. \tag{23}$$

In fact, (23) implies

$$v_k\big(k^{\alpha}(\hat{x} - x)_+\big) \leqslant C^*|x|^{\frac{p-1}{m(p-1)-q}},$$

which gives

$$\int_x^{+\infty} v(s,0,K,\hat{x})\,ds = \int_x^{\hat{x}} v_k\big(K^{\alpha}(\hat{x} - s)\big)\,ds \leqslant C|x|^{((m+1)(p-1)-q)/(m(p-1)-q)} \leqslant \int_x^{+\infty} u_0(s)\,ds.$$

We now turn to prove (23). For simplicity, denote the right-hand side of (23) by I_4 which satisfies

$$\begin{aligned} I_4 &\geqslant \int_{\eta}^{C^*|x|^{\frac{p-1}{m(p-1)-q}}} \frac{ms^{m-1-\frac{q}{p-1}}}{(K^{1-\alpha}\eta^{1-q} + \lambda)^{1/(p-1)}}\,ds \\ &= \frac{m(p-1)}{m(p-1)-q} \frac{1}{(K^{1-\alpha}\eta^{1-q} + \lambda)^{1/(p-1)}} \big(C^{*\frac{m(p-1)-q}{p-1}}|x| - \eta^{\frac{m(p-1)-q}{p-1}}\big). \end{aligned}$$

Let $\sigma = K^{1-\alpha}\eta^{1-q}$. Direct calculation shows that when σ is suitably small, the coefficient of $|x|$ can be chosen to be 1. By taking

$$K = \sigma\left(\frac{m(p-1)-q}{m(p-1)}(\sigma + \lambda)^{1/(p-1)}\right)^{\frac{(p-1)(q-1)}{m(p-1)-q}} |\hat{x}|^{\frac{(p-1)(q-1)}{m(p-1)-q}},$$

one has

$$I_4 \geqslant |x| - |\hat{x}| = (\hat{x} - x)_+.$$

Moreover, there exist $\theta, \tau > 0$ such that $\int_{-\delta}^{+\infty} u(s,t)\,ds \geqslant \theta$ for any $t \in (0,\tau)$. We choose $|\hat{x}|$ sufficiently small, such that

$$\int_{-\delta}^{+\infty} v(s,t,K,\hat{x})\,ds = \int_{-\delta}^{K^{1-\alpha}t+\hat{x}} v_k\big(Kt - K^{\alpha}(s-\hat{x})\big)_+\,ds = \int_{0}^{Kt+K^{\alpha}(\delta+\hat{x})} v_k(s)\,ds \leqslant \theta.$$

Therefore, we can conclude that

$$\int_{x}^{+\infty} v(s,t,K,\hat{x})\,ds \leqslant \int_{x}^{+\infty} u(s,t)\,ds, \quad (x,t) \in [-\delta,+\infty) \times [0,\tau].$$

By Lemma 2.3, we further obtain

$$\xi_+(t) \geqslant \underline{\xi}_+(t,\hat{x}). \tag{24}$$

Furthermore, $v(0,t,K,\hat{x}) > 0$ holds if

$$t > \hat{t}(\hat{x}) = K^{\alpha-1}|\hat{x}| = \frac{1}{\sigma}\left(\frac{m(p-1)}{m(p-1)-q}\right)^{\frac{(q-1)(p-1)}{m(p-1)-q}} (\sigma+\lambda)^{-\frac{q-1}{m(p-1)-q}} |\hat{x}|^{1-\frac{(q-1)(p-1)}{m(p-1)-q}}.$$

Noticing that $1 - \frac{(q-1)(p-1)}{m(p-1)-q} > 0$, then $\hat{t}(\hat{x}) \to 0$ as $\hat{x} \to 0^-$. Recall (24), and take $|\hat{x}| = \beta t_0^{\frac{m(p-1)-q}{(m+1)(p-1)-qp}}$ for $t_0 \in (0,\tau]$. Then, if β is sufficiently small, we have

$$\begin{aligned}
\xi_+(t_0) &\geqslant K^{1-\alpha}t_0 - |\hat{x}| \\
&= t_0^{\frac{m(p-1)-q}{(m+1)(p-1)-qp}} \beta^{\frac{(q-1)(p-1)}{m(p-1)-q}} \left(C_\sigma - \beta^{1-\frac{(q-1)(p-1)}{m(p-1)-q}}\right) \\
&\geqslant c t_0^{\frac{m(p-1)-q}{(m+1)(p-1)-qp}}.
\end{aligned}$$

The proof of Theorem 4.2 is completed. □

5. Strong convection when $q \leqslant q_0$

In this section, we consider the case of $q \leqslant 1$. We shall see that, influenced by the convection, it is impossible of the waiting time phenomenon to happen for the left interface, especially, $\xi_-(t)$ doesn't exist any more if $q < 1$. In fact, if $q = 1$, let $v(x,t) = u(x - \lambda t, t)$. Then we see that v satisfies

$$\frac{\partial v}{\partial t} = \left(\left|(v^m)'\right|^{p-2}(v^m)'\right)'.$$

Therefore

$$\xi_-^u(t) = \xi_-^v(t) - \lambda t \leqslant \xi_-^v(0) - \lambda t.$$

But, if $q < 1$, for any large $M > 0$, there always exists $\delta > 0$ such that $\lambda q \delta^{q-1} > M$, which means that the solution of the problem (1) is an upper solution of the following problem

$$\frac{\partial u}{\partial t} = \left(\left|(u^m)_x\right|^{p-2}(u^m)_x\right)_x + M u_x$$

near $\xi_-(0)$. Then it follows that $\xi_-(t) \leqslant \xi_-(0) - Mt$. By the arbitrariness of M, it is easy to see that $\xi_-(t)$ doesn't exist. However, different from the left sides, the right interface always exists, and may decrease or increase.

Theorem 5.1. *Let $u(x,t)$ be the solution of the problem* (1)–(2)*. Let $\widehat{M}(q,u_0)$ and $\widehat{M}_c(q)$ be defined as in Theorem* 4.2*.*

(i) *If $\widehat{M}(q,u_0) < \widehat{M}_c(q)$, then there exist $\hat{t}, \hat{c} > 0$ such that*

$$\xi_+(t) \leqslant \xi_+(0) - \hat{c}t^{\frac{m(p-1)-q}{(m+1)(p-1)-pq}} \quad \textit{for any } t \in (0, \hat{t}]. \tag{25}$$

(ii) *While if $\widehat{M}(q,u_0) > \widehat{M}_c(q)$, then there exist $\tilde{c}, \tilde{t} > 0$ such that*

$$\xi_+(t) \geqslant \xi_+(0) + \tilde{c}t^{\frac{m(p-1)-q}{(m+1)(p-1)-qp}} \quad \textit{for any } t \in [0, \tilde{t}]. \tag{26}$$

Remark 5.1. The above theorem shows the strong influence on the right interface. For small initial datum, namely $\widehat{M}(q,u_0) < \widehat{M}_c(q)$, the interface is no longer increasing, but strictly decreasing. While for large initial datum, namely $\widehat{M}(q,u_0) > \widehat{M}_c(q)$, the convection has no ability to change the original monotonicity as for the case without convection.

Proof of Theorem 5.1. As discussed in the previous sections, without loss of generality, we still assume that $\xi_+(0) = 0$.

(i) Recalling a family of traveling wave solutions of Eq. (1)

$$v(x,t,K) = v_k\big([-x-Kt]_+\big),$$

where K is to be determined, and v_k, with $v_k < (\frac{\lambda}{K})^{1/(1-q)}$ if $q < 1$; $K < \lambda$ if $q = 1$, is defined by

$$\int_0^{v_k(y)} \frac{ms^{m-1}}{(\lambda s^q - Ks)^{1/(p-1)}}\, ds = y.$$

From the condition (i) and combining with the continuity of u, we see that there exist $C, \delta, \tau > 0$ with $\widehat{M} < C < C^* \frac{m(p-1)-q}{(m+1)(p-1)-q}$ such that

$$\int_x^{+\infty} u(s,t)\, ds \leqslant C|x|^r \quad \text{for any } (x,t) \in [-\delta, +\infty) \times [0,\tau].$$

Moreover, when $x \in [-\delta, +\infty)$,

$$\int_x^{+\infty} v(s,0,K)\, ds = \int_x^0 v_k\big((-s)_+\big)\, ds \geqslant \int_x^{+\infty} u_0(s)\, ds$$

is ensured by

$$v_k\big((-x)_+\big) \geqslant rC|x|^{r-1}, \quad -\delta \leqslant x < 0.$$

While the above inequality holds if $I_5 \leqslant |x|$, where

$$I_5 = \int_0^{rC|x|^{r-1}} \frac{m s^{m-1-\frac{q}{p-1}}}{(\lambda - K s^{1-q})^{1/(p-1)}}\, ds.$$

A direct calculation yields

$$\begin{aligned} I_5 &\leqslant \int_0^{rC|x|^{r-1}} \frac{m s^{m-1-\frac{q}{p-1}}}{(\lambda - K(rC|x|^{r-1})^{1-q})^{1/(p-1)}}\, ds \\ &\leqslant \int_0^{rC|x|^{r-1}} \frac{m s^{m-1-\frac{q}{p-1}}}{(\lambda - K(rC\delta^{r-1})^{1-q})^{1/(p-1)}}\, ds \\ &= \frac{m(p-1)}{m(p-1)-q} \frac{(rC)^{m-\frac{q}{p-1}}}{(\lambda - K(rC\delta^{r-1})^{1-q})^{1/(p-1)}} |x|. \end{aligned}$$

Notice that $rC < C^*$, and denote $\sigma = K(rC\delta^{r-1})^{1-q}$. Then there exist positive constants σ, η small enough, such that

$$\frac{m(p-1)}{m(p-1)-q} \frac{(rC)^{m-\frac{q}{p-1}}}{(\lambda - \sigma)^{1/(p-1)}} = 1 - \eta.$$

Then it follows that $I_5 \leqslant (1-\eta)|x|$. Taking

$$K = \sigma \left(rC\delta^{r-1}\right)^{q-1},$$

then one has

$$\int_x^{+\infty} v(s,0,K)\, ds \geqslant \int_x^{+\infty} u_0(s)\, ds.$$

Furthermore, from the above results, we arrive at

$$\int_{-\delta}^{+\infty} v(s,t,K)\, ds = \int_0^{\delta - Kt} v_k(s)\, ds \geqslant \int_0^{\delta - Kt} \frac{rC}{(1-\eta)^{r-1}} s^{r-1}\, ds = \frac{C}{(1-\eta)^{r-1}} (\delta - Kt)^r.$$

Choosing $\tau^* = \min\{\frac{\delta}{K}(1-(1-\eta)^{(r-1)/r}), \tau\}$, then for $t \leqslant \tau^*$, we have $\frac{C}{(1-\eta)^{r-1}}(\delta - Kt)^r \geqslant C\delta^r$, namely

$$\int_{-\delta}^{+\infty} v(s,t,K)\, ds \geqslant \int_{-\delta}^{+\infty} u(s,t)\, ds \quad \text{for any } t \in \left[0, \tau^*\right].$$

By means of Lemma 2.2 and Remark 2.1, we arrive at

$$\int_x^{+\infty} v(s,t,K)\,ds \geqslant \int_x^{+\infty} u(s,t)\,ds, \quad (x,t) \in [-\delta,+\infty) \times \big[0,\tau^*\big].$$

Recalling Lemma 2.3, we further obtain

$$\xi_+(t_0) \leqslant \bar{\xi}_+(t_0,K) = -\sigma(rC)^{q-1}\delta^{(r-1)(q-1)}t_0 \quad \text{for any } t_0 \in \big[0,\tau^*\big].$$

Notice that the above equality holds for any $\hat{\delta} \in (0,\delta]$. In particular, we choose

$$\hat{\delta} = \left(\frac{\sigma(rC)^{q-1}t_0}{1-(1-\eta)^{(r-1)/r}}\right)^{1/(1+(r-1)(1-q))},$$

and thus (25) holds.

(ii) Similar to (i), we recall a family of traveling wave solutions of Eq. (1) as follows

$$v(x,t,K) = v_k\big([Kt-x]_+\big),$$

where K is to be determined, and v_k is defined by

$$\int_0^{v_k(y)} \frac{ms^{m-1}}{(\lambda s^q + Ks)^{1/(p-1)}}\,ds = y.$$

From condition (ii) and combining with the continuity of u, we see that there exist $C,\delta,\tau > 0$ with $C^*\frac{m(p-1)-q}{(m+1)(p-1)-q} < C < \widehat{M}$ such that

$$\int_x^{+\infty} u(s,t)\,ds \geqslant C|x|^r \quad \text{for any } (x,t) \in [-\delta,+\infty) \times [0,\tau].$$

In addition, we note that for $x \in [-\delta,+\infty)$,

$$\int_x^{+\infty} v(s,0,K)\,ds = \int_x^{0} v_k\big((-s)_+\big)\,ds \leqslant \int_x^{+\infty} u_0(s)\,ds$$

holds if

$$v_k\big((-x)_+\big) \leqslant rC|x|^{r-1}, \quad -\delta \leqslant x < 0.$$

While the above inequality is ensured by $I_6 \geqslant |x|$, where

$$I_6 = \int_0^{rC|x|^{r-1}} \frac{ms^{m-1-\frac{q}{p-1}}}{(\lambda + Ks^{1-q})^{1/(p-1)}}\,ds.$$

A simple calculation yields

$$I_6 \geqslant \int_0^{rC|x|^{r-1}} \frac{ms^{m-1-\frac{q}{p-1}}}{(\lambda + K(rC\delta^{r-1})^{1-q})^{1/(p-1)}}\,ds$$

$$= \frac{m(p-1)}{m(p-1)-q}\frac{(rC)^{m-\frac{q}{p-1}}}{(\lambda + K(rC\delta^{r-1})^{1-q})^{1/(p-1)}}|x|.$$

It is easy to verify that $rC > C^*$. By taking $\sigma = K(rC\delta^{r-1})^{1-q}$, there exist positive constants σ, η small enough, such that

$$\frac{m(p-1)}{m(p-1)-q}\frac{(rC)^{m-\frac{q}{p-1}}}{(\lambda+\sigma)^{1/(p-1)}} = 1+\eta.$$

That is $I_6 \geqslant (1+\eta)|x|$. Taking

$$K = \sigma\left(rC\delta^{r-1}\right)^{q-1},$$

then one has

$$\int_x^{+\infty} v(s,0,K)\,ds \leqslant \int_x^{+\infty} u_0(s)\,ds.$$

Furthermore, we conclude that

$$\int_{-\delta}^{+\infty} v(s,t,K)\,ds = \int_0^{\delta+Kt} v_k(s)\,ds \leqslant \int_0^{\delta+Kt} \frac{rC}{(1+\eta)^{r-1}}s^{r-1}\,ds = \frac{C}{(1+\eta)^{r-1}}(\delta+Kt)^r.$$

Choose $\tau^* = \min\{\frac{\delta}{K}((1+\eta)^{(r-1)/r}-1), \tau\}$, and then we have $\frac{C}{(1+\eta)^{r-1}}(\delta+Kt)^r \leqslant C\delta^r$ for $t \leqslant \tau^*$, namely

$$\int_{-\delta}^{+\infty} v(s,t,K)\,ds \leqslant \int_{-\delta}^{+\infty} u(s,t)\,ds \quad \text{for any } t \in \left[0,\tau^*\right].$$

By means of Lemma 2.2 and Remark 2.1, we arrive at

$$\int_x^{+\infty} v(s,t,K)\,ds \leqslant \int_x^{+\infty} u(s,t)\,ds, \quad (x,t) \in [-\delta,+\infty) \times \left[0,\tau^*\right].$$

While by Lemma 2.3, we further conclude that

$$\xi_+(t_0) \geqslant \underline{\xi}_+(t_0,K) = \sigma(rC)^{q-1}\delta^{(r-1)(q-1)}t_0 \quad \text{for any } t_0 \in \left[0,\tau^*\right].$$

The same conclusion holds if δ is replaced by any $\hat{\delta} \in (0,\delta]$. Specially, after choosing

$$\hat{\delta} = \left(\frac{\sigma(rC)^{q-1}t_0}{(1+\eta)^{(r-1)/r}-1}\right)^{\frac{1}{1+(r-1)(1-q)}},$$

then (26) holds. This completes the proof of Theorem 5.1. □

Table 1
Summary for $\lambda > 0$.

q	Waiting time	Left interface	Right interface
$q \geqslant q_c$	exist	$\lim_{x\to\xi_-(0)^+} \hat{u}_0^1(x) < \infty$	$\lim_{x\to\xi_+(0)^-} \tilde{u}_0^1(x) < \infty$
	not exist	$\lim_{x\to\xi_-(0)^+} \hat{u}_0^1(x) = \infty$	$\lim_{x\to\xi_+(0)^-} \tilde{u}_0^1(x) = \infty$
$q_0 < q < q_c$	exist	$\lim_{x\to\xi_-(0)^+} \hat{u}_0^2(x) < \infty$	$\lim_{x\to\xi_+(0)^-} \tilde{u}_0^3(x) < \widehat{M}_c(q)$
	not exist	$\lim_{x\to\xi_-(0)^+} \hat{u}_0^2(x) = +\infty$	$\lim_{x\to\xi_+(0)^-} \tilde{u}_0^3(x) > \widehat{M}_c(q)$
$q \leqslant q_0$	shrink		$\lim_{x\to\xi_+(0)^-} \tilde{u}_0^3(x) < \widehat{M}_c(q)$
	expand	always	$\lim_{x\to\xi_+(0)^-} \tilde{u}_0^3(x) > \widehat{M}_c(q)$

For readers' convenience, we give Table 1 to list all the results. Here $q_0 = 1$, $q_c = \frac{(m+1)(p-1)}{p}$, $\hat{u}_0^i(x) = |x - \xi_-(0)|^{-\hat{q}_i} \int_{-\infty}^{x} u_0(s)\,ds$, $\tilde{u}_0^i(x) = |x - \xi_+(0)|^{-\hat{q}_i} \int_{x}^{+\infty} u_0(s)\,ds$ $(i = 1, 2, 3)$, where $\hat{q}_1 = \frac{(m+1)(p-1)}{m(p-1)-1}$, $\hat{q}_2 = \frac{q}{q-1}$, $\hat{q}_3 = \frac{(m+1)(p-1)-q}{m(p-1)-q}$.

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