Stability of Contact Waves for Navier-Stokes Equations and Boltzmann Equation

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Contents

Abstract

As a typical example of hyperbolic conservation laws, the system of Euler equations representing the conservation of mass, momentum and energy has three basic wave patterns. They are two nonlinear waves called shock and rarefaction wave, and one linearly degenerate wave called contact discontinuity. These basic wave patterns have their counterparts both in other physical models for gas motions in equilibrium involving viscosity and heat conductivity, and gas motion in non-equilibrium. The stability of these basic wave patterns in the system of Navier-Stokes equations and the Boltzmann equation has been an active research topic. Even though the stability of the two nonlinear wave patterns has been extensively studied, the stability of the linearly degenerate contact wave was not solved until recently. In this paper, we will briefly present our recent results in [32] on the stability of the contact wave patterns for both the Navier-Stokes equations and the Boltzmann equation.

1 Introduction

The study of hyperbolic conservation laws in the form of

$$U_t + F(U)_x = 0, \tag{1.1}$$

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has a long history and the earliest mathematical work can be traced back to Euler in 1755 on the study of acoustic waves. And the pioneer nonlinear formulation on the fluid dynamics was done by Riemann through the consideration of two stationary gases separated by a membrane when the membrane is suddenly removed. This fundamental work bared the name of Riemann problem is so essential that it plays an important role on the existence and stability theory.

As a typical example of hyperbolic conservation laws, the system of Euler euqations has three basic wave patterns in the solution to the Riemann problem. They are two nonlinear waves, called shock and rarefaction wave, and a linearly degenerate wave called contact discontinuity. These dilation invariant solutions [66], [15], and their linear superposition in the increasing order of characteristic speeds, called Riemann solutions, govern both the local and large time asymptotic behavior of general solutions to the inviscid Euler system [45]. Since the inviscid system is an idealization when the dissipative effects are neglected, it is of great importance to study the large time asymptotic behavior of solutions to the corresponding viscous systems in the form of

$$U_t + F(U)_x = (B(U)U_x)_x, \tag{1.2}$$

toward the viscous versions of these basic wave patterns. As a basic system for the viscous fluid, the Navier-Stokes equations which include the effects of viscosity and heat conductivity, have the above wave phenomena which are smoothed out by the dissipative effect. Furthermore, coming from statistics physics for rarefied gas, the Boltzmann equation which describes the macroscopic and microscopic aspects in the non-equilibrium gas motion, has similar wave phenomena in the macroscopic level.

In this paper, we will briefly present our recent results on the stability of contact waves for the system of Navier-Stokes equations and the Boltzmann equation. It is worthy to pointing out that even though the stability of the basic wave patterns individually is now well understood, the stability of the wave pattern to the Riemann problem consisting of these basic wave patterns are still not known. It is somehow due to the differences in the analytic techniques used for different wave patterns and the different properties of the basic wave patterns in terms of monotonicity and decay rates.

In the next section, we will consider the Navier-Stokes equations. Indeed, there have been great interests and intensive studies in the respect of wave phenomena in the development of the mathematical theory for viscous systems of conservation laws since 1985, started with studies on the nonlinear stability of viscous shock profiles by Goodman [21] and Matsumura-Nishihara [56]. Deeper understanding has been achieved on the asymptotic stability toward nonlinear waves, viscous shock profiles and viscous rarefaction waves; and the linearly degenerate wave, contact discontinuities which have been shown to be nonlinearly stable with quite general perturbations for the compressible Navier-Stokes system and more general system of viscous strictly hyperbolic conservation laws (1.2). Moreover, some new phenomena have been discovered and new techniques, such as weighted characteristic energy methods and uniform approximate Green's functions, have been developed based on the intrinsic properties of the underlying wave patterns see [32], [39], [41], [48], [68], [46] [69], [58], [62] and the references therein.

Precisely, when the solution to the corresponding Riemann problem of Euler equations consists of only a shock wave, the smooth solution profile to the Navier-Stokes equations is the so called shock profile satisfying a system of differential equations with two given end states. Since shock wave is a compression wave, the monotone decreasing property of the characteristic speed in the shock profile plays an crucial role in the stability analysis. In different settings, the nonlinear stability of the shock profiles with smallness assumption on its wave strength has been established in [39], [47], and [56], *etc.*

When the solution to the corresponding Riemann problem consists of only a rarefaction wave, the corresponding nonlinear stability results are obtained in [41], [48], [58], and [60] for different settings. Notice that the rarefaction wave is an expansion wave and the monotone increasing property of the characteristics is also crucially used in the stability analysis. In particular, the stability of strong rarefaction wave was studied in [62]. Moreover, it shows that, for the general gas, a global stability result can also be established for the non-isentropic ideal polytropic gas provided that the adiabatic exponent γ is close to 1. Furthermore, for the isentropic compressible Navier-Stokes equations, the corresponding global stability result holds provided that the resulting compressible Euler equations is strictly hyperbolic and both characteristical fields are genuinely nonlinear. Here, global stability means that the initial perturbation can be large. Since it does not require the strength of the rarefaction waves to be small, these results give the nonlinear stability of strong rarefaction waves for the one-dimensional compressible Navier-Stokes equations.

For the study on the stability of the linearly degenerate wave, i.e., damped contact discontinuity, with general perturbations for both the Navier-Stokes equations, it is somehow surprising that the energy method can be applied to capture the coupling of the contact discontinuity with the diffusion waves created by the perturbations in the sound wave families so that a priori estimate can be closed with a convergence rate on the solution to the wave profile time asymptotically.

Compared to the works on the stability of nonlinear waves, the problem of stability of contact discontinuities is more subtle so that the previous studies in [28], [30], [33], [51] and [76] are less satisfactory. One of the main reasons is the contact discontinuities are associated with linearly degenerate fields and are less stable compared with the nonlinear waves for the inviscid system (1.1), [45]. Thus the stabilizing effects around a damped contact wave pattern for Navier-Stokes equations should come mainly from the viscosity and heat conductivity. A general perturbation of a contact wave may introduce waves in the nonlinear sound wave families, and interactions of these waves with the linear contact wave are the major difficulties to overcome, see [76], [51] and [30]. Another technical difficulty is that the viscosity matrix for the compressible Navier-Stokes equations is only semi-positive definite.

The mathematical aspect on the stability toward contact waves for solutions to systems of viscous conservation laws was first studied by Xin in [76], where the metastability of a weak contact discontinuity for the compressible Euler equations with uniform viscosity, was proved by showing that although a contact discontinuity is not an asymptotic attractor for the viscous system, yet a viscous contact wave, which approximates the contact discontinuity on any finite time interval, is asymptotically nonlinear stable for small generic perturbations and the detailed asymptotic behavior can be determined a priorily by initial mass distribution. This was later generalized by Liu-Xin in [51] to show the metastability of contact discontinuities for a class of general systems of nonlinear conservation laws with uniform viscosity, and obtain pointwise asymptotic behavior toward viscous contact wave by approximate fundamental solutions, which also leads to the nonlinear stability of the viscous contact wave in L^p-norms for all $p \geq 1$. However, the theory in [51] and [76] does not apply to the compressible Navier-Stokes system since its viscosity matrix $B(U)$ in (1.2) is only semi-positive definite.

For a free boundary value problem to the Navier-Stokes equations with a particle path as free boundary, the nonlinear stability of a viscous contact wave is proved in the super-norm through the energy method by Huang-Matsumura-Shi in [28]. However, the approach can not be applied here to study the asymptotic behavior toward contact waves for solutions to Cauchy problems of the Navier-Stokes equations since the analysis in [28] depends crucially on the availability of Poincaré type inequality, which does not hold in the whole space. Recently, a more satisfactory answer was obtained in [30] which shows that for a weak contact discontinuity for the compressible Euler system, one can construct a smooth viscous contact wave for the Navier-Stokes system solves Euler equations asymptotically, and approximates the given contact discontinuity on any finite time interval, and such a viscous contact wave is nonlinearly stable under small initial perturbation with zero mass condition. There the stability is in sup-norm and a rate of convergence is also obtained. Notice that the convergence rate to either the viscous shock wave or viscous rarefaction wave has not been achieved yet for the compressible Navier-Stokes system, see [39], [48], [62]. However, the rate of decay obtained in the form of $(1+t)^{-\frac{1}{4}}$ may not be optimal. Motivated by the pointwise behavior toward viscous contact waves for solutions to the Euler system with uniform viscosity (see [76] and [51]), one would conjecture that its decay rate could be improved to $(1+t)^{-\frac{1}{2}}$.

In [30], the major assumption in the stability theory is the initial zero excessive mass condition which excludes the possible presence of diffusion waves in the sound wave families. As it is shown in [76] and [51] for the compressible Euler equations with uniform viscosity, a generic perturbation of a viscous contact wave introduces not only a shift with center of the viscous contact wave, but also nonlinear and linear diffusion waves. Note that the fine accurate asymptotic ansatz as in [76] and [51] may not be necessary for the stability theory toward contact waves in the super-norm. The main results in [32] are to overcome the difficulty for the excessive mass and obtain the stability and convergence rate for the viscous contact waves. Therefore, it gives a satisfactory answer to the problem on the stability of contact discontinuity in the gas motion.

In the last section, we consider the stability of contact waves for the Boltzmann equation. The Boltzmann equation is a fundamental equation, which gives a statistical description of the time evolution of particles in rarefied gas. It takes the form of

$$f_t + \xi \cdot \nabla f = Q(f,f), \ (f,x,t,\xi) \in \mathbf{R} \times \mathbf{R^3} \times \mathbf{R}^+ \times \mathbf{R}^3, \tag{1.3}$$

where f is the distribution function of the particles and $Q(f,f)$ is the collision operator which gives the gain and loss rate of the particle distribution function through collision.

Since its derivation by Boltzmann in 1872, the mathematical problems on (1.3) have been extensively studied with fruitful results. Among them, we mention a few as: the renormalized solution, fluid dynamic limits, global existence around a global Maxwellian, regularity of the solutions, cf. [5], [6], [16], [19], [43] and references therein. Since they are not directly related to our problem considered here, we will not discuss them in details. Notice that the energy method making use of the spectrum properties of the linearized operator from Grad to Ukai gives a good description of the perturbation of a global Maxwellian, cf. [22], [65], [70], [71]. Recently, the energy method based on the decomposition of the solution and the equation has been developed and used for the study of existence, stability and large time behavior of the solutions, cf. [54] for stability of shock profiles and later in [52] for decomposition around the local Maxwellian and the works after that. Notice also that in [23] the decompositon around a global Maxwellian is also used for the problems on space periodic solutions. It is worthy to pointing out that by decomposing the solution and the equation around the local Maxwellian, the techniques used for fluid dynamics can be applied and the solutions around non-trivial time asymptotic solution profile can be studied more clearly.

One of the most important properties of the Boltzmann equation is its asymptotic equivalence to the macroscopic fluid dynamics equations. In fact, the first order of the Hilbert expansion for the Boltzmann equation is the system of Euler equation and the second order of the Chapman-Enskog expansion gives the system of the Navier-Stokes equations. Hence, one can expect the wave phenomena for the macroscopic fluid dynamics also exist in the solutions to the Boltzmann equation. In fact, there is a series of work on the wave phenomena for the Boltzmann equation starting from the existence of shock profile proved by [9]. Recently, the nonlinear stability of shock profiles, rarefaction wave profiles and contact waves for the Boltzmann equation are also studied through energy method, cf. [54], [53], [31] and [32].

Another the fundamental properties of the Boltzmann equation is the celebrated H-theorem which implies that the solution is time irreversible so that the mathematical entropy is decreasing in time for non-equilibrium gas. There are two ways to view this dissipative effect. One is from the linearized version of the collision operator which dissipates on the sub-space(non-fluid components) orthogonal to the null space(fluid components) of this linearized operator. This in some sense implies that the gas approaches to equilibrium as time tends to infinity. Another consideration comes from the dissipation through the fluid entropy in the nonlinear setting. In this case, the dissipative effect indeed corresponds to those from the viscosity and heat conductivity as for the Navier-Stokes equations.

Now we come back to the stability of a wave pattern. For a non-trivial solution profile connecting two different global Maxwellians at $x = \pm\infty$, it is reasonable and better to decompose the Boltzmann equation and its solution with respect to the local Maxwellian. This kind of decomposition was introduced in [52] by rewritting the Boltzmann equation into a fluid-type dynamics system with the non-fluid component appearing in the source terms, coupled with an equation for the time evolution of the non-fluid component. In fact, set, cf. [54, 52, 61],

$$f(x,t,\xi) = M(x,t,\xi) + G(x,t,\xi),$$

where the local Maxwellian M and G represent the fluid and non-fluid components in the solution respectively. Here, the local Maxwellian M is defined by the five conserved quantities, that is, the mass density $\rho(x,t)$, momentum $m(x,t) = \rho(x,t)u(x,t)$, and energy density $(E(x,t) + \frac{1}{2}|u(x,t)|^2)$. As presented in the last section, the governing system for the fluid components is of fluid-type so that the techniques for Navier-Stokes equations can be applied with some extra terms coming from the non-fluid component. Moreover, the dissipative effect of the linearized operator on the non-fluid component helps to close the energy estimate for the Boltzmann equation. Similar to the Navier-Stokes equations, the dissipative effect in the Boltzmann equation also spreads out the basic wave patterns so that the energy method can be applied. Therefore, the stability of contact waves can also be established for this equation for non-equilibrium rarefied gas in [31, 32].

2 Navier-Stokes equations

The one dimensional compressible Navier-Stokes equations in Lagrangian coordinates take the form:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = \mu(\frac{u_x}{v})_x, \\ (e + \frac{u^2}{2})_t + (pu)_x = (\kappa\frac{\theta_x}{v} + \mu\frac{uu_x}{v})_x, \end{cases} \tag{2.1}$$

where $v(x,t) > 0$ denotes the specific volume, $u(x,t)$ the velocity, $\theta(x,t) > 0$ the absolute temperature, $\mu > 0$ the viscosity and $\kappa > 0$ the coefficient of heat conduction. Here we study the perfect gas so that the pressure p and the internal energy e are given respectively by

$$p = \frac{R\theta}{v}, \quad e = \frac{R}{\gamma - 1}\theta + const. \tag{2.2}$$

where $\gamma > 1$ is the adiabatic exponent and $R > 0$ is the gas constant. The initial data $(v_0, u_0, \theta_0)(x)$ is assume to satisfy

$$(v_0, u_0, \theta_0)(x) \to (v_\pm, 0, \theta_\pm) \quad \text{as} \quad x \to \pm\infty. \tag{2.3}$$

To make the two far field states be connected by a single contact discontinuity for the system of Euler equations, we also need to impose

$$p_- := \frac{R\theta_-}{v_-} = \frac{R\theta_+}{v_+} := p_+. \tag{2.4}$$

We are interested in the asymptotic behavior toward to contact discontinuity for solutions to the compressible Navier-Stokes system with general initial perturbation. First, we need to construct the background profile consisting the contact wave for the Navier-Stokes equations and the two diffusion waves in the first and third families respectively.

In the setting of compressible Navier-Stokes equations, the contact wave profile $(\bar{v}, \bar{u}, \bar{\theta})(x,t)$ becomes smooth and behaves as a diffusion wave due to the dissipation effect from the viscosity and heat conductivity. From [30], the pressure of the profile $(\bar{v}, \bar{u}, \bar{\theta})(x,t)$ is almost constant, i.e.

$$\bar{p} = \frac{R\bar{\theta}}{\bar{v}} \approx p_+, \tag{2.5}$$

which indicates the leading part of the energy equation $(2.1)_3$ is

$$\frac{R}{\gamma-1}\theta_t + p_+ u_x = \kappa(\frac{\theta_x}{v})_x. \tag{2.6}$$

By (2.6) and the mass equation $(2.1)_1$, we obtain a nonlinear diffusion equation,

$$\theta_t = a(\frac{\theta_x}{\theta})_x, \quad a = \frac{\kappa p_+(\gamma-1)}{\gamma R^2} > 0. \tag{2.7}$$

From [2] and [17], (2.7) has a unique self similarity solution $\Theta(\xi), \xi = \frac{x}{\sqrt{1+t}}$ with the boundary conditions

$$\Theta(-\infty, t) = \theta_-, \quad \Theta(+\infty, t) = \theta_+.$$

With Θ defined above, the contact wave profile $(\bar{v}, \bar{u}, \bar{\theta})$ is then defined by:

$$\bar{v} = \frac{R}{p_+}\Theta, \quad \bar{u} = \frac{Ra}{p_+\Theta}\Theta_x, \quad \bar{\theta} = \Theta - \frac{\gamma-1}{2R}\bar{u}^2. \tag{2.8}$$

It is straightforward to check that this contact wave profile approximates the contact discontinuity to the Euler equation in L^p norm, for $p \geq 1$ on any finite time interval as the heat conductivity coefficient κ tends to zero.

To construct the diffusion waves in the other two families, we first denote the conserved quantities by

$$m(x,t) = (v, u, \theta + \frac{\gamma-1}{2R}u^2)^t, \quad \bar{m}(x,t) = (\bar{v}, \bar{u}, \bar{\theta} + \frac{\gamma-1}{2R}\bar{u}^2)^t, \tag{2.9}$$

where $()^t$ means the transpose of the vector $()$. At the far fields $x = \pm\infty$, the vectors m and $\bar{m}$ are the same, that is $m_\pm = (v_\pm, 0, \theta_\pm)^t$. Since we consider the general initial perturbation here, the integral $\int_{-\infty}^{\infty}(m(x,0) - \bar{m}(x,0))dx$ may not be zero in general. Hence, the mass distributes in all characteristic fields when time evolves, which introduces diffusion waves in the two nonlinear sound wave families as in [76] and [51]. Thus we need to construct two diffusion waves θ_1 and θ_3 to carry the exceed mass in the first and third characteristic fields respectively. In fact, let

$$A(v, u, \theta) = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{p}{v} & 0 & \frac{R}{v} \\ -\frac{(\gamma-1)pu}{Rv} & \frac{\gamma-1}{R}p & \frac{(\gamma-1)u}{v} \end{pmatrix}, \tag{2.10}$$

be the Jacobi matrix of the flux $(-u, p, \frac{\gamma-1}{R}pu)^t$. Then, it is straightforward to check that the first eigenvalue of $A(v_-, 0, \theta_-)$ is $\lambda_1^- = -\sqrt{\frac{\gamma p_-}{v_-}}$ with right eigenvector

$$r_1^- = (-1, \lambda_1^-, \frac{\gamma-1}{R}p_-)^t. \tag{2.11}$$

Similarly, the third eigenvalue and right eigenvector of $A(v_+, 0, \theta_+)$ are respectively $\lambda_3^+ = \sqrt{\frac{\gamma p_+}{v_+}}$ and

$$r_3^+ = (-1, \lambda_3^+, \frac{\gamma-1}{R}p_+)^t. \tag{2.12}$$

By strict hyperbolicity, the vectors r_1^-, $m_+ - m_- = (v_+ - v_-, 0, \theta_+ - \theta_-)^t$ and r_3^+ are linearly independent in R^3. Thus, the integral $\int_{-\infty}^{\infty}(m(x,0) - \bar{m}(x,0))dx$ can be distributed as follows

$$\int_{-\infty}^{\infty}(m(x,0) - \bar{m}(x,0))dx = \bar{\theta}_1 r_1^- + \bar{\theta}_2(m_+ - m_-) + \bar{\theta}_3 r_3^+, \tag{2.13}$$

with some constants $\bar{\theta}_i, i = 1, 2, 3$. Now we can define the ansatz $\tilde{m}(x,t)$ by

$$\tilde{m}(x,t) = \bar{m}(x + \bar{\theta}_2, t) + \bar{\theta}_1\theta_1 r_1^- + \bar{\theta}_3\theta_3 r_3^+, \tag{2.14}$$

where

$$\theta_1(x,t) = \frac{1}{\sqrt{4\pi(1+t)}}e^{-\frac{(x-\lambda_1^-(1+t))^2}{4(1+t)}}, \quad \theta_3(x,t) = \frac{1}{\sqrt{4\pi(1+t)}}e^{-\frac{(x-\lambda_3^+(1+t))^2}{4(1+t)}}, \tag{2.15}$$

satisfying

$$\theta_{1t} + \lambda_1^-\theta_{1x} = \theta_{1xx}, \quad \theta_{3t} + \lambda_3^+\theta_{3x} = \theta_{3xx}, \tag{2.16}$$

and $\int_{-\infty}^{\infty}\theta_i(x,t)dx = 1$ for $i = 1, 3$, and all $t \geq 0$ respectively. More precisely, the ansatz $\tilde{m}$ has the following expression

$$\tilde{m}(x,t) = (\tilde{v}, \tilde{u}, \tilde{\theta} + \frac{\gamma-1}{2R}\tilde{u}^2)^t(x,t), \tag{2.17}$$

with

$$\begin{aligned}
&\tilde{v}(x,t) = \bar{v}(x + \bar{\theta}_2, t) - \bar{\theta}_1\theta_1 - \bar{\theta}_3\theta_3, \\
&\tilde{u}(x,t) = \bar{u}(x + \bar{\theta}_2, t) + \lambda_1^-\bar{\theta}_1\theta_1 + \lambda_3^+\bar{\theta}_3\theta_3, \\
&\tilde{\theta}(x,t) = \bar{\theta}(x + \bar{\theta}_2, t) + \frac{\gamma-1}{2R}\bar{u}^2(x + \bar{\theta}_2, t) + \frac{\gamma-1}{R}p_+(\bar{\theta}_1\theta_1 + \bar{\theta}_3\theta_3) - \frac{\gamma-1}{2R}\tilde{u}^2.
\end{aligned} \tag{2.18}$$

Furthermore, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}(m(x,0) - \tilde{m}(x,0))dx \\
&= \int_{-\infty}^{\infty}(m(x,0) - \bar{m}(x,0))dx + \int_{-\infty}^{\infty}(\bar{m}(x,0) - \tilde{m}(x,0))dx \\
&= \bar{\theta}_2(m_+ - m_-) + \int_{-\infty}^{\infty}(\bar{m}(x,0) - \bar{m}(x + \bar{\theta}_2, 0))dx = 0.
\end{aligned} \tag{2.19}$$

We can now state the main result for the compressible Navier-Stokes (2.1)-(2.2). First, we denote the perturbation around the ansatz $(\tilde{v}, \tilde{u}, \tilde{\theta})$ by

$$\phi(x,t) = v - \tilde{v}, \quad \psi(x,t) = u - \tilde{u}, \quad \zeta(x,t) = \theta - \tilde{\theta}. \tag{2.20}$$

Then set

$$\begin{aligned}
&\Phi(x,t) = \int_{-\infty}^{x}\phi(y,t)dy, \qquad \Psi(x,t) = \int_{-\infty}^{x}\psi(y,t)dy, \\
&\bar{W}(x,t) = \int_{-\infty}^{x}(e + \frac{|u|^2}{2} - \tilde{e} - \frac{|\tilde{u}|^2}{2})(y,t)dy.
\end{aligned} \tag{2.21}$$

Notice that the quantities $(\Phi, \Psi, \bar{W})$ can be well defined in some Sobolev space since the compressible Navier-Stokes equations (2.1) and $(\Phi, \Psi, \bar{W})(\pm\infty, 0) = 0$ due to (2.19).

The precise statement of our first result is as follows.

Theorem 1. Let $(\tilde{v},\tilde{u},\tilde{\theta})(x,t)$ be defined in (2.18) and $\delta=|\theta_+-\theta_-|$. Then there exist positive constants δ_0 and ϵ, such that if $\delta\le\delta_0$ and the initial data (v_0,u_0,θ_0) satisfies

$$\|(\Phi,\Psi,\bar{W})\|_{L^2}+\|m-\bar{m}\|_{H^1}\le\epsilon,$$

then the system (2.1) admits a unique global solution $(v,u,\theta)(x,t)$ satisfying

$$(\Phi,\Psi,\bar{W})\in C(0,+\infty;H^2),$$

$$\phi\in L^2(0,+\infty;H^1),\quad (\psi,\zeta)\in L^2(0,+\infty;H^2).$$

Furthermore, the solution satisfies

$$\|(v-\tilde{v},u-\tilde{u},\theta-\tilde{\theta}\|_{L^\infty}\le C(\epsilon+\delta_0^{\frac{1}{2}})(1+t)^{-\frac{1}{4}}.$$

3 Boltzmann equation

Since the profile studied is in one space dimension, we consider the Boltzmann equation as follows

$$f_t+\xi_1 f_x=Q(f,f),\ (f,x,t,\xi)\in\mathbf{R}\times\mathbf{R}\times\mathbf{R}^+\times\mathbf{R}^3, \tag{3.1}$$

where $f(x,t,\xi)$ represents the distributional density of particles at space-time (x,t) with velocity ξ. For monatomic gas, the rotational invariance of the molecules leads to the collision operator $Q(f,f)$ as a bilinear collision operator in the form of, cf. [7]:

$$Q(f,g)(\xi)\equiv\frac{1}{2}\int_{\mathbf{R}^3}\int_{\mathbf{S}^2_+}\Big(f(\xi')g(\xi'_*)+f(\xi'_*)g(\xi')-f(\xi)g(\xi_*)-f(\xi_*)g(\xi)\Big)B(|\xi-\xi_*|,\theta)\ d\xi_* d\Omega,$$

with θ being the angle between the relative velocity and the unit vector Ω. Here $\mathbf{S}^2_+=\{\Omega\in\mathbf{S}^2:\ (\xi-\xi_*)\cdot\Omega\ge 0\}$. The conservation of momentum and energy gives the following relation between velocities before and after collision:

$$\begin{cases}\xi'=\xi-[(\xi-\xi_*)\cdot\Omega]\,\Omega,\\ \xi'_*=\xi_*+[(\xi-\xi_*)\cdot\Omega]\,\Omega.\end{cases}$$

Here we consider the Boltzmann equation for the two basic models, i.e., the hard sphere model and the hard potential with angular cut-off. In these two cases, the collision kernel $B(|\xi-\xi_*|,\theta)$ takes the forms

$$B(|\xi-\xi_*|,\theta)=|(\xi-\xi_*,\Omega)|,$$

and

$$B(|\xi-\xi_*|,\theta)=|\xi-\xi_*|^{\frac{n-5}{n-1}}b(\theta),\quad b(\theta)\in L^1([0,\pi]),\ n>5,$$

respectively. Here, n is the index in the inverse power potentials proportional to r^{1-n} with r being the distance between two particles. The analysis can be generalized to other kernels with similar property. But we will not discuss them here.

For a non-trivial solution profile connecting two different global Maxwellians at $x=\pm\infty$, we decompose the Boltzmann equation and its solution with respect to the local Maxwellian. This kind of decomposition was introduced in [52], [54] by rewriting the Boltzmann equation into a fluid-type dynamics system with the non-fluid component appearing in the source terms, coupled with an equation for the time evolution of the non-fluid component. In fact, set, cf. [52], [61],

$$f(x,t,\xi)=M(x,t,\xi)+G(x,t,\xi),$$

where the local Maxwellian M and G represent the fluid and non-fluid components in the solution respectively. Here, the local Maxwellian M is defined by the five conserved quantities, that is, the mass density $\rho(x,t)$, momentum $m(x,t) = \rho(x,t)u(x,t)$, and energy density $(E(x,t) + \frac{1}{2}|u(x,t)|^2)$:

$$\begin{cases} \rho(x,t) \equiv \int_{\mathbf{R}^3} f(x,t,\xi)d\xi, \\ m^i(x,t) \equiv \int_{\mathbf{R}^3} \psi_i(\xi)f(x,t,\xi)d\xi \text{ for } i=1,2,3, \\ \left[\rho\left(E+\frac{1}{2}|u|^2\right)\right](x,t) \equiv \int_{\mathbf{R}^3} \psi_4(\xi)f(x,t,\xi)d\xi, \end{cases} \tag{3.2}$$

as

$$M \equiv M_{[\rho,u,\theta]}(x,t,\xi) \equiv \frac{\rho(x,t)}{\sqrt{(2\pi R\theta(x,t))^3}} \exp\left(-\frac{|\xi-u(x,t)|^2}{2R\theta(x,t)}\right). \tag{3.3}$$

Here $\theta(x,t)$ is the temperature which is related to the internal energy E by $E = \frac{3}{2}R\theta$ with R being the gas constant, and $u(x,t)$ is the fluid velocity. It is well known that the collision invariants $\psi_\alpha(\xi)$ are given by, cf. [7]:

$$\begin{cases} \psi_0(\xi) \equiv 1, \\ \psi_i(\xi) \equiv \xi_i \quad \text{for} \quad i=1,2,3, \\ \psi_4(\xi) \equiv \frac{1}{2}|\xi|^2, \end{cases}$$

satisfying

$$\int_{\mathbf{R}^3} \psi_j(\xi)Q(h,g)d\xi = 0, \quad \text{for} \quad j=0,1,2,3,4.$$

In the sequel, the inner product of h and g in $L^2_\xi(\mathbf{R}^3)$ with respect to a given Maxwellian $\tilde{M}$ is defined by:

$$\langle h,g\rangle_{\tilde{M}} \equiv \int_{\mathbf{R}^3} \frac{1}{\tilde{M}} h(\xi)g(\xi)d\xi,$$

when the integral is well defined. If $\tilde{M}$ is the local Maxwellian M, with respect to the corresponding inner product, the macroscopic space is spanned by the following five pairwise orthogonal functions

$$\begin{cases} \chi_0(\xi) \equiv \frac{1}{\sqrt{\rho}}M, \\ \chi_i(\xi) \equiv \frac{\xi_i-u_i}{\sqrt{R\theta\rho}}M \quad \text{for} \quad i=1,2,3, \\ \chi_4(\xi) \equiv \frac{1}{\sqrt{6\rho}}(\frac{|\xi-u|^2}{R\theta}-3)M, \\ <\chi_i,\chi_j>=\delta_{ij}, \ i,j=0,1,2,3,4. \end{cases}$$

Using these five basic functions, we define the macroscopic projection P_0 and microscopic projection P_1 as follows:

$$\begin{cases} P_0h \equiv \sum_{j=0}^{4} <h,\chi_j>\chi_j, \\ P_1h \equiv h - P_0h. \end{cases}$$

The projections P_0 and P_1 are orthogonal and satisfy

$$P_0P_0 = P_0, P_1P_1 = P_1, P_0P_1 = P_1P_0 = 0.$$

A function $h(\xi)$ is called microscopic or non-fluid if

$$\int h(\xi)\psi_j(\xi)d\xi = 0, \ j=0,1,2,3,4.$$

Under this decomposition, the solution $f(x,t,\xi)$ of the Boltzmann equation satisfies

$$P_0 f = M,\ P_1 f = G,$$

and the Boltzmann equation becomes

$$(M+G)_t + \xi_1 (M+G)_x = 2Q(M,G) + Q(G,G),$$

which is equivalent to the following fluid-type system for the fluid components (see [52], [53], [54] for details):

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = -\int \xi_1^2 G_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = -\int \xi_1 \xi_i G_x d\xi,\ i = 2,3 \\ (\rho(e + \frac{|u|^2}{2}))_t + (\rho u_1 (e + \frac{|u|^2}{2}) + p u_1)_x = -\int \frac{1}{2} \xi_1 |\xi|^2 G_x d\xi, \end{cases} \tag{3.4}$$

or more precisely,

$$\begin{cases} \rho_t + (\rho u_1)_x = 0, \\ (\rho u_1)_t + (\rho u_1^2 + p)_x = \frac{4}{3}(\mu(\theta) u_{1x})_x - \int \xi_1^2 \Theta_x d\xi, \\ (\rho u_i)_t + (\rho u_1 u_i)_x = (\mu(\theta) u_{ix})_x - \int \xi_1 \xi_i \Theta_x d\xi,\ i = 2,3 \\ (\rho(e + \frac{|u|^2}{2}))_t + (\rho u_1 (e + \frac{|u|^2}{2}) + p u_1)_x = (\lambda(\theta)\theta_x)_x \\ \quad + \frac{4}{3}(\mu(\theta) u_1 u_{1x})_x + \sum_{i=2}^{3} (\mu(\theta) u_i u_{ix})_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Theta_x d\xi, \end{cases} \tag{3.5}$$

together with the equation for the non-fluid component G:

$$G_t + P_1(\xi_1 M_x) + P_1(\xi_1 G_x) = L_M G + Q(G,G). \tag{3.6}$$

It follows from (3.6) that

$$G = L_M^{-1}(P_1(\xi_1 M_x)) + \Theta$$

with

$$\Theta = L_M^{-1}(G_t + P_1(\xi_1 G_x) - Q(G,G)). \tag{3.7}$$

Here L_M is the linearized operator of the collision operator with respect to the local Maxwellian M:

$$L_M h = Q(M,h) + Q(h,M),$$

and the null space N of L_M is spanned by the macroscopic variables:

$$\chi_j,\ j = 0,1,2,3,4.$$

Furthermore, there exists a positive constant $\sigma_0(\rho,u,\theta) > 0$ such that for any function $h(\xi) \in N^\perp$, see [22],

$$< h, L_M h > \le -\sigma_0 < \nu(|\xi|) h, h >,$$

where $\nu(|\xi|)$ is the collision frequency. For the hard sphere and the hard potential with angular cut-off, the collision frequency $\nu(|\xi|)$ has the following property

$$0 < \nu_0 < \nu(|\xi|) \le c(1 + |\xi|)^\beta,$$

for some positive constants ν_0, c and $0 < \beta \leq 1$. In the above presentation, we have normalized the gas constant R to be $\frac{2}{3}$ for simplicity so that $e = \theta$ and $p = \frac{2}{3}\rho\theta$. Notice also that the viscosity coefficient $\mu(\theta) > 0$ and the heat conductivity coefficient $\lambda(\theta) > 0$ are smooth functions of the temperature θ.

Since the problem is in one dimensional space $x \in \mathbf{R}$, in the macroscopic level, it is more convenient to rewrite the system and the equation by using the *Lagrangian* coordinates as in the study of conservation laws. That is, consider the coordinate transformation:

$$x \Rightarrow \int_0^x \rho(y,t)dy, \qquad t \Rightarrow t.$$

We will still denote the *Lagrangian* coordinates by (x,t) for simplicity of notation. The system (3.1) and (3.4) in the Lagrangian coordinates become, respectively,

$$f_t - \frac{u_1}{v} f_x + \frac{\xi_1}{v} f_x = Q(f,f), \tag{3.8}$$

and

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = -\int \xi_1^2 G_x d\xi, \\ u_{it} = -\int \xi_1 \xi_i G_x d\xi, \ i = 2,3 \\ (e + \frac{|u|^2}{2})_t + (pu_1)_x = -\int \frac{1}{2}\xi_1 |\xi|^2 G_x d\xi. \end{cases} \tag{3.9}$$

Moreover, (3.5) and (3.6) take the form

$$\begin{cases} v_t - u_{1x} = 0, \\ u_{1t} + p_x = \frac{4}{3}(\frac{\mu(\theta)}{v} u_{1x})_x - \int \xi_1^2 \Theta_{1x} d\xi, \\ u_{it} = (\frac{\mu(\theta)}{v} u_{ix})_x - \int \xi_1 \xi_i \Theta_{1x} d\xi, \ i = 2,3 \\ (e + \frac{|u|^2}{2})_t + (pu_1)_x = (\frac{\lambda(\theta)}{v}\theta_x)_x + \frac{4}{3}(\frac{\mu(\theta)}{v} u_1 u_{1x})_x \\ + \sum_{i=2}^{3} (\frac{\mu(\theta)}{v} u_i u_{ix})_x - \int \frac{1}{2} \xi_1 |\xi|^2 \Theta_{1x} d\xi, \end{cases} \tag{3.10}$$

and

$$G_t - \frac{u_1}{v} G_x + \frac{1}{v} P_1(\xi_1 M_x) + \frac{1}{v} P_1(\xi_1 G_x) = L_M G + Q(G,G), \tag{3.11}$$

with

$$G = L_M^{-1}(\frac{1}{v} P_1(\xi_1 M_x)) + \Theta_1,$$

and

$$\Theta_1 = L_M^{-1}(G_t - \frac{u_1}{v} G_x + \frac{1}{v} P_1(\xi_1 G_x) - Q(G,G)). \tag{3.12}$$

Notice that system (3.10) can be regarded as the compressible Navier-Stokes equations (2.1) with some source terms coming from non-fluid components. Analogous to (2.1), we construct the ansatz for the wave profile of the Boltzmann equation as follows. First, let $\Theta(\frac{x}{\sqrt{1+t}})$ be the unique self-similarity solution of the following nonlinear diffusion equation

$$\Theta_t = (a(\Theta)\Theta_x)_x, \quad \Theta(-\infty,t) = \theta_-, \quad \Theta(+\infty,t) = \theta_+, \tag{3.13}$$

where the function $a(s)=\frac{9p_+\lambda(s)}{10\theta}>0$. Notice that (3.13) is exactly the same as the diffusion equation (2.7) for the compressible Naiver-Stokes equations when $\gamma=\frac{5}{3}$, $R=\frac{2}{3}$ and $\kappa=\lambda(\theta)$. We then define

$$\bar{v}=\frac{2}{3p_+}\Theta,\quad \bar{u}_1=\frac{2a(\Theta)}{3p_+}\Theta_x,\quad \bar{u}_i=0,\ i=2,3,\quad \bar{\theta}=\Theta-\frac{1}{2}|\bar{u}|^2. \tag{3.14}$$

Let $m=(v,u_1,\theta+\frac{1}{2}|u|^2)$ and $\bar{m}=(\bar{v},\bar{u}_1,\bar{\theta}+\frac{1}{2}|\bar{u}|^2)$. Since $\int_{-\infty}^{\infty}(m(x,0)-\bar{m}(x,0))dx$ is usually not zero, we have to introduce two diffusion waves in the sound wave families as shown in the previous subsection. Let

$$A_-=\begin{pmatrix}0 & -1 & 0\\ -\frac{p_-}{v_-} & 0 & \frac{2}{3v_-}\\ 0 & p_- & 0\end{pmatrix},\quad A_+=\begin{pmatrix}0 & -1 & 0\\ -\frac{p_+}{v_+} & 0 & \frac{2}{3v_+}\\ 0 & p_+ & 0\end{pmatrix}, \tag{3.15}$$

be the Jacobi matrices of the flux $(-u,p,pu)^t$ at $(v_-,0,\theta_-)$ and $(v_+,0,\theta_+)$ respectively. It is easy to check that $\lambda_1^-=-\sqrt{\frac{5p_-}{3v_-}}$ is the first eigenvalue of A_- with $r_1^-=(-1,\lambda_1^-,p_-)^t$ being the corresponding eigenvector. And $\lambda_3^+=\sqrt{\frac{5p_+}{3v_+}}$ and $r_3^+=(-1,\lambda_3^+,p_+)^t$ are those of the third family of A_+. Since r_1^-, $(v_+-v_-,0,\theta_+-\theta_-)^t$ and r_3^+ are linearly independent in R^3 by strict hyperbolicity, we have

$$\int_{-\infty}^{\infty}(m(x,0)-\bar{m}(x,0))dx=\bar{\theta}_1r_1^-+\bar{\theta}_2(v_+-v_-,0,\theta_+-\theta_-)^t+\bar{\theta}_3r_3^+ \tag{3.16}$$

with unique constants $\bar{\theta}_i, i=1,2,3$. The ansatz $\tilde{m}(x,t)$ for m is defined as

$$\tilde{m}(x,t)=\bar{m}(x+\bar{\theta}_2,t)+\bar{\theta}_1\theta_1r_1^-+\bar{\theta}_3\theta_3r_3^+, \tag{3.17}$$

where

$$\theta_1(x,t)=\frac{1}{\sqrt{4\pi(1+t)}}e^{-\frac{(x-\lambda_1^-(1+t))^2}{4(1+t)}},\quad \theta_3(x,t)=\frac{1}{\sqrt{4\pi(1+t)}}e^{-\frac{(x-\lambda_3^+(1+t))^2}{4(1+t)}}, \tag{3.18}$$

satisfying $\theta_{1t}+\lambda_1^-\theta_{1x}=\theta_{1xx},\ \theta_{3t}+\lambda_3^+\theta_{3x}=\theta_{3xx}$ and $\int_{-\infty}^{\infty}\theta_i(x,t)dx=1$ for $i=1,3$ and all $t\geq 0$. Similar to (2.19), we now have $\int_{-\infty}^{\infty}(m(x,0)-\tilde{m}(x,0))dx=0$. Notice that $\int_{-\infty}^{\infty}u_i(x,0)dx$ may not be zero either for $i=2,3$. For this, we define

$$\bar{u}_i(x,t)=\bar{\theta}_{i+2}\frac{1}{\sqrt{4\pi(1+t)}}e^{-\frac{x^2}{4(1+t)}},\ i=2,3, \tag{3.19}$$

where $\bar{\theta}_{i+2}=\int_{-\infty}^{\infty}u_i(x,0)dx$. It is obvious that $\int_{-\infty}^{\infty}(u_i(x,0)-\tilde{u}_i(x,0))dx=0, i=2,3$.

Finally, the ansatz is defined as

$$\begin{aligned}
&\tilde{v}(x,t)=\bar{v}(x+\bar{\theta}_2,t)-\bar{\theta}_1\theta_1-\bar{\theta}_3\theta_3,\\
&\tilde{u}_1(x,t)=\bar{u}_1(x+\bar{\theta}_2,t)+\lambda_1^-\bar{\theta}_1\theta_1+\lambda_3^+\bar{\theta}_3\theta_3,\\
&\tilde{u}_i=\frac{\bar{\theta}_{i+2}}{\sqrt{4\pi(1+t)}}e^{-\frac{x^2}{4(1+t)}},\ i=2,3,\\
&\tilde{\theta}(x,t)=\bar{\theta}(x+\bar{\theta}_2,t)+\frac{1}{2}|\bar{u}|^2(x+\bar{\theta}_2,t)+p_+(\bar{\theta}_1\theta_1+\bar{\theta}_3\theta_3)-\frac{1}{2}|\tilde{u}|^2.
\end{aligned} \tag{3.20}$$

Here $(\tilde{v},\tilde{u},\tilde{\theta})$ satisfies

$$\tilde{m}(x,t)=(\tilde{v},\tilde{u}_1,\tilde{\theta}+\frac{1}{2}|\tilde{u}|^2)^t(x,t). \tag{3.21}$$

With above preparation, we are ready to state the result on the stability of the contact wave pattern for the Boltzmann equation (3.1). Denote the perturbation around the ansatz $(\tilde{v}, \tilde{u}, \tilde{\theta})$ by

$$\phi(x,t) = v - \tilde{v}, \quad \psi(x,t) = u - \tilde{u}, \quad \zeta(x,t) = \theta - \tilde{\theta}. \tag{3.22}$$

Then set

$$\begin{aligned} &\Phi(x,t) = \int_{-\infty}^{x} \phi(y,t)dy, \qquad \Psi(x,t) = \int_{-\infty}^{x} \psi(y,t)dy, \\ &\bar{W}(x,t) = \int_{-\infty}^{x} (e + \frac{|u|^2}{2} - \tilde{e} - \frac{|\tilde{u}|^2}{2})(y,t)dy, \end{aligned} \tag{3.23}$$

so that the quantities Φ, Ψ ad $\bar{W}$ can be well defined in some Sobolev space. The second main theorem is as follows.

Theorem 2. Let $(\tilde{v}, \tilde{u}, \tilde{\theta})(x,t)$ be the ansatz defined in (3.20) with $\delta = |\theta_+ - \theta_-|$. Then there exist small positive constants δ_0, ϵ and global Maxwellian $M_* = M_{[\rho_*, u_*, \theta_*]}$, such that if $\delta \leq \delta_0$ and the initial data satisfies

$$\begin{aligned} &\{\|(\Phi, \Psi, \bar{W})\|_{L^2_x} + \|(\phi, \psi, \zeta)\|_{H^1_x} + \sum_{|\alpha|=2} \|\partial^\alpha f\|_{L^2_x(L^2_\xi(\frac{1}{\sqrt{M_*}}))} \\ &+ \sum_{0\leq|\alpha|\leq 1} \|\partial^\alpha G\|_{L^2_x(L^2_\xi(\frac{1}{\sqrt{M_*}}))}\}|_{t=0} \leq \epsilon, \end{aligned}$$

then the Cauchy problem (3.8) admits a unique global solution $f(x,t,\xi)$ satisfying

$$\|f(x,t,\xi) - M_{[\bar{v},\bar{u},\bar{\theta}]}\|(t)_{L^\infty_x(L^2_\xi(\frac{1}{\sqrt{M_*}}))} \leq C(\epsilon + \delta_0^{\frac{1}{2}})(1+t)^{-\frac{1}{4}}.$$

Here $f(\xi) \in L^2_\xi(\frac{1}{\sqrt{M_*}})$ means that $\frac{f(\xi)}{\sqrt{M_*}} \in L^2_\xi(\mathbf{R}^3)$.

Remark The estimate for the higher order derivatives on the solution can be obtained similarly, provided that the initial data has the same order regularity.

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