

# GLOBAL WELL-POSEDNESS AND A DECAY ESTIMATE FOR THE CRITICAL DISSIPATIVE QUASI-GEOSTROPHIC EQUATION IN THE WHOLE SPACE

HONGJIE DONG AND DAPENG DU

ABSTRACT. We study the critical dissipative quasi-geostrophic equations in  $\mathbb{R}^2$  with arbitrary  $H^1$  initial data. After showing certain decay estimate, a global well-posedness result is proved by adapting the method in [11] with a suitable modification. A decay in time estimate for higher order homogeneous Sobolev norms of solutions is also discussed.

## 1. INTRODUCTION

In this note, we consider the initial value problem of 2D dissipative quasi-geostrophic equations

$$\begin{cases} \theta_t + u \cdot \nabla \theta + (-\Delta)^{\gamma/2} \theta = 0 & \text{on } \mathbb{R}^2 \times (0, \infty), \\ \theta(0, x) = \theta_0(x) & x \in \mathbb{R}^2, \end{cases} \quad (1.1)$$

where  $\gamma \in (0, 2]$  is a fixed parameter and the velocity  $u = (u_1, u_2)$  is divergence free and determined by the Riesz transforms of the potential temperature  $\theta$ :

$$u = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) = (-\partial_{x_2} (-\Delta)^{1/2} \theta, \partial_{x_1} (-\Delta)^{1/2} \theta).$$

The main problem addressed here is the global regularity of (1.1) with  $\gamma = 1$  and arbitrary  $H^1$  initial data.

Equation (1.1) is an important model in geophysical fluid dynamics. It is derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency. When  $\gamma = 1$ , it is the dimensionally correct analogue of the 3D incompressible Navier-Stokes equations. The critical dissipative quasi-geostrophic equations is an interesting model for investigating existence issues on genuine

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3D Navier-Stokes equations. Recently, this equation has been studied intensively, see [4], [5], [6], [7], [13], [14], [8], [16], [17], [18], [19], [20], [21] and references therein.

The cases  $\gamma > 1$ ,  $\gamma = 1$  and  $\gamma < 1$  are called sub-critical, critical and super-critical respectively. The sub-critical case is better understood and the global well-posedness result is well-known. For this case, we refer the readers to Wu [18], Carrilo and Ferreira [2], Constantin and Wu [6], Dong and Li [10] and reference therein.

The cases of critical and super-critical dissipative quasi-geostrophic equations still have quite a few unsolved problems. One major problem is the issue of global regularity or breakdown of regular solutions, which was suggested by Klainerman [12] seven years ago as one of the most challenging PDE problems of the twenty-first Century. In the critical case, Constantin, Córdoba and Wu [5] gave a construction of global regular solutions for initial data in  $H^1$  under a smallness assumption of  $L^\infty$  norm of the data. For other results about local well-posedness and small-data global well-posedness in various function spaces, see also Chae and Lee [3], Ju [13], [15], Miura [16] and reference therein. For the issue of global regularity with large initial data, breakthrough only occurred recently. Caffarelli and Vasseur [1] constructed a global regular Leray-Hopf type weak solution for the critical quasi-geostrophic equations with merely  $L^2$  initial data. The global well-posedness for the critical quasi-geostrophic equations with periodic  $C^\infty$  data was proved by Kiselev, Nazarov and Volberg in an elegant paper [11]. Their argument is based on a certain non-local maximum principle for a suitable chosen modulus of continuity.

Miura [16] recently established the local in time existence of a unique regular solution for large initial data in the critical Sobolev space  $H^{2-\gamma}$ . A similar result was also obtain independently in Ju [15] by using a different approach. Very recently, the first author showed that the solutions by Miura and Ju have higher regularity and are global in time with periodic  $H^1$  data. Roughly speaking, it is proved that the smoothing effect of the equations in spaces is the same for the corresponding linear equations. For other results about the critical and super-critical dissipative quasi-geostrophic equations, we also refer the readers to [7], [14], [8], [19], [20] and [21]. However, at present the following aforementioned problem suggested by Klainerman is still open:

*For  $\gamma = 1$ , is (1.1) globally well-posed with arbitrary smooth in  $\mathbb{R}^2$  ?*

We give an affirmative answer to this question (Theorem 2.3) with initial data in the critical Sobolev space  $H^1$ . Our strategy is to apply a local smoothing result proved in [9] and then adapt the idea in

[11] with a proper modification. One essential difference between periodic and non-periodic settings is that in periodic domains, one can appeal certain compactness property, which is not valid in the whole space. We circumvent this difficult by showing some decay estimate of solutions as space variables go to infinity. Moreover, we show that the solution is actually a smooth classical solution to (1.1) and higher order homogeneous Sobolev norms of the solution decay polynomially as  $t$  goes to infinity (Theorem 4.1).

The remaining part of the note is organized as follows: after reviewing some local well-posedness and smoothing results, the main theorem is given in the following section. We prove the theorem in Section 3. Section 4 is devoted to a decay in time estimate of higher order homogeneous Sobolev norms of solutions.

## 2. MAIN THEOREM

The local well-posedness of (1.1) with  $H^{2-\gamma}$  data is recently established by Miura and Ju independently.

**Proposition 2.1.** *Let  $\gamma \in (0, 1]$  and  $\theta_0 \in H^{2-\gamma}(\mathbb{R}^2)$ . Then there exists  $T > 0$  such that the initial value problem for (1.1) has a unique solution*

$$\theta(t, x) \in C([0, T]; H^{2-\gamma}(\mathbb{R}^2)) \cap L^2(0, T; H^{2-\gamma/2}(\mathbb{R}^2)).$$

Moreover the solution  $\theta$  satisfies

$$\sup_{0 < t < T} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta}} < \infty,$$

for any  $\beta \in [0, \gamma)$  and

$$\lim_{t \rightarrow 0} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta}} = 0,$$

for any  $\beta \in (0, \gamma)$ .

The following proposition is the main result of [9], which says that the solution in Proposition 2.1 has higher regularities.

**Proposition 2.2.** *The solution  $\theta$  in Proposition 2.1 satisfies*

$$\sup_{0 < t < T} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta}} < \infty, \quad (2.1)$$

for any  $\beta \geq 0$  and

$$\lim_{t \rightarrow 0} t^{\beta/\gamma} \|\theta(t, \cdot)\|_{\dot{H}^{2-\gamma+\beta}} = 0, \quad (2.2)$$

for any  $\beta > 0$ .

By the Sobolev imbedding theorem, the previous proposition implies that the solution in Proposition 2.1 is infinitely differentiable in  $x$  with bounded derivatives for any  $t \in (0, T)$ . Then because of the first equation in (1.1), it is infinitely differentiable in both  $x$  and  $t$  with bounded derivatives for any  $t \in (0, T)$ . Therefore,  $\theta$  is actually a classical solution of (1.1).

In the sequel, we always assume  $\gamma = 1$ , i.e. the critical case. Next we state our main theorem.

**Theorem 2.3.** *With  $\theta_0 \in H^1(\mathbb{R}^2)$ , the initial value problem (1.1) has a unique global solution*

$$\theta(t, x) \in C_b([0, \infty); H^1(\mathbb{R}^2)) \cap L^2(0, \infty; H^{3/2}(\mathbb{R}^2)). \quad (2.3)$$

*Remark 2.4.* In [9], it is shown that with zero-mean periodic  $H^1$  data, the solution and all its derivatives decay exponentially as  $t$  goes to infinity. This is certainly not the case for equations with non-periodic data. Moreover, in the periodic setting  $\theta(t, \cdot)$  is spatial periodic for sufficiently large  $t$ . We conjecture this is still true for the solution in Theorem 2.3.

### 3. PROOF OF THEOREM 2.3

This section is devoted to the proof of Theorem 2.3. The argument is mainly based on a decay estimate of  $|\nabla_x \theta|$  as  $x \rightarrow \infty$  and a non-local maximum principle as in [11] with a proper modification.

**Definition 3.1.** We say a function  $f$  has modulus of continuity  $\omega$  if  $|f(x) - f(y)| \leq \omega(|x - y|)$ , where  $\omega$  is an unbounded increasing continuous concave function  $\omega : [0, +\infty) \rightarrow [0, +\infty)$ . We say  $f$  has strict modulus of continuity  $\omega$  if the inequality is strict for  $x \neq y$ .

Recall the remark before Theorem 2.3. After fixing a time  $t_1 \in (0, T)$  and considering  $\theta(t - t_1)$  instead of  $\theta$ , we may assume  $\theta_0 \in H^1 \cap C^\infty$  and  $\theta_0$  is bounded along with all its derivatives. Therefore, for any unbounded continuous concave function  $\omega$  satisfying

$$\omega(0) = 0, \quad \omega' > 0, \quad \omega'(0) < +\infty, \quad \lim_{\xi \rightarrow 0^+} \omega''(\xi) = -\infty, \quad (3.1)$$

we can find a constant  $C > 0$  such that  $\omega(\xi)$  is a *strict* modulus of continuity of  $\theta_0(Cx)$ . Due to the scaling property of (1.1),  $\theta_c(t, x) = \theta(Ct, Cx)$  is the solution of (1.1) with initial data  $\theta_0(Cx)$ . Thus if we can show that suitable modulus of continuity  $\omega$  is preserved by the dissipative evolution so that  $\theta_c$  is a global solution, the same is true for  $\theta$ .

With aforementioned property (3.1) of  $w$ , if  $f \in C^2$  has modulus of continuity  $\omega$ , it is easy to show that pointwisely  $|\nabla f(x)| < \omega'(0)$  (see [11]). Assume further that  $f \in H^4(\mathbb{R}^2)$ , due to the Sobolev imbedding theorem,  $\nabla f(x)$  is a uniformly continuous function and goes to zero as  $|x| \rightarrow \infty$ . Thus, we get

**Lemma 3.2.** *If  $f \in H^4(\mathbb{R}^2)$  has modulus of continuity  $\omega$  satisfying (3.1), we have  $\|\nabla f\|_{L^\infty} < \omega'(0)$ .*

Next we show that strict modulus of continuity is preserved at least for a short time.

**Lemma 3.3.** *Assume  $\theta(t, \cdot)$  has strict modulus of continuity  $\omega$  for all  $t \in [0, T_1]$ . Then there exists  $\delta > 0$  such that  $\theta(t, \cdot)$  has strict modulus of continuity  $\omega$  for all  $t \in [0, T_1 + \delta]$ .*

*Proof.* By the assumption,

$$\sup_{0 \leq t \leq T_1} \|\theta(t, \cdot)\|_{H_x^k} < \infty$$

for any  $k \geq 0$  and  $\theta$  is smooth up to time  $T_1$ . Owing to the local existence and regularity theorem, there exists a number  $\delta_1 \in (0, 1)$ , such that we can continue  $\theta$  up to time  $T_1 + \delta_1$  and

$$\sup_{0 \leq t \leq T_1 + \delta_1} \|\theta(t, \cdot)\|_{H_x^{20}} < \infty.$$

By the Sobolev imbedding theorem and the first equation in (1.1),  $\theta \in C^2([0, T_1 + \delta_1] \times \mathbb{R}^2)$  with bounded derivatives up to order two. Since  $\omega$  is unbounded and by Lemma 3.2, there exists  $\delta_2 \in (0, \delta_1)$  so that

$$|\theta(t, x) - \theta(t, y)| < \omega(|x - y|) \quad (3.2)$$

for any  $t \in [T_1, T_1 + \delta_2]$  and  $|x - y| \in (0, \delta_2] \cup [\delta_2^{-1}, \infty)$ .

In what follows we always assume  $|x - y| \in (\delta_2, \delta_2^{-1})$ . Note that in  $[T_1, T_1 + \delta_2] \times \mathbb{R}^2$   $|\nabla_x \theta|$  is uniformly continuous and belongs to  $L^2([T_1, T_1 + \delta_2] \times \mathbb{R}^2)$ . Thus, it goes to zero as  $|x| \rightarrow \infty$  uniformly in  $t$ , and we can find a constant  $N > 0$  such that

$$|\nabla_x \theta(t, x)| < \delta_2 \omega(\delta_2^{-1})$$

for any  $t \in [T_1, T_1 + \delta_1]$  and any  $x$  satisfying  $|x| \geq N$ . Now by the concavity of  $\omega$ , it is easily seen that (3.2) holds for any  $x, y$  satisfying  $|x| \geq N$ ,  $|y| \geq N$ . Finally, if either  $|x| < N$  or  $|y| < N$ , we must have  $(x, y) \in \Omega$ , where

$$\Omega := \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \max[|x|, |y|] \leq N + \delta_2^{-1}, |x - y| \geq \delta_2^{-1}\}.$$

Because of this and since  $\theta \in (T_1, \cdot)$  has strict modulus of continuity  $\omega$  and  $\theta$  is a  $C^2$  function, there exists  $\delta \in (0, \delta_2)$  so that (3.2) in  $[T_1, T_1 + \delta) \times \Omega$ . The lemma is proved.  $\square$

Notice that if  $\theta(t, \cdot)$  has strict modulus of continuity  $\omega$  for all  $t \in [0, T_1)$ , then  $\theta$  is smooth up to  $T_1$  and  $\theta(T_1, \cdot)$  has modulus of continuity  $\omega$  by continuity. Therefore, to show that the modulus of continuity is preserved for all the time, it suffices to rule out the case that

$$\sup_{x \neq y} \frac{\theta(T_1, x) - \theta(T_1, y)}{\omega(|x - y|)} = 1.$$

**Lemma 3.4.** *Under the conditions above, there exist two different points  $x, y \in \mathbb{R}^2$  satisfying*

$$\theta(T_1, x) - \theta(T_1, y) = \omega(|x - y|).$$

Assume for a moment that Lemma 3.4 is proved. We choose a suitable  $\omega$  by letting for  $r > 0$

$$\omega''(r) = -\frac{\delta_3}{r^{1/2} + r^2 \log r}, \quad \omega'(r) = -\int_r^\infty \omega''(s) ds, \quad \omega(0) = 0.$$

Then following the argument in [11], for a sufficiently small  $\delta_3$ , we reach a contradiction:

$$\frac{\partial}{\partial t}(\theta(T_1, x) - \theta(T_2, y)) < 0.$$

Instead of rewriting the proof, we refer the readers to [11], where a slightly different modulus of continuity is constructed.

Now it remains to prove Lemma 3.4.

*Proof.* By the same argument as in the proof of Lemma 3.3, there exist  $\delta_2, N > 0$  such that (3.2) holds in  $\mathbb{R}^2 \setminus \Omega$ . Then the lemma follows from the compactness of  $\Omega$ .  $\square$

We have shown that  $\omega$  is a strict modulus of continuity for all the time and  $\theta$  doesn't have gradient blow-up. Consequently,  $\theta \in L_{\text{loc}}^\infty((0, T), H^k(\mathbb{R}^2))$  for any  $k \geq 0$  and it is a smooth global solution to (1.1). Due to the boundedness of the Riesz transforms and the Sobolev imbedding theorem,  $u$  is also smooth. The uniqueness then follows in a standard way from the local uniqueness result (see, e.g. [16]).

In the last part of this section, we shall prove (2.3). By using Theorem 4.1 of Córdoba and Córdoba [4] we obtain the following decay estimate for  $L^\infty$  norm of the solution.

**Lemma 3.5.** *Under the assumptions of Theorem 2.3, there exists a positive constant  $C$  depending only on  $\theta_0$  so that*

$$\|\theta(t, \cdot)\|_{L^\infty} \leq \frac{C}{1+t}$$

for any  $t \geq t_1$ .

After multiplying the first equation of (1.1) by  $\Delta\theta$ , integrating by parts and using the boundedness of Riesz transforms and the Gagliardo-Nirenberg inequality, we get (see Theorem 2.5 in [5] for details)

$$\frac{1}{2} \frac{d}{dt} \|\nabla\theta(t, \cdot)\|_{L_x^2}^2 + \|\Lambda^{3/2}\theta(t, \cdot)\|_{L_x^2}^2 \leq C_1 \|\theta(t, \cdot)\|_{L_x^\infty} \|\Lambda^{3/2}\theta(t, \cdot)\|_{L_x^2}^2, \quad (3.3)$$

for some constant  $C_1 > 0$ . By Lemma 3.5, there exists  $T_2 \geq t_1$  such that

$$\|\theta(t, \cdot)\|_{L^\infty} \leq \frac{1}{2C_1}$$

holds for any  $t \geq T_2$ . The inequality above and (3.3) yield  $\|\nabla\theta(t, \cdot)\|_{L_x^2}$  is non-increasing in  $[T_2, \infty)$  and  $\|\Lambda^{3/2}\theta(t, \cdot)\|_{L_x^2} \in L^2([T_2, \infty))$ . This together with the  $L^p$ -maximum principle for the quasi-geostrophic equations completes the proof of Theorem 2.3.

#### 4. A DECAY ESTIMATE

The last section is for a decay estimate of the solution. We show that higher order homogeneous Sobolev norms of  $\theta$  decay in time polynomially. A similar decay estimate under a smallness assumption can be found in [9].

By multiplying the first equation of (1.1) by  $\Delta^2\theta$  instead of  $\Delta\theta$ , in a same fashion as (3.3) one can get

$$\frac{1}{2} \frac{d}{dt} \|\Delta\theta(t, \cdot)\|_{L_x^2}^2 + \|\Lambda^{5/2}\theta(t, \cdot)\|_{L_x^2}^2 \leq C_2 \|\theta(t, \cdot)\|_{L_x^\infty} \|\Lambda^{5/2}\theta(t, \cdot)\|_{L_x^2}^2, \quad (4.1)$$

for some constant  $C_2 > 0$ . By Lemma 3.5, there exists  $T_3 \geq t_1$  such that

$$\|\theta(t, \cdot)\|_{L^\infty} \leq \frac{1}{2C_2}$$

holds for any  $t \geq T_3$ . Then (4.1) implies that  $\|\Delta\theta(t, \cdot)\|_{L_x^2}$  is non-increasing in  $[T_3, \infty)$  and thus bounded in  $[1, \infty)$ . Consequently, by interpolation we have

$$\|\theta(t, \cdot)\|_{\dot{H}_x^{1+\varepsilon}} \in L^\infty([1, \infty)) \quad (4.2)$$

for any  $\varepsilon \in [0, 1]$ . Then we can adapt the method in [9] to obtain the following decay in time estimate.

**Theorem 4.1.** *For any  $\varepsilon \in (0, 1)$  and any  $\beta \geq \varepsilon$ , we have*

$$\sup_{1 \leq t < \infty} t^{\beta-\varepsilon} \|\theta(t, \cdot)\|_{\dot{H}^{1+\beta}} < \infty, \quad (4.3)$$

Indeed by using a commutator estimate in Lemma 2.5 of [9], without much more work one can bootstrap from the boundedness (4.2) to get (4.3). Since the proof essentially follows that of Theorem 1.3 in [9], we omit the detail and leave it to interested readers.

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(H. Dong) SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY,  
EINSTEIN DRIVE, PRINCETON, NJ 08540, USA

*E-mail address:* `hjdong@math.ias.edu`

(D. Du) SCHOOL OF MATHEMATICAL SCIENCES, FUDAN UNIVERSITY, SHANGHAI  
200433, P.R. CHINA

*E-mail address:* `dpdu@fudan.edu.cn`