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Regularity Criteria for the Dissipative Quasi-Geostrophic Equations in Hölder Spaces

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Abstract: We study regularity criteria for weak solutions of the dissipative quasi-geostrophic equation (with dissipation $(-\Delta)^{\gamma/2}$, $0 < \gamma \leq 1$). We show in this paper that if $\theta \in C((0, T); C^{1-\gamma})$, or $\theta \in L^r((0, T); C^\alpha)$ with $\alpha = 1 - \gamma + \frac{\gamma}{r}$ is a weak solution of the 2D quasi-geostrophic equation, then θ is a classical solution in $(0, T] \times \mathbb{R}^2$. This result improves our previous result in [18].

1. Introduction

In this paper we present two regularity results for weak solutions of the 2D dissipative quasi-geostrophic equation, that extend our previous work [18]. We consider the following initial value problem:

$$\begin{cases} \theta_t + u \cdot \nabla\theta + (-\Delta)^{\gamma/2}\theta = 0, & x \in \mathbb{R}^2, t \in (0, \infty), \\ \theta(0, x) = \theta_0(x), \end{cases} \tag{1.1}$$

where $\gamma \in (0, 2]$ is a fixed parameter and the velocity $u = (u_1, u_2)$ is divergence free and determined by the Riesz transforms of the potential temperature θ:

$$u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta) = (-\partial_{x_2}(-\Delta)^{-1/2}\theta, \partial_{x_1}(-\Delta)^{-1/2}\theta).$$

The 2D dissipative quasi-geostrophic equation appears in geophysical studies of strongly rotating fluids (see, for example, Pedlosky [24]).

The central mathematical question related to the initial value problem (1.1) is whether there exists a global in time smooth solution to (1.1) evolving from any given smooth initial data. In order to recall known results to this question, we note that cases $\gamma > 1$,

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$\gamma = 1$ and $\gamma < 1$ are called subcritical, critical and supercritical, respectively. Resnick [25] established existence of a global weak solution in both dissipative and non-dissipative cases. The existence of solutions is fully understood in the subcritical case: Constantin and Wu [10] proved that every sufficiently smooth initial data give a rise to a unique global smooth solution. In the critical case, $\gamma = 1$, Constantin, Cordoba and Wu [8] established existence of a unique global classical solution corresponding to any initial data that are small in L^∞. The hypothesis requiring smallness in L^∞ has been removed recently in two elegant papers [1 and 20]. More precisely, Kiselev, Nazarov and Volberg [20] proved persistence of a global solution in C^∞ corresponding to any C^∞ periodic initial data. Their proof is based on a maximum modulus of the continuity principle. In [17] Dong and Du adapted the method of [20] to obtain global well-posedness for the critical 2D dissipative quasi-geostrophic equations with H^1 initial data in the whole space. On the other hand, Caffarelli and Vasseur [1] used harmonic extension to establish regularity of the Leray-Hopf weak solution. More precisely, their approach consists of establishing the following three claims:

(1) Every Leray-Hopf weak solution corresponding to initial data $\theta_0 \in L^2$ is in $L^\infty_{\text{loc}}(\mathbb{R}^2 \times (0, \infty))$.
(2) The L^∞ solutions are Hölder regular, i.e. they are in C^γ for some $\gamma > 0$.
(3) Every Hölder regular solution is a classical solution in $C^{1,\beta}$.

However the question addressing global in time existence of a solution still remains open in the supercritical case, $\gamma < 1$. We note that, in this case Chae and Lee [4], Wu [26,29], Chen, Miao and Zhang [5] and Hmidi and Keraani [19] established existence of a global solution in Besov spaces evolving from small initial data (see also [21,23]). Also recently, Constantin and Wu in [11] implemented the approach of [1] in the super-critical case. They proved that the claim (1) is valid in the super-critical case. Towards addressing the claim (2), Constantin and Wu in [11] proved that L^∞ solutions are Hölder continuous under the additional assumption that the velocity $u \in C^{1-\gamma}$. The claim (3) is considered by Constantin and Wu in a separate paper [12] where they obtained a conditional regularity result of the type: if a Leray-Hopf solution is in the sub-critical space $L^\infty((t_0, t_1); C^\delta(\mathbb{R}^2))$ for some $\delta > 1 - \gamma$ on the time interval $[t_0, t_1]$, then such a solution is a classical solution on $(t_0, t_1]$.

In [18] we extended the result of Constantin and Wu [12] to scaling invariant mixed time-space Besov spaces. More precisely, in [18] we proved that if

$$\theta \in L^r((0, T); B^\alpha_{p,\infty}(\mathbb{R}^2)), \tag{1.2}$$

for any $\gamma \in (0, 1]$, $p \in [2, \infty)$, $T \in (0, \infty)$, $r \in [2, \infty)$ with $\alpha = \frac{2}{p} + 1 - \gamma + \frac{\gamma}{r}$, is a weak solution of the 2D quasi-geostrophic equation (1.1), then θ is a classical solution of (1.1) in $(0, T] \times \mathbb{R}^2$. The significance of this space is that it is scaling invariant under the scaling transformation

$$\theta_\lambda = \lambda^{\gamma-1}\, \theta(\lambda x, \lambda^\gamma t).$$

It is natural to ask whether the result of [18] can be extended to include the case $r = \infty$, $p = \infty$ in (1.2). In this paper we explore that question and prove that if

$$\theta \in C((0, T); C^{1-\gamma}(\mathbb{R}^2))$$

with $\gamma \in (0, 1)$ is a weak solution of (1.1), then θ is a classical solution of (1.1) in the region $(0, T] \times \mathbb{R}^2$. Since $\dot{B}^{\delta}_{\infty,\infty} \cap L^{\infty} = C^{\delta}$, this regularity result extends our previous result [18] to include the case $p = \infty$ and not quite $r = \infty$ (since we require continuity in time). The importance of the space $C^{1-\gamma}(\mathbb{R}^2)$ is in the fact that it is the largest scaling invariant space for the 2D quasi-geostrophic equations (1.1). We note that this new regularity result is inspired by the analogous conditional regularity result for the Navier-Stokes equations that was recently obtained by Cheskidov and Shvydkoy [6]. For the precise statement of our result, see Theorem 3.4. We remark that, as in [6], from the proof it is clear that we allow small jump discontinuities of $\theta(t, \cdot)$ in the $C^{1-\gamma}$ norm.

The proof of Theorem 3.4 relies on a regularity criterion, stated in Lemma 4.1, which exploits a certain cancellation property of the bilinear term. We identify such a cancellation property by means of Bony's paraproduct formula for Littlewood-Paley operators and use of a certain commutator estimate involving Littlewood-Paley operators. The approach that we use to identify the cancellation property differs on a technical level from the approach employed in [6], where the authors followed [7] in order to identify the cancellation property.

Thanks to the above mentioned cancellation property of the nonlinear term, we present another conditional regularity result too (see Theorem 3.5), which extends our previous result [18] to include the case $p = \infty, r \in [1, \infty)$ in (1.2).

Organization of the paper. The paper is organized as follows. In Sect. 2 we introduce the notation and we review known estimates that shall be used throughout the paper. In Sect. 3 we state the main results of the paper. In Sect. 4 we present a proof of the crucial regularity criterion (Lemma 4.1) which is based on the cancellation property. Also in Sect. 4 we give a proof of Theorem 3.4. Then in Sect. 5 we give a proof of Theorem 3.5.

2. Notation and Preliminaries

2.1. Notation and spaces. We recall that for any $\beta \in \mathbb{R}$ the fractional Laplacian $(-\Delta)^{\beta}$ is defined via its Fourier transform:

$$\widehat{(-\Delta)^{\beta} f}(\xi) = |\xi|^{2\beta} \hat{f}(\xi).$$

We note that by a weak solution to (1.1) we mean $\theta(t, x)$ in $(0, \infty) \times \mathbb{R}^2$ such that for any smooth function $\phi(t, x)$ satisfying $\phi(t, \cdot) \in \mathcal{S}$ for each t, the identity

$$\int_{\mathbb{R}^2} \theta(T, \cdot)\phi(T, \cdot)\, dx - \int_{\mathbb{R}^2} \theta(0, \cdot)\phi(0, \cdot)\, dx - \int_0^T \int_{\mathbb{R}^2} \theta\phi_t\, dx\, dt$$
$$- \int_0^T \int_{\mathbb{R}^2} u\theta\nabla\phi\, dx\, dt + \int_0^T \int_{\mathbb{R}^2} \theta\Lambda^{\gamma}\phi\, dx\, dt = 0$$

holds for any $T > 0$.

Before we give the definition of the spaces that will be used throughout the paper, we shall review the Littlewood-Paley decomposition. For any integer j, define Δ_j to be the Littlewood-Paley projection operator with $\Delta_j v = \phi_j * v$, where

$$\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi), \quad \hat{\phi} \in C_0^{\infty}(\mathbb{R}^2\backslash\{0\}), \quad \hat{\phi} \geq 0,$$
$$\operatorname{supp}\hat{\phi} \subset \{\xi \in \mathbb{R}^2 \mid 1/2 \leq |\xi| \leq 2\}, \quad \sum_{j\in\mathbb{Z}} \hat{\phi}_j(\xi) = 1 \text{ for } \xi \neq 0.$$

Formally, we have the Littlewood-Paley decomposition

$$v(\cdot,t)=\sum_{j\in\mathbb{Z}}\Delta_j v(\cdot,t).$$

Also denote

$$\Lambda=(-\Delta)^{1/2},\quad \bar{\Delta}_{-1}=\sum_{k<0}\Delta_k,\quad \Delta_{\le j}=\sum_{k\le j}\Delta_k,$$

$$v_j=\Delta_j v,\quad v_{\le j}=\sum_{k=-\infty}^{j}v_k,\quad v_{\ge j}=\sum_{k=j}^{\infty}v_k,\quad v_{i\le\cdot\le j}=\sum_{k=i}^{j}v_k.$$

For any $p,q\in[1,\infty]$ and $s\in\mathbb{R}$, we denote by $\dot{B}^s_{p,q}$ and $B^s_{p,q}$, respectively the homogeneous and inhomogeneous Besov spaces equipped with norms

$$\|v\|_{\dot{B}^s_{p,q}}:=\begin{cases}\left(\sum_{j\in\mathbb{Z}}2^{jsq}\|\Delta_j v\|^q_{L^p}\right)^{1/q}, & \text{for } q<\infty,\\ \sup_{j\in\mathbb{Z}}2^{js}\|\Delta_j v\|_{L^p}, & \text{for } q=\infty,\end{cases}$$

$$\|v\|_{B^s_{p,q}}:=\begin{cases}\left(\sum_{j\ge 0}2^{jsq}\|\Delta_j v\|^q_{L^p}\right)^{1/q}+\|\bar{\Delta}_{-1}v\|_{L^p}, & \text{for } q<\infty,\\ \sup_{j\ge 0}2^{js}\|\Delta_j v\|_{L^p}+\|\bar{\Delta}_{-1}v\|_{L^p}, & \text{for } q=\infty.\end{cases}$$

If $s>0$, we have

$$B^s_{p,q}=\dot{B}^s_{p,q}\cap L^p,\quad \|v\|_{B^s_{p,q}}\sim\|v\|_{\dot{B}^s_{p,q}}+\|v\|_{L^p},\quad C^s=B^s_{\infty,\infty}.$$

2.2. Preliminaries. The following Bernstein's inequality is well-known.

Lemma 2.1. i) *Let $p\in[1,\infty]$ and $s\in\mathbb{R}$. Then for any $j\in\mathbb{Z}$, we have*

$$\lambda 2^{js}\|\Delta_j v\|_{L^p}\le\|\Lambda^s\Delta_j v\|_{L^p}\le\lambda' 2^{js}\|\Delta_j v\|_{L^p} \tag{2.1}$$

with some constants λ and λ' depending only on p and s.

ii) *Moreover, for $1\le p\le q\le\infty$, there exists a positive constant C depending only on p and q such that*

$$\|\Delta_j v\|_{L^q}\le C2^{(1/p-1/q)dj}\|\Delta_j v\|_{L^p}. \tag{2.2}$$

Now we recall the generalized Bernstein's inequality and a lower bound for an integral involving a fractional Laplacian which will be used in the paper. They can be found in [21,28] and [5].

Lemma 2.2. i) *Let $p\in[2,\infty)$ and $\gamma\in[0,2]$. Then for any $j\in\mathbb{Z}$, we have*

$$\lambda 2^{\gamma j/p}\|\Delta_j v\|_{L^p}\le\|\Lambda^{\gamma/2}(|\Delta_j v|^{p/2})\|^{2/p}_{L^2}\le\lambda' 2^{\gamma j/p}\|\Delta_j v\|_{L^p}, \tag{2.3}$$

with some positive constants λ and λ' depending only on p and γ.

ii) *Moreover, we have*

$$\int_{\mathbb{R}^2} (\Lambda^\gamma v)|v|^{p-2}v \geq c\|\Lambda^{\gamma/2}|v|^{p/2}\|_{L^2}^2, \tag{2.4}$$

and

$$\int_{\mathbb{R}^2} (\Lambda^\gamma \Delta_j v)|\Delta_j v|^{p-2}\Delta_j v \geq c2^{\gamma j}\|\Delta_j v\|_{L^p}^p, \tag{2.5}$$

with some positive constant c depending only on p and γ.

Also we will use the following commutator estimate on the Littlewood-Paley projection operator.

Lemma 2.3. *Let $d \geq 1$ be an integer, $r, r_1, r_2 \in [1,\infty]$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1$. Then for any $j \in \mathbb{Z}$ we have*

$$\|[u, \Delta_j]v\|_{L^r(\mathbb{R}^d)} \leq C2^{-j}\|\nabla u\|_{L^{r_1}(\mathbb{R}^d)}\|v\|_{L^{r_2}(\mathbb{R}^d)} \tag{2.6}$$

as long as the right-hand side is finite. Here C is a positive constant independent of j, and

$$[u, \Delta_j]v = u\Delta_j(v) - \Delta_j(uv).$$

Proof. This follows easily from the integral representation of the Littlewood-Paley projection, Minkowski inequality and Hölder's inequality. □

Finally we recall the following regularity criterion for (1.1), which is the main result of [18].

Theorem 2.4. *Let $\gamma \in (0,1]$, $p \in [2,\infty)$, $T \in (0,\infty)$ and $r \in [2,\infty)$. Denote by $\alpha = \frac{2}{p} + 1 - \gamma + \frac{\gamma}{r}$. If*

$$\theta \in L^r((0,T); B^\alpha_{p,\infty}(\mathbb{R}^2))$$

is a weak solution of (1.1), then θ is in $C^\infty((0,T] \times \mathbb{R}^2)$, and thus it is a classical solution of (1.1) in the region $(0,T] \times \mathbb{R}^2$.

3. Formulation of Results

Assumption 3.1. In the sequel, we always assume that θ is regular at time $t = 0$. More precisely, we assume $\theta(0,\cdot)$ is in $B^{\delta_0}_{p_0,q_0}$ for some $p_0 \in [1,\infty)$, $q_0 \in [1,\infty]$ and $\delta_0 > 1 - \gamma + \frac{2}{p_0}$.

Remark 3.2. Assumption 3.1 seems quite natural due to the local smoothing effect of (1.1) (see, for instance, [15 and 16]).

Remark 3.3. Because of the well-known embedding relations of Besov spaces, we have $\theta(0,\cdot) \in B^\delta_{p,q}$ for any $p \in [p_0,\infty]$, $q \in [1,\infty]$ and some $\delta > 1 - \gamma + \frac{2}{p}$. Moreover, by the L^p maximum principle for (1.1), it holds that $\theta \in L^\infty([0,\infty); L^p)$ for any $p \in [p_0,\infty]$.

Now we state the main results of the paper. The first theorem states that weak solutions in certain critical Hölder spaces are regular.

Theorem 3.4. *Let $\gamma \in (0, 1)$ and $T \in (0, \infty)$. If*

$$\theta \in C((0, T); C^{1-\gamma}(\mathbb{R}^2))$$

is a weak solution of (1.1), then θ is in $C^\infty((0, T] \times \mathbb{R}^2)$, and thus it is a classical solution of (1.1) in the region $(0, T] \times \mathbb{R}^2$.

The second theorem extends Theorem 2.4 to the limiting case $p = \infty$.

Theorem 3.5. *Let $\gamma \in (0, 1)$, $T \in (0, \infty)$, $r \in [1, \infty)$ and $\alpha = 1 - \gamma + \frac{\gamma}{r}$. If*

$$\theta \in L^r((0, T); C^\alpha(\mathbb{R}^2))$$

is a weak solution of (1.1), then θ is in $C^\infty((0, T] \times \mathbb{R}^2)$, and thus it is a classical solution of (1.1) in the region $(0, T] \times \mathbb{R}^2$.

Remark 3.6. It is not clear to us if the result of Theorem 3.5 still holds true when $r = \infty$. In some sense, Theorem 3.4 gives a partial answer to this open problem (see also Lemma 4.2). On the other hand, for the critical dissipative quasi-geostrophic equation, i.e. $\gamma = 1$, Caffarelli and Vasseur [1] established that any weak solution in $L^\infty_{\text{loc}}((0, \infty) \times \mathbb{R}^2)$ is regular.

4. Proof of Theorem 3.4

As we mentioned in the introduction, Theorem 3.4 is inspired by the analogous theorem for the Navier-Stokes equations presented in [6]. The proof of Theorem 3.4 relies on a regularity criterion stated in Lemma 4.2. On the other hand the proof of Lemma 4.2 is based on the regularity criterion formulated in Lemma 4.1 which exploits a certain cancellation property of the nonlinear term. We identify such a cancellation property by using Bony's paraproduct formula for Littlewood-Paley operators and a commutator estimate involving Littlewood-Paley operators. Hence on a technical level our approach differs from the approach employed in [6].

Lemma 4.1. *Let θ be a weak solution of (1.1) in $[0, T] \times \mathbb{R}^2$. Then there exists a positive constant ε_0 such that if θ satisfies*

$$\limsup_{j \to \infty} \sup_{t \in (0,T)} 2^{j(1-\gamma)} \|\theta_j(t, \cdot)\|_{L^\infty_x} < \varepsilon_0, \tag{4.1}$$

then $\theta(t, x)$ is regular in $[0, T] \times \mathbb{R}^2$.

Proof. We prove the lemma by contradiction. Suppose θ blows up in $(0, T]$. Without loss of generality, one may assume T is the first blow-up time. Let us start by applying the operator Δ_j, $j > 0$ to the first equation in (1.1) and use the divergence-free property of u to obtain

$$\partial_t \theta_j + \nabla \cdot \Delta_j(u\theta) + \Lambda^\gamma \theta_j = 0. \tag{4.2}$$

We multiply (4.2) by $|\theta_j|^{p-2}\theta_j$, where p is an even number to be specified later, and integrate in x to obtain

$$\frac{1}{p}\frac{d}{dt}\|\theta_j\|_{L^p}^p + \int_{\mathbb{R}^2} (\Lambda^\gamma \theta_j)\ |\theta_j|^{p-2}\theta_j\, dx = \int_{\mathbb{R}^2} \nabla\cdot\Delta_j(u\theta)\ |\theta_j|^{p-2}\theta_j\, dx. \tag{4.3}$$

Fix an integer $N \geq 10$ and fix an $\varepsilon \in (0, 1)$. In order to simplify the notation we will denote by β,

$$\beta = 2 + p(1-\gamma). \tag{4.4}$$

Now we use Lemma 2.2 to obtain a lower bound on the second term on the left-hand side of (4.3) to derive

$$\frac{1}{p}\frac{d}{dt}\|\theta_j\|_{L^p}^p + \lambda 2^{\gamma j}\|\theta_j\|_{L^p}^p \leq \int_{\mathbb{R}^2} \nabla\cdot\Delta_j(u\theta)\ |\theta_j|^{p-2}\theta_j\, dx,$$

which after being multiplied by $2^{j(\beta+\varepsilon)}$ and summed over $j \geq N$ gives:

$$\begin{aligned} &\frac{1}{p}\sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)}\frac{d}{dt}\|\theta_j\|_{L^p}^p + \lambda\sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon+\gamma)}\|\theta_j\|_{L^p}^p \\ &\quad\leq \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)}\int_{\mathbb{R}^2} \nabla\cdot\Delta_j(u\theta)\ |\theta_j|^{p-2}\theta_j\, dx. \end{aligned} \tag{4.5}$$

In order to bound the term on the right-hand side of (4.5) we split the nonlinear term $\Delta_j(u\theta)$ by applying Bony's decomposition and the localization properties of the Littlewood-Paley operators as follows:

$$\Delta_j(u\theta) = N_{j,lh} + N_{j,hl} + N_{j,hh},$$

where

$$\begin{aligned} N_{j,lh} &= \Delta_j(u_{\leq j+4}\theta_{j-2\leq\cdot\leq j+2}), \\ N_{j,hl} &= \Delta_j(u_{j-2\leq\cdot\leq j+2}\theta_{\leq j-3}), \\ N_{j,hh} &= \sum_{k=j+3}^{\infty}\Delta_j(u_{k-2\leq\cdot\leq k+2}\theta_k). \end{aligned}$$

Hence we can write

$$\sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)}\int_{\mathbb{R}^2} \nabla\cdot\Delta_j(u\theta)\ |\theta_j|^{p-2}\theta_j\, dx = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)}\int_{\mathbb{R}^2} \nabla\cdot N_{j,lh}\ |\theta_j|^{p-2}\theta_j\, dx\,, \\ I_2 &= \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)}\int_{\mathbb{R}^2} \nabla\cdot N_{j,hl}\ |\theta_j|^{p-2}\theta_j\, dx\,, \\ I_3 &= \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)}\int_{\mathbb{R}^2} \nabla\cdot N_{j,hh}\ |\theta_j|^{p-2}\theta_j\, dx. \end{aligned}$$

In what follows we denote $f_{j-2\leq\cdot\leq j+2}$ by $\tilde{f}_j$.

We start by estimating I_1. We use localization properties of Littlewood-Paley operators and the divergence-free property of u to notice that

$$\begin{aligned} I_1 &= \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)} \int_{\mathbb{R}^2} \nabla\cdot\Delta_j(u_{\leq j+4}\tilde{\theta}_j)\ |\theta_j|^{p-2}\theta_j\,dx \\ &= \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)} \int_{\mathbb{R}^2} \nabla\cdot\Delta_{\leq j+6}\left(\Delta_j(u_{\leq j+4}\tilde{\theta}_j) - u_{\leq j+4}\theta_j\right)\ |\theta_j|^{p-2}\theta_j\,dx \\ &= \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)} \int_{\mathbb{R}^2} \nabla\cdot\Delta_{\leq j+6}[\Delta_j, u_{\leq j+4}]\tilde{\theta}_j\ |\theta_j|^{p-2}\theta_j\,dx, \end{aligned}$$

thanks to which we can use Hölder's inequality to get

$$I_1 \lesssim \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)} \left\|\nabla\cdot\Delta_{\leq j+6}[\Delta_j, u_{\leq j+4}]\tilde{\theta}_j\right\|_{L^{\frac{p+1}{2}}} \ \|\theta_j\|_{L^{p+1}}^{p-1}. \tag{4.6}$$

We then apply the commutator estimate stated in Lemma 2.3 to obtain

$$\begin{aligned} \left\|\nabla\cdot\Delta_{\leq j+6}[\Delta_j, u_{\leq j+4}]\tilde{\theta}_j\right\|_{L^{\frac{p+1}{2}}} &\lesssim \|\nabla u_{\leq j+4}\|_{L^{p+1}} \|\tilde{\theta}_j\|_{L^{p+1}} \\ &\lesssim \sum_{k\leq j+4} 2^k \|u_k\|_{L^{p+1}} \|\tilde{\theta}_j\|_{L^{p+1}}. \end{aligned} \tag{4.7}$$

Now we combine (4.6) and (4.7) and use the properties of Riesz transforms as follows:

$$\begin{aligned} I_1 &\lesssim \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)} \|\theta_j\|_{L^{p+1}}^{p-1} \sum_{k\leq j+4} 2^k \|u_k\|_{L^{p+1}} \|\tilde{\theta}_j\|_{L^{p+1}} \\ &\lesssim \sum_{j=N-2}^{\infty} 2^{j(\beta+\varepsilon)} \|\theta_j\|_{L^{p+1}}^{p} \sum_{k\leq j+4} 2^k \|\theta_k\|_{L^{p+1}} \\ &\lesssim \sum_{j=N-2}^{\infty} 2^{j(\beta+\varepsilon+1)\frac{p}{p+1}} \|\theta_j\|_{L^{p+1}}^{p} \sum_{k\leq j+4} \left(\frac{2^k}{2^j}\right)^{1-\frac{\beta+\varepsilon+1}{p+1}} 2^{k(\beta+\varepsilon+1)\frac{1}{p+1}} \|\theta_k\|_{L^{p+1}} \\ &\lesssim \sum_{j=N-2}^{\infty} 2^{j(\beta+\varepsilon+1)} \|\theta_j\|_{L^{p+1}}^{p+1} + \sum_{j=N-2}^{\infty} \sum_{k\leq j+4} \left(\frac{2^k}{2^j}\right)^{\frac{p-\beta-\varepsilon}{2}} 2^{k(\beta+\varepsilon+1)} \|\theta_k\|_{L^{p+1}}^{p+1} \end{aligned} \tag{4.8}$$

$$\lesssim \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon+1)} \|\theta_j\|_{L^{p+1}}^{p+1} + R(N), \tag{4.9}$$

where

$$R(N) = \sup_{t\in(0,T)} \sum_{l=-1}^{N-1} 2^{l(\beta+\varepsilon+1)} \|\theta_l\|_{L^{p+1}}^{p+1}. \tag{4.10}$$

We note that in order to obtain (4.8) we use Young's inequality, Hölder's inequality and we require that p satisfies $p - \beta - \varepsilon > 0$. Hence we choose p such that

$$p > \frac{2+\varepsilon}{\gamma}. \tag{4.11}$$

In an analogous way we obtain the following upper bound on I_2:

$$I_2 \lesssim \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon+1)} \|\theta_j\|_{L^{p+1}}^{p+1} + R(N). \tag{4.12}$$

Now we obtain an upper bound on I_3. We start by applying the Hölder inequality:

$$I_3 \lesssim \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)} \|\nabla \cdot N_{j,hh}\|_{L^{\frac{p+1}{2}}} \|\theta_j\|_{L^{p+1}}^{p-1}. \tag{4.13}$$

To estimate $\|\nabla \cdot N_{j,hh}\|_{L^{\frac{p+1}{2}}}$ we apply the Hölder inequality and properties of the Riesz transform to obtain

$$\begin{aligned}\|\nabla \cdot N_{j,hh}\|_{L^{\frac{p+1}{2}}} &= \|\nabla \cdot \sum_{k=j+3}^{\infty} \Delta_j (u_{k-2\leq\cdot\leq k+2}\theta_k)\|_{L^{\frac{p+1}{2}}} \\ &\lesssim 2^j \sum_{k=j+3}^{\infty} \|\tilde{\theta}_k\|_{L^{p+1}} \|\theta_k\|_{L^{p+1}},\end{aligned}$$

which combined with (4.13) gives

$$\begin{aligned} I_3 &\lesssim \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon+1)} \|\theta_j\|_{L^{p+1}}^{p-1} \sum_{k=j+3}^{\infty} \|\tilde{\theta}_k\|_{L^{p+1}} \|\theta_k\|_{L^{p+1}} \\ &\leq \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon+1)\,\frac{p-1}{p+1}} \|\theta_j\|_{L^{p+1}}^{p-1} \sum_{k=j+3}^{\infty} \left(\frac{2^j}{2^k}\right)^{\frac{2(\beta+\varepsilon+1)}{p+1}} 2^{k\frac{2(\beta+\varepsilon+1)}{p+1}} \|\tilde{\theta}_k\|_{L^{p+1}}^2 \\ &\lesssim \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon+1)} \|\theta_j\|_{L^{p+1}}^{p+1} + \sum_{j=N}^{\infty} \sum_{k=j+3}^{\infty} \left(\frac{2^j}{2^k}\right)^{\frac{\beta+\varepsilon+1}{2}} 2^{k(\beta+\varepsilon+1)} \|\tilde{\theta}_k\|_{L^{p+1}}^{p+1} \qquad (4.14) \\ &\lesssim \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon+1)} \|\theta_j\|_{L^{p+1}}^{p+1}, \qquad (4.15)\end{aligned}$$

where to obtain (4.14) we used Young's inequality.

Now we combine (4.5), (4.9), (4.12) and (4.15) to obtain

$$\frac{1}{p}\frac{d}{dt} \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)} \|\theta_j\|_{L^p}^p + \lambda \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon+\gamma)} \|\theta_j\|_{L^p}^p \lesssim \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon+1)} \|\theta_j\|_{L^{p+1}}^{p+1} + R(N),$$

which thanks to the following interpolation inequality

$$\|\theta_j\|_{L^{p+1}}^{p+1} \leq \|\theta_j\|_{L^p}^p \|\theta_j\|_{L^\infty},$$

implies

$$\frac{1}{p}\frac{d}{dt}\sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)}\|\theta_j\|_{L^p}^p + \lambda \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon+\gamma)}\|\theta_j\|_{L^p}^p$$
$$\le C_1 \sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon+\gamma)}\|\theta_j\|_{L^p}^p \left(2^{j(1-\gamma)}\|\theta_j\|_{L^\infty}\right) + C_1 R(N), \tag{4.16}$$

where C_1 is some universal constant. We choose N such that $2^{j(1-\gamma)}\|\theta_j\|_{L^\infty} < \frac{\lambda}{C_1}$ for all $j \ge N$ and all $t \in (0, T)$. Hence (4.16) implies that for any $t \in (0, T)$,

$$\frac{1}{p}\frac{d}{dt}\sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)}\|\theta_j\|_{L^p}^p \le C_1 R(N).$$

Since β is given via (4.4) and $R(N) < \infty$, we can use Theorem 2.4 to conclude that $\theta(t)$ is regular on $(0, T]$, which gives a contradiction. □

Lemma 4.2. *Let θ be a weak solution of* (1.1) *in* $[0, T] \times \mathbb{R}^2$. *There exists a positive constant ε_1 such that if θ satisfies*

$$\sup_{t\in(0,T]} \limsup_{s\to t^-} \|\theta(t,\cdot) - \theta(s,\cdot)\|_{C^{1-\gamma}} < \varepsilon_1, \tag{4.17}$$

then $\theta(t, x)$ is regular in $[0, T] \times \mathbb{R}^2$.

Proof. We prove the lemma by contradiction and without loss of generality assume T be the first blow-up time of θ. Because of (4.17), there exists $t_1 \in (0, T)$ such that $\|\theta(T,\cdot) - \theta(t_1,\cdot)\|_{C^{1-\gamma}} < \varepsilon_1$. Since $\theta(t_1,\cdot)$ is regular, we have

$$\limsup_{j\to\infty} 2^{j(1-\gamma)}\|\theta_j(t_1,\cdot)\|_{L_x^\infty} = 0,$$

and therefore,

$$\limsup_{j\to\infty} 2^{j(1-\gamma)}\|\theta_j(T,\cdot)\|_{L_x^\infty} < \varepsilon_1.$$

This and (4.17) imply that for some $t_2 \in (0, T)$,

$$\limsup_{j\to\infty} \sup_{s\in(t_2,T)} 2^{j(1-\gamma)}\|\theta_j(s,\cdot)\|_{L_x^\infty} < 2\varepsilon_1.$$

To get a contradiction, it suffices to set $\varepsilon_1 = \varepsilon_0/2$ and apply Lemma 4.1. □

Proof of Theorem 3.4. It follows directly from Lemma 4.2. □

5. Proof of Theorem 3.5

By applying Young's inequality, we get

$$2^{j(\beta+\varepsilon+1)}\|\theta_j\|_{L^\infty_x} \le C_2 2^{j(\beta+\varepsilon+\gamma+r(1-\gamma))}\|\theta_j\|^r_{L^\infty_x} + \frac{\lambda}{2C_1}2^{j(\beta+\gamma+\varepsilon)},$$

for some constant $C_2 > 0$ depending only on λ, p and r. This together with (4.16) yields

$$\frac{1}{p}\frac{d}{dt}\sum_{j=N}^{\infty} 2^{j(\beta+\varepsilon)}\|\theta_j\|^p_{L^p_x} + \frac{\lambda}{2}\sum_{j=N}^{\infty} 2^{j(\beta+\gamma+\varepsilon)}\|\theta_j\|^p_{L^p_x}$$
$$\le C_2 \sum_{j=N}^{\infty}\left(2^{j(\beta+\varepsilon)}\|\theta_j\|^p_{L^p_x}\right)\left(2^{j(1-\gamma+\frac{\gamma}{r})}\|\theta_j\|_{L^\infty_x}\right)^r + R(N). \qquad (5.1)$$

Due to the definition of Besov spaces, the right-hand side of (5.1) is less than or equal to

$$C_2\sum_{j=N}^{\infty}\left(2^{j(\beta+\varepsilon)}\|\theta_j\|^p_{L^p_x}\right)\|\theta\|^r_{C^\alpha} + R(N).$$

To finish the proof of Theorem 3.5, it suffices to use Gronwall's inequality and Theorem 2.4 keeping in mind that β is given via (4.4).

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