A REMARK ON THE HARD LEFSCHETZ THEOREM FOR KÄHLER ORBIFOLDS

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ABSTRACT. We give a proof of the Hard Lefschetz Theorem for orbifolds that does not involve intersection homology. We use a foliated version of the Hard Lefschetz Theorem due to El Kacimi.

INTRODUCTION

Even though orbifolds are not smooth manifolds in general, these mildly singular objects retain a strong flavor of smoothness. For example, for a compact orientable orbifold V, Poincaré duality holds over the reals. If V is moreover Kähler, it satisfies the Hard Lefschetz Theorem (HLT).

The former was proved by Satake, by adapting the classical proof. However, the only known proof of the HLT for orbifolds uses the far-reaching version of the HLT in intersection homology. Fulton raises the problem of finding a more direct proof (*cf.* [Ful], p.105). We propose a solution based on a result of El Kacimi (*cf.* [EK1]). We emphasize that the methods that we are using here are well known to the specialists of Riemannian foliations (*cf.* [EK1], [EK2], [GHS], [Mo], [Rum], [Ton]).

1. THE RESOLUTION OF AN ORBIFOLD AS A SEIFERT BUNDLE

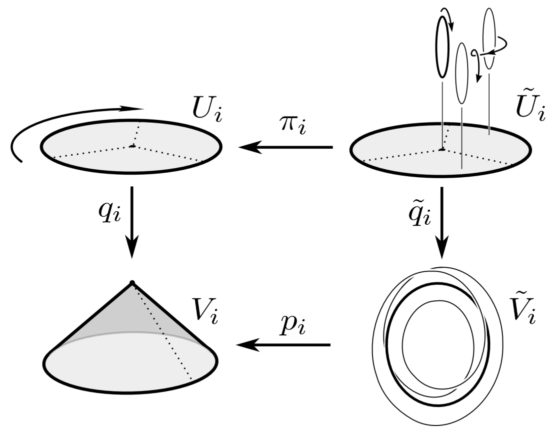
Definition 1.1. Let $|V|$ be a Hausdorff topological space. A *Kähler orbifold* structure V on $|V|$ is given by the following:
(i) An open cover $\{V_i\}_i$ of $|V|$.
(ii) For each $i \in I$, a connected and open $U_i \subset \mathbb{C}^n$ with a Kähler metric h_i; a finite subgroup Γ_i of holomorphic isometries of U_i; a continuous map $q_i : U_i \to V_i$, called a *local uniformization*, inducing a homeomorphism from $\Gamma_i \setminus U_i$ onto V_i.
(iii) For all $x_i \in U_i$ and $x_j \in U_j$ such that $q_i(x_i) = q_j(x_j)$, there exist $W_i \subset U_i$ and $W_j \subset U_j$, open connected neighbourhoods of x_i and x_j, and a holomorphic isometry $\phi_{ji} : W_i \to W_j$, called a *change of charts*, such that $q_j \phi_{ji} = q_i$ on W_i.

Remark: it is true, but not trivial, that V is Kähler in the above "orbifold" sense if and only if, as a complex space, V is Kähler in the sense of Grauert (defined in [G]).

Let V be a compact Kähler orbifold. The representation of V as the leaf space of a smooth foliation is due to [GHS], Prop. 4.1. We explain their construction, and check that the foliation is transversely Kähler.

Fix i. For any local uniformization $q_i : U_i \to V_i$, we denote by $\pi_i : \tilde{U}_i \to U_i$ the principal bundle of unitary frames. The Γ_i-action on U_i lifts naturally to

2000 *Mathematics Subject Classification.* Primary 14F25; Secondary 53C12.



$\tilde{U}_i$. Indeed, let $\xi_i = (\xi_i^1, \ldots, \xi_i^n) \in \tilde{U}_i$ be a unitary frame at x_i and let $g_i \in \Gamma_i$. Then $g_i\xi_i := \big((g_i)_*\xi_i^1, \ldots, (g_i)_*\xi_i^n\big)$ is a unitary frame at $g_i x_i$. As an isometry is determined by its derivative at one point, this lifted action is *free*. The quotient $\tilde{V}_i := \Gamma_i \setminus \tilde{U}_i$ is a $\mathcal{C}^\infty$*-smooth manifold*, and $\tilde{q}_i : \tilde{U}_i \to \tilde{V}_i$ is a finite non-ramified covering.

Fix i and j. Any change of charts $\phi_{ji} : W_i \subset U_i \to W_j \subset U_j$ lifts, as above, to a diffeomorphism $\tilde{\phi}_{ji} : \tilde{W}_i \subset \tilde{U}_i \to \tilde{W}_j \subset \tilde{U}_j,\ \xi_i \mapsto \big((\phi_{ji})_*\xi_i^1, \ldots, (\phi_{ji})_*\xi_i^n\big)$.

Remark 1.2. For any two changes of charts ϕ_{ji} and ϕ'_{ji} defined on the same $W_i \subset U_i$, there exists $g_j \in \Gamma_j$ such that $\phi'_{ji} = g_j\phi_{ji}$ (this follows from the classical case of a non-ramified q_j). Accordingly, $\tilde{\phi}'_{ji} = g_j\tilde{\phi}_{ji}$ on $\tilde{W}_i \subset \tilde{U}_i$.

Denote $V_{ji} := V_i \cap V_j$. We want to define a lifting of the obvious gluing map

$$f_{ji} : V_{ji} \subset V_i \to V_{ji} \subset V_j$$

to a gluing diffeomorphism

$$\tilde{f}_{ji} : p_i^{-1}(V_{ji}) \subset \tilde{V}_i \to p_j^{-1}(V_{ji}) \subset \tilde{V}_j.$$

Let $z_i \in p_i^{-1}(V_{ji})$. Let $\phi_{ji} : W_i \to W_j$ be a change of charts with $y_i \in \tilde{W}_i$ such that $\tilde{q}_i(y_i) = z_i$. Take $\tilde{f}_{ji}(z_i) := \tilde{q}_j\tilde{\phi}_{ji}(y_i)$. This is well-defined: if we choose another point $g_i y_i$ and another change of charts $\phi'_{ji} : g_iW_i \to W'_j$, then both ϕ_{ji} and $\phi'_{ji}g_i$ are changes of charts on W_i, so, by *Rem.* 1.2, there is a $g_j \in \Gamma_j$ such that $\tilde{\phi}'_{ji}g_i = g_j\tilde{\phi}_{ji}$. Thus $\tilde{q}_j\tilde{\phi}'_{ji}g_i(y_i) = \tilde{q}_jg_j\tilde{\phi}_{ji}(y_i) = \tilde{q}_j\tilde{\phi}_{ji}(y_i)$.

Also from *Rem.* 1.2, for all $i, j, k,\ \tilde{f}_{ki} = \tilde{f}_{kj}\tilde{f}_{ji}$. Therefore, the disjoint collection $\{\tilde{V}_i\}_i$ glued according to $\{\tilde{f}_{ji}\}_{ij}$ yields a smooth compact manifold that we call $\tilde{V}$. It is Hausdorff as a consequence of the following

Proposition 1.3. *For any compact orbifold V, the manifold $\tilde{V}$ constructed as above admits a $U(n)$-action, turning $p : \tilde{V} \to V$ into a Seifert principal bundle.*

Proof. First we define, for any fixed i, a right $U(n)$-action on $\tilde{U}_i$. Let $\xi_i = (\xi_i^1, \ldots, \xi_i^n) \in \tilde{U}_i$, and $A = (A^{\alpha\beta})_{1\leq\alpha,\beta\leq n} \in U(n)$. We define ξ_iA as the "matrix product" $(\xi_i^1, \ldots, \xi_i^n)A$.

This means that $\xi_i A := (\zeta^1, \dots, \zeta^n)$ where for all β, $\zeta^\beta := \sum_\alpha A^{\alpha\beta}\xi_i^\alpha$. Now, by linearity of the derivative, for all g_i, A and ξ_i, $\quad g_i(\xi_i A) = (g_i\xi_i)A$. So the action descends to $\tilde{V}_i$. Similarly, the $U(n)$-actions on $\tilde{V}_i$ and $\tilde{V}_j$ commute with $\tilde{f}_{ji}$, endowing $\tilde{V}$ with a global action.

The Seifert bundle structure follows from the construction. □

Proposition 1.4. *The foliation $\mathfrak{F}$ of $\tilde{V}$ by $U(n)$-orbits is transversely Kähler, i.e., $\mathfrak{F}$ is transversely holomorphic (so the normal bundle admits a complex structure J), and there exists a closed real 2-form ω on $\tilde{V}$ such that:*

(i) the tangent bundle of $\mathfrak{F}$ is the kernel of ω;

(ii) The quadratic form $h(-,-) := \omega(J-,-) + \sqrt{-1}\omega(-,-)$ defines a Hermitian metric on the normal bundle of $\mathfrak{F}$.

Proof. For an open subset $O \subset \tilde{V}$ small enough, we can assume that $O \subset \tilde{q}_i(\tilde{V}_i)$ for some i, and take a smooth section $\sigma : O \to \tilde{V}_i$ of $\tilde{q}_i$. Then, on O, the foliation $\mathfrak{F}$ is defined by the submersion $\pi_i\sigma : O \to U_i$, which induces an isomorphism between the normal bundle of $\mathfrak{F}$ and the tangent bundle of U_i. By hypothesis, we have a Γ_i-invariant Kähler form ω_i on U_i. Define $\omega := \pi_i^*\omega_i$, a well-defined form on $\tilde{V}$ that satisfies the above properties. □

2. A TAUT AND TRANSVERSELY KÄHLER METRIC

Let g_0 be a Riemannian metric on $\tilde{V}$. Up to averaging it over the action, we can assume that it is $U(n)$-invariant. Denote $m = \dim U(n)$. Fix a global frame on $U(n)$ given by right-invariant vector fields $E_1, \dots, E_m$. Let $e_1, \dots, e_m$ be the associated fundamental vector fields on $\tilde{V}$, and for any $[\xi] \in \tilde{V}$, let $M_0[\xi] := \Big(g_0\big(e_k[\xi], e_l[\xi]\big)\Big)_{1\le k,l\le m} \in \mathbb{R}^{m\times m}$, and define a smooth function u on $\tilde{V}$ by

$$u_0[\xi] := \big(\det M_0[\xi]\big)^{-\frac{1}{m}}.$$

By invariance, M_0 and u_0 are constant on each orbit. Now, we define a new Riemannian metric on $\tilde{V}$ by the conformal change $g_1 = u_0 g_0$.

Proposition 2.1. *With respect to g_1, the $U(n)$-orbits are minimal submanifolds.*

Proof. (inspired by the proof of [Ton] Prop 7.6.) Since this is a local statement, we can assume that we are on $\tilde{V}_i$ for some i. Using the covering map $\tilde{q}_i$, we lift the metric g_1 to a metric h_1 on $\tilde{U}_i$. Let $X_0 \subset \tilde{U}_i$ be any $U(n)$-orbit, and let Z be an arbitrary vector field that is compactly supported and normal to X_0. Let $\{\varphi_t\}_{t\in\mathbb{R}}$ be the flow associated to Z, and define $X_t := \varphi_t(X_0)$. By the first variational formula (or [Jos] 3.6.4), it is enough to prove that

$$\frac{d}{dt}\big(\text{Vol } X_t\big)_{|t=0} = 0.$$

Because the metric h_1 is $U(n)$-invariant, we can replace Z with its average over the action. This new Z is still orthogonal to X_0, and is $U(n)$-invariant, so X_t is a $U(n)$-orbit for every t. On $\tilde{V}$, the volume form of a $U(n)$-orbit is

$$\big(\det M_1\big)^{\frac{1}{2}}\ \check{e}_1 \wedge \cdots \wedge \check{e}_m,$$

where $M_1 = \Big(g_1\big(e_k, e_l\big)\Big)_{1\le k,l\le m}$, and $\check{e}_1, \dots, \check{e}_m$ is the dual basis. On the other hand, $\big(\det M_1\big)^{\frac{1}{2}} = u_0^{\frac{m}{2}}\big(\det M_0\big)^{\frac{1}{2}} = \big((\det M_0)^{-\frac{1}{m}}\big)^{\frac{m}{2}}\big(\det M_0\big)^{\frac{1}{2}} = 1$.

Fix an arbitrary t. Choosing any point on M_t, we get a one-to-one a parameterization $f : U(n) \to X_t$. Then

$$\text{Vol } X_t = \int_{U(n)} f^* \tilde{q}_i^* \Big(\check{e}_1 \wedge \cdots \wedge \check{e}_m\Big) = \int_{U(n)} \check{E}_1 \wedge \cdots \wedge \check{E}_m,$$

which does not depend on t. □

The metric g_1 determines an embedding $N \subset T\tilde{V}$ of the normal bundle of $\mathfrak{F}$. Modifying g_1, we define a metric g on $\tilde{V}$ by replacing $g_{1|N}$ with the metric h obtained in Prop. 1.4. The proof of Prop. 2.1 shows that the leaves of $\mathfrak{F}$ are still minimal submanifolds with respect to g.

3. Main result

Let M be any compact smooth manifold, with a foliation $\mathfrak{F}$. A form $\alpha \in \Omega^*(M, \mathbb{C})$ is called *basic* when for all vectors Z tangent to the foliation, $\iota_Z\alpha = 0 = \iota_Z d\alpha$, where ι_Z denotes the interior product. The basic forms determine a subcomplex of the usual de Rham complex of M. The associated cohomology groups are called the basic cohomology groups, and are denoted $H_B^*(\mathfrak{F}, \mathbb{C})$. They can be thought of as a substitute for the de Rham cohomology of the leaf space, which can be very singular and/or not Hausdorff in general. When the leaf space is an orbifold V, the basic cohomology is isomorphic to the singular cohomology of V over $\mathbb{C}$ (*cf.* [Pfl] 5.3).

Let V be a compact Kähler orbifold, with $\tilde{V}$, $\mathfrak{F}$ and g as described in the previous sections. We want apply El Kacimi's Basic Hard Lefschetz Theorem: if $\mathfrak{F}$ is transversely Kähler with respect to ω, and $H_B^{2n}(\mathfrak{F}, \mathbb{C}) \neq 0$, then for any k, $\alpha \mapsto \alpha \wedge \omega^k$ is an isomorphism $H_B^{n-k}(\mathfrak{F}, \mathbb{C}) \to H_B^{n+k}(\mathfrak{F}, \mathbb{C})$, where n is the complex codimension of $\mathfrak{F}$. We refer to [EK1] for a more complete statement of the theorem.

By a theorem of Masa (*cf.* [Ma]), the existence of a metric turning the leaves into minimal submanifolds is equivalent to $H_B^{2n}(\mathfrak{F}, \mathbb{C}) \neq 0$. This completes the proof of the Hard Lefschetz Theorem for V.

Remark: in order to avoid the use of [Ma], one can easily adapt the proof of [EK1] (written when Masa's theorem was only a conjecture). The assumption $H_B^{2n}(\mathfrak{F}, \mathbb{C}) \neq 0$ is used in [EK1] to prove the classical Hodge-theoretic adjoint formula: $\delta = (-1)^i * \partial *$. Due to the vanishing of the mean curvature associated to our metric g, this formula follows from [Ton] Th. 7.10.

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