AN INTERIOR ESTIMATE FOR CONVEX SOLUTIONS AND A RIGIDITY THEOREM

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ABSTRACT. We establish an interior C^2 estimate for $k+1$ convex solutions to Dirichlet problems of k-Hessian equations. We also use such estimate to obtain a rigidity theorem for $k+1$ convex entire solutions of k-Hessian equations in Euclidean space.

1. INTRODUCTION

In this paper, we consider an interior C^2 estimate for the following Dirichlet problem for k-Hessian equations,

$$\begin{cases}\sigma_k(D^2u) &= f(x,u,\nabla u), \quad \text{in } \Omega,\\ u &= 0, \quad \text{on } \partial\Omega.\end{cases} \tag{1.1}$$

Here, u is a function defined in some domain Ω. ∇u is the gradient of u and D^2u is the Hessian of u. We also require $f>0$ and smooth enough respect to every variables.

The interior C^2 estimates for Monge-Ampère equations were studied at first by A.V. Pogorelov [14], [9]. Then, K.S. Chou and X.-J. Wang extended Pogorelov's estimates to the case of k-Hessian equations of [8], [16]. Explicitly, in their paper, for any function f not depending ∇u in (1.1), they have proved that, for any small positive constant ε, the following estimates hold,

$$(-u)^{1+\varepsilon}\Delta u \leqslant C. \tag{1.2}$$

Here, constant C depends on the domain Ω, k, f and $\sup_\Omega|\nabla u|$.

Maybe a natural question is whether these interior estimates are still valid for that f does depend on the gradient term ∇u, namely, interior estimates for (1.1). For the 2-Hessian equation, we can get this type of interior estimates.

Theorem 1. *For 2-Hessian equations, i.e. $k=2$ in (1.1), there is some constant $\beta>0$, such that*

$$(-u)^{\beta}\Delta u \leqslant C. \tag{1.3}$$

Here positive constants β and C depend on the domain Ω, the function f, $\sup_\Omega|u|$ and $\sup_\Omega|\nabla u|$.

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By some reasons, the small constant ε should not be zero in Chou-Wang's proof. On the other hand, for Monge-Ampère equation case, namely, $k = n$ in (1.1), we can drop the small ε [14], [9]. It reminds us that if the convexity is better, estimate (1.2) can be improved. Using techniques developing in [11], we can get the following theorem,

Theorem 2. *Suppose that function u is a $k+1$ convex solution for the Dirichlet problem of k-Hessian equations* (1.1)*. Namely, function u is in $k+1$ convex cone. We have,*

$$(-u)\Delta u \leqslant C. \tag{1.4}$$

Here, positive constant C depends on $\sup_\Omega |\nabla u|$, $\sup_\Omega |u|$*, the function f and the domain Ω.*

Here the definition of the k-convex cone is following Caffarelli-Nirenberg-Spruck [3],

Definition 3. *For a domain $\Omega \subset \mathbb{R}^n$, a function $v \in C^2(\Omega)$ is called k-convex if the eigenvalues $\kappa(x) = (\kappa_1(x), \cdots, \kappa_n(x))$ of the hessian $\nabla^2 v(x)$ is in Γ_k for all $x \in \Omega$, where Γ_k is the Garding's cone*

$$\Gamma_k = \{\kappa \in \mathbb{R}^n \mid \quad \sigma_m(\kappa) > 0, \quad m = 1, \cdots, k\}.$$

Note that the constant β is large in Theorem 1. We can not improve β to be 1 or $1+\varepsilon$ as Chou-Wang's paper [8] or Theorem 2.

An application of the interior estimates may to prove rigidity theorems for k-Hessian equations. Consider the entire solutions u in n-dimensional Euclidean spaces of the following equations,

$$\sigma_k(D^2 u) = 1. \tag{1.5}$$

S.-Y. A. Chang and Y. Yuan in [6] proposed a problem that: Are the entire solutions of (1.5) with lower bound only quadratic polynomials ?

Let's review known results related the above problem. For $k = 1$, (1.5) is a linear equation. It is a obvious result coming from the Liouville property of the harmonic functions. For $k = n$, Monge-Ampère equation case, it is a well know theorem. For $n = 2$, K. Jörgens [12] proved that every entire strictly convex solution is a quadratic polynomial. Then, E. Calabi [4] obtained the same result for $n = 3, 4, 5$. At last, A.V. Pogorelov [13],[14] gave a proof for all dimensions. Then, S.Y. Cheng and S.T. Yau [7] gave another more geometry proof. In 2003, L. Caffarelli and Y. Li, [5] extended the theorem of Jörgens, Calabi and Pogorelov.

For $k = 2$, S.-Y. A. Chang and Y. Yuan [6] have proved that, if

$$D^2 u \geqslant \delta - \sqrt{\frac{2n}{n-1}},$$

for any $\delta > 0$, then the entire solution of the equation (1.5) only have quadratic polynomials. For general k, it is still open, but J. Bao, J.Y. Chen, B. Guan and M. Ji in [2] obtained that, strictly convex entire solutions of (1.5), satisfying a quadric growth are quadratic polynomials. Here, the quadratic growth means that, there is some positive constant c, b and sufficiently large R, such that,

$$u(x) \geqslant c|x|^2 - b, \text{ for } |x| \geqslant R. \tag{1.6}$$

Note that, our interior estimates Theorem 2 holds for $k+1$ convex solutions. Hence, we can relax their restriction. In deed, we have proved,

Theorem 4. *The entire solutions in $k+1$ convex cone of the equations* (1.5) *defined in $\mathbb{R}^n$ with quadratic growth are quadratic polynomials.*

In our proof, we don't need the assumption of strictly convexity. Hence, we do not use the estimates of L. Caffaralli. Now, we give the following two Lemmas, which will be needed in our proof.

Lemma 5. *Set $k > l$. For $\alpha = \dfrac{1}{k-l}$, we have,*

$$\begin{aligned} & -\frac{\sigma_k^{pp,qq}}{\sigma_k} u_{pph} u_{qqh} + \frac{\sigma_l^{pp,qq}}{\sigma_l} u_{pph} u_{qqh} \\ \geqslant\ & \left(\frac{(\sigma_k)_h}{\sigma_k} - \frac{(\sigma_l)_h}{\sigma_l} \right) \left((\alpha - 1) \frac{(\sigma_k)_h}{\sigma_k} - (\alpha+1) \frac{(\sigma_l)_h}{\sigma_l} \right). \end{aligned} \tag{1.7}$$

further more, for sufficiently small $\delta > 0$, we have,

$$\begin{aligned} & -\sigma_k^{pp,qq} u_{pph} u_{qqh} + (1 - \alpha + \frac{\alpha}{\delta}) \frac{(\sigma_k)_h^2}{\sigma_k} \\ \geqslant\ & \sigma_k(\alpha + 1 - \delta\alpha) \left[\frac{(\sigma_l)_h}{\sigma_l} \right]^2 - \frac{\sigma_k}{\sigma_l} \sigma_l^{pp,qq} u_{pph} u_{qqh}. \end{aligned} \tag{1.8}$$

The another one is,

Lemma 6. *Denote $Sym(n)$ the set of all $n \times n$ symmetric matrices. Let F be a C^2 symmetric function defined in some open subset $\Psi \subset Sym(n)$. At any diagonal matrix $A \in \Psi$ with distinct eigenvalues, let $\ddot{F}(B, B)$ be the second derivative of C^2 symmetric function F in direction $B \in Sym(n)$, then*

$$\ddot{F}(B, B) = \sum_{j,k=1}^{n} \ddot{f}^{jk} B_{jj} B_{kk} + 2 \sum_{j<k} \frac{\dot{f}^j - \dot{f}^k}{\kappa_j - \kappa_k} B_{jk}^2. \tag{1.9}$$

The proof of the first Lemma can be found in [10] and [11]. The second Lemma can be found in [1] and [3].

The paper is organized by three sections. The first section gives the interior estimates of 2-Hessian case. The second section gives the interior estimates for $k+1$ convex solutions. The last section proves the rigidity theorem.

2. An interior C^2 estimate for σ_2 equations

In this section, we prove Theorem 1. We consider the following test function,

$$M = \max_{|\xi|=1, x\in\Omega} (-u)^{\beta} exp\{\frac{\varepsilon}{2}|Du|^2 + \frac{a}{2}|x|^2\} u_{\xi\xi},$$

where β, ε and a are three constants which we will be determined later. Suppose that M achieve its maximum value in Ω at some point x_0 along some direction ξ. We can assume that $\xi = (1, 0, \cdots, 0)$. By rotating the coordinate, we diagonal the matrix (u_{ij}), and we also can assume that $u_{11} \geqslant u_{22} \cdots \geqslant u_{nn}$.

Hence, at x_0, differentiating the test function twice, we have

$$\frac{\beta u_i}{u} + \frac{u_{11i}}{u_{11}} + \varepsilon u_i u_{ii} + a x_i = 0, \tag{2.1}$$

and,

$$\frac{\beta u_{ii}}{u} - \frac{\beta u_i^2}{u^2} + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} + \sum_k \varepsilon u_k u_{kii} + \varepsilon u_{ii}^2 + a \leqslant 0.$$

In the above inequality, contracting with σ_2^{ii}, we have,

$$\begin{aligned}\frac{2\beta\sigma_2}{u} - \frac{\beta\sigma_2^{ii}u_i^2}{u^2} + \frac{\sigma_2^{ii}u_{11ii}}{u_{11}} - \frac{\sigma_2^{ii}u_{11i}^2}{u_{11}^2}& \\ + \sum_k \varepsilon u_k \sigma_2^{ii} u_{kii} + \varepsilon \sigma_2^{ii} u_{ii}^2 + (n-1)a\sigma_1 &\leqslant 0.\end{aligned} \tag{2.2}$$

At x_0, differentiating equation (1.1) twice, we have,

$$\sigma_2^{ii} u_{iij} = f_j + f_u u_j + f_{p_j} u_{jj}, \tag{2.3}$$

and

$$\sigma_2^{ii} u_{iijj} + \sigma_2^{pq,rs} u_{pqj} u_{rsj} \geqslant -C - C u_{jj}^2 + \sum_k f_{p_k} u_{kjj}. \tag{2.4}$$

Inserting (2.4) into (2.2), we have,

$$\begin{aligned}0 \geqslant& \frac{2\beta\sigma_2}{u} - \frac{\beta\sigma_2^{ii}u_i^2}{u^2} + \frac{1}{u_{11}}[-C - Cu_{11}^2 + \sum_k f_{p_k} u_{k11} - K(\sigma_2)_1^2 + K(\sigma_2)_1^2 \\ &- \sigma_2^{pq,rs} u_{pq1} u_{rs1}] - \frac{\sigma_2^{ii} u_{11i}^2}{u_{11}^2} + \sum_k \varepsilon u_k \sigma_2^{ii} u_{kii} + \varepsilon \sigma_2^{ii} u_{ii}^2 + (n-1)a\sigma_1.\end{aligned} \tag{2.5}$$

Using (2.1) and (2.3), we have,

$$\frac{1}{u_{11}} \sum_k f_{p_k} u_{k11} + \sum_k \varepsilon u_k \sigma_k^{ii} u_{kii} \geqslant -C - \sum_k \frac{\beta u_k f_{p_k}}{u}.$$

Note that

$$\begin{aligned}-\sigma_2^{pq,rs} u_{pq1} u_{rs1} =& - \sigma_2^{pp,qq} u_{pp1} u_{qq1} + \sum_{p \neq q} u_{pq1}^2 \\ \geqslant& - \sigma_2^{pp,qq} u_{pp1} u_{qq1} + 2 \sum_{i \neq 1} u_{11i}^2.\end{aligned}$$

Using Lemma 5, there exists some sufficiently large constant K depending on f, such that,

$$K(\sigma_2)_1^2 - \sigma_2^{pp,qq} u_{pp1} u_{qq1} \geqslant 0.$$

Using the above two formulas, inequality (2.5) becomes,

$$-\frac{C}{u} \geqslant -\frac{\beta\sigma_2^{ii}u_i^2}{u^2} + \frac{2}{u_{11}}\sum_{i\neq 1} u_{11i}^2 - \frac{\sigma_2^{ii}u_{11i}^2}{u_{11}^2} + \varepsilon\sigma_2^{ii}u_{ii}^2 + (n-1)a\sigma_1 - Cu_{11} - C. \tag{2.6}$$

Take a sufficiently large a such that,

$$(n-1)a\sigma_1 - Cu_{11} - C \geqslant a\sigma_1.$$

Here, we always assume that u_{11} is sufficiently large. Now we should divide into two cases to deal with other third order derivatives.

(A) Suppose $\sum_{i=2}^{n-1}\lambda_i \leqslant \lambda_1/3$. In this case, using (2.1), we have,

$$-\frac{\beta\sigma_2^{ii}u_i^2}{u^2} \geqslant -\frac{2\sigma_2^{ii}}{\beta}\frac{u_{11i}^2}{u_{11}^2} - \frac{2\sigma_2^{ii}}{\beta}(\varepsilon u_i u_{ii} + ax_i)^2. \tag{2.7}$$

Using (2.6) and (2.7), we have,

$$\begin{aligned}-\frac{C}{u} \geqslant& -\frac{\beta\sigma_2^{11}u_1^2}{u^2} + \frac{2}{u_{11}}\sum_{i\neq 1}u_{11i}^2 - (1+\frac{2}{\beta})\sum_{i\neq 1}\frac{\sigma_2^{ii}u_{11i}^2}{u_{11}^2} \\ &- \frac{\sigma_2^{11}u_{111}^2}{u_{11}^2} + \varepsilon\sigma_2^{ii}u_{ii}^2 - \sum_{i\neq 1}\frac{2\sigma_2^{ii}}{\beta}(\varepsilon u_i u_{ii} + ax_i)^2 + a\sigma_1.\end{aligned} \tag{2.8}$$

Since, $\sum_{i=2}^{n-1}\lambda_i \leqslant \frac{\lambda_1}{3}$, we have, for sufficiently large β,

$$\frac{2}{u_{11}}\sum_{i\neq 1}u_{11i}^2 - (1+\frac{2}{\beta})\sum_{i\neq 1}\frac{\sigma_2^{ii}u_{11i}^2}{u_{11}^2} \geqslant 0.$$

Again, using (2.1), we have,

$$-\frac{\sigma_2^{11}u_{111}^2}{u_{11}^2} \geqslant -2\sigma_2^{11}(\frac{\beta u_1}{u})^2 - 2\sigma_2^{11}(\varepsilon u_1 u_{11} + ax_1)^2.$$

Then we obtain,

$$\begin{aligned}&-\frac{C}{u} + \frac{(\beta+2\beta^2)\sigma_2^{11}u_1^2}{u^2} \\ \geqslant& \varepsilon\sigma_2^{ii}u_{ii}^2 - \sum_{i\neq 1}\frac{2\sigma_2^{ii}}{\beta}(\varepsilon u_i u_{ii} + ax_i)^2 - 2\sigma_2^{11}(\varepsilon u_1 u_{11} + ax_1)^2 + a\sigma_1 \\ \geqslant& \sum_i \varepsilon\sigma_2^{ii}u_{ii}^2 - \sum_{i\neq 1}\frac{4\sigma_2^{ii}}{\beta}\varepsilon^2 u_i^2 u_{ii}^2 - \sum_{i\neq 1}\frac{4\sigma_2^{ii}}{\beta}a^2x_i^2 - 4\sigma_2^{11}\varepsilon^2u_1^2u_{11}^2 \\ &- 4\sigma_2^{11}a^2x_1^2 + au_{11}.\end{aligned} \tag{2.9}$$

We choose ε and β, such that

$$\varepsilon > 8\varepsilon^2 \max_{\Omega} |Du|^2, \text{ and } \beta > a^2.$$

Hence, (2.9) becomes,

$$-\frac{C}{u} + \frac{C\sigma_2^{11}}{u^2} \geqslant \frac{\varepsilon}{2}\sigma_2^{11}u_{11}^2 - 4\sigma_2^{11}a^2x_1^2 + (a-C)u_{11}. \tag{2.10}$$

Taking a and u_{11} sufficiently large, we obtain (1.3).

(B) If $\sum_{i=2}^{n-1}\lambda_i \geqslant \frac{\lambda_1}{3}$, then we have $\frac{\lambda_1}{3(n-2)} \leqslant \lambda_2 \leqslant \lambda_1$. Using (2.7), (2.6) becomes,

$$\begin{aligned}(2.11)\qquad & -\frac{C}{u} + \sum_i \frac{(\beta+2\beta^2)\sigma_2^{ii}u_i^2}{u^2} \\ \geqslant & \sum_i \varepsilon\sigma_2^{ii}u_{ii}^2 - 4\sum_i \sigma_2^{ii}\varepsilon^2u_i^2u_{ii}^2 - 4\sum_i \sigma_2^{ii}a^2x_i^2 + a\sigma_1.\end{aligned}$$

We should divide this case into two subcases, (B1) $\sigma_2^{22} \geqslant 1$ and (B2) $\sigma_2^{22} < 1$. We also take a sufficiently small ε, such that $\varepsilon > 8\varepsilon^2 \max_\Omega |Du|^2$. In both subcases, the right hand side of the above inequality always has high order term u_{11}^2 or u_{11}^3, then we have (1.3). See [11] for detail.

3. An interior C^2 estimate for $k+1$ convex solutions

In this section, we consider the interior estimates for k Hessian equations (1.1). We will prove Theorem 2. Before we start our proof, we need the following fact.

Lemma 7. *Suppose u is a $k+1$ convex solution for equation* (1.1)*. Then, there is some constant $K_0 > 0$ depending on the diameter of the domain Ω, $\sup_\Omega |u|$ and $\sup_\Omega |\nabla u|$, such that,*

$$D^2u + K_0 I \geqslant 0.$$

Here "$\geqslant 0$" means the matrix is semi positive definite.

Proof. We choose K_0 satisfying

$$(\frac{K_0}{n})^k \geqslant \sup_\Omega f(x,u,\nabla u).$$

Suppose $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ is the eigenvalues of the Hessian D^2u. Then, we have, using $u \in \Gamma_{k+1}$,

$$\begin{aligned}\sigma_k &= \sigma_{k-1}(\lambda|1)\lambda_1 + \sigma_k(\lambda|1) \geqslant \sigma_{k-1}(\lambda|1)\lambda_1 \\ &= \sigma_{k-2}(\lambda|12)\lambda_1\lambda_2 + \lambda_1\sigma_{k-1}(\lambda|12) \geqslant \sigma_{k-2}(\lambda|12)\lambda_1\lambda_2 \\ &= \cdots \geqslant \cdots \\ &= \lambda_1\lambda_2\cdots\lambda_k \geqslant \lambda_k^k.\end{aligned}$$

Hence, $\lambda_k \leqslant K_0/n$. Since, $u \in \Gamma_k$, we have,

$$\sum_{i=k}^n \lambda_i > 0,$$

which implies that $\lambda_n + K_0 \geqslant 0$. We obtain the Lemma. $\square$

We use the m-polynomials. Here, m should be sufficiently large to give more convexity, since we have more negative terms. Let's consider the following test function,

$$\varphi = m\log(-u) + \log P_m + \frac{mN}{2}|Du|^2, \tag{3.1}$$

where

$$P_m = \sum_j \kappa_j^m, \text{ and } \kappa_j = \lambda_j + K_0,$$

and N is some undetermined constant. The $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of the Hessian D^2u. By Lemma 7, $\kappa_1, \kappa_2, \cdots, \kappa_n$ are non negative. Suppose that function φ achieves its maximum value in Ω at some point x_0. Rotating the coordinates, we assume that (u_{ij}) is diagonal matrix at x_0, and $\kappa_1 \geqslant \kappa_2 \cdots \geqslant \kappa_n$.

Differentiating our test function twice and using Lemma 6, at x_0, we have,

$$\frac{\sum_j \kappa_j^{m-1} u_{jji}}{P_m} + Nu_iu_{ii} + \frac{u_i}{u} = 0, \tag{3.2}$$

and,

$$\begin{aligned} 0 \geqslant & \frac{1}{P_m}[\sum_j \kappa_j^{m-1} u_{jjii} + (m-1)\sum_j \kappa_j^{m-2} u_{jji}^2 + \sum_{p\neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} u_{pqi}^2] \\ & - \frac{m}{P_m^2}(\sum_j \kappa_j^{m-1} u_{jji})^2 + \sum_s Nu_su_{sii} + Nu_{ii}^2 + \frac{u_{ii}}{u} - \frac{u_i^2}{u^2}. \end{aligned} \tag{3.3}$$

At x_0, differentiating the equation(1.1) twice, we have,

$$\sigma_k^{ii} u_{iij} = \psi_{p_j} u_{jj} + \psi_u u_j + \psi_j, \tag{3.4}$$

and

$$\sigma_k^{ii} u_{iijj} + \sigma_k^{pq,rs} u_{pqj} u_{rsj} \geqslant -C - Cu_{11}^2 + \sum_s \psi_{p_s} u_{sjj}. \tag{3.5}$$

Here, C is a constant depending on f, the diameter of the domain Ω, $\sup_\Omega |u|$ and $\sup_\Omega |\nabla u|$. Contacting σ_k^{ii} in both side of (3.3), and using (3.4)(3.5), we get,

(3.6)

$$\begin{aligned} 0 \geqslant\ & \frac{1}{P_m}[\sum_l \kappa_l^{m-1}(-C - Cu_{11}^2 + \sum_s \psi_{p_s} u_{sll} - K(\sigma_k)_l^2 + K(\sigma_k)_l^2 - \sigma_k^{pq,rs} u_{pql} u_{rsl}) \\ & +(m-1)\sigma_k^{ii}\sum_j \kappa_j^{m-2} u_{jji}^2 + \sigma_k^{ii}\sum_{p\neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} u_{pqi}^2] - \frac{m\sigma_k^{ii}}{P_m^2}(\sum_j \kappa_j^{m-1} u_{jji})^2 \\ & +\sum_s Nu_su_{sii}\sigma_k^{ii} + Nu_{ii}^2\sigma_k^{ii} + \frac{k\sigma_k}{u} - \frac{\sigma_k^{ii}u_i^2}{u^2}. \end{aligned}$$

Using (3.2) and (3.4), we have,

$$\frac{1}{P_m}\sum_l\sum_s \kappa_l^{m-1}\psi_{p_s}u_{sll}+\sum_s Nu_s\sigma_k^{ii}u_{sii}\geqslant -\sum_s\psi_{p_s}\frac{u_s}{u}-C.$$

On the other hand, we have,

$$-\sigma_k^{pq,rs}u_{pql}u_{rsl}=-\sigma_k^{pp,qq}u_{ppl}u_{qql}+\sigma_k^{pp,qq}u_{pql}^2.$$

Then, using the previous two formulas, (2.8) becomes,

(3.7)

$$\begin{aligned}0 \;\geqslant\; & \frac{1}{P_m}[\sum_l \kappa_l^{m-1}(-C-Cu_{11}^2-K\psi_{p_l}^2u_{ll}^2+K(\sigma_k)_l^2-\sigma_k^{pp,qq}u_{ppl}u_{qql}+\sigma_k^{pp,qq}u_{pql}^2)\\ &+(m-1)\sigma_k^{ii}\sum_j\kappa_j^{m-2}u_{jji}^2+\sigma_k^{ii}\sum_{p\neq q}\frac{\kappa_p^{m-1}-\kappa_q^{m-1}}{\kappa_p-\kappa_q}u_{pqi}^2]\\ &-\frac{m\sigma_k^{ii}}{P_m^2}(\sum_j\kappa_j^{m-1}u_{jji})^2+Nu_{ii}^2\sigma_k^{ii}+\frac{k\sigma_k}{u}-\frac{\sigma_k^{ii}u_i^2}{u^2}-\sum_s\psi_{p_s}\frac{u_s}{u}.\end{aligned}$$

Let's deal with the third order derivatives. Denote,

$A_i=\dfrac{\kappa_i^{m-1}}{P_m}(K(\sigma_k)_i^2-\sum_{p,q}\sigma_k^{pp,qq}u_{ppi}u_{qqi})$, $B_i=\dfrac{2\kappa_j^{m-1}}{P_m}\sum_j\sigma_k^{jj,ii}u_{jji}^2$,

$C_i=\dfrac{m-1}{P_m}\sigma_k^{ii}\sum_j\kappa_j^{m-2}u_{jji}^2$, $D_i=\dfrac{2\sigma_k^{jj}}{P_m}\sum_{j\neq i}\dfrac{\kappa_j^{m-1}-\kappa_i^{m-1}}{\kappa_j-\kappa_i}u_{jji}^2$,

$E_i=\dfrac{m\sigma_k^{ii}}{P_m^2}(\sum_j\kappa_j^{m-1}u_{jji})^2$.

We divide two cases to deal with the third order deriavatives, $i\neq 1$ and $i=1$.

Lemma 8. *For any $i\neq 1$, we have*

$$A_i+B_i+C_i+D_i-(1+\frac{1}{m})E_i\geqslant 0,$$

for sufficiently large m.

Proof. At first, by Lemma 5, for sufficiently large K, we have,

(3.8) $$K(\sigma_k)_l^2-\sigma_k^{pp,qq}u_{ppl}u_{qql}\geqslant\sigma_k(1+\frac{\alpha}{2})[\frac{(\sigma_1)_l}{\sigma_1}]^2\geqslant 0.$$

Hence, $A_i\geqslant 0$.

Then, we also have,

$$(3.9)\qquad P_m^2[B_i+C_i+D_i-(1+\frac{1}{m})E_i]$$

$$\begin{aligned}
=\ & \sum_{j\neq i} P_m[2\kappa_j^{m-1}\sigma_k^{jj,ii}+(m-1)\kappa_j^{m-2}\sigma_k^{ii}+2\sigma_k^{jj}\sum_{l=0}^{m-2}\kappa_i^{m-2-l}\kappa_j^l]u_{jji}^2\\
& +P_m(m-1)\sigma_k^{ii}\kappa_i^{m-2}u_{iii}^2\\
& -(m+1)\sigma_k^{ii}(\sum_{j\neq i}\kappa_j^{2m-2}u_{jji}^2+\kappa_i^{2m-2}u_{iii}^2+\sum_{p\neq q}\kappa_p^{m-1}\kappa_q^{m-1}u_{ppi}u_{qqi}).
\end{aligned}$$

Note that

(3.10)

$$\begin{aligned}
\kappa_j\sigma_k^{jj,ii}+\sigma_k^{jj} &= (\lambda_j+K_0)\sigma_k^{jj,ii}+\sigma_k^{jj}\\
&= K_0\sigma_k^{jj,ii}+\sigma_k^{ii}-\sigma_{k-1}(\lambda|ij)+\lambda_i\sigma_{k-2}(\lambda|ij)+\sigma_{k-1}(\lambda|ij)\\
&= (K_0+\lambda_i)\sigma_k^{jj,ii}+\sigma_k^{ii}\\
&\geqslant \sigma_k^{ii}.
\end{aligned}$$

For any index $j\neq i$, using the above inequality, we have,

$$(3.11)\qquad \begin{aligned}
& P_m[2\kappa_j^{m-1}\sigma_k^{jj,ii}+(m-1)\kappa_j^{m-2}\sigma_k^{ii}+2\sigma_k^{jj}\sum_{l=0}^{m-2}\kappa_i^{m-2-l}\kappa_j^l]u_{jji}^2\\
& -(m+1)\sigma_k^{ii}\kappa_j^{2m-2}u_{jji}^2\\
\geqslant\ & P_m(m+1)\sigma_k^{ii}\kappa_j^{m-2}u_{jji}^2-(m+1)\sigma_k^{ii}\kappa_j^{2m-2}u_{jji}^2\\
& +2P_m\sigma_k^{jj}(\sum_{l=0}^{m-3}\kappa_i^{m-2-l}\kappa_j^l)u_{jji}^2\\
\geqslant\ & (m+1)(P_m-\kappa_j^m)\sigma_k^{ii}\kappa_j^{m-2}u_{jji}^2+2P_m\sigma_k^{jj}(\sum_{l=0}^{m-3}\kappa_i^{m-2-l}\kappa_j^l)u_{jji}^2
\end{aligned}$$

Using Cauchy-Schwarz inequalities, we have,

$$(3.12)\qquad \begin{aligned}
& 2\sum_{j\neq i}\sum_{p\neq i,j}\kappa_j^{m-2}\kappa_p^m u_{jji}^2\\
=\ & \sum_{p\neq i}\sum_{q\neq i,p}\kappa_p^{m-2}\kappa_q^m u_{ppi}^2+\sum_{q\neq i}\sum_{p\neq i,q}\kappa_q^{m-2}\kappa_p^m u_{qqi}^2\\
\geqslant\ & 2\sum_{p\neq q;p,q\neq i}\kappa_p^{m-1}\kappa_q^{m-1}u_{ppi}u_{qqi}.
\end{aligned}$$

Hence, by (3.9), (3.11) and (3.12), we obtain,

$$
\begin{aligned}
(3.13)\ & P_m^2(B_i+C_i+D_i-(1+\frac{1}{m})E_i)\\
\geqslant\ & \sum_{j\neq i}(m+1)\kappa_i^m\kappa_j^{m-2}\sigma_k^{ii}u_{jji}^2+((m-1)(P_m-\kappa_i^m)-2\kappa_i^m)\kappa_i^{m-2}\sigma_k^{ii}u_{iii}^2\\
& -2(m+1)\sigma_k^{ii}\kappa_i^{m-1}u_{iii}\sum_{j\neq i}\kappa_j^{m-1}u_{jji}+2P_m\sum_{j\neq i}\sigma_k^{jj}(\sum_{l=0}^{m-3}\kappa_i^{m-2-l}\kappa_j^l)u_{jji}^2\\
\geqslant\ & \sum_{j\neq i}[(m+1)\kappa_i^m\kappa_j^{m-2}\sigma_k^{ii}+2\kappa_1^m\sigma_k^{jj}\sum_{l=0}^{m-3}\kappa_i^{m-2-l}\kappa_j^l]u_{jji}^2\\
& +((m-1)(P_m-\kappa_i^m)-2\kappa_i^m)\kappa_i^{m-2}\sigma_k^{ii}u_{iii}^2-2(m+1)\sigma_k^{ii}\kappa_i^{m-1}u_{iii}\sum_{j\neq i}\kappa_j^{m-1}u_{jji}.
\end{aligned}
$$

We divide two cases to discuss.
Case(A) For $\lambda_j\geqslant\lambda_i$, we divide into two sub cases to discuss. If $\lambda_i\geqslant K_0$, for $1\leqslant l\leqslant m-3$, we have,

$$
\begin{aligned}
(3.14)\qquad 2\kappa_1^m\sigma_k^{jj}\kappa_i^{m-2-l}\kappa_j^l &= 2\kappa_1^m(\lambda_i\sigma_k^{ii,jj}+\sigma_{k-1}(\lambda|ij))\kappa_i^{m-2-l}\kappa_j^l\\
&\geqslant \kappa_1^m(\kappa_i\sigma_k^{ii,jj}+\sigma_{k-1}(\lambda|ij))\kappa_i^{m-2-l}\kappa_j^l\\
&\geqslant \kappa_1^m(\kappa_j\sigma_k^{ii,jj}+\sigma_{k-1}(\lambda|ij))\kappa_i^{m-l-1}\kappa_j^{l-1}\\
&\geqslant \kappa_1^m(\lambda_j\sigma_k^{ii,jj}+\sigma_{k-1}(\lambda|ij))\kappa_i^{m-l-1}\kappa_j^{l-1}\\
&= \kappa_1^m\kappa_i^{m-1-l}\kappa_j^{l-1}\sigma_k^{ii}.
\end{aligned}
$$

Here, we have used $\sigma_{k-1}(\lambda|ij)>0$ since u is a $k+1$ convex solution.

If $\lambda_i<K_0$, for all $k\leqslant l\leqslant k+8$, we have,

$$\kappa_1^{l+1}\sigma_k^{jj}\geqslant\kappa_1^l\lambda_1\sigma_k^{11}\geqslant c_0\sigma_k\kappa_1\lambda_1^{l-1}\geqslant\sigma_k^{ii}$$

when λ_1 is sufficiently large. Here, we have used $\lambda_1\sigma_k^{11}\geqslant c_0\sigma_k$. Hence, we have,

$$
\begin{aligned}
(3.15)\quad \kappa_1^m\sigma_k^{jj}\kappa_i^{m-2-l}\kappa_j^l &\geqslant \kappa_1^{l+1}\sigma_k^{jj}\kappa_1^{m-l-2}\kappa_j^l\kappa_i^{m-l-2}\kappa_1\geqslant\sigma_k^{ii}\kappa_j^{m-2}\kappa_i^m\frac{\kappa_1}{\kappa_i^{l+2}}\\
&\geqslant \sigma_k^{ii}\kappa_j^{m-2}\kappa_i^m.
\end{aligned}
$$

Since $\lambda_i<K_0$, we have used $\kappa_1\geqslant\kappa_i^{l+2}$ for sufficiently large λ_1.
Case (B) For $\lambda_j<\lambda_i$, obviously, we have,

$$2\kappa_1^m\sigma_k^{jj}\kappa_i^{m-2-l}\kappa_j^l\geqslant2\kappa_1^m\kappa_i^{m-2-l}\kappa_j^l\sigma_k^{ii}.$$

Combing the above two cases, we get, for $k\leqslant l\leqslant k+8$,

$$(3.16)\qquad 2\kappa_1^m\sigma_k^{jj}\kappa_i^{m-2-l}\kappa_j^l\ \geqslant\ \kappa_i^m\kappa_j^{m-2}\sigma_k^{ii}.$$

Thus, (3.13) becomes,

$$
\begin{aligned}
(3.17)\qquad & P_m^2(B_i+C_i+D_i-(1+\frac{1}{m})E_i)\\
\geqslant\ & \sum_{j\neq i}(m+8)\kappa_i^m\kappa_j^{m-2}\sigma_k^{ii}u_{jji}^2+((m-1)(P_m-\kappa_i^m)-2\kappa_i^m)\kappa_i^{m-2}\sigma_k^{ii}u_{iii}^2\\
& -2(m+1)\sigma_k^{ii}\kappa_i^{m-1}u_{iii}\sum_{j\neq i}\kappa_j^{m-1}u_{jji}\\
\geqslant\ & (m+8)\kappa_i^m\kappa_1^{m-2}\sigma_k^{ii}u_{11i}^2+((m-1)\kappa_1^m-2\kappa_i^m)\kappa_i^{m-2}\sigma_k^{ii}u_{iii}^2\\
& -2(m+1)\sigma_k^{ii}\kappa_i^{m-1}u_{iii}\kappa_1^{m-1}u_{11i}\\
\geqslant\ & (m+8)\kappa_i^m\kappa_1^{m-2}\sigma_k^{ii}u_{11i}^2+(m-3)\kappa_1^m\kappa_i^{m-2}\sigma_k^{ii}u_{iii}^2\\
& -2(m+1)\sigma_k^{ii}\kappa_i^{m-1}u_{iii}\kappa_1^{m-1}u_{11i}\\
\geqslant\ & 0.
\end{aligned}
$$

Here, we have used, for $m\geqslant 10$,

$$(m+8)(m-3)\geqslant(m+1)^2.$$

So, we take

$$m=\max\{10,k+11\},$$

which is sufficiently large. □

The left case is $i=1$. Let's begin with the following Lemma which is modified from [11].

Lemma 9. *For $\mu=1,\cdots,k-1$, if there exists some positive constant $\delta\leqslant 1$, such that $\lambda_\mu/\lambda_1\geqslant\delta$. Then there exits two sufficiently small positive constants η,δ' depending on δ, such that, if $\lambda_{\mu+1}/\lambda_1\leqslant\delta'$, we have,*

$$A_1+B_1+C_1+D_1-(1+\frac{\eta}{m})E_1\geqslant 0.$$

Proof. At first, we have,

$$
\begin{aligned}
(3.18)\qquad & P_m^2(B_1+C_1+D_1-(1+\frac{\eta}{m})E_1)\\
\geqslant\ & \sum_{j\neq 1}((1-\eta)P_m+(m+\eta)\kappa_1^m)\kappa_j^{m-2}\sigma_k^{11}u_{jj1}^2\\
& +((m-1)(P_m-\kappa_1^m)-(1+\eta)\kappa_1^m)\kappa_1^{m-2}\sigma_k^{11}u_{111}^2\\
& -2(m+\eta)\sigma_k^{11}\kappa_1^{m-1}u_{111}\sum_{j\neq 1}\kappa_j^{m-1}u_{jj1}+2P_m\sum_{j\neq 1}\sigma_k^{jj}(\sum_{l=0}^{m-3}\kappa_1^{m-2-l}\kappa_j^l)u_{jj1}^2.
\end{aligned}
$$

Since $\sigma_k^{jj}\geqslant\sigma_k^{11}$ for any $j\neq 1$, for $m\geqslant 5$, it is obvious,

$$2P_m\sum_{j\neq 1}\sigma_k^{jj}(\sum_{l=0}^{m-3}\kappa_1^{m-2-l}\kappa_j^l)u_{jj1}^2\geqslant 3\sum_{j\neq 1}\kappa_1^m\kappa_j^{m-2}\sigma_k^{11}u_{jj1}^2+2P_m\kappa_1^{m-2}\sum_{j\neq 1}\sigma_k^{jj}u_{jj1}^2.$$

Hence, by (3.18), we obtain,

$$
\begin{aligned}
(3.19)\qquad & P_m^2(B_1+C_1+D_1-(1+\frac{\eta}{m})E_1) \\
\geqslant\ & \sum_{j\neq 1}(m+4)\kappa_1^m\kappa_j^{m-2}\sigma_k^{11}u_{jj1}^2+(m-1)\sum_{j\neq 1}\kappa_j^m\kappa_1^{m-2}\sigma_k^{11}u_{111}^2 \\
& -2(m+\eta)\sigma_k^{11}\kappa_1^{m-1}u_{111}\sum_{j\neq 1}\kappa_j^{m-1}u_{jj1} \\
& -(1+\eta)\kappa_1^{2m-2}\sigma_k^{11}u_{111}^2+2P_m\kappa_1^{m-2}\sum_{j\neq 1}\sigma_k^{jj}u_{jj1}^2 \\
\geqslant\ & -(1+\eta)\kappa_1^{2m-2}\sigma_k^{11}u_{111}^2+2P_m\kappa_1^{m-2}\sum_{j\neq 1}\sigma_k^{jj}u_{jj1}^2.
\end{aligned}
$$

Here, we have used

$$
(m+4)(m-1)\geqslant (m+1)^2,
$$

for $m\geqslant 5$. By Lemma 5, we have,

$$
\begin{aligned}
(3.20)\qquad A_1\ \geqslant\ & \frac{\kappa_1^{m-1}}{P_m}[\sigma_k(1+\frac{\alpha}{2})\frac{(\sigma_\mu)_1^2}{\sigma_\mu^2}-\frac{\sigma_k}{\sigma_\mu}\sigma_\mu^{pp,qq}u_{pp1}u_{qq1}] \\
\geqslant\ & \frac{\kappa_1^{m-1}\sigma_k}{P_m\sigma_\mu^2}[(1+\frac{\alpha}{2})\sum_a(\sigma_\mu^{aa}u_{aa1})^2+\frac{\alpha}{2}\sum_{a\neq b}\sigma_\mu^{aa}\sigma_\mu^{bb}u_{aa1}u_{bb1} \\
& +\sum_{a\neq b}(\sigma_\mu^{aa}\sigma_\mu^{bb}-\sigma_\mu\sigma_\mu^{aa,bb})u_{aa1}u_{bb1}].
\end{aligned}
$$

For $\mu=1$, notice that $\sigma_1^{aa}=1$ and $\sigma_1^{aa,bb}=0$. Then, we have,

$$
\begin{aligned}
(3.21)\qquad (1+\frac{\alpha}{2})\sum_{a,b}u_{aa1}u_{bb1}\ \geqslant\ & 2(1+\frac{\alpha}{2})\sum_{a\neq 1}u_{aa1}u_{111}+(1+\frac{\alpha}{2})u_{111}^2 \\
\geqslant\ & (1+\frac{\alpha}{4})u_{111}^2-C_\alpha\sum_{a\neq 1}u_{aa1}^2.
\end{aligned}
$$

Then, we get,

$$
\begin{aligned}
(3.22)\qquad P_m^2A_1\ \geqslant\ & \frac{P_m\kappa_1^{m-1}\sigma_k}{\sigma_1^2}(1+\frac{\alpha}{4})u_{111}^2-\frac{\kappa_1^{m-1}P_mC_\alpha}{\sigma_1^2}\sum_{a\neq 1}u_{aa1}^2 \\
\geqslant\ & \frac{P_m\kappa_1^{m-2}\sigma_k^{11}}{(1+\sum_{j\neq 1}\lambda_j/\lambda_1)^2}(1+\frac{\alpha}{4})u_{111}^2-\frac{C_\alpha P_m\kappa_1^{m-1}}{\sigma_1^2}\sum_{a\neq 1}u_{aa1}^2 \\
\geqslant\ & (1+\eta)P_m\kappa_1^{m-2}\sigma_k^{11}u_{111}^2-\frac{C_\alpha P_m\kappa_1^{m-1}}{\sigma_1^2}\sum_{a\neq 1}u_{aa1}^2.
\end{aligned}
$$

The last two inequalities come from,

$$
\sigma_k\geqslant \lambda_1\sigma_k^{11},
$$

for sufficiently large λ_1, and

$$1+\frac{\alpha}{4} \geqslant (1+\eta)(1+(n-1)\delta')^2. \tag{3.23}$$

For $\mu \geqslant 2$, obviously, for $a \neq b$, we have,

$$\begin{aligned}
&\sigma_\mu^{aa}\sigma_\mu^{bb} - \sigma_\mu\sigma_\mu^{aa,bb} \\
= \;& (\lambda_b\sigma_{\mu-2}(\lambda|ab) + \sigma_{\mu-1}(\lambda|ab))(\lambda_a\sigma_{\mu-2}(\lambda|ab) + \sigma_{\mu-1}(\lambda|ab)) \\
& -(\lambda_a\lambda_b\sigma_{\mu-2}(\lambda|ab) + \lambda_a\sigma_{\mu-1}(\lambda|ab) + \lambda_b\sigma_{\mu-1}(\lambda|ab) + \sigma_\mu(\lambda|ab))\sigma_{\mu-2}(\lambda|ab) \\
= \;& \sigma_{\mu-1}^2(\lambda|ab) - \sigma_\mu(\lambda|ab)\sigma_{\mu-2}(\lambda|ab) \\
\geqslant \;& 0.
\end{aligned} \tag{3.24}$$

The last inequality comes from Newton inequality. Since $u \in \Gamma_{k+1} \subset \Gamma_{\mu+2}$, we have, for any $a \leqslant \mu$,

$$\sigma_\mu^{aa} \geqslant \frac{\lambda_1 \cdots \lambda_\mu}{\lambda_a}. \tag{3.25}$$

For $a, b \leqslant \mu$, we claim,

$$\begin{aligned}
&\sigma_{\mu-1}(\lambda|ab) \leqslant C\frac{\lambda_1 \cdots \lambda_{\mu+1}}{\lambda_a\lambda_b}, \quad \sigma_\mu(\lambda|ab) \leqslant C\frac{\lambda_1 \cdots \lambda_{\mu+2}}{\lambda_a\lambda_b} \\
&\sigma_{\mu-2}(\lambda|ab) \leqslant C\frac{\lambda_1 \cdots \lambda_\mu}{\lambda_a\lambda_b}.
\end{aligned} \tag{3.26}$$

The proof of the above three inequalities are same. We only give more detail for the first one. Since, $u \in \Gamma_{\mu+2}$, then, for any index $i \geqslant \mu+1$, there is some constant C such that,

$$|\lambda_i| \leqslant C\lambda_{\mu+1}.$$

We write down the expression of σ_μ and replace any λ_i for $i \geqslant \mu+1$ by $\lambda_{\mu+1}$, then we obtain the first inequality. Using (3.26) and (3.25), we get, for $a, b \leqslant \mu$,

$$\sigma_{\mu-1}^2(\lambda|ab) - \sigma_\mu(\lambda|ab)\sigma_{\mu-2}(\lambda|ab) \leqslant C_1(\frac{\lambda_{\mu+1}}{\lambda_b}\sigma_\mu^{aa})^2. \tag{3.27}$$

Then, by (3.27), we have, for any undetermined positive constant ϵ,

$$\begin{aligned}
&\sum_{a\neq b; a,b\leqslant\mu} (\sigma_\mu^{aa}\sigma_\mu^{bb} - \sigma_\mu\sigma_\mu^{aa,bb})u_{aa1}u_{bb1} \\
\geqslant \;& -\sum_{a\neq b; a,b\leqslant\mu} (\sigma_{\mu-1}^2(\lambda|ab) - \sigma_\mu(\lambda|ab)\sigma_{\mu-2}(\lambda|ab))u_{aa1}^2 \\
\geqslant \;& -\sum_{a\neq b; a,b\leqslant\mu} C_1(\frac{\lambda_{\mu+1}}{\lambda_b})^2(\sigma_\mu^{aa}u_{aa1})^2 \\
\geqslant \;& -\frac{C_2}{\delta^2}(\frac{\lambda_{\mu+1}}{\lambda_1})^2\sum_{a\leqslant\mu}(\sigma_\mu^{aa}u_{aa1})^2 \quad \geqslant \quad -\epsilon\sum_{a\leqslant\mu}(\sigma_\mu^{aa}u_{aa1})^2.
\end{aligned} \tag{3.28}$$

Here, we choose a sufficiently small δ', such that,

$$\delta' \leqslant \delta\sqrt{\epsilon/C_2}. \tag{3.29}$$

By (3.27), we also have,

$$
\begin{aligned}
(3.30)\qquad & 2\sum_{a\leqslant\mu;b>\mu}(\sigma_\mu^{aa}\sigma_\mu^{bb}-\sigma_\mu\sigma_\mu^{aa,bb})u_{aa1}u_{bb1}\\
\geqslant\ & -2\sum_{a\leqslant\mu;b>\mu}\sigma_\mu^{aa}\sigma_\mu^{bb}|u_{aa1}u_{bb1}|\\
\geqslant\ & -\epsilon\sum_{a\leqslant\mu;b>\mu}(\sigma_\mu^{aa}u_{aa1})^2-\frac{1}{\epsilon}\sum_{a\leqslant\mu;b>\mu}(\sigma_\mu^{bb}u_{bb1})^2.
\end{aligned}
$$

Again by (3.27), we have,

$$
\begin{aligned}
(3.31)\qquad \sum_{a\neq b;a,b>\mu}(\sigma_\mu^{aa}\sigma_\mu^{bb}-\sigma_\mu\sigma_\mu^{aa,bb})u_{aa1}u_{bb1} &\geqslant -\sum_{a\neq b;a,b>\mu}\sigma_\mu^{aa}\sigma_\mu^{bb}|u_{aa1}u_{bb1}|\\
&\geqslant -\sum_{a\neq b;a,b>\mu}(\sigma_\mu^{aa}u_{aa1})^2.
\end{aligned}
$$

Hence, combing (3.20), (3.28), (3.30) and (3.31), then taking $\alpha=0$ in (3.20), we get,

$$
(3.32)\qquad A_1 \geqslant \frac{\kappa_1^{m-1}\sigma_k}{P_m\sigma_\mu^2}[(1-2\epsilon)\sum_{a\leqslant\mu}(\sigma_\mu^{aa}u_{aa1})^2-C_\epsilon\sum_{a>\mu}(\sigma_\mu^{aa}u_{aa1})^2].
$$

For $a>\mu$, we have,

$$
\sigma_\mu^{aa}\leqslant C\lambda_1\cdots\lambda_{\mu-1},\quad \text{and } \sigma_\mu\geqslant\lambda_1\cdots\lambda_\mu.
$$

For $a\leqslant\mu$, we have,

$$
\sigma_\mu(\lambda|a)\leqslant C\frac{\lambda_1\cdots\lambda_{\mu+1}}{\lambda_a}
$$

Then, we have, for $\lambda_1\geqslant K_0$,

$$
\begin{aligned}
(3.33)\qquad & P_m^2A_1\\
\geqslant\ & \frac{P_m\kappa_1^{m-1}\lambda_1\sigma_k^{11}}{\sigma_\mu^2}(1-2\epsilon)\sum_{a\leqslant\mu}(\sigma_\mu^{aa}u_{aa1})^2-\frac{P_m\kappa_1^{m-1}\sigma_kC_\epsilon}{\sigma_\mu^2}\sum_{a>\mu}(\sigma_\mu^{aa}u_{aa1})^2\\
\geqslant\ & \frac{P_m\kappa_1^{m-1}\sigma_k^{11}}{\lambda_1}(1-2\epsilon)\sum_{a\leqslant\mu}(\frac{\lambda_a\sigma_\mu^{aa}}{\sigma_\mu})^2u_{aa1}^2-\frac{P_m\kappa_1^{m-3}\lambda_1^2C_\epsilon}{\sigma_\mu^2}\sum_{a>\mu}(\sigma_\mu^{aa}u_{aa1})^2\\
\geqslant\ & \kappa_1^{2m-2}\sigma_k^{11}(1-2\epsilon)(1+\delta^m)\sum_{a\leqslant\mu}(1-\frac{C_3\lambda_{\mu+1}}{\lambda_a})^2u_{aa1}^2-\frac{P_m\kappa_1^{m-3}\lambda_\mu^2C_\epsilon}{\delta^2\sigma_\mu^2}\sum_{a>\mu}(\sigma_\mu^{aa}u_{aa1})^2\\
\geqslant\ & \kappa_1^{2m-2}\sigma_k^{11}(1-2\epsilon)(1+\delta^m)(1-\frac{C_3\lambda_{\mu+1}}{\delta\lambda_1})^2\sum_{a\leqslant\mu}u_{aa1}^2-\frac{P_m\kappa_1^{m-3}C_\epsilon}{\delta^2}\sum_{a>\mu}u_{aa1}^2\\
\geqslant\ & (1+\eta)\kappa_1^{2m-2}\sigma_k^{11}\sum_{a\leqslant\mu}u_{aa1}^2-\frac{P_m\kappa_1^{m-3}C_\epsilon}{\delta^2}\sum_{a>\mu}u_{aa1}^2.
\end{aligned}
$$

Here, the last inequality comes from that we choose δ', η and ϵ satisfying

$$\delta' C_3 \leqslant 2\epsilon\delta, \quad (1-2\epsilon)^2(1+\delta^m) \geqslant 1+\eta. \tag{3.34}$$

Using (3.19) and (3.22) or (3.33), we have,

$$\begin{aligned} & P_m^2(A_1+B_1+C_1+D_1-(1+\frac{\eta}{m})E_1) \\ \geqslant\ & 2P_m\kappa_1^{m-2}\sum_{j\neq 1}\sigma_k^{jj}u_{jj1}^2 - \frac{C_\epsilon P_m \kappa_1^{m-3}}{\delta^2}\sum_{j>\mu}u_{jj1}^2. \end{aligned} \tag{3.35}$$

Now, for $k \geqslant j > \mu$, we have,

$$\kappa_1\sigma_{k-1}(\kappa|j) \geqslant \frac{\lambda_1\cdots\lambda_k\cdot\kappa_1}{\lambda_j} \geqslant \frac{\sigma_k\lambda_1}{C_4\lambda_j} \geqslant \frac{\sigma_k}{C_4\delta'}.$$

For $j \geqslant k+1$, we have,

$$\kappa_1\sigma_{k-1}(\kappa|j) \geqslant \frac{\sigma_k\lambda_1}{C_4\lambda_k} \geqslant \frac{\sigma_k}{C_4\delta'}.$$

For both cases, chose δ' small enough such that,

$$\delta' < \frac{\sigma_k\delta^2}{C_4C_\epsilon},$$

then (3.35) is nonnegative. We complete the proof. □

Hence, a directly corollary of Lemma 8 and Lemma 9 is the following.

Corollary 10. *There exists two finite sequence of positive numbers* $\{\delta_i\}_{i=1}^k$ *and* $\{\varepsilon_i\}_{i=1}^k$*, such that, if the following inequality holds for some index* $1 \leqslant r \leqslant k-1$,

$$\frac{\lambda_r}{\lambda_1} \quad \geqslant \quad \delta_r, \ \textit{and}\ \frac{\lambda_{r+1}}{\lambda_1} \leqslant \delta_{r+1},$$

then, for sufficiently large K*, we have,*

$$A_1+B_1+C_1+D_1-(1+\frac{\varepsilon_r}{m})E_1 \geqslant 0. \tag{3.36}$$

Proof. We use induction to find the sequence $\{\delta_i\}_{i=1}^k$ and $\{\varepsilon_i\}_{i=1}^k$. Let $\delta_1 = 1/2$. Then $\lambda_1/\lambda_1 = 1 > \delta_1$. Assume that we have determined δ_r for $1 \leqslant r \leqslant k-1$. We want to search for δ_{r+1}. In Lemma 9, we may choose $\mu = r$ and $\delta = \delta_r$. Then there is some δ_{r+1} and ε_r such that, if $\lambda_{r+1} \leqslant \delta_{r+1}\lambda_1$, we have (3.36). We have δ_{r+1} and ε_r. □

Now, we continue to prove Theorem 2.

By Corollary 10, there exists some sequence $\{\delta_i\}_{i=1}^k$. We divide two cases to deal with.

Case(A): $\lambda_k \geqslant \delta_k\lambda_1$. Then, obviously we have,

$$f = \sigma_k > \lambda_1\cdots\lambda_k \geqslant \delta_k^{k-1}\lambda_1^k,$$

which implies $\lambda_1 \leqslant C$. Hence, we have proved Theorem 2.

Case(B): There exists some index $1 \leqslant r \leqslant k-1$ such that,

$$\lambda_r \geqslant \delta_r \lambda_1 \text{ and } \lambda_{r+1} \leqslant \delta_{r+1}\lambda_1.$$

By Corollary 10, and Lemma 8 we have,

$$\sum_{i=1}^n (A_i + B_i + C_i + D_i) - E_1 - (1+\frac{1}{m})\sum_{i=2}^n E_i \geqslant 0.$$

Using the definitions of A_i, B_i, C_i, D_i, E_i and (3.7), we have,

$$(3.37)\quad 0 \geqslant \frac{1}{P_m}\sum_l \kappa_l^{m-1}(-C - Cu_{11}^2 - K\psi_{p_l}^2 u_{ll}^2) + \sum_{i=2}^n \frac{\sigma_k^{ii}}{P_m^2}(\sum_j \kappa_j^{m-1}u_{jji})^2$$
$$+Nu_{ii}^2\sigma_k^{ii} + \frac{k\sigma_k}{u} - \frac{\sigma_k^{ii}u_i^2}{u^2} - \sum_s \psi_{p_s}\frac{u_s}{u}.$$

By (3.2), we have, for any fixed $i \geqslant 2$,

$$-\frac{\sigma_k^{ii}u_i^2}{u^2} = -\frac{\sigma_k^{ii}}{P_m^2}(\sum_j \kappa_j^{m-1}u_{jji})^2 + \sigma_k^{ii}N^2u_i^2u_{ii}^2 + \frac{2N\sigma_k^{ii}u_i^2u_{ii}}{u}.$$

Hence, (3.37) becomes,

$$(3.38)\qquad 0 \geqslant -C(K)\lambda_1 + \sum_{i=2}^n(\sigma_k^{ii}N^2u_i^2u_{ii}^2 + \frac{2N\sigma_k^{ii}u_i^2u_{ii}}{u})$$
$$+Nu_{ii}^2\sigma_k^{ii} + \frac{k\sigma_k}{u} - \frac{\sigma_k^{11}u_1^2}{u^2} - \sum_s \psi_{p_s}\frac{u_s}{u}.$$

Since, there is some positive constant c_0 such that,

$$u_{11}\sigma_k^{11} \geqslant c_0 > 0,$$

then we have,

$$0 \geqslant (\frac{c_0 N}{2} - C(K))\lambda_1 + \sum_{i=2}^n \frac{2N\sigma_k u_i^2}{u} + \frac{N}{2}\sigma_k^{11}\lambda_1^2 + \frac{k\sigma_k}{u} - \frac{\sigma_k^{11}u_1^2}{u^2} - \sum_s \psi_{p_s}\frac{u_s}{u}.$$

Here, we have used

$$\sigma_k = \lambda_i\sigma_k^{ii} + \sigma_k(\lambda|i) \geqslant \lambda_i\sigma_k^{ii}.$$

Hence, we obtain, for $N \geqslant \dfrac{4C(K)}{c_0}$,

$$-\frac{C}{u} + \frac{C\sigma_k^{11}}{u^2} \geqslant \frac{N}{4}\lambda_1 + \frac{N}{2}\sigma_k^{11}\lambda_1^2$$

If at maximum value point p, $-u \geqslant \sigma_k^{11}$, the above inequality becomes,

$$\frac{2C}{-u} \geqslant \frac{N}{4}\lambda_1,$$

which implies our result. If $-u \leqslant \sigma_k^{11}$, the inequality becomes,

$$\frac{2C\sigma_k^{11}}{(-u)^2} \geqslant \frac{N}{2}\sigma_k^{11}\lambda_1^2,$$

which also implies our result. We complete the proof of Theorem 2.

4. A rigidity theorem for $k+1$ convex solutions

In this section, we prove Theorem 4. At first we have the following Lemma.

Lemma 11. *We consider the Dirichlet problem of the k-Hessian equations,*

$$\begin{cases} \sigma_k(D^2u) &= f(x) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}. \tag{4.1}$$

Here, f is a smooth function defined in Ω. For $k+1$ convex solutions, we have the following type of interior estimates,

$$(-u)^\beta \Delta u \leqslant C. \tag{4.2}$$

for sufficiently large $\beta > 0$. Here constant C and β only depends on the diameters of the domains Ω and k.

Proof. Obviously, for sufficiently large a and b, the function $w = \frac{a}{2}|x|^2 - b$ can control u by comparison principal (see [3] for detail), namely,

$$w \leqslant u \leqslant 0.$$

Here a, b depends on the diameter of the domain Ω. Hence, in the following proof, the constant β, C in (4.2) can contains $\sup_\Omega |u|$.

Since u is a $k+1$ convex solution, by Lemma 7, there is some constant $K_0 > 0$, such that $D^2u + K_0 I \geqslant 0$. We consider the following test functions,

$$\varphi = m\beta \log(-u) + \log P_m + \frac{m}{2}|x|^2.$$

where $P_m = \sum_j \kappa_j^m$, $\kappa_i = \lambda_i + K_0 > 0$. Suppose φ achieves its maximal value at $x_0 \in \Omega$. We may assume (u_{ij}) is diagonal by rotating the coordinate and $u_{11} \geqslant u_{22} \cdots \geqslant u_{nn}$. We always denote $u_{ii} = \lambda_i$.

At the point x_0, we differentiate the test function twice and using Lemma 6. We have,

$$\frac{\sum_j \kappa_j^{m-1} u_{jji}}{P_m} + x_i + \frac{\beta u_i}{u} = 0, \tag{4.3}$$

and,

$$\begin{aligned} 0 \geqslant\ & \frac{1}{P_m}[\sum_j \kappa_j^{m-1} u_{jjii} + (m-1)\sum_j \kappa_j^{m-2} u_{jji}^2 + \sum_{p\neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} u_{pqi}^2] \\ & - \frac{m}{P_m^2}(\sum_j \kappa_j^{m-1} u_{jji})^2 + \frac{\beta u_{ii}}{u} - \frac{\beta u_i^2}{u^2} + 1. \end{aligned} \tag{4.4}$$

Differentating the equation (4.1) twice at x_0, we have,

$$(\sigma_k)_j = \sigma_k^{ii} u_{iij} = f_j, \tag{4.5}$$

and

$$\sigma_k^{ii} u_{iijj} + \sigma_k^{pq,rs} u_{pqj} u_{rsj} = f_{jj}, \tag{4.6}$$

Then, contracting σ_k^{ii} in (4.4) and using the previous two equalities, we have,

$$\begin{aligned} 0 \geqslant \; & \frac{1}{P_m}[\sum_l \kappa_l^{m-1}(f_{ll} - \sigma_k^{pq,rs} u_{pql} u_{rsl}) + (m-1)\sigma_k^{ii} \sum_j \kappa_j^{m-2} u_{jji}^2 \\ & + \sigma_k^{ii} \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} u_{pqi}^2] - \frac{m\sigma_k^{ii}}{P_m^2}(\sum_j \kappa_j^{m-1} u_{jji})^2 \\ & + \frac{\beta k}{u} - \frac{\beta \sigma_k^{ii} u_i^2}{u^2} + (n-k+1)\sigma_{k-1}. \end{aligned} \tag{4.7}$$

Using (4.3), we have,

$$-\frac{\beta \sigma_k^{ii} u_i^2}{u^2} \geqslant -\frac{2\sigma_k^{ii}}{\beta} \frac{(\sum_j \kappa_j^{m-1} u_{jji})^2}{P_m^2} - 2\frac{\sigma_k^{ii} x_i^2}{\beta}$$

Note that,

$$-\sigma_k^{pq,rs} u_{pql} u_{rsl} = -\sigma_k^{pp,qq} u_{ppl} u_{qql} + \sigma_k^{pp,qq} u_{pql}^2.$$

For sequence $\{\varepsilon_i\}_{i=1}^k$ appears in Corollary 10, Let

$$\varepsilon_\beta = \frac{2}{\beta} < \min\{\frac{1}{10}, \varepsilon_1 \cdots, \varepsilon_k\},$$

then, (4.7) becomes

$$\begin{aligned} 0 \geqslant \; & \frac{1}{P_m}[\sum_l \kappa_l^{m-1}(f_{ll} - \sigma_k^{pp,qq} u_{ppl} u_{qql}) + 2\sum_{j \neq i} \kappa_j^{m-2} \sigma_k^{jj,ii} u_{jji}^2 \\ & + (m-1)\sigma_k^{ii} \sum_j \kappa_j^{m-2} u_{jji}^2 + \sigma_k^{ii} \sum_{p \neq q} \frac{\kappa_p^{m-1} - \kappa_q^{m-1}}{\kappa_p - \kappa_q} u_{pqi}^2] \\ & - \frac{(m+\varepsilon_\beta)\sigma_k^{ii}}{P_m^2}(\sum_j \kappa_j^{m-1} u_{jji})^2 + \frac{\beta k}{u} - 2\frac{\sigma_k^{ii} x_i^2}{\beta} + (n-k+1)\sigma_{k-1}. \end{aligned} \tag{4.8}$$

Next we mainly deal with the third order derivative terms. We divide into two case: $i \neq 1$ and $i = 1$. By Lemma 8, we have, for sufficiently large K,

$$
\begin{aligned}
(4.9)\quad 0 \;\leqslant\;& \frac{1}{P_m}\Big[\sum_{l=2}^{n}\kappa_l^{m-1}(K(\sigma_k)_l^2-\sigma_k^{pp,qq}u_{ppl}u_{qql})+2\sum_{i=2}^{n}\sum_{j\neq i}\kappa_j^{m-2}\sigma_k^{jj,ii}u_{jji}^2\\
&+(m-1)\sum_{i=2}^{n}\sigma_k^{ii}\sum_j\kappa_j^{m-2}u_{jji}^2+2\sum_{i=2}^{n}\sigma_k^{ii}\sum_{j\neq i}\frac{\kappa_j^{m-1}-\kappa_i^{m-1}}{\kappa_j-\kappa_i}u_{jji}^2\Big]\\
&-\frac{m+1}{P_m^2}\sum_{i=2}^{n}\sigma_k^{ii}(\sum_j\kappa_j^{m-1}u_{jji})^2.
\end{aligned}
$$

Hence, (4.8) becomes,

$$
\begin{aligned}
(4.10)\,0 \;\geqslant\;& \frac{1}{P_m}\Big[\kappa_1^{m-1}(-C+K(\sigma_k)_1^2-\sigma_k^{pp,qq}u_{pp1}u_{qq1})+2\sum_{j\neq 1}\kappa_j^{m-2}\sigma_k^{jj,11}u_{jj1}^2\\
&+(m-1)\sigma_k^{11}\sum_j\kappa_j^{m-2}u_{jj1}^2+2\sigma_k^{11}\sum_{j\neq 1}\frac{\kappa_j^{m-1}-\kappa_1^{m-1}}{\kappa_j-\kappa_1}u_{jj1}^2\Big]\\
&-\frac{(m+\varepsilon_\beta)\sigma_k^{11}}{P_m^2}(\sum_j\kappa_j^{m-1}u_{jj1})^2+\frac{\beta k}{u}-2\frac{\sigma_k^{ii}x_i^2}{\beta}+C_0\sigma_{k-1}.
\end{aligned}
$$

Now, we divide two sub-cases to continue. By Corollary 10, there exists some sequence $\{\delta_i\}_{i=1}^k$.

Case(A): $\lambda_k \geqslant \delta_k\lambda_1$. Then, obviously we have,

$$f=\sigma_k>\lambda_1\cdots\lambda_k\geqslant \delta_k^{k-1}\lambda_1^k,$$

which implies $\lambda_1 \leqslant C$. Hence, we have proved Lemma 11.

Case(B): There exists some index $1 \leqslant r \leqslant k-1$ such that,

$$\lambda_r \geqslant \delta_r\lambda_1 \text{ and } \lambda_{r+1} \leqslant \delta_{r+1}\lambda_1.$$

By Corollary 10, (4.10) becomes,

$$0 \;\geqslant\; \frac{\beta k}{u}-2\frac{\sigma_k^{ii}x_i^2}{\beta}+C_0\sigma_{k-1}-C.$$

We take β sufficiently large, then, we have

$$C \geqslant \frac{\beta k}{u}+\frac{C_0}{2}\sigma_{k-1}\geqslant \frac{\beta k}{u}+c_0\sigma_1^{\frac{1}{k-1}}\sigma_k^{\frac{k-2}{k-1}},$$

where we have used Newton-Maclaurin in the last inequality. Hence, we obtain Lemma 11. □

Proof of Theorem 4 The proof is classical [15]. Suppose u is an entrie solution of the equation (1.5). For arbitrary positive constant $R > 1$, we consider the set

$$\Omega_R=\{y\in\mathbb{R}^n;u(Ry)\leqslant R^2\}.$$

Let

$$v(y) = \frac{u(Ry) - R^2}{R^2}.$$

We consider the following Dirichlet problem,

$$\begin{cases} \sigma_k[D^2 v] & = & 1 & \text{in } \Omega_R \\ v & = & 0 & \text{on } \partial\Omega_R \end{cases}. \tag{4.11}$$

Using Lemma 11, we have the following type estimates,

$$(-v)^\beta \Delta v \leqslant C. \tag{4.12}$$

Here β and C depend on k, diameter of the Ω_R. Now using the quadratic growth condition appears in Theorem 4, we have

$$c|Ry|^2 - b \leqslant u(Ry) \leqslant R^2,$$

which implies

$$|y|^2 \leqslant \frac{1+b}{c}.$$

Thus Ω_R is bounded. Hence, the constant C, β become two absolutely constants. We now consider the domain

$$\Omega'_R = \{y; u(Ry) \leqslant R^2/2\} \subset \Omega_R.$$

In Ω'_R, we have,

$$v(y) \leqslant -\frac{1}{2}.$$

Hence, (4.12) implies that in Ω'_R, we have,

$$\Delta v \leqslant 2^\beta C.$$

Note that,

$$\nabla^2_y v = \nabla^2_x u.$$

Thus, using previous two formulas, we have, in $\Omega'_R = \{x; u(x) \leqslant R^2/2\}$,

$$\Delta u \leqslant C, \tag{4.13}$$

where C is a absolutely constant. Since R is arbitrary, we have the above inequality in whole $\mathbb{R}^n$. Using Evans-Krylov theory [9], we have

$$|D^2 u|_{C^\alpha(B_R)} \leqslant C \frac{|D^2 u|_{C^0(B_R)}}{R^\alpha} \leqslant \frac{C}{R^\alpha}.$$

Hence, we obtain our theorem letting $R \to +\infty$.

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References

[1] J. Ball, *Differentiability properties of symmetric and isotropic functions*. Duke Math. J. **51**, (1984), 699-728.

[2] J. Bao, J. Chen, B. Guan and M. Ji, *Liouvell property and regularity of a Hessian quotient equation*, Amer. J. Math. 125(2003), No.2, 301-316.

[3] L. Caffarelli, L. Nirenberg and J. Spruck, *The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian* Acta Math. **155**, (1985),261 - 301.

[4] E. Calabi, *Improper affine hyperspheres of convex type and a generalization of a theoem by K. Jörgens*, Michigen Math. J. 5(1958),105-126.

[5] L. Caffarelli, and Y. Li, *An extension to a theorem of Jrgens, Calabi, and Pogorelov*. Comm. Pure Appl. Math. 56 (2003), no. 5, 549-583.

[6] S.-Y. A. Chang and Y. Yuan, *A Liouvelll problem for sigma-2 equation*, Discrete Contin. Dyn. Syst. 28 (2010), no. 2, 659-664.

[7] S.Y. Cheng and S.T. Yau, *Complete affine hypersurfaces, part I. The completeness of affine metrics*, Comm. Pure Appl. Math., **39**, (1986), 839-866.

[8] K.S. Chou and X.-J. Wang, *A variational theory of the Hessian equation*. Comm. Pure Appl. Math. **54**, (2001), 1029-1064.

[9] D. Gilbarge and N.S. Trudinger, *Elliptic partial differential equations of the second order*, second edition, Springer, 1998.

[10] P. Guan, J. Li, and Y.Y. Li, *Hypersurfaces of Prescribed Curvature Measure*, Duke Math. J. **161**, (2012), 1927-1942.

[11] P. Guan, C. Ren and Z. Wang, *Global C^2 estimates for convex solutions of curvature equations*, to appear in Comm. Pure Appl. Math.

[12] K. Jögens, *Über die Lösungen der Differentialgleichung $rs-t^2=1$*, Math. Ann. 127(1954),130-134.

[13] A.V. Pogorelov, *On the improper convex affine hyperspheres*. Geom. Dedi. No.1 (1972), 33-46.

[14] A.V. Pogorelov, *The Minkowski Multidimensional Problem*, John Wiley, 1978.

[15] N.S. Trudinger and X.-J. Wang, *The Monge-Ampere equation and its geometric applications*, Handbook of geometric analysis, International Press, 2008, Vol. I, pp. 467-524.

[16] X.-J. Wang, *k Hessian equation*. Geometric Analysis and PDEs, Lecture Notes in Mathematics 2009, pp 177-252.

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