



# A proof of Alexandrov’s uniqueness theorem for convex surfaces in $\mathbb{R}^3$

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## Abstract

We give a new proof of a classical uniqueness theorem of Alexandrov [4] using the weak uniqueness continuation theorem of Bers–Nirenberg [8]. We prove a version of this theorem with the minimal regularity assumption: the spherical Hessians of the corresponding convex bodies as Radon measures are nonsingular.

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We give a new proof of the following uniqueness theorem of Alexandrov, using the weak unique continuation theorem of Bers–Nirenberg [8].

**Theorem 1.** (See Theorem 9 in [4].) Suppose  $M_1$  and  $M_2$  are two closed strictly convex  $C^2$  surfaces in  $\mathbb{R}^3$ , suppose  $f(y_1, y_2) \in C^1$  is a function such that  $\frac{\partial f}{\partial y_1} \frac{\partial f}{\partial y_2} > 0$ . Denote by  $\kappa_1 \geq \kappa_2$  the principal curvatures of surfaces, and denote by  $v_{M_1}$  and  $v_{M_2}$  the Gauss maps of  $M_1$  and  $M_2$  respectively. If

$$f(\kappa_1(v_{M_1}^{-1}(x), \kappa_2(v_{M_1}^{-1}(x)))) = f(\kappa_1(v_{M_2}^{-1}(x), \kappa_2(v_{M_2}^{-1}(x))))), \quad \forall x \in \mathbb{S}^2 \tag{1}$$

then  $M_1$  is equal to  $M_2$  up to a translation.

This classical result was first proved for analytical surfaces by Alexandrov in [3], for  $C^4$  surfaces by Pogorelov in [20], and Hartman and Wintner [14] reduced regularity to  $C^3$ , see also [21]. Pogorelov [22,23] published certain uniqueness results for  $C^2$  surfaces, these general results would imply Theorem 1 in  $C^2$  case. It was pointed out

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in [19] that the proof of Pogorelov is erroneous, it contains an uncorrectable mistake (see pp. 301–302 in [19]). There is a counter-example of Martinez-Maure [15] (see also [19]) to the main claims in [22,23]. The results by Han–Nadirashvili–Yuan [13] imply two proofs of **Theorem 1**, one for  $C^2$  surfaces and another for  $C^{2,\alpha}$  surfaces. The problem is often reduced to a uniqueness problem for linear elliptic equations in appropriate settings, either on  $S^2$  or in  $\mathbb{R}^3$ , we refer to [4,21]. Here we will concentrate on the corresponding equation on  $S^2$ , as in [11]. The advantage in this setting is that it is globally defined.

If  $M$  is a strictly convex surface with support function  $u$ , then the principal curvatures at  $v^{-1}(x)$  are the reciprocals of the principal radii  $\lambda_1, \lambda_2$  of  $M$ , which are the eigenvalues of spherical Hessian  $W_u(x) = (u_{ij}(x) + u(x)\delta_{ij})$  where  $u_{ij}$  are the covariant derivatives with respect to any given local orthonormal frame on  $S^2$ . Set

$$\tilde{F}(W_u) =: f\left(\frac{1}{\lambda_1(W_u)}, \frac{1}{\lambda_2(W_u)}\right) = f(\kappa_1, \kappa_2). \tag{2}$$

In view of Lemma 1 in [5], if  $f$  satisfies the conditions in **Theorem 1**, then  $\tilde{F}^{ij} = \frac{\partial \tilde{F}}{\partial w_{ij}} \in L^\infty$  is uniformly elliptic. In the case  $n = 2$ , it can be read off from the explicit formulas

$$\lambda_1 = \frac{\sigma_1(W_u) - \sqrt{\sigma_1(W_u)^2 - 4\sigma_2(W_u)}}{2}, \quad \lambda_2 = \frac{\sigma_1(W_u) + \sqrt{\sigma_1(W_u)^2 - 4\sigma_2(W_u)}}{2}.$$

As noted by Alexandrov in [5],  $\tilde{F}^{ij}$  in general is not continuous if  $f(y_1, y_2)$  is not symmetric (even  $f$  is analytic).

We want to address when **Theorem 1** remains true for convex bodies in  $\mathbb{R}^3$  with weakened regularity assumption. In the Brunn–Minkowski theory, the uniqueness of Alexandrov–Fenchel–Jessen [1,2,10] states that, if two bounded convex bodies in  $\mathbb{R}^{n+1}$  have the same  $k$ th area measures on  $S^n$ , then these two bodies are the same up to a rigidity motion in  $\mathbb{R}^{n+1}$ . Though for a general convex body, the principal curvatures of its boundary may not be defined. But one can always define the support function  $u$ , which is a function on  $S^2$ . By the convexity, then  $W_u = (u_{ij} + u\delta_{ij})$  is a Radon measure on  $S^2$ . Also, by Alexandrov’s theorem for the differentiability of convex functions,  $W_u$  is defined for almost every point  $x \in S^2$ . Denote  $\mathcal{N}$  to be the space of all positive definite  $2 \times 2$  matrices, and let  $G$  be a function defined on  $\mathcal{N}$ . For a support function  $u$  of a bounded convex body  $\Omega_u$ ,  $G(W_u)$  is defined for a.e.  $x \in S^2$ . For fixed support functions  $u^l$  of  $\Omega_{u^l}$ ,  $l = 1, 2$ , there is  $\Omega \subset S^2$  with  $|S^2 \setminus \Omega| = 0$  such that  $W_{u^1}, W_{u^2}$  are pointwise finite in  $\Omega$ . Set  $P_{u^1, u^2} = \{W \in \mathcal{N} \mid \exists x \in \Omega, W = W_{u^1}(x), \text{ or } W = W_{u^2}(x)\}$ , let  $\mathcal{P}_{u^1, u^2}$  be the convex hull of  $P_{u^1, u^2}$  in  $\mathcal{N}$ .

We establish the following slightly more general version of **Theorem 1**.

**Theorem 2.** *Suppose  $\Omega_1$  and  $\Omega_2$  are two bounded convex bodies in  $\mathbb{R}^3$ . Let  $u^l$ ,  $l = 1, 2$  be the corresponding supporting functions respectively. Suppose the spherical Hessians  $W_{u^l} = (u^l_{ij} + \delta_{ij}u^l)$  (in the weak sense) are two non-singular Radon measures. Let  $G : \mathcal{N} \rightarrow \mathbb{R}$  be a  $C^{0,1}$  function such that*

$$\Lambda I \geq (G^{ij})(W) := \left(\frac{\partial G}{\partial W_{ij}}\right)(W) \geq \lambda I > 0, \quad \forall W \in \mathcal{P}_{u^1, u^2},$$

for some positive constants  $\Lambda, \lambda$ . If

$$G(W_{u^1}) = G(W_{u^2}), \tag{3}$$

at almost every parallel normal  $x \in S^2$ , then  $\Omega_1$  is equal to  $\Omega_2$  up to a translation.

Suppose  $u^1, u^2$  are the support functions of two convex bodies  $\Omega_1, \Omega_2$  respectively, and suppose  $W_{u^l}, l = 1, 2$  are defined and they satisfy Eq. (3) at some point  $x \in S^2$ . Then, for  $u = u^1 - u^2$ ,  $W_u(x)$  satisfies equation

$$F^{ij}(x)(W_u(x)) = 0, \tag{4}$$

with  $F^{ij}(x) = \int_0^1 \frac{\partial \tilde{F}}{\partial w_{ij}}(tW_{u^1}(x) + (1-t)W_{u^2}(x))dt$ . By the convexity,  $W_{u^l}, l = 1, 2$  exist almost everywhere on  $S^2$ . If they satisfy Eq. (3) almost everywhere, Eq. (4) is verified almost everywhere. Note that  $u$  may not be a solution (even in a weak sense) of partial differential equation (4). The classical elliptic theory (e.g., [16,18,8]) requires  $u \in W^{2,2}$  in order to make sense of  $u$  as a weak solution of (4). A main step in the proof of **Theorem 2** is to show that with the assumptions in the theorem,  $u = u^1 - u^2$  is indeed in  $W^{2,2}(S^2)$ . The proof will appear in the last part of the paper.

Let’s now focus on  $W^{2,2}$  solutions of differential equation (4), with general uniformly elliptic condition on tensor  $F^{ij}$  on  $\mathbb{S}^2$ :

$$\lambda|\xi|^2 \leq F^{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall x \in \mathbb{S}^2, \xi \in \mathbb{R}^2, \tag{5}$$

for some positive numbers  $\lambda, \Lambda$ . The aforementioned proofs of Theorem 1 [20,14,21,13] all reduce to the statement that any solution of (5) is a linear function, under various regularity assumptions on  $F^{ij}$  and  $u$ . Eq. (4) is also related to minimal cone equation in  $\mathbb{R}^3$  [13]. The following result was proved in [13].

**Theorem 3.** (See Theorem 1.1 in [13].) Suppose  $F^{ij}(x) \in L^\infty(\mathbb{S}^2)$  satisfies (5), suppose  $u \in W^{2,2}(\mathbb{S}^2)$  is a solution of (4). Then,  $u(x) = a_1x_1 + a_2x_2 + a_3x_3$  for some  $a_i \in \mathbb{R}$ .

There the original statement in [13] is for 1-homogeneous  $W_{loc}^{2,2}(\mathbb{R}^3)$  solution  $v$  of equation

$$\sum_{i,j=1}^3 a^{ij}(X)v_{ij}(X) = 0. \tag{6}$$

These two statements are equivalent. To see this, set  $u(x) = \frac{v(X)}{|X|}$  with  $x = \frac{X}{|X|}$ . By the homogeneity assumption, the radial direction corresponds to null eigenvalue of  $\nabla^2 v$ , the other two eigenvalues coincide the eigenvalues of the spherical Hessian of  $W = (u_{ij} + u\delta_{ij})$ .  $v(X) \in W_{loc}^{2,2}(\mathbb{R}^3)$  is a solution to (6) if and only if  $u \in W^{2,2}(\mathbb{S}^2)$  is a solution to (4) with  $F^{ij}(x) = \langle e_i, Ae_j \rangle$ , where  $A = (a^{ij}(\frac{X}{|X|}))$  and  $(e_1, e_2)$  is any orthonormal frame on  $\mathbb{S}^2$ .

The proof in [13] uses gradient maps and support planes introduced by Alexandrov, as in [3,20,21]. We give a different proof of Theorem 3 using the maximum principle for smooth solutions and the unique continuation theorem of Bers–Nirenberg [8], working purely on solutions of Eq. (4) on  $\mathbb{S}^2$ .

Note that  $F$  in Theorem 2 (and Theorem 1) is not assumed to be symmetric. The weak assumption  $F^{ij} \in L^\infty$  is needed to deal with this case. This assumption also fits well with the weak unique continuation theorem of Bers–Nirenberg. This beautiful result of Bers–Nirenberg will be used in a crucial way in our proof. If  $u \in W^{2,2}(\mathbb{S}^2)$ ,  $u \in C^\alpha(\mathbb{S}^2)$  for some  $0 < \alpha < 1$  by the Sobolev embedding theorem. Eq. (4) and  $C^{1,\alpha}$  estimates for 2-d linear elliptic PDE (e.g., [16,18,8]) imply that  $u$  is in  $C^{1,\alpha}(\mathbb{S}^2)$  for some  $\alpha > 0$  depending only on  $\|u\|_{C^0}$  and the ellipticity constants of  $F^{ij}$ . This fact will be assumed in the rest of the paper.

The following lemma is elementary.

**Lemma 4.** Suppose  $F^{ij} \in L^\infty(\mathbb{S}^2)$  satisfies (5), suppose at some point  $x \in \mathbb{S}^2$ ,  $W_u(x) = (u_{ij}(x) + u(x)\delta_{ij})$  satisfies (4). Then,

$$|W_u|^2(x) \leq -\frac{2\Lambda}{\lambda} \det W_u(x).$$

**Proof.** At  $x$ , by Eq. (4),

$$\det W_u = -\frac{1}{F^{22}}(F^{11}W_{11}^2 + 2F^{12}W_{11}W_{12} + F^{22}W_{12}^2) \leq -\frac{\lambda}{\Lambda}(W_{11}^2 + W_{12}^2), \tag{7}$$

and similarly,  $\det W_u \leq -\frac{\lambda}{\Lambda}(W_{22}^2 + W_{21}^2)$ . Thus,

$$(W_{11}^2 + W_{12}^2 + W_{21}^2 + W_{22}^2) \leq -\frac{2\Lambda}{\lambda} \det W_u. \quad \square \tag{8}$$

For each  $u \in C^1(\mathbb{S}^2)$ , set  $X_u = \sum_i u_i e_i + u e_{n+1}$ . For any unit vector  $E$  in  $\mathbb{R}^3$ , define

$$\phi_E(x) = \langle E, X_u(x) \rangle, \quad \text{and} \quad \rho_u(x) = |X_u(x)|^2, \tag{9}$$

where  $\langle \cdot, \cdot \rangle$  is the standard inner product in  $\mathbb{R}^3$ . The function  $\rho$  was introduced by Weyl in his study of Weyl’s problem [25]. It played important role in Nirenberg’s solution of Weyl’s problem in [17]. Our basic observation is that there is a maximum principle for  $\rho_u$  and  $\phi_E$ .

**Lemma 5.** Suppose  $U \subset \mathbb{S}^2$  is an open set,  $F^{ij} \in C^1(U)$  is a tensor in  $U$  and  $u \in C^3(U)$  satisfies Eq. (4), then there are two constants  $C_1, C_2$  depending only on the  $C^1$ -norm of  $F^{ij}$  such that

$$F^{ij}(\rho_u)_{ij} \geq -C_1|\nabla\rho_u|, \quad F^{ij}(\phi_E)_{ij} \geq -C_2|\nabla\phi_E| \quad \text{in } U. \tag{10}$$

**Proof.** Picking any orthonormal frame  $e_1, e_2$ , we have

$$(X_u)_i = W_{ij}e_j, \quad (X_u)_{ij} = W_{ijk}e_k - W_{ij}\vec{x}. \tag{11}$$

By Codazzi property of  $W$  and (4),

$$\frac{1}{2}F^{ij}(\rho_u)_{ij} = \langle X_u, F^{ij}W_{ijk}e_k \rangle + F^{ij}W_{ik}W_{kj} = -u_k F^{ij}_{,k}W_{ij} + F^{ij}W_{ik}W_{kj}.$$

On the other hand,  $\nabla\rho_u = 2W \cdot (\nabla u)$ . At the non-degenerate points (i.e.,  $\det W \neq 0$ ),  $\nabla u = \frac{1}{2}W^{-1} \cdot \nabla\rho_u$ , where  $W^{-1}$  denotes the inverse matrix of  $W$ . Now,

$$2u_k F^{ij}_{,k}W_{ij} = W^{kl}(\rho_u)_l F^{ij}_{,k}W_{ij} = (\rho_u)_l F^{ij}_{,k} \frac{A^{kl}W_{ij}}{\det W}, \tag{12}$$

where  $A^{kl}$  denotes the co-factor of  $W_{kl}$ .

The first inequality in (10) follows from (8) and (12).

The proof for  $\phi_E$  follows the same argument and the following facts:

$$F^{ij}(\phi_E)_{ij} = -\langle E, e_k \rangle F^{ij}_{,k}W_{ij}, \quad \nabla\phi_E = W \cdot \langle E, e_k \rangle. \quad \square$$

Lemma 5 yields immediately Theorem 1 in  $C^3$  case, which corresponds to the Hartman–Wintner theorem [14].

**Corollary 6.** Suppose  $f \in C^2$  and is symmetric,  $M_1, M_2$  are two closed convex  $C^3$  surfaces satisfy conditions in Theorem 1, then the surfaces are the same up to a translation.

**Proof.** Since  $f \in C^2$  is symmetric,  $F^{ij}$  in (4) is in  $C^1(\mathbb{S}^2)$  and  $u \in C^3(\mathbb{S}^2)$ . By Lemma 5 and the strong maximum principle,  $X_u$  is a constant vector.  $\square$

To precede further, set

$$\mathcal{M} = \left\{ p \in \mathbb{S}^2 : \rho_u(p) = \max_{q \in \mathbb{S}^2} \rho_u(q) \right\},$$

for each unit vector  $E \in \mathbb{R}^3$ ,

$$\mathcal{M}_E = \left\{ p \in \mathbb{S}^2 : \phi_E(p) = \max_{q \in \mathbb{S}^2} \phi_E(q) \right\}.$$

**Lemma 7.**  $\mathcal{M}$  and  $\mathcal{M}_E$  have no isolated points.

**Proof.** We prove the lemma for  $\mathcal{M}$ , the proof for  $\mathcal{M}_E$  is the same. If point  $p_0 \in \mathcal{M}$  is an isolated point, we may assume  $p_0 = (0, 0, 1)$ . Pick  $\bar{U}$  a small open geodesic ball centered at  $p_0$  such that  $\bar{U}$  is properly contained in  $\mathbb{S}^2_+$ , and pick a sequence of smooth 2-tensor  $(F^\epsilon)^{ij} > 0$  which is convergent to  $(F^{ij})$  in  $L^\infty$ -norm in  $\bar{U}$ . Consider

$$\begin{cases} F^\epsilon{}^{ij}(u^\epsilon_{ij} + u^\epsilon\delta_{ij}) = 0 & \text{in } \bar{U} \\ u^\epsilon = u & \text{on } \partial\bar{U}. \end{cases} \tag{13}$$

Since  $x_3 > 0$  in  $\mathbb{S}^2_+$ , one may write  $u^\epsilon = x_3v^\epsilon$  in  $\bar{U}$ . As  $(x_3)_{ij} = -x_3\delta_{ij}$ , it easy to check that  $v^\epsilon$  satisfies

$$F^\epsilon{}^{ij}v^\epsilon_{ij} + b_kv^\epsilon_k = 0 \quad \text{in } \bar{U}.$$

Therefore, (13) is uniquely solvable.

Since  $p_0 \in \mathcal{M}$  is an isolated point, there are open geodesic balls  $\bar{U}' \subset \bar{U}$  centered at  $p_0$  and a small  $\delta > 0$  such that

$$\rho_u(p_0) - \rho_u(p) \geq \delta \quad \text{for } \forall p \in \partial\bar{U}'. \tag{14}$$

By the  $C^{1,\alpha}$  estimates for linear elliptic equation in dimension two and the uniqueness of the Dirichlet problem [16, 8, 18],  $\exists \epsilon_k$  such that

$$\|u - u^{\epsilon_k}\|_{C^{1,\alpha}(\bar{U}')} \rightarrow 0, \quad \|\rho_u - \rho_{u^{\epsilon_k}}\|_{C^\alpha(\bar{U}')} \rightarrow 0.$$

Together with (14), if  $\epsilon_k$  is small enough, there is a local maximal point of  $\rho_{u^{\epsilon_k}}$  in  $\bar{U}' \subset \bar{U}$ . Since  $u^{\epsilon_k}, F_\epsilon^{ij} \in C^\infty(\bar{U}')$  satisfy (13), it follows from Lemma 5 and the strong maximum principle that  $\rho_{u^{\epsilon_k}}$  must be constant in  $\bar{U}'$ , when  $\epsilon_k$  is small enough. This implies that  $\rho$  is constant in  $\bar{U}'$ . A contradiction.  $\square$

We now prove Theorem 3.

**Proof of Theorem 3.** For any  $p_0 \in \mathcal{M}$ , if  $\rho_u(p_0) = 0$ , then  $u \equiv 0$ . We may assume  $\rho_u(p_0) > 0$ . Set  $E := \frac{X_u(p_0)}{|X_u(p_0)|}$ . Choose another two unit constant vectors  $\beta_1, \beta_2$  with  $\langle \beta_i, \beta_j \rangle = \delta_{ij}, \beta_i \perp E$  for  $i, j = 1, 2$ . Under these orthogonal coordinates in  $\mathbb{R}^3$ ,

$$X_u(p) = a(p)E + b_1(p)\beta_1 + b_2(p)\beta_2, \quad \forall p \in \mathcal{M}_E. \tag{15}$$

On the other hand,  $\phi_E(p) = \rho_u^{1/2}(p_0), \forall p \in \mathcal{M}_E$ . Thus,

$$a(p) = \rho_u^{1/2}(p_0), \quad b_1(p) = b_2(p) = 0, \quad \forall p \in \mathcal{M}_E. \tag{16}$$

Consider the function  $\tilde{u}(x) = u(x) - \rho_u^{1/2}(p_0)E \cdot x$ . (15) and (16) yield,  $\forall p \in \mathcal{M}_E$ ,

$$\nabla_{e_i} \tilde{u}(p) = \nabla_{e_i} u(p) - \rho_u^{1/2}(p_0) \langle E, e_i \rangle = \langle X_u(p), e_i \rangle - \rho_u^{1/2}(p_0) \langle E, e_i \rangle = 0. \tag{17}$$

Moreover,  $\tilde{u}(x)$  also satisfies Eq. (4). As pointed out in [8], if  $\tilde{u}$  satisfies an elliptic equation,  $\nabla \tilde{u}$  satisfies an elliptic system of equations. Lemma 7, (17) and the unique continuation theorem of Bers–Nirenberg (p. 113 in [7]) imply  $\nabla \tilde{u} \equiv 0$ . Thus,  $\tilde{u}(x) \equiv \tilde{u}(p_0) = 0$  and  $u(x)$  is a linear function on  $\mathbb{S}^2$ .  $\square$

Theorem 1 is a direct consequence of Theorem 3. We now prove Theorem 2.

**Proof of Theorem 2.** The main step is to show  $u = u^1 - u^2 \in W^{2,2}(\mathbb{S}^2)$ , using the assumption that  $W_{u^l}, l = 1, 2$  are non-singular Radon measures. It follows from the convexity, the spherical Hessians  $W_{u^l}, l = 1, 2$  and  $W_u$  are defined almost everywhere on  $\mathbb{S}^2$  (Alexandrov’s theorem). So, we can define  $G(W_{u^l}), l = 1, 2$  almost everywhere in  $\mathbb{S}^2$ . As  $W_{u^l}, l = 1, 2$  are nonsingular Radon measures,  $W_{u^l} \in L^1(\mathbb{S}^2)$  (see [9]), we also have  $W_u \in L^1(\mathbb{S}^2)$ . Since  $u^1, u^2$  satisfy  $G(W_{u^1}) = G(W_{u^2})$  for almost every parallel normal  $x \in \mathbb{S}^2$ , there is  $\Omega \subset \mathbb{S}^2$  with  $|\mathbb{S}^2 \setminus \Omega| = 0$ , such that  $W_u$  satisfies the following equation pointwise in  $\Omega$ ,

$$G^{ij}(x)(u_{ij}(x) + u(x)\delta_{ij}) = 0, \quad x \in \Omega,$$

where  $G^{ij} = \int_0^1 \frac{\partial G}{\partial w_{ij}}(tW_u^1 + (1-t)W_u^2)dt$ . By Lemma 4, we can obtain that

$$|W_u|^2 = W_{11}^2 + W_{12}^2 + W_{21}^2 + W_{22}^2 \leq -\frac{2\Lambda}{\lambda} \det W_u, \quad x \in \Omega.$$

On the other hand,

$$\det W_u \leq \det W_{\tilde{u}},$$

where  $\tilde{u} = u^1 + u^2$ . Thus, to prove  $u \in W^{2,2}(\mathbb{S}^2)$ , it suffices to get an upper bound for  $\int_{\mathbb{S}^2} \det W_{\tilde{u}}$ .

Recall that  $W_{u^l} \in L^1(\mathbb{S}^2)$ , so  $u^l \in W^{2,1}(\mathbb{S}^2), l = 1, 2$  and the same for  $\tilde{u}$ . This allows us to choose two sequences of smooth convex bodies  $\Omega_\epsilon^l$  with supporting functions  $u_\epsilon^l$  such that  $\|\tilde{u}_\epsilon - \tilde{u}\|_{W^{2,1}(\mathbb{S}^2)} \rightarrow 0$  as  $\epsilon \rightarrow 0$ . By Fatou’s Lemma and continuity of the area measures,

$$\int_{\mathbb{S}^2} \det W_{\tilde{u}} = \int_{\Omega} \det W_{\tilde{u}} \leq \liminf_{\epsilon \rightarrow 0} \int_{\mathbb{S}^2} \det W_{\tilde{u}_\epsilon} \leq V(\Omega^1) + V(\Omega^2) + 2V(\Omega^1, \Omega^2),$$

where  $V(\Omega^1)$ ,  $V(\Omega^2)$  denote the volumes of the convex bodies  $\Omega^1$  and  $\Omega^2$  respectively and  $V(\Omega^1, \Omega^2)$  is the mixed volume.

It follows that  $W_u \in L^2(\mathbb{S}^2)$  and thus,  $u \in W^{2,2}(\mathbb{S}^2)$ . This implies that  $u$  is a  $W^{2,2}$  weak solution of the differential equation

$$G^{ij}(x)(u_{ij}(x) + u(x)\delta_{ij}) = 0, \quad \forall x \in \mathbb{S}^2.$$

Finally, the theorem follows directly from [Theorem 3](#).  $\square$

**Remark 8.** Alexandrov proved in [\[3\]](#) that, if  $u$  is a homogeneous degree 1 analytic function in  $\mathbb{R}^3$  with  $\nabla^2 u$  definite nowhere, then  $u$  is a linear function. As a consequence, Alexandrov proved in [\[6\]](#) that if an analytic closed convex surface in  $\mathbb{R}^3$  satisfies the condition  $(\kappa_1 - c)(\kappa_2 - c) \leq 0$  at every point for some constant  $c$ , then it is a sphere. Martinez-Maure gave a  $C^2$  counter-example in [\[15\]](#) to this statement, see also [\[19\]](#). The counter-examples in [\[15,19\]](#) indicate that [Theorem 3](#) is not true if  $F^{ij}$  is merely assumed to be degenerate elliptic. It is an interesting question that under what degeneracy condition on  $F^{ij}$  so that [Theorem 3](#) is still true, even in smooth case. This question is related to similar questions in this nature posted by Alexandrov [\[4\]](#) and Pogorelov [\[21\]](#).

We shall wrap up this paper by mention a stability type result related with uniqueness. Indeed, by using the uniqueness property proved in [Theorem 3](#), we can prove the following stability theorem via compactness argument.

**Proposition 9.** *Suppose  $F^{ij}(x) \in L^\infty(\mathbb{S}^2)$  satisfies (5), and  $u(x) \in W^{2,2}(\mathbb{S}^2)$  is a solution of the following equation*

$$F^{ij}(x)(W_u)_{ij} = f(x), \quad \forall x \in \mathbb{S}^2. \tag{18}$$

Assume that  $f(x) \in L^\infty(\mathbb{S}^2)$  and there exists a point  $x_0 \in \mathbb{S}^2$  such that  $\rho_u(x_0) = 0$  (see [\(9\)](#) for the definition of  $\rho_u$ ). Then,

$$\|u\|_{L^\infty(\mathbb{S}^2)} \leq C_3 \|f\|_{L^\infty(\mathbb{S}^2)} \tag{19}$$

holds for some positive constant  $C_3$  only depending on the ellipticity constants  $\lambda, \Lambda$ .

**Proof.** As mentioned above, we will prove this proposition by a compactness argument. Suppose the desired estimate [\(19\)](#) does not hold, then there exists a sequence of functions  $\{f_n(x)\}_{n=1}^\infty$  on  $\mathbb{S}^2$  with  $\|f\|_{L^\infty(\mathbb{S}^2)} \leq C_4$  and a sequence of points  $\{x_n\}_{n=1}^\infty \subset \mathbb{S}^2$  such that  $\rho_{u_n}(x_n) = 0$  and  $K_n := \frac{\|u\|_{L^\infty(\mathbb{S}^2)}}{\|f\|_{L^\infty(\mathbb{S}^2)}} \rightarrow +\infty$ , where  $u_n(x)$  is the solution of Eq. [\(18\)](#) with right hand side replaced by  $f_n(x)$ .

Let  $v_n(x) = \frac{u_n(x)}{K_n \|f\|_{L^\infty(\mathbb{S}^2)}}$ , then  $\|v_n\|_{L^\infty(\mathbb{S}^2)} = 1$  and  $v_n(x)$  satisfies

$$F^{ij}(x)(W_{v_n})_{ij} = \tilde{f}_n := \frac{f_n(x)}{K_n \|f_n\|_{L^\infty(\mathbb{S}^2)}}. \tag{20}$$

By the interior  $C^{1,\alpha}$  estimates for linear elliptic equation in dimension two [\[16,8,18\]](#), we have

$$\|v_n\|_{C^{1,\alpha}(\mathbb{S}^2)} \leq C_5 (\|v_n\|_{L^\infty(\mathbb{S}^2)} + \|\tilde{f}_n\|_{L^\infty(\mathbb{S}^2)}) \leq 2C_5$$

for some positive constant  $C_5 = C_5(\lambda, \Lambda)$ . In particular, this gives that  $\|\nabla v_n\|_{L^\infty(\mathbb{S}^2)} \leq C_6$ . Now, apply the *a priori*  $W^{2,2}$  estimate for linear elliptic equation in dimension two [\[16,8,18,12\]](#), we see that  $\|v_n\|_{W^{2,2}(\mathbb{S}^2)} \leq C_7$  for some constant  $C_7 = C_7(\lambda, \Lambda, C_6)$ . It follows from this uniform estimate that, up to a subsequence,  $\{v_n(x)\}_{n=1}^\infty$  converges to some function  $v(x) \in W^{2,2}(\mathbb{S}^2)$  and  $v(x)$  satisfies

$$F^{ij}(x)(W_v)_{ij} = 0, \quad a.e. x \in \mathbb{S}^2.$$

Then, the previous uniqueness result [Theorem 3](#) tells that  $v(x)$  must be a linear function, i.e., there exists a constant vector  $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$  such that  $v(x) = a_1x_1 + a_2x_2 + a_3x_3$ .

On the other hand, recall that, by the assumption at the beginning, there exists  $x_n \in \mathbb{S}^2$  such that  $\rho_{v_n}(x_n) = 0$ . Then, up to a subsequence,  $x_n \rightarrow x_\infty \in \mathbb{S}^2$  and  $\rho_v(x_\infty) = 0$ . This together with the linear property of  $v(x)$  imply that  $v(x) \equiv 0$ . However, this contradicts with the fact that  $\|v\|_{L^\infty(\mathbb{S}^2)} = 1$  as  $\|v_n\|_{L^\infty(\mathbb{S}^2)} = 1$ .  $\square$

As a direct corollary, we have the following stability property for convex surfaces.

**Theorem 10.** *Suppose  $M_1, M_2$  and  $f$  satisfy the same assumptions as in Theorem 3. Define  $\mu_1(x) := f(\kappa_1(v_{M_1}^{-1}(x), \kappa_2(v_{M_1}^{-1}(x))))$  and  $\mu_2(x) := f(\kappa_1(v_{M_2}^{-1}(x), \kappa_2(v_{M_2}^{-1}(x))))$  for  $\forall x \in \mathbb{S}^2$ . If  $\|\mu_1 - \mu_2\|_{L^\infty(\mathbb{S}^2)} < \epsilon$ , then, module a linear translation,  $M_1$  is very close to  $M_2$ . More precisely, suppose  $u_1, u_2$  are the supporting functions of  $M_1$  and  $M_2$  after module the linear translation, then there exists a constant  $C$  such that*

$$\|u_1 - u_2\|_{L^\infty(\mathbb{S}^2)} \leq C \|\mu_1 - \mu_2\|_{L^\infty(\mathbb{S}^2)}. \tag{21}$$

Finally, it is worth to remark that there are many stability type results for convex surfaces proved in the literature (see [24]). However, almost all the proofs need to use the assumption that  $f(\kappa_1, \kappa_2, \dots, \kappa_n)$  satisfies divergence property. Here, we do not make such kind assumption in this dimension two case. There is one drawback in the above stability result: one could not get the sharp constant via the compactness argument. It would be an interesting question to derive a sharp estimate for (21).

**Conflict of interest statement**

There is no conflict of interest.

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