Categorical traces and a relative Lefschetz-Verdier formula

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Abstract

We prove a relative Lefschetz-Verdier theorem for locally acyclic objects over a Noetherian base scheme. This is done by studying duals and traces in the symmetric monoidal 2-category of cohomological correspondences. We show that local acyclicity is equivalent to dualizability and deduce that duality preserves local acyclicity. As another application of the category of cohomological correspondences, we show that the nearby cycle functor over a Henselian valuation ring preserves duals, generalizing a theorem of Gabber.

Introduction

The notions of dual and trace in symmetric monoidal categories were introduced by Dold and Puppe [DP]. They have been extended to higher categories and have found important applications in algebraic geometry and other contexts (see [BZN] by Ben-Zvi and Nadler and the references therein).

The goal of the present paper is to record several applications of the formalism of duals and traces to the symmetric monoidal 2-category of cohomological correspondences in étale cohomology. One of our main results is the following relative Lefschetz-Verdier theorem.

Theorem 0.1. *Let S be a Noetherian scheme and let Λ be a Noetherian commutative ring with $m\Lambda = 0$ for some m invertible on S. Let*

$$\begin{tikzcd} X \arrow[d, "f"'] & C \arrow[l, "\overleftarrow{c}"'] \arrow[r, "\overrightarrow{c}"] \arrow[d, "p"] & Y \arrow[d, "g"] & D \arrow[l, "\overleftarrow{d}"'] \arrow[r, "\overrightarrow{d}"] \arrow[d, "q"] & X \arrow[d, "f"] \\ X' & C' \arrow[l] \arrow[r] & Y' & D' \arrow[l] \arrow[r] & X' \end{tikzcd}$$

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be a commutative diagram of schemes separated of finite type over S, with p and $D \to D' \times_{Y'} Y$ proper. Let $L \in D_{\mathrm{cft}}(X, \Lambda)$ such that L and $f_! L$ are locally acyclic over S. Let $M \in D(Y, \Lambda)$, $u\colon \overleftarrow{c}^* L \to \overrightarrow{c}^! M$, $v\colon \overleftarrow{d}^* M \to \overrightarrow{d}^! L$. Then $s\colon C \times_{X \times_S Y} D \to C' \times_{X' \times_S Y'} D'$ is proper and

$$s_*\langle u, v\rangle = \langle (f, p, g)_! u, (g, q, f)_! v\rangle.$$

Here $D_{\mathrm{cft}}(X, \Lambda) \subseteq D(X, \Lambda)$ denotes the full subcategory spanned by objects of finite tor-dimension and of constructible cohomology sheaves, and $\langle u, v\rangle$ is the relative Lefschetz-Verdier pairing.

Remark 0.2. In the case where S is the spectrum of a field, local acyclicity is trivial and the theorem generalizes [SGA5, III Corollaire 4.5] and (the scheme case of) [V1, Proposition 1.2.5]. For S smooth over a perfect field and under additional assumptions of smoothness and transversality, Theorem 0.1 was proved by Yang and Zhao [YZ, Corollary 3.10]. The original proof in [SGA5] and its adaptation in [YZ] require the verification of a large amount of commutative diagrams. The categorical interpretation we adopt makes our proof arguably more conceptual.

It was observed by Lurie that Grothendieck's cohomological operations can be encoded by a (pseudo) functor $\mathcal{B} \to \mathcal{C}at$, where $\mathcal{B}$ denotes the category of correspondences and $\mathcal{C}at$ denotes the 2-category of categories. Contrary to the situation of [BZN, Definition 2.15], in the context of étale cohomology, the functor has a *right-lax* symmetric monoidal structure that is not expected to be symmetric monoidal even after enhancement to higher categories. Instead, we apply the formalism of traces to the corresponding cofibered category produced by the Grothendieck construction, which is the category $\mathcal{C}$ of cohomological correspondences. The relative Lefschetz-Verdier formula follows from the functoriality of traces for dualizable objects (X, L) of $\mathcal{C}$.

To complete the proof, we show that under the assumption $L \in D_{\mathrm{cft}}(X, \Lambda)$, dualizability is equivalent to local acyclicity (Theorem 2.16). As a byproduct of this equivalence, we deduce immediately that local acyclicity is preserved by duality (Corollary 2.18). Note that this last statement does not involve cohomological correspondences.

We also give applications to the nearby cycle functor Ψ over a Henselian valuation ring. The functor Ψ extends the usual nearby cycle functor over a Henselian *discrete* valuation ring and was studied by Huber [H, Section 4.2]. By studying specialization of cohomological correspondences, we generalize Gabber's theorem that Ψ preserves duals and a fixed point theorem of Vidal to Henselian valuation rings (Corollaries 3.8 and 3.13). We hope that the latter can be used to study ramification over higher-dimensional bases.

Scholze remarked that our arguments also apply in the étale cohomology of diamonds and imply the equivalence between dualizability and universal local acyclicity in this situation. This fact and applications will be discussed in his work with Fargues on the geometrization of the Langlands correspondence.

Let us briefly mention some other categorical approaches to Lefschetz type theorems. In [DP, Section 4], the Lefschetz fixed point theorem is deduced from the functoriality of traces by passing to suspension spectra. In [P], a categorical framework is set up for Lefschetz-Lunts type formulas. In May 2019, as a first draft of this paper was being written, Varshavsky informed us that he had a different strategy to deduce the Lefschetz-Verdier formula, using categorical traces in $(\infty, 2)$-categories.

This paper is organized as follows. In Section 1, we review duals and traces in symmetric monoidal 2-categories and the Grothendieck construction. In Section 2, we define the symmetric monoidal 2-category of cohomological correspondences and prove the relative Lefschetz-Verdier theorem. In Section 3, we discuss applications to the nearby cycle functor over a Henselian valuation ring.

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1 Pairings in symmetric monoidal 2-categories

We review duals, traces, and pairings in symmetric monoidal 2-categories. We give the definitions in Subsection 1.1 and discuss the functoriality of pairings in Subsection 1.2. These two subsections are mostly standard (see [BZN] and [HSS] for generalizations to higher categories). In Subsection 1.3 we review the Grothendieck construction in the symmetric monoidal context, which will be used to interpret the category of cohomological correspondences later.

By a 2-category, we mean a weak 2-category (also known as a bicategory in the literature).

1.1 Pairings

Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be a symmetric monoidal 2-category.

Definition 1.1 (dual)**.** An object X of $\mathcal{C}$ is *dualizable* if there exist an object $X^\vee$ of $\mathcal{C}$, called the *dual* of X, and morphisms $\mathrm{ev}_X\colon X^\vee \otimes X \to 1_{\mathcal{C}}$, $\mathrm{coev}_X\colon 1_{\mathcal{C}} \to X \otimes X^\vee$, called evaluation and coevaluation, respectively, such that the composites

$$X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} X \otimes X^\vee \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X, \quad X^\vee \xrightarrow{\mathrm{id}_{X^\vee} \otimes \mathrm{coev}_X} X^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^\vee}} X^\vee$$

are isomorphic to identities.

Remark 1.2. For X dualizable, $X^\vee$ is dualizable of dual X. For X and Y dualizable, $X \otimes Y$ is dualizable of dual $X^\vee \otimes Y^\vee$.

For X and Y in $\mathcal{C}$, we let $\mathcal{H}om(X, Y)$ denote the internal mapping object if it exists.

Remark 1.3. Assume that X is dualizable of dual $X^\vee$.

(a) The morphisms coev_X and ev_X exhibit $- \otimes X^\vee$ as right (and left) adjoint to $- \otimes X$. Thus, for every object Y, $\mathcal{H}om(X, Y)$ exists and is equivalent to $Y \otimes X^\vee$. In particular, $\mathcal{H}om(X, 1_{\mathcal{C}})$ exists and is equivalent to $X^\vee$.

(b) If, moreover, $\mathcal{H}om(Y, 1_{\mathcal{C}})$ exists, then we have equivalences

$$\begin{aligned}\mathcal{H}om(X \otimes Y, 1_{\mathcal{C}}) \simeq \mathcal{H}om(X, \mathcal{H}om(Y, 1_{\mathcal{C}})) &\overset{\text{(a)}}{\simeq} \mathcal{H}om(Y, 1_{\mathcal{C}}) \otimes \mathcal{H}om(X, 1_{\mathcal{C}}),\\ \mathcal{H}om(Y, X) \simeq \mathcal{H}om(Y, \mathcal{H}om(X^\vee, 1_{\mathcal{C}})) &\simeq \mathcal{H}om(X^\vee \otimes Y, 1_{\mathcal{C}})\\ \simeq \mathcal{H}om(Y, 1_{\mathcal{C}}) \otimes \mathcal{H}om(X^\vee, 1_{\mathcal{C}}) &\simeq \mathcal{H}om(Y, 1_{\mathcal{C}}) \otimes X.\end{aligned}$$

Lemma 1.4. *An object X is dualizable if and only if $\mathcal{H}om(X, 1_{\mathcal{C}})$ and $\mathcal{H}om(X, X)$ exist and the morphism $m\colon X \otimes \mathcal{H}om(X, 1_{\mathcal{C}}) \to \mathcal{H}om(X, X)$ adjoint to*

$$X \otimes \mathcal{H}om(X, 1_{\mathcal{C}}) \otimes X \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$

is a split epimorphism. Here $\mathrm{ev}_X\colon \mathcal{H}om(X, 1_{\mathcal{C}}) \otimes X \to 1_{\mathcal{C}}$ denotes the counit.

Proof. The "only if" part is a special case of Remark 1.3. For the "if" part, we define $\mathrm{coev}_X\colon 1_{\mathcal{C}} \to X \otimes \mathcal{H}om(X, 1_{\mathcal{C}})$ to be the composite of a section of m and the morphism $1_{\mathcal{C}} \to \mathcal{H}om(X, X)$ corresponding to id_X. It is easy to see that ev_X and coev_X exhibit $\mathcal{H}om(X, 1_{\mathcal{C}})$ as a dual of X. $\square$

For X and Y dualizable, the dual of a morphism $u\colon X \to Y$ is the composite

$$u^\vee\colon Y^\vee \xrightarrow{\mathrm{id}_{Y^\vee} \otimes \mathrm{coev}_X} Y^\vee \otimes X \otimes X^\vee \xrightarrow{\mathrm{id}_{Y^\vee} \otimes u \otimes \mathrm{id}_{X^\vee}} Y^\vee \otimes Y \otimes X^\vee \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{id}_{X^\vee}} X^\vee.$$

This construction gives rise to a functor $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(Y^\vee, X^\vee)$. We have commutative squares with invertible 2-morphisms

$$\begin{tikzcd} 1_{\mathcal{C}} \arrow[r, "\mathrm{coev}_X"] \arrow[d, "\mathrm{coev}_Y"'] & X \otimes X^\vee \arrow[d, "u \otimes \mathrm{id}"] \\ Y \otimes Y^\vee \arrow[r, "\mathrm{id} \otimes u^\vee"] & Y \otimes X^\vee \end{tikzcd} \qquad \begin{tikzcd} X \otimes Y^\vee \arrow[r, "u \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes u^\vee"'] & Y \otimes Y^\vee \arrow[d, "\mathrm{ev}_Y"] \\ X \otimes X^\vee \arrow[r, "\mathrm{ev}_X"] & 1_{\mathcal{C}}. \end{tikzcd} \tag{1.1}$$

Moreover, for $X \xrightarrow{u} Y \xrightarrow{v} Z$ with X, Y, Z dualizable, we have $(vu)^\vee \simeq u^\vee v^\vee$.

Notation 1.5. We let $\Omega\mathcal{C}$ denote the category $\mathrm{End}(1_{\mathcal{C}})$.

Construction 1.6 (dimension, trace, and pairing)**.** Let X be a dualizable object of $\mathcal{C}$ and let $e\colon X \to X$ be an endomorphism. We define the trace $\mathrm{tr}(e)$ to be the object of $\Omega\mathcal{C}$ given by the composite

$$1_{\mathcal{C}} \xrightarrow{\mathrm{coev}_X} X \otimes X^\vee \xrightarrow{e \otimes \mathrm{id}_{X^\vee}} X \otimes X^\vee \xrightarrow{\mathrm{ev}_X} 1_{\mathcal{C}},$$

where in the last arrow we used the commutativity constraint.

Let $u\colon X \to Y$ and $v\colon Y \to X$ be morphisms. We define the pairing by $\langle u, v\rangle = \mathrm{tr}(v \circ u)$.

We define the *dimension* of a dualizable object X to be $\dim(X) := \langle \mathrm{id}_X, \mathrm{id}_X\rangle$, which is the composite $1_{\mathcal{C}} \xrightarrow{\mathrm{coev}_X} X \otimes X^\vee \xrightarrow{\mathrm{ev}_X} 1_{\mathcal{C}}$.

If X and Y are both dualizable, then $\langle u, v\rangle$ is isomorphic to the composite

$$1_{\mathcal{C}} \xrightarrow{\mathrm{coev}_X} X \otimes X^\vee \xrightarrow{u \otimes v^\vee} Y \otimes Y^\vee \xrightarrow{\mathrm{ev}_Y} 1_{\mathcal{C}}.$$

In this case, we have an isomorphism $\langle u, v\rangle \simeq \langle v, u\rangle$. In fact, by (1.1), we have commutative squares with invertible 2-morphisms

$$\begin{tikzcd}
 & X \otimes X^\vee \arrow[dr, "u\otimes \mathrm{id}"] & & Y \otimes Y^\vee \arrow[dr, "\mathrm{ev}_Y"] & \\
1_{\mathcal{C}} \arrow[ur, "\mathrm{coev}_X"] \arrow[dr, "\mathrm{coev}_Y"'] & & Y \otimes X^\vee \arrow[ur, "\mathrm{id}\otimes v^\vee"] \arrow[dr, "v\otimes \mathrm{id}"'] & & 1_{\mathcal{C}}. \\
 & Y \otimes Y^\vee \arrow[ur, "\mathrm{id}\otimes u^\vee"'] & & X \otimes X^\vee \arrow[ur, "\mathrm{ev}_X"'] &
\end{tikzcd}$$

The definition and construction above holds in particular for symmetric monoidal 1-categories. In the next subsection, 2-morphisms will play an important role.

1.2 Functoriality of pairings

A morphism $f\colon X \to X'$ in a 2-category is said to be *right adjointable* if there exist a morphism $f^!\colon X' \to X$, called the right adjoint of f, and 2-morphisms $\eta\colon \mathrm{id}_X \to f^! \circ f$ and $\epsilon\colon f \circ f^! \to \mathrm{id}_{X'}$ such that the composites

$$f \xrightarrow{\mathrm{id}\circ\eta} f \circ f^! \circ f \xrightarrow{\epsilon\circ\mathrm{id}} f, \quad f^! \xrightarrow{\eta\circ\mathrm{id}} f^! \circ f \circ f^! \xrightarrow{\mathrm{id}\circ\epsilon} f^!$$

are identities.

Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be a symmetric monoidal 2-category.

Construction 1.7. Consider a diagram in $\mathcal{C}$

(1.2)
$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow_\alpha"] & Y \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "\Swarrow_\beta"] & X \arrow[d, "f"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & X'
\end{tikzcd}$$

with X and X' dualizable and f right adjointable. We will construct a morphism $\langle u, v\rangle \to \langle u', v'\rangle$ in $\Omega\mathcal{C}$.

In the case where Y and Y' are also dualizable and g is also right adjointable, we define $\langle u, v\rangle \to \langle u', v'\rangle$ by the diagram

$$\begin{tikzcd}
1_{\mathcal{C}} \arrow[r, "\mathrm{coev}_X"] \arrow[dr, "\mathrm{coev}_{X'}"'] \arrow[dr, phantom, bend left, "\Swarrow"] & X \otimes X^\vee \arrow[r, "u\otimes v^\vee"] \arrow[d, "f\otimes f^{!\vee}"] \arrow[dr, phantom, "\alpha\otimes\beta^{!\vee}\Downarrow"] & Y \otimes Y^\vee \arrow[d, "g\otimes g^{!\vee}"'] \arrow[dr, "\mathrm{ev}_Y"] \arrow[dr, phantom, bend right, "\Swarrow"] & \\
 & X' \otimes X'^\vee \arrow[r, "u'\otimes v'^\vee"'] & Y' \otimes Y'^\vee \arrow[r, "\mathrm{ev}_{Y'}"] & 1_{\mathcal{C}}
\end{tikzcd}$$

where $\beta^!$ is the composite

$$v \circ g^! \xrightarrow{\eta_f} f^! \circ f \circ v \circ g^! \xrightarrow{\mathrm{id}\circ\beta\circ\mathrm{id}} f^! \circ v' \circ g \circ g^! \xrightarrow{\epsilon_g} f^! \circ v',$$

and the 2-morphisms in the triangles are

$$(f \otimes f^{!\vee}) \circ \mathrm{coev}_X \simeq ((f \circ f^!) \otimes \mathrm{id}) \circ \mathrm{coev}_{X'} \xrightarrow{\epsilon_f} \mathrm{coev}_{X'},$$
$$\mathrm{ev}_Y \xrightarrow{\eta_g} \mathrm{ev}_Y \circ ((g^! \circ g) \otimes \mathrm{id}) \simeq \mathrm{ev}_{Y'} \circ (g \otimes g^{!\vee}).$$

In particular, a morphism $\mathrm{tr}(e) \to \mathrm{tr}(e')$ is defined for every diagram in $\mathcal{C}$ of the form

(1.3)
$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow"] & X \arrow[d, "f"] \\ X' \arrow[r, "e'"] & X'. \end{tikzcd}$$

In general, we define $\langle u, v\rangle \to \langle u', v'\rangle$ as the morphism $\mathrm{tr}(v \circ u) \to \mathrm{tr}(v' \circ u')$ associated to the composite down-square of (1.2).

Trace can be made into a functor $\mathrm{End}(\mathcal{C}) \to \Omega\mathcal{C}$, where $\mathrm{End}(\mathcal{C})$ is a $(2,1)$-category whose objects are pairs $(X, e\colon X \to X)$ with X dualizable and morphisms are diagrams (1.3) with f right adjointable [HSS, Section 2.1]. Composition in $\mathrm{End}(\mathcal{C})$ is given by vertical composition of diagrams.

For the case of Theorem 0.1 where f is not proper, we will need to relax the adjointability condition in Construction 1.7 as follows. In a 2-category, a *down-square equipped with a splitting* is a diagram

(1.4)
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X' \arrow[ur, "w"] \arrow[r, "u'"'] & Y'. \end{tikzcd}$$
with $\Downarrow$ in each of the two triangles.

Note that the composition of (1.4) with a down-square on the left or on the right is a down-square equipped with a splitting. Moreover, a down-square with one vertical arrow f right adjointable is equipped with a splitting induced by the diagram

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d, "f"] \\ X' \arrow[ur, "f^!"] \arrow[r, equal] & X'. \end{tikzcd}$$
with $\eta\Downarrow$ in the upper triangle and $\Downarrow\epsilon$ in the lower triangle.

Construction 1.8. Consider a diagram in $\mathcal{C}$

(1.5)
$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] \arrow[dr, phantom, "\swarrow_\beta"] & X \arrow[d, "f"] \\ X' \arrow[ur, "w"] \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & X' \end{tikzcd}$$
with $\gamma\Downarrow$ in the upper triangle and $\Downarrow\delta$ in the lower triangle of the left square,

with X and X' dualizable. We will construct a morphism $\langle u, v\rangle \to \langle u', v'\rangle$ in $\Omega\mathcal{C}$.

In the case where Y is also dualizable, we decompose (1.5) into

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow_\gamma"] & Y \arrow[r, "v"] \arrow[d, equal] \arrow[dr, phantom, "="] & X \arrow[d, "f"] \\ X' \arrow[r, "w"] \arrow[d, equal] \arrow[dr, phantom, "\swarrow_\delta"] & Y \arrow[r, "fv"] \arrow[d, "g"] \arrow[dr, phantom, "\swarrow_\beta"] & X' \arrow[d, equal] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & X' \end{tikzcd}$$

and take the composite

$$\langle u, v\rangle \simeq \langle v, u\rangle \to \langle fv, w\rangle \simeq \langle w, fv\rangle \to \langle u', v'\rangle.$$

Here the two arrows are given by the case $f = \mathrm{id}$ of Construction 1.7. In particular, a morphism $\mathrm{tr}(e) \to \mathrm{tr}(e')$ is defined for every diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow", near start] & X \arrow[d, "f"] \\
X' \arrow[r, "e'"'] \arrow[ur] \arrow[r, phantom, "\Downarrow", shift left=2] & X'.
\end{tikzcd}$$

In general, we define $\langle u, v\rangle \to \langle u', v'\rangle$ as the morphism $\mathrm{tr}(v \circ u) \to \mathrm{tr}(v' \circ u')$ associated to the horizontal composition of (1.5).

Remark 1.9. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal 2-categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor. Then F preserves duals, pairings, and functoriality of pairings.

1.3 The Grothendieck construction

Given a category B and a (pseudo) functor $F\colon B \to \mathcal{C}at$, Grothendieck constructed a category cofibered over B, whose strict fiber at an object X of B is $F(X)$ [SGA1, VI]. We review Grothendieck's construction in the context of symmetric monoidal 2-categories. Our convention on 2-morphisms is made with applications to categorical correspondences in mind.

Let $(\mathcal{B}, \otimes, 1_{\mathcal{B}})$ be a symmetric monoidal 2-category.

Construction 1.10. Let $F\colon (\mathcal{B}, \otimes, 1_{\mathcal{B}}) \to (\mathcal{C}at^{\mathrm{co}}, \times, *)$ be a right-lax symmetric monoidal functor, where $\mathcal{C}at^{\mathrm{co}}$ denotes the 2-category obtained from the 2-category $\mathcal{C}at$ of categories by reversing the 2-morphisms. We have an object e_F of $F(1_{\mathcal{B}})$ and functors $F(X) \times F(X') \xrightarrow{\boxtimes} F(X \otimes X')$ for objects X and X' of $\mathcal{B}$. Given morphisms $c\colon X \to Y$ and $c'\colon X' \to Y'$ in $\mathcal{B}$, we have a natural transformation

$$\begin{tikzcd}
F(X) \times F(X') \arrow[r, "\boxtimes"] \arrow[d, "F(c)\times F(c')"'] \arrow[dr, phantom, "F_{c,c'}\ \Swarrow"] & F(X \otimes X') \arrow[d, "F(c\otimes c')"] \\
F(Y) \times F(Y') \arrow[r, "\boxtimes"'] & F(Y \otimes Y').
\end{tikzcd} \tag{1.6}$$

The Grothendieck construction provides a symmetric monoidal 2-category $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ as follows.

An object of $\mathcal{C} = \mathcal{C}_F$ is a pair (X, L), where $X \in \mathcal{B}$ and $L \in F(X)$. A morphism $(X, L) \to (Y, M)$ in $\mathcal{C}$ is a pair (c, u), where $c\colon X \to Y$ is a morphism in $\mathcal{B}$ and $u\colon F(c)(L) \to M$ is a morphism in $F(Y)$. A 2-morphism $(c, u) \to (d, v)$ is a 2-morphism $p\colon c \to d$ such that the following diagram commutes:

$$\begin{tikzcd}
F(c)(L) \arrow[r, "u"] & M. \\
F(d)(L) \arrow[u, "F(p)(L)"] \arrow[ur, "v"'] &
\end{tikzcd}$$

We take $1_{\mathcal{C}} = (1_{\mathcal{B}}, e_F)$. We put $(X, L) \otimes (X', L') := (X \otimes X', L \boxtimes L')$. For morphisms $(c, u)\colon (X, L) \to (Y, M)$ and $(c', u')\colon (X', L') \to (Y', M')$, we put $(c, u) \otimes (c', u') := (c \otimes c', v)$, where

$$v\colon F(c \otimes c')(L \boxtimes L') \xrightarrow{F_{c,c'}} F(c)L \boxtimes F(c')L' \xrightarrow{u \boxtimes u'} M \boxtimes M'.$$

In applications in later sections, $F_{c,c'}$ will be a natural isomorphism.

Given a morphism $f\colon X \to Y$ in $\mathcal{B}$ and an object L of $F(X)$, we write $f_\natural = (f, \mathrm{id}_{F(f)L})\colon (X, L) \to (Y, F(f)L)$.

Lemma 1.11. *Given a* 2*-morphism*

(1.7)
$$\begin{tikzcd} X \arrow[r, "c"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow p"] & Y \arrow[d, "g"] \\ X' \arrow[r, "c'"'] & Y' \end{tikzcd}$$

in $\mathcal{B}$ and a morphism $(c, u)\colon (X, L) \to (Y, M)$ in $\mathcal{C}$ above c, there exists a unique morphism $(c', u')\colon (X', F(f)L) \to (Y', F(g)M)$ in $\mathcal{C}$ above c' such that p defines a 2*-morphism in $\mathcal{C}$:*

$$\begin{tikzcd} (X, L) \arrow[r, "{(c,u)}"] \arrow[d, "f_\natural"'] \arrow[dr, phantom, "\swarrow p"] & (Y, M) \arrow[d, "g_\natural"] \\ (X', F(f)L) \arrow[r, "{(c',u')}"'] & (Y', F(g)M). \end{tikzcd}$$

Proof. By definition, u' is the morphism $F(c')F(f)L \simeq F(c'f)L \xrightarrow{F(p)} F(gc)L \simeq F(g)F(c)L \xrightarrow{u} F(g)M$. □

Construction 1.12. Let $F, G\colon (\mathcal{B}, \otimes, 1_{\mathcal{B}}) \to (\mathcal{C}at^{\mathrm{co}}, \times, *)$ be right-lax symmetric monoidal functors. Let $\alpha\colon F \to G$ be a right-lax symmetric monoidal natural transformation, which consists of the following data:

- For every object X of $\mathcal{B}$, a functor $\alpha_X\colon F(X) \to G(X)$;
- For every morphism $c\colon X \to Y$, a natural transformation

$$\begin{tikzcd} F(X) \arrow[r, "F(c)"] \arrow[d, "\alpha_X"'] \arrow[dr, phantom, "\alpha_c \nearrow"] & F(Y) \arrow[d, "\alpha_Y"] \\ G(X) \arrow[r, "G(c)"'] & G(Y); \end{tikzcd}$$

- A morphism $e_\alpha\colon e_G \to \alpha_{1_{\mathcal{B}}}(e_F)$ in $F(1_{\mathcal{B}})$;
- For objects X and X' of $\mathcal{B}$, a natural transformation

$$\begin{tikzcd} F(X) \times F(X') \arrow[r, "\boxtimes"] \arrow[d, "\alpha_X \times \alpha_{X'}"'] \arrow[dr, phantom, "\alpha_{X,X'} \nearrow"] & F(X \otimes X') \arrow[d, "\alpha_{X \otimes X'}"] \\ G(X) \times G(X') \arrow[r, "\boxtimes"'] & G(X \otimes X') \end{tikzcd}$$

subject to various compatibilities. We construct a right-lax symmetric monoidal functor $\psi\colon (\mathcal{C}_F, \otimes, 1) \to (\mathcal{C}_G, \otimes, 1)$ as follows.

We take $\psi(X, L) = (X, \alpha_X(L))$ and $\psi(c, u) = (c, \psi u)$, where

$$\psi u\colon G(c)(\alpha_X(L)) \xrightarrow{\alpha_c} \alpha_Y(F(c)L) \xrightarrow{u} \alpha_Y(M).$$

We let ψ send every 2-morphism p to p. The right-lax symmetric monoidal structure on ψ is given by

$$\begin{gathered}
(\mathrm{id}, e_\alpha)\colon (1_\mathcal{B}, e_G) \to (1_\mathcal{B}, \alpha_{1_\mathcal{B}}(e_F)) = \psi(1_\mathcal{B}, e_F),\\
\psi(X, L) \otimes \psi(X', L') = (X \otimes X', \alpha_X(L) \boxtimes \alpha_{X'}(L'))\\
\xrightarrow{(\mathrm{id}, \alpha_{X,X'})} (X \otimes X', \alpha_{X\otimes X'}(L \boxtimes L')) = \psi((X, L) \otimes (X', L')),
\end{gathered}$$

$$\begin{tikzcd}
\psi(X, L) \otimes \psi(X', L') \arrow[r, "{(\mathrm{id},\alpha_{X,X'})}"] \arrow[d, "\psi(c,u)\otimes\psi(c',u')"'] \arrow[dr, phantom, "\Downarrow\mathrm{id}"] & \psi((X, L) \otimes (X', L')) \arrow[d, "\psi((c,u)\otimes(c',u'))"] \\
\psi(Y, M) \otimes \psi(Y', M') \arrow[r, "{(\mathrm{id},\alpha_{Y,Y'})}"] & \psi((Y, M) \otimes (Y', M')).
\end{tikzcd}$$

This is a symmetric monoidal structure if e_α and $\alpha_{X,Y}$ are isomorphisms (which will be the case in our applications).

Construction 1.13. Let $(\mathcal{B}, \otimes, 1_\mathcal{B}) \xrightarrow{H} (\mathcal{B}', \otimes, 1_{\mathcal{B}'}) \xrightarrow{G} (\mathcal{C}at^{\mathrm{co}}, \times, *)$ be right-lax symmetric monoidal functors. Then we have an obvious right-lax symmetric monoidal functor $\mathcal{C}_{GH} \to \mathcal{C}_G$ sending (X, L) to (HX, L), (c, u) to (Hc, u), and every 2-morphism p to Hp. This is a symmetric monoidal functor if H is.

Construction 1.14. Let

$$\begin{tikzcd}
(\mathcal{B}, \otimes, 1_\mathcal{B}) \arrow[d, "H"'] \arrow[dr, "F"] \arrow[dr, phantom, "\Downarrow", shift right=3] & \\
(\mathcal{B}', \otimes, 1_{\mathcal{B}'}) \arrow[r, "G"'] & (\mathcal{C}at^{\mathrm{co}}, \times, *)
\end{tikzcd}$$

be a diagram of right-lax symmetric monoidal functors and right-lax symmetric monoidal transformation. Combining the two preceding constructions, we obtain right-lax symmetric monoidal functors $\mathcal{C}_F \to \mathcal{C}_{GH} \to \mathcal{C}_G$.

Lemma 1.15. *Consider a 2-morphism* (1.7) *in $\mathcal{B}$ and a morphism $(c, u)\colon (X, L) \to (Y, M)$ in $\mathcal{C}$ above c. Let $(c', u')\colon (X', F(f)L) \to (Y', G(g)M)$ be the morphism associated to (c, u) and let $(c', (\psi u)')\colon (X', G(f)\alpha_X L) \to (Y', G(g)\alpha_Y M)$ be the morphism associated to $(c, \psi u)$. Then the following square commutes:*

$$\begin{tikzcd}
G(c')G(f)\alpha_X L \arrow[r, "(\psi u)'"] \arrow[d, "\alpha_f"'] & G(g)\alpha_Y M \arrow[d, "\alpha_g"] \\
G(c')\alpha_{X'} F(f)L \arrow[r, "\psi u'"] & \alpha_Y F(g)M.
\end{tikzcd}$$

Proof. The square decomposes into

$$\begin{tikzcd}
G(c')G(f)\alpha_X L \arrow[r, "G(p)"] \arrow[d, "\alpha_f"'] & G(g)G(c)\alpha_X L \arrow[d, "\alpha_c"] & \\
G(c')\alpha_{X'}F(f)L \arrow[d, "\alpha_{c'}"'] & G(g)\alpha_Y F(c)L \arrow[r, "u"] \arrow[d, "\alpha_g"] & G(g)\alpha_Y M \arrow[d, "\alpha_g"] \\
\alpha_{Y'}F(c')F(f)L \arrow[r, "F(p)"] & \alpha_{Y'}F(g)F(c)L \arrow[r, "u"] & \alpha_Y F(g)M
\end{tikzcd}$$

where the inner cells commute. $\square$

2 A relative Lefschetz-Verdier formula

We apply the formalism of duals and pairings to the symmetric monoidal 2-category of cohomological correspondences, which we define in Subsection 2.2. We prove relative Künneth formulas in Subsection 2.1 and use them to show the equivalence of dualizability and local acyclicity (Theorem 2.16) in Subsection 2.3. We prove the relative Lefschetz-Verdier theorem for dualizable objects (Theorem 2.21) in Subsection 2.4. Together, the two theorems imply Theorem 0.1. In Subsection 2.5, we prove, as an application of Theorem 2.16, that base change preserves duals of locally acyclic objects (Corollary 2.27).

We will often drop the letters L and R from the notation of derived functors.

2.1 Relative Künneth formulas

We extend some Künneth formulas over fields [SGA5, III 1.6, Proposition 1.7.4, (3.1.1)] to Noetherian base schemes under the assumption of universal local acyclicity. Some special cases over a smooth scheme over a perfect field were previously known [YZ, Corollary 3.3, Proposition 3.5].

Let S be a coherent scheme and let Λ be a torsion commutative ring. Let X be a scheme over S. We let $D(X,\Lambda)$ denote the unbounded derived category of the category of étale sheaves of Λ-modules on X. Recall from [D, Th. finitude, DÃĺfinition 2.12] that $L \in D^+(X,\Lambda)$ is said to be *locally acyclic* over S if the canonical map $L_x \to R\Gamma(X_{(x)t}, L)$ is an isomorphism for every geometric point $x \to X$ and every algebraic geometric point $t \to S_{(x)}$. Here $X_{(x)t} := X_{(x)} \times_{S_{(x)}} t$ denotes the Milnor fiber. For X of finite type over S, local acyclicity coincides with strong local acyclicity [LZ, Lemma 4.7].

Notation 2.1. For $a_X\colon X \to S$ separated of finite type, we write $K_X = a_X^!\Lambda_S$ and $D_X = R\mathcal{H}om(-,K_X)$. Note that $K_S = \Lambda_S$ is in general not a dualizing complex.

Assume in the rest of Subsection 2.1 that S and Λ are Noetherian. We let $D_{\mathrm{ft}}(X,\Lambda)$ denote the full subcategory of $D(X,\Lambda)$ consisting of complexes of finite tor-amplitude.

Proposition 2.2. *Let X', X, Y be schemes of finite type over S and let $f: X \to X'$ be a morphism over S. Let $M \in D_{\mathrm{ft}}(Y, \Lambda)$ universally locally acyclic over S, $L \in D^+(X, \Lambda)$. Then the canonical morphism $f_*L \boxtimes_S M \to (f \times_S \mathrm{id}_Y)_*(L \boxtimes_S M)$ is an isomorphism.*

Proof. By cohomological descent for a Zariski open cover, we may assume f separated. By Nagata compactification, we are reduced to two cases: either f is proper, in which case we apply proper base change, or f is an open immersion, in which case we apply [D, Th. finitude, App., Proposition 2.10] (with $i = \mathrm{id}_{X'}$). $\square$

In the rest of Subsection 2.1, assume that $m\Lambda = 0$ for some integer m invertible on S.

Proposition 2.3. *Let X', X, Y be schemes of finite type over S and let $f: X \to X'$ be a separated morphism over S. Let $M \in D_{\mathrm{ft}}(Y, \Lambda)$ universally locally acyclic over S, $L \in D^+(X', \Lambda)$. Then the canonical morphism $f^!L \boxtimes_S M \to (f \times_S \mathrm{id}_Y)^!(L \boxtimes_S M)$ is an isomorphism.*

The morphism is adjoint to

$$(f \times_S \mathrm{id}_Y)_!(f^!L \boxtimes_S M) \simeq f_!f^!L \boxtimes_S M \xrightarrow{\mathrm{adj}\boxtimes_S \mathrm{id}_M} L \boxtimes_S M,$$

where $\mathrm{adj}\colon f_!f^!L \to L$ denotes the adjunction.

Proof. We may assume that f is smooth or a closed immersion. For f smooth of dimension d, $f^*(d)[2d] \simeq f^!$ and the assertion is clear. Assume that f is a closed immersion, and let j be the complementary open immersion. Let $f_Y = f \times_S \mathrm{id}_Y$ and $j_Y = j \times_S \mathrm{id}_Y$. Then we have a morphism of distinguished triangles

$$\begin{tikzcd}
f^!L \boxtimes_S M \arrow[r] \arrow[d, "\alpha"'] & f^*L \boxtimes_S M \arrow[r] \arrow[d, "\simeq"] & f^*j_*j^*L \boxtimes_S M \arrow[r] \arrow[d, "\beta"] & {} \\
f_Y^!(L \boxtimes_S M) \arrow[r] & f_Y^*(L \boxtimes_S M) \arrow[r] & f_Y^*j_{Y*}j_Y^*(L \boxtimes_S M) \arrow[r] & {} ,
\end{tikzcd}$$

where β is an isomorphism by Proposition 2.2. It follows that α is an isomorphism. $\square$

The following is a variant of [S, Corollary 8.10] and [LZ, Theorem 6.8]. Here we do not require smoothness or regularity.

Corollary 2.4. *Let X and Y be schemes of finite type over S, with X separated over S. Let $M \in D_{\mathrm{ft}}(Y, \Lambda)$ universally locally acyclic over S. Then the canonical morphism $K_X \boxtimes_S M \to p_Y^!M$ is an isomorphism, where $p_Y: X \times_S Y \to Y$ is the projection.*

Proof. This is Proposition 2.3 applied to $X' = S$ and $L = \Lambda_S$. $\square$

Proposition 2.5. *Let X and Y be schemes of finite type over S, with X separated over S. Let $M \in D_{\mathrm{ft}}(Y, \Lambda)$ universally locally acyclic over S, $L \in D_c^-(X, \Lambda)$. Then the canonical morphism $D_X L \boxtimes_S M \to R\mathcal{H}om(p_X^*L, p_Y^!M)$ is an isomorphism. Here $p_X: X \times_S Y \to X$ and $p_Y: X \times_S Y \to Y$ are the projections.*

The morphism is adjoint to $(D_X L \otimes L) \boxtimes_S M \to K_X \boxtimes_S M \to p_Y^! M$.

Proof. By [SGA4, IX Proposition 2.7], we may assume $L = j_!\Lambda$ for $j\colon U \to X$ étale with U affine. Then the morphism can be identified with

$$j_* D_U \Lambda_U \boxtimes_S M \to j_{Y*}(D_U \Lambda_U \boxtimes M) \to j_{Y*} R\mathcal{H}om(\Lambda_{U\times_S Y}, j_Y^! p_Y^! M) \simeq R\mathcal{H}om(j_{Y!}\Lambda_{U\times_S Y}, p_Y^! M),$$

where $j_Y = j \times_S \mathrm{id}_Y \colon U \times_S Y \to X \times_S Y$. The first arrow is an isomorphism by Proposition 2.2. The second arrow is an isomorphism by Corollary 2.4. ☐

2.2 The category of cohomological correspondences

Let S be a coherent scheme and let Λ be a torsion commutative ring.

Construction 2.6. We define the 2-category of cohomological correspondences $\mathcal{C} = \mathcal{C}_{S,\Lambda}$ as follows. An object of $\mathcal{C}$ is a pair (X, L), where X is a scheme separated of finite type over S and $L \in D(X, \Lambda)$. A correspondence over S is a pair of morphisms $X \xleftarrow{\overleftarrow{c}} C \xrightarrow{\overrightarrow{c}} Y$ of schemes over S. A morphism $(X, L) \to (Y, M)$ in $\mathcal{C}$ is a cohomological correspondence over S, namely a pair (c, u), where $c = (\overleftarrow{c}, \overrightarrow{c})$ is a correspondence over S and $u\colon \overleftarrow{c}^* L \to \overrightarrow{c}^! M$ is a morphism in $D(C, \Lambda)$. Given cohomological correspondences $(X, L) \xrightarrow{(c,u)} (Y, M) \xrightarrow{(d,v)} (Z, N)$, the composite is (e, w), where e is the composite correspondence given by the diagram

(2.1)
$$\begin{tikzcd}[column sep=small]
 & & C \times_Y D \arrow[dl, "\overleftarrow{d}'"'] \arrow[dr, "\overrightarrow{c}'"] & & \\
 & C \arrow[dl, "\overleftarrow{c}"'] \arrow[dr, "\overrightarrow{c}"] & & D \arrow[dl, "\overleftarrow{d}"'] \arrow[dr, "\overrightarrow{d}"] & \\
X & & Y & & Z,
\end{tikzcd}$$

and w is given by the composite

$$\overleftarrow{d}'^* \overleftarrow{c}^* L \xrightarrow{u} \overleftarrow{d}'^* \overrightarrow{c}^! M \xrightarrow{\alpha} \overrightarrow{c}'^! \overleftarrow{d}^* M \xrightarrow{v} \overrightarrow{c}'^! \overrightarrow{d}^! N,$$

where α is adjoint to the base change isomorphism $\overrightarrow{c}'_! \overleftarrow{d}'^* \simeq \overleftarrow{d}^* \overrightarrow{c}_!$. Given (c, u) and (d, v) from (X, L) to (Y, M), a 2-morphism $(c, u) \to (d, v)$ is a *proper* morphism of schemes $p\colon C \to D$ satisfying $\overleftarrow{d} p = \overleftarrow{c}$ and $\overrightarrow{d} p = \overrightarrow{c}$ and such that v is equal to

$$\overleftarrow{d}^* L \xrightarrow{\mathrm{adj}} p_* p^* \overleftarrow{d}^* L \simeq p_! \overleftarrow{c}^* L \xrightarrow{u} p_! \overrightarrow{c}^! M \simeq p_! p^! \overrightarrow{d}^! M \xrightarrow{\mathrm{adj}} \overrightarrow{d}^! M.$$

Here we used the canonical isomorphism $p_! \simeq p_*$. Composition of 2-morphisms is given by composition of morphisms of schemes.

The 2-category admits a symmetric monoidal structure. We put $(X, L) \otimes (X', L') := (X \times_S X', L \boxtimes_S L')$. Given $(c, u)\colon (X, L) \to (Y, M)$ and $(c', u')\colon (X', L') \to (Y', M')$, we define $(c, u) \otimes (c', u')$ to be (d, v), where $d = (\overleftarrow{c} \times_S \overleftarrow{c'}, \overrightarrow{c} \times_S \overrightarrow{c'})$ and v is the composite

$$\overleftarrow{d}^*(L \boxtimes_S L') \simeq \overleftarrow{c}^* L \boxtimes_S \overleftarrow{c'}^* L' \xrightarrow{u \boxtimes_S u'} \overrightarrow{c}^! M \boxtimes_S \overrightarrow{c'}^! M' \xrightarrow{\alpha} \overrightarrow{d}^!(M \boxtimes_S M'),$$

where α is adjoint to the Künneth formula $\overrightarrow{d}_!(- \boxtimes_S -) \simeq \overrightarrow{c}_! - \boxtimes_S \overrightarrow{c'}_! -$. Tensor product of 2-morphisms is given by product of morphisms of schemes over S. The monoidal unit of $\mathcal{C}$ is (S, Λ_S).

Remark 2.7. Let $\mathcal{B}_S$ be the symmetric monoidal 2-category of correspondences obtained by omitting L from the above construction. The symmetric monoidal structure on $\mathcal{B}_S$ is given by fiber product of schemes over S (which is not the product in $\mathcal{B}_S$ for S nonempty). Consider the functor $F\colon \mathcal{B}_S \to \mathcal{C}at^{\mathrm{co}}$ carrying X to $D(X,\Lambda)$ and $c = (\overleftarrow{c}, \overrightarrow{c})$ to $\overrightarrow{c}_! \overleftarrow{c}^*$, and a 2-morphism $p\colon c \to d$ to the natural transformation $\overrightarrow{d}_! \overleftarrow{d}^* \xrightarrow{\mathrm{adj}} \overrightarrow{d}_! p_* p^* \overleftarrow{d}^* \simeq \overrightarrow{c}_! \overleftarrow{c}^*$. The compatibility of F with composition (2.1) is given by the base change isomorphism $\overleftarrow{d}^* \overrightarrow{c}_! \simeq \overrightarrow{c}'_! \overleftarrow{d}'^*$. The functor F admits a right-lax symmetric monoidal structure given by $e_F = \Lambda_S$ and $\boxtimes$, with Künneth formula for !-pushforward providing a natural isomorphism $F_{c,c'}$ (1.6). The Grothendieck construction (Construction 1.10) then produces $\mathcal{C}_{S,\Lambda}$.

The category $\Omega\mathcal{C}$ consists of pairs (X, α), where X is a scheme separated of finite type over S and $\alpha \in H^0(X, K_X)$. A morphism $(X,\alpha) \to (Y,\beta)$ is a proper morphism $X \to Y$ of schemes over S such that $\beta = p_*\alpha$, where

$$p_*\colon H^0(X, K_X) \to H^0(Y, K_Y) \tag{2.2}$$

is given by adjunction $p_*p^! \simeq p_!p^! \to \mathrm{id}$.

Lemma 2.8. *The symmetric monoidal structure $\otimes$ on $\mathcal{C}$ is closed, with internal mapping object $\mathcal{H}om((X,L),(Y,M)) = (X \times_S Y, R\mathcal{H}om(p_X^*L, p_Y^! M))$.*

Proof. We construct an isomorphism of categories

$$F\colon \mathrm{Hom}((X,L)\otimes(Y,M),(Z,N)) \simeq \mathrm{Hom}((X,L),\mathcal{H}om((Y,M),(Z,N)))$$

as follows. An object of the source (resp. target) is a pair $(C \xrightarrow{c} X \times_S Y \times_S Z, u)$, where u belongs to $H^0(C, c^! -)$ applied to left-hand (resp. right-hand) side of the isomorphism

$$\alpha\colon R\mathcal{H}om(p_X^*L \otimes p_Y^*M, p_Z^!N) \simeq R\mathcal{H}om(p_X^*L, R\mathcal{H}om(p_Y^*M, p_Z^!N)).$$

Here p_X, p_Y, p_Z denote the projections from $X \times_S Y \times_S Z$. We define F by $F(c,u) = (c,u')$, where u' is the image of u under the map induced by α, and $F(p) = p$ for every morphism p in the source of F. $\square$

For an object (X,L) of $\mathcal{C}$ and a morphism $f\colon X \to X'$ of schemes separated of finite type over S, we let $f_\natural = (\mathrm{id}_X, f)_\natural = ((\mathrm{id}_X, f), L \xrightarrow{\mathrm{adj}} f^! f_! L)\colon (X,L) \to (X', f_!L)$.

Lemma 2.9. *Let (X,L) be an object of $\mathcal{C}$ and let $f\colon X \to X'$ be a proper morphism of schemes separated of finite type over S. Then $f_\natural\colon (X,L) \to (X', f_*L)$ admits a right adjoint $f^\natural\colon ((f,\mathrm{id}_X), f^*f_*L \xrightarrow{\mathrm{adj}} L)\colon (X', f_*L) \to (X,L)$.*

Proof. The counit $f_\natural f^\natural \to \mathrm{id}_{(X', f_*L)}$ is given by f and the unit $\mathrm{id}_{(X,L)} \to f^\natural f_\natural$ is given by the diagonal $X \to X \times_{X'} X$. $\square$

Construction 2.10 (!-pushforward)**.** Consider a commutative diagram of schemes separated of finite type over S

(2.3)
$$\begin{tikzcd} X \arrow[d, "f"'] & C \arrow[l, "\overleftarrow{c}"'] \arrow[r, "\overrightarrow{c}"] \arrow[d, "p"] & Y \arrow[d, "g"] \\ X' & C' \arrow[l, "\overleftarrow{c'}"'] \arrow[r, "\overrightarrow{c'}"] & Y' \end{tikzcd}$$

such that $q\colon C \to X \times_{X'} C'$ is proper. Let $(c, u)\colon (X, L) \to (Y, M)$ be a cohomological correspondence above c. Let $p^\sharp = (f, p, g)$. By Lemma 1.11, we have a unique cohomological correspondence $(c', p^\sharp_! u)\colon (X', f_! L') \to (Y', g_! M')$ above c' such that q defines a 2-morphism in $\mathcal{C}$:

$$\begin{tikzcd} (X, L) \arrow[r, "{(c,u)}"] \arrow[d, "f_\natural"'] \arrow[dr, phantom, "\not\Downarrow q"] & (Y, M) \arrow[d, "g_\natural"] \\ (X', f_! L) \arrow[r, "{(c', p^\sharp_! u)}"] & (Y', g_! M). \end{tikzcd}$$

For a more explicit construction of $p^\sharp_! u$, see [Z, Construction 7.16]. We will often be interested in the case where f, g, and p are proper. In this case we write $p^\sharp_* u$ for $p^\sharp_! u$.

This construction is compatible with horizontal and vertical compositions.

2.3 Dualizable objects

Let S and Λ be as in Subsection 2.2. Next we study dualizable objects of $\mathcal{C} = \mathcal{C}_{S,\Lambda}$.

Proposition 2.11. *Let (X, L) be a dualizable object of $\mathcal{C}$.*

(a) The dual of (X, L) is $(X, D_X L)$ and the biduality morphism $L \to D_X D_X L$ is an isomorphism. Moreover, for any object (Y, M) of $\mathcal{C}$, the canonical morphisms

(2.4)
$$\begin{aligned} D_X L \boxtimes_S M &\to R\mathcal{H}om(p_X^* L, p_Y^! M), \\ L \boxtimes_S D_Y M &\to R\mathcal{H}om(p_Y^* M, p_X^! L), \\ D_X L \boxtimes_S D_Y M &\to D_{X\times_S Y}(L \boxtimes_S M) \end{aligned}$$

are isomorphisms.

(b) For every morphism of schemes $g\colon Y \to Y'$ separated of finite type over S and all $M \in D(Y, \Lambda)$, $M' \in D(Y', \Lambda)$ the canonical morphisms

$$\begin{aligned} L \boxtimes_S g_* M &\to (\mathrm{id}_X \times_S g)_*(L \boxtimes_S M), \\ L \boxtimes_S g^! M' &\to (\mathrm{id}_X \times_S g)^!(L \boxtimes_S M') \end{aligned}$$

are isomorphisms.

(c) If $L \in D^+(X, \Lambda)$, then L is locally acyclic over S.

(d) If S is Noetherian finite-dimensional, $m\Lambda = 0$ with m invertible on S, and for every scheme Y of finite type over X, (Y, Λ) has finite cohomological dimension, then L is c-perfect.

Recall that for any Noetherian scheme X, $L \in D(X,\Lambda)$ is said to be *c-perfect* [ILO, XVII Définition 7.7.1] if for there exists a finite partition (X_i) of X by subschemes such that for each i, $L|_{X_i} \in D(X_i,\Lambda)$ is a perfect complex in the sense of [SGA6, I Exemple 4.8], namely locally isomorphic to a perfect complex of constant Λ-modules. Here we used the fact that locally constant Λ-modules with finitely presented stalks on Noetherian schemes are stable under direct summands [SGA4, IX Proposition 2.13 (i)]. For Λ Noetherian, "c-perfect" is equivalent to "$\in D_{\mathrm{cft}}$".

Proof. (a) follows from Remarks 1.2, 1.3 and the identification of internal mapping objects (Lemma 2.8). Via biduality and (2.4), the morphisms in (b) can be identified with the isomorphisms

$$R\mathcal{H}om(p'^*_X L^\vee, p^!_{Y'} g_* M) \simeq R\mathcal{H}om(p'^*_X L^\vee, f_* p^!_Y M) \simeq f_* R\mathcal{H}om(p^*_X L^\vee, p^!_Y M),$$
$$R\mathcal{H}om(p^*_X L^\vee, p^!_Y g^! M') \simeq R\mathcal{H}om(p^*_X L^\vee, f^! p^!_{Y'} M') \simeq f^! R\mathcal{H}om(p'^*_X L^\vee, p^!_{Y'} M'),$$

where $f = \mathrm{id}_X \times_S g$, $p'_X \colon X \times Y' \to X$ is the projection. (c) follows from (b) and Lemma 2.12 below. For (d), note that for $M \in D(X,\Lambda)$, $\mathrm{Hom}(\Lambda_X, \Delta^!(D_X L \boxtimes_S M)) \simeq \mathrm{Hom}(L, M)$ by (2.4), where $\Delta \colon X \to X \times_S X$ is the diagonal. Since Λ_X is a compact object of $D(X,\Lambda)$ and $\Delta^!$ commutes with small direct sums by Lemma 2.13 below, it follows that L is a compact object. We conclude the proof of (d) by Lemma 2.14 below. □

The following is a variant of [S, Proposition 8.11].

Lemma 2.12. *Let $X \to S$ be a morphism of coherent schemes and let $L \in D^+(X,\Lambda)$. Assume that for every quasi-finite morphism $g \colon Y \to Y'$ of affine schemes with Y' étale over S, the canonical morphism $L \boxtimes_S g_*\Lambda_Y \to (\mathrm{id}_X \times_S g)_*(L \boxtimes_S \Lambda_Y)$ is an isomorphism. Then L is locally acyclic over S.*

Proof. Let $s \to S$ be a geometric point and let $g \colon t \to S_{(s)}$ be an algebraic geometric point. Consider the diagram

$$\begin{tikzcd} X_t \arrow[r, "g_X"] \arrow[d] & X_{(s)} \arrow[d] & X_s \arrow[l, "i_X"'] \arrow[d] \\ t \arrow[r, "g"] & S_{(s)} & s \arrow[l, "i"'] \end{tikzcd}$$

obtained by base change. By the assumption and passing to the limit, the morphism $L|_{X_s} \to i_X^* g_{X*}(L|_{X_t})$ can be identified with $L \boxtimes_S -$ applied to $\Lambda_s \to i^* g_* \Lambda_t$, which is an isomorphism. □

Lemma 2.13. *Let $f \colon X \to Y$ be a separated morphism of finite type between finite-dimensional Noetherian schemes. Assume that $m\Lambda = 0$ with m invertible on Y. Then $Rf^!$ commutes with small direct sums.*

Proof. We may assume that f is a closed immersion. Let j be the complementary open immersion. Since Rj_* has finite cohomological dimension [ILO, XVIIIA Corollary 1.4], Rj_* commutes with small direct sums by [LZ, Lemma 1.10]. □

The following is an analogue of [GL, Propositions 2.2.4.5, 2.2.6.2].

Lemma 2.14. *Let X be a coherent scheme such that for every scheme U étale of finite presentation over X, (U,Λ) has finite cohomological dimension. Then $D(X,\Lambda)$ is compactly generated. If, moreover, X is Noetherian and for every scheme Y of finite type over X, (Y,Λ) has finite cohomological dimension, then the compact objects of $D(X,\Lambda)$ are precisely c-perfect complexes.*

Proof. For every étale morphism of finite presentation $j\colon U\to X$ and every integer n, $j_!\Lambda_U[n]$ is a compact object. These objects form an (essentially small) family of generators. Indeed, if $H^{-n}(U,L)\simeq \mathrm{Hom}(j_!\Lambda_U[n],L)=0$ for all j and $n\ge 0$, then $L_x=0$ for every geometric point $x\to X$.

To prove the last assertion, note that every c-perfect complex is a successive extension of objects of the form $v_!\mathcal{L}$, where $v\colon V\to X$ is an immersion and $\mathcal{L}\in D(V,\Lambda)$ is a perfect complex. It follows that c-perfect complexes are compact. Moreover, they form a thick subcategory of $D(X,\Lambda)$ containing $j_!\Lambda_U[n]$ for all j and $n\ge 0$. We conclude by Neeman's version of the Thomason localization theorem [N, Statements 2.1.2 and 2.1.3] (applied to $\mathcal{S}=D(X,\Lambda)$ and R the collection of c-perfect complexes). $\square$

Remark 2.15. The evaluation and coevaluation maps for a dualizable object (X,L) of $\mathcal{C}$ can be given explicitly as follows. The evaluation map $(X\times_S X, D_XL\boxtimes_S L)\to (S,\Lambda)$ is given by $X\times_S X\xleftarrow{\Delta} X\to S$ and the usual evaluation map $\Delta^*(D_XL\boxtimes_S L)\simeq D_XL\otimes L\to K_X$, where Δ denotes the diagonal. The coevaluation map $(S,\Lambda)\to(X\times_S X, L\boxtimes_S D_XL)$ is given by $S\leftarrow X\xrightarrow{\Delta} X\times_S X$ and id_L considered as a morphism

$$\Lambda_X\to R\mathcal{H}om(L,L)\simeq \Delta^! R\mathcal{H}om(p_2^*L,p_1^!L)\simeq \Delta^!(L\boxtimes_S D_XL).$$

We can identify dualizable objects of $\mathcal{C}$ under mild assumptions.

Theorem 2.16. *Let S be a Noetherian scheme, Λ a Noetherian commutative ring with $m\Lambda=0$ for m invertible on S. Let X be a scheme separated of finite type over S, $L\in D_{\mathrm{cft}}(X,\Lambda)$. Then (X,L) is a dualizable object of $\mathcal{C}$ if and only if L is locally acyclic over S. In this case the dual of (X,L) is (X,D_XL).*

We will use Gabber's theorem that for X of finite type over S, $L\in D^b_c(X,\Lambda)$ is locally acyclic if and only if it is universally locally acyclic [LZ, Corollary 6.6].

Proof. We have already seen the last assertion and the "only if" part of the first assertion in Parts (a) and (c) of Proposition 2.11. Now assume L locally acyclic over S. By Lemmas 1.4 and 2.8, it suffices to show that the canonical morphism $L\boxtimes_S D_XL\to R\mathcal{H}om(p_2^*L,p_1^!L)$ is an isomorphism, which is Proposition 2.5. $\square$

Remark 2.17. Without invoking Gabber's theorem, our proof and Proposition 2.26 show that for $L\in D_{\mathrm{cft}}(X,L)$, (X,L) is dualizable if and only if L is universally locally acyclic over S.

Corollary 2.18. *For S, Λ, and X as in Theorem 2.16 and $L\in D_{\mathrm{cft}}(X,\Lambda)$ locally acyclic over S, D_XL is locally acyclic over S.*

This was known under the additional assumption that S is regular (and excellent) [LZ, Corollary 5.13] (see also [BG, B.6 2)] for S smooth over a field). Our proof here is different from the one in [LZ]. In fact, without invoking Gabber's theorem, our proof here shows that D_X preserves universal local acyclicity and makes no use of oriented topoi.

Proof. This follows immediately from Theorem 2.16 and Remark 1.2. $\square$

Corollary 2.19. *Let S be an Artinian scheme, Λ and X as in Theorem 2.16, and $L \in D(X, \Lambda)$. Then (X, L) is a dualizable object of $\mathcal{C}$ if and only if $L \in D_{\mathrm{cft}}(X, \Lambda)$.*

Proof. For $L \in D_{\mathrm{cft}}(X, \Lambda)$, L is locally acyclic over S by [D, Th. finitude, Corollaire 2.16] and thus (X, L) is dualizable by the theorem. (Alternatively one can apply Lemmas 1.4 and 2.8 and [SGA5, III (3.1.1)].) For the converse, we may assume that S is the spectrum of a separably closed field by Proposition 2.26. In this case, Proposition 2.11 (d) applies. $\square$

2.4 The relative Lefschetz-Verdier pairing

Let S be a coherent scheme and Λ a torsion commutative ring.

Notation 2.20. For objects (X, L) and (Y, M) of $\mathcal{C}$ with (X, L) dualizable and morphisms $(c, u)\colon (X, L) \to (Y, M)$ and $(d, v)\colon (Y, M) \to (X, L)$, we write the pairing $\langle (c, u), (d, v)\rangle \in \Omega\mathcal{C}$ in Construction 1.6 as $(F, \langle u, v\rangle)$, where $F = C \times_{X \times_S Y} D$. We call $\langle u, v\rangle \in H^0(F, K_F)$ the relative Lefschetz-Verdier pairing. The pairing is symmetric: $\langle u, v\rangle$ can be identified with $\langle v, u\rangle$ via the canonical isomorphism $\langle c, d\rangle \simeq \langle d, c\rangle$.

For an endomorphism (e, w) of a dualizable object (X, L) of $\mathcal{C}$, we write $\mathrm{tr}(e, w) = (X^e, \mathrm{tr}(w))$, where $X^e = E \times_{e, X \times_S X, \Delta} X$ and $\mathrm{tr}(w) = \langle w, \mathrm{id}_L\rangle \in H^0(X^e, K_{X^e})$. We define the *characteristic class* $\mathrm{cc}_{X/S}(L)$ to be $\mathrm{tr}(\mathrm{id}_L) = \langle \mathrm{id}_L, \mathrm{id}_L\rangle \in H^0(X, K_X)$. In other words, $\dim(X, L) = (X, \mathrm{cc}_{X/S}(L))$.

Theorem 2.21 (Relative Lefschetz-Verdier)**.** *Let*

(2.5)
$$\begin{tikzcd}
X \arrow[d, "f"'] & C \arrow[l, "\overleftarrow{c}"'] \arrow[r, "\overrightarrow{c}"] \arrow[d, "p"] & Y \arrow[d, "g"] & D \arrow[l, "\overleftarrow{d}"'] \arrow[r, "\overrightarrow{d}"] \arrow[d, "q"] & X \arrow[d, "f"] \\
X' & C' \arrow[l, "\overleftarrow{c'}"'] \arrow[r, "\overrightarrow{c'}"] & Y' & D' \arrow[l, "\overleftarrow{d'}"'] \arrow[r, "\overrightarrow{d'}"] & X'
\end{tikzcd}$$

be a commutative diagram of schemes separated of finite type over S, with p and $D \to D' \times_{Y'} Y$ proper. Let $L \in D(X, \Lambda)$ such that (X, L) and $(X', f_! L)$ are dualizable objects of $\mathcal{C}$. Let $M \in D(Y, \Lambda)$, $u\colon \overleftarrow{c}^* L \to \overrightarrow{c}^! M$, $v\colon \overleftarrow{d}^* M \to \overrightarrow{d}^! L$. Then $s\colon C \times_{X \times_S Y} D \to C' \times_{X' \times_S Y'} D'$ is proper and

$$s_* \langle u, v\rangle = \langle p_!^\sharp u, q_!^\sharp v\rangle.$$

Combining this with Theorem 2.16, we obtain Theorem 0.1.

Proof. By Construction 2.10 applied to the right half of (2.5) and to the decomposition (which was used in the proof of [Z, Proposition 8.11])

$$\begin{tikzcd}
X \arrow[d, "f"'] & C \arrow[l, "\overleftarrow{c}"'] \arrow[r, "\overrightarrow{c}"] \arrow[d, equal] & Y \arrow[d, equal] \\
X' \arrow[d, equal] & C \arrow[l, "f\overleftarrow{c}"'] \arrow[r, "\overrightarrow{c}"] \arrow[d, "p"] & Y \arrow[d, "g"] \\
X' & C' \arrow[l, "\overleftarrow{c'}"'] \arrow[r, "\overrightarrow{c'}"] & Y'
\end{tikzcd}$$

of the left half of (2.5), we get a diagram in $\mathcal{C}$

$$\begin{tikzcd}
(X, L) \arrow[r, "{(c,u)}"] \arrow[d, "f_\natural"'] & (Y, M) \arrow[r, "{(d,v)}"] \arrow[d, "g_\natural"] \arrow[dr, phantom, "\Swarrow"] & (X, L) \arrow[d, "f_\natural"] \\
(X', f_! L) \arrow[r, "{(c', p_!^\sharp u)}"'] \arrow[ur, "{(e,w)}" description] & (Y', g_! M) \arrow[r, "{(d', q_!^\sharp v)}"'] & (X', f_! L)
\end{tikzcd}$$

where $e = (f\overleftarrow{c}, \overrightarrow{c})$ and $w = (f, \mathrm{id}_C, \mathrm{id}_Y)_! u$. By Construction 1.8, we then get a morphism $(F, \langle u, v\rangle) \to (F', \langle p_!^\sharp u, q_!^\sharp v\rangle)$ in $\Omega\mathcal{C}$ given by $s\colon F \to F'$. □

In the case where f is proper, the dualizability of $(X', f_* L)$ follows from that of (X, L) by Proposition 2.23 below. Moreover, in this case, by Lemma 2.9, $f_\natural$ is right adjointable and it suffices in the above proof to apply the more direct Construction 1.7 in place of Construction 1.8.

Corollary 2.22. *Let $f\colon X \to X'$ be a proper morphism of schemes separated of finite type over S and let $L \in D(X, \Lambda)$ such that (X, L) is a dualizable object of $\mathcal{C}$. Then $f_* \mathrm{cc}_{X/S}(L) = \mathrm{cc}_{X'/S}(f_* L)$.*

Proof. This follows from Theorem 2.21 applied to $c = d = (\mathrm{id}_X, \mathrm{id}_X)$, $c' = d' = (\mathrm{id}_{X'}, \mathrm{id}_{X'})$ and $u = v = \mathrm{id}_L$. □

Proposition 2.23. *Let $f\colon X \to Y$ be a proper morphism of schemes separated of finite type over S. Let (X, L) be a dualizable object of $\mathcal{C}$. Then $(Y, f_* L)$ is dualizable.*

Proof. By Remark 1.3 and Lemmas 1.4 and 2.8, the canonical morphism

$$\alpha\colon D_X L \boxtimes_S M \to R\mathcal{H}om(p_X^* L, p_Z^! M)$$

is an isomorphism for every object (Z, M) of $\mathcal{C}$ and it suffices to show that the canonical morphism

$$\beta\colon D_Y f_* L \boxtimes_S M \to R\mathcal{H}om(q_Y^* f_* L, q_Z^! M)$$

is an isomorphism. Here p_X, p_Z, q_Y, q_Z are the projections as shown in the commutative diagram

$$\begin{tikzcd}
X \arrow[d, "f"'] & X \times_S Z \arrow[l, "p_X"'] \arrow[d, "f \times_S \mathrm{id}_Z"'] \arrow[dr, "p_Z"] & \\
Y & Y \times_S Z \arrow[l, "q_Y"] \arrow[r, "q_Z"] & Z.
\end{tikzcd}$$

Via the isomorphisms $D_Y f_* L \boxtimes_S M \simeq (f \times_S \mathrm{id}_Z)_*(D_X L \boxtimes_S M)$ and

$$R\mathcal{H}om(q_Y^* f_* L, q_Z^! M) \simeq R\mathcal{H}om((f \times_S \mathrm{id}_Z)_* p_X^* L, q_Z^! M) \simeq (f \times_S \mathrm{id}_Z)_* R\mathcal{H}om(p_X^* L, p_Z^! M),$$

β can be identified with $(f \times_S \mathrm{id}_Z)_* \alpha$. □

Remark 2.24. The relative Lefschetz-Verdier formula and the proof given above hold for Artin stacks of finite type over an Artin stack S, with proper morphisms replaced by a suitable class of morphisms equipped with canonical isomorphisms $f_! \simeq f_*$ (such as proper representable morphisms). The characteristic class lives in $H^0(I_{X/S}, K_{I_{X/S}})$, where $I_{X/S} = X \times_{\Delta, X \times_S X, \Delta} X$ is the inertia stack of X over S.

Theorem 2.21 does not cover the twisted Lefschetz-Verdier formula in [XZ, A.2.19]

Remark 2.25 (Scholze). Arguments of this paper also apply in the étale cohomology of diamonds and imply the equivalence between dualizability and universal local acyclicity in this situation. This fact and applications will be discussed in the work of Fargues and Scholze on the geometrization of the Langlands correspondence. Previously Kaletha and Weinstein proved a Lefschetz-Verdier formula for diamonds and v-stacks [KW, Theorem 4.4.1] under an assumption which is equivalent (see [DP, Theorem 1.3]) to dualizability in the category of cohomological correspondences, although the latter was not discussed in [KW].

2.5 Base change and duals

We conclude this section with a result on the preservation of duals by base change.

Let $g\colon S \to T$ be a morphism of coherent schemes and let Λ be a torsion commutative ring. For a scheme X separated of finite type over S, we write $D_{X/S} = R\mathcal{H}om(-, a^! \Lambda)$, where $a\colon X \to S$. For a scheme Y separated of finite type over T, we write $D_{Y/T} = R\mathcal{H}om(-, b^! \Lambda)$, where $b\colon Y \to T$.

Proposition 2.26. *Let (Y, M) be a dualizable object of $\mathcal{C}_{T,\Lambda}$. Then $(Y_S, g_Y^* M)$ is a dualizable object of $\mathcal{C}_{S,\Lambda}$ and the canonical morphism $g_Y^* D_{Y/T} M \to D_{Y_S/S} g_Y^* M$ is an isomorphism. Here $Y_S = Y \times_T S$ and $g_Y\colon Y_S \to Y$ is the projection.*

We prove the proposition by constructing a symmetric monoidal functor $g^*\colon \mathcal{C}_{T,\Lambda} \to \mathcal{C}_{S,\Lambda}$ as follows. We take $g^*(Y, M) = (Y_S, g_Y^* M)$. For $(d, v)\colon (Y, M) \to (Z, N)$, we take $g^*(d, v) = (d_S, v_S)$, here d_S is the base change of d by g and v_S is the composite

$$\overleftarrow{d}_S^* g_Y^* M \simeq g_D^* \overleftarrow{d}^* M \xrightarrow{g_D^* v} g_D^* \overrightarrow{d}^! M \to \overrightarrow{d}_S^! g_Z^* M.$$

For every 2-morphism p of $\mathcal{C}_{T,\Lambda}$, we take $g^*(p) = p \times_T S$. The symmetric monoidal structure on g^* is obvious. Proposition 2.26 then follows from the fact that $g^*\colon \mathcal{C}_{T,\Lambda} \to \mathcal{C}_{S,\Lambda}$ preserves duals (Remark 1.9).

The construction above is a special case of Construction 1.14 (applied to $H\colon \mathcal{B}_T \to \mathcal{B}_S$ given by base change by g).

Corollary 2.27. *Let $g\colon S \to T$ be a morphism of coherent schemes with T Noetherian and let Λ be a Noetherian commutative ring with $m\Lambda = 0$ for m invertible on T. Then for any scheme Y separated of finite type over T and any $M \in D_{\mathrm{cft}}(Y, \Lambda)$ locally acyclic over T, the canonical morphism $g_Y^* D_{Y/T} M \to D_{Y_S/S} g_Y^* M$ is an isomorphism. Here $Y_S = Y \times_T S$ and $g_Y\colon Y_S \to Y$ is the projection.*

Note that the statement does not involve cohomological correspondences.

Proof. This follows from Proposition 2.26 and Theorem 2.16. □

3 Nearby cycles over Henselian valuation rings

Let R be a Henselian valuation ring and let $S = \mathrm{Spec}(R)$. We do not assume that the valuation is discrete. In other words, we do not assume S Noetherian. Let η be the generic point and let s be the closed point. Let X be a scheme of finite type over S. Let $X_\eta = X \times_S \eta$, $X_s = X \times_S s$. We consider the morphisms of topoi

$$X_\eta \xrightarrow{\overleftarrow{\Psi}_X} X \overleftarrow{\times}_S \eta \xleftarrow{i_X} X_s \overleftarrow{\times}_S \eta \simeq X_s \bar{\times}_s \eta,$$

where $\overleftarrow{\times}$ denotes the oriented product of topoi [ILO, XI] and $\bar{\times}$ denotes the fiber product of topoi. Let Λ be a commutative ring such that $m\Lambda = 0$ for some m invertible on S. We will study the composite functor

$$\Psi_X\colon D(X_\eta, \Lambda) \xrightarrow{\overleftarrow{\Psi}_X} D(X \overleftarrow{\times}_S \eta, \Lambda) \xrightarrow{i_X^*} D(X_s \bar{\times}_s \eta, \Lambda).$$

Let $\bar{s}$ be an algebraic geometric point above s and let $\bar{\eta} \to S_{(\bar{s})}$ be an algebraic geometric point above η. The restriction of $\Psi_X L$ to $X_{\bar{s}} \simeq X_{\bar{s}} \bar{\times}_{\bar{s}} \bar{\eta}$ can be identified with $(j_* L)|_{X_{\bar{s}}}$, where $j\colon X_{\bar{\eta}} \to X_{(\bar{s})}$, and was studied by Huber [H, Section 4.2]. We do not need Huber's results in this paper.

In Subsection 3.1, we study the symmetric monoidal functor given by Ψ and cohomological correspondences. We deduce that Ψ commutes with duals (Corollary 3.8), generalizing a theorem of Gabber. We also obtain a new proof of the theorems of Deligne and Huber that Ψ preserves constructibility (Corollary 3.9). In Subsection 3.2, extending results of Vidal, we use the compatibility of specialization with proper pushforward to deduce a fixed point result.

3.1 Künneth formulas and duals

Proposition 3.1 (Künneth formulas)**.** *Let X and Y be schemes of finite type over S and let $L \in D(X_\eta, \Lambda)$, $M \in D(Y_\eta, \Lambda)$, then the canonical morphisms*

$$\overleftarrow{\Psi}_X L \boxtimes \overleftarrow{\Psi}_Y M \to \overleftarrow{\Psi}_{X \times_S Y}(L \boxtimes M), \quad \Psi_X L \boxtimes \Psi_Y M \to \Psi_{X \times_S Y}(L \boxtimes M),$$

are isomorphisms.

The Künneth formula for Ψ over a *discrete* Henselian valuation ring was a theorem of Gabber ([I1, Théorème 4.7], [BB, Lemma 5.1.1]).

Proof. It suffices to show that the first morphism is an isomorphism. By passing to the limit and the finiteness of cohomological dimensions, it suffices to show that $\Psi_{X,U/S}\colon X_U \to X \overset{\leftarrow}{\times}_S U$ satisfies Künneth formula for each open subscheme $U \subseteq S$. We then reduce to the case $U = S$, where the Künneth formula is [I2, Theorem A.3]. The Ψ-goodness is satisfied by Orgogozo's theorem ([O, Théorème 2.1], [LZ, Example 4.26 (2)]). $\square$

Construction 3.2. Let $f\colon X \to Y$ be a separated morphism of schemes of finite type over S. Then we have canonical natural transformations

$$f_s^*\Psi_Y \to \Psi_X f_\eta^*, \tag{3.1}$$

$$\Psi_Y f_{\eta *} \to f_{s*}\Psi_X, \tag{3.2}$$

$$f_{s!}\Psi_X \to \Psi_Y f_{\eta !}, \tag{3.3}$$

$$\Psi_Y f_\eta^! \to f_s^!\Psi_Y. \tag{3.4}$$

Here we denoted $f_s \bar{\times}_s \eta$ by f_s. (3.1) is the base change

$$f_s^* i_Y^* \overleftarrow{\Psi}_Y \simeq i_X^*(f \overset{\leftarrow}{\times}_S \mathrm{id}_\eta)^* \overleftarrow{\Psi}_Y \to i_X^* \overleftarrow{\Psi}_X f_\eta^*$$

and (3.4) is defined similarly to [LZ, (4.9)] as

$$i_X^* \overleftarrow{\Psi}_X f_\eta^! \simeq i_X^*(f \overset{\leftarrow}{\times}_S \mathrm{id}_\eta)^! \overleftarrow{\Psi}_Y \to f_\eta^! i_Y^* \overleftarrow{\Psi}_Y.$$

(3.1) and (3.2) correspond to each other by adjunction. The same holds for (3.3) and (3.4). For f proper, (3.2) and (3.3) are inverse to each other.

Construction 3.3. We construct symmetric monoidal 2-categories $\mathcal{C}_1$ and $\mathcal{C}_2$ and a symmetric monoidal functor $\psi\colon \mathcal{C}_1 \to \mathcal{C}_2$ as follows.

The construction of $\mathcal{C}_1$ is identical to that of $\mathcal{C}_{S,\Lambda}$ (Construction 2.6) except that we replace the derived category $D(-,\Lambda)$ by $D((-)_\eta,\Lambda)$. Thus an object of $\mathcal{C}_1$ is a pair (X, L), where X is a scheme separated of finite type over S and $L \in D(X_\eta,\Lambda)$. A morphism $(X, L) \to (Y, M)$ is a pair (c, u), where $c\colon X \to Y$ is a correspondence and (c_η, u) is a cohomological correspondence over η. A 2-morphism $(c, u) \to (d, v)$ is a 2-morphism $p\colon c \to d$ such that p_η is a 2-morphism $(c_\eta, u) \to (d_\eta, v)$. We have $(X, L) \boxtimes (Y, M) = (X \times_S Y, L \boxtimes_\eta M)$. The monoidal unit is (S, Λ_η).

The construction of $\mathcal{C}_2$ is identical to that of $\mathcal{C}_{s,\Lambda}$ except that we replace the derived category $D(-,\Lambda)$ by $D((-) \bar{\times}_s \eta, \Lambda)$. Thus an object of $\mathcal{C}_2$ is a pair (X, L), where X is a scheme separated of finite type over s and $L \in D(X \bar{\times}_s \eta, \Lambda)$. The monoidal unit is (s, Λ_η).

We define ψ by $\psi(X, L) = (X_s, \Psi_X L)$, $\psi(c, u) = (c_s, \psi u)$, where ψu is specialization of u defined as the composite

$$\overleftarrow{c}_s^*\Psi_X L \xrightarrow{(3.1)} \Psi_C \overleftarrow{c}_\eta^* L \xrightarrow{\Psi_C(u)} \Psi_C \overrightarrow{c}_\eta^! M \xrightarrow{(3.4)} \overrightarrow{c}_s^! \Psi_Y M.$$

For every 2-morphism p, $\psi p = p_s$. The symmetric monoidal structure is given by the Künneth formula (Proposition 3.1) and the canonical isomorphism $\Psi_S \Lambda_S \simeq \Lambda_\eta$.

Remark 3.4. The symmetric monoidal 2-category $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) is obtained via the Grothendieck construction (Construction 1.10) from the right-lax symmetric monoidal functor $\mathcal{B}_S \to \mathcal{C}at^{\mathrm{co}}$ (resp. $\mathcal{B}_s \to \mathcal{C}at^{\mathrm{co}}$) carrying X to $D(X_\eta, \Lambda)$ (resp. $D(X \bar{\times}_s \eta, \Lambda)$). The symmetric monoidal functor ψ is a special case of Construction 1.14 (with $H\colon \mathcal{B}_S \to \mathcal{B}_s$ given by taking special fiber). More explicitly, if $\mathcal{C}_2'$ denotes the symmetric monoidal 2-category obtained from the right-lax symmetric monoidal functor $\mathcal{B}_S \to \mathcal{C}at^{\mathrm{co}}$ carrying X to $D(X_s \bar{\times}_s \eta, \Lambda)$, then ψ decomposes into $\mathcal{C}_1 \xrightarrow{\psi_1} \mathcal{C}_2' \xrightarrow{\psi_2} \mathcal{C}_2$, where ψ_1 carries (X, L) to $(X, \Psi_X L)$ and ψ_2 carries (X, L) to (X_s, L).

The proof the following lemma is identical to that of Lemma 2.8.

Lemma 3.5. *The symmetric monoidal structures $\otimes$ on $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) is closed, with mapping object $\mathcal{H}om((X,L),(Y,M)) = (X \times_S Y, R\mathcal{H}om(p_{X_\eta}^* L, p_{Y_\eta}^! M))$ (resp. $\mathcal{H}om((X,L),(Y,M)) = (X \times_s Y, R\mathcal{H}om(p_X^* L, p_Y^! M))$).*

Remark 3.6. It follows from Remark 1.3 and Lemma 3.5 that the dual of a dualizable object (X, L) in $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) is $(X, D_{X_\eta} L)$ (resp. $(X, D_{X \bar{\times}_s \eta} L)$). Here for $a\colon U \to \eta$ and $b\colon V \to s$ separated of finite type, we write $K_U = a^! \Lambda_\eta$, $D_U = R\mathcal{H}om(-, K_U)$ and $K_{V \bar{\times}_s \eta} = b^! \Lambda_\eta$, $D_{V \bar{\times}_s \eta} = R\mathcal{H}om(-, K_{V \bar{\times}_s \eta})$.

In the rest of Subsection 3.1, we assume that Λ is Noetherian.

Proposition 3.7. *An object (X, L) in $\mathcal{C}_1$ or $\mathcal{C}_2$ is dualizable if and only if $L \in D_{\mathrm{cft}}$.*

Proof. By Lemma 1.4 and the identification of internal mapping objects (Lemmas 2.8 and 3.5), (X, L) in $\mathcal{C}_1$ is dualizable if and only if (X_η, L) in $\mathcal{C}_\eta$ is dualizable. The latter condition is equivalent to $L \in D_{\mathrm{cft}}$ by Corollary 2.19.

Similarly, (X, L) in $\mathcal{C}_2$ is dualizable if and only if $(X_{\bar{s}}, L|_{X_{\bar{s}}})$ in $\mathcal{C}_{\bar{s}}$ is dualizable, by [LZ, Lemma 1.29]. The latter condition is equivalent to $L|_{X_{\bar{s}}} \in D_{\mathrm{cft}}$, which is in turn equivalent to $L \in D_{\mathrm{cft}}$. $\square$

Corollary 3.8. *Let X be a scheme separated of finite type over S and let $L \in D_c^-(X_\eta, \Lambda)$. The canonical morphism $\Psi_X D_{X_\eta} L \to D_{X_s \bar{\times}_s \eta} \Psi_X L$ is an isomorphism in $D(X_s \bar{\times}_s \eta, \Lambda)$.*

This generalizes a theorem of Gabber for Henselian *discrete* valuation rings [I1, Théorème 4.2]. Our proof here is different from that of Gabber. One can also deduce Corollary 3.8 from the commutation of duality with sliced nearby cycles over general bases [LZ, Theorem 0.1].

Proof. The cohomological dimension of Ψ_X is bounded by $\dim(X_\eta)$. Thus we may assume that L is of the form $u_! \Lambda_U$, where $u\colon U \to X_\eta$ is an étale morphism of finite type. In particular, we may assume $L \in D_{\mathrm{cft}}(X_\eta, \Lambda)$. In this case, (X, L) is dualizable by Proposition 3.7. We conclude by the fact that ψ preserve duals (Remark 1.9) and the identification of duals (Remark 3.6). $\square$

We also deduce a new proof of the following finiteness theorem of Deligne (for Henselian *discrete* valuation rings) [D, Th. finitude, Théorème 3.2] and Huber [H, Proposition 4.2.5]. Our proof relies on Deligne's theorem on local acyclicity over a field [D, Corollaire 2.16].

Corollary 3.9. *Let X be a scheme of finite type over S. Then Ψ_X preserves D^b_c and D_{cft}.*

Proof. We may assume that X is separated. As in the proof of Corollary 3.8, we are reduced to the case of D_{cft}. This case follows from Proposition 3.7 and the fact that ψ preserves dualizable objects (Remark 1.9). $\square$

By Remark 1.9, ψ also preserves pairings, and we obtain the following generalization of [V1, Proposition 1.3.5].

Corollary 3.10. *Consider morphisms of schemes separated of finite type over S:*

$$\begin{tikzcd} X & C \arrow[l, "\overleftarrow{c}"'] \arrow[r, "\overrightarrow{c}"] & Y & D \arrow[l, "\overleftarrow{d}"'] \arrow[r, "\overrightarrow{d}"] & X. \end{tikzcd}$$

Let $L \in D_{\mathrm{cft}}(X_\eta, \Lambda)$, $M \in D(Y_\eta, \Lambda)$, $u\colon \overleftarrow{c}^*_\eta L \to \overrightarrow{c}^!_\eta M$, $v\colon \overleftarrow{d}^*_\eta M \to \overrightarrow{d}^!_\eta L$. Then $\mathrm{sp}\langle u, v\rangle = \langle \psi u, \psi v\rangle$, *where* sp *is the composition*

$$H^0(F_\eta, K_{F_\eta}) \to H^0(F_s \bar{\times}_s \eta, \Psi_F K_{F_\eta}) \to H^0(F_s \bar{\times}_s \eta, K_{F_s \bar{\times}_s \eta})$$

and $F = C \times_{X \times Y} D$.

3.2 Pushforward and fixed points

Construction 3.11 (!-Pushforward in $\mathcal{C}_2$)**.** Consider a commutative diagram (2.3) in $\mathcal{B}_s$ such that $q\colon C \to X \times_{X'} C'$ is proper. Let $(c, u)\colon (X, L) \to (Y, M)$ be a morphism in $\mathcal{C}_2$ above c. By Lemma 1.11, we have a unique morphism $(c', p^\sharp_! u)\colon (X', f_! L') \to (Y', g_! M')$ in $\mathcal{C}_2$ above c' such that q defines a 2-morphism in $\mathcal{C}_2$:

$$\begin{tikzcd} (X, L) \arrow[r, "{(c,u)}"] \arrow[d, "f_\natural"'] \arrow[dr, phantom, "\Swarrow q"] & (Y, M) \arrow[d, "g_\natural"] \\ (X', f_! L) \arrow[r, "{(c', p^\sharp_! u)}"'] & (Y', g_! M). \end{tikzcd}$$

For f, g, p proper, we write $p^\sharp_* u$ for $p^\sharp_! u$.

Applying Lemma 1.15 to the functor ψ_1, we obtain the following.

Proposition 3.12. *Consider a commutative diagram of schemes separated of finite type over S*

$$\begin{tikzcd} X \arrow[d, "f"'] & C \arrow[l, "\overleftarrow{c}"'] \arrow[r, "\overrightarrow{c}"] \arrow[d, "p"] & Y \arrow[d, "g"] \\ X' & C' \arrow[l, "\overleftarrow{c}'"] \arrow[r, "\overrightarrow{c}'"'] & Y' \end{tikzcd}$$

such that $C \to X \times_{X'} C'$ is proper. Let $L \in D(X_\eta, \Lambda)$, $M \in D(Y_\eta, \Lambda)$, $u\colon \overleftarrow{c}^*_\eta L \to \overrightarrow{c}^!_\eta M$. Then the square

$$\begin{tikzcd} \overleftarrow{c}'^*_s f_{s!} \Psi_X L \arrow[r, "p^\sharp_{s!}\psi u"] \arrow[d] & \overrightarrow{c}'^!_s g_{s!} \Psi_Y M \arrow[d] \\ \overleftarrow{c}'^*_s \Psi_{X'} f_{\eta!} L \arrow[r, "\psi p^\sharp_{\eta!} u"'] & \overrightarrow{c}'^!_s \Psi_{Y'} g_{\eta!} M \end{tikzcd}$$

commutes. Here the vertical arrows are given by (3.3)*. In particular, in the case where f, g, p are proper, $p^\sharp_{s*}\psi u$ can be identified with $\psi p^\sharp_{\eta*}u$ via the isomorphisms $f_{s*}\Psi_X \simeq \Psi_{X'} f_{\eta*}$ and $g_{s*}\Psi_Y \simeq \Psi_{Y'} g_{\eta*}$.*

This generalizes a result of Vidal [V2, Théorème 7.5.1] for certain Henselian valuation rings of rank 1. As in [V2, Sections 7.5, 7.6], Proposition 3.12 implies the following fixed point result, generalizing [V2, Proposition 5.1, Corollaire 7.5.3].

Corollary 3.13. *Assume that η is separably closed. Consider a commutative diagram of schemes*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & S \arrow[d, "\sigma"', "\simeq"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

with f proper and σ fixing s. Assume that g_s does not fix any point of X_s. Then $\mathrm{tr}(g, R\Gamma(X_\eta, \Lambda)) = 0$*. If, moreover, g is an isomorphism and $U \subseteq X_\eta$ is an open subscheme such that $g(U) = U$, then* $\mathrm{tr}(g, R\Gamma_c(U, \Lambda)) = 0$.

Proof. For completeness, we recall the arguments of [V2, Corollaire 7.5.2]. We may assume $\Lambda = \mathbb{Z}/m\mathbb{Z}$. We decompose the commutative diagram into

$$\begin{tikzcd} X \arrow[r, "\gamma"] \arrow[dr, "g"'] & \sigma^* X \arrow[r] \arrow[d, "\sigma"] & S \arrow[d, "\sigma"] \\ & X \arrow[r, "f"] & S. \end{tikzcd}$$

Consider the cohomological correspondences $(\mathrm{id}, \sigma)\colon (X_s, \Psi\Lambda) \to (\sigma^* X_s, \Psi\Lambda)$ and $(c, u)\colon (\sigma^* X_\eta, \Lambda) \to (X_\eta, \Lambda)$, where $c = (\mathrm{id}_{X_\eta}, \gamma_\eta)$ and $u = \mathrm{id}_\Lambda$. We have a commutative diagram

$$\begin{tikzcd} R\Gamma(X_\eta, \Lambda) \arrow[r, "\sigma"] \arrow[d, "\simeq"'] & R\Gamma(\sigma^* X_\eta, \Lambda) \arrow[r, "f^\sharp_{\eta*}u"] \arrow[d, "\simeq"] & R\Gamma(X_\eta, \Lambda) \arrow[d, "\simeq"] \\ R\Gamma(X_s, \Psi_X\Lambda) \arrow[r, "\sigma"] & R\Gamma(X_s, \Psi_X\Lambda) \arrow[r, "f^\sharp_{s*}\psi u"] & R\Gamma(X_s, \Psi_X\Lambda) \end{tikzcd}$$

where the square on the right commutes by Proposition 3.12. The composite of the upper horizontal arrows is the action of g. Thus, by the Lefschetz-Verdier formula over s, we have

$$\mathrm{tr}(g, R\Gamma(X_\eta, \Lambda)) = \int_{X_s^{g_s}} \langle \sigma, \psi u \rangle = 0,$$

where $\int_F\colon H^0(F, K_F) \to \Lambda$ denotes the trace map. For the last assertion of the corollary, it suffices to note that

$$\mathrm{tr}(g, R\Gamma_c(U, \Lambda)) = \mathrm{tr}(g, R\Gamma(X_\eta, \Lambda)) - \mathrm{tr}(g, R\Gamma(Z_\eta, \Lambda)) = 0,$$

where Z is the closure of $X_\eta \backslash U$ in X, equipped with the reduced subscheme structure. □

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