Quotient stacks and equivariant étale cohomology algebras: Quillen's theory revisited

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To the memory of Daniel Quillen

Abstract

Let k be an algebraically closed field. Let Λ be a noetherian commutative ring annihilated by an integer invertible in k and let ℓ be a prime number different from the characteristic of k. We prove that if X is a separated algebraic space of finite type over k endowed with an action of a k-algebraic group G, the equivariant étale cohomology algebra $H^*([X/G],\Lambda)$, where $[X/G]$ is the quotient stack of X by G, is finitely generated over Λ. Moreover, for coefficients $K \in D_c^+([X/G],\mathbb{F}_\ell)$ endowed with a commutative multiplicative structure, we establish a structure theorem for $H^*([X/G],K)$, involving fixed points of elementary abelian ℓ-subgroups of G, which is similar to Quillen's theorem [36, Theorem 6.2] in the case $K = \mathbb{F}_\ell$. One key ingredient in our proof of the structure theorem is an analysis of specialization of points of the quotient stack. We also discuss variants and generalizations for certain Artin stacks.

Introduction

In [36], Quillen developed a theory for mod ℓ equivariant cohomology algebras $H_G^*(X,\mathbb{F}_\ell)$, where ℓ is a prime number, G is a compact Lie group, and X is a topological space endowed with an action of G. Recall that, for $r \in \mathbb{N}$, an *elementary abelian ℓ-group of rank r* is defined to be a group isomorphic to the direct product of r cyclic groups of order ℓ [36, Section 4]. Quillen showed that $H_G^*(X,\Lambda)$ is a finitely generated Λ-algebra for any noetherian commutative ring Λ [36, Corollary 2.2] and established structure theorems ([36, Theorem 6.2], [37, Theorem 8.10]) relating the ring structure of $H_G^*(X,\mathbb{F}_\ell)$ to the elementary abelian ℓ-subgroups A of G and the components of the fixed points set X^A. We refer the reader to [24, Section 1] for a summary of Quillen's theory.

In this article, we establish an algebraic analogue. Let k be an algebraically closed field of characteristic $\neq \ell$ and let Λ be noetherian commutative ring annihilated by an integer invertible in k. Let G be an algebraic group over k (*not* necessarily affine) and let X be a separated algebraic space of finite type over k endowed with an action of G. We consider the étale cohomology ring $H^*([X/G],\Lambda)$ of the quotient stack $[X/G]$. One of our main results is that this ring is a finitely generated Λ-algebra (Theorem 4.6) and the ring homomorphism

$$H^*([X/G],\mathbb{F}_\ell) \to \varprojlim_{\mathcal{A}} H^*(BA,\mathbb{F}_\ell)$$

given by restriction maps is a uniform F-isomorphism (Theorem 6.11), i.e. has kernel and cokernel killed by a power of $F\colon a \mapsto a^\ell$ (see Definition 6.10 for a review of this notion introduced by Quillen [36, Section 3]). Here $\mathcal{A}$ is the category of pairs (A,C), where A is an elementary abelian ℓ-subgroup of G and C is a connected component of X^A. The morphisms $(A,C)\to(A',C')$ of $\mathcal{A}$ are given by elements $g\in G$ such that $Cg \supset C'$ and $g^{-1}Ag \subset A'$. We also establish a generalization (Theorem 6.17) for $H^*([X/G],K)$, where $K \in D_c^+([X/G],\mathbb{F}_\ell)$ is a constructible complex of sheaves on $[X/G]$ endowed with a commutative ring structure.

A key ingredient in Quillen's original proofs is the continuity property [36, Proposition 5.6]. In the algebraic setting, this property is replaced by an analysis of the specialization of points of the quotient stack $[X/G]$. In order to make sense of this, we introduce the notions of geometric points and of ℓ-elementary points of Artin stacks. Our structure theorems for equivariant cohomology algebras are consequences of the following general structure theorem (Theorem 8.3): if $\mathcal{X} = [X/G]$

or $\mathcal{X}$ is a Deligne-Mumford stack of finite presentation and finite inertia over k, and if $K \in D_c^+(\mathcal{X}, \mathbb{F}_\ell)$ is endowed with a commutative ring structure, then the ring homomorphism

$$H^*(\mathcal{X}, K) \to \varprojlim_{x\colon \mathcal{S}\to\mathcal{X}} H^*(\mathcal{S}, x^*K)$$

given by restriction maps is a uniform F-isomorphism. Here the limit is taken over the category of ℓ-elementary points of $\mathcal{X}$.

In [26] we established an algebraic analogue [26, Theorem 8.1] of a localization theorem of Quillen [36, Theorem 4.2], which he had deduced from his structure theorems for equivariant cohomology algebras. This was one of the motivations for us to investigate algebraic analogues of these theorems. We refer the reader to [25] for a report on the present article and on some results of [26].

In Part I we review background material on quotient and classifying stacks (Section 1), and collect results on the cohomology of Artin stacks (Section 2) that are used at different places in this article. The ring structures of the cohomology algebras we are considering reflect ring structures on objects of derived categories. We discuss this in Section 3.

The reader familiar with the general nonsense recalled in Part I could skip it and move directly to Part II, which contains the main results of the paper. In Section 4, we prove the above-mentioned finiteness theorem (Theorem 4.6) for equivariant cohomology algebras. One key step of the proof amounts to replacing an abelian variety by its ℓ-divisible group, which was communicated to us by Deligne. In Section 5, we present a crucial result on the finiteness of orbit types, which is an analogue of [36, Lemma 6.3] and was communicated to us by Serre.

In Section 6, we state the above-mentioned structure theorems (Theorems 6.11, 6.17) for equivariant cohomology algebras. In Section 7, we introduce and discuss the notions of geometric points and of ℓ-elementary points of Artin stacks. Using them we state in Section 8 the main result of this paper, the structure theorem (Theorem 8.3) for cohomology algebras of certain Artin stacks, and show that it implies the structure theorems of the equivariant case. In Section 9, we establish some Künneth formulas needed in the proof of Theorem 8.3, which is given in Section 10. Finally, in Section 11 we prove an analogue of Quillen's stratification theorem [37, Theorems 10.2, 12.1] for the reduced spectrum of mod ℓ étale equivariant cohomology algebras.

The results of this paper have applications to the structure of varieties of supports. We hope to return to this in a future article.

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Conventions

We fix a universe $\mathcal{U}$, which we will occasionally enlarge. We say "small" instead of "$\mathcal{U}$-small" when there is no ambiguity. We say that a category is *essentially small* (resp. *essentially finite*) if it is equivalent to a small (resp. finite) category. Schemes are assumed to be small. Presheaves take values in the category of $\mathcal{U}$-sets. For any category $\mathcal{C}$, we denote by $\widehat{\mathcal{C}}$ the category of presheaves on $\mathcal{C}$, which is a $\mathcal{U}$-topos if $\mathcal{C}$ is essentially small. If $f\colon \mathcal{C} \to \mathcal{D}$ is a fibered category, we denote by $\mathcal{C}(U)$ (or sometimes $\mathcal{C}_U$) the fiber category of f over an object U of $\mathcal{D}$.

By a *stack* over a $\mathcal{U}$-site C we mean a stack in groupoids over C [44, 02ZI] whose fiber categories are essentially small.[1] By a stack, we mean a stack over the big fppf site of $\operatorname{Spec}(\mathbb{Z})$. Unlike [31], we do not assume algebraic spaces and Artin stacks to be quasi-separated. We say that a morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is *representable* (this property is called "representable by an algebraic space" in [44, 02ZW]) if for every scheme U and every morphism $y\colon U \to \mathcal{Y}$, the 2-fiber product $U \times_{y,\mathcal{Y},f} \mathcal{X}$ is representable by an algebraic space. By an *Artin stack* (resp. *Deligne-Mumford stack*), we mean an "algebraic stack" (resp. Deligne-Mumford stack) over $\operatorname{Spec}(\mathbb{Z})$ in the sense of [44, 026O] (resp. [44, 03YO]), namely a stack $\mathcal{X}$ such that the diagonal $\Delta_{\mathcal{X}}\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable and such that there exists an algebraic space X and a smooth (resp. étale) surjective morphism $X \to \mathcal{X}$.

By an *algebraic group* over a field k, we mean a group scheme over k of finite type. Unless otherwise stated, groups act on the right.

Contents

Part I
Preliminaries

1 Groupoids and quotient stacks

Classically, if G is a compact Lie group, a classifying space BG for G is the base of a contractible (right) G-torsor PG. Such a classifying space exists and is essentially unique (up to homotopy equivalence). If X is a G-space (i.e. a topological space endowed with a continuous (right) action of G), one can twist X by PG and get a space $PG \wedge^G X$, defined as the quotient of $PG \times X$ by

[1] The fiber categories of prestacks over C are also assumed to be essentially small.

the diagonal action of G, $((p,x),g) \mapsto (pg, xg)$. This space $PG \wedge^G X$ is a fiber bundle over BG of fiber X, and $PG \times X$ is a G-torsor over $PG \wedge^G X$. If Λ is a ring, the equivariant cohomology of X with value in Λ is defined by

$$H^*_G(X, \Lambda) := H^*(PG \wedge^G X, \Lambda) \simeq H^*(BG, R\pi_*\Lambda)$$

where $\pi \colon PG \wedge^G X \to BG$ is the projection. The functorial properties of this cohomology, introduced by Borel, are discussed by Quillen in [36, Section 1].

A well-known similar formalism exists in algebraic geometry, with classifying spaces replaced by classifying stacks. We review this formalism in this section.

Construction 1.1. Let C be a category in which finite limits are representable. We define the category

$$\mathrm{Eq}(C)$$

of equivariant objects in C as follows. The objects of $\mathrm{Eq}(C)$ are pairs (X, G) consisting of a group object G of C and an object X of C endowed with an action of G, namely a morphism $X \times G \to X$ satisfying the usual axioms for composition and identity. A morphism $(X, G) \to (Y, H)$ in $\mathrm{Eq}(C)$ is a pair (f, u) consisting of a homomorphism $u \colon G \to H$ and a u-equivariant morphism $f \colon X \to Y$. Here the u-equivariance of f is the commutativity of the following diagram in C:

$$\begin{array}{ccc} X \times G & \longrightarrow & X \\ {\scriptstyle f \times u}\downarrow & & \downarrow{\scriptstyle f} \\ Y \times H & \longrightarrow & Y. \end{array}$$

While $\mathrm{Eq}(C)$ is a category, groupoids in C form a (2,1)-category[2]

$$\mathrm{Grpd}(C).$$

We regard groupoids $X_\bullet$ in C as internal categories, consisting of two objects X_0 and X_1 of C, called respectively the object of objects and the object of morphisms, together with four morphisms in C,

$$e \colon X_0 \to X_1, \quad s, t \colon X_1 \to X_0, \quad m \colon X_1 \times_{s_X, X_0, t_X} X_1 \to X_1,$$

called respectively identity, source, target, and composition. A 1-morphism of groupoids $f_\bullet \colon X_\bullet \to Y_\bullet$ is an internal functor between the underlying internal categories, namely a pair of morphisms $f_0 \colon X_0 \to Y_0$, $f_1 \colon X_1 \to Y_1$, compatible with e, s, t, m. For 1-morphisms of groupoids $f_\bullet, g_\bullet \colon X_\bullet \to Y_\bullet$, a 2-morphism $f_\bullet \to g_\bullet$ is an internal natural isomorphism, namely, a morphism $r \colon X_0 \to Y_1$ of C such that $s_Y r = f_0$, $t_Y r = g_0$, and $m_Y(g_1, r s_X) = m_Y(r t_X, f_1)$. The last identity can be stated informally as for any $(u \colon a \to b) \in X_1$, $g_1(u) r(a) = r(b) f_1(u)$.

We define a functor

$$\mathrm{Eq}(C) \to \mathrm{Grpd}(C). \tag{1.1.1}$$

as follows. To an object (X, G) of $\mathrm{Eq}(C)$, we assign a groupoid in C

$$(X, G)_\bullet$$

with $(X, G)_0 = X$, $(X, G)_1 = X \times G$, $e(x) = (x, 1)$, $s(x, g) = xg$, $t(x, g) = x$, and composition given by $(x, g)(xg, h) = (x, gh)$. The inverse-assigning morphism is $(x, g) \mapsto (xg, g^{-1})$. Here we follow the conventions of [31, 3.4.3] (see also [44, 0444] where groups act on the left). A morphism $(f, u) \colon (X, G) \to (Y, H)$ in $\mathrm{Eq}(C)$ gives a morphism of groupoids $(f, u)_\bullet \colon (X, G)_\bullet \to (Y, H)_\bullet$, $(f, u)_0 = f$, $(f, u)_1 = f \times u \colon (x, g) \mapsto (f(x), u(g))$.

The functor (1.1.1) is faithful, but not fully faithful. The maximal 2-subcategory $\mathrm{Grpd}^{\mathrm{Eq}}(C)$ of $\mathrm{Grpd}(C)$ spanned by the objects in the image of (1.1.1) can be described as follows.

Proposition 1.2. *Let (X, G), (Y, H), and (Z, I) be objects of* $\mathrm{Eq}(C)$.

[2] A (2,1)-category is a 2-category whose 2-morphisms are invertible.

(a) *For any morphism of groupoids $\varphi = (\varphi_0, \varphi_1): (X,G)_\bullet \to (Y,H)_\bullet$, there exist a unique pair of morphisms $f: X \to Y$, $u: X \times G \to H$ such that $\varphi_1(x,g) = (f(x), u(x,g))$, and the pair (f,u) satisfies the following relations:*
   (i) *f is u-*equivariant*, i.e. $f(xg) = f(x)u(x,g)$,*
   (ii) *$u(x,g)u(xg,g') = u(x,gg')$.*

   Conversely, any pair (f,u) satisfying (i), (ii) defines a morphism of groupoids $\varphi_\bullet$. Moreover, if $(a,u): (X,G)_\bullet \to (Y,H)_\bullet$ and $(b,v): (Y,H)_\bullet \to (Z,I)_\bullet$ are morphisms of groupoids, the composition is given by (ba,w), where $w: X \times G \to I$ is given by $w(x,g) = v(a(x), u(x,g))$.

(b) *Let $\varphi_i = (f_i, u_i): (X,G)_\bullet \to (Y,H)_\bullet$ ($i = 1,2$) be 1-morphisms of groupoids. Then a 2-morphism from φ_1 to φ_2 is a morphism $r: X \to H$ satisfying the relations*
   (i) *$f_1(x) = f_2(x)r(x)$,*
   (ii) *$r(x)u_1(x,g) = u_2(x,g)r(xg)$.*

   Composition of 2-morphisms is given by multiplication in H.

We will sometimes call a morphism $u: X \times G \to H$ satisfying (a) (ii) a *crossed homomorphism.*

Proof. In (a), the uniqueness of (f,u) are clear, while the existence (resp. (i), resp. (ii)) expresses the compatibility of φ with the target (resp. source, resp. composition) morphism. The other statements are straightforward. □

Definition 1.3. We say that a pseudofunctor $F: \mathcal{C} \to \mathcal{D}$ between 2-categories is *faithful* (resp. *fully faithful*) if for every pair of objects X and Y in $\mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(FX,FY)$ induced by F is fully faithful (resp. an equivalence of categories). We say that F is *essentially surjective* if for every object Y of $\mathcal{D}$, there exists an object X of $\mathcal{C}$ and an equivalence $FX \simeq Y$ in $\mathcal{D}$.

Construction 1.4. Let E be a $\mathcal{U}$-topos (we will be mostly interested in the case where E is the topos of fppf sheaves on some algebraic space), endowed with its canonical topology. A groupoid $X_\bullet$ in E defines a category $[X_\bullet]'$ fibered in groupoids over E whose fiber at U is $X_\bullet(U)$. This is an E-prestack, and, as in [31, 3.4.3], we denote the associated E-stack [44, 02ZP] by $[X_\bullet]$. If π denotes the canonical composite morphism

$$\pi: X_0 \to [X_\bullet]' \to [X_\bullet],$$

the groupoid can be recovered from π: there is a natural isomorphism

$$X_1 \xrightarrow{\sim} X_0 \times_{[X_\bullet]} X_0 \tag{1.4.1}$$

identifying the projections $p_1, p_2: X_0 \times_{[X_\bullet]} X_0 \to X_0$ with s and t, and identifying the second projection $\mathrm{id} \times \pi \times \mathrm{id}: X_0 \times_{[X_\bullet]} X_0 \times_{[X_\bullet]} X_0 \to X_0 \times_{[X_\bullet]} X_0$ with m. Here $X_0 \times_{[X_\bullet]} X_0$ denotes the sheaf carrying U to the set of isomorphism classes of triples (x,y,α), $x,y \in X_0(U)$, $\alpha: \pi(x) \simeq \pi(y)$. More generally, there is a natural isomorphism of simplicial objects

$$\mathrm{Ner}(X_\bullet) \xrightarrow{\sim} \mathrm{cosk}_0(\pi) \tag{1.4.2}$$

between the nerve of the groupoid $X_\bullet$ and the 0-th coskeleton of π.

We denote by $\mathrm{Stack}(E)$ (resp. $\mathrm{PreStack}(E)$) the (2,1)-category of E-stacks (E-prestacks). The pseudofunctor $\mathrm{Grpd}(E) \to \mathrm{PreStack}(E)$ sending $X_\bullet$ to $[X_\bullet]'$ is fully faithful and the pseudofunctor $\mathrm{PreStack}(E) \to \mathrm{Stack}(E)$ sending an E-prestack to its associated E-stack is faithful. Therefore, the composite pseudofunctor

$$\mathrm{Grpd}(E) \to \mathrm{Stack}(E) \tag{1.4.3}$$

sending $X_\bullet$ to its associated E-stack $[X_\bullet]$ is faithful. In other words, if $X_\bullet$, $Y_\bullet$ are groupoids in E, and $\varphi_i: X_\bullet \to Y_\bullet$ ($i = 1,2$) is a morphism of groupoids, then the natural map

$$\mathrm{Hom}(\varphi_1, \varphi_2) \to \mathrm{Hom}([\varphi_1], [\varphi_2])$$

is bijective. However, in general, not every morphism $f: [X_\bullet] \to [Y_\bullet]$ is of the form $[\varphi]$ for a morphism of groupoids $\varphi: X_\bullet \to Y_\bullet$ (see Remark 1.7 below). On the other hand, (1.4.3) is essentially surjective.

Notation 1.5. In the case of the groupoid $(X,G)_\bullet$ associated with a G-object X of E, the stack $[(X,G)_\bullet]$ is denoted by

$$[X/G] \tag{1.5.1}$$

and called the *quotient stack* of X by G. For $X = e$ the final object of E (with the trivial action of G), it is called the *classifying stack* of G and denoted by

$$BG \coloneqq [e/G]. \tag{1.5.2}$$

Recall ([31, 2.4.2], [44, 04WM]) that the projection $X \to [X/G]$ makes X into a universal G-torsor over $[X/G]$, i.e. for U in E, the groupoid $[X/G](U)$ is canonically equivalent to the category of pairs (P,a), where P is a right G_U-torsor and a is a G-equivariant morphism from P to X; morphisms from (P,a) to (Q,b) are G-equivariant morphisms $c\colon P \to Q$ such that $a = bc$.

The action of G on X is recovered from π: the isomorphism (1.4.1) takes the form

$$X \times G \xrightarrow{\sim} X \times_{[X/G]} X, \tag{1.5.3}$$

identifying the projections p_1, p_2 with $(x,g) \mapsto xg$, $(x,g) \mapsto x$.

For $X = e$, $BG(U)$ is the groupoid of G-torsors on U for U in E, which justifies the terminology "classifying stack". For general X, the projection $[X/G] \to BG$ induces $X \to e$ by the base change $B\{1\} \to BG$, so that one can think of $[X/G] \to BG$ as a "fibration" with fiber X. In other words, $[X/G]$ plays the role of the object $PG \wedge^G X$ recalled at the beginning of Section 1.

In order to describe morphisms from $[X/G]$ to $[Y/H]$ associated to morphisms of groupoids from $(X,G)_\bullet$ to $(Y,H)_\bullet$, we need to introduce the following notation. Let (X,G) be an object of $\mathrm{Eq}(E)$, and let $u\colon X \times G \to H$ be a crossed homomorphism (Proposition 1.2). We denote by

$$X \wedge^{G,u} H \tag{1.5.4}$$

the quotient of $X \times H$ by G acting by $(x,h)g = (xg, u(x,g)^{-1}h)$. This is an H-object of E, the action of H on it being deduced from its action by right translations on $X \times H$. For any H-object Y of E, the map

$$\mathrm{Hom}_u(X,Y) \to \mathrm{Hom}_H(X \wedge^{G,u} H, Y) \tag{1.5.5}$$

sending a u-equivariant morphism f (Proposition 1.2 (a) (i)) to the morphism $f^u\colon (x,h) \mapsto f(x)h$ is bijective.

When $u\colon X \times G \to H$ is defined by $u(x,g) = u_0(g)$ for a group homomorphism $u_0\colon G \to H$, $X \wedge^{G,u} H$ coincides with the usual contracted product [20, Définition III.1.3.1], i.e. the quotient of $X \times H$ by the diagonal action of G, $(x,h)g \coloneqq (xg, u_0(g)^{-1}h)$.

The following proposition, whose verification is straightforward, describes the restriction of (1.4.3) to $\mathrm{Grpd}^{\mathrm{Eq}}(E)$.

Proposition 1.6. *Let (X,G) and (Y,H) be objects of* $\mathrm{Eq}(E)$.

(a) Let $(f,u)\colon (X,G)_\bullet \to (Y,H)_\bullet$ be a morphism of groupoids (Proposition 1.2), and let

$$[f/u]\colon [X/G] \to [Y/H]$$

be the associated morphism of stacks. For $(P,a) \in [X/G](U)$, $[f/u](P,a)$ is the pair consisting of the H-torsor $P \wedge^{G,v} H$ (where v is the composition of $a \times \mathrm{id}_G\colon P \times G \to X \times G$ and u) and the H-equivariant morphism $a^v\colon P \wedge^{G,v} H \to Y$ defined by a via (1.5.5)*.*

(b) Let φ_1, φ_2, r be as in Proposition 1.2 (b). Then the 2-morphism $[r]\colon [f_1/u_1] \to [f_2/u_2]$ induced by r is given by the Y-morphism $P \wedge^{G,v_1} H \to P \wedge^{G,v_2} H$ sending (p,h) to $(p, r(a(p))^{-1}h)$.

For a crossed homomorphism $u\colon X \times G \to H$, the unit section of H defines a u-equivariant morphism

$$X \to X \wedge^{G,u} H. \tag{1.6.1}$$

The morphism of E-stacks

$$[X/G] \to [(X \wedge^{G,u} H)/H] \tag{1.6.2}$$

induced by (1.6.1) sends $T \to X$ to $T \wedge^{G,u} H \to X \wedge^{G,u} H$.

Remark 1.7. The restriction of (1.4.3) to $\mathrm{Grpd}^{\mathrm{Eq}}(E)$ is not fully faithful in general. In other words, for objects (X,G), (Y,H) of $\mathrm{Eq}(E)$, a morphism of stacks $[X/G] \to [Y/H]$ does not necessarily come from a morphism of groupoids $(X,G)_\bullet \to (Y,H)_\bullet$. In fact, in the case $G = \{1\}$ and Y is a nontrivial H-torsor over X, any quasi-inverse of the equivalence $[Y/H] \to X$ does not come from a morphism of groupoids. See Proposition 1.19 for a useful criterion. See also [47, Proposition 5.1] for a calculus of fractions for the composite functor $\mathrm{Eq}(E) \to \mathrm{Stack}(E)$ of (1.1.1) and (1.4.3).

Definition 1.8. We say that a morphism $X \to Y$ in a 2-category $\mathcal{C}$ is *faithful* (resp. a *monomorphism*) if for every object U of $\mathcal{C}$, the functor $\mathrm{Hom}(U,X) \to \mathrm{Hom}(U,Y)$ is faithful (resp. fully faithful).

In a 2-category, we need to distinguish between 2-limits [18, Definition 1.4.26] and strict 2-limits (called "2-limits" in [4, Definition 7.4.1]). Strict 2-products are 2-products. If a diagram $X \to Y \leftarrow X'$ in $\mathcal{C}$ admits a 2-fiber product $X \times_Y X'$ and a strict 2-fiber product Z, the canonical morphism $Z \to X \times_Y X'$ is a monomorphism.

In a (2,1)-category $\mathcal{C}$ admitting 2-fiber products, a morphism $X \to Y$ is faithful (resp. a monomorphism) if and only if its diagonal morphism $X \to X \times_Y X$ is a monomorphism (resp. an equivalence).

A morphism of E-prestacks $\mathcal{X} \to \mathcal{Y}$ is faithful (resp. a monomorphism, resp. an equivalence) if and only if $\mathcal{X}(U) \to \mathcal{Y}(U)$ is a faithful functor (resp. a fully faithful functor, resp. an equivalence of categories) for every U in E. If $\mathcal{X}'$ is an E-prestack and $\mathcal{X}$ is its associated E-stack, then the canonical morphism $\mathcal{X}' \to \mathcal{X}$ is a monomorphism.

Let $(f,u)\colon (X,G) \to (Y,H)$ be a morphism of $\mathrm{Eq}(E)$. If u is a monomorphism, then $[f/u]\colon [X/G] \to [Y/H]$ is faithful.

Remark 1.9. The category $\mathrm{Eq}(C)$ admits finite limits, whose formation commutes with the projection functors $(X,G) \mapsto X$ and $(X,G) \mapsto G$ from $\mathrm{Eq}(C)$ to C and to the category of group objects of C, respectively. The 2-category $\mathrm{Grpd}(C)$ admits finite strict 2-limits, whose formation commutes with the projection 2-functors $X_\bullet \mapsto X_0$ and $X_\bullet \mapsto X_1$ from $\mathrm{Grpd}(C)$ to C. The functor $\mathrm{Eq}(C) \to \mathrm{Grpd}(C)$ (1.1.1) sending (X,G) to $(X,G)_\bullet$ carries finite limits to finite *strict* 2-limits.

The 2-category $\mathrm{Grpd}(C)$ admits finite 2-limits as well. The 2-fiber product of a diagram $X_\bullet \xrightarrow{f} Y_\bullet \xleftarrow{g} Y'_\bullet$ in $\mathrm{Grpd}(C)$ is the groupoid $W_\bullet$ of triples (x,y,α), where $x \in X_0$, $y \in Y'_0$, and $(\alpha\colon f(x) \xrightarrow{\sim} g(y)) \in Y_1$. More formally, $W_0 = X_0 \times_{Y_0,s_Y} Y_1 \times_{t_Y,Y_0} Y'_0$ and W_1 is the limit of the diagram

$$X_1 \to Y_1 \xleftarrow{p_1} Y_1 \times_{Y_0} Y_1 \xrightarrow{m} Y_1 \xleftarrow{m} Y_1 \times_{Y_0} Y_1 \xrightarrow{p_2} Y_1 \leftarrow Y'_1.$$

A morphism $X_\bullet \to Y_\bullet$ in $\mathrm{Grpd}(C)$ is faithful (resp. a monomorphism) if and only if the morphism $X_1 \to (X_0 \times X_0) \times_{Y_0 \times Y_0,(s_Y,t_Y)} Y_1$ is a monomorphism (resp. isomorphism).

The category $\mathrm{Stack}(E)$ admits small 2-limits. The pseudofunctor $\mathrm{Grpd}(E) \to \mathrm{Stack}(E)$ (1.4.3) preserves finite 2-limits and thus preserves faithful morphisms and monomorphisms.

Remark 1.10. A commutative square in $\mathrm{Eq}(E)$,

(1.10.1)
$$\begin{array}{ccc} (X',G') & \xrightarrow{(f',\gamma')} & (Y',H') \\ \downarrow{\scriptstyle (p,u)} & & \downarrow{\scriptstyle (q,v)} \\ (X,G) & \xrightarrow{(f,\gamma)} & (Y,H) \end{array}$$

induces a 2-commutative square of E-stacks

(1.10.2)
$$\begin{array}{ccc} [X'/G'] & \longrightarrow & [Y'/H'] \\ \downarrow & & \downarrow \\ [X/G] & \longrightarrow & [Y/H]. \end{array}$$

It is not true in general that if (1.10.1) is cartesian, (1.10.2) is 2-cartesian, as (1.5.3) already shows. However, we have the following result, which is a partial generalization of [47, Proposition 5.4].

Proposition 1.11. *Consider a cartesian square* (1.10.1) *in* $\mathrm{Eq}(E)$*. If the morphism*

(1.11.1) $$H' \times G \to H$$

in E given by $(h,g) \mapsto v(h)\gamma(g)$ is an epimorphism, then (1.10.2) *is 2-cartesian.*

Proof. Let
$$\alpha\colon [X'/G'] \to \mathcal{X} := [X/G] \times_{[Y/H]} [Y'/H']$$
be the induced morphism of E-stacks. By Remark 1.9 and the remark following Definition 1.8, α is a monomorphism. We need to show that for any object V of E, the functor $\alpha_V\colon [X'/G']_V \to \mathcal{X}_V$ is essentially surjective. By definition, $\mathcal{X}_V$ is the category of triples $((T,t),(T',t'),s)$, where $(T, t\colon T \to X)$ is an object of $[X/G]_V$, $(T', t'\colon T' \to Y')$ is an object of $[Y'/H']_V$, and $s\colon [f/u]_V(T,t) \to [q/v]_V(T',t')$ is an isomorphism. In other words (Proposition 1.6 (b)), $s\colon T \wedge^{G,\gamma} H \to T' \wedge^{H',v} H$ is an isomorphism of H-torsors over V, compatible with the morphisms to Y (induced by qt' and ft). The functor α_V sends an object (P,w) of $[X'/G']_V$ to $([p/u]_V(P,w), [f'/\gamma']_V(P,w), \sigma)$, where $\sigma\colon [fp/\gamma u]_V(P,w) \to [qf'/v\gamma']_V(P,w)$ is the obvious isomorphism. Let $a = ((T,t),(T',t'),s)$ be an object of $\mathcal{X}_V$. It remains to show that there exist a cover $(V_i \to V)_{i\in I}$ and, for every $i \in I$, an object (P_i, w_i) of $[X'/G']_{V_i}$ such that $\alpha(P_i,w_i) \simeq a_{V_i}$. Take a cover $(V_i \to V)_{i\in I}$ such that for every i, T_{V_i} and T'_{V_i} are both trivial and choose trivializations of them. Then s_{V_i} is represented by the left multiplication by some $h_i \in H(V_i)$. By the assumption on (1.11.1), we may assume $h_i = v(h'_i)\gamma(g_i)$, $h'_i \in H'(V_i)$, $g_i \in G(V_i)$. In this case, the square

(1.11.2) $$\begin{array}{ccc} H_{V_i} & \xrightarrow{s_{V_i}} & H_{V_i} \\ {\scriptstyle \lambda_{\gamma(g_i)}}\downarrow & & \downarrow{\scriptstyle \lambda^{-1}_{v(h'_i)}} \\ H_{V_i} & \xrightarrow{1} & H_{V_i} \end{array}$$

commutes, where λ_h is the left multiplication by h. Thus (1.11.2) gives an isomorphism $a_{V_i} \simeq b_i$, where $b_i = ((G_{V_i}, t\lambda^{-1}_{g_i}), (H'_{V_i}, t'\lambda_{h'_i}), 1)$. Taking the product of $(G_{V_i}, t\lambda^{-1}_{g_i})$ and $(H'_{V_i}, t'\lambda_{h'_i})$ over $(H_{V_i}, (t\lambda^{-1}_{g_i})^\gamma = (t'\lambda_{h'_i})^v)$ gives us an element (P_i, w_i) of $[X/G]_{V_i}$ whose image under α is b_i. □

Corollary 1.12. *Suppose $u\colon G \to Q$ is an epimorphism of groups of E, with kernel K. Then the natural morphism*
$$BK \xrightarrow{\sim} e \times_{BQ} BG$$
is an equivalence.

In other words, we can view $Bu\colon BG \to BQ$ as a fibration of fiber BK.

Definition 1.13. We say that a groupoid $X_\bullet$ in E is an *equivalence relation* if $(s_X, t_X)\colon X_1 \to X_0 \times X_0$ is a monomorphism. In this case, the associated E-stack $[X_\bullet]$ is represented by the quotient sheaf in E. We say that the action of G on X is *free* if the associated groupoid $(X,G)_\bullet$ is an equivalence relation. In this case, $[X/G]$ is represented by the sheaf X/G.

Proposition 1.14. *Let (X,G) be an object in* $\mathrm{Eq}(E)$*, and let K be a normal subgroup of G acting freely on X. Then the morphism $f\colon [X/G] \to [(X/K)/(G/K)]$ is an equivalence.*

Proof. Indeed, for every U in E, $[(X/K)/(G/K)]_U$ is the category of pairs (T,α), where T is a G/K-torsor and $\alpha\colon T \to X/K$ is a G-equivariant map, and the functor f_U admits a quasi-inverse carrying (T,α) to its base change by the projection $X \to X/K$. □

The following *induction formula* will be useful later in the calculation of equivariant cohomology groups (cf. [36, (1.7)]).

Corollary 1.15. *Let (X,G) be an object of* $\mathrm{Eq}(E)$ *and let $u\colon X \times G \to H$ be a crossed homomorphism. Assume that the action of G on $X \times H$, as defined in Notation 1.5, is free (Definition 1.8). Then $f\colon [X/G] \to [X \wedge^{G,u} H/H]$* (1.6.2) *is an equivalence.*

Proof. The morphism f can be decomposed as

$$[X/G] \xrightarrow{\alpha} [X \times H/G \times H] \xrightarrow{\beta} [X \wedge^{G,u} H/H],$$

where β is an equivalence by Proposition 1.14, and α is induced by the morphism $X \to X \times H$ given by the unit section of H and the crossed homomorphism $X \times G \to G \times H$ sending (x, g) to $(g, u(x, g))$. Since α is a 2-section of the morphism $[X \times H/G \times H] \to [X/G]$, which is an equivalence by Proposition 1.14, α is also an equivalence. $\square$

Corollary 1.16. *Let $u\colon H \hookrightarrow G$ be a monomorphism of group objects in E. Then*
(a) The morphism of stacks $BH \to [(H\backslash G)/G]$ is an equivalence.
(b) The natural morphism $H\backslash G \to e \times_{BG} BH$ is an isomorphism.

In other words, (a) says that, for any homogeneous space X of group G, if H is the stabilizer of a section x of X, then the morphism $BH \to [X/G]$ given by $x\colon e \to X$ is an equivalence, while (b) can be thought as saying that $BH \to BG$ is a fibration of fiber $H\backslash G$.

Proof. Assertion (a) follows from Corollary 1.15. Assertion (b) follows from Proposition 1.11 applied to the cartesian square

$$\begin{array}{ccc} (H\backslash G, \{1\}) & \longrightarrow & (e, \{1\}) \\ \downarrow & & \downarrow \\ (H\backslash G, G) & \longrightarrow & (e, G) \end{array}$$

(cf. the paragraph following (1.5.3)) and from (a). $\square$

Construction 1.17. We will apply the above formalism to a relative situation, which we now describe. Let $\mathcal{X}$ be an E-stack. We denote by $\mathrm{Stack}_{/\mathcal{X}}$ the (2,1)-category of E-stacks over $\mathcal{X}$. An object of $\mathrm{Stack}_{/\mathcal{X}}$ is a pair $(\mathcal{Y}, y)$, where $\mathcal{Y}$ is an E-stack and $y\colon \mathcal{Y} \to \mathcal{X}$ is a morphism of E-stacks. A morphism in $\mathrm{Stack}_{/\mathcal{X}}$ from $(\mathcal{Y}, y)$ to $(\mathcal{Z}, z)$ is a pair (f, α), where $f\colon \mathcal{Y} \to \mathcal{Z}$ is a morphism of E-stacks and $\alpha\colon y \to zf$ is a 2-morphism:

(1.17.1)
$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{Z} \\ & {\scriptstyle y}\searrow \;\Nearrow & \downarrow{\scriptstyle z} \\ & & \mathcal{X}. \end{array}$$

A 2-morphism $(f, \alpha) \to (g, \beta)$ in $\mathrm{Stack}_{/\mathcal{X}}$ is a 2-morphism $\eta\colon f \to g$ in the (2,1)-category $\mathrm{Stack}(E)$ such that $\beta = (z * \eta) \circ \alpha$.

A morphism $y\colon \mathcal{Y} \to \mathcal{X}$ of E-stacks is faithful (Definition 1.8) if and only if for any object U of E and any morphism $x\colon U \to \mathcal{X}$, the 2-fiber product $U \times_{x,\mathcal{X},y} \mathcal{Y}$ is isomorphic to a sheaf. Consider the 2-subcategory $\mathcal{S}$ of $\mathrm{Stack}_{/\mathcal{X}}$ spanned by objects $(\mathcal{Y}, y)$ with y faithful. For any morphism $(f, \alpha)\colon (\mathcal{Y}, y) \to (\mathcal{Z}, z)$ in $\mathcal{S}$, f is necessarily faithful. A 2-morphism $\eta\colon (f, \alpha) \to (g, \beta)$ in $\mathcal{S}$, if it exists, is uniquely determined by (f, α) and (g, β). In other words, if we denote by $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$ the category obtained from $\mathcal{S}$ by identifying isomorphic morphisms, then the 2-functor $\mathcal{S} \to \mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$ is a 2-equivalence.

For any morphism $\phi\colon \mathcal{X} \to \mathcal{Y}$ of E-stacks, base change by ϕ induces a functor $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{Y}} \to \mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$. If S is an object of E, $\mathrm{Stack}^{\mathrm{faith}}_{/S}$ is equivalent to $E_{/S}$. More generally, if $U_\bullet$ is a groupoid in E, $\mathrm{Stack}^{\mathrm{faith}}_{/[U_\bullet]}$ is equivalent to the category of descent data relative to $U_\bullet$. In particular, if (X, G) is an object of $\mathrm{Eq}(E)$, $\mathrm{Stack}^{\mathrm{faith}}_{/[X/G]}$ is equivalent to the category of G-objects of E, equivariant over X. For example, $\mathrm{Stack}^{\mathrm{faith}}_{/BG}$ is equivalent to the topos B_G of Grothendieck.

Proposition 1.18.
(a) The category $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$ is a $\mathcal{U}$-topos.

(b) *Let $\mathcal{X}$ be a stack. For any stack $\mathcal{Y}$ over $\mathcal{X}$, associating to any stack $\mathcal{Z}$ faithful over $\mathcal{X}$ the groupoid* $\mathrm{Hom}_{\mathcal{X}}(\mathcal{Z},\mathcal{Y})$ *defines a stack* $\underline{\mathcal{Y}}$ *over* $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$. *The 2-functor*

$$\mathrm{Stack}_{/\mathcal{X}} \to \mathrm{Stack}(\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}), \quad \mathcal{Y} \mapsto \underline{\mathcal{Y}} \tag{1.18.1}$$

is a 2-equivalence.

Proof. (a) We apply Giraud's criterion [50, IV Théorème 1.2]. If $\mathcal{T}$ is a small generating family of E, then $\coprod_{U\in\mathcal{T}}\mathrm{Ob}(\mathcal{X}(U))$ is an essentially small generating family of $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$. Let us now show that every sheaf $\mathcal{F}$ on $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$ for the canonical topology is representable. Consider, for every object U of E, the category of pairs (x,s) consisting of $x\in\mathcal{X}(U)$ and $s\in\Gamma(x,\mathcal{F})$, where the last occurrence of x is to be understood as the object $x\colon U\to\mathcal{X}$ in $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$. A morphism $(x,s)\mapsto(y,t)$ is a morphism $\alpha\colon x\to y$ in $\mathcal{X}(U)$ such that $\alpha^*t=s$. This defines an E-stack $\mathcal{X}'$. The faithful morphism $\mathcal{X}'\to\mathcal{X}$ of E-stacks defined by the first projection $(x,s)\mapsto x$ represents $\mathcal{F}$. The other conditions in Giraud's criterion are trivially satisfied. Thus $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$ is a $\mathcal{U}$-topos.

(b) We construct a 2-quasi-inverse to (1.18.1) as follows. Let $\mathcal{C}$ be a stack over $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$. For every object U of E, consider the category of pairs (x,s) consisting of $x\in\mathcal{X}(U)$ and $s\in\mathcal{C}(x)$. A morphism $(x,s)\to(y,t)$ is a pair (α,β) consisting of a morphism $\alpha\colon x\to y$ in $\mathcal{X}(U)$ and a morphism $\beta\colon\alpha^*t\to s$ in $\mathcal{C}(x)$. This defines an E-stack $\mathcal{Y}$. The first projection $(x,s)\mapsto x$ defines a morphism $\mathcal{Y}\to\mathcal{X}$ of E-stacks. The construction $\mathcal{C}\mapsto(\mathcal{Y}\to\mathcal{X})$ defines a pseudofunctor

$$\mathrm{Stack}(\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}) \to \mathrm{Stack}_{/\mathcal{X}}, \tag{1.18.2}$$

which is a 2-quasi-inverse to (1.18.1). □

The composition of (1.4.3) and (1.18.2) is a faithful and essentially surjective (Definition 1.3) pseudofunctor

$$\mathrm{Grpd}(\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}) \to \mathrm{Stack}_{/\mathcal{X}}. \tag{1.18.3}$$

We denote the image of a groupoid $X_\bullet$ in $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$ under (1.18.3) by $[X_\bullet/\mathcal{X}]$, and the image of a morphism $f_\bullet$ of groupoids under (1.18.3) by $[f_\bullet/\mathcal{X}]$. For (X,G) in $\mathrm{Eq}(\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}})$, we denote the image of $(X,G)_\bullet$ under (1.18.3) by $[X/G/\mathcal{X}]$. For $(f,u)\colon(X,G)_\bullet\to(Y,H)_\bullet$, we denote the image under (1.18.3) by $[f/u/\mathcal{X}]$.

We now apply the above formalism to the big fppf topoi of algebraic spaces. Recall that a stack is a stack over the big fppf site of $\mathrm{Spec}\,\mathbb{Z}$. The following result will be useful in Sections 7 and 8.

Proposition 1.19. *Let $\mathcal{X}$ be a stack, and let $X_\bullet$, $Y_\bullet$ be objects in* $\mathrm{Grpd}(\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}})$. *Assume that X_0 is a strictly local scheme and the morphisms $Y_1 \rightrightarrows Y_0$ are representable and smooth. Then the functor induced by* (1.18.3)*:*

$$F\colon \mathrm{Hom}_{\mathrm{Grpd}(\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}})}(X_\bullet,Y_\bullet)\to\mathrm{Hom}_{\mathrm{Stack}_{/\mathcal{X}}}([X_\bullet/\mathcal{X}],[Y_\bullet/\mathcal{X}])$$

is an equivalence of categories.

Proof. It remains to show that F is essentially surjective. Let $\phi\colon[X_\bullet/\mathcal{X}]\to[Y_\bullet/\mathcal{X}]$ be a morphism in $\mathrm{Stack}_{/\mathcal{X}}$. For the 2-cartesian square

$$\begin{array}{ccc} X'_0 & \longrightarrow & Y_0 \\ \downarrow & & \downarrow \\ X_0 \longrightarrow [X_\bullet/\mathcal{X}] & \xrightarrow{\phi} & [Y_\bullet/\mathcal{X}]. \end{array}$$

Since X'_0 is representable and smooth over X_0, it admits a section by [22, Corollaire 17.16.3 (ii), Proposition 18.8.1], which induces a 2-commutative square

$$\begin{array}{ccc} X_0 & \xrightarrow{f_0} & Y_0 \\ \downarrow & & \downarrow \\ [X_\bullet/\mathcal{X}] & \xrightarrow{\phi} & [Y_\bullet/\mathcal{X}]. \end{array}$$

Let $f_1 = f_0 \times_\phi f_0 \colon X_1 \to Y_1$. Then $f_\bullet \colon X_\bullet \to Y_\bullet$ is a morphism of groupoids in $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$ and $\phi \simeq [f_\bullet/\mathcal{X}]$. □

Remark 1.20. Let $\mathcal{X}$ be a stack. We denote by $\mathrm{Stack}^{\mathrm{rep}}_{/\mathcal{X}}$ the full subcategory of $\mathrm{Stack}^{\mathrm{faith}}_{/\mathcal{X}}$ consisting of representable morphisms $X \to \mathcal{X}$. A morphism in this category from $X \to \mathcal{X}$ to $Y \to \mathcal{X}$ is an isomorphism class of pairs (f, α) (1.17.1). The morphisms $f \colon X \to Y$ are necessarily representable. Assume that $\mathcal{X}$ is an Artin stack. For any object $X \to \mathcal{X}$ of $\mathrm{Stack}^{\mathrm{rep}}_{/\mathcal{X}}$, X is necessarily an Artin stack. For any object $X_\bullet$ in $\mathrm{Grpd}(\mathrm{Stack}^{\mathrm{rep}}_{/\mathcal{X}})$, if s_X and t_X are flat and locally of finite presentation, then $[X_\bullet/\mathcal{X}]$ is an Artin stack. In particular, for any object (X, G) in $\mathrm{Eq}(\mathrm{Stack}^{\mathrm{rep}}_{/\mathcal{X}})$ with G flat of and locally of finite presentation over $\mathcal{X}$, $[X/G/\mathcal{X}]$ is an Artin stack.

2 Miscellany on the étale cohomology of Artin stacks

Notation 2.1. Let $\mathcal{X}$ be an Artin stack. We denote by $\mathrm{AlgSp}_{/\mathcal{X}}$ the full subcategory of $\mathrm{Stack}^{\mathrm{rep}}_{/\mathcal{X}}$ (Remark 1.20) consisting of morphisms $U \to \mathcal{X}$ with U an algebraic space. We let $\mathrm{Sp}^{\mathrm{sm}}_{/\mathcal{X}}$ denote the full subcategory of $\mathrm{AlgSp}_{/\mathcal{X}}$ spanned by smooth morphisms $U \to \mathcal{X}$. The covering families of the smooth pretopology on $\mathrm{Sp}^{\mathrm{sm}}_{/\mathcal{X}}$ are those $(U_i \to U)_{i\in I}$ such that $\coprod_{i\in I} U_i \to U$ is smooth and surjective. The covering families for the étale pretopology on $\mathrm{Sp}^{\mathrm{sm}}_{/\mathcal{X}}$ are those $(U_i \to U)_{i\in I}$ such that $\coprod_{i\in I} U_i \to U$ is étale and surjective. Since every smooth cover in $\mathrm{Sp}^{\mathrm{sm}}_{/\mathcal{X}}$ has an étale refinement by [22, Corollaire 17.16.3 (ii)], the smooth pretopology and the étale pretopology generate the same topology on $\mathrm{Sp}^{\mathrm{sm}}_{/\mathcal{X}}$ (cf. [31, Définition 12.1]). We let $\mathcal{X}_{\mathrm{sm}}$ denote the associated topos, and call it the *smooth topos* of $\mathcal{X}$.

Notation 2.2. The category of sheaves in $\mathcal{X}_{\mathrm{sm}}$ is equivalent to the category of systems $(\mathcal{F}_u, \theta_\phi)$, where $u \colon U \to \mathcal{X}$ runs through objects of $\mathrm{Sp}^{\mathrm{sm}}_{/\mathcal{X}}$, $\phi \colon u \to v$ runs through morphisms of $\mathrm{Sp}^{\mathrm{sm}}_{/\mathcal{X}}$, $\mathcal{F}_u$ is an étale sheaf on U, and $\theta_\phi \colon \phi^*\mathcal{F}_v \to \mathcal{F}_u$, satisfying a cocycle condition [31, 12.2] and such that θ_ϕ is an isomorphism for ϕ étale. Following [31, Définition 12.3], we say that a sheaf $\mathcal{F}$ on $\mathcal{X}$ is *cartesian* if θ_ϕ is an isomorphism for all ϕ, or, equivalently, for all ϕ smooth (cf. [34, Lemma 3.8]). We denote by $\mathrm{Sh}_{\mathrm{cart}}(\mathcal{X})$ the full subcategory of $\mathrm{Sh}(\mathcal{X}_{\mathrm{sm}})$ consisting of cartesian sheaves.

Let Λ be a commutative ring. Following [31, Définition 18.1.4], we say, if Λ is noetherian, that a sheaf $\mathcal{F}$ of Λ-modules on $\mathcal{X}$ is *constructible* if $\mathcal{F}$ is cartesian and if $\mathcal{F}_u$ is constructible for some smooth atlas $u \colon U \to \mathcal{X}$, or equivalently, for every smooth atlas $u \colon U \to \mathcal{X}$. We denote by $\mathrm{Mod}_{\mathrm{cart}}(\mathcal{X}, \Lambda)$ (resp. $\mathrm{Mod}_c(\mathcal{X}, \Lambda)$) the full subcategory of $\mathrm{Mod}(\mathcal{X}_{\mathrm{sm}}, \Lambda)$ consisting of cartesian (resp. constructible) sheaves.

We denote by $D_{\mathrm{cart}}(\mathcal{X}, \Lambda)$ (resp. $D_c(\mathcal{X}, \Lambda)$) the full subcategory of $D(\mathcal{X}_{\mathrm{sm}}, \Lambda)$ consisting of complexes with cartesian (resp. constructible) cohomology sheaves. We have $D_c(\mathcal{X}, \Lambda) \subset D_{\mathrm{cart}}(\mathcal{X}, \Lambda)$. We will work exclusively with $D_{\mathrm{cart}}(\mathcal{X}, \Lambda)$ rather than $D(\mathcal{X}_{\mathrm{sm}}, \Lambda)$. We have functors

$$\otimes^L_\Lambda \colon D_{\mathrm{cart}}(\mathcal{X}, \Lambda) \times D_{\mathrm{cart}}(\mathcal{X}, \Lambda) \to D_{\mathrm{cart}}(\mathcal{X}, \Lambda), \quad R\mathcal{H}om \colon D_{\mathrm{cart}}(\mathcal{X}, \Lambda)^{\mathrm{op}} \times D_{\mathrm{cart}}(\mathcal{X}, \Lambda) \to D_{\mathrm{cart}}(\mathcal{X}, \Lambda)$$

defined on unbounded derived categories.

If $\mathcal{X}$ is a Deligne-Mumford stack, we denote by $\mathcal{X}_{\mathrm{et}}$ or simply $\mathcal{X}$ its étale topos. The inclusion of the étale site in the smooth site induces a morphism of topoi $(\epsilon_*, \epsilon^*) \colon \mathcal{X}_{\mathrm{sm}} \to \mathcal{X}_{\mathrm{et}}$. Note that ϵ_* is exact and ϵ^* induces an equivalence from $\mathcal{X}_{\mathrm{et}}$ to $\mathrm{Sh}_{\mathrm{cart}}(\mathcal{X}_{\mathrm{sm}})$. For any commutative ring Λ, ϵ^* induces $D(\mathcal{X}, \Lambda) \xrightarrow{\sim} D_{\mathrm{cart}}(\mathcal{X}, \Lambda)$.

Notation 2.3. Let $f \colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Artin stacks and let Λ be a commutative ring. Although the smooth topos is not functorial, we have a pair of adjoint functors

$$f^* \colon \mathrm{Sh}_{\mathrm{cart}}(\mathcal{Y}) \to \mathrm{Sh}_{\mathrm{cart}}(\mathcal{X}), \quad f_* \colon \mathrm{Sh}_{\mathrm{cart}}(\mathcal{X}) \to \mathrm{Sh}_{\mathrm{cart}}(\mathcal{Y}).$$

and a pair of adjoint functors [32]

$$f^* \colon D_{\mathrm{cart}}(\mathcal{Y}, \Lambda) \to D_{\mathrm{cart}}(\mathcal{X}, \Lambda), \quad Rf_* \colon D_{\mathrm{cart}}(\mathcal{X}, \Lambda) \to D_{\mathrm{cart}}(\mathcal{Y}, \Lambda),$$

where f^* is t-exact and Rf_* is left t-exact for the canonical t-structures. Note that Rf_* is defined on the whole category D_{cart}, not just on D^+_{cart}. For $M, N \in D_{\mathrm{cart}}(\mathcal{Y}, \Lambda)$, we have a natural isomorphism

$$f^*(M \otimes^L_\Lambda N) \xrightarrow{\sim} f^*M \otimes^L_\Lambda f^*N.$$

If f is a surjective morphism, then the functors f^* are conservative and the functor $f^*\colon \mathrm{Sh}_{\mathrm{cart}}(\mathcal{Y}) \to \mathrm{Sh}_{\mathrm{cart}}(\mathcal{X})$ is faithful.

A 2-morphism $\alpha\colon f \to g$ of morphisms of Artin stacks $\mathcal{X} \to \mathcal{Y}$ induces natural isomorphisms $\alpha^*\colon g^* \to f^*$ and $R\alpha_*\colon Rf_* \to Rg_*$. The following squares commute

$$\begin{CD} \mathbf{1}_{D_{\mathrm{cart}}(\mathcal{Y},\Lambda)} @>>> Rf_*f^* \\ @VVV @VV{R\alpha_*}V \\ Rg_*g^* @>{\alpha^*}>> Rg_*f^* \end{CD} \qquad \begin{CD} g^*Rf_* @>{R\alpha_*}>> g^*Rg_* \\ @V{\alpha^*}VV @VVV \\ f^*Rf_* @>>> \mathbf{1}_{D_{\mathrm{cart}}(\mathcal{X},\Lambda)}. \end{CD}$$

Recall that a morphism of Artin stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is *universally submersive* [44, 06U6] if for every morphism of Artin stacks $\mathcal{Y}' \to \mathcal{Y}$, the base change $\mathcal{Y}' \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}'$ is submersive (on the underlying topological spaces).

Proposition 2.4. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Artin stacks. Assume that f is universally submersive (resp. faithfully flat and locally of finite presentation). Then f is of descent (resp. effective descent) for cartesian sheaves.*

Here effective descent means f^* induces an equivalence $\mathrm{Sh}_{\mathrm{cart}}(\mathcal{Y}) \xrightarrow{\sim} \mathrm{DD}(f)$ to the category of descent data, whose objects are cartesian sheaves $\mathcal{F}$ on $\mathcal{X}$ endowed with an isomorphism $p_1^*\mathcal{F} \to p_2^*\mathcal{F}$ satisfying the cocycle condition, where $p_1, p_2\colon \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ are the two projections.

Proof. By general properties of descent [19, Proposition 6.25, Théorème 10.4] and the case of schemes [50, VIII Proposition 9.1] (resp. [50, VIII Théorème 9.4]), it suffices to show that smooth atlases are of effective descent for cartesian sheaves. In other words we may assume f is smooth and $\mathcal{X}$ is an algebraic space. In this case, we construct a quasi-inverse F of $\mathrm{Sh}_{\mathrm{cart}}(\mathcal{Y}) \to \mathrm{DD}(f)$ as follows. Let A be a descent datum for f. For every object $u\colon U \to \mathcal{Y}$ of $\mathrm{Sp}^{\mathrm{sm}}_{/\mathcal{Y}}$, A induces a descent datum A_u for étale sheaves for the base change $f_u\colon \mathcal{X} \times_{\mathcal{Y}} U \to U$ of f by u, and we take $(FA)_u$ to be the corresponding étale sheaf on U. For a morphism $\phi\colon u \to v$ in $\mathrm{Sp}^{\mathrm{sm}}_{/\mathcal{Y}}$, we take $\phi^*(FA)_v \to (FA)_u$ to be the isomorphism induced by the isomorphism of descent data $\phi^*A_v \to A_u$ for étale sheaves for f_u. □

Corollary 2.5. *Let S be an algebraic space, let G be a flat group algebraic S-space locally of finite presentation, and let X be an algebraic space over S, endowed with an action of G. Denote by $\alpha\colon G \times_S X \to X$ the action and by $p\colon G \times_S X \to X$ the projection, and let $f\colon X \to [X/G]$ be the canonical morphism. Then f^* induces an equivalence of categories from* $\mathrm{Mod}_{\mathrm{cart}}([X/G])$ *to the category of pairs $(\mathcal{F}, a)$, where $\mathcal{F} \in \mathrm{Sh}(X)$ and $a\colon \alpha^*\mathcal{F} \to p^*\mathcal{F}$ is a map satisfying the usual cocycle condition.*

Such pairs are called G-equivariant sheaves on X. The cocycle condition implies that $i^*a\colon \mathcal{F} \to \mathcal{F}$ is the identity, where $i\colon X \to G \times_S X$ is the morphism induced by the unit section of G.

Proof. This follows from Proposition 2.4 and the fact that f is faithfully flat of finite presentation. □

Corollary 2.6. *Let S and G be as in Corollary 2.5. Assume that G has* connected *geometric fibers. Let $f\colon S \to BG$ be a morphism corresponding to a G-torsor T on S. Then the functor*

$$f^*\colon \mathrm{Sh}_{\mathrm{cart}}(BG) \to \mathrm{Sh}(S),$$

is an equivalence.

Proof. By Proposition 2.4, since f is faithfully flat locally of finite presentation, f^* induces an equivalence of categories from $\mathrm{Sh}_{\mathrm{cart}}(BG)$ to the category of pairs $(\mathcal{F}, a)$, where $\mathcal{F}$ is a sheaf on S and $a\colon p^*\mathcal{F} \to p^*\mathcal{F}$ is a descent datum with respect to f. As $S \times_{f,BG,f} S$ is the sheaf H on S of G-automorphisms of T, and $p_1 = p_2$ is the projection $p\colon H \to S$, a corresponds to an action of H on $\mathcal{F}$. This action is trivial. Indeed, this can be checked over geometric points $s \to S$, so we may assume that S is the spectrum of an algebraically closed field. In this case, $H \simeq G$. As p^*F is constant and G is connected, and as the restriction of a to the unit section is the identity, a is the identity. □

Remark 2.7. Corollary 2.6 implies that f^* and f_* are quasi-inverse to each other and the natural transformations $\mathrm{id}_{\mathrm{Sh}_{\mathrm{cart}}(BG)} \to f_*f^*$, $f^*f_* \to \mathrm{id}_{\mathrm{Sh}(S)}$ are natural isomorphisms. Since f is a 2-section of the projection $\pi\colon BG \to S$, we get natural isomorphisms

$$\pi_* \simeq \pi_*f_*f^* \simeq f^*, \quad \pi^* \simeq f_*f^*\pi^* \simeq f_*.$$

In particular, we have natural isomorphisms $f_*\pi_* \simeq \mathrm{id}$ and $\pi^*f^* \simeq \mathrm{id}$.

Lemma 2.8. *Let $\mathcal{X}$ be an Artin stack, let Λ be a commutative ring, and let $I \subset \mathbb{Z}$ be an interval. For $M \in D_{\mathrm{cart}}(\mathcal{X}, \Lambda)$, the following conditions are equivalent:*

(a) For every $N \in \mathrm{Mod}_{\mathrm{cart}}(\mathcal{X}, \Lambda)$, $\mathcal{H}^q(M \otimes^L_\Lambda N) = 0$ for all $q \in \mathbb{Z} - I$.

(b) For every finitely presented Λ-module N, $\mathcal{H}^q(M \otimes^L_\Lambda N) = 0$ for all $q \in \mathbb{Z} - I$.

(c) For every geometric point $i\colon x \to X$, i^*M as an element of $D(x, \Lambda)$ is of tor-amplitude contained in I.

If the conditions of the lemma are satisfied, we say M is of *cartesian tor-amplitude* contained in I. If $M \in D_{\mathrm{cart}}(\mathcal{X}, \Lambda)$ has cartesian tor-amplitude contained in $[a, +\infty)$ and $N \in D^{\geq b}_{\mathrm{cart}}(\mathcal{X}, \Lambda)$, then $M \otimes^L_\Lambda N$ is in $D^{\geq a+b}_{\mathrm{cart}}(\mathcal{X}, \Lambda)$.

Proof. Obviously (a) implies (b) and (b) implies (c). Since the family of functors $i^*\colon D_{\mathrm{cart}}(\mathcal{X}, \Lambda) \to D(x, \Lambda)$ is conservative, where i runs through all geometric points of $\mathcal{X}$, (c) implies (a). □

Proposition 2.9 (Projection formula). *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Artin stacks and let Λ be a commutative ring. Let $L \in D_{\mathrm{cart}}(\mathcal{X}, \Lambda)$, and let $K \in D_{\mathrm{cart}}(\mathcal{Y}, \Lambda)$ such that $\mathcal{H}^qK$ is constant for all q. Assume one of the following:*

(a) Λ is noetherian regular and $K \in D^+_c$, $L \in D^+$.

(b) Λ is noetherian and $K \in D^b_c(\Lambda)$ has finite cartesian tor-amplitude.

(c) $Rf_*\colon D_{\mathrm{cart}}(\mathcal{X}, \Lambda) \to D_{\mathrm{cart}}(\mathcal{Y}, \Lambda)$ has finite cohomological amplitude, Λ is noetherian, $K \in D_c$, and either $K, L \in D^-$ or L has finite cartesian tor-amplitude.

(d) f is quasi-compact quasi-separated, Λ is annihilated by an integer invertible on $\mathcal{Y}$, $K \in D^+$, $L \in D^+$, and either Λ is noetherian regular or K has finite cartesian tor-amplitude.

(e) f is quasi-compact quasi-separated, Λ is annihilated by an integer invertible on $\mathcal{Y}$, and $Rf_*\colon D_{\mathrm{cart}}(\mathcal{X}, \Lambda) \to D_{\mathrm{cart}}(\mathcal{Y}, \Lambda)$ has finite cohomological amplitude.

Then the map

$$K \otimes^L_\Lambda Rf_*L \to Rf_*(f^*K \otimes^L_\Lambda L)$$

induced by the composite map

$$f^*(K \otimes^L_\Lambda Rf_*L) \xrightarrow{\sim} f^*K \otimes^L_\Lambda f^*Rf_*L \to f^*K \otimes^L_\Lambda L$$

is an isomorphism.

Proof. In case (a), we may assume that K is a (constant) Λ-module and we are then in case (b). In case (b), we may assume that Λ is local and it then suffices to take a finite resolution of K by finite projective Λ-modules. In the first case of (c), we may assume K is a constant Λ-module. It then suffices to take a resolution of K by finite free Λ-modules. In the second case of (c), we reduce to the first case of (c) using Corollary 2.10 below of the first case of (c). In the first case of (d), we may assume $K \in D^b_c$ and we are in the second case of (d). In the second case of (d), we may assume that K is a flat Λ-module, thus a filtered colimit of finite free Λ-modules. Since R^qf_* commutes with filtered colimits, we are reduced to the trivial case where K is a finite free Λ-module. In case (e), since Rf_* preserves small coproducts, we may assume that $L \in D^-$ and K is represented by a complex in $C^-(\Lambda)$ of flat Λ-modules. We may further assume that $L \in D^b$ and K is a flat Λ-module. We are thus reduced to the second case of (d). □

Corollary 2.10. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Artin stacks and let Λ be a noetherian commutative ring. Assume that the functor $Rf_*\colon D_{\mathrm{cart}}(\mathcal{X}, \Lambda) \to D_{\mathrm{cart}}(\mathcal{Y}, \Lambda)$ has finite cohomological amplitude. Then, for every $L \in D^-_{\mathrm{cart}}(\mathcal{X}, \Lambda)$ of cartesian tor-amplitude contained in $[a, +\infty)$, Rf_*L has cartesian tor-amplitude contained in $[a, +\infty)$.*

Proof. This follows immediately from the first case of Proposition 2.9 (c) and Lemma 2.8. □

The following statement on generic constructibility and generic base change generalizes [34, Theorem 9.10].

Proposition 2.11. *Let $\mathcal{Z}$ be an Artin stack and let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Artin stacks of finite type over $\mathcal{Z}$. Let Λ be a noetherian commutative ring annihilated by an integer invertible on $\mathcal{Z}$, and let $L \in D_c^+(\mathcal{X}, \Lambda)$. Then for every integer i there exists a dense open substack $\mathcal{Z}^\circ$ of $\mathcal{Z}$ such that*

(a) The restriction of $R^i f_* L$ to $\mathcal{Z}^\circ \times_{\mathcal{Z}} \mathcal{Y} \subset \mathcal{Y}$ is constructible.

(b) $R^i f_* L$ is compatible with arbitrary base change of Artin stacks $\mathcal{Z}' \to \mathcal{Z}^\circ \subset \mathcal{Z}$.

Proof. Recall first that for any 2-commutative diagram of Artin stacks of the form

$$\begin{array}{ccccc} \mathcal{X}'' & \xrightarrow{h'} & \mathcal{X}' & \xrightarrow{g'} & \mathcal{X} \\ {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}'' & \xrightarrow{h} & \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

the following diagram commutes:

(2.11.1)
$$\begin{array}{ccccc} (gh)^* Rf_* L & & \xrightarrow{\quad b_{gh} \quad} & & Rf''_*(g'h')^* L \\ {\scriptstyle \simeq}\downarrow & & & & \downarrow{\scriptstyle \simeq} \\ h^* g^* Rf_* L & \xrightarrow{h^* b_g} & h^* Rf'_* g'^* L & \xrightarrow{b_h} & Rf''_* h'^* g'^* L \end{array}$$

where b_{gh}, b_g, b_h are base change maps.

If $\mathcal{Z}$ is a scheme, then, as in [34, Theorem 9.10], cohomological descent and the case of schemes [11, Th. finitude 1.9] imply that there exists a dense open subscheme $\mathcal{Z}^\circ$ of $\mathcal{Z}$ such that (a) holds and that $R^i f_* L$ is compatible with arbitrary base change of schemes $Z' \to \mathcal{Z}^\circ \subset \mathcal{Z}$. This implies (b). In fact, for any base change of Artin stacks $g\colon \mathcal{Z}' \to \mathcal{Z}^\circ \subset \mathcal{Z}$, take a smooth atlas $p\colon Z' \to \mathcal{Z}'$ where Z' is a scheme. Then b_p is an isomorphism and b_{gp} is an isomorphism by assumption. It follows that $p^* b_g$ and hence b_g are isomorphisms.

In the general case, let $p\colon Z \to \mathcal{Z}$ be a smooth atlas. By the preceding case, there exists a dense open subscheme $Z^\circ \subset Z$ such that after forming the 2-commutative diagram with 2-cartesian squares

$$\begin{array}{ccccc} \mathcal{X}_Z & \xrightarrow{f_Z} & \mathcal{Y}_Z & \longrightarrow & Z \\ {\scriptstyle p_{\mathcal{X}}}\downarrow & & \downarrow{\scriptstyle p_{\mathcal{Y}}} & & \downarrow{\scriptstyle p} \\ \mathcal{X} & \xrightarrow{f} & \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

the restriction of $R^i f_{Z*} p_{\mathcal{X}}^* L$ to $Z^\circ \times_Z \mathcal{Y}_Z$ is constructible and that $R^i f_{Z*} p_{\mathcal{X}}^* L$ commutes with arbitrary base change of Artin stacks $\mathcal{W} \to Z^\circ \subset Z$. We claim that $\mathcal{Z}^\circ = p(Z^\circ)$ satisfies (a) and (b). To see this, let $p^\circ\colon Z^\circ \to \mathcal{Z}^\circ$ be the restriction of p. By definition p° is surjective. Then (a) follows from the fact that

$$p_{\mathcal{Y}}^{\circ *}(R^i f_* L | \mathcal{Z}^\circ \times_{\mathcal{Z}} \mathcal{Y}) \simeq R^i f_{Z*} p_{\mathcal{X}}^* L | Z^\circ \times_Z \mathcal{Y}_Z$$

is constructible. For any base change of Artin stacks $\mathcal{Z}' \to \mathcal{Z}^\circ$, form the following 2-cartesian square:

$$\begin{array}{ccc} Z' & \xrightarrow{h} & Z^\circ \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p^\circ} \\ \mathcal{Z}' & \xrightarrow{g} & \mathcal{Z}^\circ. \end{array}$$

By (2.11.1), $b_{p'}(p'^* b_g)$ can be identified with $b_h(h^* b_{p^\circ})$. Since p° and p' are smooth, b_{p° and $b_{p'}$ are isomorphisms. By the construction of p°, b_h is an isomorphism. It follows that $p'^* b_g$ and hence b_g are isomorphisms. □

Remark 2.12. For $\mathcal{Z} = BG$, where G is an algebraic group over a field k, $f\colon \mathcal{X} \to \mathcal{Y}$ a quasi-compact and quasi-separated morphism of Artin stacks over $\mathcal{Z}$, and Λ is a commutative ring annihilated by an integer invertible in k, the above proof combined with the remark following [11, Th. finitude 1.9] shows that $Rf_*\colon D^+_{\mathrm{cart}}(\mathcal{X}, \Lambda) \to D^+_{\mathrm{cart}}(\mathcal{Y}, \Lambda)$ commutes with arbitrary base change of Artin stacks $\mathcal{Z}' \to \mathcal{Z}$.

3 Multiplicative structures in derived categories

Definition 3.1. For us, a $\otimes$-*category* is a symmetric monoidal category [33, Section VII.7], that is, a category $\mathcal{T}$ endowed with a bifunctor $\otimes\colon \mathcal{T}\times\mathcal{T} \to \mathcal{T}$, a unit object $\mathbf{1}$ and functorial isomorphisms

$$a_{LMN}\colon L \otimes (M \otimes N) \to (L \otimes M) \otimes N,$$
$$c_{MN}\colon M \otimes N \to N \otimes M,$$
$$u_M\colon M \otimes \mathbf{1} \to M, \quad v_M\colon \mathbf{1} \otimes M \to M,$$

satisfying the axioms of *loc. cit.*. We define a *pseudo-ring* in $\mathcal{T}$ to be an object K of $\mathcal{T}$ endowed with a morphism $m\colon K \otimes K \to K$ such that the following associativity diagram commutes:

$$\begin{array}{ccccc} K \otimes (K \otimes K) & \xrightarrow{\mathrm{id}_K \otimes m} & K \otimes K & & \\ \downarrow{\scriptstyle a_{KKK}} & & & \searrow{\scriptstyle m} & \\ (K \otimes K) \otimes K & \xrightarrow{m \otimes \mathrm{id}_K} & K \otimes K & \xrightarrow{m} & K. \end{array}$$

A pseudo-ring (K, m) is called *commutative* if the following diagram commutes

$$\begin{array}{ccc} K \otimes K & & \\ \downarrow{\scriptstyle c_{KK}} & \searrow{\scriptstyle m} & \\ K \otimes K & \xrightarrow{m} & K. \end{array}$$

A *homomorphism of pseudo-rings* $(K, m) \to (K', m')$ is a morphism $f\colon K \to K'$ of $\mathcal{T}$ such that the following diagram commutes:

$$\begin{array}{ccc} K \otimes K & \xrightarrow{m} & K \\ \downarrow{\scriptstyle f \otimes f} & & \downarrow{\scriptstyle f} \\ K' \otimes K' & \xrightarrow{m'} & K'. \end{array}$$

We define a *left* (K, m)*-pseudomodule* to be an object M of $\mathcal{T}$ endowed with a morphism $n\colon K \otimes M \to M$ such that the following diagram commutes

$$\begin{array}{ccccc} K \otimes (K \otimes M) & \xrightarrow{\mathrm{id}_K \otimes n} & K \otimes M & & \\ \downarrow{\scriptstyle a_{KKM}} & & & \searrow{\scriptstyle n} & \\ (K \otimes K) \otimes M & \xrightarrow{m \otimes \mathrm{id}_M} & K \otimes M & \xrightarrow{n} & M. \end{array}$$

A *homomorphism of left* (K, m)*-pseudomodules* $(M, n) \to (M', n')$ is a morphism $h\colon M \to M'$ of $\mathcal{T}$ such that the following diagram commutes

$$\begin{array}{ccc} K \otimes M & \xrightarrow{n} & M \\ \downarrow{\scriptstyle \mathrm{id}_K \otimes h} & & \downarrow{\scriptstyle h} \\ K \otimes M' & \xrightarrow{n'} & M'. \end{array}$$

Definition 3.2. Let $f\colon (K,m) \to (K',m')$ be a homomorphism of pseudo-rings. We define a *splitting of* f to be a morphism $n\colon K'\otimes K \to K$, making K into a (K',m')-pseudomodule and such that the following diagram commutes

$$\begin{array}{ccccc}
K\otimes K & \xrightarrow{f\otimes \mathrm{id}_K} & K'\otimes K & \xrightarrow{\mathrm{id}_{K'}\otimes f} & K'\otimes K' \\
 & \searrow^{m} & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle m'} \\
 & & K & \xrightarrow{f} & K'.
\end{array}$$

Definition 3.3. We define a *ring* in $\mathcal{T}$ to be a pseudo-ring (K,m) in $\mathcal{T}$ endowed with a morphism $e\colon \mathbf{1} \to K$ such that the following diagrams commute:

$$\begin{array}{ccc}
K\otimes \mathbf{1} & \xrightarrow{\mathrm{id}_K\otimes e} & K\otimes K \\
 & \searrow^{u_K} & \downarrow{\scriptstyle m} \\
 & & K
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{1}\otimes K & \xrightarrow{e\otimes \mathrm{id}_K} & K\otimes K \\
 & \searrow^{v_K} & \downarrow{\scriptstyle m} \\
 & & K.
\end{array}$$

(Thus a ring in our sense is a "monoid" in the terminology of [33, Section VII.3].) The unit $\mathbf{1}$ endowed with $u_{\mathbf{1}}\colon \mathbf{1}\otimes\mathbf{1} \to \mathbf{1}$ and $\mathrm{id}_{\mathbf{1}}\colon \mathbf{1}\to\mathbf{1}$ is a commutative ring in $\mathcal{T}$. A *ring homomorphism* $(K,m,e) \to (K',m',e')$ is a homomorphism of pseudo-rings $f\colon (K,m) \to (K',m')$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{1} & \xrightarrow{e} & K \\
 & \searrow^{e'} & \downarrow{\scriptstyle f} \\
 & & K'.
\end{array}$$

A *left* (K,m,e)*-module* is a left (K,m)-pseudomodule (M,n) such that the following diagram commutes

$$\begin{array}{ccc}
\mathbf{1}\otimes M & \xrightarrow{e\otimes M} & K\otimes M \\
 & \searrow^{v_M} & \downarrow{\scriptstyle n} \\
 & & M.
\end{array}$$

A *homomorphism of left* (K,m,e)*-modules* $(M,n) \to (M',n')$ is a homomorphism between the underlying left (K,m)-pseudomodules.

Construction 3.4. Let $\mathcal{T} = (\mathcal{T},\otimes,a,c,u,v)$ and $\mathcal{T}' = (\mathcal{T}',\otimes,a',c',u',v')$ be $\otimes$-categories, and let $\omega\colon \mathcal{T} \to \mathcal{T}'$ be a functor. A *left-lax* $\otimes$*-structure* on ω is a natural transformation of functors $\mathcal{T}\times\mathcal{T} \to \mathcal{T}'$ consisting of morphisms of $\mathcal{T}'$

$$o_{MN}\colon \omega(M\otimes N) \to \omega(M)\otimes\omega(N),$$

such that the following diagrams commute:

$$\begin{array}{ccccc}
\omega(L\otimes(M\otimes N)) & \xrightarrow{o_{L,M\otimes N}} & \omega(L)\otimes\omega(M\otimes N) & \xrightarrow{\omega(L)\otimes o_{MN}} & \omega(L)\otimes(\omega(M)\otimes\omega(N)) \\
\downarrow{\scriptstyle \omega(a_{LMN})} & & & & \downarrow{\scriptstyle a'_{\omega(L)\omega(M)\omega(N)}} \\
\omega((L\otimes M)\otimes N) & \xrightarrow{o_{L\otimes M,N}} & \omega(L\otimes M)\otimes\omega(N) & \xrightarrow{o_{LM}\otimes\omega(N)} & (\omega(L)\otimes\omega(M))\otimes\omega(N)
\end{array}$$

$$\begin{array}{ccc}
\omega(M\otimes N) & \xrightarrow{o_{MN}} & \omega(M)\otimes\omega(N) \\
\downarrow{\scriptstyle \omega(c_{MN})} & & \downarrow{\scriptstyle c'_{\omega(M)\omega(N)}} \\
\omega(N\otimes M) & \xrightarrow{o_{NM}} & \omega(N)\otimes\omega(M).
\end{array}$$

A *right-lax* $\otimes$*-structure* on ω is a left-lax $\otimes$-structure on $\omega^{\mathrm{op}}\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{T}'^{\mathrm{op}}$. It is given by functorial morphisms

$$t_{MN}\colon \omega(M)\otimes\omega(N) \to \omega(M\otimes N),$$

such that the above diagrams with arrows o inverted and replaced by t commute. A $\otimes$-*structure* on ω is a left-lax $\otimes$-structure o such that o_{MN} is an isomorphism for all M and N. In this case $t_{MN} = o_{MN}^{-1}$ defines a right-lax $\otimes$-structure. If t is a right-lax $\otimes$-structure on ω and (K, m) is a pseudo-ring in $\mathcal{T}$, we endow $\omega(K)$ with the pseudo-ring structure

$$\omega(K) \otimes \omega(K) \xrightarrow{t_{KK}} \omega(K \otimes K) \xrightarrow{\omega(m)} \omega(K).$$

If, moreover, (M, n) is a left (K, m)-pseudomodule, we endow $\omega(M)$ with the left $\omega(K)$-pseudomodule structure

$$\omega(K) \otimes \omega(M) \xrightarrow{t_{KM}} \omega(K \otimes M) \xrightarrow{\omega(n)} \omega(M).$$

If (K, m) is commutative, then $\omega(K)$ is commutative. This construction sends homomorphisms of pseudo-rings to homomorphisms of pseudo-rings and homomorphisms of left pseudomodules to homomorphisms of left pseudomodules.

If (ω, t), (ω', t') are functors endowed with right-lax $\otimes$-structures, we say that a natural transformation $\alpha\colon \omega \to \omega'$ preserves the right-lax $\otimes$-structures if the following diagram commutes

$$\begin{array}{ccc} \omega(M) \otimes \omega(N) & \xrightarrow{t_{MN}} & \omega(M \otimes N) \\ \downarrow{\scriptstyle \alpha_M \otimes \alpha_N} & & \downarrow{\scriptstyle \alpha_{M \otimes N}} \\ \omega'(M) \otimes \omega'(N) & \xrightarrow{t'_{MN}} & \omega'(M \otimes N). \end{array}$$

In this case, for any pseudo-ring K in $\mathcal{T}$, $\alpha_K\colon \omega(K) \to \omega'(K)$ is a homomorphism of pseudo-rings.

Construction 3.5. Now suppose that $\omega\colon \mathcal{T} \to \mathcal{T}'$ admits a right adjoint $\tau\colon \mathcal{T}' \to \mathcal{T}$. For any left-lax $\otimes$-structure o on ω, endow τ with the right-lax $\otimes$-structure t such that $t_{MN}\colon \tau(M) \otimes \tau(N) \to \tau(M \otimes N)$ is adjoint to the composition

$$\omega(\tau(M) \otimes \tau(N)) \xrightarrow{o_{\tau(M)\tau(N)}} \omega(\tau(M)) \otimes \omega(\tau(N)) \xrightarrow{\alpha_M \otimes \alpha_N} M \otimes N,$$

where $\alpha_M\colon \omega(\tau(M)) \to M$, $\alpha_N\colon \omega(\tau(N)) \to N$ are adjunction morphisms. It is straightforward to check that this construction defines a bijection from the set of left-lax $\otimes$-structures on ω to the set of right-lax $\otimes$-structures on τ.

In the above construction, if o is a $\otimes$-structure on ω, then the adjunction morphisms $\alpha\colon \omega\tau \to \mathrm{id}_{\mathcal{T}'}$ and $\beta\colon \mathrm{id}_{\mathcal{T}} \to \tau\omega$ preserve the resulting right-lax $\otimes$-structures.

Construction 3.6. This formalism has a unital variant. A *left-lax unital* $\otimes$-*structure* on a functor $\omega\colon \mathcal{T} \to \mathcal{T}'$ is a left-lax $\otimes$-structure endowed with a morphism $p\colon \omega(\mathbf{1}) \to \mathbf{1}'$ in $\mathcal{T}'$ such that the following diagrams commute

$$\begin{array}{ccc} \omega(M \otimes \mathbf{1}) & \xrightarrow{o_{M\mathbf{1}}} & \omega(M) \otimes \omega(\mathbf{1}) \\ \downarrow{\scriptstyle \omega(u_M)} & & \downarrow{\scriptstyle \mathrm{id}_{\omega(M)} \otimes p} \\ \omega(M) & \xleftarrow[u'_{\omega(M)}]{} & \omega(M) \otimes \mathbf{1}' \end{array} \qquad \begin{array}{ccc} \omega(\mathbf{1} \otimes M) & \xrightarrow{o_{\mathbf{1}M}} & \omega(\mathbf{1}) \otimes \omega(M) \\ \downarrow{\scriptstyle \omega(v_M)} & & \downarrow{\scriptstyle p \otimes \mathrm{id}_{\omega(M)}} \\ \omega(M) & \xleftarrow[v'_{\omega(M)}]{} & \mathbf{1}' \otimes \omega(M) \end{array}$$

A *right-lax unital* $\otimes$-*structure* is a left-lax unital $\otimes$-structure on $\omega^{\mathrm{op}}\colon \mathcal{T}^{\mathrm{op}} \to \mathcal{T}'^{\mathrm{op}}$. It consists of a right-lax $\otimes$-structure endowed with a morphism $s\colon \mathbf{1}' \to \omega(\mathbf{1})$ in $\mathcal{T}'$ such that the above diagrams, with arrows o inverted and replaced by t, arrows p inverted and replaced by s, commute. A *unital* $\otimes$-*structure* is a left-lax unital $\otimes$-structure (o, p) such that o is a $\otimes$-structure and p is invertible. Constructions 3.4 and 3.5 can be carried over to the unital case.

Let $\mathcal{T}$ be a $\otimes$-category, and let $\mathcal{C}$ be a category. Then the category $\mathcal{T}^{\mathcal{C}}$ of functors $\mathcal{C} \to \mathcal{T}$ has a natural $\otimes$-structure. The constant functor $\mathcal{T} \to \mathcal{T}^{\mathcal{C}}$ defined by $M \mapsto (M)_{\mathcal{C}}$ has a natural unital $\otimes$-structure.

Construction 3.7. Let $X = (X, \mathcal{O}_X)$ be a commutatively ringed topos. Two $\otimes$-categories will be of interest to us:

(a) The (unbounded) derived category $D(X) = D(X, \mathcal{O}_X)$, equipped with $\otimes^L_{\mathcal{O}_X} \colon D(X)\times D(X) \to D(X)$ [27, Theorem 18.6.4].
(b) The category $\mathrm{GrMod}(X) = \mathrm{GrMod}(X, \mathcal{O}_X)$ of graded $\mathcal{O}_X$-modules $H = \bigoplus_{n\in\mathbb{Z}} H^n$, with $\otimes$ given by $(H \otimes K)^n = \bigoplus_{i+j=n} H^i \otimes_{\mathcal{O}_X} K^j$, the isomorphism $c\colon H \otimes K \to K \otimes H$ being given by the usual sign rule.

The cohomology functor

$$\mathcal{H}^*\colon D(X) \to \mathrm{GrMod}(X)$$

has a natural right-lax unital $\otimes$-structure given by the canonical maps $\mathcal{H}^*L\otimes\mathcal{H}^*M \to \mathcal{H}^*(L\otimes^L M)$. (This is a unital $\otimes$-structure when $\mathcal{O}_X$ is a constant field, which is the case we are mostly interested in).

Let $f\colon X = (X, \mathcal{O}_X) \to Y = (Y, \mathcal{O}_Y)$ be a morphism of commutatively ringed topoi. We endow $f^*\colon \mathrm{GrMod}(Y) \to \mathrm{GrMod}(X)$ with the unital $\otimes$-structure defined by the functorial isomorphisms

$$f^*(M \otimes N) \to f^*M \otimes f^*N, \quad f^*\mathcal{O}_Y \to \mathcal{O}_X.$$

We endow $Lf^*\colon D(Y) \to D(X)$ [27, Theorem 18.6.9] with the unital $\otimes$-structure defined by the functorial isomorphisms

$$Lf^*(M \otimes^L N) \to Lf^*M \otimes^L f^*N, \quad Lf^*\mathcal{O}_Y \to \mathcal{O}_X.$$

We endow the right adjoint functors $f_*\colon \mathrm{GrMod}(X) \to \mathrm{GrMod}(Y)$ and $Rf_*\colon D(X) \to D(Y)$ with the induced right-lax unital $\otimes$-structures.

Construction 3.8. Let $\mathcal{X}$ be an Artin stack, and let Λ be a commutative ring. We consider the $\otimes$-categories $D_{\mathrm{cart}}(\mathcal{X}, \Lambda)$ and $\mathrm{GrMod}_{\mathrm{cart}}(\mathcal{X}, \Lambda)$, the category of graded cartesian sheaves of Λ-modules.

Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Artin stacks. As in Construction 3.7, we endow the functors $f^*\colon \mathrm{GrMod}_{\mathrm{cart}}(\mathcal{Y}, \Lambda) \to \mathrm{GrMod}_{\mathrm{cart}}(\mathcal{X}, \Lambda)$ and $f^*\colon D_{\mathrm{cart}}(\mathcal{Y}, \Lambda) \to D_{\mathrm{cart}}(\mathcal{X}, \Lambda)$ with the natural unital $\otimes$-structures. We endow the right adjoint functors $f_*\colon \mathrm{GrMod}_{\mathrm{cart}}(\mathcal{Y}, \Lambda) \to \mathrm{GrMod}_{\mathrm{cart}}(\mathcal{X}, \Lambda)$ and $Rf_*\colon D_{\mathrm{cart}}(\mathcal{X}, \Lambda) \to D_{\mathrm{cart}}(\mathcal{Y}, \Lambda)$ with the induced right-lax unital $\otimes$-structures.

Assume that Λ is annihilated by an integer n invertible on $\mathcal{Y}$ and f is locally of finite presentation. Then we have $Rf^!\colon D_{\mathrm{cart}}(\mathcal{Y}, \Lambda) \to D_{\mathrm{cart}}(\mathcal{X}, \Lambda)$. As in [11, Cycle (1.2.2.3)], for M and N in $D_{\mathrm{cart}}(\mathcal{Y}, \Lambda)$, we have a morphism

$$f^*M \otimes^L Rf^!N \to Rf^!(M \otimes^L N)$$

given by the morphism $Rf^!N \to Rf^!R\mathcal{H}om(M, M \otimes^L N) \simeq R\mathcal{H}om(f^*M, Rf^!(M \otimes^L N))$. For a pseudo-ring (L, m) in $D_{\mathrm{cart}}(\mathcal{Y}, \Lambda)$, we endow $Rf^!L$ with the left f^*L-pseudomodule structure given by the composition

$$f^*L \otimes^L Rf^!L \to Rf^!(L \otimes^L L) \xrightarrow{Rf^!m} Rf^!L$$

Assume moreover that $f = i$ is a closed immersion. Then the right-lax $\otimes$-structure on $i_* = Ri_*$ is an isomorphism and its inverse is a $\otimes$-structure consisting of a functorial isomorphism

$$i_*(M \otimes^L N) \to i_*M \otimes^L i_*N.$$

We endow the right adjoint functor $Ri^!$ of i_* with the induced right-lax $\otimes$-structure. Note that the right *unital* $\otimes$-structure on i_* is not invertible in general. For a pseudo-ring (L, m) in $D_{\mathrm{cart}}(\mathcal{Y}, \Lambda)$, the above left i^*L-pseudomodule structure on $Ri^!L$ is a splitting of the homomorphism of pseudo-rings $Ri^!L \to i^*L$ (Definition 3.2).

In the rest of this section, we discuss multiplicative structures on spectral objects. We will only consider spectral objects of type $\tilde{\mathbb{Z}}$, where $\tilde{\mathbb{Z}}$ is the category associated to the ordered set $\mathbb{Z}\cup\{\pm\infty\}$.

Definition 3.9. Let $\mathcal{T}$ be category endowed with a bifunctor $\otimes\colon \mathcal{T}\times\mathcal{T} \to \mathcal{T}$. Let J be a category endowed with a bifunctor $*\colon J \times J \to J$. Let X, X', X'' be functors $J \to \mathcal{T}$. A *pairing* from X,X' to X'' is a natural transformation of functors $J \times J \to \mathcal{T}$ consisting of morphisms of $\mathcal{T}$

$$X(j) \otimes X'(j') \to X''(j * j').$$

Assume moreover that $(\mathcal{T}, \otimes)$ and $(J, *)$ are endowed with structures of $\otimes$-categories. A pairing from X, X to X is called *associative* if for $j, j', j'' \in J$, the following diagram commutes

$$\begin{array}{ccccc} X(j) \otimes (X(j') \otimes X(j'')) & \longrightarrow & X(j) \otimes X(j' * j'') & \longrightarrow & X(j * (j' * j'')) \\ \downarrow{\scriptstyle a} & & & & \downarrow{\scriptstyle a} \\ (X(j) \otimes X(j')) \otimes X(j'') & \longrightarrow & X(j * j') \otimes X(j'') & \longrightarrow & X((j * j') * j''), \end{array}$$

and is called *commutative* if for $j, j' \in J$, the following diagram commutes

$$\begin{array}{ccc} X(j) \otimes X(j') & \longrightarrow & X(j * j') \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ X(j') \otimes X(j) & \longrightarrow & X(j' * j). \end{array} \tag{3.9.1}$$

Assume moreover that $\mathcal{T}$ is additive and $\otimes$ is an additive bifunctor. Let $\mathfrak{S}$ be the $\otimes$-category given by the discrete category $\{\pm 1\}$ and the ordinary product. Let $\sigma\colon J \to \mathfrak{S}$ be a $\otimes$-functor. A pairing from X, X to X is called *σ-commutative* if for $j, j' \in J$, the diagram (3.9.1) is $\max\{\sigma(j), \sigma(j')\}$-commutative.

Construction 3.10. Let $\mathrm{Ar}(\tilde{\mathbb{Z}})$ be the category of morphisms of $\tilde{\mathbb{Z}} = \mathbb{Z} \cup \{\pm\infty\}$. We represent objects of $\mathrm{Ar}(\tilde{\mathbb{Z}})$ by pairs (p, q), $p, q \in \tilde{\mathbb{Z}}$, $p \le q$. We endow $\mathrm{Ar}(\tilde{\mathbb{Z}})$ with a structure of $\otimes$-category by the formula

$$(p, q) * (p', q') = (\max\{p + q' - 1, p' + q - 1\}, q + q' - 1).$$

Here we adopt the convention that $(-\infty) + (+\infty) = -\infty = (+\infty) + (-\infty)$.

Definition 3.11. Let $\mathcal{D}$ be a triangulated category endowed with a triangulated bifunctor $\otimes\colon \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ [27, Definition 10.3.6]. Let (X, δ), (X', δ), (X'', δ'') be spectral objects with values in $\mathcal{D}$ [46, II 4.1.2]. A *pairing* from (X, δ), (X', δ') to (X'', δ'') consists of a pairing from X, X' to X'', namely a natural transformation of functors $\mathrm{Ar}(\tilde{\mathbb{Z}}) \times \mathrm{Ar}(\tilde{\mathbb{Z}}) \to \mathcal{D}$ consisting of morphisms of $\mathcal{D}$

$$X(p, q) \otimes X'(p', q') \to X''((p, q) * (p', q')),$$

such that for $p \le q \le r$, $p' \le q' \le r'$ in $\tilde{\mathbb{Z}}$ satisfying $q + r' = q' + r$ and $p + r' = p' + r$, the diagram

$$\begin{array}{ccc} X(q, r) \otimes X'(q', r') & \longrightarrow & X''(q'', r'') \\ \downarrow{\scriptstyle (\delta \otimes \mathrm{id}, \mathrm{id} \otimes \delta')} & & \downarrow{\scriptstyle \delta''} \\ (X(p, q)[1] \otimes X'(q', r')) \oplus (X(q, r) \otimes X'(p', q')[1]) & \longrightarrow & X''(p'', q'')[1] \end{array}$$

commutes. Here $(q'', r'') = (q, r) * (q', r')$, $(p'', q'') = (p, q) * (q', r') = (q, r) * (p', q')$.

Assume moreover that $(\mathcal{D}, \otimes)$ is endowed with a structure of $\otimes$-category[3]. A pairing from (X, δ), (X, δ) to (X, δ) is called *associative* (resp. *commutative*) if the underlying pairing from X, X to X is.

Example 3.12. Let X be a commutatively ringed topos, and let $K, K', K'' \in D(X)$. We consider the second spectral object (K, δ) associated to K [46, III 4.3.1, 4.3.4], with $K(p, q) = \tau^{[p, q-1]} K$, where $\tau^{[p, q-1]}$ is the canonical truncation functor. Similarly, we have spectral objects (K', δ'), (K'', δ''). A map $K \otimes^L K' \to K''$ in $D(X)$ defines a pairing from (K, δ), (K', δ') to (K'', δ'') given by

$$\begin{aligned} \tau^{[p, q-1]} K \otimes^L \tau^{[p', q'-1]} K' \to \tau^{\ge p''} (\tau^{[p, q-1]} K \otimes^L \tau^{[p', q'-1]} K') \simeq \tau^{\ge p''} (\tau^{\le q-1} K \otimes^L \tau^{\le q'-1} K') \\ \xrightarrow{\alpha} \tau^{[p'', q''-1]} (K \otimes^L K') \to \tau^{[p'', q''-1]} K'', \end{aligned}$$

[3] Here we do not assume that the constraints of the $\otimes$-category are natural transformations of triangulated functors [27, Definition 10.1.9 (ii)] in each variable.

where $(p'',q'')=(p,q)*(p',q')$, α is given by the map $\tau^{\leq q-1}K\otimes^L\tau^{\leq q'-1}K'\to\tau^{\leq q''-1}(K\otimes^L K'')$ induced by adjunction from the map $\tau^{\leq q-1}K\otimes^L\tau^{\leq q'-1}K'\to K\otimes^L K'$. Moreover, if K is a pseudo-ring (resp. commutative pseudo-ring), then the induced pairing from (K,δ), (K,δ) to (K,δ) is associative (resp. commutative).

The above also holds with $D(X)$ replaced by $D_{\mathrm{cart}}(\mathcal{X},\Lambda)$, where $\mathcal{X}$ is an Artin stack and Λ is a commutative ring.

Definition 3.13. Let $\mathcal{A}$ be an abelian category endowed with an additive bifunctor $\otimes\colon\mathcal{A}\times\mathcal{A}\to\mathcal{A}$. Let (H^n,δ), (H'^n,δ'), (H''^n,δ'') be spectral objects with values in $\mathcal{A}$ [46, II 4.1.4]. A *pairing* from (H^n,δ), (H'^n,δ') to (H''^n,δ'') consists of a pairing from H^*, H'^* to H''^*, namely a natural transformation of functors $(\mathbb{Z}\times\mathrm{Ar}(\tilde{\mathbb{Z}}))\times(\mathbb{Z}\times\mathrm{Ar}(\tilde{\mathbb{Z}}))\to\mathcal{A}$ consisting of morphisms of $\mathcal{A}$

$$H^n(p,q)\otimes H'^{n'}(p',q')\to H''^{n+n'}((p,q)*(p',q')),$$

such that for $p\leq q\leq r$, $p'\leq q'\leq r'$ in $\tilde{\mathbb{Z}}$ satisfying $q+r'=q'+r$ and $p+r'=p'+r$, the diagram

$$\begin{array}{ccc} H^n(q,r)\otimes H'^{n'}(q',r') & \longrightarrow & H''^{n+n'}(q'',r'') \\ \downarrow{\scriptstyle(\delta\otimes\mathrm{id},(-1)^n\mathrm{id}\otimes\delta')} & & \downarrow{\scriptstyle\delta''} \\ (H^{n+1}(p,q)\otimes H'^{n'}(q',r'))\oplus(H^n(q,r)\otimes H'^{n'+1}(p',q')) & \longrightarrow & H''^{n+n'+1}(p'',q'') \end{array}$$

commutes. Here $(q'',r'')=(q,r)*(q',r')$, $(p'',q'')=(p,q)*(q',r')=(q,r)*(p',q')$. Note that if (H^n,δ), (H'^n,δ'), and (H''^n,δ'') are stationary [46, II 4.4.2], then the pairing from H^*, H'^* to H''^* is uniquely determined by the pairing from $H^*\mid\mathrm{Ar}^-$, $H'^*\mid\mathrm{Ar}^-$ to $H''^*\mid\mathrm{Ar}^-$, where $\mathrm{Ar}^-=\mathrm{Ar}(\mathbb{Z}\cup\{-\infty\})$. In fact, in this case, for every n, there exists an integer $u(n)$ such that for every $q\geq u(n)$, the morphism $H^n(-\infty,q)\to H^n(-\infty,+\infty)$ is an isomorphism.

Consider the induced spectral sequences $(E_2^{pq}\Rightarrow H^n)$, $(E_2'^{pq}\Rightarrow H'^n)$, $(E_2''^{pq}\Rightarrow H''^n)$ given by [46, II (4.3.3.2)]. A pairing from (H^n,δ), (H'^n,δ') to (H''^n,δ'') induces compatible pairings of differential bigraded objects of $\mathcal{A}$

$$E_r^{pq}\otimes E_r'^{p'q'}\to E_r''^{p+p',q+q'}$$

for $2\leq r\leq\infty$ (satisfying $d_r''(xy)=d_r(x)y+(-1)^{p+q}xd_r'(y)$ for $x\in E_r^{pq}$, $y\in E_r'^{p'q'}$) and a pairing of filtered[4] graded objects of $\mathcal{A}$

$$F^pH^n\otimes F^{p'}H'^{n'}\to F^{p+p'}H''^{n+n'},$$

compatible with the pairing on E_∞.

Assume moreover that $(\mathcal{A},\otimes)$ is endowed with a structure of $\otimes$-category. A pairing from (H^n,δ), (H^n,δ) to (H^n,δ) is called *associative* (resp. *commutative*) if the underlying pairing from H^*, H^* to H^* is associative (resp. σ-commutative, where $\sigma\colon\mathbb{Z}\times\mathrm{Ar}(\tilde{\mathbb{Z}})\to\mathfrak{S}$ is given by $(n,(p,q))\mapsto(-1)^n$). An associative (resp. commutative) pairing from (H^n,δ), (H^n,δ) to (H^n,δ) induces associative (resp. commutative) pairings on E_r^{pq} and F^pH^n. Here the commutativity for E_r^{pq} and F^pH^n are relative to the functors $\mathbb{Z}\times\mathbb{Z}\to\mathfrak{S}$ given by $(p,q)\mapsto(-1)^{p+q}$ and $(p,n)\mapsto(-1)^n$, respectively.

Remark 3.14. Let $\mathcal{D}$, $\mathcal{D}'$ be triangulated categories endowed with triangulated bifunctors $\otimes\colon\mathcal{D}\times\mathcal{D}\to\mathcal{D}$, $\otimes\colon\mathcal{D}'\times\mathcal{D}'\to\mathcal{D}'$. Let $\tau\colon\mathcal{D}\to\mathcal{D}'$ be a triangulated functor endowed with a natural transformation of functors $\mathcal{D}\times\mathcal{D}\to\mathcal{D}'$ consisting of morphisms $\tau(M)\otimes\tau(N)\to\tau(M\otimes N)$ of $\mathcal{D}'$ that is a natural transformation of triangulated functors in each variable. Let (X,δ), (X',δ'), (X'',δ'') be spectral objects with values in $\mathcal{D}$. Then a pairing from (X,δ), (X',δ') to (X'',δ'') induces a pairing from $\tau(X,\delta)$, $\tau(X',\delta')$ to $\tau(X'',\delta'')$. If $(\mathcal{D},\otimes)$, $(\mathcal{D}',\otimes)$ are endowed with structures of $\otimes$-categories and τ is a right-lax $\otimes$-functor (Construction 3.4), then an associative (resp. commutative) pairing from (X,δ), (X,δ) to (X,δ) induces an associative (resp. commutative) pairing from $\tau(X,\delta)$, $\tau(X,\delta)$ to $\tau(X,\delta)$.

Similarly, let $\mathcal{A}$ be an abelian category endowed with an additive bifunctor $\otimes\colon\mathcal{A}\times\mathcal{A}\to\mathcal{A}$ and let $H\colon\mathcal{D}\to\mathcal{A}$ be a cohomological functor endowed with a natural transformation of functors

[4] For the filtration, we use the convention $F^pH^n=\mathrm{Im}(H^n(-\infty,n-p+1)\to H^n(-\infty,\infty))$. In particular, in Example 3.15 below, $F^pH^n(X,K)=\mathrm{Im}(H^n(X,\tau^{\leq n-p}K)\to H^n(X,K))$.

$\mathcal{D}\times\mathcal{D}\to\mathcal{A}$ consisting of morphisms $H(M)\otimes H(N)\to H(M\otimes N)$ of $\mathcal{A}$. We adopt the convention that for $p\le q\le r$ in $\tilde{\mathbb{Z}}$, the map $\delta^n\colon H^n(X(q,r))\to H^{n+1}(X(p,q))$ is $(-1)^n$ times the map obtained by applying H to $\delta[n]\colon X(q,r)[n]\to X(p,q)[n+1]$. Then a pairing from (X,δ), (X',δ') to (X'',δ'') induces a pairing from $H^*(X,\delta)$, $H^*(X',\delta')$ to $H^*(X'',\delta'')$ given by

$$\begin{aligned}H(X(p,q)[n])\otimes H(X'(p',q')[n'])&\to H(X(p,q)[n]\otimes X'(p',q')[n'])\\ &\simeq H((X(p,q)\otimes X'(p',q'))[n+n'])\to H(X''((p,q)*(p',q'))[n+n']).\end{aligned}$$

Here we have used the composite of the isomorphisms

$$M[m]\otimes N[n]\simeq (M\otimes N[n])[m]\simeq (M\otimes N)[m+n]$$

given by the structure of bifunctor of additive categories with translation [27, Definition 10.1.1 (v)] on $\otimes\colon\mathcal{D}\times\mathcal{D}\to\mathcal{D}$. If $(\mathcal{D},\otimes)$, $(\mathcal{A},\otimes)$ are endowed with structures of $\otimes$-categories and H is a right-lax $\otimes$-functor, and if the associativity (resp. commutativity) constraint of $(\mathcal{D},\otimes)$ is a natural transformation of triangulated functors in each variable, then an associative (resp. commutative) pairing from (X,δ), (X,δ) to (X,δ) induces an associative (resp. commutative) pairing from $H^*(X,\delta)$, $H^*(X,\delta)$ to $H^*(X,\delta)$. Indeed, the assumption on the commutativity constraint implies the $(-1)^{mn}$-commutativity of the following diagram

$$\begin{array}{ccccc}M[m]\otimes N[n] & \xrightarrow{\sim} & (M\otimes N[n])[m] & \xrightarrow{\sim} & (M\otimes N)[m+n]\\ \downarrow\simeq & & & & \downarrow\simeq\\ N[n]\otimes M[m] & \xrightarrow{\sim} & (N\otimes M[m])[n] & \xrightarrow{\sim} & (N\otimes M)[m+n].\end{array}$$

Example 3.15. Let X be a commutatively ringed topos and let K be an object of $D(X)$. The second spectral sequence of hypercohomology

$$E_2^{pq}=H^p(X,\mathcal{H}^qK)\Rightarrow H^{p+q}(X,K)$$

is induced from the spectral object $H^*(K,\delta)$, where (K,δ) is the second spectral object associated to K. If K is a pseudo-ring in $D(X)$, then Remark 3.14 applied to Example 3.12 endows the spectral sequence with an associative multiplicative structure, which is graded commutative when K is commutative.

Part II
Main results

4 Finiteness theorems for equivariant cohomology rings

We will first discuss Chern classes of vector bundles on Artin stacks. Let $\mathcal{X}$ be an Artin stack, let $n\ge 2$ be an integer invertible on $\mathcal{X}$, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. The isomorphism class of $\mathcal{L}$ defines an element in $H^1(\mathcal{X},\mathbb{G}_m)$. We denote by

(4.0.1) $$c_1(\mathcal{L})\in H^2(\mathcal{X},\mathbb{Z}/n\mathbb{Z}(1))$$

the image of this element by the homomorphism $H^1(\mathcal{X},\mathbb{G}_m)\to H^2(\mathcal{X},\mathbb{Z}/n\mathbb{Z}(1))$ induced by the short exact sequence

$$1\to\mathbb{Z}/n\mathbb{Z}(1)\to\mathbb{G}_m\xrightarrow{n}\mathbb{G}_m\to 1,$$

where the map marked by n is raising to the n-th power. For any integer i, we write $\mathbb{Z}/n\mathbb{Z}(i)=\mathbb{Z}/n\mathbb{Z}(1)^{\otimes i}$. We say a quasi-coherent sheaf [44, 06WG] $\mathcal{E}$ on $\mathcal{X}$ is a *vector bundle* if there exists a smooth atlas $p\colon X\to\mathcal{X}$ such that $p^*\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of finite rank. The following theorem generalizes the construction of Chern classes of vector bundles on schemes ([39, Théorème 1.3] and [51, VII 3.4, 3.5]). If $\mathcal{X}$ is a Deligne-Mumford stack, it yields the Chern classes over the étale topos of $\mathcal{X}$ locally ringed by $\mathcal{O}_{\mathcal{X}}$, defined by Grothendieck in [21, (1.4)]. In particular, it also generalizes [21, (2.3)].

Theorem 4.1. *There exists a unique way to define, for every Artin stack $\mathcal{X}$ over $\mathbb{Z}[1/n]$ and every vector bundle $\mathcal{E}$ on $\mathcal{X}$, elements $c_i(\mathcal{E}) \in H^{2i}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}(i))$ for all $i \geq 0$ such that the formal power series $c_t(\mathcal{E}) = \sum_{i\geq 0} c_i(\mathcal{E})t^i$ satisfies the following conditions:*

(a) (Functoriality) If $f: \mathcal{Y} \to \mathcal{X}$ is a morphism of stacks over $\mathbb{Z}[1/n]$, then $f^*(c_t(\mathcal{E})) = c_t(f^*\mathcal{E})$;

(b) (Additivity) If $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is an exact sequence of vector bundles, then $c_t(\mathcal{E}) = c_t(\mathcal{E}')c_t(\mathcal{E}'')$;

(c) (Normalization) If $\mathcal{L}$ is a line bundle on $\mathcal{X}$, then $c_1(\mathcal{L})$ coincides with the class defined in (4.0.1) and $c_t(\mathcal{L}) = 1_{\mathcal{X}} + c_1(\mathcal{L})t$. Here $1_{\mathcal{X}}$ denotes the image of 1 by the adjunction homomorphism $\mathbb{Z}/n\mathbb{Z} \to H^0(\mathcal{X}, \mathbb{Z}/n\mathbb{Z})$.

Moreover, we have:

(d) $c_0(\mathcal{E}) = 1_{\mathcal{X}}$ and $c_i(\mathcal{E}) = 0$ for $i > \mathrm{rk}(\mathcal{E})$.

The $c_i(\mathcal{E})$ are called the (étale) *Chern classes* of $\mathcal{E}$. It follows from (b) and (d) that $c_t(\mathcal{E})$ only depends on the isomorphism class of $\mathcal{E}$.

To prove Theorem 4.1, we need the following result, which generalizes [51, VII Théorème 2.2.1] and [39, Théorème 1.2].

Proposition 4.2. *Let $\mathcal{X}$ be an Artin stack and let $\mathcal{E}$ be a vector bundle of constant rank r on $\mathcal{X}$. Let n be an integer invertible on $\mathcal{X}$ and let Λ be a commutative ring over $\mathbb{Z}/n\mathbb{Z}$. We denote by $\pi: \mathbb{P}(\mathcal{E}) \to \mathcal{X}$ the projective bundle of $\mathcal{E}$. Let $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \in H^2(\mathbb{P}(\mathcal{E}), \Lambda(1))$ as in (4.0.1). Then the powers $\xi^i \in H^{2i}(\mathbb{P}(\mathcal{E}), \Lambda(i))$ of ξ define an isomorphism in $D(\mathcal{X}, \Lambda)$*

$$(1, \xi, \ldots, \xi^{r-1}) \colon \bigoplus_{i=0}^{r-1} \Lambda(-i)[-2i] \xrightarrow{\sim} R\pi_*\Lambda. \tag{4.2.1}$$

Proof. By base change [32], we reduce to the case of schemes, which is proven in [51, VII Théorème 2.2.1]. □

The uniqueness of Chern classes is a consequence of the following lemma, which generalizes [39, Propositions 1.4, 1.5].

Lemma 4.3. *Let $\mathcal{X}$ be an Artin stack, let n be an integer invertible on $\mathcal{X}$, and let Λ be a commutative ring over $\mathbb{Z}/n\mathbb{Z}$.*

(a) (Splitting principle) Let $\mathcal{E}$ be a vector bundle on $\mathcal{X}$ of rank r and let $\pi: \mathcal{F}lag(\mathcal{E}) \to \mathcal{X}$ be the fibration of complete flags of $\mathcal{E}$. Then $\pi^*\mathcal{E}$ admits a canonical filtration by vector bundles such that the graded pieces are line bundles, and the morphism $\Lambda \to R\pi_*\Lambda$ is a split monomorphism.

(b) Let $E: 0 \to \mathcal{E}' \to \mathcal{E} \xrightarrow{p} \mathcal{E}'' \to 0$ be a short exact sequence of vector bundles and let $\pi: \mathcal{S}ect(E) \to \mathcal{X}$ be the fibration of sections of p. Then $\mathcal{S}ect(E)$ is a torsor under $\mathcal{H}om(\mathcal{E}'', \mathcal{E}')$ and π^*E is canonically split. Moreover, the morphism $\Lambda \to R\pi_*\Lambda$ is an isomorphism.

Proof. (a) follows from Proposition 4.2, as π is a composite of r successive projective bundles. For (b), up to replacing $\mathcal{X}$ by an atlas, we may assume that π is the projection from an affine space. In this case the assertion follows from [50, XV Corollaire 2.2]. □

To define $c_i(\mathcal{E})$, we may assume $\mathcal{E}$ is of constant rank r. As usual, we define

$$c_i(\mathcal{E}) \in H^{2i}(\mathcal{X}, \mathbb{Z}/n\mathbb{Z}(i)), \quad 1 \leq i \leq r,$$

as the unique elements satisfying

$$\xi^r + \sum_{1\leq i\leq r} (-1)^i c_i(\mathcal{E})\xi^{r-i} = 0,$$

where $\xi = c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)) \in H^2(\mathbb{P}(\mathcal{E}), \mathbb{Z}/n\mathbb{Z}(1))$. We put $c_0(\mathcal{E}) = 1$ and $c_i(\mathcal{E}) = 0$ for $i > r$. The properties (a) to (d) follow from the case of schemes. If $c_i(\mathcal{E}) = 0$ for all $i > 0$, in particular if $\mathcal{E}$ is trivial, then (4.2.1) is an isomorphism of rings in $D(\mathcal{X}, \Lambda)$.

Theorem 4.4. *Let S be an algebraic space, let n be an integer invertible on S, and let Λ be a commutative ring over $\mathbb{Z}/n\mathbb{Z}$. Let $N \geq 1$ be an integer, let $G = GL_{N,S}$, and let $T = \prod_{i=1}^{N} T_i \subset G$ be the subgroup of diagonal matrices, where $T_i = \mathbb{G}_{m,S}$. Let $\pi\colon BG \to S$, $\tau\colon BT \to S$, $f'\colon G/T \to S$ be the projections, let $k\colon S \to BG$ be the canonical section, let $f\colon BT \to BG$ be the morphism induced by the inclusion $T \to G$ and let $h\colon G/T \to BT$ be the morphism induced by the projection $G \to S$, as shown in the following 2-commutative diagram*

(4.4.1)
$$\begin{array}{ccccc} G/T & \xrightarrow{h} & BT & & \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & \searrow{\scriptstyle \tau} & \\ S & \xrightarrow{k} & BG & \xrightarrow{\pi} & S. \end{array}$$

Let $\mathcal{E}$ be the standard vector bundle of rank N on BG, corresponding to the natural representation of G in $\mathcal{O}_S^N$. The i-th Chern class $c_i(\mathcal{E})$ of $\mathcal{E}$ induces a morphism

$$\alpha_i\colon K_i = \Lambda_S(-i)[-2i] \to R\pi_*\Lambda.$$

Let $\mathcal{L}_i$ be the inverse image on BT of the standard line bundle on BT_i. Its first Chern class $c_1(\mathcal{L}_i)$ induces a morphism

$$\beta_i\colon L_i = \Lambda_S(-1)[-2] \to R\tau_*\Lambda.$$

For a graded sheaf of Λ-modules $M = \bigoplus_{i\in\mathbb{Z}} M_i$ on S, we let $M^\Delta = \bigoplus_i M_i(-i)[-2i] \in D(S,\Lambda)$. Let $\Lambda_S[x_1,\dots,x_N]$ (resp. $\Lambda_S[t_1,\dots,t_N]$) be a polynomial algebra on generators x_i of degree i (resp. t_i of degree 1). The corresponding object $\Lambda_S[x_1,\dots,x_N]^\Delta$ (resp. $\Lambda_S[t_1,\dots,t_N]^\Delta$) is naturally identified with $\mathrm{S}_\Lambda(\bigoplus_{1\leq i\leq N} K_i)$ (resp. $\mathrm{S}_\Lambda(\bigoplus_{1\leq i\leq N} L_i)$). Then the ring homomorphisms

(4.4.2)
$$\alpha\colon \Lambda_S[x_1,\dots,x_N]^\Delta \to R\pi_*\Lambda,$$

(4.4.3)
$$\beta\colon \Lambda_S[t_1,\dots,t_N]^\Delta \to R\tau_*\Lambda,$$

defined respectively by α_i and β_i, are isomorphisms of rings in $D(S,\Lambda)$, and fit into a commutative diagram of rings in $D(S,\Lambda)$

(4.4.4)
$$\begin{array}{ccccc} \Lambda_S[x_1,\dots,x_N]^\Delta & \xrightarrow{\sigma} & \Lambda_S[t_1,\dots,t_N]^\Delta & \xrightarrow{\rho} & (\Lambda_S[t_1,\dots,t_N]/(\sigma_1,\dots,\sigma_N))^\Delta \\ {\scriptstyle \alpha}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \beta}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \gamma}\downarrow{\scriptstyle \simeq} \\ R\pi_*\Lambda & \xrightarrow{a_f} & R\tau_*\Lambda & \xrightarrow{a_h} & Rf'_*\Lambda, \end{array}$$

which commutes with arbitrary base change of algebraic spaces $S' \to S$. Here σ sends x_i to the i-th elementary symmetric polynomial σ_i in $t_1,\dots,t_N$, ρ is the projection, a_f is induced by adjunction by f and a_h is induced by adjunction by h. Moreover, as graded module over $R^{2*}\pi_*\Lambda(*)$, $R^{2*}\tau_*\Lambda(*)$ is free of rank $N!$.

In particular, we have canonical decompositions

$$R\pi_*\Lambda \simeq \bigoplus_q R^{2q}\pi_*\Lambda[-2q], \quad R\tau_*\Lambda \simeq \bigoplus_q R^{2q}\tau_*\Lambda[-2q], \quad Rf'_*\Lambda \simeq \bigoplus_q R^{2q}f'_*\Lambda[-2q],$$

a_h induces an epimorphism $R^*\tau_*\Lambda \to R^*f'_*\Lambda$ and a_f induces an isomorphism $R^*\pi_*\Lambda \xrightarrow{\sim} (R^*\tau_*\Lambda)^{\mathfrak{S}_N}$, where $\mathfrak{S}_N$ is the symmetric group on N letters. Moreover, (4.4.4) induces a commutative diagram of sheaves of Λ-algebras on S

(4.4.5)
$$\begin{array}{ccccc} \Lambda_S[x_1,\dots,x_N] & \xrightarrow{\sigma} & \Lambda_S[t_1,\dots,t_N] & \xrightarrow{\rho} & \Lambda_S[t_1,\dots,t_N]/(\sigma_1,\dots,\sigma_N) \\ {\scriptstyle \alpha}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \beta}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \gamma}\downarrow{\scriptstyle \simeq} \\ R^{2*}\pi_*\Lambda(*) & \xrightarrow{a_f} & R^{2*}\tau_*\Lambda(*) & \xrightarrow{a_h} & R^{2*}f'_*\Lambda(*), \end{array}$$

where α carries x_i to the image of $c_i(\mathcal{E})$ under the edge homomorphism

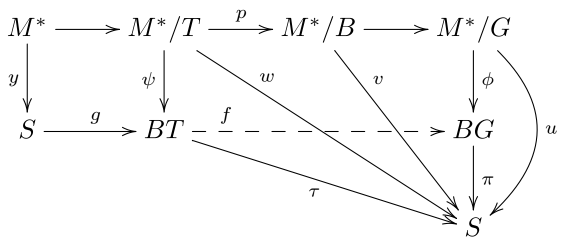
$$H^{2i}(BG,\Lambda(i)) \to H^0(S, R^{2i}\pi_*\Lambda(i)),$$

and β carries t_i to the image of $c_1(\mathcal{L}_i)$ under the edge homomorphism

$$H^{2i}(BT, \Lambda(i)) \to H^0(S, R^{2i}\tau_*\Lambda(i)).$$

We will derive from Theorem 4.4 a formula for Rf_* (see Corollary 4.5).

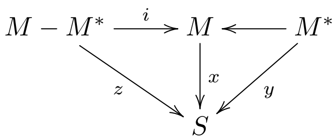
Proof. As in [2, Lemma 2.3.1], we approximate BG by a finite Grassmannian $G(N, N') = M^*/G$,[5] where $N' \geq N$, M is the algebraic S-space of $N' \times N$ matrices $(a_{ij})_{\substack{1\leq i\leq N' \\ 1\leq j\leq N}}$, M^* is the open subspace of M consisting of matrices of rank N. Let $B \subset G$ be the subgroup of upper triangular matrices. The square on the right of the diagram with 2-cartesian squares



induces a commutative square

(4.4.6)
$$\begin{array}{ccccc} R\pi_*\Lambda & \longrightarrow & & & R\tau_*\Lambda \\ \downarrow & & & & \downarrow \\ Ru_*\Lambda & \longrightarrow & Rv_*\Lambda & \xrightarrow{a_p} & Rw_*\Lambda. \end{array}$$

Here a_p is induced by the adjunction $\Lambda \to Rp_*\Lambda$. The latter is an isomorphism by [50, XV Corollaire 2.2], because p is a (B/T)-torsor and B/T is isomorphic to the unipotent radical of B, which is an affine space over S. The diagram



induces an exact triangle

$$Rz_*Ri^!\Lambda \to Rx_*\Lambda \to Ry_*\Lambda \to .$$

Since M is an affine space over S, the adjunction $\Lambda \to Rx_*\Lambda$ is an isomorphism [50, XV Corollaire 2.2]. Since x is smooth and the fibers of z are of codimension $N' - N + 1$, we have $Ri^!\Lambda \in D^{\geq 2(N'-N+1)}$ by semi-purity [11, Cycle 2.2.8]. It follows that the adjunction $\Lambda \to \tau^{\leq 2(N'-N)}Ry_*\Lambda$ is an isomorphism. By smooth base change by g (resp. fg) [32], this implies that the adjunction $\Lambda \to \tau^{\leq 2(N'-N)}R\psi_*\Lambda$ (resp. $\Lambda \to \tau^{\leq 2(N'-N)}R\phi_*\Lambda$) is an isomorphism, so that the right (resp. left) vertical arrow of $\tau^{\leq 2(N'-N)}$(4.4.6) is an isomorphism.

The assertions then follow from an explicit computation of $Ru_*\Lambda$ and $Rv_*\Lambda$. Note that M^*/B is a partial flag variety of the free $\mathcal{O}_S$-module $\mathcal{O}_S^{N'}$ of type $(1, \dots, 1, N'-N)$. By [51, VII Propositions 5.2, 5.6 (a)] applied to u and v, we have a commutative square

$$\begin{array}{ccc} A^\Delta & \xrightarrow{\sim} & Ru_*\Lambda \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ C^\Delta & \xrightarrow{\sim} & Rv_*\Lambda. \end{array}$$

[5] This approximation argument was explained by Deligne to the first author in the context of de Rham cohomology in 1967.

Here

$$A = \Lambda_S[x_1, \ldots, x_N, y_1, \ldots, y_{N'-N}]/(\sum_{i+j=m} x_i y_j)_{m \geq 1},$$
$$C = \Lambda_S[t_1, \ldots, t_N, y_1, \ldots, y_{N'-N}]/(\sum_{i+j=m} \sigma_i y_j)_{m \geq 1},$$

the upper horizontal arrow sends x_i to the i-th Chern class $c_i(\mathcal{E}_{N'})$ of the canonical bundle $\mathcal{E}_{N'}$ of rank N on the Grassmannian M^*/G, the lower horizontal arrow sends t_i to the first Chern class $c_1(\mathcal{L}_{i,N'})$ of the i-th standard line bundle $\mathcal{L}_{i,N'}$ of the partial flag variety M^*/B, and the upper (resp. lower) horizontal arrow sends y_i to the i-th Chern class $c_i(\mathcal{E}'_{N'})$ of the canonical bundle $\mathcal{E}'_{N'}$ of rank $N' - N$ on M^*/G (resp. on M^*/B). In the definition of the ideals, we put $x_0 = y_0 = 1$, $x_i = 0$ for $i > N$ and $y_i = 0$ for $i > N' - N$, and we used the fact that c_m of the trivial bundle of rank N' is zero for $m \geq 1$. As $\mathcal{E}_{N'}$ (resp. $\mathcal{L}_{i,N'}$) is induced from $\mathcal{E}$ (resp. $\mathcal{L}_i$), by the functoriality of Chern classes (Theorem 4.1), these isomorphisms are compatible with the morphisms α (4.4.2) and β (4.4.3). We can rewrite A as $\Lambda[x_1, \ldots, x_N]/(P_m(x_1, \ldots, x_m))_{m>N'-N}$ and rewrite C as $\Lambda[t_1, \ldots, t_N]/(Q_m(t_1, \ldots, t_m))_{m>N'-N}$, where P_m is an isobaric polynomial of weight m in $x_1, \ldots, x_m$, x_i being of weight i, and Q_m is a homogeneous polynomial of degree m in $t_1, \ldots, t_m$. As the vertical arrows of (4.4.6) induce isomorphisms after application of the truncation functor $\tau^{\leq 2(N'-N)}$, it follows that $\tau^{\leq 2(N'-N)}$ of the square on the left of (4.4.4) is commutative and the vertical arrows induce isomorphisms after application of $\tau^{\leq 2(N'-N)}$. To get the square on the right of (4.4.4), it suffices to apply the preceding computation of $Rw_*\Lambda$ (via $Rv_*\Lambda$) to the case $N' = N$, because, in this case, $f' = w$. The fact that (4.4.4) commutes with base change follows from the functoriality of Chern classes. The last assertion of the theorem then follows from [51, VII Lemme 5.4.1]. □

Corollary 4.5. *With assumptions and notation as in Theorem 4.4:*

(a) For every locally constant Λ-module $\mathcal{F}$ on S, the projection formula maps

$$\mathcal{F} \otimes^L_\Lambda R\pi_*\Lambda \to R\pi_*\pi^*\mathcal{F}, \quad \mathcal{F} \otimes^L_\Lambda R\tau_*\Lambda \to R\tau_*\tau^*\mathcal{F}$$

are isomorphisms.

(b) The classes $c_1(\mathcal{L}_i)$ induce an isomorphism of rings $\Lambda_{BG}[t_1, \ldots, t_N]^\Delta/J \to Rf_*\Lambda$ in $D(BG, \Lambda)$, where J is the ideal generated by $\sigma_i - c_i(\mathcal{E})$. Moreover, the left square of (4.4.1) *induces an isomorphism of $R\pi_*\Lambda$-modules $R\tau_*\Lambda \simeq R\pi_*\Lambda \otimes^L_\Lambda Rf'_*\Lambda$.*

Proof. (a) We may assume that $\mathcal{F}$ is a constant Λ-module of value F. Then the assertion follows from Theorem 4.4 applied to Λ and to the ring of dual numbers $\Lambda \oplus F$ (with $m_1 m_2 = 0$ for $m_1, m_2 \in F$).

(b) Since $f^*\mathcal{E} \simeq \bigoplus_{i=1}^N \mathcal{L}_i$, $f^*c_i(\mathcal{E}) = c_i(f^*\mathcal{E})$ is the i-th elementary symmetric polynomial in $c_1(\mathcal{L}_1), \ldots, c_1(\mathcal{L}_N)$. Thus the ring homomorphism $\Lambda_{BG}[t_1, \ldots, t_N]^\Delta \to Rf_*\Lambda$ induced by $c_1(\mathcal{L}_i)$ factorizes through a ring homomorphism $\Lambda_{BG}[t_1, \ldots, t_N]^\Delta/J \to Rf_*\Lambda$. By Proposition 1.11 applied to the square

$$\begin{array}{ccc} (G,T) & \longrightarrow & (S,T) \\ \downarrow & & \downarrow \\ (G,G) & \longrightarrow & (S,G), \end{array}$$

the left square of (4.4.1) is 2-cartesian. By smooth base change by k, we have $k^*Rf_*\Lambda \simeq Rf'_*\Lambda$. The first assertion then follows from Theorem 4.4. By Remark 2.7, it follows that $Rf_*\Lambda \simeq \pi^*k^*Rf_*\Lambda \simeq \pi^*Rf'_*\Lambda$. Thus $R\tau_*\Lambda \simeq R\pi_*Rf_*\Lambda \simeq R\pi_*\pi^*Rf'_*\Lambda$ and the second assertion follows from (a). □

Let k be a separably closed field, let n be an integer invertible in k, and let Λ be a noetherian commutative ring over $\mathbb{Z}/n\mathbb{Z}$. The next sequence of results are analogues of Quillen's finiteness theorem [36, Theorem 2.1, Corollaries 2.2, 2.3]. Recall that an algebraic space over $\operatorname{Spec} k$ is of finite presentation if and only if it is quasi-separated and of finite type.

Theorem 4.6. *Let G be an algebraic group over k, let X be an algebraic space of finite presentation over* $\operatorname{Spec} k$ *equipped with an action of G, and let K be an object of $D_c^b([X/G],\Lambda)$ (see Notation 2.2). Then $H^*(BG,\Lambda)$ is a finitely generated Λ-algebra and $H^*([X/G],K)$ is a finite $H^*(BG,\Lambda)$-module. In particular, if K is a ring in the sense of Definition 3.3, then the graded center $ZH^*([X/G],K)$ of $H^*([X/G],K)$ is a finitely generated Λ-algebra.*

Initially the authors established Theorem 4.6 for G either a linear algebraic group or a semi-abelian variety. The finiteness of $H^*(BG,\Lambda)$ in the general case was proved by Deligne in [12].

Corollary 4.7. *Let G be an algebraic group over k and let $f\colon \mathcal{X}\to BG$ be a representable morphism of Artin stacks of finite presentation over* $\operatorname{Spec} k$*, and let $K\in D_c^b(\mathcal{X},\Lambda)$. Consider $H^*(\mathcal{X},K)$ as an $H^*(BG,\Lambda)$-module by restriction of scalars via the map $f^*\colon H^*(BG,\Lambda)\to H^*(\mathcal{X},\Lambda)$. Then $H^*(\mathcal{X},K)$ is a finite $H^*(BG,\Lambda)$-module.*

Proof. It suffices to apply Theorem 4.6 to $Rf_*K\in D_c^b(BG,\Lambda)$. □

Corollary 4.8. *Let X (resp. Y) be an algebraic space of finite presentation over* $\operatorname{Spec} k$*, equipped with an action of an algebraic group G (resp. H) over k. Let $(f,u)\colon (X,G)\to(Y,H)$ be an equivariant morphism. Assume that u is a monomorphism. Then the map $[f/u]^*$ makes $H^*([X/G],\Lambda)$ a finite $H^*([Y/H],\Lambda)$-module.*

Indeed, since the map $[X/G]\to BH$ induced by u is representable, $H^*([X/G],\Lambda)$ is a finite $H^*(BH,\Lambda)$-module by Corollary 4.7, hence a finite $H^*([Y/H],\Lambda)$-module.

Proof of Theorem 4.6. By the invariance of étale cohomology under schematic universal homeomorphisms, we may assume k algebraically closed and G reduced (hence smooth). Then G is an extension $1\to G^0\to G\to F\to 1$, where F is the finite group $\pi_0(G)$ and G^0 is the identity component of G. By Chevalley's theorem (cf. [8, Theorem 1.1.1] or [9, Theorem 1.1]), G^0 is an extension $1\to L\to G^0\to A\to 1$, where A is an abelian variety and $L=G_{\mathrm{aff}}$ is the largest connected affine normal subgroup of G^0. Then L is also normal in G, and if $E=G/L$, then E is an extension $1\to A\to E\to F\to 1$. We will sum up this dévissage by saying that G is an iterated extension $G=L\cdot A\cdot F$.

By [20, VIII 7.1.5, 7.3.7], for every algebraic group H over k, the extensions of F by H with given action of F on H by conjugation are classified by $H^2(BF,H)$. In particular, the extension E of F by A defines an action of F on A and a class in $H^2(BF,A)$, which comes from a class α in $H^2(BF,A[m])$, where m is the order of F and $A[m]$ denotes the kernel of $m\colon A\to A$. Indeed the second arrow in the exact sequence

$$H^2(BF,A[m])\to H^2(BF,A)\xrightarrow{\times m} H^2(BF,A)$$

is equal to zero. This allows us to define an inductive system of subgroups $E_i=A[mn^i]\cdot F$ of E, given by the image of α in $H^2(F,A[mn^i])$. This induces an inductive system of subgroups $G_i=L\cdot A[mn^i]\cdot F$ of G, fitting into short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L & \longrightarrow & G_i & \longrightarrow & E_i & \longrightarrow & 1\\ & & \| & & \downarrow & \square & \downarrow & & \\ 1 & \longrightarrow & L & \longrightarrow & G & \longrightarrow & E & \longrightarrow & 1. \end{array}$$

Form the diagram with cartesian squares

$$\begin{array}{ccccc} [X/G_i] & \longrightarrow & BG_i & \longleftarrow & G/G_i\\ {\scriptstyle f_i}\downarrow & & \downarrow & & \downarrow\\ [X/G] & \longrightarrow & BG & \longleftarrow & \operatorname{Spec} k. \end{array}$$

Note that $G/G_i=A/A[mn^i]$ and the vertical arrows in the above diagram are proper representable. By the classical projection formula [50, XVII (5.2.2.1)], $Rf_{i*}f_i^*K\simeq K\otimes_\Lambda^L Rf_{i*}\Lambda$. Moreover, $f_{i*}\Lambda\simeq\Lambda$. Thus we have a distinguished triangle

$$K\to Rf_{i*}f_i^*K\to K\otimes_\Lambda^L\tau^{\geq 1}Rf_{i*}\Lambda\to . \qquad (4.8.1)$$

The first term forms a constant system and the third term $N_i = K \otimes^L_\Lambda \tau^{\geq 1} Rf_{i*}\Lambda$ forms an AR-null system of level $2d$ in the sense that $N_{i+2d} \to N_i$ is zero for all i, where $d = \dim A$. Indeed the stalks of $R^q f_{i*}\Lambda$ are $H^q(A/A[mn^i], \Lambda)$, which is zero for $q > 2d$. For $q = 0$, the transition maps of $(H^0(A/A[mn^i], \Lambda))$ are id_Λ and for $q > 0$, the transition maps of $(H^q(A/A[mn^i], \Lambda))$ are zero. Thus, in the induced long exact sequence of (4.8.1)

$$H^{*-1}([X/G], N_i) \to H^*([X/G], K) \xrightarrow{\alpha_i} H^*([X/G_i], f_i^* K) \to H^*([X/G], N_i),$$

the system $(H^*([X/G], N_i))$ is AR-null of level $2d$. Therefore, α_i is injective for $i \geq 2d$ and $\operatorname{Im} \alpha_i = \operatorname{Im}(H^*([X/G_{i+2d}], f^*_{i+2d} K) \to H^*([X/G_i], f_i^* K))$ for all i. Taking $i = 2d$, we get $H^*([X/G], K) = \operatorname{Im}(H^*([X/G_{4d}], f^*_{4d} K) \to H^*([X/G_{2d}], f^*_{2d} K))$. In particular, $H^*(BG, \Lambda)$ is a quotient Λ-algebra of $H^*(BG_{4d}, \Lambda)$, and $H^*([X/G], K)$ is a quotient $H^*(BG, \Lambda)$-module of $H^*([X/G_{4d}], f^*_{4d} K)$. Therefore, it suffices to show the theorem with G replaced by G_{4d}. In particular, we may assume that G is a linear algebraic group.

Let $G \to \mathrm{GL}_r$ be an embedding into a general linear group. By Corollary 1.15, the morphism of Artin stacks over $B\mathrm{GL}_r$,

$$[X/G] \to [(X \wedge^G \mathrm{GL}_r)/\mathrm{GL}_r],$$

is an equivalence. Replacing G by GL_r and X by $X \wedge^G \mathrm{GL}_r$, we may assume that $G = \mathrm{GL}_r$. Let $f\colon [X/G] \to BG$. Then $Rf_* K \in D^b_c(BG, \Lambda)$. Thus we may assume $X = \operatorname{Spec} k$. The full subcategory of objects K satisfying the theorem is a triangulated category. Thus we may further assume $K \in \mathrm{Mod}_c(BG, \Lambda)$. In this case, since G is connected, K is necessarily constant (Corollary 2.6) so that $K \simeq \pi^* M$ for some finite Λ-module M, where $\pi\colon BG \to \operatorname{Spec} k$. In this case, by Theorem 4.4, $H^*(BG, \Lambda) \simeq \Lambda[c_1, \dots, c_r]$ is a noetherian ring and $H^*(BG, K) \simeq M \otimes_\Lambda \Lambda[c_1, \dots, c_r]$ is a finite $H^*(BG, \Lambda)$-module. □

Remark 4.9. We have shown in the proof of Theorem 4.6 that $H^*([X/G], K) \simeq \operatorname{Im}(H^*([X/G_{4d}], f^*_{4d} K) \to H^*([X/G_{2d}], f^*_{2d} K))$. In particular, $H^*([X/G], K)$ is a quotient $H^*(BG, \Lambda)$-module of $H^*([X/G_{4d}], f^*_{4d} K)$. Here $G_{2d} < G_{4d}$ are affine subgroups of G, independent of X and K, and $f_{2d}\colon [X/G_{2d}] \to [X/G]$, $f_{4d}\colon [X/G_{4d}] \to [X/G]$.

In the following examples, we write $H^*(-)$ for $H^*(-, \Lambda)$, with Λ as in Theorem 4.6.

Example 4.10. Let G/k be an extension of an abelian variety A of dimension g by a torus T of dimension r. Then

(a) $H^1(A)$, $H^1(T)$, $H^1(G)$ are free over Λ of ranks $2g$, r and $2g+r$ respectively, and the sequence $0 \to H^1(A) \to H^1(G) \to H^1(T) \to 0$ is exact. The inclusion $H^1(G) \hookrightarrow H^*(G)$ induces an isomorphism of Λ-modules

$$\wedge H^1(G) \xrightarrow{\sim} H^*(G). \tag{4.10.1}$$

(b) The homomorphism

$$d_2^{01}\colon H^1(G) \to H^2(BG)$$

in the spectral sequence

$$E_2^{pq} = H^p(BG) \otimes H^q(G) \Rightarrow H^{p+q}(\operatorname{Spec} k)$$

of the fibration $\operatorname{Spec} k \to BG$ is an isomorphism.

(c) We have $H^{2i+1}(BG) = 0$ for all i, and the inclusion $H^2(BG) \hookrightarrow H^*(BG)$ extends to an isomorphism of Λ-algebras

$$\mathrm{S}(H^2(BG)) \xrightarrow{\sim} H^*(BG).$$

Let us briefly sketch a proof.

Assertion (a) is standard. By projection formula, we may assume $\Lambda = \mathbb{Z}/n\mathbb{Z}$. As the multiplication by n on T is surjective, the sequence $0 \to T[n] \to G[n] \to A[n] \to 0$ is exact. The surjection $\pi_1(G) \to G[n]$ induces an injection $\operatorname{Hom}(G[n], \mathbb{Z}/n\mathbb{Z}) \to H^1(G)$. The fact that this injection and (4.10.1) are isomorphisms follows (after reducing to $n = \ell$ prime) from the structure of Hopf algebra of $H^*(G)$, as $H^{2g+r}(G) \xrightarrow{\sim} H^r(T) \otimes H^{2g}(A)$ is of rank 1 (cf. [42, Chapter VII, Proposition 16]).

Assertion (b) follows immediately from (a).

To prove (c) we calculate $H^*(BG)$ using the nerve $B_\bullet G$ of G (cf. [10, 6.1.5]):

$$H^*(BG) = H^*(B_\bullet G),$$

which gives the Eilenberg-Moore spectral sequence:

$$E_1^{ij} = H^j(B_iG) \Rightarrow H^{i+j}(BG). \tag{4.10.2}$$

One finds that

$$E_1^{\bullet,j} \simeq L \wedge^j (H^1(G)[-1]).$$

By [23, I 4.3.2.1 (i)] we get

$$E_1^{\bullet,j} \simeq L\mathrm{S}^j(H^1(G))[-j].$$

Thus

$$E_2^{ij} \simeq \begin{cases} \mathrm{S}^j(H^1(G)) & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The E_2 term is concentrated on the diagonal, hence (4.10.2) degenerates at E_2, and we get an isomorphism

$$H^*(BG) = \mathrm{S}(H^1(G)[-2]),$$

from which (c) follows.

Example 4.11. Let G be a connected algebraic group over k. Assume that for every prime number ℓ dividing n, $H^i(G, \mathbb{Z}_\ell)$ is torsion-free for all i. Classical results due to Borel [5] can be adapted as follows.

(a) $H^*(G)$ is the exterior algebra over a free Λ-module having a basis of elements of odd degree [5, Propositions 7.2, 7.3].

(b) In the spectral sequence of the fibration $\operatorname{Spec} k \to BG$,

$$E_2^{ij} = H^i(BG) \otimes H^j(G) \Rightarrow H^{i+j}(\operatorname{Spec} k),$$

primitive and transgressive elements coincide [5, Proposition 20.2], and the transgression gives an isomorphism $d_{q+1}\colon P^q \xrightarrow{\sim} Q^{q+1}$ from the transgressive part $P^q = E_{q+1}^{0q}$ of $H^q(G) \simeq E_2^{0q}$ to the quotient $Q^{q+1} = E_{q+1}^{q+1,0}$ of $H^{q+1}(BG) \simeq E_2^{q+1,0}$. Moreover, Q^* is a free Λ-module having a basis of elements of even degrees, and every section of $H^*(BG) \to Q^*$ provides an isomorphism between $H^*(BG)$ and the polynomial algebra $\mathrm{S}_\Lambda(Q^*)$ [5, Théorèmes 13.1, 19.1].

Now assume that G is a connected reductive group over k. Let T be *the* maximal torus in G, and $W = \mathrm{Norm}_G(T)/T$ *the* Weyl group. Recall that G is ℓ-torsion-free if ℓ does not divide the order of W, cf. [6], [43, Section 1.3]. As in [11, Sommes trig., 8.2], the following results can be deduced from the classical results on compact Lie groups by lifting G to characteristic zero.

(c) The spectral sequence

$$E_2^{ij} = H^i(BG) \otimes H^j(G/T) \Rightarrow H^{i+j}(BT) \tag{4.11.1}$$

degenerates at E_2, E_2^{ij} being zero if i or j is odd[6]. In particular, the homomorphism

$$H^*(BT) \to H^*(G/T)$$

induced by the projection $G/T \to BT$ is surjective. In other words, in view of Theorem 4.4, $H^*(G/T)$ is generated by the Chern classes of the invertible sheaves L_χ obtained by pushing out the T-torsor G over G/T by the characters $\chi\colon T \to \mathbb{G}_{\mathrm{m}}$.

(d) The Weyl group W acts on (4.11.1), trivially on $H^*(BG)$, and $H^*(G/T)$ is the regular representation of W [5, Lemme 27.1]. In particular, the homomorphism $H^*(BG) \to H^*(BT)$ induced by the projection $BT \to BG$ induces an isomorphism

$$H^*(BG) \xrightarrow{\sim} H^*(BT)^W. \tag{4.11.2}$$

[6]The vanishing of $H^j(G/T)$ for j odd follows for example from the Bruhat decomposition of G/B for a Borel B containing T.

5 Finiteness of orbit types

Let k be a field of characteristic $p \geq 0$, let G be an algebraic group over k, and let A be a finite group. The presheaf of sets $\mathcal{H}om_{\mathrm{group}}(A, G)$ on $\mathrm{AlgSp}_{/k}$ is represented by a closed subscheme X of the product $\prod_{a \in A} G$ of copies of G indexed by A. In the case where $A \simeq (\mathbb{Z}/\ell\mathbb{Z})^r$ is an elementary abelian ℓ-group of rank r, $\mathcal{H}om_{\mathrm{group}}(A, G)(T)$ can be identified with the set of commuting r-tuples of ℓ-torsion elements of $G(T)$. The group G acts on X by conjugation. Let $x \in X(k)$ be a rational point of X and let $c\colon G \to X$ be the G-equivariant morphism sending g to xg, where $xg\colon a \mapsto g^{-1}x(a)g$. Let $H = c^{-1}(x) \subset G$ be the *inertia* subgroup at x. The morphism c decomposes into

$$G \to H\backslash G \xrightarrow{f} X,$$

where f is an immersion [13, III, § 3, Proposition 5.2]. The *orbit* of x under G is the (scheme-theoretic) image of f, which is a subscheme of X. The orbits of X are disjoint with each other.

The following result is probably well known. It was communicated to us by Serre.

Theorem 5.1 (Serre)**.** *Assume that the order of A is not divisible by p. Then the orbits of X under the action of G are open subschemes. Moreover, if G is smooth, then X is smooth.*

The condition on the order of A is essential. For example, if $p > 0$, $A = \mathbb{Z}/p\mathbb{Z}$ and $G = \mathbb{G}_a$ is the additive group, then G acts trivially on $X \simeq G$.

Note that for any field extension k' of k, if Y is an orbit of X under G, then $Y_{k'}$ is an orbit of $X_{k'}$ under $G_{k'}$.

Corollary 5.2. *The orbits are closed and the number of orbits is finite. Moreover, if k is algebraically closed, then the orbits form a disjoint open covering of X.*

Proof. It suffices to consider the case when k is algebraically closed. In this case, rational points of X form a dense subset [22, Corollaire 10.4.8]. Thus, by Theorem 5.1, the orbits form a disjoint open covering of the quasi-compact topological space X. Therefore, the orbits are also closed and the number of orbits is finite. □

Corollary 5.3. *Let G be an algebraic group over k and let ℓ be a prime number distinct from p. There are finitely many conjugacy classes of elementary abelian ℓ-subgroups of G. Moreover, if k is algebraically closed and k' is an algebraically closed extension of k, then the natural map $S_k \to S_{k'}$ from the set S_k of conjugacy classes of elementary abelian ℓ-subgroups of G to the set $S_{k'}$ of conjugacy classes of elementary abelian ℓ-subgroups of $G_{k'}$ is a bijection.*

Proof. By Corollary 5.2, it suffices to show that the ranks of the elementary abelian ℓ-subgroups of G are bounded. For this, we may assume k algebraically closed, and G smooth. As in the proof of Theorem 4.6, let L be the maximal connected affine normal subgroup of the identity component G^0 of G. Let d be the dimension of the abelian variety G^0/L, and let m be the maximal integer such that $\ell^m | [G : G^0]$. Choose an embedding of L into some GL_n. Then every elementary abelian subgroup of G has rank $\leq n + 2d + m$. □

To prove the theorem, we need a lemma on tangent spaces. Let S be an algebraic space, and let X be an S-functor, that is, a presheaf of sets on $\mathrm{AlgSp}_{/S}$. Recall [49, II 3.1] that the *tangent bundle* to X is defined to be the S-functor

$$T_{X/S} = \mathcal{H}om_S(\mathrm{Spec}(\mathcal{O}_S[\epsilon]/(\epsilon^2)), X),$$

which is endowed with a projection to X. For every point $u \in X(S)$, the *tangent space* to X at u is the S-functor [49, II 3.2]

$$T^u_{X/S} = T_{X/S} \times_{X,u} S.$$

Recall [49, II 3.11] that, for S-functors Y and Z, we have an isomorphism

$$T_{\mathcal{H}om_S(Y,Z)/S} \simeq \mathcal{H}om_S(Y, T_{Z/S}).$$

For a morphism $f\colon Y \to Z$ of S-functors, this induces an isomorphism

$$T^f_{\mathcal{H}om_S(Y,Z)/S} \simeq \mathcal{H}om_{Z/S}((Y, f), T_{Z/S}). \tag{5.3.1}$$

Assume that Z is an S-group, that is, a presheaf of groups on $\mathrm{AlgSp}_{/S}$. Then we have an isomorphism of schemes $T_{Z/S} \simeq Z \times_S \mathrm{Lie}(Z/S)$, where $\mathrm{Lie}(Z/S) = T^1_{Z/S}$. Thus (5.3.1) induces an isomorphism

$$T^f_{\mathcal{H}om_S(Y,Z)/S} \xrightarrow{\sim} \mathcal{H}om_S(Y, \mathrm{Lie}(Z/S)). \tag{5.3.2}$$

Furthermore, if Y is an S-group and f is a homomorphism of S-groups, then the image of $T^f_{\mathcal{H}om_{S\text{-group}}(Y,Z)/S}$ by (5.3.2) is $\mathcal{Z}^1_S(Y, \mathrm{Lie}(Z/S))$ [49, II 4.2], where Y acts on $\mathrm{Lie}(Z/S)$ by the formula $y \mapsto \mathrm{Ad}(f(y))$.[7]

Lemma 5.4. *Let $f\colon Y \to Z$ be a homomorphism of S-groups as above. Let $c\colon Z \to \mathcal{H}om_{S\text{-group}}(Y, Z)$ be the morphism given by*

$$z \mapsto (y \mapsto (z^{-1}f(y)z)).$$

Then the composition

$$\mathrm{Lie}(Z/S) \xrightarrow{T^1_{c/S}} T^f_{\mathcal{H}om_{S\text{-group}}(Y,Z)/S} \to \mathcal{H}om_S(Y, \mathrm{Lie}(Z/S))$$

is given by $t \mapsto (y \mapsto \mathrm{Ad}(f(y))t - t)$, and the image is $\mathcal{B}^1_S(Y, \mathrm{Lie}(Z/S))$.

Proof. The exact sequence

$$1 \to \mathrm{Lie}(Z/S) \to T_{Z/S} \to Z \to 1$$

has a canonical splitting, which allows one to identify $T_{Z/S}$ with the semidirect product $\mathrm{Lie}(Z/S) \rtimes Z$. An element (t, z) of the semidirect product (evaluated at an S-scheme S') corresponds to the image of $dR_z(t) \in T^z_{Z/S}(S')$, where $R_z\colon Z \times_S S' \to Z \times_S S'$ is the right translation by z. Multiplication in the semidirect product is given by

$$(t, z)(t', z') = (t + \mathrm{Ad}(z)t', zz').$$

The image of t under $T^1_{c/S}$ in $T^f_{\mathcal{H}om_S(Y,Z)/S} \xrightarrow{\sim} \mathcal{H}om_{Z/S}(Y, T_{Z/S})$ is $T_{c/S}(t, 1)$, given by

$$y \mapsto (t, 1)^{-1}(0, f(y))(t, 1) = (\mathrm{Ad}(f(y))t - t, f(y)).$$

Hence the image in $\mathcal{H}om_S(Y, \mathrm{Lie}(Z/S))$ is $y \mapsto \mathrm{Ad}(f(y))t - t$. □

Proof of Theorem 5.1. We may assume k algebraically closed and G smooth. As in the beginning of Section 5, let $u\colon A \to G$ be a rational point of X, let H be the inertia at u, let $Y = H\backslash G$, and let $c\colon G \to X$ be the G-equivariant morphism sending g to ug, which factorizes through an immersion $j\colon Y \to X$. Since $H^1(A, \mathrm{Lie}(G)) = 0$ for any action of A on $\mathrm{Lie}(G)$, it follows from Lemma 5.4 that $T^1_c\colon \mathrm{Lie}(G) \to T^u_X$ is an epimorphism. Thus the map $T^{\{H\}}_j\colon T^{\{H\}}_Y \to T^u_X$ is an isomorphism. Since Y is smooth [49, $\mathrm{VI_B}$ 9.2], j is étale [22, Corollaire 17.11.2] and hence an open immersion at this point. In other words, the orbit of u contains an open neighborhood of u. Since the rational points of X form a dense subset [22, Corollaire 10.4.8], the orbits of rational points form an open covering of X, which implies that X is smooth. □

6 Structure theorems for equivariant cohomology rings

Throughout this section κ is a field and k is an algebraically closed field.

Definition 6.1. For a functor $F\colon \mathcal{C} \to \mathcal{D}$ and an object d of $\mathcal{D}$, let $(d \downarrow F) = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/}$ (strict fiber product) be the category whose objects are pairs (c, ϕ) of an object c of $\mathcal{C}$ and a morphism $\phi\colon d \to F(c)$ in $\mathcal{D}$, and arrows are defined in the natural way. Recall that F is said to be *cofinal* if, for every object d of $\mathcal{D}$, the category $(d \downarrow F)$ is nonempty and connected.

If F is cofinal and $G\colon \mathcal{D} \to \mathcal{E}$ is a functor such that $\varinjlim GF$ exists, then $\varinjlim G$ exists and the morphism $\varinjlim GF \to \varinjlim G$ is an isomorphism [33, Theorem IX.3.1].

[7] For compatibility with [49, II 4.1], we write the adjoint action as left action.

Lemma 6.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a full and essentially surjective functor. Then F is cofinal.*

Proof. Let d be an object of $\mathcal{D}$. As F is essentially surjective, there exist an object c of $\mathcal{C}$ and an isomorphism $f\colon d \xrightarrow{\sim} F(c)$ in $\mathcal{D}$, which give an object of $(d \downarrow F)$. As F is full, for any morphism $g\colon d \to F(c')$, with c' an object of $\mathcal{C}$, there exists a morphism $h\colon c \to c'$ in $\mathcal{C}$ such that $F(h) = gf^{-1}$, which gives a morphism $(c, f) \to (c', g)$ in $(d \downarrow F)$. □

We now introduce some enriched categories, which will be of use in the structure theorems, especially Theorem 6.17.

Definition 6.3. Let $\mathcal{D}$ be a category enriched in the category $\mathrm{AlgSp}_{/\kappa}$ of algebraic κ-spaces, with Cartesian product as the monoidal operation [29, Section 1.2]. For objects X and Y of $\mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(X,Y)$ is an algebraic κ-space and composition of morphisms in $\mathcal{D}$ is given by morphisms of algebraic κ-spaces. We denote by $\mathcal{D}(\kappa)$ the category having the same objects as $\mathcal{D}$, in which

$$\mathrm{Hom}_{\mathcal{D}(\kappa)}(X,Y) = (\mathrm{Hom}_{\mathcal{D}}(X,Y))(\kappa).$$

Assume that κ is separably closed. We denote by $\mathcal{D}(\pi_0)$ the category having the same objects as $\mathcal{D}$, in which

$$\mathrm{Hom}_{\mathcal{D}(\pi_0)}(X,Y) = \pi_0(\mathrm{Hom}_{\mathcal{D}}(X,Y)).$$

Note that, if $\mathrm{Hom}_{\mathcal{D}}(X,Y)$ is of finite type, $\mathrm{Hom}_{\mathcal{D}(\pi_0)}(X,Y)$ is a finite set. We have a functor

$$\eta\colon \mathcal{D}(\kappa) \to \mathcal{D}(\pi_0),$$

which is the identity on objects, and sends $f \in \mathrm{Hom}_{\mathcal{D}}(X,Y)(\kappa)$ to the connected component containing it. Assume that $\mathrm{Hom}_{\mathcal{D}}(X,Y)$ is locally of finite type. If κ is algebraically closed, or if for all X, Y in $\mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(X,Y)$ is smooth over κ, then η is full, hence cofinal by Lemma 6.2.

Construction 6.4. Let G be an algebraic group over k, let X be an algebraic space of finite presentation over k, endowed with an action of G, and let ℓ be a prime number. We define a category enriched in the category $\mathrm{Sch}^{\mathrm{ft}}_{/k}$ of schemes of finite type over k,

$$\mathcal{A}_{G,X,\ell},$$

as follows. Objects of $\mathcal{A}_{G,X,\ell}$ are pairs (A, C) where A is an elementary abelian ℓ-subgroup of G and C is a connected component of the algebraic space of fixed points X^A (which is a closed algebraic subspace of X if X is separated). For objects (A, C) and (A', C') of $\mathcal{A}_{G,X}$, we denote by $\mathrm{Trans}_G((A,C),(A',C'))$ the *transporter* of (A, C) into (A', C'), namely the closed subgroup scheme of G representing the functor

$$S \mapsto \{g \in G(S) \mid g^{-1}A_S g \subset A'_S, C_S g \supset C'_S\}.$$

In fact, $\mathrm{Trans}_G((A,C),(A',C'))$ is a closed and open subscheme of the scheme $\mathrm{Trans}_G(A,A')$ defined by the cartesian square

$$\begin{array}{ccc} \mathrm{Trans}_G(A,A') & \longrightarrow & \prod_{a\in A} A' \\ \downarrow & & \downarrow \\ G & \longrightarrow & \prod_{a\in A} G \end{array}$$

where the lower horizontal arrow is given by $g \mapsto (g^{-1}ag)_{a\in A}$. Indeed, if we consider the morphism

$$F\colon \mathrm{Trans}_G(A,A') \times X^{A'} \to X^A \quad (g,x) \mapsto xg^{-1}$$

and the induced map

$$\phi\colon \pi_0(\mathrm{Trans}_G(A,A')) \to \pi_0(X^A) \quad \Gamma \mapsto \pi_0(F)(\Gamma, C'),$$

then $\mathrm{Trans}_G((A,C),(A',C'))$ is the union of the connected components of $\mathrm{Trans}_G(A,A')$ corresponding to $\phi^{-1}(C)$. We define

$$\mathrm{Hom}_{\mathcal{A}_{G,X,\ell}}((A,C),(A',C')) \coloneqq \mathrm{Trans}_G((A,C),(A',C')).$$

Composition of morphisms is given by the composition of transporters

$$\mathrm{Trans}_G((A',C'),(A'',C'')) \times \mathrm{Trans}_G((A,C),(A',C')) \to \mathrm{Trans}_G((A,C),(A'',C'')),$$

which is a morphism of k-schemes. When no confusion arises, we omit ℓ from the notation. We will denote $\mathcal{A}_{G,\mathrm{Spec}(k)}$ by $\mathcal{A}_G$.

For an object (A,C) of $\mathcal{A}_{G,X}$, we denote by $\mathrm{Cent}_G(A,C)$ its *centralizer*, namely the closed subscheme of G representing the functor

$$S \mapsto \{g \in G(S) \mid C_S g = C_S \text{ and } g^{-1}ag = a \text{ for all } a \in A\}.$$

For objects (A,C), (A',C') of $\mathcal{A}_{G,X}$, we have natural injections (cf. [37, (8.2)])

$$\mathrm{Cent}_G(A,C)\backslash\mathrm{Trans}_G((A,C),(A',C')) \to \mathrm{Cent}_G(A)\backslash\mathrm{Trans}_G(A,A') \to \mathrm{Hom}(A,A'). \tag{6.4.1}$$

We let $\mathcal{A}^\flat_{G,X}$ denote the category having the same objects as $\mathcal{A}_{G,X}$, but with morphisms defined by the left hand side of (6.4.1). We call the finite group

$$W_G(A,C) \coloneqq \mathrm{Cent}_G(A,C)\backslash\mathrm{Trans}_G((A,C),(A,C)) \tag{6.4.2}$$

the *Weyl group* of (A,C). This is a subgroup of the finite group

$$W_G(A) = \mathrm{Cent}_G(A)\backslash\mathrm{Norm}_G(A) \subset \mathrm{Aut}(A).$$

The functors

$$\mathcal{A}_{G,X}(k) \to \mathcal{A}_{G,X}(\pi_0) \to \mathcal{A}^\flat_{G,X}$$

(the second one defined via (6.4.1)) are cofinal by Lemma 6.2.

Let k' be an algebraically closed extension of k. We have a functor $\mathcal{A}_{G,X}(k) \to \mathcal{A}_{G_{k'},X_{k'}}(k')$ carrying (A,C) to $(A,C_{k'})$. Since the map

$$\pi_0(\mathrm{Trans}_G((A,C),(A',C')) \to \pi_0(\mathrm{Trans}_{G_{k'}}((A,C_{k'}),(A',C'_{k'})))$$

is a bijection, this induces a functor $\mathcal{A}_{G,X}(\pi_0) \to \mathcal{A}_{G_{k'},X_{k'}}(\pi_0)$.

In the rest of the section we assume ℓ invertible in k.

Lemma 6.5. *The category $\mathcal{A}_{G,X}(\pi_0)$ is essentially finite, and the functor $\mathcal{A}_{G,X}(\pi_0) \to \mathcal{A}_{G_{k'},X_{k'}}(\pi_0)$ is an equivalence. In particular, $\mathcal{A}_G(\pi_0)$ is essentially finite.*

Proof. Let S be a set of representatives of isomorphisms classes of objects of $\mathcal{A}_G(\pi_0)$. In other words, S is a set of representatives of conjugacy classes of elementary abelian ℓ-subgroups of G. By Corollary 5.3, this is a finite set. Let T be the set of objects (A,C) of $\mathcal{A}_{G,X}(\pi_0)$ such that $A \in S$. Then T is a finite set. The conclusion follows from the following facts:

(a) For (A,C) and (A',C') in $\mathcal{A}_{G,X}$, $\mathrm{Hom}_{\mathcal{A}_{G,X}(\pi_0)}((A,C),(A',C'))$ is finite (Definition 6.3), and, by Construction 6.4,

$$\mathrm{Hom}_{\mathcal{A}_{G,X}(\pi_0)}((A,C),(A',C')) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}_{G_{k'},X_{k'}}(\pi_0)}((A,C_{k'}),(A',C'_{k'})).$$

(b) The finite set T is a set of representatives of isomorphism classes of objects of $\mathcal{A}_{G,X}(\pi_0)$, and $\{(A,C_{k'}) \mid (A,C) \in T\}$ is a set of representatives of isomorphism classes of objects of $\mathcal{A}_{G_{k'},X_{k'}}(\pi_0)$.

Indeed, (b) follows from the following obvious lemma. □

Lemma 6.6. *Let B, C be sets endowed with equivalence relations denoted by $\simeq$ and let $f\colon B \to C$ be a map such that $b \simeq b'$ implies $f(b) \simeq f(b')$. Let S be a set of representatives of C. For every $s \in S$, let T_s be a set of representatives of $f^{-1}(s)$. Then $\bigcup_{s\in S} T_s$ is a set of representatives of B if and only if for every $b \in B$ and every $c \in S$ such that $f(b) \simeq c$, there exists $b' \in f^{-1}(c)$ such that $b \simeq b'$.*

Remark 6.7. Let G be an algebraic group over k and let T be a subtorus of G. Then $W_G(T) = \mathrm{Cent}_G(T)\backslash\mathrm{Norm}_G(T)$ is a finite subgroup of $\mathrm{Aut}(T)$. The inclusions

$$\mathrm{Norm}_G(T) \subset \mathrm{Norm}_G(T[\ell]), \quad \mathrm{Cent}_G(T) \subset \mathrm{Cent}_G(T[\ell])$$

induce a homomorphism $\rho\colon W_G(T) \to W_G(T[\ell])$. Via the isomorphisms $\mathrm{Aut}(T) \simeq \mathrm{Aut}(M)$ and $\mathrm{Aut}(T[\ell]) \simeq \mathrm{Aut}(M/\ell M)$, where $M = \mathrm{X}^*(T)$, ρ is compatible with the reduction homomorphism $\mathrm{Aut}(M) \to \mathrm{Aut}(M/\ell M)$. If T is a maximal torus, then ρ is surjective by the proof of [43, 1.1.1].

For $\ell > 2$, ρ is injective. In fact, for an element g of $\mathrm{Ker}(\mathrm{Aut}(M) \to \mathrm{Aut}(M/\ell M))$ and arbitrary ℓ, the ℓ-adic logarithm $\log(g) := \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m}(g-1)^m \in \ell\mathrm{End}(M) \otimes \mathbb{Z}_\ell$ is well defined. If $g^n = \mathrm{id}$ for some $n \geq 1$, then $n\log(g) = \log(g^n) = 0$, so that $\log(g) = 0$. In the case $\ell > 2$, we then have $g = \exp\log(g) = \mathrm{id}$. For $\ell = 2$, ρ is not injective in general. For example, if $G = \mathrm{SL}_2$ and T is a maximal torus, then $W_G(T) \simeq \mathbb{Z}/2$ and $W_G(T[2]) = \{1\}$.

If $G = \mathrm{GL}_n$ and T is a maximal torus, then ρ is an isomorphism for arbitrary ℓ. In fact, in this case, $\mathrm{Norm}_G(T) = \mathrm{Norm}_G(T[\ell])$ and $\mathrm{Cent}_G(T) = \mathrm{Cent}_G(T[\ell])$.

Notation 6.8. We will sometimes omit the constant coefficient $\mathbb{F}_\ell$ from the notation. We will sometimes write H_G^* for $H^*(BG) = H^*(BG, \mathbb{F}_\ell)$.

Construction 6.9. Let $T = \mathrm{Trans}_G(A, A')$, let $g \in T(k)$, and let $c_g\colon A \to A'$ be the map $a \mapsto g^{-1}ag$. In the above notation, the morphism $Bc_g\colon BA \to BA'$ induces a homomorphism $\theta_g\colon H_{A'}^* \to H_A^*$. This defines a presheaf (H_A^*, θ_g) on $\mathcal{A}_G^\flat$, hence on $\mathcal{A}_{G,X}^\flat$.

If (A, C) is an object of $\mathcal{A}_{G,X}$, we have

$$H^*([C/A]) = H_A^* \otimes H^*(C).$$

The restriction $H^*([X/G]) \to H^*([C/A])$ induced by the inclusion $(C, A) \to (X, G)$, composed with the projection

$$H^*([C/A]) \to H_A^* \tag{6.9.1}$$

induced by $H^*(C) \to H^0(C) = \mathbb{F}_\ell$, defines a homomorphism

$$(A, C)^*\colon H^*([X/G]) \to H_A^*. \tag{6.9.2}$$

For $g \in \mathrm{Trans}((A, C), (A', C'))(k) \subset T(k)$, we have the following 2-commutative square of groupoids in the category $\mathrm{AlgSp}_{/U}$ (Construction 1.1):

$$\begin{array}{ccc} (C', A)_\bullet & \xrightarrow{(\mathrm{id}, c_g)} & (C', A')_\bullet \\ {\scriptstyle (g^{-1}, \mathrm{id})}\downarrow & \not\swarrow & \downarrow \\ (C, A)_\bullet & \longrightarrow & (X, G)_\bullet \end{array}$$

(with trivial action of A and A' on C' and trivial action of A on C), where the 2-morphism is given by g. The corresponding 2-commutative square of Artin stacks

$$\begin{array}{ccc} BA \times C' & \longrightarrow & BA' \times C' \\ \downarrow & & \downarrow \\ BA \times C & \longrightarrow & [X/G] \end{array}$$

induces by adjunction (Notation 2.3) the following commutative square:

$$\begin{array}{ccc} H^*([X/G]) & \longrightarrow & H^*([C/A]) \\ \downarrow & & \downarrow {\scriptstyle [g^{-1}/\mathrm{id}]^*} \\ H^*([C'/A']) & \xrightarrow{[\mathrm{id}/c_g]^*} & H^*([C'/A]). \end{array}$$

Composing with the projections (6.9.1), we obtain the following commutative diagram:

$$\begin{array}{ccc} H^*([X/G]) & & \\ {\scriptstyle (A',C')^*}\downarrow & \searrow^{(A,C)^*} & \\ H^*_{A'} & \xrightarrow[\theta]{} & H^*_A. \end{array}$$

Therefore the maps $(A,C)^*$ (6.9.2) define a homomorphism

$$a(G,X)\colon H^*([X/G]) \to \varprojlim_{\mathcal{A}^\flat_{G,X}} (H^*_A, \theta_g). \tag{6.9.3}$$

Note that the right-hand side is the equalizer of

$$(j_1, j_2)\colon \prod_{(A,C)\in\mathcal{A}_{G,X}} H^*_A \rightrightarrows \prod_{g\colon (A,C)\to(A',C')} H^*_A,$$

where g runs through morphisms in $\mathcal{A}^\flat_{G,X}$, $j_1(h_{(A,C)}) = (h_{(A,C)})_g$, $j_2(h_{(A,C)}) = (\theta_g h_{(A',C')})_g$. Moreover, by the finiteness results Corollary 4.8 and Lemma 6.5, the right-hand side of (6.9.3) is a finite $H^*(BG)$-module, and, in particular, a finitely generated $\mathbb{F}_\ell$-algebra.

To state our main result for the map $a(G,X)$ (6.9.3), we need to recall the notion of uniform F-isomorphism. For future reference, we give a slightly extended definition as follows.

Definition 6.10. Let GrVec be the category of graded $\mathbb{F}_\ell$-vector spaces. It is an $\mathbb{F}_\ell$-linear $\otimes$-category. The commutativity constraint of GrVec follows Koszul's rule of signs, such that a (pseudo-)ring in GrVec is an anti-commutative graded $\mathbb{F}_\ell$-(pseudo-)algebra.

Let $\mathcal{C}$ be a category. As a special case of Construction 3.7, the functor category $\mathrm{GrVec}^{\mathcal{C}} := \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{GrVec})$ is a $\mathbb{F}_\ell$-linear $\otimes$-category. The functor $\varprojlim_{\mathcal{C}}\colon \mathrm{GrVec}^{\mathcal{C}} \to \mathrm{GrVec}$ is the right adjoint to the unital $\otimes$-functor $\mathrm{GrVec} \to \mathrm{GrVec}^{\mathcal{C}}$, thus has a right unital $\otimes$-structure. If $u\colon R \to S$ is a homomorphism of pseudo-rings in $\mathrm{GrVec}^{\mathcal{C}}$, we say that u is a *uniform F-injection* (resp. *uniform F-surjection*) if there exists an integer $n \geq 0$ such that for any object i of $\mathcal{C}$ and any homogeneous element (or, equivalently, any element) a in the kernel of u_i (resp. in S_i), $a^{\ell^n} = 0$ (resp. a^{ℓ^n} is in the image of u_i). Note that $a^{\ell^n} = 0$ for some $n \geq 0$ is equivalent to $a^m = 0$ for some $m \geq 1$. We say u is a *uniform F-isomorphism* if it is both a uniform F-injection and a uniform F-surjection. These definitions apply in particular to GrVec by taking $\mathcal{C}$ to be a discrete category of one object, in which case the notion of a uniform F-isomorphism coincides with the definition in [36, Section 3].

The following result is an analogue of Quillen's theorem ([36, Theorem 6.2], [37, Theorem 8.10]):

Theorem 6.11. *Let X be a separated algebraic space of finite type over k, and let G be an algebraic group over k acting on X. Then the homomorphism $a(G,X)$ (6.9.3) is a uniform F-isomorphism (Definition 6.10).*

Remark 6.12. Let A be an elementary abelian ℓ-group of rank $r \geq 0$. We identify $H^1(BA, \mathbb{F}_\ell)$ with $\check{A} = \mathrm{Hom}(A, \mathbb{F}_\ell)$. Recall [36, Section 4] that we have a natural identification of $\mathbb{F}_\ell$-graded algebras

$$H^*(BA, \mathbb{F}_\ell) = \begin{cases} \mathrm{S}(\check{A}) & \text{if } \ell = 2 \\ \wedge(\check{A}) \otimes \mathrm{S}(\beta\check{A}) & \text{if } \ell > 2, \end{cases}$$

where S (resp. $\wedge$) denotes a symmetric (resp. exterior) algebra over $\mathbb{F}_\ell$, and $\beta\colon \check{A} \to H^2(BA, \mathbb{F}_\ell)$ is the Bockstein operator. In particular, if $\{x_1, \dots, x_r\}$ is a basis of $\check{A}$ over $\mathbb{F}_\ell$, then

$$H^*(BA, \mathbb{F}_\ell) = \begin{cases} \mathbb{F}_\ell[x_1, \dots, x_r] & \text{if } \ell = 2 \\ \wedge(x_1, \dots, x_r) \otimes \mathbb{F}_\ell[y_1, \dots, y_r] & \text{if } \ell > 2 \end{cases}$$

where $y_i = \beta x_i$.

Corollary 6.13. *With X and G as in Theorem 6.11, let $K \in D_c^b([X/G], \mathbb{F}_\ell)$. The Poincaré series*

$$\mathrm{PS}_t(H^*([X/G], K)) = \sum_{i\geq 0} \dim_{\mathbb{F}_\ell} H^i([X/G], K)t^i$$

is a rational function of t of the form $P(t)/\prod_{1\leq i\leq n}(1-t^{2i})$, with $P(t) \in \mathbb{Z}[t]$. The order of the pole of $\mathrm{PS}_t(H^*([X/G]))$ at $t = 1$ is the maximum rank of an elementary abelian ℓ-subgroup A of G such that $X^A \neq \emptyset$.

Proof. By Theorem 4.6, $H^*([X/G], K)$ is a finitely generated module over $H^*([X/G])$, which is a finitely generated algebra over $\mathbb{F}_\ell$. Therefore the Poincaré series $\mathrm{PS}_t(H^*([X/G], K))$ is a rational function of t, and the order of the pole at $t = 1$ of $\mathrm{PS}_t(H^*([X/G]))$ is equal to the dimension of the commutative ring $H^{2*}([X/G])$. To show that $\mathrm{PS}_t(H^*([X/G], K))$ is of the form given in Corollary 6.13, recall (Remark 4.9) that we have shown in the proof of Theorem 4.6 that $H^*([X/G], K)$ is a quotient $H^*(BH)$-module of $H^*([X/H], f^*K)$ for a certain affine subgroup H of G, f denoting the canonical morphism $[X/H] \to [X/G]$. Embedding H into some GL_n and applying Corollary 4.7, we deduce that $H^*([X/G], K)$ is a finite $H^*(B\mathrm{GL}_n)$-module. Since $H^*(B\mathrm{GL}_n) \simeq \mathbb{F}_\ell[c_1, \dots, c_n]$, where c_i is of degree $2i$ (Theorem 4.4), $\mathrm{PS}_t(M^*)$ is of the form $P(t)/\prod_{1\leq i\leq n}(1-t^{2i})$ with $P(t) \in \mathbb{Z}[t]$ for every finite graded $H^*(B\mathrm{GL}_n)$-module M^*. The last assertion of Corollary 6.13 is derived from Theorem 6.11 as in [36, Theorem 7.7]. One can also see it in a more geometric way, observing that the reduced spectrum of $H^{\varepsilon*}([X/G])$ (where $\varepsilon = 1$ if $\ell = 2$ and 2 otherwise) is homeomorphic to an amalgamation of standard affine spaces $\underline{A} = \mathrm{Spec}(H_A^{\varepsilon*})_{\mathrm{red}}$ associated with the objects (A, C) of $\mathcal{A}_{G,X}$ (see Construction 11.1). ☐

Example 6.14. Let G be a connected reductive group over k with no ℓ-torsion, and let T be a maximal torus of G. Let $\iota\colon \mathcal{A}' \hookrightarrow \mathcal{A}_G^\flat$ be the full subcategory spanned by $T[\ell]$. The functor ι is cofinal. Indeed, for every object A of $\mathcal{A}_G^\flat$, since A is toral, there exists a morphism $c_g\colon A \to T[\ell]$ in $\mathcal{A}_G^\flat$. Moreover, for morphisms $c_g\colon A \to T[\ell]$, $c_{g'}\colon A \to T[\ell]$ in $\mathcal{A}_G^\flat$, there exists an isomorphism $c_h\colon T[\ell] \to T[\ell]$ such that $c_h c_g = c_{g'}$ in $\mathcal{A}_G^\flat$, by [43, 1.1.1] applied to the conjugation $c_{g^{-1}g'}\colon c_g(A) \to c_{g'}(A)$. Let $W = W_G(T)$. The map $a(G, \mathrm{Spec}(k))$ can be identified with the injective F-isomorphism

$$H_G^* \simeq (H_T^*)^W \to (H_{T[\ell]}^*)^W$$

induced by restriction (where the isomorphism is (4.11.2)). In particular,

$$\varprojlim_{A\in\mathcal{A}_G^\flat} (H_A^{\varepsilon*})_{\mathrm{red}} \simeq \mathrm{S}(T[\ell]^\vee)^W,$$

where $\varepsilon = 1$ if $\ell = 2$ and $\varepsilon = 2$ if $\ell > 2$. Moreover, for $\ell > 2$, $a(G, \mathrm{Spec}(k))$ induces an isomorphism $H_G^{2*} \simeq ((H_{T[\ell]}^{2*})_{\mathrm{red}})^W$.

Example 6.15. Let $X = X(\Sigma)$ be a toric variety over k with torus T, where Σ is a fan in $N \otimes \mathbb{R}$ and $N = \mathrm{X}_*(T)$. We identify $T[\ell]$ with $N \otimes \mu_\ell$. The inertia $I_\sigma \subset T$ of the orbit O_σ corresponding to a cone $\sigma \in \Sigma$ is $N_\sigma \otimes \mathbb{G}_\mathrm{m}$, where N_σ is the sublattice of N generated by $N \cap \sigma$, so that $A_\sigma = I_\sigma[\ell] \simeq N_\sigma \otimes \mu_\ell$. The latter can be identified with the image of $N\cap\sigma$ in $N\otimes\mathbb{F}_\ell$. This defines an object (A_σ, C_σ) of $\mathcal{A}_{T,X}$, where C_σ is the connected component of X^{A_σ} containing O_σ. The functor $\Sigma \to \mathcal{A}_{T,X}^\flat$ is cofinal. Thus we have a canonical isomorphism

$$\varprojlim_{A\in\mathcal{A}_{T,X}^\flat} (H_A^{\varepsilon*})_{\mathrm{red}} \simeq \varprojlim_{\sigma\in\Sigma} (H_{A_\sigma}^{\varepsilon*})_{\mathrm{red}}.$$

Note that $(H_{A_\sigma}^*)_{\mathrm{red}}$ can be canonically identified with $\mathrm{S}(M_\sigma) \otimes \mathbb{F}_\ell$, where $M_\sigma = M/(M \cap \sigma^\perp)$ and $\mathrm{S}(M_\sigma)$ is the algebra of integral polynomial functions on σ. In particular, we have a canonical isomorphism

$$\varprojlim_{A\in\mathcal{A}_{T,X}^\flat} (H_A^{\varepsilon*})_{\mathrm{red}} \simeq \mathrm{PP}^*(\Sigma) \otimes \mathbb{F}_\ell, \tag{6.15.1}$$

where

$$\mathrm{PP}^*(\Sigma) = \{f\colon \mathrm{Supp}(\Sigma) \to \mathbb{R} \mid (f \mid \sigma) \in \mathrm{S}(M_\sigma) \text{ for each } \sigma \in \Sigma\}$$

is the algebra of piecewise polynomial functions on Σ. Recall that Payne established an isomorphism from the integral equivariant Chow cohomology ring $A^*_T(X)$ of Edidin and Graham [14, 2.6] onto $\mathrm{PP}^*(\Sigma)$ [35, Theorem 1]. Combining Theorem 6.11 and (6.15.1), we obtain a uniform F-isomorphism

$$H^*([X/T], \mathbb{F}_\ell) \to \mathrm{PP}^*(\Sigma) \otimes \mathbb{F}_\ell.$$

If X is smooth, this is an isomorphism, and $\mathrm{PP}^*(\Sigma)$ is isomorphic to the Stanley-Reisner ring of Σ [3, Section 4].

In the rest of this section, we state an analogue of Theorem 6.11 with coefficients.

Construction 6.16. Let G be an algebraic group over k, X an algebraic k-space endowed with an action of G, and $K \in D^+_{\mathrm{cart}}([X/G], \mathbb{F}_\ell)$.

If A, A' are elementary abelian ℓ-subgroups of G and $g \in G(k)$ conjugates A into A' (i.e. $g^{-1}Ag \subset A'$), A acts trivially on $X^{A'}$ via $c_g = A \to A'$ (where c_g is the conjugation $s \mapsto g^{-1}sg$), and we have an equivariant morphism $(1, c_g)\colon (X^{A'}, A) \to (X, G)$, where 1 denotes the inclusion $X^{A'} \subset X$, inducing

$$[1/c_g]\colon [X^{A'}/A] = BA \times X^{A'} \to [X/G].$$

We thus have, for all q, a restriction map

$$H^q([X/G], K) \to H^q([X^{A'}/A], [1/c_g]^*K).$$

On the other hand, we have a natural projection

$$\pi\colon [X^{A'}/A] = BA \times X^{A'} \to X^{A'},$$

hence an edge homomorphism for the corresponding Leray spectral sequence

$$H^q([X^{A'}/A], [1/c_g]^*K) \to H^0(X^{A'}, R^q\pi_*[1/c_g]^*K).$$

By composition we get a homomorphism

$$a^q(A, A', g)\colon H^q([X/G], K) \to H^0(X^{A'}, R^q\pi_*[1/c_g]^*K). \tag{6.16.1}$$

Since $R^*\pi_*\mathbb{F}_\ell = \bigoplus_q R^q\pi_*\mathbb{F}_\ell$ is a constant sheaf of value $H^*(BA, \mathbb{F}_\ell)$, $R^*\pi_*[1/c_g]^*K = \bigoplus_q R^q\pi_*[1/c_g]^*K$ is endowed with a $H^*(BA, \mathbb{F}_\ell)$-module structure by Constructions 3.4 and 3.7, which induces a $H^*(BG, \mathbb{F}_\ell)$-module structure via the ring homomorphism $[1/c_g]^*\colon H^*(BG, \mathbb{F}_\ell) \to H^*(BA, \mathbb{F}_\ell)$. The map $a(A, A', g) = \bigoplus_q a^q(A, A', g)$ is $H^*(BG, \mathbb{F}_\ell)$-linear.

If (Z, Z', h) is a second triple consisting of elementary abelian ℓ-subgroups Z, Z', and $h \in G(k)$ such that $c_h\colon Z \to Z'$, the datum of elements a and b of $G(k)$ such that $g = ahb$ and $c_a\colon A \to Z$, $c_b\colon Z' \to A'$, defines a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{c_g} & A' \\ {\scriptstyle c_a}\downarrow & & \uparrow{\scriptstyle c_b} \\ Z & \xrightarrow{c_h} & Z', \end{array} \tag{6.16.2}$$

hence a morphism $[b^{-1}/c_a]\colon [X^{A'}/A] \to [X^{Z'}/Z]$, fitting into a 2-commutative diagram

$$\begin{array}{ccccc} & & [X^{A'}/A] & \xrightarrow{\pi} & X^{A'} \\ & {\scriptstyle [1/c_g]}\swarrow \ \Downarrow & \downarrow{\scriptstyle [b^{-1}/c_a]} & & \downarrow{\scriptstyle b^{-1}} \\ [X/G] & \xleftarrow[{[1/c_h]}]{} & [X^{Z'}/Z] & \xrightarrow{\pi} & X^{Z'}, \end{array} \tag{6.16.3}$$

where the 2-morphism of the triangle is induced by b. Consider the homomorphism

(6.16.4)

$$(a,b)^*\colon H^0(X^{Z'}, R^q\pi_*[1/c_h]^*K) \to H^0(X^{A'}, (b^{-1})^*R^q\pi_*[1/c_h]^*K) \to H^0(X^{A'}, R^q\pi_*[1/c_g]^*K),$$

where the first map is adjunction by b^{-1} and the second map is base change map for the square in (6.16.3). This fits into a commutative triangle

$$\begin{array}{ccc} H^q([X/G],K) & & \\ \downarrow & \searrow & \\ H^0(X^{Z'}, R^q\pi_*[1/c_h]^*K) & \xrightarrow{(a,b)^*} & H^0(X^{A'}, R^q\pi_*[1/c_g]^*K), \end{array} \tag{6.16.5}$$

where the vertical and oblique maps are given by (6.16.1). Denote by

$$\mathcal{A}_G(k)^\natural \tag{6.16.6}$$

the following category. Objects of $\mathcal{A}_G(k)^\natural$ are triples (A, A', g) as above, morphisms $(A, A', g) \to (Z, Z', h)$ are pairs $(a, b) \in G(k) \times G(k)$ such that $g = ahb$ and $c_a\colon A \to Z$, $c_b\colon Z' \to A'$. Via the maps $(a, b)^*$ (6.16.4), the groups $H^0(X^{A'}, R^q\pi_*[1/c_g]^*K)$ form a projective system indexed by $\mathcal{A}_G(k)^\natural$, and by the commutativity of (6.16.5) we get a homomorphism

$$a^q_G(X, K)\colon H^q([X/G], K) \to R^q_G(X, K), \tag{6.16.7}$$

where

$$R^q_G(X, K) := \varprojlim_{(A,A',g)\in\mathcal{A}_G(k)^\natural} H^0(X^{A'}, R^q\pi_*[1/c_g]^*K). \tag{6.16.8}$$

Since $\bigoplus_q (a, b)^*$ is $H^*(BG, \mathbb{F}_\ell)$-linear, $R^*_G(X, K) := \bigoplus_q R^q_G(X, K)$ is endowed with a structure of $H^*(BG, \mathbb{F}_\ell)$-module. The map

$$a_G(X, K) = \bigoplus_q a^q_G(X, K)\colon H^*([X/G], K) \to R^*_G(X, K) \tag{6.16.9}$$

induced by (6.16.7) is a homomorphism of $H^*(BG, \mathbb{F}_\ell)$-modules. If K is a (pseudo-)ring in $D^+_{\mathrm{cart}}([X/G], \mathbb{F}_\ell)$, $R^*_G(X, K)$ is a $\mathbb{F}_\ell$-(pseudo-)algebra and $a_G(X, K)$ is a homomorphism of $\mathbb{F}_\ell$-(pseudo-)algebras.

Theorem 6.17. *Let G be an algebraic group over k, X a separated algebraic space of finite type over k endowed with an action of G, and $K \in D^+_c([X/G], \mathbb{F}_\ell)$.*

(a) $R^q_G(X, K)$ is a finite-dimensional $\mathbb{F}_\ell$-vector space for all q; if $K \in D^b_c([X/G], \mathbb{F}_\ell)$, $R^*_G(X, K)$ is a finite module over $H^*(BG, \mathbb{F}_\ell)$.

(b) If K is a pseudo-ring in $D^+_c([X/G], \mathbb{F}_\ell)$ (Construction 3.8), the kernel of the homomorphism $a_G(X, K)$ (6.16.9) is a nilpotent ideal of $H^*([X/G], K)$. If, moreover, K is commutative, then $a_G(X, K)$ is a uniform F-isomorphism (Definition 6.10).

Remark 6.18. The projective limit in (6.16.7) is the equalizer of the double arrow

$$(j_1, j_2)\colon \prod_{A\in\mathcal{A}_G} \Gamma(X^A, R^q\pi_{A*}[1/c_1]^*K) \rightrightarrows \prod_{(A,A',g)\in\mathcal{A}_G(k)^\natural} \Gamma(X^{A'}, R^q\pi_{(A,A',g)*}[1/c_g]^*K),$$

where $\pi_A = \pi_{(A,A,1)}$, $[1/c_1]\colon [X^A/A] \to [X/G]$, j_1 is induced by $(1, g)\colon (A, A', g) \to (A, A, 1)$ and j_2 is induced by $(g, 1)\colon (A, A', g) \to (A', A', 1)$.

This is a consequence of the following general fact (applied to $\mathcal{C} = \mathcal{A}_G(k)$). Let $\mathcal{C}$ be a category. Define a category $\mathcal{C}^\natural$ as follows. The objects of $\mathcal{C}^\natural$ are the morphisms $A \to A'$ of $\mathcal{C}$. A morphism in $\mathcal{C}^\natural$ from $A \to A'$ to $Z \to Z'$ is a pair of morphisms $(A \to Z, Z' \to A')$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \uparrow \\ Z & \longrightarrow & Z'. \end{array}$$

Let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}^\natural$. Then the sequence

$$\Gamma(\widehat{\mathcal{C}^\natural}, \mathcal{F}) \to \prod_{A\in\mathcal{C}} \mathcal{F}(\mathrm{id}_A) \rightrightarrows \prod_{(a\colon A\to A')\in\mathcal{C}^\natural} \mathcal{F}(a)$$

is exact. Here the two projections are induced by $(\mathrm{id}_A, a)\colon a \to \mathrm{id}_A$ and $(a, \mathrm{id}_{A'})\colon a \to \mathrm{id}_{A'}$, respectively.

Indeed, because the two compositions are equal, we have a map $s\colon \Gamma(\widehat{\mathcal{C}^\natural}, \mathcal{F}) \to K$, where K is the equalizer of the double arrow. It is straightforward to check that the map $K \to \prod_{a\in\mathcal{C}^\natural} \mathcal{F}(a)$ factors through $\Gamma(\widehat{\mathcal{C}^\natural}, \mathcal{F})$ to give the inverse of s.

Note that this statement generalizes the calculation of *ends* $\int_{A\in\mathcal{C}} F(A, A)$ [33, Section IX.5] of a functor F from $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ to the category of sets. More generally, for any category $\mathcal{D}$ and any functor $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$, $\int_{A\in\mathcal{C}} F(A, A)$ can be identified with the limit $\varprojlim_{a\colon A\to A'} F(A, A')$ indexed by $\mathcal{C}^\natural$.

Remark 6.19. For $K = \mathbb{F}_\ell$, the commutative diagram

$$\begin{array}{ccccc}
H^*([X/G], \mathbb{F}_\ell) & \longrightarrow & \prod_{(A,C)\in\mathcal{A}_{G,X}} H^*_A & \rightrightarrows & \prod_{g\colon (A,C)\to(A',C')} H^*_A \\
& \searrow & \downarrow \simeq & & \downarrow \simeq \\
& & \prod_{A\in\mathcal{A}_G} \Gamma(X^A, R^q\pi_{A*}\mathbb{F}_\ell) & \rightrightarrows & \prod_{(A,A',g)\in\mathcal{A}_G(k)^\natural} \Gamma(X^{A'}, R^q\pi_{(A,A',g)*}\mathbb{F}_\ell)
\end{array}$$

induces a commutative diagram

$$\begin{array}{ccc}
H^*([X/G], \mathbb{F}_\ell) & \xrightarrow{a(G,X)} & \varprojlim_{\mathcal{A}^\flat_{G,X}} H^*_A \\
& \searrow^{a_G(X,\mathbb{F}_\ell)} & \downarrow \simeq \\
& & R^*_G(X, \mathbb{F}_\ell).
\end{array}$$

Therefore Theorem 6.17 generalizes Theorem 6.11.

Part (b) of Theorem 6.17 will be proved as a corollary of a more general structure theorem (Theorem 8.3). Part (a) will follow from the next lemma.

Lemma 6.20. *Let $\mathcal{E}_G$ be the category enriched in $\mathrm{Sch}^{\mathrm{ft}}_{/k}$ having the same objects as $\mathcal{A}_G(k)^\natural$ and in which $\mathrm{Hom}_{\mathcal{E}_G}((A, A', g), (Z, Z', h))$ is the subscheme of $G \times G$ representing the presheaf of sets on $\mathrm{AlgSp}_{/k}$:*

$$S \mapsto \{(a, b) \in (G \times G)(S) \mid a^{-1}A_S a \subset Z_S, b^{-1}Z'_S b \subset A'_S, g = ahb\}$$

(so that by definition $\mathcal{E}_G(k) = \mathcal{A}_G(k)^\natural$).

(a) The functor $F\colon \mathcal{E}_G(\pi_0) \to \mathcal{A}_G(\pi_0)^\natural$ carrying (A, A', g) to (A, A', γ), where γ is the connected component of $\mathrm{Trans}_G(A, A')$ containing g, is an equivalence of categories. In particular, $\mathcal{E}_G(\pi_0)$ is equivalent to a finite category, and for every algebraically closed extension k' of k, the natural functor $\mathcal{E}_G(\pi_0) \to \mathcal{E}_{G_{k'}}(\pi_0)$ is an equivalence of categories.

(b) The projective system $H^0(X^{A'}, R^q\pi_*[1/c_g]^*K)$ indexed by $(A, A', g) \in \mathcal{A}_G(k)^\natural$ factors through $\mathcal{E}_G(\pi_0)$.

Remark 6.21. The projective system in Lemma 6.20 (b) does not factor through $(\mathcal{A}^\flat_G)^\natural$ in general. Indeed, if G is a finite discrete group of order prime to ℓ, then $\mathcal{A}_G(\pi_0)$ and $\mathcal{A}_G(\pi_0)^\natural$ are both connected groupoids of fundamental group G, while $\mathcal{A}^\flat_G$ is a simply connected groupoid. If $K \in \mathrm{Mod}_c(BG, \mathbb{F}_\ell)$, then the projective system in Lemma 6.20 (b) can be identified with the $\mathbb{F}_\ell$-representation of G corresponding to K.

The proof of Lemma 6.20 will be given after Remark 6.26. We will exploit the fact that the family of stacks $[X^{A'}/A]$ parameterized by $(A, A', g) \in \mathcal{A}_G(k)^\natural$ underlies a family "algebraically parameterized" by $\mathcal{E}_G$. To make sense of this, the following general framework will be convenient.

Definition 6.22. Let $\mathcal{D}$ be a category enriched in $\mathrm{AlgSp}_{/\kappa}$ (Definition 6.3). By a *family of Artin κ-stacks parameterized by $\mathcal{D}$*, or, for short, an *Artin $\mathcal{D}$-stack*, we mean a collection

$$X = (X_A, x_{A,B}, \sigma_A, \gamma_{A,B,C})_{A,B,C\in\mathcal{D}},$$

where X_A is an Artin stack over κ, $x_{A,B}\colon X_A \times \mathrm{Hom}_{\mathcal{D}}(A,B) \to X_B$ is a morphism of Artin stacks over κ, σ_A and $\gamma_{A,B,C}$ are 2-morphisms:

$$\begin{array}{ccc} X_A & \xrightarrow{\mathrm{id}_{X_A}\times i} & X_A \times \mathrm{Hom}_{\mathcal{D}}(A,A) \\ & \searrow^{\mathrm{id}} \quad \sigma_A \Uparrow & \downarrow{\scriptstyle x_{A,A}} \\ & & X_A \end{array}$$

$$\begin{array}{ccc} X_A \times \mathrm{Hom}_{\mathcal{D}}(A,B) \times \mathrm{Hom}_{\mathcal{D}}(B,C) & \xrightarrow{\mathrm{id}_{X_A}\times c} & X_A \times \mathrm{Hom}_{\mathcal{D}}(A,C) \\ {\scriptstyle x_{A,B}\times \mathrm{id}_{\mathrm{Hom}_{\mathcal{D}}(B,C)}}\downarrow & \gamma_{A,B,C} \Uparrow & \downarrow{\scriptstyle x_{A,C}} \\ X_B \times \mathrm{Hom}_{\mathcal{D}}(B,C) & \xrightarrow{x_{B,C}} & X_C, \end{array}$$

satisfying identities of 2-morphisms expressing the unit and associativity axioms. Here $i\colon \mathrm{Spec}(\kappa) \to \mathrm{Hom}_{\mathcal{D}}(A,A)$ is the unit section and $c\colon \mathrm{Hom}_{\mathcal{D}}(A,B) \times \mathrm{Hom}_{\mathcal{D}}(B,C) \to \mathrm{Hom}_{\mathcal{D}}(A,C)$ is the composition.

A morphism $f\colon X \to Y$ of Artin $\mathcal{D}$-stacks is a collection $((f_A)_{A\in\mathcal{D}}, (\phi_{A,B})_{A,B\in\mathcal{D}})$, where $f_A\colon X_A \to Y_A$ is a morphism of Artin stacks over κ and $\phi_{A,B}$ is a 2-morphism:

$$\begin{array}{ccc} X_A \times_{\mathrm{Spec}(\kappa)} \mathrm{Hom}_{\mathcal{D}}(A,B) & \xrightarrow{x_{A,B}} & X_B \\ {\scriptstyle f_A\times\mathrm{id}}\downarrow & \Swarrow \phi_{A,B} & \downarrow{\scriptstyle f_B} \\ Y_A \times_{\mathrm{Spec}(\kappa)} \mathrm{Hom}_{\mathcal{D}}(A,B) & \xrightarrow{y_{A,B}} & Y_B \end{array}$$

satisfying certain identities of 2-morphisms with respect to the unit section i and the composition c.

Definition 6.23. Let Λ be a commutative ring and let X be an Artin $\mathcal{D}$-stack. We define a category $D_{\mathrm{cart}}(X,\Lambda)$ as follows. An object of $D_{\mathrm{cart}}(X,\Lambda)$ is a collection $((K_A)_{A\in\mathcal{D}}, (\alpha_{A,B})_{A,B\in\mathcal{D}})$, where $K_A \in D_{\mathrm{cart}}(X_A,\Lambda)$, $\alpha_{A,B}\colon x_{A,B}^*K_B \to p^*K_A$, $p\colon X_A \times \mathrm{Hom}_{\mathcal{D}}(A,B) \to X_A$ is the projection, such that the following diagrams commute

$$\begin{array}{ccc} i^*x_{A,A}^*K_A & \xrightarrow{i^*\alpha_{A,A}} & i^*p^*K_A \\ & \searrow^{\simeq}_{\sigma_A^*} & \downarrow\simeq \\ & & K_A \end{array} \qquad \begin{array}{ccc} x_{A,C}^*K_C & \xrightarrow{\alpha_{A,C}} & p^*K_A \\ {\scriptstyle \gamma_{A,B,C}^*}\downarrow\simeq & & \uparrow{\scriptstyle \alpha_{A,B}} \\ x_{A,B}^*x_{B,C}^*K_C & \xrightarrow{\alpha_{B,C}} & p^*x_{A,B}^*K_B. \end{array}$$

A morphism $K \to L$ in $D_{\mathrm{cart}}(X,\Lambda)$ is a collection $(K_A \to L_A)_{A\in\mathcal{D}}$ of morphisms in $D_{\mathrm{cart}}(X_A,\Lambda)$ commuting with $\alpha_{A,B}$. If S is an Artin stack over κ, we denote by $S_{\mathcal{D}}$ the constant Artin $\mathcal{D}$-stack. If κ is separably closed and $\mathrm{Hom}_{\mathcal{D}}(A,B)$ is noetherian for every A and every B in $\mathcal{D}$, then $\mathrm{Mod}_{\mathrm{cart}}(\mathrm{Spec}(\kappa)_{\mathcal{D}},\Lambda)$ is equivalent to the category of projective systems of Λ-modules indexed by $\mathcal{D}(\pi_0)$. Indeed, in this case, $\alpha_{A,B}\colon p^*K_B \to p^*K_A$ is a morphism between constant sheaves on $\mathrm{Hom}_{\mathcal{D}}(A,B)$, and has to be constant on every connected component of $\mathrm{Hom}_{\mathcal{D}}(A,B)$.

Remark 6.24. If $\mathcal{D}$ is discrete (i.e. induced from a usual category) and X is a $\mathcal{D}$-scheme, i.e. a functor from $\mathcal{D}$ to the category of κ-schemes, the category $D_{\mathrm{cart}}(X,\Lambda)$ consists of families of objects $K_A \in D(X_A,\Lambda)$ and compatible transition maps $X_f^*K_B \to K_A$ for $f\colon A \to B$, and should not be confused with the derived category of sheaves of Λ-modules on the total étale topos of X over $\mathcal{D}$.

Construction 6.25. Let $f = ((f_A)_{A\in\mathcal{D}}, (\phi_{A,B})_{A,B\in\mathcal{D}})$ be a morphism of Artin $\mathcal{D}$-stacks. The functors f_A^* induce a functor $f^*\colon D_{\mathrm{cart}}(Y,\Lambda) \to D_{\mathrm{cart}}(X,\Lambda)$. On the other hand, for $K \in D_{\mathrm{cart}}(X,\Lambda)$ we have a diagram

$$\begin{array}{ccc} y_{A,B}^* Rf_{B*}K_B & & p^*Rf_{A*}K_A \\ \downarrow & & \downarrow \\ R(f_A\times \mathrm{id})_* x_{A,B}^* K_B & \xrightarrow{\alpha_{A,B}} & R(f_A\times\mathrm{id})_* p^* K_A \end{array} \tag{6.25.1}$$

where the left (resp. right) vertical arrow is base change for the square $\phi_{A,B}$ (resp. for the obvious cartesian square).

Assume that Λ is annihilated by an integer invertible in κ, and that the condition (a) (resp. (b)) below holds:

(a) $\mathrm{Hom}_{\mathcal{D}}(A,B)$ is smooth over κ for all objects A, B in $\mathcal{D}$;

(b) f_A is quasi-compact and quasi-separated and $K_A \in D^+_{\mathrm{cart}}$ for every object A of $\mathcal{D}$.

Then the right vertical arrow is an isomorphism by smooth base change (resp. generic base change (Remark 2.12)) from $\mathrm{Spec}(\kappa)$ to $\mathrm{Hom}_{\mathcal{D}}(A,B)$, and thus the diagram (6.25.1) defines a map $y_{A,B}^*Rf_{B*}K_B \to p^*Rf_{A*}K_A$. These maps endow $(Rf_{A*}K_A)$ with a structure of object of $D_{\mathrm{cart}}(Y,\Lambda)$. We thus get a functor

$$Rf_*\colon D_{\mathrm{cart}}(X,\Lambda) \to D_{\mathrm{cart}}(Y,\Lambda) \quad (\text{resp. } D^+_{\mathrm{cart}}(X,\Lambda) \to D^+_{\mathrm{cart}}(Y,\Lambda)).$$

The adjunctions $\mathrm{id}_{D_{\mathrm{cart}}(X_A,\Lambda)} \to Rf_{A*}f_A^*$ induce a natural transformation $\mathrm{id} \to Rf_*f^*$.

Remark 6.26. The construction of Rf_* above encodes the homotopy-invariance of étale cohomology [52, XV Lemme 2.1.3]. More precisely, assume κ separably closed. Let Y, Y' be two Artin stacks over κ, $L \in D_{\mathrm{cart}}(Y,\Lambda)$, $L' \in D_{\mathrm{cart}}(Y',\Lambda)$. A morphism $c\colon (Y,L) \to (Y',L')$ is a pair (g,ϕ), where $g\colon Y \to Y'$, $\phi\colon g^*L' \to L$. Following [52, XV Section 2.1], we say that two morphisms $c_0, c_1\colon (Y,L) \to (Y',L')$ are *homotopic* if there exists a connected scheme T of finite type over κ, two points $0, 1 \in T(\kappa)$, a morphism $(Y \times_{\mathrm{Spec}(\kappa)} T, \mathrm{pr}_1^*L) \to (Y,L')$ inducing c_0 and c_1 by taking fibers at 0 and 1, respectively. This is equivalent to the existence of an Artin $\mathcal{D}_T$-stack X and an object $K \in D_{\mathrm{cart}}(X,\Lambda)$ such that $X_A = Y$, $X_{A'} = Y'$, $K_A = L$, $K_{A'} = L'$ and inducing c_0 and c_1 by taking fibers at 0 and 1. Here $\mathcal{D}_T$ is the $\mathrm{Sch}^{\mathrm{ft}}_{/\kappa}$-enriched category with $\mathrm{Ob}(\mathcal{D}_T) = \{A, A'\}$, $\mathrm{Hom}_{\mathcal{D}_T}(A,A) = \mathrm{Hom}_{\mathcal{D}_T}(A',A') = \mathrm{Spec}(\kappa)$, $\mathrm{Hom}_{\mathcal{D}_T}(A',A) = \emptyset$ and $\mathrm{Hom}_{\mathcal{D}_T}(A,A') = T$. If c_0 and c_1 are homotopic, then $c_0^* = c_1^*\colon H^*(Y',L') \to H^*(Y,L)$. To prove this, we may assume that T is a smooth curve as in [52, XV Lemme 2.1.3]. Let $a\colon X \to \mathrm{Spec}(\kappa)_{\mathcal{D}_T}$ be the projection. By the above, R^*a_*K is a projective system of graded Λ-modules indexed by $\mathcal{D}_T(\pi_0)$, and $c_0^* = c_1^*$ is the image of the nontrivial arrow of $\mathcal{D}_T(\pi_0)$.

Proof of Lemma 6.20. By construction, F is essentially surjective. Consider the morphism of schemes $\phi\colon \mathrm{Hom}_{\mathcal{E}_G}((A,A',g),(Z,Z',h)) \to \mathrm{Hom}_{\mathcal{A}_G}(Z',A') = \mathrm{Trans}_G(Z',A')$ given by $(a,b) \mapsto b$. It fits into the following Cartesian diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{E}_G}((A,A',g),(Z,Z',h)) & \xrightarrow{\phi} & \mathrm{Trans}_G(Z',A') \\ \downarrow & & \downarrow \\ \{t \in \mathrm{Hom}(Z',A') \mid t(c_h(Z)) \supset c_g(A)\} & \hookrightarrow & \mathrm{Hom}(Z',A'). \end{array}$$

In particular, ϕ is an open and closed immersion and induces an injection on

$$\mathrm{Hom}_{\mathcal{E}_G(\pi_0)}((A,A',g),(Z,Z',h)) \to \mathrm{Hom}_{\mathcal{A}_G(\pi_0)}(Z',A').$$

In other words, the composite functor $p_2\circ F\colon \mathcal{E}_G(\pi_0) \to \mathcal{A}_G(\pi_0)^{\mathrm{op}}$ is faithful, where $p_2\colon \mathcal{A}_G(\pi_0)^\natural \to \mathcal{A}_G(\pi_0)^{\mathrm{op}}$. Therefore, F is faithful. To show that F is full, let $(\alpha,\beta)\colon F(A,A',g) \to F(Z,Z',h)$ be a morphism in $\mathcal{A}_G(\pi_0)^\natural$. Choose $b \in \beta(k) \subset G(k)$. Then we have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Trans}_G(A,Z) & \xrightarrow{\psi} & \mathrm{Trans}_G(A,A') \\ \downarrow & & \downarrow \\ \mathrm{Hom}(A,Z) & \hookrightarrow & \mathrm{Hom}(A,A'), \end{array}$$

where $\psi\colon a \mapsto ahb$. In particular, ψ is an open and closed immersion. The map $\pi_0(\mathrm{Trans}_G(A,Z)) \to \pi_0(\mathrm{Trans}_G(A,A'))$ induced by ψ carries α to $\gamma = \alpha\eta\beta$, where $\gamma \in \pi_0(\mathrm{Trans}_G(A,A'))$ and $\eta \in \pi_0(\mathrm{Trans}_G(Z,Z'))$ are the connected components of g and h, respectively. Thus there exists $a \in \alpha(k) \subset G(k)$ such that $g = \psi(a) = ahb$. Then $(a,b)\colon (A,A',g) \to (Z,Z',h)$ is a morphism in $\mathcal{E}_G(k) = \mathcal{A}_G(k)^\natural$, and induces a morphism τ in $\mathcal{E}_G(\pi_0)$ such that $F(\tau) = (\alpha,\beta)$. Therefore, F is an equivalence of categories. The second assertion of (a) follows from this and Lemma 6.5.

Let us prove (b). For (A,A',g) and (Z,Z',h) in $\mathcal{E}_G(k)$, consider the scheme

$$T = \mathrm{Hom}_{\mathcal{E}_G}((A,A',g),(Z,Z',h))$$

and the tautological section $t = (\underline{a},\underline{b}) \in T(T)$. Then, if $[X^{A'}/A]_T$ (resp. $[X^{Z'}/Z]_T$) denotes the product of $[X^{A'}/A]$ (resp. $[X^{Z'}/Z]$) with T over $\mathrm{Spec}(k)$, t defines a morphism of stacks $[\underline{b}^{-1}/c_{\underline{a}}]\colon [X^{A'}/A]_T \to [X^{Z'}/Z]_T$ over T, whose fiber at (a,b) is $[b^{-1}/c_a]$. These morphisms are compatible with composition of morphisms up to 2-morphisms, and define a structure of $\mathcal{E}_G$-stack (Definition 6.22) on the family of stacks $[X^{A'}/A]$ for $(A,A',g) \in \mathcal{E}_G(k)$. Moreover, we have a diagram over T

(6.26.1)
$$\begin{array}{ccccc} & & [X^{A'}/A]_T & \xrightarrow{\pi} & X_T^{A'} \\ & \swarrow^{[1/c_g]} & \downarrow{\scriptstyle [\underline{b}^{-1}/c_{\underline{a}}]} & & \downarrow{\scriptstyle \underline{b}^{-1}} \\ [X/G]_T & \xleftarrow[{[1/c_h]}]{} & [X^{Z'}/Z]_T & \xrightarrow{\pi} & X_T^{Z'}, \end{array}$$

where the 2-morphism of the triangle is induced by $\underline{b}$. The fiber of (6.26.1) at (a,b) is (6.16.3). Therefore we get morphisms of Artin $\mathcal{E}_G$-stacks

$$[X/G]_{\mathcal{E}_G} \leftarrow ([X^{A'}/A])_{(A,A',g)} \xrightarrow{\pi} (X^{A'})_{(A,A',g)}.$$

Thus the system $H^0(X^{A'}, R^q\pi_*[1/c_g]^*K)$ indexed by $(A,A',g) \in \mathcal{A}_G(k)^\natural$ can be extended to an object of $\mathrm{Mod}_{\mathrm{cart}}(\mathrm{Spec}(k)_{\mathcal{E}_G}, \mathbb{F}_\ell)$, which amounts to a system indexed by $\mathcal{E}_G(\pi_0)$. More concretely, the morphism $(a,b)^*$ (6.16.4) is the stalk at (a,b) of a morphism of constant sheaves on T

(6.26.2)
$$(\underline{a},\underline{b})^*\colon H^0(X^{Z'}, R^q\pi_*[1/c_h]^*K)_T \to H^0(X^{A'}, R^q\pi_*[1/c_g]^*K)_T,$$

defined by $(\underline{a},\underline{b})$ via (6.26.1). Therefore it depends only on the connected component of (a,b) in T. □

We need the following lemma for the proof of Theorem 6.17 (a).

Lemma 6.27. *Let Y be an algebraic space over k, and let A be a finite discrete group. Let $L \in D_c^b([Y/A],\mathbb{F}_\ell)$, where A acts trivially on Y. Let $\pi\colon [Y/A] = BA \times Y \to Y$ be the second projection. Consider the structure of $H^*(BA,\mathbb{F}_\ell)$-module on $R^*\pi_*L$ given by Constructions 3.4 and 3.7, as $R^*\pi_*\mathbb{F}_\ell$ is a constant sheaf of value $H^*(BA,\mathbb{F}_\ell)$. Then $R^*\pi_*L$ is a sheaf of constructible $H^*(BA,\mathbb{F}_\ell)$-modules.*

Proof. We may assume L concentrated in degree zero. Suppose first that L is locally constant. Then $R^*\pi_*L$ is a locally constant, constructible sheaf of $H^*(BA,\mathbb{F}_\ell)$-modules. Indeed, by definition there is an étale covering (U_α) of Y such that $L \mid [U_\alpha/A]$ (considered as a sheaf of $\mathbb{F}_\ell[A]$-modules on U_α) is a constant $\mathbb{F}_\ell[A]$-module of finite dimension over $\mathbb{F}_\ell$ of value L_α. Then $R^*\pi_*L \mid U_\alpha$ is a constant $H^*(BA,\mathbb{F}_\ell)$-module of value $H^*(BA,L_\alpha)$. By Theorem 4.6, $H^*(BA,L_\alpha)$ is a finite $H^*(BA,\mathbb{F}_\ell)$-module, so the lemma is proved in this case. In general, we may assume Y to be an affine scheme. Take a finite stratification $Y = \bigcup Y_\alpha$ into disjoint locally closed constructible subsets such that $L \mid Y_\alpha$ is locally constant, or equivalently, that $L \mid [Y_\alpha/A]$ is locally constant. Then, if $\pi_\alpha = \pi \mid [Y_\alpha/A] \to Y_\alpha$, $(R^*\pi_*L) \mid Y_\alpha \simeq R\pi_{\alpha*}(L \mid Y_\alpha)$ by the finiteness of A, and we conclude by the preceding case. □

Proof of Theorem 6.17 (a). By Lemma 6.20 (b) we can rewrite $R_G^q(X,K)$ in the form

$$R_G^q(X,K) \coloneqq \varprojlim_{(A,A',g)\in\mathcal{E}_G(\pi_0)} H^0(X^{A'}, R^q\pi_*[1/c_g]^*K).$$

As $\mathcal{E}_G(\pi_0)$ is essentially finite (Lemma 6.20 (a)) and $R^q\pi_*[1/c_g]^*K$ is constructible, the first assertion follows. Let us now prove the second assertion. As $\mathcal{E}_G(\pi_0)$ is equivalent to a finite category, it is enough to show that, for all (A, A', g), $H^0(X^{A'}, R^*\pi_*[1/c_g]^*K)$ is a finite $H^*(BG, \mathbb{F}_\ell)$-module. As A acts trivially on $X^{A'}$, $R^*\pi_*[1/c_g]^*K$ is a constructible sheaf of $H^*(BA, \mathbb{F}_\ell)$-modules by Lemma 6.27. Therefore $H^0(X^{A'}, R^*\pi_*[1/c_g]^*K)$ is a finite $H^*(BA, \mathbb{F}_\ell)$-module, thus, by Corollary 4.8, a finite $H^*(BG, \mathbb{F}_\ell)$-module. □

7 Points of Artin stacks

In this section we discuss two kinds of points of Artin stacks which will be of use to us:

(a) *geometric points*, which generalize the usual geometric points of schemes,

(b) *ℓ-elementary points*, which depend on a prime number ℓ, and are adapted to the study of the maps $a(G, X)$ (6.9.3) and $a_G(X, K)$ (Theorem 6.17).

The statement of the main structure theorem on Artin stacks (Theorem 8.3) requires only the notion (b). The notion (a) is a technical tool used in the proof.

Definition 7.1. Let $\mathcal{X}$ be a Deligne-Mumford stack. By a *geometric point* of $\mathcal{X}$ we mean a morphism $x \to \mathcal{X}$, where x is the spectrum of a separably closed field. The geometric points of $\mathcal{X}$ form a category

$$P_{\mathcal{X}},$$

where a morphism from $x \to \mathcal{X}$ to $y \to \mathcal{X}$ is defined as an $\mathcal{X}$-morphism $\mathcal{X}_{(x)} \to \mathcal{X}_{(y)}$ of the corresponding strict henselizations [31, Remarque 6.2.1]. The category $P_{\mathcal{X}}$ is essentially $\mathcal{U}$-small. One shows as in [50, VIII Théorème 7.9] that the functor $(x \to \mathcal{X}) \mapsto (\mathcal{F} \mapsto \mathcal{F}_x)$ from $P_{\mathcal{X}}$ to the category of points of the étale topos $\mathcal{X}_{\text{et}}$ is an equivalence of categories.

When $\mathcal{X}$ is a scheme, $P_{\mathcal{X}}$ is the usual category of geometric points of $\mathcal{X}$. If $\mathcal{X} = \operatorname{Spec} k$, k a field, $P_{\mathcal{X}}$ is a connected groupoid whose fundamental group is isomorphic to the Galois group of k.

As $P_{\mathcal{X}}$ is an essentially $\mathcal{U}$-small category, we have a morphism of topoi

$$p\colon \widehat{P_{\mathcal{X}}} \to \mathcal{X}_{\text{et}}, \tag{7.1.1}$$

where $\widehat{P_{\mathcal{X}}}$ denotes the topos of presheaves on $P_{\mathcal{X}}$. For a sheaf $\mathcal{F}$ on $\mathcal{X}$, $p^*\mathcal{F}$ is the presheaf $(x \to \mathcal{X}) \mapsto \mathcal{F}_x$ on $P_{\mathcal{X}}$, and p_* applied to a presheaf $(K_x)_{x\in P_{\mathcal{X}}}$ is the sheaf whose set of sections on U is $\varprojlim_{x\in P_U} K_x$. In particular we have an adjunction map

$$b_{\mathcal{X},\mathcal{F}}\colon \mathcal{F} \to p_*p^*\mathcal{F}, \tag{7.1.2}$$

which is a monomorphism, as $\mathcal{X}_{\text{et}}$ has enough points and $p^*b_{\mathcal{X},\mathcal{F}}$ is a split monomorphism (this fact holds of course more generally for any topos $\mathcal{X}$ with an essentially small conservative family of points $P_{\mathcal{X}}$, cf. [50, IV 6.7]).

Proposition 7.2. *Let $\mathcal{X}$ be a locally noetherian Deligne-Mumford stack, Λ a noetherian commutative ring, $\mathcal{F}$ a constructible sheaf of Λ-modules on $\mathcal{X}$. Then the adjunction map $b_{\mathcal{X},\mathcal{F}}\colon \mathcal{F} \to p_*p^*\mathcal{F}$ (7.1.2) is an isomorphism. In particular, the homomorphism*

$$\phi\colon \mathcal{F}(\mathcal{X}) \to \varprojlim_{x\in P_{\mathcal{X}}} \mathcal{F}_x \tag{7.2.1}$$

is an isomorphism.

Proof. If $f\colon \mathcal{Y} \to \mathcal{X}$ is a morphism of Deligne-Mumford stacks, the square of topoi

$$\begin{array}{ccc} \widehat{P_{\mathcal{Y}}} & \xrightarrow{p_{\mathcal{Y}}} & \mathcal{Y}_{\text{et}} \\ {\scriptstyle P_f}\downarrow & & \downarrow{\scriptstyle f_{\text{et}}} \\ \widehat{P_{\mathcal{X}}} & \xrightarrow{p_{\mathcal{X}}} & \mathcal{X}_{\text{et}} \end{array}$$

commutes and induces base change morphisms

$$p_{\mathcal{X}}^* f_* \to P_{f*} p_{\mathcal{Y}}^* \tag{7.2.2}$$

and

$$f^* p_{\mathcal{X}*} \to p_{\mathcal{Y}*} P_f^* \tag{7.2.3}$$

and commutative diagrams

$$\begin{array}{ccc} f_*\mathcal{G} & \xrightarrow{f_* b_{\mathcal{Y},\mathcal{G}}} & f_* p_{\mathcal{Y}*} p_{\mathcal{Y}}^* \mathcal{G} \\ {\scriptstyle b_{\mathcal{X}, f_*\mathcal{G}}}\downarrow & & \downarrow\simeq \\ p_{\mathcal{X}*} p_{\mathcal{X}}^* f_* \mathcal{G} & \xrightarrow{(7.2.2)} & p_{\mathcal{X}*} P_{f*} p_{\mathcal{Y}}^* \mathcal{G} \end{array} \qquad \begin{array}{ccc} f^*\mathcal{F} & \xrightarrow{b_{\mathcal{Y}, f^*\mathcal{F}}} & p_{\mathcal{Y}*} p_{\mathcal{Y}}^* f^* \mathcal{F} \\ {\scriptstyle f^* b_{\mathcal{X},\mathcal{F}}}\downarrow & & \downarrow\simeq \\ f^* p_{\mathcal{X}*} p_{\mathcal{X}}^* \mathcal{F} & \xrightarrow{(7.2.3)} & p_{\mathcal{Y}*} P_f^* p_{\mathcal{X}}^* \mathcal{F}. \end{array} \tag{7.2.4}$$

If f is a closed immersion, (7.2.2) is an isomorphism. If f is étale, (7.2.3) is an isomorphism.

Let $i\colon \mathcal{Z} \to \mathcal{X}$ be a closed immersion, and let $j\colon \mathcal{U} \to \mathcal{X}$ be the complementary open immersion. Then the following diagram with exact rows commutes (where we write p for $p_{\mathcal{X}}$):

$$\begin{array}{ccccccccc} 0 & \to & j_! j^* \mathcal{F} & \to & \mathcal{F} & \to & i_* i^* \mathcal{F} & \to & 0 \\ & & {\scriptstyle b_{\mathcal{X}, j_! j^* \mathcal{F}}}\downarrow & & \downarrow{\scriptstyle b_{\mathcal{X},\mathcal{F}}} & & \downarrow{\scriptstyle b_{\mathcal{X}, i_* i^* \mathcal{F}}} & & \\ 0 & \to & p_* p^* j_! j^* \mathcal{F} & \to & p_* p^* \mathcal{F} & \to & p_* p^* i_* i^* \mathcal{F}. & & \end{array}$$

Thus, to show that $b_{\mathcal{X},\mathcal{F}}$ is an isomorphism, it suffices to show that both $b_{\mathcal{X},i_*i^*\mathcal{F}}$ and $b_{\mathcal{X},j_!j^*\mathcal{F}}$ are isomorphisms. By the square on the left of (7.2.4) applied to $\mathcal{G} = i^*\mathcal{F}$, $b_{\mathcal{X},i_*i^*\mathcal{F}}$ is an isomorphism if $b_{\mathcal{Z},i^*\mathcal{F}}$ is an isomorphism. On the other hand, the following diagram commutes:

$$\begin{array}{ccccc} j^*\mathcal{F} & \xleftarrow{\sim} & j^* j_! j^* \mathcal{F} & \xrightarrow{j^* b_{\mathcal{X}, j_! j^* \mathcal{F}}} & j^* p_* p^* j_! j^* \mathcal{F} \\ {\scriptstyle b_{\mathcal{U}, j^*\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle b_{\mathcal{U}, j^* j_! j^* \mathcal{F}}} & & {\scriptstyle \simeq}\downarrow{\scriptstyle (7.2.3)} \\ p_{\mathcal{U}*} p_{\mathcal{U}}^* j^* \mathcal{F} & \xleftarrow{\sim} & p_{\mathcal{U}*} p_{\mathcal{U}}^* j^* j_! j^* \mathcal{F} & \xrightarrow{\sim} & p_{\mathcal{U}*} P_j^* p^* j_! j^* \mathcal{F}. \end{array} \tag{7.2.5}$$

We now prove that

$$i^*(p_* p^* j_! j^* \mathcal{F}) = 0. \tag{7.2.6}$$

By the commutativity of (7.2.5), this will imply that $b_{\mathcal{X},j_!j^*\mathcal{F}}$ is an isomorphism if $b_{\mathcal{U},j^*\mathcal{F}}$ is an isomorphism. For any geometric point $z \to \mathcal{Z}$,

$$(p_* p^* j_! j^* \mathcal{F})_z \simeq \varinjlim_{U \in N_{\mathcal{X}}(z)^{\mathrm{op}}} \varprojlim_{u \in P_U} (j_! j^* \mathcal{F})_u, \tag{7.2.7}$$

where $N_{\mathcal{X}}(z)$ is the category of étale neighborhoods of z in $\mathcal{X}$ that are quasi-compact and quasi-separated schemes. Let U be any such neighborhood. Take a finite stratification $(U_\alpha)_{\alpha\in A}$ of U by connected locally closed constructible subschemes such that the restrictions $\mathcal{F} \mid U_\alpha$ are locally constant. Let $P_{U,(U_\alpha)_{\alpha\in A}}$ be the category obtained from P_U by inverting all arrows in the full subcategories P_{U_α}. Geometric points of the same stratum are isomorphic in $P_{U,(U_\alpha)_{\alpha\in A}}$. Let $B \subset A$ be the subset of indices α such that there exists a morphism from a geometric point of U_α to z in $P_{U,(U_\alpha)_{\alpha\in A}}$. Let $V = \bigcup_{\alpha\in B} U_\alpha$. Since the geometric points of V are closed under generization in U, V is an open subset of U. Since specialization maps on the same stratum are isomorphisms, the projective system $((j_!j^*\mathcal{F})_v)_{v\in P_V}$ factors uniquely through a projective system $((j_!j^*\mathcal{F})_x)_{x\in P_{V,(U_\alpha)_{\alpha\in B}}}$ (where on each stratum U_α, $\alpha \in B$ all specialization maps are isomorphisms) and

$$\varprojlim_{v\in P_V} (j_! j^* \mathcal{F})_v \simeq \varprojlim_{x \in P_{V,(U_\alpha)_{\alpha\in B}}} (j_! j^* \mathcal{F})_x$$

by Lemma 7.3 below. Note that P_V contains z and that for any object x of $P_{V,(U_\alpha)_{\alpha\in B}}$ there exists a morphism from x to z. Therefore, as $(j_!j^*\mathcal{F})_z = 0$, this limit is zero. This implies that the full subcategory of $N_{\mathcal{X}}(z)^{\mathrm{op}}$ consisting of the neighborhoods U such that $\varprojlim_{v\in P_U}(j_!j^*\mathcal{F})_v = 0$ is cofinal. It follows that the limit (7.2.7) is zero and hence (7.2.6) holds, as claimed. To sum up, we have shown that $b_{\mathcal{X},\mathcal{F}}$ is an isomorphism if both $b_{\mathcal{Z},i^*\mathcal{F}}$ and $b_{\mathcal{U},j^*\mathcal{F}}$ are isomorphisms.

By induction, we may therefore assume $\mathcal{F}$ locally constant. Using the square on the right of (7.2.4), we may assume $\mathcal{F}$ constant. In this case it suffices to show that (7.2.1) is an isomorphism. We may further assume that $\mathcal{X}$ is connected and noetherian. Then $P_{\mathcal{X}}$ is a connected category and the assertion is trivial. □

Lemma 7.3. *Let $\mathcal{C}$ be a category and let S be a set of morphisms in $\mathcal{C}$. If we denote by $F\colon \mathcal{C} \to S^{-1}\mathcal{C}$ the localization functor, then F and F^{op} are cofinal (Definition 6.1).*

Proof. It suffices to show that F is cofinal. Let X be an object of $S^{-1}\mathcal{C}$, let Y be an object of $\mathcal{C}$ and let $f\colon X \to FY$ be a morphism in $S^{-1}\mathcal{C}$. Then $f = t_n s_n^{-1} \dots t_1 s_1^{-1}$ with t_i in $\mathcal{C}$ and $s_i \in S$. Using t_i and s_i, f can be connected to $\mathrm{id}_X\colon X \to FX$ in $(X \downarrow S^{-1}\mathcal{C})$. □

Remark 7.4. If, in Proposition 7.2, the sheaf $\mathcal{F}$ is not assumed constructible, then the monomorphism ϕ is not an isomorphism in general, as shown by the following example. Let X be a scheme of dimension ≥ 1 of finite type over a separably closed field k and let $\mathcal{F} = \bigoplus_{x\in|X|} i_{x*}\Lambda$, where $|X|$ is the set of closed points of X and let $i_x\colon \{x\} \to X$ be the inclusion. Then $\Gamma(X,\mathcal{F}) \simeq \Lambda^{(|X|)}$ (by commutation of $\Gamma(X,-)$ with filtered inductive limits). On the other hand, for $x \in P_X$, $\mathcal{F}_x = \Lambda$ if the image of x is a closed point, and $\mathcal{F}_x = 0$ otherwise, hence $\varprojlim_{x\in P_X} \Lambda \simeq \Lambda^{|X|}$. The monomorphism φ in Proposition 7.2 can be identified with the inclusion $\Lambda^{(|X|)} \subset \Lambda^{|X|}$, which is not an isomorphism, as $|X|$ is infinite.

Remark 7.5. In the situation of Proposition 7.2, the morphism

$$R\Gamma(\mathcal{X},\mathcal{F}) \to R \varprojlim_{x\in P_{\mathcal{X}}} \mathcal{F}_x$$

is not an isomorphism in general. In fact, if $\mathcal{X} = \mathrm{Spec}(k)$, then the left hand side computes the continuous cohomology of the Galois group G of k while the right hand side computes the cohomology of G as a discrete group.

Definition 7.6. Let $\mathcal{X}$ be an Artin stack. By a *geometric point* of $\mathcal{X}$ we mean a morphism $a\colon S \to \mathcal{X}$, where S is a strictly local scheme. If $a\colon S \to \mathcal{X}$ and $b\colon S \to \mathcal{X}$ are geometric points of $\mathcal{X}$, a morphism $(a\colon S \to \mathcal{X}) \to (b\colon T \to \mathcal{X})$ is a morphism $u\colon S \to T$ together with a 2-morphism

(7.6.1)
$$\begin{array}{ccc} S & \longrightarrow & T \\ & \searrow \;\nearrow\!\!\!\nearrow & \downarrow \\ & & \mathcal{X}. \end{array}$$

We thus get a full subcategory $\mathcal{P}'_{\mathcal{X}}$ of $\mathrm{AlgSp}_{/\mathcal{X}}$ (Notation 2.1). We define the *category of geometric points* of $\mathcal{X}$ as the category

$$\mathcal{P}_{\mathcal{X}} = M_{\mathcal{X}}^{-1}\mathcal{P}'_{\mathcal{X}},$$

localization of $\mathcal{P}'_{\mathcal{X}}$ by the set $M_{\mathcal{X}}$ of morphisms $(a \to b)$ in $\mathcal{P}'_{\mathcal{X}}$ sending the closed point of S to the closed point of T.

Although $\mathcal{P}'_{\mathcal{X}}$ is a $\mathcal{U}$-category and not essentially small in general, we will show in Proposition 7.9 that $\mathcal{P}_{\mathcal{X}}$ is essentially small. The next proposition shows that the definition above is consistent with Definition 7.1.

Proposition 7.7. *For any Deligne-Mumford stack $\mathcal{X}$, the functor $P_{\mathcal{X}} \to \mathcal{P}'_{\mathcal{X}}$ sending every geometric point $x \to \mathcal{X}$ to the strict henselization $\mathcal{X}_{(x)} \to \mathcal{X}$ induces an equivalence of categories $\iota\colon P_{\mathcal{X}} \to \mathcal{P}_{\mathcal{X}}$.*

Proof. Consider the functor $F'\colon \mathcal{P}'_{\mathcal{X}} \to P_{\mathcal{X}}$ sending $S \to \mathcal{X}$ to its closed point $s \to \mathcal{X}$. For any morphism in $\mathcal{P}'_{\mathcal{X}}$ as in (7.6.1), its image under F' is the induced morphism $\mathcal{X}_{(s)} \to \mathcal{X}_{(t)}$, where s and t are the closed points of S and T, respectively. The functor $F\colon \mathcal{P}_{\mathcal{X}} \to P_{\mathcal{X}}$ induced by F' gives a quasi-inverse to $\iota\colon P_{\mathcal{X}} \to \mathcal{P}_{\mathcal{X}}$. In fact $F\iota = \mathrm{id}_{P_{\mathcal{X}}}$ and we have a natural isomorphism $\mathrm{id}_{\mathcal{P}_{\mathcal{X}}} \to \iota F$ given by the morphism $S \to \mathcal{X}_{(s)}$ in $M_{\mathcal{X}}$ for $S \to \mathcal{X}$ in $\mathcal{P}_{\mathcal{X}}$ of closed point s. □

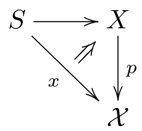
Remark 7.8. The reason why we do not consider the category of points $\mathrm{Point}(\mathcal{X}_{\mathrm{sm}})$ of the smooth topos $\mathcal{X}_{\mathrm{sm}}$ is that already in the case $\mathcal{X}$ is an algebraic space, the functor $\mathrm{Point}(\mathcal{X}_{\mathrm{sm}}) \to \mathrm{Point}(\mathcal{X}_{\mathrm{et}})$ induced by the morphism of topoi $\epsilon\colon \mathcal{X}_{\mathrm{sm}} \to \mathcal{X}_{\mathrm{et}}$ is not an equivalence. For example, if $U \to \mathcal{X}$ is a smooth morphism and y is a geometric point of U lying above a geometric point x of $\mathcal{X}$ such that the image of y in the fiber $U \times_{\mathcal{X}} x$ is not a closed point, then the points $\tilde{x}\colon \mathcal{F} \mapsto (\mathcal{F}_{\mathcal{X}})_x$ and $\tilde{y}\colon \mathcal{F} \mapsto (\mathcal{F}_U)_y$ of $\mathcal{X}_{\mathrm{sm}}$ are not equivalent, but have equivalent images in $\mathrm{Point}(\mathcal{X}_{\mathrm{et}})$. Indeed, if we denote by $\epsilon^!\colon \mathcal{X}_{\mathrm{et}} \to \mathcal{X}_{\mathrm{sm}}$ the right adjoint of ϵ_*, then the stalk of $\epsilon^!\mathcal{G}$ is $\mathcal{G}_x$ at $\tilde{x}$, but is e at $\tilde{y}$.

Proposition 7.9. *Let $\mathcal{X}$ be an Artin stack, and let $\tilde{\mathcal{P}}'_{\mathcal{X}}$ be the full subcategory of $\mathcal{P}'_{\mathcal{X}}$ consisting of morphisms $S \to \mathcal{X}$, such that $S \to \mathcal{X}$ is the strict henselization of some smooth atlas $X \to \mathcal{X}$ at some geometric point of X. Let $\tilde{M}_{\mathcal{X}} = M_{\mathcal{X}} \cap \mathrm{Ar}(\tilde{\mathcal{P}}'_{\mathcal{X}})$. Then the inclusion $\tilde{\mathcal{P}}'_{\mathcal{X}} \subset \mathcal{P}'_{\mathcal{X}}$ induces an equivalence of categories $\tilde{M}_{\mathcal{X}}^{-1}\tilde{\mathcal{P}}'_{\mathcal{X}} \to \mathcal{P}_{\mathcal{X}}$.*

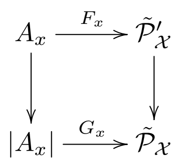
Note that $\tilde{\mathcal{P}}'_{\mathcal{X}}$ and hence $\tilde{\mathcal{P}}_{\mathcal{X}}$ are essentially small. Thus Proposition 7.9 shows that $\mathcal{P}_{\mathcal{X}}$ is essentially small,

Proof. We write $\tilde{\mathcal{P}}_{\mathcal{X}} = \tilde{M}_{\mathcal{X}}^{-1}\tilde{\mathcal{P}}'_{\mathcal{X}}$. For $x\colon S \to \mathcal{X}$ in $\mathcal{P}'_{\mathcal{X}}$, let A_x be the full subcategory of $(\mathrm{AlgSp}_{/\mathcal{X}})_{x/}$ consisting of diagrams

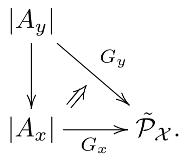
(7.9.1)



such that p is a smooth atlas. Then A_x is nonempty since every smooth surjection to S admits a section [22, Corollaire 17.16.3 (ii)]. Moreover, A_x admits finite nonempty products. Consider the functor $F_x\colon A_x \to \tilde{\mathcal{P}}'_{\mathcal{X}}$ sending (7.9.1) to the strict localization $X_{(s)} \to \mathcal{X}$ at the closed point s of S. For any pair of morphisms $(f, g)\colon X \rightrightarrows Y$ with the same source and target in A_x, $F_x(f)$ and $F_x(g)$ have the same image in $\tilde{\mathcal{P}}_{\mathcal{X}}$. Indeed, $f \mid S = g \mid S$ implies $F_x(f)t = F_x(g)t$, where $t \in \tilde{M}_{\mathcal{X}}$ is the inclusion of the closed point of $X_{(s)}$. Thus there exists a unique functor G_x making the following diagram commutative



where $|A_x|$ is the simply connected groupoid having the same objects as A_x. This construction is functorial in x, in the sense that for $x \to y$ in $\mathcal{P}'_{\mathcal{X}}$, we have a natural transformation



Choosing an object X in A_x for every x, we obtain a functor $\mathcal{P}'_{\mathcal{X}} \to \tilde{\mathcal{P}}_{\mathcal{X}}$ sending x to $X_{(s)}$. This functor factors through $\mathcal{P}_{\mathcal{X}} \to \tilde{\mathcal{P}}_{\mathcal{X}}$ and defines a quasi-inverse of $\tilde{\mathcal{P}}_{\mathcal{X}} \to \mathcal{P}_{\mathcal{X}}$. □

Remark 7.10. For any morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Artin stacks, composition with f defines a functor $\mathcal{P}'_f\colon \mathcal{P}'_{\mathcal{X}} \to \mathcal{P}'_{\mathcal{Y}}$, which induces $\mathcal{P}_f\colon \mathcal{P}_{\mathcal{X}} \to \mathcal{P}_{\mathcal{Y}}$.

(a) If f is a schematic universal homeomorphism, then $\mathcal{P}_f$ is an equivalence of categories. In fact, for any object $T \to \mathcal{Y}$ of $\mathcal{P}'_{\mathcal{Y}}$, the base change $S = T \times_{\mathcal{Y}} \mathcal{X} \to T$ is a schematic universal homeomorphism, so that S is a strictly local scheme by [22, Proposition 18.8.18 (i)]. The functor $\mathcal{P}'_{\mathcal{Y}} \to \mathcal{P}'_{\mathcal{X}}$ carrying $T \to \mathcal{Y}$ to $T \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ carries $M_{\mathcal{Y}}$ to $M_{\mathcal{X}}$ and induces a quasi-inverse of $\mathcal{P}_f$.

(b) For morphisms $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ of Artin stacks, the functor $\mathcal{P}'_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{Z}} \to \mathcal{P}'_{\mathcal{X}} \times_{\mathcal{P}'_{\mathcal{Y}}} \mathcal{P}'_{\mathcal{Z}}$ is an equivalence of categories.

Example 7.11. Let k be a separably closed field, and let G be an algebraic group over k. Then $\mathcal{P}_{BG}$ is a connected groupoid whose fundamental group is isomorphic to $\pi_0(G)$.

To prove this, by Remark 7.10 (a), we may assume k algebraically closed and G smooth. Then, for every object $S \to BG$ of $\mathcal{P}'_{BG}$, the corresponding G_S-torsor is trivial and we fix a trivialization. For any strictly local scheme S over $\mathrm{Spec}(k)$, we denote by $p_S\colon S \to BG$ the object of $\mathcal{P}_{BG}$ corresponding to the trivial G_S-torsor and by $a_S\colon S \to \mathrm{Spec}(k)$ the projection. By the definition of BG (1.5.2), morphisms $p_S \to p_T$ in $\mathcal{P}'_{BG}$ correspond bijectively to pairs (f,r), where $f\colon S \to T$ is a morphism of schemes and $r \in G(S)$. We denote the morphism corresponding to (f,r) by $\theta(f,r)$. If s is the closed point of S, $r(s) \in G(s)$ belongs to the inverse image of a unique connected component of G, denoted $[r]$. Let Π be the groupoid with one object and fundamental group $\pi_0(G)$. The above construction defines a full functor $\mathcal{P}'_{BG} \to \Pi$ sending $\theta(f,r)$ to $[r]$, which induces a functor still denoted by $F\colon \mathcal{P}_{BG} \to \Pi$. Since $\theta(a_S,r)\colon p_S \to p_{\mathrm{Spec}(k)}$ is in M_{BG} and $\theta(f,r)\theta(a_T,1) = \theta(a_S,r)$, $\theta(f,r)$ is an isomorphism in $\mathcal{P}_{BG}$. Thus $\mathcal{P}_{BG}$ is a connected groupoid. To show that F is an equivalence of categories, it suffices to check that for all $r \in G^0(S)$, $\theta(a_S,r) \equiv \theta(a_S,1)$. Here $\equiv$ stands for equality in $\mathcal{P}_{BG}$. For this, we may assume that S is a point, say $S = \mathrm{Spec}(k')$. We regard $r\colon \mathrm{Spec}(k') \to G$ as a geometric point of G. Since G^0 is irreducible, $X = G_{(1)} \times_G G_{(r)}$ is nonempty. Let x be a geometric point of X, and let $t \in G(G)$ be the tautological section. Then

$$\theta(a_{G_{(1)}},t)\theta(s_1,1) = \theta(a_{\mathrm{Spec}(k)},1) = \theta(a_{G_{(1)}},1)\theta(s_1,1),$$

where $s_1\colon \mathrm{Spec}(k) \to G_{(1)}$ is the closed point. It follows that $\theta(a_{G_{(1)}},t) \equiv \theta(a_{G_{(1)}},1)$, $\theta(a_x,t) \equiv \theta(a_x,1)$, and hence $\theta(a_{G_{(r)}},t) \equiv \theta(a_{G_{(r)}},1)$. Therefore, if $s_r\colon \mathrm{Spec}(k') \to G_{(r)}$ denotes the closed point, we have

$$\theta(a_{\mathrm{Spec}(k')},r) = \theta(a_{G_{(r)}},t)\theta(s_r,1) \equiv \theta(a_{G_{(r)}},1)\theta(s_r,1) = \theta(a_{\mathrm{Spec}(k')},1).$$

Construction 7.12. Let $\mathcal{X}$ be a locally noetherian Artin stack. If $\mathcal{F}$ is a cartesian sheaf on $\mathcal{X}$, then the presheaf

$$\mathfrak{p}'\mathcal{F}\colon (a\colon S \to \mathcal{X}) \mapsto \Gamma(S, a^*\mathcal{F}) \simeq \mathcal{F}_s$$

(where s is the closed point of S) on $\mathcal{P}'_{\mathcal{X}}$ defines a presheaf on $\mathcal{P}_{\mathcal{X}}$, which will denote by $\mathfrak{p}\mathcal{F}$. We thus get an exact functor

$$\mathfrak{p}\colon \mathrm{Sh}_{\mathrm{cart}}(\mathcal{X}) \to \hat{\mathcal{P}}_{\mathcal{X}}. \tag{7.12.1}$$

If $\mathcal{X}$ is a Deligne-Mumford stack, then $p^* \simeq \iota^*\mathfrak{p}$, where $p\colon \hat{P}_{\mathcal{X}} \to \mathcal{X}_{\mathrm{et}}$ is the projection (7.1.1) and $\iota\colon P_{\mathcal{X}} \to \mathcal{P}_{\mathcal{X}}$ the equivalence of Proposition 7.7.

The following result generalizes Proposition 7.2.

Proposition 7.13. *Let $\mathcal{X}$ be an Artin stack, let Λ be a noetherian commutative ring, and let $\mathcal{F}$ be a constructible sheaf of Λ-modules on $\mathcal{X}$. Then the map*

$$\Gamma(\mathcal{X},\mathcal{F}) \to \varprojlim_{x\in\mathcal{P}_{\mathcal{X}}} \mathcal{F}_x \tag{7.13.1}$$

defined by the restriction maps $\Gamma(\mathcal{X},\mathcal{F}) \to (\mathfrak{p}\mathcal{F})(x) = \mathcal{F}_x$ is an isomorphism.

The proof will be given after a couple of lemmas.

Lemma 7.14. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between small categories. Assume that for any morphism $f\colon X \to Y$ in $\mathcal{D}$, there exists a morphism $a\colon A \to B$ in $\mathcal{C}$ and a commutative square in $\mathcal{D}$ of the following form:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \simeq\downarrow & & \downarrow\simeq \\ F(A) & \xrightarrow{F(a)} & F(B). \end{array}$$

Then F is of descent for presheaves. More precisely, for any presheaf $\mathcal{F}$ on $\mathcal{D}$, with the notation of [50, IV 4.6], the sequence

$$\mathcal{F} \to F_*F^*\mathcal{F} \rightrightarrows F_{2*}F_2^*\mathcal{F}$$

is exact, where $\mathcal{C} \times_{\mathcal{D}} \mathcal{C}$ is the 2-fiber product, $F_2\colon \mathcal{C} \times_{\mathcal{D}} \mathcal{C} \to \mathcal{D}$ is the projection, and the double arrow is induced by the two projections from $\mathcal{C} \times_{\mathcal{D}} \mathcal{C}$ to $\mathcal{C}$. In particular, the sequence

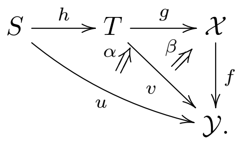
$$\Gamma(\hat{\mathcal{D}}, \mathcal{F}) \to \Gamma(\hat{\mathcal{C}}, F^*\mathcal{F}) \rightrightarrows \Gamma(\widehat{\mathcal{C} \times_{\mathcal{D}} \mathcal{C}}, F_2^*\mathcal{F}) \tag{7.14.1}$$

is exact.

Proof. For any X in $\mathcal{D}$, $\mathcal{F}(X) \to (F_*F^*\mathcal{F})(X) \rightrightarrows (F_{2*}F_2^*\mathcal{F})(X)$ is (7.14.1) applied to the functor $F'\colon \mathcal{C}_{/X} \to \mathcal{D}_{/X}$ induced by F and the presheaf $\mathcal{F}|(\mathcal{D}_{/X})$. Since F' also satisfies the assumption of the lemma, it suffices to prove that (7.14.1) is exact. By definition, $\Gamma(\hat{\mathcal{C}}, F^*\mathcal{F})$ consists of families $s = (s_X) \in \varprojlim_{X \in \mathcal{C}} \mathcal{F}(F(X))$. Similarly, $\Gamma(\widehat{\mathcal{C} \times_{\mathcal{D}} \mathcal{C}}, F_2^*\mathcal{F}) = \varprojlim_{(Y,Z,\alpha)\in\mathcal{C}\times_{\mathcal{D}}\mathcal{C}} \mathcal{F}(F_2(Y,Z,\alpha))$. Let E be the equalizer of the double arrow in (7.14.1). We construct $\epsilon\colon E \to \Gamma(\hat{\mathcal{D}}, \mathcal{F})$ as follows. Let $s \in E$. For any object X of $\mathcal{D}$, put $\epsilon(s)_X = \mathcal{F}(e)(s_A) \in \mathcal{F}(X)$, for a choice of $e\colon X \xrightarrow{\sim} F(A)$. This does not depend on the choice of e, because if $e'\colon X \xrightarrow{\sim} F(A')$, then $(A, A', e'e^{-1})$ defines an object of $\mathcal{C} \times_{\mathcal{D}} \mathcal{C}$, and $s \in E$ implies $\mathcal{F}(e)(s_A) = \mathcal{F}(e')(s_{A'})$. For any morphism $f\colon X \to Y$ in $\mathcal{D}$, the hypothesis implies that $\mathcal{F}(f)(\epsilon(s)_Y) = \epsilon(s)_X$. This finishes the construction of ϵ It is straightforward to check that ϵ is an inverse of $\Gamma(\hat{\mathcal{D}}, \mathcal{F}) \to E$. ☐

Lemma 7.15. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a smooth surjective morphism of Artin stacks. If $\mathcal{U}$ is a universe containing $\mathcal{P}'_{\mathcal{X}}$ and $\mathcal{P}'_{\mathcal{Y}}$, then the functor $\mathcal{P}'_f\colon \mathcal{P}'_{\mathcal{X}} \to \mathcal{P}'_{\mathcal{Y}}$ satisfies the condition of Lemma 7.14 for $\mathcal{U}$.*

Proof. Let $(h, \alpha)\colon (S, u) \to (T, v)$ be a morphism in $\mathcal{P}'_{\mathcal{Y}}$. Since $\mathcal{X} \times_{\mathcal{Y}} T$ is an Artin stack smooth over T, it admits a section, giving rise to the following 2-commutative diagram



Then the following diagram commutes

$$\begin{array}{ccc} (S,u) & \xrightarrow{(h,\alpha)} & (T,v) \\ \downarrow & & \downarrow \\ \mathcal{P}'_f((S,gh)) & \xrightarrow{\mathcal{P}'_f((h,\mathrm{id}))} & \mathcal{P}'_f((T,g)), \end{array}$$

where the left (resp. right) vertical arrow is the isomorphism $(\mathrm{id}_S, \beta\alpha\colon u \to fgh)$ (resp. $(\mathrm{id}_T, \beta\colon v \to fg)$). ☐

Proof of Proposition 7.13. Note that $\varprojlim_{x\in\mathcal{P}_{\mathcal{X}}} \mathcal{F}_x \to \varprojlim_{x\in\mathcal{P}'_{\mathcal{X}}} \mathcal{F}_x$ is an isomorphism by Lemma 7.3. Let $f\colon X \to \mathcal{X}$ be a smooth atlas. The following diagram commutes:

$$\begin{array}{ccccc} \Gamma(\mathcal{X}, \mathcal{F}) & \longrightarrow & \Gamma(X, f^*\mathcal{F}) & \rightrightarrows & \Gamma(X \times_{\mathcal{X}} X, g^*\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow \\ \varprojlim_{x\in\mathcal{P}'_{\mathcal{X}}} \mathcal{F}_x & \longrightarrow & \varprojlim_{x\in\mathcal{P}'_X} \mathcal{F}_x & \rightrightarrows & \varprojlim_{x\in\mathcal{P}'_{X\times_{\mathcal{X}}X}} \mathcal{F}_x. \end{array}$$

Here $g\colon X \times_{\mathcal{X}} X \to \mathcal{X}$ and the double arrows are induced by the two projections from $X \times_{\mathcal{X}} X$ to X. The top row is exact by the definition of a sheaf. The bottom row is exact by Lemmas 7.14, 7.15 and Remark 7.10 (b). The middle and right vertical arrows are isomorphisms by Propositions 7.2 and 7.7. It follows that the left vertical arrow is also an isomorphism. □

Example 7.16. For $\mathcal{X} = BG$ as in Example 7.11, $\mathcal{F}$ corresponds (by Corollaries 2.5 and 2.6) to a Λ-module of finite type M equipped with an action of $\pi_0(G)$. Thus $\Gamma(BG, \mathcal{F})$ is the module of invariants $M^{\pi_0(G)}$. By Example 7.11, $\varprojlim_{x\in\mathcal{P}_{BG}} \mathcal{F}_x$ is the set of zero cycles $Z^0(\pi_0(G), M)$, and (7.13.1) is the tautological isomorphism.

If G is finite, the isomorphism (7.13.1) extends to an isomorphism

$$R\Gamma(BG,\mathcal{F}) \xrightarrow{\sim} R \varprojlim_{x\in\mathcal{P}_{BG}} \mathcal{F}_x = R\Gamma(B\pi_0(G), M).$$

However, this no longer holds for G general, as the example of $G = \mathbb{G}_{\mathrm{m}}$ and $\mathcal{F} = \Lambda$ already shows (Theorem 4.4).

In the rest of this section, we fix a prime number ℓ.

Definition 7.17. Let $\mathcal{X}$ be an Artin stack. By an *ℓ-elementary point* of $\mathcal{X}$ we mean a *representable* morphism $x\colon \mathcal{S} \to \mathcal{X}$, where $\mathcal{S}$ is isomorphic to a quotient stack $[S/A]$, where S is a strictly local scheme endowed with an action of an elementary abelian ℓ-group A acting trivially on the closed point of S. If $x\colon [S/A] \to \mathcal{X}$, $y\colon [T/B] \to \mathcal{X}$ are ℓ-elementary points of $\mathcal{X}$, a morphism from x to y is an isomorphism class of pairs (φ, α), where $\varphi\colon [S/A] \to [T/B]$ is a morphism and $\alpha\colon x \to y\varphi$ is a 2-morphism. An isomorphism between two pairs $(\varphi,\alpha) \to (\psi,\beta)$ is a 2-morphism $c\colon \varphi \to \psi$ such that $\beta = (y * c) \circ \alpha$. We thus get a category $\mathcal{C}'_{\mathcal{X},\ell}$, full subcategory of $\mathrm{Stack}^{\mathrm{rep}}_{/\mathcal{X}}$ (Remark 1.20).

Proposition 7.18. *Let $\mathcal{X}$ be an Artin stack.*

(a) Let $x\colon \mathcal{S} = [S/A] \to \mathcal{X}$ be an ℓ-elementary point of $\mathcal{X}$, let s be the closed point of S, and let ε be the composition $s \to S \to \mathcal{S}$. Then $\mathrm{Aut}_{\mathcal{S}(s)}(\varepsilon) = A$, and the morphism x induces an injection

$$\mathrm{Aut}_{\mathcal{S}(s)}(\varepsilon) \hookrightarrow \mathrm{Aut}_{\mathcal{X}(s)}(x).$$

(b) Let $x\colon [S/A] \to \mathcal{X}$, $y\colon [T/B] \to \mathcal{X}$ be ℓ-elementary points of $\mathcal{X}$, and let $(\varphi,\alpha)\colon x \to y$ be a morphism in $\mathcal{C}'_{\mathcal{X},\ell}$. Then there exists a pair (f,u), where $u\colon A \to B$ is a group monomorphism and $f\colon S \to T$ is a u-equivariant morphism of $\mathcal{X}$-schemes, such that the morphism of $\mathcal{X}$-stacks (φ,α) is induced by the morphism of groupoids $(f,u)\colon (S,A)_\bullet \to (T,B)_\bullet$ over $\mathcal{X}$. If (f,u) is such a pair and $r \in B$, then (fr,u) is also such a pair. If (f_1,u_1) and (f_2,u_2) are two such pairs, then $u_1 = u_2$ and there exists a unique $r \in B$ such that $f_1 = f_2 r$.

(c) Assume that $\mathcal{X} = [X/G]$ for an algebraic space X over a base algebraic space U, endowed with an action of a smooth group algebraic space G over U. Then every ℓ-elementary point $x\colon [S/A] \to [X/G]$ lifts to a morphism of U-groupoids $(x_0, i)\colon (S,A)_\bullet \to (X,G)_\bullet$, where $x_0\colon S \to X$ and $i\colon S \times A \to G$. Moreover, in the situation of (b), if (x_0,i), (y_0,j), (f,u) are liftings of x, y, φ to U-groupoids, respectively, then there exists a unique 2-morphism of U-groupoids (Proposition 1.2) lifting α

(7.18.1)
$$\begin{array}{ccc} (S,A)_\bullet & \xrightarrow{(f,u)} & (T,B)_\bullet \\ & {\scriptstyle (x_0,i)} \searrow \; \Nearrow & \downarrow {\scriptstyle (y_0,j)} \\ & & (X,G)_\bullet \end{array}$$

given by $r\colon S \to G$ satisfying $x_0(z) = (y_0 f)(z) r(z)$ and $i(z,a) = r(z)^{-1} j(f(z), u(a)) r(za)$.

Proof. (a) The first assertion follows from the definition of $[S/A]$ (Notation 1.5), and the second one from the assumption that x is representable, hence faithful.

(b) Applying Proposition 1.19 to the groupoids $(S,A)_\bullet$ and $(T,B)_\bullet$ over $\mathcal{X}$, we get a pair (f,u), with $u\colon S \times A \to B$ given by Proposition 1.2 (a), such that $[f/u] = (\varphi,\alpha)$. The morphism u is constant on S, hence induced by a homomorphism, still denoted u, from A to B. Since φ is representable, u is a monomorphism. Such a pair (f, u) is unique up to a unique 2-isomorphism. If (f_1, u_1) and (f_2, u_2) are two choices, a 2-isomorphism from $(f_1, u_1)_\bullet$ to $(f_2, u_2)_\bullet$ is given by $r\colon S \to B$ (Proposition 1.2 (b)), which is necessarily constant, of value denoted again $r \in B$. Then we have $f_1 = f_2 r$ and $ru_1 = u_2 r$, hence $u_1 = u_2$.

(c) The existence of the liftings follows from Proposition 1.19 applied to the three groupoids. The description of the morphisms and the 2-morphism of groupoids comes from Proposition 1.2 (b). □

Remark 7.19. As the referee points out, Definition 7.17 is related to the ℓ-torsion inertia stack $I(\mathcal{X}, \ell)$ considered in [1, Proposition 3.1.3]. Indeed, $\mathcal{P}'_{I(\mathcal{X},\ell)}$ can be identified with the subcategory of $\mathcal{C}'_{\mathcal{X},\ell}$ spanned by ℓ-elementary points of the form $S \times BA \to \mathcal{X}$ and morphisms inducing id_A, where $A = \mathbb{Z}/\ell$.

Definition 7.20. For a stack of the form $\mathcal{S} = [S/A]$ as in Definition 7.17, the group A is, in view of Proposition 7.18 (a), uniquely determined by $\mathcal{S}$ (up to an isomorphism). We define the *rank* of $\mathcal{S}$ to be the rank of A, and for an ℓ-elementary point $x\colon \mathcal{S} \to \mathcal{X}$, we define the *rank* of x to be the rank of $\mathcal{S}$. ℓ-elementary points of rank zero are just geometric points (Definition 7.6). The full subcategory of $\mathcal{C}'_{\mathcal{X}}$ (Definition 7.17) spanned by ℓ-elementary points of rank zero is the category $\mathcal{P}'_{\mathcal{X}}$ (Definition 7.6).

Definition 7.21. We define the *category of ℓ-elementary points of $\mathcal{X}$* to be the category

$$\mathcal{C}_{\mathcal{X},\ell} = N^{-1}_{\mathcal{X},\ell}\mathcal{C}'_{\mathcal{X},\ell} \tag{7.21.1}$$

deduced from $\mathcal{C}'_{\mathcal{X},\ell}$ by inverting the set $N_{\mathcal{X},\ell}$ of morphisms given by pairs (f, u) (Proposition 7.18 (b)) such that $f\colon S \to T$ carries the closed point of S to the closed point of T and u is a group isomorphism. When no ambiguity can arise, we will remove the subscript ℓ from the notation.

Although $\mathcal{C}'_{\mathcal{X},\ell}$ is only a $\mathcal{U}$-category, we will see that $\mathcal{C}_{\mathcal{X},\ell}$ is essentially small if $\mathcal{X}$ is a Deligne-Mumford stack of finite inertia or a global quotient stack (Proposition 7.26 and Remark 8.10).

We may interpret $\mathcal{C}_{\mathcal{X},\ell}$ with the help of an auxiliary category $\bar{\mathcal{C}}'_{\mathcal{X},\ell}$ as follows.

Construction 7.22. Objects of $\bar{\mathcal{C}}'_{\mathcal{X},\ell}$ are pairs (x, A) such that $x\colon S \to \mathcal{X}$ is a geometric point of $\mathcal{X}$, A is an elementary abelian ℓ-group acting on x by $\mathcal{X}$-automorphisms with trivial action on the closed point of S, and the morphism $[S/A] \to \mathcal{X}$ is representable. Morphisms of $\bar{\mathcal{C}}'_{\mathcal{X},\ell}$ are pairs $(f, u)\colon (x, A) \to (y, B)$, where $u\colon A \to B$ is a homomorphism and $f\colon x \to y$ is an equivariant morphism in $\mathcal{P}'_{\mathcal{X}}$. Note that u is necessarily a monomorphism. By definition, $\bar{\mathcal{C}}'_{\mathcal{X},\ell}$ is a full subcategory of $\mathrm{Eq}(\mathrm{Stack}^{\mathrm{rep}}_{/\mathcal{X}})$.

We have a natural functor $\rho'\colon \bar{\mathcal{C}}'_{\mathcal{X},\ell} \to \mathcal{C}'_{\mathcal{X},\ell}$ sending (x, A) to $[S/A] \to \mathcal{X}$. By Proposition 7.18 (a) and (b), the functor is full and essentially surjective, and in particular cofinal (Lemma 6.2). If $\varpi'\colon \mathcal{P}'_{\mathcal{X}} \hookrightarrow \mathcal{C}'_{\mathcal{X},\ell}$ is the inclusion functor, and $\bar{\varpi}'\colon \mathcal{P}'_{\mathcal{X}} \to \bar{\mathcal{C}}'_{\mathcal{X},\ell}$ is the functor sending x to $(x, \{1\})$, which is also fully faithful, we have a 2-commutative diagram

$$\begin{array}{ccc} & & \bar{\mathcal{C}}'_{\mathcal{X},\ell} \\ & \overset{\bar{\varpi}'}{\nearrow} & \downarrow{\scriptstyle \rho'} \\ \mathcal{P}'_{\mathcal{X}} & \xrightarrow{\varpi'} & \mathcal{C}'_{\mathcal{X},\ell}. \end{array} \tag{7.22.1}$$

Let

$$\bar{\mathcal{C}}_{\mathcal{X},\ell} = \bar{N}^{-1}_{\mathcal{X},\ell}\bar{\mathcal{C}}'_{\mathcal{X},\ell} \tag{7.22.2}$$

be the category deduced from $\bar{\mathcal{C}}'_{\mathcal{X},\ell}$ by inverting the set $\bar{N}_{\mathcal{X},\ell}$ of morphisms $(f, u)\colon (S, A) \to (T, B)$ such that f sends the closed point s of S to the closed point t of T and $u\colon A \to B$ is an isomorphism. The diagram (7.22.1) induces a diagram

$$\begin{array}{ccc} & & \bar{\mathcal{C}}_{\mathcal{X},\ell} \\ & \overset{\bar{\varpi}}{\nearrow} & \downarrow{\scriptstyle \rho} \\ \mathcal{P}_{\mathcal{X}} & \xrightarrow{\varpi} & \mathcal{C}_{\mathcal{X},\ell}. \end{array} \tag{7.22.3}$$

The functor ρ is essentially surjective, and its effects on morphisms can be described as follows. Let (x,A) and (y,B) be objects of $\bar{\mathcal{C}}_{\mathcal{X}}$. The action of B on (y,B) by automorphisms in $\bar{\mathcal{C}}'_{\mathcal{X}}$ induces an action of B on (y,B) by automorphisms in $\bar{\mathcal{C}}_{\mathcal{X}}$, and, in turn, an action of B on $\mathrm{Hom}_{\bar{\mathcal{C}}_{\mathcal{X}}}((x,A),(y,B))$. This action is compatible with composition in the sense that if $f\colon (x,A)\to (y,B)$, $g\colon (y,B)\to(z,C)$ are morphisms of $\bar{\mathcal{C}}_{\mathcal{X}}$, and $b\in B$, then $g\circ(fb)=(g(\theta(g)(b)))\circ f$, where $\theta\colon \bar{\mathcal{C}}_{\mathcal{X}}\to\mathcal{A}$ is the functor induced by the functor $\bar{\mathcal{C}}'_{\mathcal{X}}\to\mathcal{A}$ carrying (x,A) to A. Here $\mathcal{A}$ denotes the category whose objects are elementary abelian ℓ-groups and whose morphisms are monomorphisms.

Proposition 7.23.

(a) The functor ρ induces a bijection

$$\mathrm{Hom}_{\bar{\mathcal{C}}_{\mathcal{X}}}((x,A),(y,B))/B \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_{\mathcal{X}}}(\rho(x,A),\rho(y,B)). \tag{7.23.1}$$

(b) The functors $\bar{\varpi}$ and ϖ are fully faithful.

Proof. (a) Indeed, consider the quotient category $\mathcal{C}^{\sharp}_{\mathcal{X}}$ having the same objects as $\bar{\mathcal{C}}_{\mathcal{X}}$ with morphisms defined by

$$\mathrm{Hom}_{\mathcal{C}^{\sharp}_{\mathcal{X}}}((x,A),(y,B))=\mathrm{Hom}_{\bar{\mathcal{C}}_{\mathcal{X}}}((x,A),(y,B))/B,$$

and the quotient functor $\rho^{\sharp}\colon\bar{\mathcal{C}}_{\mathcal{X}}\to\mathcal{C}^{\sharp}_{\mathcal{X}}$. By the universal properties of ρ, $\rho^{\sharp}$, and the localization functors $\bar{\mathcal{C}}'_{\mathcal{X}}\to\bar{\mathcal{C}}_{\mathcal{X}}$, $\mathcal{C}'_{\mathcal{X}}\to\mathcal{C}_{\mathcal{X}}$, we obtain an equivalence between $\mathcal{C}^{\sharp}_{\mathcal{X}}$ and $\mathcal{C}_{\mathcal{X}}$, compatible with ρ and $\rho^{\sharp}$.

(b) It follows from (a) that ρ induces an equivalence of categories from the full subcategory of $\bar{\mathcal{C}}_{\mathcal{X}}$ spanned by the image of $\mathcal{P}_{\mathcal{X}}$ to the full subcategory of $\mathcal{C}_{\mathcal{X}}$ spanned by the image of $\mathcal{P}_{\mathcal{X}}$. Thus it suffices to show that $\bar{\varpi}$ is fully faithful. The functor $\bar{\mathcal{C}}'_{\mathcal{X}}\to\mathcal{P}'_{\mathcal{X}}$ sending (x,A) to x is a quasi-retraction of $\bar{\varpi}'$, and induces a quasi-retraction of $\bar{\varpi}$. Here, by a quasi-retraction of a functor F, we mean a functor G endowed with a natural isomorphism $GF\simeq\mathrm{id}$. Thus $\bar{\varpi}$ is faithful. Let us show that $\bar{\varpi}$ is full. Let x,x' be geometric points of $\mathcal{X}$. By definition, any morphism $f\colon x\to x'$ in $\bar{\mathcal{C}}_{\mathcal{X}}$ is of the form $(t_n,v_n)(s_n,u_n)^{-1}\dots(t_1,v_1)(s_1,u_1)^{-1}$, where $(t_i,v_i)\colon(x_i,A_i)\to(y_{i+1},B_{i+1})$ is in $\bar{\mathcal{C}}'_{\mathcal{X}}$ and $(s_i,u_i)\colon(x_i,A_i)\to(y_i,B_i)$ is in $\bar{N}_{\mathcal{X}}$ for $1\le i\le n$, $y_1=x$, $y_{n+1}=x'$, $B_1=B_{n+1}=\{1\}$. Then $u_i\colon A_i\to B_i$ is an isomorphism and $v_i\colon A_i\to B_{i+1}$ is a monomorphism. Thus $A_i=B_i=\{1\}$. Moreover, t_i is in $\mathcal{P}'_{\mathcal{X}}$ and s_i is in $M_{\mathcal{X}}$. It follows that $f=\bar{\varpi}(a)$, where $a=t_ns_n^{-1}\dots t_1s_1^{-1}$ is in $\mathcal{P}_{\mathcal{X}}$. □

Remark 7.24. For any *representable* morphism $f\colon\mathcal{X}\to\mathcal{Y}$ of Artin stacks, composition with f induces functors $\mathcal{C}_f\colon\mathcal{C}_{\mathcal{X}}\to\mathcal{C}_{\mathcal{Y}}$ and $\bar{\mathcal{C}}_f\colon\bar{\mathcal{C}}_{\mathcal{X}}\to\bar{\mathcal{C}}_{\mathcal{Y}}$. As in Remark 7.10 (a), $\mathcal{C}_f$ and $\bar{\mathcal{C}}_f$ are equivalences of categories if f is a schematic universal homeomorphism.

Definition 7.25. Morphisms in the categories $\bar{\mathcal{C}}_{\mathcal{X},\ell}$ and $\mathcal{C}_{\mathcal{X},\ell}$ are in general difficult to describe. When $\mathcal{X}$ is a Deligne-Mumford stack of finite inertia, the categories $\bar{\mathcal{C}}_{\mathcal{X},\ell}$ and $\mathcal{C}_{\mathcal{X},\ell}$ admit simpler descriptions, as in Proposition 7.7. Let us call a *DM ℓ-elementary point of $\mathcal{X}$* a pair (x,A), where $x\colon s\to\mathcal{X}$ is a geometric point of $\mathcal{X}$ and A an ℓ-elementary abelian subgroup of $\mathrm{Aut}_{\mathcal{X}(s)}(x)$. Define a morphism from $(x\colon s\to\mathcal{X},A)$ to $(y\colon t\to\mathcal{X},B)$ to be an $\mathcal{X}$-morphism $\mathcal{X}_{(x)}\to\mathcal{X}_{(y)}$ such that $f(A)\subset B$, where $f\colon\mathrm{Aut}_{\mathcal{X}(s)}(x)\to\mathrm{Aut}_{\mathcal{X}(t)}(y)$ is defined as follows. Note that $I_{(y)}\coloneqq I_{\mathcal{X}}\times_{\mathcal{X}}\mathcal{X}_{(y)}$ is finite and unramified over $\mathcal{X}_{(y)}$, thus is a finite disjoint union of closed subschemes of $\mathcal{X}_{(y)}$ by [22, Corollaire 18.4.7]. For $a\in\mathrm{Aut}_{\mathcal{X}(s)}(x)$, the point $s\to I_{(y)}$ given by a lies in same component as the point $t\to I_{(y)}$ given by $f(a)$. We thus get a category $\bar{C}_{\mathcal{X},\ell}$. We define the *category of DM ℓ-elementary points of $\mathcal{X}$* to be the category $C_{\mathcal{X},\ell}$ having the same objects as $\bar{C}_{\mathcal{X},\ell}$ and such that $\mathrm{Hom}_{C_{\mathcal{X},\ell}}((x,A),(y,B))=\mathrm{Hom}_{\bar{C}_{\mathcal{X},\ell}}((x,A),(y,B))/B$. We omit the subscript ℓ from the notation when no ambiguity arises.

Note that for $(x,A)\in\bar{C}_{\mathcal{X}}$, the morphism $[\mathcal{X}_{(x)}/A]\to\mathcal{X}$ is representable.

Proposition 7.26. *Let $\mathcal{X}$ be a Deligne-Mumford stack of finite inertia. Then the functor $C_{\mathcal{X}}\to\bar{\mathcal{C}}'_{\mathcal{X}}$ carrying (x,A) to $(\mathcal{X}_{(x)}\to\mathcal{X},A)$ induces an equivalence of categories $\iota\colon\bar{C}_{\mathcal{X}}\to\bar{\mathcal{C}}_{\mathcal{X}}$ and, in turn, an equivalence of categories $\mathcal{C}_{\mathcal{X}}\to C_{\mathcal{X}}$.*

Proof. A quasi-inverse of ι is induced by the functor $\bar{\mathcal{C}}'_{\mathcal{X}}\to\bar{C}_{\mathcal{X}}$ carrying $(S\to\mathcal{X},A)$ to $(s\to\mathcal{X},A)$, where s is the closed point of S. □

In the sequel, for $\mathcal{X}$ a Deligne-Mumford stack of finite inertia, we will often identify the categories $\mathcal{C}_{\mathcal{X}}$ and $C_{\mathcal{X}}$ by the equivalence of Proposition 7.26 and call *DM ℓ-elementary points* just ℓ-elementary points.

Construction 7.27. Let $\mathcal{X}$ be an Artin stack. Let $\mathcal{F}$ be a cartesian sheaf on $\mathcal{X}$. If $x\colon [S/A] \to \mathcal{X}$ is an ℓ-elementary point of $\mathcal{X}$, let $\mathcal{F}_x := x^*\mathcal{F}$, and

$$\Gamma(x, \mathcal{F}_x) := \Gamma([S/A], \mathcal{F}_x) \simeq \Gamma(BA, \mathcal{F}_s) \simeq \mathcal{F}_s^A.$$

If $(\varphi, \alpha)\colon [S/A] \to [Y/B]$ is a morphism in $\mathcal{C}'_{\mathcal{X}}$, we have a natural map $\Gamma(x, \mathcal{F}_x) \to \Gamma(y, \mathcal{F}_y)$ given by restriction, and in this way we get a presheaf $\mathfrak{q}'\mathcal{F}\colon x \mapsto \Gamma(x, \mathcal{F}_x)$ on $\mathcal{C}'_{\mathcal{X}}$, which factors through a presheaf $\mathfrak{q}\mathcal{F}$ on $\mathcal{C}_{\mathcal{X}}$. The canonical restriction maps $\Gamma(X, \mathcal{F}) \to \Gamma(x, \mathcal{F}_x)$ yield a map

$$\Gamma(X, \mathcal{F}) \to \varprojlim_{x \in \mathcal{C}_{\mathcal{X}}} \Gamma(x, \mathcal{F}_x). \tag{7.27.1}$$

If $x\colon [S/A] \to \mathcal{X}$ is an elementary point of rank zero, i.e. a geometric point of X (Definition 7.17), $\Gamma(x, \mathcal{F}_x) = \mathcal{F}_x$, and by restriction via $\varpi\colon \mathcal{P}_{\mathcal{X}} \hookrightarrow \mathcal{C}_{\mathcal{X}}$, the presheaf $\mathfrak{q}\mathcal{F}$ induces the presheaf $\mathfrak{p}\mathcal{F}$ (Construction 7.12). Therefore we have a commutative diagram

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \longrightarrow & \varprojlim_{x \in \mathcal{C}_{\mathcal{X}}} \Gamma(x, \mathcal{F}_x) \\ & \searrow & \downarrow \\ & & \varprojlim_{x \in \mathcal{P}_{\mathcal{X}}} \mathcal{F}_x, \end{array} \tag{7.27.2}$$

where the horizontal (resp. oblique) map is (7.27.1) (resp. (7.13.1)), and the vertical one is restriction via ϖ.

Proposition 7.28. *Let $\mathcal{X}$ be a locally noetherian Artin stack, Λ a noetherian commutative ring, and $\mathcal{F}$ a constructible sheaf of Λ-modules on $\mathcal{X}$. Then* (7.27.1) *is an isomorphism.*

Proof. The oblique map of (7.27.2) is an isomorphism by Proposition 7.13. By Lemma 7.3, the vertical map is obtained by applying the functor $\Gamma(\widehat{\mathcal{C}'_{\mathcal{X}}}, -) = \varprojlim_{\mathcal{C}'_{\mathcal{X}}}(-)$ to the adjunction map

$$\alpha\colon \mathfrak{q}'\mathcal{F} \to \varpi'_*\varpi'^*\mathfrak{q}'\mathcal{F} = \varpi'_*\mathfrak{p}'\mathcal{F},$$

where $\varpi'\colon \mathcal{P}'_{\mathcal{X}} \to \mathcal{C}'_{\mathcal{X}}$. Thus it suffices to show that α is an isomorphism. Here

$$(\varpi'_*, \varpi'^*)\colon \widehat{\mathcal{P}'_{\mathcal{X}}} \to \widehat{\mathcal{C}'_{\mathcal{X}}}$$

is the morphism of topoi defined by $(\varpi'^*\mathcal{E})(z) = \mathcal{E}(\varpi'(z))$ and $(\varpi'_*\mathcal{G})(x) = \varprojlim_{(t,\phi)\in(\varpi'\downarrow x)} \mathcal{G}_t$, where for an ℓ-elementary point $x\colon [S/A] \to \mathcal{X}$, $(\varpi' \downarrow x)$ is the category of pairs (t, ϕ), where t is a geometric point of $\mathcal{X}$ and $\phi\colon \varpi' t \to x$ is a morphism in $\mathcal{C}'_{\mathcal{X}}$, which is equivalent to $\mathcal{P}'_{[S/A]}$. Let $\mathcal{A}$ be the groupoid with one object $*$ and fundamental group A. Consider the functor $F\colon \mathcal{A} \to (\varpi' \downarrow x)$ sending $*$ to $(x\varepsilon, \varepsilon)$, where $\varepsilon\colon S \to [S/A]$, and $a \in A$ to the morphism $x\varepsilon \to x\varepsilon$ induced by the action of a. For any object (t, ϕ) of $(\varpi' \downarrow x)$, the category $((t, \phi) \downarrow F)$ is a simply connected groupoid. Therefore, F is cofinal and

$$\alpha(x)\colon \Gamma(x, \mathcal{F}_x) \to \varprojlim_{(t,\phi)\in(\varpi'\downarrow x)} \mathcal{F}_t \xrightarrow{\sim} \varprojlim_{\mathcal{A}} \mathcal{F}_{x\varepsilon} \simeq \mathcal{F}_s^A$$

is an isomorphism. Here s is the closed point of S. □

In the next section we study higher cohomological variants of (7.27.1).

8 A generalization of the structure theorems to Artin stacks

In this section we fix an algebraically closed field k and a prime number ℓ invertible in k.

Construction 8.1. Let $\mathcal{X}$ be an Artin stack, and let $K \in D_{\mathrm{cart}}(\mathcal{X}, \mathbb{F}_\ell)$. For $q \in \mathbb{Z}$, consider the presheaf of $\mathbb{F}_\ell$-vector spaces on $\mathcal{C}_\mathcal{X}$ (7.21.1)

$$(x\colon [S/A] \to \mathcal{X}) \mapsto H^q([S/A], K_x) \simeq H^q(BA, K_s)$$

(where $K_x := x^*K$ and s is the closed point of S), and let

$$R^q(\mathcal{X}, K) := \varprojlim_{(x\colon \mathcal{S}\to\mathcal{X})\in\mathcal{C}_\mathcal{X}} H^q(\mathcal{S}, K_x). \tag{8.1.1}$$

The restriction maps $H^q(\mathcal{X}, K) \to H^q(\mathcal{S}, K_x)$ define a map

$$a^q_{\mathcal{X},K}\colon H^q(\mathcal{X}, K) \to R^q(\mathcal{X}, K). \tag{8.1.2}$$

We denote by $a_{\mathcal{X},K}$ the direct sum of these maps:

$$a_{\mathcal{X},K} = \bigoplus_q a^q_{\mathcal{X},K}\colon H^*(\mathcal{X}, K) \to R^*(\mathcal{X}, K). \tag{8.1.3}$$

If K has a (pseudo-)ring structure (Construction 3.8), then both sides of (8.1.3) are $\mathbb{F}_\ell$-(pseudo-)algebras, and $a_{\mathcal{X},K}$ is a homomorphism of $\mathbb{F}_\ell$-(pseudo-)algebras.

Definition 8.2. We say that an Artin stack $\mathcal{X}$ over k is a *global quotient stack* if $\mathcal{X}$ is equivalent to a stack of the form $[X/G]$ for X a *separated* algebraic space of finite type over k and G an algebraic group over k. We say that an Artin stack $\mathcal{X}$ of finite presentation over k *has a stratification by global quotients* if there exists a stratification of $\mathcal{X}_{\mathrm{red}}$ by locally closed substacks such that each stratum is a global quotient stack.

Recall that an Artin stack over k is of finite presentation if and only if it is quasi-separated and of finite type over k. Note that our Definition 8.2 differs from [15, Definition 2.9] and [30, Definition 3.5.3] because we allow quotients by non-affine algebraic groups.

The following theorem is our main result.

Theorem 8.3. *Let $\mathcal{X}$ be an Artin stack of finite presentation over k admitting a stratification by global quotients, $K \in D^+_c(\mathcal{X}, \mathbb{F}_\ell)$.*

- *(a) $R^q(\mathcal{X}, K)$ is a finite-dimensional $\mathbb{F}_\ell$-vector space for all q. Moreover, $R^*(\mathcal{X}, \mathbb{F}_\ell)$ is a finitely generated $\mathbb{F}_\ell$-algebra and, for K in $D^b_c(\mathcal{X}, \mathbb{F}_\ell)$, $R^*(\mathcal{X}, K)$ is a finitely generated $R^*(\mathcal{X}, \mathbb{F}_\ell)$-module.*
- *(b) If K is a pseudo-ring in $D^+_c(\mathcal{X}, \mathbb{F}_\ell)$, then* $\operatorname{Ker} a_{\mathcal{X},K}$ *(8.1.2) is a nilpotent ideal of $H^*(\mathcal{X}, K)$. If, moreover, K is commutative and $\mathcal{X}$ is a Deligne-Mumford stack with finite inertia or a global quotient stack, then $a_{\mathcal{X},K}$ is a uniform F-isomorphism (Definition 6.10).*

Remark 8.4. (a) A non separated scheme of finite presentation over k is not a global quotient stack in the sense of Definition 8.2 in general. Michel Raynaud gave the example of an affine plane with doubled origin. More generally, if Y is a separated smooth scheme of finite type over k, and Y' is obtained by gluing two copies of Y, $Y^{(1)}$ and $Y^{(2)}$, along the complement of a nonempty closed subset of codimension ≥ 2, then, for any algebraic group G, every G-torsor X over Y' is non separated. To see this, we may assume G smooth. By étale localization on Y, we may further assume that X admits a section s_i over $Y^{(i)}$, $i = 1, 2$. Assume that X is separated. The restrictions of s_1 and s_2 to $V = Y^{(1)} \cap Y^{(2)}$ provide a section of $G \times V$, which extends by Weil's extension theorem (see [7, Theorem 4.4.1] for a generalization) to a section of $G \times Y^{(2)}$. Via this section, s_1 and s_2 can be glued to give a trivialization of X over Y', contradicting the separation assumptions.

(b) Recall [30, Proposition 3.5.9] that, if for every geometric point $\eta \to \mathcal{X}$, the inertia $I_\eta = \eta \times_\mathcal{X} I_\mathcal{X}$ is affine, where $I_\mathcal{X} = \mathcal{X} \times_{\Delta_\mathcal{X}, \mathcal{X}\times\mathcal{X}, \Delta_\mathcal{X}} \mathcal{X}$, then $\mathcal{X}$ has a stratification by global quotients in the sense of [30, Definition 3.5.3], and a fortiori in the sense of Definition 8.2.

(c) On the other hand, the fact that $\mathcal{X}$ has a stratification by global quotients in the sense of Definition 8.2 imposes restrictions on its inertia groups. In fact, if k has characteristic zero, then, for any geometric point $\eta = \operatorname{Spec}(K) \to \mathcal{X}$ with K algebraically closed, $I_\eta^0/(I_\eta)_{\mathrm{aff}}$ is an abelian variety over K defined over k. Here I_η^0 is the identity component of I_η and $(I_\eta)_{\mathrm{aff}}$ is the largest connected affine normal subgroup of I_η. Indeed, if $\mathcal{X} = [X/G]$, then I_η is a subgroup of $G \otimes_k K$, so that $I_\eta^0/(I_\eta)_{\mathrm{aff}}$ is isogenous to an abelian subvariety of $(G^0/G_{\mathrm{aff}}) \otimes_k K$, hence is defined over k (for an abelian variety A over k, torsion points of order invertible in k of $A \otimes_k K$ are defined over k as k is algebraically closed).

(d) For an Artin stack $\mathcal{X}$ of finite presentation over k and a commutative ring K in $D_c^b(\mathcal{X}, \mathbb{F}_\ell)$, we do not know whether $H^*(\mathcal{X}, K)$ is a finitely generated $\mathbb{F}_\ell$-algebra or whether $a_{\mathcal{X},K}$ is a uniform F-isomorphism in general, even under the assumption that $\mathcal{X}$ has a stratification by global quotients. It may be the case that to treat the general case we would need to reformulate the theory in a relative setting.

The proof of Theorem 8.3 will be given in Section 10. In the rest of this section we show that Theorem 8.3 (b) implies Theorem 6.17 (b).

Construction 8.5. Let G be an algebraic group over k and X an algebraic space over k endowed with an action of G (here we do not assume X to be of finite type over k). To show that Theorem 8.3 (b) implies Theorem 6.17 (b), we will proceed in two steps.

(1) For $K \in D^+_{\mathrm{cart}}([X/G], \mathbb{F}_\ell)$ we will construct a homomorphism

(8.5.1)
$$\alpha\colon R^*([X/G], K) \to R^*_G(X, K),$$

which will be a homomorphism of $\mathbb{F}_\ell$-(pseudo-)algebras if K has a (pseudo-)ring structure, and whose composition with $a_{[X/G],K}\colon H^*([X/G], K) \to R^*([X/G], K)$ will be $a_G(X, K)$ (6.16.7).

(2) We will show that α is an isomorphism.

Let us construct α. Recall that

$$R^q([X/G], K) = \varprojlim_{(x\colon \mathcal{S} \to [X/G]) \in \mathcal{C}_{[X/G]}} H^q(\mathcal{S}, K_x),$$

and $R^q_G(X, K) = \varprojlim_{(A,A',g) \in \mathcal{A}_G(k)^\natural} H^0(X^{A'}, R^q\pi_*[1/c_g]^*K)$ (6.16.8). We first compare the categories $\mathcal{A}_G(k)^\natural$ and $\mathcal{C}_{[X/G]}$ by means of a third category $\mathcal{C}_{X,G}$ mapping to them by functors E and Π:

(8.5.2)
$$\begin{array}{ccc} & \mathcal{C}_{X,G} & \\ {\scriptstyle E}\swarrow & & \searrow{\scriptstyle \Pi} \\ \mathcal{C}_{[X/G]} & & \mathcal{A}_G(k)^\natural. \end{array}$$

The category $\mathcal{C}_{X,G}$ is cofibered over $\mathcal{A}_G(k)^\natural$ by Π. The fiber category of $\mathcal{C}_{X,G}$ at an object (A, A', g) of $\mathcal{A}_G(k)^\natural$ is the category of points $P_{X^{A'}}$ of the fixed point space of A' in X. If $(a, b)\colon (A, A', g) \to (Z, Z', h)$ is a morphism in $\mathcal{A}_G(k)^\natural$ (cf. (6.16.2)), we define the pushout functor $P_{b^{-1}}\colon P_{X^{A'}} \to P_{X^{Z'}}$ to be the functor induced by $b^{-1}\colon X^{A'} \to (X^{A'})b^{-1} = X^{bA'b^{-1}} \subset X^{Z'}$. If $x\colon s \to X^{A'}$ is a geometric point of $X^{A'}$, let $E_{(A,A',g)}(x)\colon [s/A] \to [X/G]$ be the ℓ-elementary point of $[X/G]$ defined by the composition

$$E_{(A,A',g)}(x)\colon [s/A] \xrightarrow{[x/A]} [X^{A'}/A] \xrightarrow{[1/c_g]} [X/G].$$

For $(x\colon s \to X^{A'}) \in P_{X^{A'}}$, $(y\colon t \to X^{Z'}) \in P_{X^{Z'}}$, let $u\colon x \to y$ be a morphism in $\mathcal{C}_{X,G}$ above $(a, b)\colon (A, A', g) \to (Z, Z', h)$. The morphism

$$E(u)\colon E_{(A,A',g)}(x) \to E_{(Z,Z',h)}(y)$$

is defined as follows. By definition, u is a commutative square

$$\begin{array}{ccc} X^{A'} & \xleftarrow{x} & (X^{A'})_{(x)} \\ {\scriptstyle b^{-1}}\downarrow & & \downarrow{\scriptstyle f} \\ X^{Z'} & \xleftarrow{y} & (X^{Z'})_{(y)}, \end{array}$$

where the horizontal arrows denote by abuse of notation the morphisms induced by strict localizations. It gives the (2-commutative) square on the right of the diagram

$$\begin{array}{ccccc} & & [X^{A'}/A] & \xleftarrow{[x/A]} & [(X^{A'})_{(x)}/A] \\ & \swarrow{\scriptstyle [1/c_g]} \ \Downarrow & \downarrow{\scriptstyle [b^{-1}/c_a]} & & \downarrow{\scriptstyle [f/c_a]} \\ [X/G] & \xleftarrow[{[1/c_h]}]{} & [X^{Z'}/Z] & \xleftarrow{[y/Z]} & [(X^{Z'})_{(y)}/Z], \end{array} \tag{8.5.3}$$

whose composition with the 2-morphism (given by b) in the left triangle of (8.5.3) (appearing in (6.16.3)) is the morphism $E(u)$. This defines the functor E in (8.5.2).

Fix $q \in \mathbb{Z}$. Denote by $H^q(K_\bullet)$ the projective system $((\xi\colon \mathcal{S} \to [X/G]) \mapsto H^q(\mathcal{S}, K_\xi))$ on $\mathcal{C}_{[X/G]}$, whose projective limit is $R^q([X/G], K)$ (8.1.1). In other words, $R^q([X/G], K) = \Gamma(\widehat{\mathcal{C}_{[X/G]}}, H^q(K_\bullet))$. We have an inverse image map

$$\Gamma(\widehat{\mathcal{C}_{[X/G]}}, H^q(K_\bullet)) \to \Gamma(\widehat{\mathcal{C}_{X,G}}, E^* H^q(K_\bullet)) \simeq \Gamma(\widehat{\mathcal{A}_G(k)^\natural}, \Pi_* E^* H^q(K_\bullet)). \tag{8.5.4}$$

By the cofinality lemma (Lemma 8.6) below,

$$(\Pi_* E^* H^q(K_\bullet))_{(A,A',g)} \simeq \varprojlim_{x \in P_{X^{A'}}} H^q([x/A], K_x).$$

By Proposition 7.2 (applied to the algebraic space $X^{A'}$), we have a natural isomorphism

$$\varprojlim_{x \in P_{X^{A'}}} H^q([x/A], K_x) \xrightarrow{\sim} H^0(X^{A'}, R^q\pi_*([1/c_g]^* K))$$

where $\pi\colon [X^{A'}/A] = BA \times X^{A'} \to X^{A'}$ is the projection, and $[1/c_g]\colon [X^{A'}/A] \to [X/G]$ is the morphism in (6.16.3). Finally, we find a natural isomorphism

$$\Gamma(\widehat{\mathcal{A}_G(k)^\natural}, \Pi_* E^* H^q(K_\bullet)) \xrightarrow{\sim} \varprojlim_{(A,A',g) \in \mathcal{A}_G(k)^\natural} H^0(X^{A'}, R^q\pi_*([1/c_g]^* K)),$$

which, by the definition of $R^q_G(X, K)$ (6.16.8), can be rewritten

$$\Gamma(\widehat{\mathcal{A}_G(k)^\natural}, \Pi_* E^* H^q(K_\bullet)) \xrightarrow{\sim} R^q_G(X, K). \tag{8.5.5}$$

The composition of (8.5.4) and (8.5.5) yields the desired map α (8.5.1).

Lemma 8.6. *Let $\Pi\colon \mathcal{C} \to \mathcal{E}$ be a cofibered category, let e be an object of $\mathcal{E}$, and let Π_e be the fiber category of Π above e. Then the functor $F\colon \Pi_e \to (\Pi \downarrow e)$ is cofinal. In particular, for every presheaf $\mathcal{F}$ on $\mathcal{C}$, $(\Pi_*\mathcal{F})(e) \xrightarrow{\sim} \varprojlim_{c \in \Pi_e} \mathcal{F}(c)$.*

Proof. For every object $(c, f\colon \Pi c \to e)$ of $(\Pi \downarrow e)$, $(c, f) \to F(f_* c)$ is an initial object of $((c, f) \downarrow F)$. Thus $((c, f) \downarrow F)$ is connected. □

Proposition 8.7. *Under the assumptions of Construction 8.5, the functor E is cofinal. In particular, (8.5.1) is an isomorphism.*

Corollary 8.8. *Theorem 8.3 (b) implies Theorem 6.17 (b).*

Proof of Proposition 8.7. The second assertion follows from the first assertion and the construction of (8.5.1). To show the first assertion, since the functor $\mathcal{C}_{X,G_{\mathrm{red}}} \to \mathcal{C}_{X,G}$ is an isomorphism and the functor $\mathcal{C}_{[X/G_{\mathrm{red}}]} \to \mathcal{C}_{[X/G]}$ is an equivalence of categories by Remark 7.24, we may assume G smooth.

Let N be the set of morphisms in $\mathcal{C}_{X,G}$ whose image under E is an isomorphism in $\mathcal{C}_{[X/G]}$. Then E factors as

$$\mathcal{C}_{X,G} \to \mathcal{B} \coloneqq N^{-1}\mathcal{C}_{X,G} \xrightarrow{F} \mathcal{C}_{[X/G]}.$$

By Lemma 7.3, $\mathcal{C}_{X,G} \to \mathcal{B}$ is cofinal. Thus it suffices to show that F is cofinal. We will show that:

(a) F is essentially surjective;

(b) F is full.

This will imply that F is cofinal by Lemma 6.2. For the proof it is convenient to use the following notation. For an object x of $P_{X^{A'}}$ above an object (A, A', g) of $\mathcal{A}_G(k)^\natural$, we will denote the resulting object of $\mathcal{B}$ by the notation

$$(x, (A, A', g)).$$

Let us prove (a). For every ℓ-elementary point $\xi\colon [S/A] \to [X/G]$, we choose an algebraic closure $\bar{s}$ of the closed point s of S and we let $\bar{\xi}$ denote the composite $[\bar{s}/A] \to [S/A] \xrightarrow{\xi} [X/G]$. We say that a lifting

$$\sigma = (a \in X(\bar{s}), \alpha \in \mathcal{H}om(A, G)(\bar{s}), \iota\colon [a/\alpha] \simeq \bar{\xi})$$

of $\bar{\xi}$ (Proposition 7.18 (c)) is *rational* if $\alpha \in \mathcal{H}om(A, G)(k)$. Recall that α is injective. Here $\mathcal{H}om(A, G)$ is the scheme of group homomorphisms from A to G (Section 5). A rational lifting σ of $\bar{\xi}$ defines an object

$$\omega_\sigma = \omega_{a,\alpha} = (a, (\alpha(A), \alpha(A), 1))$$

of $\mathcal{B}$ and an isomorphism

$$\psi_{\xi,\sigma}\colon F(\omega_\sigma) \to \bar{\xi} \to \xi$$

in $\mathcal{C}_{[X/G]}$. By Corollary 5.2, every element of $\mathcal{H}om(A, G)(\bar{s})$ is conjugate by an element of $G(\bar{s})$ to an element of $\mathcal{H}om(A, G)(k)$. Thus every $\bar{\xi}$ admits a rational lifting. It follows that F is essentially surjective.

Let us prove (b). For any object $\mu = (x, (A, A', g))$ of $\mathcal{B}$, $\sigma_\mu = (\bar{x}, c_g\colon A \to G, \mathrm{id})$ is a rational lifting of $\overline{F(\mu)}$ and $\psi_{F(\mu),\sigma_\mu} = F(m_\mu)$, where

$$m_\mu\colon \omega_{\sigma_\mu} = (\bar{x}, (g^{-1}Ag, g^{-1}Ag, 1)) \to (x, (A, A', g)) = \mu$$

is the inverse of the obvious morphism in N above the morphism $(g, 1)\colon (g^{-1}Ag, g^{-1}Ag, 1) \leftarrow (A, A', g)$ of $\mathcal{A}_G(k)^\natural$. Now if μ and ν are objects of $\mathcal{B}$ and $f\colon F(\mu) \to F(\nu)$ is a morphism in $\mathcal{C}_{X,G}$, then $f = F(m_\nu u m_\mu^{-1})$, where u is obtained from the following lemma applied to f, $\sigma = \sigma_\mu$, $\tau = \sigma_\nu$. Thus F is full. □

Lemma 8.9. *Let $\xi\colon [S/A] \to [X/G]$, and let $\eta\colon [T/B] \to [X/G]$ be ℓ-elementary points of $[X/G]$. For every morphism $f\colon \xi \to \eta$ in $\mathcal{C}_{[X/G]}$, every rational lifting σ of $\bar{\xi}$, and every rational lifting τ of $\bar{\eta}$, there exists a morphism $u\colon \omega_\sigma \to \omega_\tau$ in $\mathcal{B}$ making the following diagram commute:*

$$\begin{array}{ccc} F(\omega_\sigma) & \xrightarrow{F(u)} & F(\omega_\tau) \\ \downarrow{\scriptstyle \psi_{\xi,\sigma}} & & \downarrow{\scriptstyle \psi_{\eta,\tau}} \\ \xi & \xrightarrow{f} & \eta. \end{array}$$

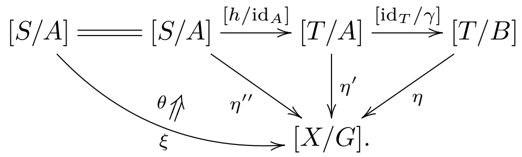
Proof. Given a triple (f, σ, τ) as in the lemma, we say that $L(f, \sigma, \tau)$ holds if there exists u satisfying the condition of the lemma. Given $f\colon \xi \to \eta$, we say that $L(f)$ holds if for every rational lifting σ of $\bar{\xi}$ and every rational lifting τ of $\bar{\eta}$, $L(f, \sigma, \tau)$ holds.

Step 1. First reductions. If $L(f\colon \xi \to \eta, \sigma, \tau)$ and $L(g\colon \eta \to \zeta, \tau, \kappa)$ hold, where σ, τ, κ are rational liftings of ξ, η, ζ, respectively, then $L(gf, \sigma, \kappa)$ holds, where (gf, σ, κ) is the composed triple $(gf, \sigma, \kappa) = (g, \tau, \kappa)(f, \sigma, \tau)$. In particular, if $L(f\colon \xi \to \eta)$ and $L(g\colon \eta \to \zeta)$ hold, then $L(gf)$

holds. Moreover, if $L(f)$ holds for an isomorphism f, then $L(f^{-1})$ holds. Thus we may assume that f is a morphism of $\mathcal{C}'_{[X/G]}$. Then $f = ([h/\gamma], \theta)$, where $(h\colon S \to T, \gamma\colon A \to B)$ is an equivariant morphism and $\theta\colon \xi \to \eta'' := \eta[h/\gamma]$ is a 2-morphism. Note that f can be decomposed as

$$\xi \xrightarrow{f_1} \eta'' \xrightarrow{f_2} \eta' \xrightarrow{f_3} \eta,$$

where $\eta' = \eta[\mathrm{id}_T/\gamma]$, $f_1 = (\mathrm{id}_{[S/A]}, \theta)$, $f_2 = ([h/\mathrm{id}_A], \mathrm{id}_{\eta''})$, $f_3 = ([\mathrm{id}_T/\gamma], \mathrm{id}_{\eta'})$, as shown by the diagram



Step 2. $L(f)$ holds for any morphism of the form $f = (\mathrm{id}_{[S/A]}, \theta)$, and in particular $L(f_1)$ holds. Let $\sigma = (a, \alpha, \iota)$ and $\tau = (b, \beta, \epsilon)$ be rational liftings of $\bar{\xi}$ and $\bar{\eta}$, respectively. Via the liftings, θ is given by $g \in J(\bar{s})$, where $J = \mathrm{Trans}_G(\beta(A), \alpha(A))$, and $a = bg$. Let $g' \in J(k)$ be a rational point of the connected component of J containing g. Then $h := g'^{-1}g \in H(\bar{s})$, where H is the identity component of $\mathrm{Norm}_G(\alpha(A))$. Let e be the generic point of $H_{\bar{s}}$. Note that $\mathcal{P}_{[H_{\bar{s}}/H_{\bar{s}}]}$ is equivalent to $P_{\bar{s}}$, and hence is a simply connected groupoid. Thus the morphism in $\mathcal{P}_{[H_{\bar{s}}/H_{\bar{s}}]}$ induced by the diagram $1 \leftarrow e \to h$ in $P_{H_{\bar{s}}}$ can be identified with the 2-morphism $i_1 \to i_h$ given by h, where $i_1, i_h\colon \bar{s} \to [H_{\bar{s}}/H_{\bar{s}}]$ are the morphisms induced by 1 and h, respectively. Then we can take u to be the morphism

$$(a, (\alpha(A), \alpha(A), 1)) \xrightarrow{v} (bg', (\alpha(A), \alpha(A), 1)) \xrightarrow{w} (b, (\beta(A), \beta(A), 1)).$$

in $\mathcal{B}$, where v is given by the diagram $1 \leftarrow e \to h$ in $P_{H_{\bar{s}}}$ via the H-equivariant morphism $H_{\bar{s}} \to X^A$ carrying 1 to a (and carrying h to bg'), and w is the obvious morphism of $\mathcal{C}_{X,G}$ above the morphism $(g'^{-1}, g')\colon (\alpha(A), \alpha(A), 1) \to (\beta(A), \beta(A), 1)$ of $\mathcal{A}_G(k)^\natural$.

Step 3. If $L(f, \sigma, \tau)$ holds for a triple (f, σ, τ), then $L(f)$ holds. Indeed, if σ' and τ' are rational liftings of $\bar{\xi}$ and $\bar{\eta}$, respectively, then, by Step 2, $L(\mathrm{id}_\xi, \sigma', \sigma)$ and $L(\mathrm{id}_\eta, \tau, \tau')$ hold, so $L(f, \sigma', \tau')$ holds because $(f, \sigma', \tau') = (\mathrm{id}_\eta, \tau, \tau')(f, \sigma, \tau)(\mathrm{id}_\xi, \sigma', \sigma)$.

Step 4. $L(f_3)$ holds. Indeed, a rational lifting $\tau = (b, \beta, \epsilon)$ of $\bar{\eta}$ induces a rational lifting of $\overline{\eta'}$, and with respect to these liftings we can take u to be the morphism in $\mathcal{B}$ induced by the diagram in $\mathcal{C}_{X,G}$

$$(b, (A, A, 1)) \leftarrow (b, (A, B, 1)) \to (b, (B, B, 1))$$

above the diagram in $\mathcal{A}_G(k)^\natural$

$$(A, A, 1) \xleftarrow{(\mathrm{id}_A, \gamma)} (A, B, 1) \xrightarrow{(\gamma, \mathrm{id}_A)} (B, B, 1).$$

Step 5. $L(f_2)$ holds. By Proposition 7.18 (c), η' can be lifted to a morphism of groupoids (b, α), where $b\colon T \to X$, and $\alpha\colon T \times A \to G$ is a crossed homomorphism, which restricts to a homomorphism $T^A \times A \to G$, corresponding to a morphism, denoted by $\alpha \mid T^A$, from the (strictly local) scheme T^A to the scheme $\mathcal{H}om(A, G)$ of group homomorphisms from A to G (Section 5). We will first show that, up to replacing T^A by a finite radicial extension, $\alpha \mid T^A$ is conjugate to a *k-rational* point of $\mathcal{H}om(A, G)$. For this, recall (Corollary 5.2) that the orbits of G acting by conjugation on $\mathcal{H}om(A, G)$ form a finite cover by open and closed subschemes. Let $C \subset \mathcal{H}om(A, G)$ be the orbit containing the image of $\alpha \mid T^A$. Choose a k-rational point $\alpha' \in \mathcal{H}om(A, G)(k)$ of C. Then the homomorphism $g \mapsto c_g(\alpha') = g^{-1}\alpha' g$ from G onto C factors through an isomorphism

$$H\backslash G \xrightarrow{\sim} C,$$

for a subgroup H of G. Let T' be defined by the cartesian square

$$\begin{array}{ccc} T' & \longrightarrow & H_{\mathrm{red}}\backslash G \\ \downarrow & & \downarrow \\ T^A & \xrightarrow{\alpha \mid T^A} & C. \end{array}$$

Since the projection $G \to H_{\mathrm{red}} \backslash G$ is smooth, the upper horizontal arrow can be lifted to a morphism $g\colon T' \to G$. Then $c_{g^{-1}}(\alpha \mid T')\colon T' \to \mathcal{H}om(A, G)$ is the constant map of value α'. Let $\pi\colon T' \to T^A \hookrightarrow T$ be the composite. We obtain a lifting $(b\pi, \alpha')\colon (T', A) \to (X, G)$ of $[T'/A] \to [X/G]$, which induces rational liftings of $\overline{\eta'}$ and $\overline{\eta''}$. With respect to these liftings, we can take u to be the morphism $\bar{s} \to \bar{t}$ in P_{X^A} above $(A, A, 1)$. □

Remark 8.10. The categories $\mathcal{C}_{X,G}$ and hence $N^{-1}\mathcal{C}_{X,G}$ are essentially small. It follows from (a) and (b) in the proof of Proposition 8.7 that $\mathcal{C}_{[X/G]}$ is essentially small.

9 Künneth formulas

The main results of this section are the Künneth formulas of Propositions 9.5 and 9.6. One may hope for more general formulas involving derived categories of modules over derived rings. We will not tackle this question. Instead, we use an elementary approach, based on module structures on spectral sequences, described in Construction 9.1 and Lemma 9.2.

Construction 9.1. Let $(\mathcal{C}, T)$ be an additive category with translation. For objects M and N in $\mathcal{C}$, the extended homomorphism group is the graded abelian group $\mathrm{Hom}^*(M, N)$ with $\mathrm{Hom}^n(M, N) = \mathrm{Hom}(M, T^n N)$. The extended endomorphism ring $\mathrm{End}^*(M) = \mathrm{Hom}^*(M, M)$ is a graded ring and $\mathrm{Hom}^*(M, N)$ is a $(\mathrm{End}^*(N), \mathrm{End}^*(M))$-bimodule. Let A^* be a graded ring. A left A^*-module structure on an object M of $\mathcal{C}$ is by definition a homomorphism $\lambda_M\colon A^* \to \mathrm{End}^*(M)$ of graded rings. More precisely, such a structure is given by morphisms $\lambda_a\colon M \to T^n M$, $a \in A^n$, $n \in \mathbb{Z}$ such that $\lambda_{a+b} = \lambda_a + \lambda_b$ for $a, b \in A^n$ and the diagram

$$\begin{array}{ccc} M & \xrightarrow{\lambda_b} & T^n M \\ & \searrow^{\lambda_{ab}} & \downarrow{\scriptstyle T^n\lambda_a} \\ & & T^{m+n}M \end{array}$$

commutes for $a \in A^m$, $b \in A^n$. A morphism $M \to M'$ in $\mathcal{C}$, with M and M' endowed with A^*-module structures, is said to preserve the A^*-module structures if it commutes with all λ_a, $a \in A^n$, $n \in \mathbb{Z}$. Let B^* be a graded right A^*-module. A morphism $B^* \otimes_{A^*} M \to N$ is by definition a homomorphism $B^* \to \mathrm{Hom}^*(M, N)$ of right A^*-modules. More precisely, it is given by a family of morphisms $f_b\colon M \to T^n N$, $b \in B^n$, $n \in \mathbb{Z}$ in $\mathcal{C}$ such that $f_{b+c} = f_b + f_c$ for $b, c \in B^n$ and the diagram

$$\begin{array}{ccc} M & \xrightarrow{\lambda_a} & T^m M \\ & \searrow^{f_{ba}} & \downarrow{\scriptstyle T^m f_b} \\ & & T^{m+n}N \end{array}$$

commutes for $a \in A^m$, $b \in B^n$. We thus get a functor $N \mapsto \mathrm{Hom}(B^* \otimes_{A^*} M, N)$ from $\mathcal{C}$ to the category of abelian groups, contravariant in M. In the category of graded abelian groups with translation given by shifting, the notion of left A^*-module coincides with the usual notion of graded left A^*-module and the above functor is represented by the usual tensor product. Let $F\colon (\mathcal{C}, T) \to (\mathcal{C}', T)$ be a functor of additive categories with translation [27, Definition 10.1.1 (ii)]. A left A^*-module structure on M induces a left A^*-module structure on FM and a morphism $B^* \otimes_{A^*} M \to N$ induces a morphism $B^* \otimes_{A^*} FM \to FN$.

Let $\mathcal{D}$ be a triangulated category, and let $\mathcal{A}$ be an abelian category. We consider the additive categories of spectral objects $\mathrm{SpOb}(\mathcal{D})$, $\mathrm{SpOb}(\mathcal{A})$ of type $\tilde{\mathbb{Z}}$ [46, II 4.1.2, 4.1.4, 4.1.6]. Here $\tilde{\mathbb{Z}}$ is the category associated to the ordered set $\mathbb{Z} \cup \{\pm\infty\}$. For $m \in \mathbb{Z}$, $(X, \delta) \in \mathrm{SpOb}(\mathcal{D})$, $(H, \delta) \in \mathrm{SpOb}(\mathcal{A})$, we put

$$(X, \delta)[m] = (X[m], (-1)^m \delta[m]), \quad (H^n, \delta^n)_n[m] = (H^{n+m}, (-1)^m \delta^{n+m})_n.$$

For $a \in \mathbb{Z} \cup \{\infty\}$, let $\mathrm{SpSeq}_a(\mathcal{A})$ be the category of spectral sequences $E_a \Rightarrow H$ in $\mathcal{A}$. We define

$$(E_a^{pq} \Rightarrow H^n)[m] = (E_a^{p+m,q} \Rightarrow H^{n+m})$$

by multiplying all d_r by $(-1)^m$. We endow $\mathrm{SpOb}(\mathcal{D})$, $\mathrm{SpOb}(\mathcal{A})$ and $\mathrm{SpSeq}_a(\mathcal{A})$ with the translation functor [1]. The resulting categories with translation are covariant in $\mathcal{D}$ and $\mathcal{A}$ for exact functors. If $H\colon \mathcal{D} \to \mathcal{A}$ is a cohomological functor, the induced functor $\mathrm{SpOb}(\mathcal{D}) \to \mathrm{SpOb}(\mathcal{A})$ commutes with translation. For $b \geq a$, the restriction functor $\mathrm{SpSeq}_a(\mathcal{A}) \to \mathrm{SpSeq}_b(\mathcal{A})$ commutes with translation. Using the notation of [46, II (4.3.3.2)], we obtain a functor $\mathrm{SpOb}(\mathcal{A}) \to \mathrm{SpSeq}_2(\mathcal{A})$, which also commutes with translation. A left A^*-module structure on an object of $\mathrm{SpSeq}_a(\mathcal{A})$ induces left A^*-module structures on H^* and E_r^{*q} for all $q \in \mathbb{Z}$ and $r \in [a, \infty]$. If we put $G_q H^n = F^{n-q} H^n$, so that the abutment is of the form $E_\infty^{pq} \xrightarrow{\sim} \mathrm{gr}_q^G H^{p+q}$, then G_q preserves the A^*-module structure. The differentials $d_r^{*q}\colon E_r^{*q} \to E_r^{*+r, q-r+1}$ and the abutment $E_\infty^{*q} \xrightarrow{\sim} \mathrm{gr}_q^G H^*$ are A^*-linear. A morphism $B^* \otimes_{A^*} (E_a \Rightarrow H) \to (E'_a \Rightarrow H')$ induces morphisms on E_r^{*q}, H^*, $G_q H^*$, $\mathrm{gr}_q^G H^*$, compatible with d_r, abutment, the projection $G_q \to \mathrm{gr}_q^G$ and the inclusions $G_{q-1} H^* \to G_q H^* \to H^*$.

Lemma 9.2. *Let H and H' be filtered graded abelian groups, H endowed with a left A^*-module structure. We let G denote the (increasing) filtrations. Assume that $G_q H^* = G_q H'^* = 0$ for q small enough and $H^n = \bigcup_{q \in \mathbb{Z}} G_q H^n$, $H'^n = \bigcup_{q\in\mathbb{Z}} G_q H'^n$ for all n. Let $B^* \otimes_{A^*} H \to H'$ be a morphism such that the homomorphism $B^* \otimes_{A^*} \mathrm{gr}_q^G H^* \to \mathrm{gr}_q^G H'^*$ is an isomorphism for all q. Then the homomorphism $B^* \otimes_{A^*} H^* \to H'^*$ is an isomorphism.*

Proof. Since $G_q H^* = G_q H'^* = 0$ for q small enough, one shows by induction that the morphism of exact sequences

$$\begin{array}{ccccccccc}
 & & B^* \otimes_{A^*} G_{q-1} H^* & \longrightarrow & B^* \otimes_{A^*} G_q H^* & \longrightarrow & B^* \otimes_{A^*} \mathrm{gr}_q^G H^* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \simeq & & \\
0 & \longrightarrow & G_{q-1} H'^* & \longrightarrow & G_q H'^* & \longrightarrow & \mathrm{gr}_q^G H'^* & \longrightarrow & 0
\end{array}$$

is an isomorphism. Then we apply the hypotheses $\varinjlim_{q\in\mathbb{Z}} G_q H^* = H^*$, $\varinjlim_{q\in\mathbb{Z}} G_q H'^* = H'^*$ and the fact that tensor product commutes with colimits. □

Construction 9.3. Let

$$\begin{array}{ccc}
X' & \xrightarrow{h} & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow{g} & Y
\end{array} \tag{9.3.1}$$

be a 2-commutative square of commutatively ringed topoi, $K \in D(\mathcal{O}_{Y'})$, $L \in D(\mathcal{O}_X)$. An element $s \in H^m(Y', K)$ corresponds to a morphism $\mathcal{O}_{Y'} \to K[m]$ in $D(\mathcal{O}_{Y'})$, and an element $t \in H^n(X, L)$ corresponds to a morphism $\mathcal{O}_X \to L[n]$ in $D(\mathcal{O}_X)$. Then

$$Lf'^* s \otimes^L_{\mathcal{O}_{X'}} Lh^* t\colon \mathcal{O}_{X'} \to Lf'^* K \otimes^L_{\mathcal{O}_{X'}} Lh^* L$$

is a morphism in $D(\mathcal{O}_{X'})$. This defines a graded map

$$H^*(Y', K) \times H^*(X, L) \to H^*(X', Lf'^* K \otimes^L_{\mathcal{O}_{X'}} Lh^* L)$$

which is $H^*(Y, \mathcal{O}_Y)$-bilinear, hence induces a homomorphism

$$H^*(Y', K) \otimes_{H^*(Y, \mathcal{O}_Y)} H^*(X, L) \to H^*(X', Lf'^* K \otimes^L_{\mathcal{O}_{X'}} Lh^* L), \tag{9.3.2}$$

which is a homomorphism of $(H^*(Y', \mathcal{O}_{Y'}), H^*(X, \mathcal{O}_X))$-bimodules.

Construction 9.4. Let $f\colon X \to Y$ be a morphism of commutatively ringed topoi, and let $L \in D(\mathcal{O}_Y)$, $K \in D(\mathcal{O}_X)$. We consider the second spectral object (L, δ) associated to L [46, III 4.3.1, 4.3.4], with $L(p, q) = \tau^{[p, q-1]} L$. For $s \in H^n(Y, \mathcal{O}_Y)$ corresponding to $\mathcal{O}_Y \to \mathcal{O}_Y[n]$, the functor $s \otimes^L_{\mathcal{O}_Y} -$ induces a morphism of spectral objects $(L, \delta) \to (L, \delta)[n]$. This endows (L, δ) with a structure of $H^*(Y, \mathcal{O}_Y)$-module (Construction 9.1). For $t \in H^n(X, K)$ corresponding to $\mathcal{O}_X \to$

$K[n]$, the functor $t\otimes^L_{\mathcal{O}_X} -$ induces a morphism of spectral objects $Lf^*(L,\delta) \to K\otimes^L_{\mathcal{O}_X} Lf^*(L,\delta)[n]$. This defines a morphism

$$H^*(X,K)\otimes_{H^*(Y,\mathcal{O}_Y)} Lf^*(L,\delta) \to K\otimes^L_{\mathcal{O}_X} Lf^*(L,\delta).$$

Applying Rf_* and composing with the adjunction $\mathrm{id}_{D(\mathcal{O}_Y)} \to Rf_*Lf^*$, we get a morphism

$$H^*(X,K)\otimes_{H^*(Y,\mathcal{O}_Y)} (L,\delta) \to Rf_*(K\otimes^L_{\mathcal{O}_X} Lf^*(L,\delta)).$$

Further applying the cohomological functor $H^0(Y,-)$, we obtain a morphism

$$H^*(X,K)\otimes_{H^*(Y,\mathcal{O}_Y)} (E_2\Rightarrow H) \to (E'_2\Rightarrow H'),$$

where the two spectral sequences are

$$E_2^{pq} = H^p(Y,\mathcal{H}^qL)\Rightarrow H^{p+q}(Y,L), \tag{9.4.1}$$

$$E'^{pq}_2 = H^p(X,K\otimes^L_{\mathcal{O}_X} Lf^*\mathcal{H}^qL)\Rightarrow H^{p+q}(X,K\otimes^L_{\mathcal{O}_X} Lf^*L). \tag{9.4.2}$$

By construction, the induced morphisms on E_2^{*q} and on H^* coincide with (9.3.2) for (9.3.1) given by id_f.

The results of Constructions 9.3 and 9.4 have obvious analogues for Artin stacks and complexes in $D_{\mathrm{cart}}(-,\Lambda)$, where Λ is a commutative ring.

Proposition 9.5. *Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & \mathcal{X} \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & \mathcal{Y} \end{array}$$

be a 2-commutative square of Artin stacks. Let $K\in D^+_{\mathrm{cart}}(\mathcal{Y}',\mathbb{F}_\ell)$ and $L\in D^+_{\mathrm{cart}}(\mathcal{X},\mathbb{F}_\ell)$. Suppose that

(a) The Leray spectral sequence for (f,L)

$$E_2^{pq} = H^p(\mathcal{Y},R^qf_*L)\Rightarrow H^{p+q}(\mathcal{X},L) \tag{9.5.1}$$

degenerates at E_2.

(b) For every q, R^qf_*L is a constant constructible $\mathbb{F}_\ell$-module on $\mathcal{Y}$.

(c) The base change morphism $\mathrm{BC}\colon g^*Rf_*L\to Rf'_*h^*L$ is an isomorphism.

(d) The morphism $\mathrm{PF}_{f'}\colon Rg_*(K\otimes Rf'_*h^*L)\to Rg_*Rf'_*(f'^*K\otimes h^*L)$ deduced from the projection formula morphism $K\otimes Rf'_*h^*L\to Rf'_*(f'^*K\otimes h^*L)$ is an isomorphism.

Then the spectral sequence (of type (9.4.2)*)*

$$E_2^{pq} = H^p(\mathcal{Y}',K\otimes R^qf'_*h^*L)\Rightarrow H^{p+q}(\mathcal{Y}',K\otimes Rf'_*h^*L) \tag{9.5.2}$$

degenerates at E_2 and the homomorphism (9.3.2)

$$H^*(\mathcal{Y}',K)\otimes_{H^*(\mathcal{Y},\mathbb{F}_\ell)} H^*(\mathcal{X},L)\to H^*(\mathcal{X}',{f'}^*K\otimes h^*L) \tag{9.5.3}$$

is an isomorphism.

Proof. Take any geometric point $t\to\mathcal{Y}'$. By (b), the E_2-term of (9.5.1) is

$$E_2^{pq} = H^p(\mathcal{Y},R^qf_*L)\simeq H^p(\mathcal{Y},\mathbb{F}_\ell)\otimes(R^qf_*L)_t.$$

By (c), (9.5.2) is isomorphic to

$$E'^{pq}_2 = H^p(\mathcal{Y}',K\otimes g^*R^qf_*L)\Rightarrow H^{p+q}(\mathcal{Y}',K\otimes g^*Rf_*L). \tag{9.5.4}$$

By (b), $E'^{pq}_2\simeq H^p(\mathcal{Y}',K)\otimes(R^qf_*L)_t$. Thus the morphism

$$H^*(\mathcal{Y}',K)\otimes_{H^*(\mathcal{Y},\mathbb{F}_\ell)} E_2^{*q}\to E'^{*q}_2$$

is an isomorphism. Il then follows from (a) and Lemma 9.2 that (9.5.4) degenerates at E_2 and the homomorphism

$$H^*(\mathcal{Y}', K) \otimes_{H^*(\mathcal{Y},\mathbb{F}_\ell)} H^*(\mathcal{X}, L) \to H^*(\mathcal{Y}', K \otimes g^* Rf_* L) \tag{9.5.5}$$

is an isomorphism. Thus (9.5.2) degenerates at E_2 and (9.5.3) is an isomorphism since it is the composition of (9.5.5) with the morphism induced by the composition

$$Rg_*(K \otimes g^* Rf_* L) \xrightarrow[\sim]{Rg_*(\mathrm{id}_K \otimes \mathrm{BC})} Rg_*(K \otimes Rf'_* h^* L) \xrightarrow[\sim]{\mathrm{PF}_{f'}} Rg_* Rf'_*(f'^* K \otimes h^* L),$$

of the isomorphisms in (c) and (d). □

In the rest of this section, let k be a separably closed field of characteristic $\neq \ell$.

Proposition 9.6. *Let G be a connected algebraic group over k, and let X be an algebraic space of finite presentation over k endowed with an action of G. Let*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{h} & [X/G] \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{Y}' & \xrightarrow{g} & BG \end{array}$$

be a 2-cartesian square of quasi-compact, quasi-separated Artin stacks, where f is the canonical projection. Let $K \in D^+_{\mathrm{cart}}(\mathcal{Y}', \mathbb{F}_\ell)$. Suppose that the map $e\colon H^*([X/G]) \to H^*(X)$ induced by the projection $X \to [X/G]$ is surjective. Then $H^*([X/G])$ is a finitely generated free $H^*(BG)$-module, the spectral sequence

$$E_2^{pq} = H^p(\mathcal{Y}', K \otimes R^q f'_* \mathbb{F}_\ell) \Rightarrow H^{p+q}(\mathcal{Y}', K \otimes Rf'_* \mathbb{F}_\ell)$$

degenerates at E_2, and the homomorphism

$$H^*(\mathcal{Y}', K) \otimes_{H^*(BG)} H^*([X/G]) \to H^*(\mathcal{X}', f'^* K)$$

is an isomorphism.

Proof. For the second and the third assertions, we apply Proposition 9.5. By Corollary 2.6 and generic base change (Remark 2.12), conditions (b) and (c) of Proposition 9.5 are satisfied. For $L \in D^+_{\mathrm{cart}}([X/G], \mathbb{F}_\ell)$, the diagram

$$\begin{array}{ccccccc} Rg_* K \otimes Rf_* L & \xrightarrow{\mathrm{PF}_g} & Rg_*(K \otimes g^* Rf_* L) & \xrightarrow{\mathrm{BC}} & Rg_*(K \otimes Rf'_* h^* L) & \xrightarrow{\mathrm{PF}_{f'}} & R(gf')_*(f'^* K \otimes h^* L) \\ & \searrow^{\mathrm{PF}_f} & & & & & \downarrow \simeq \\ & & Rf_*(f^* Rg_* K \otimes L) & \xrightarrow{\mathrm{BC}'} & Rf_*(Rh_* f'^* K \otimes L) & \xrightarrow{\mathrm{PF}_h} & R(fh)_*(f'^* K \otimes h^* L) \end{array}$$

commutes. Take $L = \mathbb{F}_\ell$. Then generic base change (Remark 2.12) and Proposition 2.9 (d) imply that BC, BC′, PF_f, PF_g, PF_h are isomorphisms, hence $\mathrm{PF}_{f'}$ is an isomorphism as well, which proves condition (d) of Proposition 9.5. Next we check condition (a) of Proposition 9.5. The Leray spectral sequence for f is

$$E_2^{pq} = H^p(BG, R^q f_* \mathbb{F}_\ell) \Rightarrow H^{p+q}([X/G]). \tag{9.6.1}$$

Since $R^q f_* \mathbb{F}_\ell$ is constant of value $H^q(X)$, we have $E_2^{pq} \simeq H^p(BG) \otimes H^q(X)$. As e is an edge homomorphism for (9.6.1), its surjectivity implies $d_r^{0q} = 0$ for all $r \geq 2$. It then follows from the $H^*(BG)$-module structure of (9.6.1) that it degenerates at E_2. The first assertion of Proposition 9.6 then follows from the fact that $H^*(X)$ is a finite-dimensional vector space. □

Proposition 9.7. *Let $G = \mathrm{GL}_{n,k}$, T be a maximal torus of G, $A = \mathrm{Ker}(-^\ell\colon T \to T)$. Then the map $H^*(BA, \mathbb{F}_\ell) \to H^*(G/A, \mathbb{F}_\ell)$ induced by the projection $G/A \to BA$ is surjective.*

Proof. Let us recall the proof on [36, page 566]. Consider the following diagram with 2-cartesian squares (Proposition 1.11):

$$\begin{array}{ccccc} G/A & \longrightarrow & G/T & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow & & \downarrow \\ BA & \longrightarrow & BT & \longrightarrow & BG. \end{array}$$

Note that the arrow $BA \to BT$ can be identified with the composition $BA \xrightarrow{\sim} [X/T] \to BT$, where $X = A\backslash T$, and the first morphism is an isomorphism by Corollary 1.16. The map $H^*(BA) \to H^*(X)$ induced by the projection $\pi\colon X = A\backslash T \to BA$ is surjective. Indeed, using Künneth formula this reduces to the case where T has dimension 1, which follows from Lemma 9.8 below. Note that π can be identified with the composition $X \to [X/T] \simeq BA$. Thus, by Proposition 9.6 applied to $f\colon [X/T] \to BT$, the map

$$H^*(G/T) \otimes_{H^*(BT)} H^*(BA) \to H^*(G/A)$$

is an isomorphism. We conclude by applying the fact that $H^*(BT) \to H^*(G/T)$ is surjective (Theorem 4.4). □

Lemma 9.8. *Let A be an elementary abelian ℓ-group, and let X be a connected algebraic space endowed with an A-action such that X is the maximal connected Galois étale cover of $[X/A]$ whose group is an elementary abelian ℓ-group. Then the homomorphism*

$$H^1(BA, \mathbb{F}_\ell) \to H^1([X/A], \mathbb{F}_\ell) \tag{9.8.1}$$

induced by the projection $[X/A] \to BA$ is an isomorphism.

Proof. For any connected Deligne-Mumford stack $\mathcal{X}$, $H^1(\mathcal{X}, \mathbb{F}_\ell)$ is canonically identified with $\operatorname{Hom}(\pi_1(\mathcal{X}), \mathbb{F}_\ell)$, and (9.8.1) is induced by the morphism

$$\pi_1([X/A]) \to \pi_1(BA) \simeq A.$$

The assumption means that A is the maximal elementary abelian ℓ-quotient of $\pi_1([X/A])$. □

Proposition 9.9. *Let X be an abelian variety over k, $A = X[\ell] = \operatorname{Ker}(\ell\colon X \to X)$. Then the map $H^*(BA, \mathbb{F}_\ell) \to H^*(X/A, \mathbb{F}_\ell)$ induced by the projection $X/A \to BA$ is surjective.*

Proof. We apply Lemma 9.8 to the morphism $\ell\colon X \to X$, which identifies the target with X/A. By Serre-Lang's theorem [48, XI Théorème 2.1], this morphism is the maximal étale Galois cover of X by an elementary abelian ℓ-group. Thus $H^1(BA) \to H^1(X/A)$ is an isomorphism. It then suffices to apply the fact that $H^*(X/A)$ is the exterior algebra of $H^1(X/A)$. □

10 Proof of the structure theorem

We proceed in several steps:

1. We first prove Theorem 8.3 (b) when $\mathcal{X}$ is a Deligne-Mumford stack with finite inertia, and whose inertia groups are elementary abelian ℓ-groups.
2. We prove Theorem 8.3 (b) for $\mathcal{X}$ a quotient stack $[X/G]$.
3. For certain quotient stacks $[X/G]$ we establish estimates for the powers of F annihilating the kernel and the cokernel of $a_G(X, K)$ (6.16.9).
4. Using (3), we prove Theorem 8.3 (b) for Deligne-Mumford stacks with finite inertia.
5. We prove Theorem 8.3 (a) and the first assertion of (b) for Artin stacks having a stratification by global quotients.

Construction 10.1. Let $f\colon X \to Y$ be a morphism of commutatively ringed topoi such that $\ell\mathcal{O}_Y = 0$, $K \in D(X)$. The Leray spectral sequence of f,

$$E_2^{ij} = H^i(Y, R^j f_* K) \Rightarrow H^{i+j}(X, K),$$

gives rise to an edge homomorphism

(10.1.1) $$e_{f,K}\colon H^*(X,K) \to H^0(Y, R^*f_*K),$$

which is a homomorphism of $\mathbb{F}_\ell$-(pseudo-)algebras if $K \in D(X)$ is a (pseudo-)ring. The following crucial lemma is similar to Quillen's result [36, Proposition 3.2].

Lemma 10.2. *Let K be a pseudo-ring in $D(X)$. Assume that $c = \mathrm{cd}(Y) < \infty$. Then $(\mathrm{Ker}\, e_{f,K})^{c+1} = 0$. Moreover, if K is commutative, then $e_{f,K}$ is a uniform F-isomorphism; more precisely, for $b \in E_2^{0,*}$, we have $b^{\ell^n} \in \mathrm{Im}\, e_{f,K}$, where $n = \max\{c-1, 0\}$.*

Proof. We imitate the proof of [36, Proposition 3.2] (for the case of finite cohomological dimension). We have $E_2^{ij} = 0$ for $i > c$. Consider the multiplicative structure on the spectral sequence (Example 3.15). As $\mathrm{Ker}\, e_{f,K} = F^1H^*(X,K)$, where $F^\bullet$ denotes the filtration on the abutment, $(\mathrm{Ker}\, e_{f,K})^{c+1} \subset F^{c+1}H^*(X,K) = 0$. If K is commutative and $b \in E_r^{0,*}$, then the formula $d_r(b^\ell) = \ell b^{\ell-1} d_r(b) = 0$ implies that $b^\ell \in E_{r+1}^{0,*}$. Thus for $b \in E_2^{0,*}$, $b^{\ell^n} \in E_{2+n}^{0,*} = E_\infty^{0,*} = \mathrm{Im}\, e_{f,K}$. □

Construction 10.3. Let $\mathcal{X}$ be a Deligne-Mumford stack of finite presentation and finite inertia over k. By Keel-Mori's theorem [28] (see [40, Theorem 6.12] for a generalization), there exists a coarse moduli space morphism
$$f\colon \mathcal{X} \to Y,$$
which is proper and quasi-finite. Let $K \in D^+_{\mathrm{cart}}(\mathcal{X}, \mathbb{F}_\ell)$. Then Construction 10.1 and Lemma 10.2 apply to f and K with $\mathrm{cd}_\ell(Y) \le 2\dim(Y)$.

For any geometric point t of Y, consider the following diagram of Artin stacks with 2-cartesian squares:

$$\begin{array}{ccccc} \mathcal{X}_t & \longrightarrow & \mathcal{X}_{(t)} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f} \\ t & \longrightarrow & Y_{(t)} & \longrightarrow & Y. \end{array}$$

We have canonical isomorphisms

(10.3.1) $$(R^qf_*K)_t \xrightarrow{\sim} H^q(\mathcal{X}_{(t)}, K) \xrightarrow{\sim} H^q(\mathcal{X}_t, K),$$

the second one by the proper base change theorem (cf. [34, Theorem 9.14]). Therefore, if we let P_Y denote the category of geometric points of Y (Definition 7.1), the map

(10.3.2) $$H^0(Y, R^qf_*K) \to \varprojlim_{t\in P_Y} H^q(\mathcal{X}_{(t)}, K) \xrightarrow{\sim} \varprojlim_{t\in P_Y} H^q(\mathcal{X}_t, K),$$

is an isomorphism if $K \in D_c^+(\mathcal{X}, \mathbb{F}_\ell)$, by Proposition 7.2. On the other hand, recall (8.1.1) that

$$R^q(\mathcal{X}, K) = \varprojlim_{(x\colon \mathcal{S}\to\mathcal{X})\in\mathcal{C}_\mathcal{X}} H^q(\mathcal{S}, K_x) = \Gamma(\widehat{\mathcal{C}_\mathcal{X}}, H^q(K_\bullet)),$$

where $K_x = x^*K$ and $H^q(K_\bullet)$ denotes the presheaf on $\mathcal{C}_\mathcal{X}$ whose value at x is $H^q(\mathcal{S}, K_x)$. We define a category $\mathcal{C}_f$ and functors

$$\begin{array}{ccc} & \mathcal{C}_f & \\ {\scriptstyle\varphi}\swarrow & & \searrow{\scriptstyle\psi} \\ \mathcal{C}_\mathcal{X} & & P_Y \end{array}$$

as follows. The category $\mathcal{C}_f$ is cofibered over P_Y by ψ. The fiber category of ψ at a geometric point $t \to Y$ is $\mathcal{C}_{\mathcal{X}_{(t)}}$. The pushout functor $\mathcal{C}_{\mathcal{X}_{(t)}} \to \mathcal{C}_{\mathcal{X}_{(z)}}$ for a morphism of geometric points $t \to z$ is induced by the morphism $\mathcal{X}_{(t)} \to \mathcal{X}_{(z)}$ (Remark 7.24). The functors $\varphi_t\colon \mathcal{C}_{\mathcal{X}_{(t)}} \to \mathcal{C}_\mathcal{X}$ induced by the morphisms $\mathcal{X}_{(t)} \to \mathcal{X}$ define φ. Thus we have an inverse image map

(10.3.3) $$\varphi^*\colon R^q(\mathcal{X}, K) \to \Gamma(\widehat{\mathcal{C}_f}, \varphi^*H^q(K_\bullet)).$$

By Lemma 8.6 we have

$$\psi_*\varphi^* H^q(K_\bullet)_t \simeq \Gamma(\widehat{\mathcal{C}_{\mathcal{X}_{(t)}}}, \varphi_t^* H^q(K_\bullet)).$$

Thus we have

$$\Gamma(\widehat{\mathcal{C}_f}, \varphi^* H^q(K_\bullet)) \simeq \Gamma(\widehat{P_Y}, \psi_*\varphi^* H^q(K_\bullet)) \xrightarrow{\sim} \varprojlim_{t\in P_Y} \varprojlim_{(x\colon \mathcal{S}\to\mathcal{X}_{(t)})\in\mathcal{C}_{\mathcal{X}_{(t)}}} H^q(\mathcal{S}, K_x) \tag{10.3.4}$$

Proposition 10.4.

(a) The following diagram commutes

$$\begin{array}{ccccc}
H^q(\mathcal{X}, K) & \xrightarrow{e^q_{f,K}} & H^0(Y, R^q f_* K) & \xrightarrow{(10.3.2)} & \varprojlim_{t\in P_Y} H^q(\mathcal{X}_{(t)}, K) \\
\downarrow{\scriptstyle a^q_{\mathcal{X},K}} & & & & \downarrow{\scriptstyle \varprojlim_{t\in P_Y} a^q_{\mathcal{X}_{(t)},K}} \\
R^q(\mathcal{X}, K) & \xrightarrow{\varphi^*} & \Gamma(\widehat{\mathcal{C}_f}, \varphi^* H^q(K_\bullet)) & \underset{\sim}{\xrightarrow{(10.3.4)}} & \varprojlim_{t\in P_Y} \varprojlim_{(x\colon \mathcal{S}\to\mathcal{X}_{(t)})\in\mathcal{C}_{\mathcal{X}_{(t)}}} H^q(\mathcal{S}, K_x).
\end{array}$$

(b) φ^* is an isomorphism.

(c) Consider the commutative square

$$\begin{array}{ccc}
H^q(\mathcal{X}_{(t)}, K) & \xrightarrow{\sim} & H^q(\mathcal{X}_t, K) \\
\downarrow{\scriptstyle a^q_{\mathcal{X}_{(t)},K}} & & \downarrow{\scriptstyle a^q_{\mathcal{X}_t,K}} \\
\varprojlim_{\mathcal{S}\in\mathcal{C}_{\mathcal{X}_{(t)}}} H^q(\mathcal{S}, K) & \xrightarrow{\iota^*} & \varprojlim_{\mathcal{S}\in\mathcal{C}_{\mathcal{X}_t}} H^q(\mathcal{S}, K)
\end{array}$$

defined by the functor $\iota\colon \mathcal{C}_{\mathcal{X}_t} \to \mathcal{C}_{\mathcal{X}_{(t)}}$ induced by the inclusion $\mathcal{X}_t \to \mathcal{X}_{\mathcal{T}}$, in which the upper horizontal map is the second isomorphism of (10.3.1)*. The map ι^* is an isomorphism.*

Proof. Assertion (a) follows from the definitions. For (b) it suffices to show that φ is cofinal. Let $\tau\colon \mathcal{C}_{\mathcal{X}} \to \mathcal{C}_f$ be the functor carrying an ℓ-elementary point $x\colon [S/A] \to \mathcal{X}$, with s the closed point of S, to the induced ℓ-elementary point $\tau(x)\colon [S/A] \to \mathcal{X}_{(f(s))}$. Then we have $\varphi\tau \simeq \mathrm{id}_{\mathcal{C}_{\mathcal{X}}}$, and a canonical natural transformation $\tau\varphi \to \mathrm{id}_{\mathcal{C}_f}$, carrying an object $\xi\colon [S/A] \to \mathcal{X}_{(t)}$ of $\mathcal{C}_f$ to the cocartesian morphism $\tau\varphi(\xi) \to \xi$ in $\mathcal{C}_f$ above the morphism $f(s) \to t$ in P_Y. These exhibit τ as a left adjoint to φ. Therefore, by Lemma 10.5 below, φ is cofinal. For (c), it suffices again to show that ι is cofinal. Let $X \to \mathcal{X}_{(t)}$ be an étale atlas. As f is quasi-finite, up to replacing X by a connected component, we may assume that X is a strictly local scheme, finite over $Y_{(t)}$. Then $\mathcal{X}_{(t)} \simeq [X/G]$, where $G = \mathrm{Aut}_{\mathcal{X}_{(t)}}(x)$, x is the closed point of X. Let $\xi\colon [S/A] \to [X/G]$ be an ℓ-elementary point of $[X/G]$. The ℓ-elementary point $[x/A] \to \mathcal{X}_t$, endowed with the morphism in $\mathcal{C}_{[X/G]}$ given by the diagram

$$[S/A] \to [X/A] \leftarrow [x/A]$$

in $\mathcal{C}'_{[X/G]}$, defines an initial object of $(\xi \downarrow \iota)$. Therefore, ι is cofinal. $\square$

Lemma 10.5. *Let $G\colon \mathcal{A} \to \mathcal{B}$ be a functor. If G has a left adjoint, then G is cofinal.*

Proof. Let $F\colon \mathcal{B} \to \mathcal{A}$ be a left adjoint to G. Then, for every object b of $\mathcal{B}$, $(Fb, b \to GFb)$ is an initial object of $(b \downarrow G)$. Thus $(b \downarrow G)$ is connected. $\square$

Corollary 10.6. *The assertion of Theorem 8.3 (b) holds if $\mathcal{X}$ is a Deligne-Mumford stack with finite inertia, whose inertia groups are elementary abelian ℓ-groups. More precisely, if $c = \mathrm{cd}_\ell(Y)$, where Y is the coarse moduli space of $\mathcal{X}$, then $(\mathrm{Ker}\, a_{\mathcal{X},K})^{c+1} = 0$ and for K commutative and $b \in E_2^{0,*}$, we have $b^{\ell^n} \in \mathrm{Im}\, a_{\mathcal{X},K}$, where $n = \max\{c-1, 0\}$.*

Proof. It suffices to show that, for all $t \in P_Y$,

$$a^q_{\mathcal{X}_t,K}\colon H^q(\mathcal{X}_t, K) \to \varprojlim_{(x\colon \mathcal{S}\to\mathcal{X}_t)\in\mathcal{C}_{\mathcal{X}_t}} H^q(\mathcal{S}, K_x)$$

is an isomorphism. Indeed, by Proposition 10.4 (c) this will imply that the right vertical arrow in the diagram of Proposition 10.4 (a) is an isomorphism. As (10.3.2) is an isomorphism, φ^* is an isomorphism (Proposition 10.4 (b)), and $e_{f,K} = \bigoplus e^q_{f,K}$ has nilpotent kernel and, if K is commutative, is an F-isomorphism (Lemma 10.2), it will follow that $a_{\mathcal{X},K} = \bigoplus a^q_{\mathcal{X},K}$ has the same properties with the same bounds for the exponents. As $f\colon \mathcal{X} \to Y$ is a coarse moduli space morphism, there exists a finite radicial extension $t' \to t$ and a geometric point y' of $\mathcal{X}$ above t' such that $(\mathcal{X}_{t'})_{\mathrm{red}} \simeq B\mathrm{Aut}_{\mathcal{X}}(y')$. Therefore we are reduced to showing that $a_{\mathcal{X},K}$ is an isomorphism for $\mathcal{X} = BA_k$, where A is an elementary abelian ℓ-group. In this case, $\mathrm{id}_{BA_k}\colon BA_k \to BA_k$ is a final object of $\mathcal{C}_{BA_k}$, so we can identify $R^q(BA_k, K)$ with $H^q(BA_k, K)$, and $a^q_{BA_k,K}$ with the identity. □

Corollary 10.7. *Suppose $\mathcal{X} = [X/G]$ is a global quotient stack (Definition 8.2), where the action of G on X satisfies the following two properties:*

(a) The morphism $\gamma\colon G \times X \to X \times X$, $(g, x) \mapsto (x, xg)$ is finite and unramified.

(b) All the inertia groups of G are elementary abelian ℓ-groups.

Then the assertions of Corollary 10.6 hold.

Proof. As γ in (a) can be identified with the morphism $X \times_{[X/G]} X \to X \times X$, which is the pull-back of the diagonal morphism $\Delta_{[X/G]}\colon [X/G] \to [X/G] \times [X/G]$ by $X \times X \to [X/G] \times [X/G]$, (a) implies that $\Delta_{[X/G]}$ is finite and unramified. In particular, $[X/G]$ is a Deligne-Mumford stack. Moreover, as the inertia stack is the pull-back of $\Delta_{[X/G]}$ by $\Delta_{[X/G]}$, $[X/G]$ has finite inertia. Taking (b) into account, we see that $[X/G]$ satisfies the assumptions of 10.6, and therefore 8.3 (b) holds for $[X/G]$. □

Proposition 10.8. *Theorem 8.3 (b) for global quotient stacks $[X/G]$ (Definition 8.2) follows from Theorem 8.3 (b) for G linear.*

Proof. Consider the system of subgroups $G_i = L \cdot A[m\ell^i] \cdot F$ of $G = L \cdot A \cdot F$ as in the proof of Theorem 4.6 (with $\Lambda = \mathbb{F}_\ell$ and $n = \ell$), where m is the order of F. Note that every elementary abelian ℓ-subgroup of $A \cdot F$ is contained in $A[m\ell] \cdot F$. As a consequence, every elementary abelian ℓ-subgroup of G is contained in G_1, so that the restriction map $R^*_G(X, K) \to R^*_{G_i}(X, K)$ is an isomorphism for $i \geq 1$. Consider the commutative diagram

$$\begin{array}{ccccc}
H^*([X/G], K) & \longrightarrow & H^*([X/G_{4d}], K) & \xrightarrow{\alpha} & H^*([X/G_{2d}], K) \\
\downarrow{\scriptstyle a_G(X,K)} & & \downarrow{\scriptstyle a_{G_{4d}}(X,K)} & & \downarrow{\scriptstyle a_{G_{2d}}(X,K)} \\
R^*_G(X, K) & \xrightarrow{\sim} & R^*_{G_{4d}}(X, K) & \xrightarrow{\sim} & R^*_{G_{2d}}(X, K),
\end{array}$$

where $d = \dim A$. By Remark 4.9, $H^*([X/G], K)$ is the image of α. Thus it suffices to show that $a_{G_{2d}}(X, K)$ has nilpotent kernel and, if K is commutative, $a_{G_{4d}}(X, K)$ is a uniform F-surjection. □

Proposition 10.9. *Theorem 8.3 (b) holds for global quotient stacks of the form $[X/G]$, where G is either a linear algebraic group, or an abelian variety.*

Proof. Although by Proposition 10.8 it would suffice to treat the case where G is linear, we prefer to treat both cases simultaneously, in order to later get better bounds for the power of F annihilating the kernel and the cokernel of the map $a_{\mathcal{X},K}$ (Corollary 10.10). We follow closely the arguments of Quillen for the proof of [36, Theorem 6.2]. If G is linear, choose an embedding of G into a linear group $L = \mathrm{GL}_n$ over k [13, Corollaire II.2.3.4], and a maximal torus T of L. If G is an abelian variety, let $L = T = G$. In both cases, denote by S the kernel of $\ell\colon T \to T$, which is an elementary abelian ℓ-group of order n. We let L act on $F = S\backslash L$ by right multiplication. If $g \in L(k)$, and if $\{S\}$ denotes the rational point of F defined by the coset S, the inertia group of L at $\{S\}g$ is $g^{-1}Sg$. Let us show that the diagonal action of G on $X \times F$ (resp. $X \times F \times F$) satisfies assumptions (a) and (b) of Corollary 10.7. It suffices to show this for $X \times F$. Consider the commutative square

$$\begin{array}{ccc}
L \times L & \xrightarrow{\sim} & L \times L \\
\downarrow & & \downarrow \\
F \times L & \longrightarrow & F \times F
\end{array}$$

where the horizontal morphisms are the morphisms $\gamma\colon (x,g)\mapsto (x,xg)$. As the vertical morphisms are finite and surjective, so is the lower horizontal morphism. Moreover, the latter is unramified. Hence the morphism $\gamma\colon F\times G\to F\times F$ is finite and unramified. The same holds for the morphism $\gamma\colon (X\times F)\times G\to (X\times F)\times(X\times F)$, $(x,y,g)\mapsto(x,y,xg,yg)$, because it is the composite $X\times F\times G\to X\times X\times F\times G\to X\times F\times X\times F$, where the first morphism $(x,y,g)\mapsto(x,xg,y,g)$ is a closed immersion by the assumption that X is separated and the second morphism $(x,x',y,g)\mapsto (x,y,x',yg)$ is a base change of $F\times G\to F\times F$. So (a) is satisfied for $X\times F$. Moreover, the inertia groups of G on $X\times F$ are conjugate in L to subgroups of S, so (b) is satisfied for $X\times F$.

As in [36, 6.2], consider the following commutative diagram

(10.9.1)

$$\begin{array}{ccccc} H^*([X/G],K) & \longrightarrow & H^*([X\times F/G],[\mathrm{pr}_1/\mathrm{id}_G]^*K) & \rightrightarrows & H^*([X\times F\times F/G],[\mathrm{pr}_1/\mathrm{id}_G]^*K) \\ \downarrow{\scriptstyle a_G(X,K)} & & \downarrow{\scriptstyle a_G(X\times F,[\mathrm{pr}_1/\mathrm{id}_G]^*K)} & & \downarrow{\scriptstyle a_G(X\times F\times F,[\mathrm{pr}_1/\mathrm{id}_G]^*K)} \\ R^*_G(X,K) & \longrightarrow & R^*_G(X\times F,[\mathrm{pr}_1/\mathrm{id}_G]^*K) & \rightrightarrows & R^*_G(X\times F\times F,[\mathrm{pr}_1/\mathrm{id}_G]^*K), \end{array}$$

in which the double horizontal arrows are defined by pr_{12} and pr_{13}. By Corollary 10.7, $a_G(X\times F,[\mathrm{pr}_1/\mathrm{id}_G]^*K)$ and $a_G(X\times F\times F,[\mathrm{pr}_1/\mathrm{id}_G]^*K)$ have nilpotent kernels and, if K is commutative, are uniform F-surjections. To show that $a_G(X,K)$ has the same properties it thus suffices to show that the rows of (10.9.1) are exact.

First consider the lower row. The component of degree q is isomorphic by definition (6.16.8) to the projective limit over $(A,A',g)\in\mathcal{A}_G(k)^\natural$ of

$$\begin{aligned}(10.9.2)\quad \Gamma(X^{A'},R^q\pi_*r^*K)&\to\Gamma(X^{A'}\times F^{A'},R^q\pi_*r^*[\mathrm{pr}_1/\mathrm{id}_G]^*K)\\ &\rightrightarrows \Gamma(X^{A'}\times F^{A'}\times F^{A'},R^q\pi_*r^*[\mathrm{pr}_1/\mathrm{id}_G]^*K),\end{aligned}$$

where we have put $r:=[1/c_g]$. In order to identify the second and third terms of (10.9.2), consider the following commutative diagram, where the middle and right squares are cartesian:

$$\begin{array}{ccccccc} [X\times F/G] & \xleftarrow{r} & BA\times X^{A'}\times F^{A'} & \xrightarrow{\pi} & X^{A'}\times F^{A'} & \xrightarrow{\mathrm{pr}_2} & F^{A'} \\ \downarrow{\scriptstyle [\mathrm{pr}_1/\mathrm{id}_G]} & & \downarrow{\scriptstyle \mathrm{id}\times\mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow \\ [X/G] & \xleftarrow{r} & BA\times X^{A'} & \xrightarrow{\pi} & X^{A'} & \longrightarrow & \operatorname{Spec} k \end{array}.$$

We have (by base change for the middle square)

$$\mathrm{pr}_1^*R^q\pi_*(r^*K)\xrightarrow{\sim}R^q\pi_*(\mathrm{id}\times\mathrm{pr}_1)^*r^*K\simeq R^q\pi_*r^*[\mathrm{pr}_1/\mathrm{id}_G]^*K.$$

By the Künneth formula for the right square, we have

$$\Gamma(X^{A'}\times F^{A'},\mathrm{pr}_1^*R^q\pi_*r^*K)\xrightarrow{\sim}\Gamma(X^{A'},R^q\pi_*r^*K)\otimes\Gamma(F^{A'},\mathbb{F}_\ell).$$

Therefore we get a canonical isomorphism

$$\Gamma(X^{A'}\times F^{A'},R^q\pi_*r^*[\mathrm{pr}_1/\mathrm{id}_G]^*K)\xrightarrow{\sim}\Gamma(X^{A'},R^q\pi_*r^*K)\otimes\Gamma(F^{A'},\mathbb{F}_\ell).$$

We have a similar identification for $X^{A'}\times F^{A'}\times F^{A'}$, and these identifications produce an isomorphism between (10.9.2) and the tensor product of $\Gamma(X^{A'},R^q\pi_*r^*K)$ with

$$(10.9.3)\qquad \Gamma(\operatorname{Spec} k,\mathbb{F}_\ell)\to\Gamma(F^{A'},\mathbb{F}_\ell)\rightrightarrows\Gamma(F^{A'}\times F^{A'},\mathbb{F}_\ell).$$

As A' is an elementary abelian ℓ-subgroup of G, A' is conjugate in L to a subgroup of S, hence $F^{A'}\neq\emptyset$. It follows that (10.9.3), (10.9.2) and hence the lower row of (10.9.1) are exact.

In order to prove the exactness of the upper row of (10.9.1), consider the square of Artin stacks with representable morphisms,

$$(10.9.4)\qquad \begin{array}{ccc} [(Y\times F)/G] & \longrightarrow & [Y/G] \\ \downarrow & & \downarrow \\ [F/L] & \longrightarrow & BL, \end{array}$$

where Y is an algebraic space of finite presentation over k endowed with an action of G, the horizontal morphisms are induced by projection from F and the vertical morphisms are induced by the embedding $G \to L$. The square is 2-cartesian by Proposition 1.11 and $BS \simeq [(S\backslash L)/L] = [F/L]$. By Propositions 9.6, 9.7 and 9.9, $H^*([F/L])$ is a finitely generated free $H^*(BL)$-module and the homomorphism

$$H^*([Y/G], K) \otimes_{H^*(BL)} H^*([F/L]) \to H^*([Y \times F/G], [\mathrm{pr}_1/\mathrm{id}_G]^*K)$$

defined by (10.9.4) is an isomorphism. Applying the above to $Y = X$ and $Y = X \times F$, we obtain an identification of the upper row of (10.9.1) with the sequence

$$H^*([Y/G], K) \to H^*([Y/G], K) \otimes_{H^*(BL)} H^*([F/L]) \rightrightarrows H^*([Y/G], K) \otimes_{H^*(BL)} H^*([F/L]) \otimes_{H^*(BL)} H^*([F/L]),$$

which is exact by the usual argument of faithfully flat descent. □

Corollary 10.10. *Let $\mathcal{X} = [X/G]$ be a global quotient stack, and assume that either (a) G is embedded in $L = \mathrm{GL}_n$, $n \geq 1$, or (b) G is an abelian variety. Let $K \in D_c^+([X/G], \mathbb{F}_\ell)$ be a pseudo-ring. Let $d = \dim X$. In case (a), let $e = \dim L/G$, $f = 2\dim L - \dim G$. In case (b), let $e = 0$, $f = \dim G$. Then*

(i) $(\operatorname{Ker} a_G(X, K))^m = 0$, *where $m = 2d + 2e + 1$,*

(ii) for K commutative and $y \in R_G^*(X, K)$, we have $y^{\ell^N} \in \operatorname{Im} a_G(X, K)$ for $N \geq \max\{2d + 2e - 1, 0\} + \log_\ell(2d + 2f + 1)$.

Proof. As in the proof of Proposition 10.9, let $F = S\backslash L$. We have $\mathrm{cd}_\ell((X \times F)/G) \leq 2\dim((X \times F)/G) = 2d + 2e$. As all inertia groups of G acting on $X \times F$ are elementary abelian ℓ-groups, by Corollary 10.7 we have $(\operatorname{Ker} a_G(X \times F, \mathrm{pr}_1^*K))^m = 0$, hence (i) by (10.9.1). For (ii), set $a_G(X, K) = a_0$, $a_G(X \times F, \mathrm{pr}_1^*K) = a_1$, $a_G(X \times F \times F, \mathrm{pr}_1^*K) = a_2$. Denote by $u_0 \colon H^*([X/G], K) \to H^*([X \times F/G], [\mathrm{pr}_1/\mathrm{id}_G]^*K)$ (resp. $v_0 \colon R_G^*(X, K) \to R_G^*(X \times F, [\mathrm{pr}_1/\mathrm{id}_G]^*K)$) the left horizontal map in (10.9.1), and $u_1 = d_0 - d_1 \colon H^*([X \times F/G], [\mathrm{pr}_1/\mathrm{id}_G]^*K) \to H^*([X \times F \times F/G], [\mathrm{pr}_1/\mathrm{id}_G]^*K)$ (resp. $v_1 = d_0 - d_1 \colon R_G^*(X \times F, [\mathrm{pr}_1/\mathrm{id}_G]^*K) \to R_G^*(X \times F \times F, \mathrm{pr}_1^*K)$), the map deduced from the double map (d_0, d_1) in (10.9.1). As d_0 and d_1 are compatible with raising to the ℓ-th power, so is u_1 (resp. v_1). Let $N_1 = \max\{2d + 2e - 1, 0\}$. By Corollary 10.7 we have $v_0(y)^{\ell^{N_1}} = a_1(x_1)$ for some $x_1 \in H^*([X \times F/G], [\mathrm{pr}_1/\mathrm{id}_G]^*K)$. By (10.9.1) we have $a_2u_1(x_1) = v_1a_1(x_1) = 0$. Let h be the least integer $\geq \log_\ell(2d + 2f + 1)$. As above we have $\mathrm{cd}_\ell((X \times F \times F)/G) \leq 2d + 2f$, so by Corollary 10.7 we get $u_1(x_1)^{\ell^h} = 0$, hence by (10.9.1) $x_1^{\ell^h} = u_0(x_0)$ for some $x_0 \in H^*([X/G], K)$, and finally $y^{\ell^{N_1+h}} = a_0(x_0)$. □

Remark 10.11.

(a) If in case (a) of Corollary 10.10, we assume moreover that X is affine, then $\mathrm{cd}_\ell((X \times F)/G) \leq d + e$ and $\mathrm{cd}_\ell((X \times F \times F)/G) \leq d + f$ by the affine Lefschetz theorem [50, XIV Corollaire 3.2]. Thus in this case (i) holds for $m = d + e + 1$ and (ii) holds for $N \geq \max\{d + e - 1, 0\} + \log_\ell(d + f + 1)$.

(b) Let $f \colon \mathcal{Y} \to \mathcal{X}$ be a finite étale morphism of Artin stacks of constant degree d. As the composite $H^*(\mathcal{X}, K) \xrightarrow{f^*} H^*(\mathcal{Y}, f^*K) \xrightarrow{\mathrm{tr}_{f,K}} H^*(\mathcal{X}, K)$ is multiplication by d, f^* is injective if d is prime to ℓ. Thus, in this case, if $\operatorname{Ker} a_{\mathcal{Y}, f^*K}$ is a nilpotent ideal, then $\operatorname{Ker} a_{\mathcal{X},K}$ is a nilpotent ideal with the same bound for the exponent. This applies in particular to the morphism $[X/H] \to [X/G]$, where $H < G$ is an open subgroup of index prime to ℓ.

Proposition 10.12. *Theorem 8.3 (b) holds if $\mathcal{X}$ is a Deligne-Mumford stack of finite inertia. More precisely, if $c = \mathrm{cd}_\ell(Y)$, where Y is the coarse moduli space of $\mathcal{X}$, and if r (resp. s) is the maximal number of elements of the inertia groups (resp. ℓ-Sylow subgroups of the inertia groups) of $\mathcal{X}$, then $(\operatorname{Ker} a(\mathcal{X}, K))^{(c+1)((s-1)^2+1)} = 0$, and for K commutative and $b \in R^*(\mathcal{X}, K)$, we have $b^{\ell^N} \in \operatorname{Im} a(\mathcal{X}, K)$ for $N \geq \max\{c-1, 0\} + \max\{r^2 - 2r, 0\} + \lceil \log_\ell(2(r-1)^2 + 1) \rceil + \lceil \log_\ell((s-1)^2 + 1) \rceil$. Here $\lceil x \rceil$ for a real number x denotes the least integer $\geq x$.*

Proof. Consider the coarse moduli space morphism $f\colon \mathcal{X} \to Y$. For every geometric point t of Y, there exists a finite radicial extension $t' \to t$ and a geometric point y' of $\mathcal{X}$ above t' such that $(\mathcal{X}_{t'})_{\mathrm{red}} \simeq B\mathrm{Aut}_{\mathcal{X}}(y')$. Note that for any field E, a finite group G of order m can be embedded into $\mathrm{GL}_m(E)$, given for example by the regular representation $E[G]$ of G. Moreover, if $m \neq 2$ or the characteristic of E is not 2, then G can be embedded into $\mathrm{GL}_{m-1}(E)$, because the subrepresentation of $E[G]$ generated by $g-h$, where $g,h \in G$, is faithful. Thus, by Remark 10.11, the map $a_{\mathcal{X}_t,K}$ in Proposition 10.4 (c) satisfies $(\operatorname{Ker} a_{\mathcal{X}_t,K})^{(s-1)^2+1} = 0$, and, for K commutative, $a_{\mathcal{X}_t,K}$ is a uniform F-surjection for all geometric points $t \to Y$ with bound for the exponent given by $\max\{r^2-2r,0\}+\lceil \log_\ell(2(r-1)^2+1)\rceil$, independent of t. Thus $(\operatorname{Ker} \varprojlim_{t\in P_Y} a_{\mathcal{X}_t,K})^{(s-1)^2+1} = 0$, and Lemma 10.13 below implies that $\varprojlim_{t\in P_Y} a_{\mathcal{X}_t,K}$ is a uniform F-surjection, with bound for the exponent given by $\max\{r^2-2r,0\}+\lceil \log_\ell(2(r-1)^2+1)\rceil + \lceil \log_\ell((s-1)^2+1)\rceil$. Hence, by Lemma 10.2 and Proposition 10.4, $a_{\mathcal{X},K}$ has the stated properties. □

Lemma 10.13. *Let $\mathcal{C}$ be a category, and let $u\colon R \to S$ be a homomorphism of pseudo-rings in* $\mathrm{GrVec}^{\mathcal{C}}$*. If u is a uniform F-injection (resp. uniform F-isomorphism) (Definition 6.10), then $\varprojlim_{\mathcal{C}} u$ is also a uniform F-injection (resp. uniform F-isomorphism). More precisely, if $m \geq 0$ is an integer such that for every object i of $\mathcal{C}$ and every $a \in \operatorname{Ker} u_i$, $a^m = 0$ (resp. and if $n \geq 0$ is an integer such that for every object i of $\mathcal{C}$ and every $b \in S_i$, $b^{\ell^n} \in \operatorname{Im} u_i$), then for every $x \in \operatorname{Ker} \varprojlim_{\mathcal{C}} u$, $x^m = 0$ (resp. for every $y \in \varprojlim_{\mathcal{C}} S$ and every integer $N \geq n + \log_\ell(m)$, $y^N \in \operatorname{Im} \varprojlim_{\mathcal{C}} u$).*

Proof. Let $x = (x_i)$ be an element in the kernel of $\varprojlim_{\mathcal{C}} u$. Since x_i is in $\operatorname{Ker} u_i$, $x^m = (x_i^m) = 0$.

Assume now that u is a uniform F-isomorphism with bounds for the exponents given by m and n, and let $y = (y_i)$ be an element of $\varprojlim_{\mathcal{C}} S$. For every object i of $\mathcal{C}$, take a_i in R_i such that $u_i(a_i) = y_i^{\ell^n}$. For every morphism $\alpha\colon i \to j$ in $\mathcal{C}$, the following diagram commutes

$$\begin{array}{ccc} R_j & \xrightarrow{u_j} & S_j \\ {\scriptstyle R_\alpha}\downarrow & & \downarrow{\scriptstyle S_\alpha} \\ R_i & \xrightarrow{u_i} & S_i. \end{array}$$

It follows that

$$u_i(R_\alpha(a_j) - a_i) = S_\alpha(u_j(a_j)) - u_i(a_i) = S_\alpha(y_j^{\ell^n}) - y_i^{\ell^n} = 0.$$

Let h be the least integer $\geq \log_\ell(m)$. Then $0 = (R_\alpha(a_j)-a_i)^{\ell^h} = R_\alpha(a_j)^{\ell^h} - a_i^{\ell^h}$, so that $w = (a_i^{\ell^h})$ is an element of $\varprojlim_{\mathcal{C}} R$. By definition, $u(w) = y^{\ell^{n+h}}$. □

In order to deal with the general case, we need the following lemma.

Lemma 10.14. *Let $u\colon R \to S$ be a homomorphism of pseudo-rings in* $\mathrm{GrVec}^{\mathcal{C}}$ *endowed with a splitting (Definition 3.2). Then* $(\operatorname{Ker} u)R = 0$*. In particular,* $(\operatorname{Ker} u)^2 = 0$*.*

Proof. Let $a \in \operatorname{Ker} u$, $b \in R$. Since $u(a) = 0$, $ab = u(a)b = 0$. □

Proposition 10.15. *The first assertion of Theorem 8.3 (b) holds.*

Proof. If $i\colon \mathcal{Y} \to \mathcal{X}$ is a closed immersion, $j\colon \mathcal{U} \to \mathcal{X}$ is the complement, then the following diagram of graded rings commutes:

$$\begin{array}{ccccc} H^*(\mathcal{Y}, Ri^!K) & \longrightarrow & H^*(\mathcal{X}, K) & \longrightarrow & H^*(\mathcal{U}, j^*K) \\ {\scriptstyle a_{\mathcal{Y},Ri^!K}}\downarrow & & \downarrow{\scriptstyle a_{\mathcal{X},K}} & & \downarrow{\scriptstyle a_{\mathcal{U},j^*K}} \\ R^*(\mathcal{Y}, Ri^!K) & \xrightarrow{u} & R^*(\mathcal{X}, K) & \longrightarrow & R^*(\mathcal{U}, K). \end{array}$$

The first row is exact and u is the composition of the inverse of the isomorphism $R^*(\mathcal{X}, i_*Ri^!K) \xrightarrow{\sim} R^*(\mathcal{Y}, Ri^!K)$ and the map $R^*(\mathcal{X}, i_*Ri^!K) \to R^*(\mathcal{X}, K)$ induced by adjunction $i_*Ri^!K \to K$. The composition

$$R^*(\mathcal{Y}, Ri^!K) \xrightarrow{u} R^*(\mathcal{X}, K) \to R^*(\mathcal{Y}, i^*K)$$

is induced by $Ri^!K \to i^*K$, hence has square-zero kernel by Lemma 10.14. Thus $(\operatorname{Ker} u)^2 = 0$. It follows that if both $a_{\mathcal{Y},Ri^!K}$ and $a_{\mathcal{U},j^*K}$ have nilpotent kernels, then $a_{\mathcal{X},K}$ has nilpotent kernel. Using this, we reduce by induction to the global quotient case. In this case, the assertion follows from Propositions 10.8 and 10.9. □

This finishes the proof of the structure theorem (Theorem 8.3 (b)).

Lemma 10.16. *Let $\mathcal{C}$ be a category having finitely many isomorphism classes of objects. Let $\mathcal{A}$ be the category whose objects are the elementary abelian ℓ-groups and whose morphisms are the monomorphisms. Let $F\colon \mathcal{C} \to \mathcal{A}$ be a functor. Let $\mathcal{F}$ be the presheaf of $\mathbb{F}_\ell$-algebras on $\mathcal{A}$ given by $\mathcal{F}(A) = \mathrm{S}(A^\vee)$. Let $\mathcal{G}$ be a presheaf of $F^*\mathcal{F}$-modules on $\mathcal{C}$. Assume that, for every object x of $\mathcal{C}$, $\mathcal{G}(x)$ is a finitely generated $\mathcal{F}(F(x))$-module. Then $R = \varprojlim_{x\in\mathcal{C}} \mathcal{F}(F(x))$ is a finitely generated $\mathbb{F}_\ell$-algebra and $S = \varprojlim_{x\in\mathcal{C}} \mathcal{G}(x)$ is a finitely generated R-module.*

Proof. We may assume that $\mathcal{C}$ has finitely many objects. For any monomorphism $u\colon A \to B$ of elementary abelian ℓ-groups, $\mathcal{F}(u)\colon \mathcal{F}(B) \to \mathcal{F}(A)$ carries $\mathrm{S}(B^\vee)^{\mathrm{GL}(B)}$ into $\mathrm{S}(A^\vee)^{\mathrm{GL}(A)}$. Thus $A \mapsto \mathcal{E}(A) = \mathrm{S}(A^\vee)^{\mathrm{GL}(A)} \subset \mathcal{F}(A)$ defines a subpresheaf $\mathcal{E}$ of $\mathbb{F}_\ell$-algebras of $\mathcal{F}$. As $\mathrm{GL}(A)$ is a finite group, by [48, V Corollaire 1.5] $\mathcal{F}(A)$ is finite over $\mathcal{E}(A)$ and $\mathcal{E}(A)$ is a finitely generated $\mathbb{F}_\ell$-algebra. For given A and B, since $\mathrm{GL}(B)$ acts transitively on the set of monomorphisms $u\colon A \to B$, the map $\mathrm{S}(B^\vee)^{\mathrm{GL}(B)} \to \mathrm{S}(A^\vee)$, restriction of $\mathcal{F}(u)$, does not depend on u. Thus $\mathcal{E}(u)$ only depends on A and B. Therefore, via the functor $\mathrm{rk}\colon \mathcal{A} \to \mathbb{N}$ carrying A to its rank, $\mathcal{E}$ factorizes through a presheaf $\mathcal{R}$ on the totally ordered set $\mathbb{N}$: we have a 2-commutative diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{A} \\ {\scriptstyle f}\downarrow & \swarrow_{\mathrm{rk}} & \downarrow{\scriptstyle \mathcal{E}} \\ \mathbb{N} & \xrightarrow{\mathcal{R}} & \mathcal{B}^{\mathrm{op}}, \end{array}$$

where $\mathcal{B}$ denotes the category of $\mathbb{F}_\ell$-algebras of finite type, and $\mathcal{R}(n) = \mathrm{S}((\mathbb{F}_\ell^n)^\vee)^{\mathrm{GL}_n(\mathbb{F}_\ell)}$, with, for $m \le n$, $\mathbb{F}_\ell^m$ included in $\mathbb{F}_\ell^n$ by any monomorphism. For a morphism $u\colon A \to B$ of $\mathcal{A}$, $\mathcal{F}(u)\colon \mathcal{F}(B) \to \mathcal{F}(A)$ is surjective, hence, as $\mathcal{F}(B)$ is finite over $\mathcal{E}(B)$, $\mathcal{F}(A)$ is finite over $\mathcal{E}(B)$, and $\mathcal{E}(A) \subset \mathcal{F}(A)$ is finite over $\mathcal{E}(B)$. By Lemma 10.19 below, for each x in $\mathcal{C}$, $\mathcal{E}(F(x))$ is finite over

$$Q = \varprojlim_{y\in\mathcal{C}} \mathcal{E}(F(y)) \simeq \varprojlim_{y\in\mathcal{C}} \mathcal{R}(f(y)).$$

The rest of the proof is similar to the proof of the last assertion of *loc. cit.* As $\mathcal{C}$ has finitely many objects, there exists a finitely generated $\mathbb{F}_\ell$-subalgebra Q_0 of Q such that, for each x in $\mathcal{C}$, $\mathcal{E}(F(x))$ is integral, hence finite over Q_0. Note that R is a Q-submodule, *a fortiori* a Q_0-submodule, of $\prod_{x\in\mathcal{C}} \mathcal{F}(F(x))$. For each x in $\mathcal{C}$, $\mathcal{F}(F(x))$ is finite over $\mathcal{E}(F(x))$, hence finite over Q_0. It follows that $\prod_{x\in\mathcal{C}} \mathcal{F}(F(x))$ is finite over Q_0. As Q_0 is a noetherian ring, R is finite over Q_0, hence a finitely generated $\mathbb{F}_\ell$-algebra. Similarly, S is a finitely generated Q_0-module, hence a finitely generated R-module. Note that Q is also finite over Q_0, hence a finitely generated $\mathbb{F}_\ell$-algebra, though we do not need this fact. □

The first step of the proof of Lemma 10.19 consists of simplifying the limit Q using cofinality. Among the functors

f_1, f_2, and f_3 are cofinal, while f_4 is not cofinal. It turns out that after making contractions of types f_1, f_2, and f_3, we obtain a rooted forest, of which the source of f_4 is a prototype.

For convenience we adopt the following order-theoretic definitions. We define a *rooted forest* to be a partially ordered set $\mathcal{P}$ such that $\mathcal{P}_{\le x} = \{y \in \mathcal{P} \mid y \le x\}$ is a finite chain for all $x \in \mathcal{P}$. We define a *rooted tree* to be a nonempty connected rooted forest. Let $\mathcal{P}$ be a rooted tree. For $x, y \in \mathcal{P}$, we say that y is a *child* of x if $x < y$ and there exists no $z \in \mathcal{P}$ such that $x < z < y$. By the connectedness of $\mathcal{P}$, $m(x) = \min \mathcal{P}_{\le x}$ is independent of $x \in \mathcal{P}$, hence $\mathcal{P}$ has a least element r, equal to $m(x)$ for all x. We call r the *root* of $\mathcal{P}$.

Remark 10.17. Although we do not need it, let us recall the comparison with graph-theoretic definitions. A *graph-theoretic rooted tree* $\mathcal{T}$ is a connected acyclic (undirected) graph with one vertex designated as the root [41, page 30]. For a graph-theoretic rooted tree $\mathcal{T}$, we let $V(\mathcal{T})$ denote the set of vertices of $\mathcal{T}$ equipped with the tree-order, with $x \le y$ if and only if the unique path from the root r to y passes through x. For any $x \in V(\mathcal{T})$, $V(\mathcal{T})_{\le x}$ consists of vertices on the path from r to x, so that $V(\mathcal{T})_{\le x}$ is a finite chain. Thus $V(\mathcal{T})$ is a rooted tree. Conversely, for any rooted tree $\mathcal{P}$, we construct a graph-theoretic rooted tree $\Gamma(\mathcal{P})$ as follows. Let G be the graph whose set of vertices is $\mathcal{P}$ and such that two vertices x and y are adjacent if and only if y is a child of x or x is a child of y. Note that each $x \le x'$ in $\mathcal{P}$ can be decomposed into a sequence $x = x_0 < x_1 < \cdots < x_n = x'$, $n \ge 0$, each x_{i+1} being a child of x_i, which defines a path from x to x' in G. Thus the connectedness of $\mathcal{P}$ implies the connectedness of G. If G admits a cycle, then there exists $y \in \mathcal{P}$ that is a child of distinct elements x and x' of $\mathcal{P}$, which contradicts the assumption that $\mathcal{P}_{\le y}$ is a chain. Let r be the root of $\mathcal{P}$. Then $\Gamma(\mathcal{P}) = (G, r)$ is a graph-theoretic rooted tree. We have $\mathcal{P} = V(\Gamma(\mathcal{P}))$ and $\mathcal{T} = \Gamma(V(\mathcal{T}))$.

The next lemma is probably standard but we could not find an adequate reference.

Lemma 10.18. *Let $\mathcal{C}$ be a category and let $f\colon \mathcal{C} \to \mathbb{N}$ be a functor. Let $\mathcal{P}$ be the set of full subcategories of $\mathcal{C}$ that are connected components of $f^{-1}(\mathbb{N}_{\ge n})$ for some $n \in \mathbb{N}$. Order $\mathcal{P}$ by inverse inclusion: for elements S and T of $\mathcal{P}$, we write $S \le T$ if $S \supset T$. Let $\psi\colon \mathcal{C} \to \mathcal{P}$ be the functor carrying an object x to the connected component $\psi(x)$ of $f^{-1}(\mathbb{N}_{\ge f(x)})$ containing x, and let $\phi\colon \mathcal{P} \to \mathbb{N}$ be the functor carrying S to* $\min f(S)$. *Then:*

- *(a) $f = \phi\psi$.*
- *(b) $\psi\colon \mathcal{C} \to \mathcal{P}$ is cofinal (Definition 6.1) and $\phi\colon \mathcal{P} \to \mathbb{N}$ is strictly increasing.*
- *(c) $\mathcal{P}$ is a rooted forest. Moreover, if $\mathcal{C}$ has finitely many isomorphism classes of objects, then $\mathcal{P}$ is a finite set.*

Proof. (a) Let x be an object of $\mathcal{C}$. As $x \in \psi(x)$, $\phi(\psi(x)) = \min f(\psi(x)) \le f(x)$. Conversely, as $\psi(x) \subset f^{-1}(\mathbb{N}_{\ge f(x)})$, $f(\psi(x)) \subset \mathbb{N}_{\ge f(x)}$, so that $\phi(\psi(x)) \ge f(x)$. Thus $\phi(\psi(x)) = f(x)$.

(b) Let $S \in \mathcal{P}$. Note that S is a connected component of $f^{-1}(\mathbb{N}_{\ge \phi(S)})$. By definition, $(S \downarrow \psi)$ is the category of pairs $(x, S \le \psi(x))$. Note that $S \supset \psi(x)$ implies that x is in S. Conversely, for x in S, S is a connected component of $f^{-1}(\mathbb{N}_{\ge n})$ for $n \le f(x)$, hence $S \supset \psi(x)$. Thus $(S \downarrow \psi)$ can be identified with S, hence is connected. This shows that ψ is cofinal. Now let $S < T$ be elements of $\mathcal{P}$. We have $\phi(S) \le \phi(T)$. If $\phi(S) = \phi(T) = n$, then S and T are both connected components of $f^{-1}(\mathbb{N}_{\ge n})$, which contradicts with the assumption $S \supsetneq T$. Thus $\phi(S) < \phi(T)$.

(c) Let $S \in \mathcal{P}$. Let $T, T' \in \mathcal{P}_{\le S}$. Then T (resp. T') is a connected components of $f^{-1}(\mathbb{N}_{\ge n})$ (resp. $f^{-1}(\mathbb{N}_{\ge n'})$), and T and T' both contain S. Thus $T \supset T'$ if $n \le n'$ and $T \subset T'$ if $n \ge n'$. Therefore, $\mathcal{P}_{\le S}$ is a chain. As ϕ is strictly increasing, ϕ induces an injection $\mathcal{P}_{\le S} \to \mathbb{N}_{\le \phi(S)}$, hence $\mathcal{P}_{\le S}$ is a finite set. Therefore, $\mathcal{P}$ is a rooted forest. Note that for $S \in \mathcal{P}$ and x in S, every object y of $\mathcal{C}$ isomorphic to x is also in S. Thus, if $\mathcal{C}$ has finitely many isomorphism classes of objects, then $\mathcal{P}$ is a finite set. □

Lemma 10.19. *Let $\mathcal{C}$ be a category having finitely many isomorphism classes of objects and let $f\colon \mathcal{C} \to \mathbb{N}$ be a functor. Let $\mathcal{R}$ be a presheaf of commutative rings on $\mathbb{N}$ such that, for each $m \le n$, $\mathcal{R}(m)$ is finite over $\mathcal{R}(n)$. Let $Q = \varprojlim_{x\in\mathcal{C}} \mathcal{R}(f(x))$. Then:*

- *(a) For each object x of $\mathcal{C}$, $\mathcal{R}(f(x))$ is finite over Q.*
- *(b) For each connected component S of $\mathcal{C}$ and each r in S satisfying $f(r) = \min f(S)$, we have*

$$\operatorname{Im}(Q \to \mathcal{R}(f(r))) = \operatorname{Im}(\mathcal{R}(\max f(S)) \to \mathcal{R}(f(r))).$$

Proof. By Lemma 10.18, we may assume that $\mathcal{C}$ is a finite rooted tree with root r. We prove this case by induction on $\#\mathcal{C}$. Let $B \subset \mathcal{C}$ be the set of children of r. For each $c \in B$, $\mathcal{C}_{\ge c}$ is a rooted tree with root c and Q is the fiber product over $\mathcal{R}(f(r))$ of the rings $Q_c = \varprojlim_{x\in\mathcal{C}_{\ge c}} \mathcal{R}(f(x))$ for $c \in B$. If B is empty, then $\mathcal{C} = \{r\}$ and the assertions are trivial. If $B = \{c\}$, then $Q \simeq Q_c$ and it suffices to apply the induction hypothesis to Q_c. Assume $\#B > 1$. Let $n = \max f(\mathcal{C})$, $n_c = \max f(\mathcal{C}_{\ge c})$, and let $c_0 \in B$ be such that $n_{c_0} = \min_{c\in B} n_c$. The complement $\mathcal{C}'$ of $\mathcal{C}_{\ge c_0}$ in $\mathcal{C}$ is a rooted tree with root r, and Q is the fiber product over $\mathcal{R}(f(r))$ of the rings Q_{c_0} and $Q' = \varprojlim_{x\in\mathcal{C}'} \mathcal{R}(f(x))$.

By the induction hypothesis, $A = \operatorname{Im}(Q_{c_0} \to \mathcal{R}(f(r))) = \operatorname{Im}(\mathcal{R}(n_{c_0}) \to \mathcal{R}(f(r)))$ and $\operatorname{Im}(Q' \to \mathcal{R}(f(r))) = \operatorname{Im}(\mathcal{R}(n) \to \mathcal{R}(f(r)))$, so that we have a cartesian square of commutative rings

$$\begin{CD} Q @>{\alpha'}>> Q' \\ @V{\beta'}VV @VV{\beta}V \\ Q_{c_0} @>{\alpha}>> A. \end{CD}$$

As α is surjective, we have $\operatorname{Ker}(\alpha') \simeq \operatorname{Ker}(\alpha)$ and α' is surjective (cf. [17, Lemme 1.3]), which implies (b). Moreover, as β is finite, β' is finite. Indeed, if $A = \sum_i a_i\beta(Q')$, then for liftings a_i' of a_i, $Q_{c_0} = \sum_i a_i'\beta'(Q)$. The assertion (a) then follows from the induction hypothesis applied to Q_{c_0} and Q'. □

Proof of Theorem 8.3 (a). Let $(j_i\colon \mathcal{X}_i \to \mathcal{X})_i$ be a finite stratification of $\mathcal{X}$ by locally closed substacks. The system of functors $(\mathcal{C}_{\mathcal{X}_i} \to \mathcal{C}_{\mathcal{X}})_i$ is essentially surjective. Thus the map

$$R^*(\mathcal{X}, K) \to \prod_i R^*(\mathcal{X}_i, j_i^*K)$$

is an injection. Thus, for the first assertion of Theorem 8.3 (a), we may assume that $\mathcal{X}$ is a global quotient stack, in which case the assertion follows from Theorem 6.17 (a) and Proposition 8.7.

Let $H^q(K_\bullet)$ denote the presheaf on $\mathcal{C}_{\mathcal{X}}$ whose value at $x\colon \mathcal{S} \to \mathcal{X}$ is $H^q(\mathcal{S}, K_x)$, where $K_x = x^*K$, so that $R^q(\mathcal{X}, K) = \varprojlim_{\mathcal{C}_{\mathcal{X}}} H^q(K_\bullet)$. Let N be the set of morphisms f in $\mathcal{C}_{\mathcal{X}}$ such that $(H^*(\mathbb{F}_{\ell\bullet}))(f)$ and $(H^*(K_\bullet))(f)$ are isomorphisms. By Lemma 7.3, $\varprojlim_{\mathcal{C}_{\mathcal{X}}} H^q(K_\bullet) \simeq \varprojlim_{N^{-1}\mathcal{C}_{\mathcal{X}}} H^q(K_\bullet)$ and similarly for $H^*(\mathbb{F}_{\ell\bullet})$. We claim that $N^{-1}\mathcal{C}_{\mathcal{X}}$ has finitely many isomorphism classes of objects. Then $\varprojlim_{N^{-1}\mathcal{C}_{\mathcal{X}}}$ commutes with direct sums, and, by Lemma 10.16, $R^*(\mathcal{X}, \mathbb{F}_\ell)$ and $R^*(\mathcal{X}, K)$ are finitely generated R-modules for a finitely-generated $\mathbb{F}_\ell$-algebra R, hence the second assertion of Theorem 8.3 (a). Using again the fact that the system of functors $(\mathcal{C}_{\mathcal{X}_i} \to \mathcal{C}_{\mathcal{X}})_i$ is essentially surjective, we may assume in the above claim that $\mathcal{X} = [X/G]$ is a global quotient. Consider the diagram (8.5.2). Note that the functor $\mathcal{A}_G(k)^\natural \to \mathcal{E}_G(\pi_0)$ induces a bijection between the sets of isomorphism classes of objects, and $\mathcal{E}_G(\pi_0)$ is essentially finite by Lemma 6.20 (a), thus $\mathcal{A}_G(k)^\natural$ has finitely many isomorphism classes of objects. Moreover, as E is essentially surjective, it suffices to show that, for every object (A, A', g) of $\mathcal{A}_G(k)^\natural$, the category $M^{-1}P_{X^{A'}}$ has finitely many isomorphism classes of objects. Here M is the set of morphisms f in $P_{X^{A'}}$ such that $(E^*_{(A,A',g)}H^*(K_\bullet))(f)$ is an isomorphism. Let (X_i) be a finite stratification of $X^{A'}$ into locally closed subschemes such that $K \mid X_i$ has locally constant cohomology sheaves. For a given i, all objects in the image of $P_{X_i} \to M^{-1}P_{X^{A'}}$ are isomorphic. Moreover, the system of functors $(P_{X_i} \to P_{X^{A'}})_i$ is essentially surjective. Therefore, $M^{-1}P_{X^{A'}}$ has finitely many isomorphism classes of objects. □

11 Stratification of the spectrum

In this section we fix an algebraically closed field k and a prime number ℓ invertible in k.

Construction 11.1. Let X be a separated algebraic space of finite type over k, and let G be an algebraic group over k acting on X. Define

$$\underline{(G,X)} \coloneqq \operatorname{Spec} H^{\varepsilon *}([X/G])_{\mathrm{red}}, \tag{11.1.1}$$

where $\varepsilon = 1$ if $\ell = 2$, and $\varepsilon = 2$ otherwise. In particular, for an elementary abelian ℓ-group A,

$$\underline{A} \coloneqq \underline{(A, \operatorname{Spec} k)} = \operatorname{Spec}(H_A^{\varepsilon *})_{\mathrm{red}}$$

is a standard affine space of dimension equal to the rank of A. The map $(A, C)^*$ (6.9.2) induces a morphism of schemes

$$(A, C)_*\colon \underline{A} \to \underline{(G, X)}, \tag{11.1.2}$$

hence $a(G,X)$ (6.9.3) induces a morphism of schemes

$$Y := \varinjlim_{(A,C)\in\mathcal{A}^\flat_{(G,X)}} \underline{A} \to \underline{(G,X)}. \tag{11.1.3}$$

It follows from Theorem 6.11 that (11.1.3) is a universal homeomorphism.

By Corollary 4.8, $(A,C)_*$ is finite. Moreover, $\mathcal{A}^\flat_{(G,X)}$ is essentially finite by Lemma 6.5. It follows that $Y \simeq \operatorname{Spec}(\varprojlim_{(A,C)\in\mathcal{A}^\flat_{(G,X)}} (H^{\varepsilon *}_A)_{\mathrm{red}})$ is finite over $\underline{(G,X)}$ and is a colimit of $\underline{A}$ in the category of locally ringed spaces and in particular a colimit of $\underline{A}$ in the category of schemes. This remark gives another proof of the second assertion of Corollary 6.13, as promised. Moreover, the $\mathbb{F}_\ell$-algebras $H^{\varepsilon *}([X/G])_{\mathrm{red}}$ and $\varprojlim_{(A,C)\in\mathcal{A}^\flat_{(G,X)}} (H^{\varepsilon *}_A)_{\mathrm{red}}$ are equipped with Steenrod operations (see Construction 11.6 below), compatible with the ring homomorphism $H^{\varepsilon *}([X/G])_{\mathrm{red}} \to \varprojlim_{(A,C)\in\mathcal{A}^\flat_{(G,X)}} (H^{\varepsilon *}_A)_{\mathrm{red}}$.

The structure of Y is described more precisely by the following *stratification theorem*, similar to [37, Theorems 10.2, 12.1].

Theorem 11.2. *Denote by $V_{(A,C)}$ the reduced subscheme of Y that is the image of the (finite) morphism $\underline{A} \to Y$ induced by (A,C). Let*

$$\underline{A}^+ := \underline{A} - \bigcup_{A'<A} \underline{A}',$$
$$V^+_{(A,C)} := V_{(A,C)} - \bigcup_{A'<A} V_{(A',C|A')},$$

where $A' < A$ means $A' \subset A$ and $A' \neq A$, and $C \mid A'$ denotes the component of $X^{A'}$ containing C. Then

- *(a) The Weyl group $W_G(A,C)$ (6.4.2) acts freely on $\underline{A}^+$ and the morphism $\underline{A}^+ \to Y$ given by (A,C) induces a homeomorphism*

$$\underline{A}^+/W_G(A,C) \to V^+_{(A,C)}. \tag{11.2.1}$$

- *(b) The subschemes $V_{(A,C)}$ of Y are the integral closed subcones of Y that are stable under the Steenrod operations on $\varprojlim_{(A,C)\in\mathcal{A}^\flat_{(G,X)}} (H^{\varepsilon *}_A)_{\mathrm{red}}$.*
- *(c) Let $(A_i,C_i)_{i\in I}$ be a finite set of representatives of isomorphism classes of objects of $\mathcal{A}^\flat_{(G,X)}$. Then the $V_{(A_i,C_i)}$ form a finite stratification of Y, namely Y is the disjoint union of the $V^+_{(A_i,C_i)}$, and $V_{(A_i,C_i)}$ is the closure of $V^+_{(A_i,C_i)}$.*

The proof is entirely analogous to that of [37, Theorems 10.2, 12.1]. One key step in the proof is the following analogue of [37, Proposition 9.6].

Proposition 11.3. *Let (A,C), (A',C') be objects of $\mathcal{A}^\flat_{(G,X)}$. The square of topological spaces*

$$\begin{array}{ccc} \underline{A}^+ \times \operatorname{Hom}_{\mathcal{A}^\flat_{G,X}}((A,C),(A',C')) & \longrightarrow & \underline{A}' \\ \downarrow \mathrm{pr}_1 & & \downarrow (A',C') \\ \underline{A}^+ & \xrightarrow{(A,C)} & Y \end{array} \tag{11.3.1}$$

is cartesian. Here the upper horizontal arrow is induced by

$$\operatorname{Hom}_{\mathcal{A}^\flat_{G,X}}((A,C),(A',C')) \to \operatorname{Hom}(\underline{A},\underline{A}'), \quad u \mapsto \operatorname{Spec}(Bu)^*_{\mathrm{red}},$$

where $(Bu)^*\colon H^{\varepsilon *}_{A'} \to H^{\varepsilon *}_A$.

As in [37, Proposition 9.6], this follows from the fact that $\mathcal{A}^\flat_{(G,X)}$ admits fiber products, whose proof is very similar to that of [37, Lemma 9.1].

Remark 11.4. The morphism (11.2.1) is not an isomorphism of schemes in general. In particular, the square (11.3.1) is not cartesian in the category of schemes. This is already shown by the example $G = \mathrm{GL}_\ell$, $X = \mathrm{Spec}(k)$, $A = \mu_\ell$ embedded diagonally in G. Let T be the standard maximal torus, and let $\{e_1, \dots, e_\ell\}$ be the standard basis of $T[\ell]$. Then $W_G(T) \simeq W_G(T[\ell]) \simeq \mathfrak{S}_\ell$ acts on $T[\ell]$ by permuting this basis, and $W_G(A) = \{1\}$. Note that $\underline{T[\ell]} \simeq \mathrm{Spec}(\mathrm{S}(T[\ell]^\vee)) \simeq \mathrm{Spec}(\mathbb{F}_\ell[t_1, \dots, t_\ell])$, and $V_{T[\ell]} = Y \simeq \underline{T[\ell]}/W_G(T[\ell])$ can be identified with the spectrum of the symmetric polynomials in $t_1, \dots, t_\ell$. As the image of the d-th fundamental symmetric polynomial in $t_1, \dots, t_\ell$ under homomorphism $\phi\colon \mathbb{F}_\ell[t_1, \dots, t_\ell] \to \mathbb{F}_\ell[t]$ carrying t_i to t is 0 for $1 \le d \le \ell - 1$ and t^ℓ for $d = \ell$, the diagram of schemes

$$\begin{array}{ccc} \underline{A} & \longrightarrow & \underline{T[\ell]} \\ \downarrow & & \downarrow \\ V_A & \longrightarrow & V_{T[\ell]} \end{array}$$

is given by the diagram of rings

$$\begin{array}{ccc} \mathbb{F}_\ell[t] & \xleftarrow{\ \phi\ } & \mathbb{F}_\ell[t_1, \dots, t_\ell] \\ \uparrow & & \uparrow \\ \mathbb{F}_\ell[t^\ell] & \longleftarrow & \mathbb{F}_\ell[t_1, \dots, t_\ell]^{\mathfrak{S}_\ell}. \end{array}$$

Therefore, (11.2.1) is given by the Frobenius map on $\mathbb{F}_\ell[t, t^{-1}]$.

Let $(f, u)\colon (X, G) \to (Y, H)$ be an equivariant morphism with (X, G) and (Y, H) as before. The induced morphism of quotient stacks $[f/u]\colon [X/G] \to [Y/H]$ induces maps

$$[f/u]^*\colon H^{\varepsilon *}([Y/H]) \to H^{\varepsilon *}([X/G]), \quad \underline{(f, u)}\colon \underline{(Y, H)} \to \underline{(X, G)}.$$

Moreover, (f, u) induces a functor $\mathcal{A}^\flat_{(u,f)}\colon \mathcal{A}^\flat_{(G,X)} \to \mathcal{A}^\flat_{(H,Y)}$ sending (A, C) to (uA, C'), where uA is the image of A under u and C' is the component of Y^{uA} containing fC, the image of C under f. We have the following analogue of [37, Proposition 10.9], with essentially the same proof.

Proposition 11.5. *The following conditions are equivalent.*
(a) $\mathcal{A}^\flat_{(u,f)}$ *is an equivalence of categories.*
(b) $[f/u]^*$ *is a uniform* F*-isomorphism.*
(c) $\underline{(f, u)}$ *is a universal homeomorphism.*

Construction 11.6. As Michèle Raynaud observed in [38, Section 4], the formalism of *Steenrod operations* [16] applies to the $\mathbb{F}_\ell$-cohomology of any topos. Let us review the construction of the operations in this case.

Let X be a topos, let K be a commutative ring in $D^+(X, \mathbb{F}_\ell)$, and let i be an integer. The Steenrod operations are $\mathbb{F}_\ell$-linear maps

$$P^i\colon H^*(X, K) \to \begin{cases} H^{*+i}(X, K) & \text{if } \ell = 2, \\ H^{*+2(\ell-1)i}(X, K) & \text{if } \ell > 2. \end{cases}$$

For $\ell = 2$, P^i is sometimes denoted Sq^i.

First note that for every complex $M \in C(X, \mathbb{F}_\ell)$, $\mathfrak{S}_\ell$ acts on $L^{\otimes \ell}$ by permutation of factors (with the usual sign rule). This induces a (non triangulated) functor

$$(-)^{\otimes \ell}\colon D(X, \mathbb{F}_\ell) \to D([X/\mathfrak{S}_\ell], \mathbb{F}_\ell). \tag{11.6.1}$$

Here we used the notation $[X/G]$ for the topos of sheaves in X endowed with an action of a finite group G. For a morphism $c\colon \mathbb{F}_\ell \to M[q]$ in $D(X, \mathbb{F}_\ell)$, applying (11.6.1), we obtain a morphism

$$c^{\otimes \ell}\colon \mathbb{F}_\ell \simeq \mathbb{F}_\ell^{\otimes \ell} \to (M[q])^{\otimes \ell} \simeq M^{\otimes \ell} \otimes S^{\otimes q}[q\ell]$$

in $D([X/\mathfrak{S}_\ell],\mathbb{F}_\ell)$, where $S\in \mathrm{Mod}([X/\mathfrak{S}_\ell],\mathbb{F}_\ell)$ is the pullback of the sheaf on $B\mathfrak{S}_\ell$ given by the sign representation $\mathrm{sgn}\colon \mathfrak{S}_\ell\to\mathbb{F}_\ell^\times$. This defines a map

$$H^q(X,M)\to H^{q\ell}([X/\mathfrak{S}_\ell],M^{\otimes\ell}\otimes S^{\otimes q}),\quad c\mapsto c^{\otimes \ell}. \tag{11.6.2}$$

Now choose a cyclic subgroup C of $\mathfrak{S}_\ell$ of order ℓ and a basis x of $H^1(BC,\mathbb{F}_\ell)\simeq \mathrm{Hom}(C,\mathbb{F}_\ell)$. Note that $\mathrm{sgn}\mid C$ is constant of value 1. Consider the composite map

$$\begin{aligned}D_{C,x}\colon H^q(X,K)\xrightarrow{(11.6.2)} H^{q\ell}([X/\mathfrak{S}_\ell],K^{\otimes\ell}\otimes S^{\otimes q})\xrightarrow{\pi} H^{q\ell}([X/\mathfrak{S}_\ell],K\otimes S^{\otimes q})\\ \to H^{q\ell}([X/C],K)\simeq\bigoplus_k H^k(BC,\mathbb{F}_\ell)\otimes H^{q\ell-k}(X,K),\end{aligned}$$

which turns out to be $\mathbb{F}_\ell$-linear. Here π is given by multiplication $K^{\otimes\ell}\to K$. Recall (Remark 6.12) that for $k\geq 0$, $H^k(BC,\mathbb{F}_\ell)=\mathbb{F}_\ell w_k$, where $w_k=x(\beta x)^{(k-1)/2}$ for k odd and $w_k=(\beta x)^{k/2}$ for k even. We define

$$D^k_{C,x}\colon H^q(X,K)\to H^{q\ell-k}(X,K)$$

by the formula $D_{C,x}u=\sum_k w_k\otimes D^k_{C,x}u$. Let $m=\frac{\ell-1}{2}$. We define

$$P^i_{C,x}=\begin{cases}(-1)^{i+m(q^2-q)/2}(m!)^{-q}D^{(q-2i)(\ell-1)}_{C,x} & \text{if } \ell>2,\ q\geq 2i,\\ D^{q-i}_{C,x} & \text{if } \ell=2,\ q\geq i,\\ 0 & \text{otherwise.}\end{cases}$$

For $\ell=2$, we have $C=\mathfrak{S}_2$ and x is unique. For $\ell>2$, $\sigma\in\mathfrak{S}_\ell$ and $a\in\mathbb{F}_\ell^\times$, $D^{2k}_{\sigma C\sigma^{-1},a(x\circ c_\sigma)}=\mathrm{sgn}(\sigma)^q a^{-k}D^{2k}_{C,x}$, where $c_\sigma\colon \sigma C\sigma^{-1}\to C$ is the homomorphism $g\mapsto\sigma^{-1}g\sigma$. Thus

$$P^i_{\sigma C\sigma^{-1},a(x\circ c_\sigma)}=\mathrm{sgn}(\sigma)^q(a^m)^qP^i_{C,x},$$

where $a^m=\pm1$. In particular, up to a sign, $P^i_{C,x}$ is independent of the choices of C and x. Let $T\in\mathfrak{S}_\ell$ be the permutation defined by $T(n)=n+1$ for $n\in\mathbb{Z}/\ell\mathbb{Z}$. In the following we will take C to be the subgroup generated by T and take x to be the dual basis of T, and omit them from the indices.

For a homomorphism of commutative rings $K\to K'$, the induced homomorphism $H^*(X,K)\to H^*(X,K')$ is compatible with Steenrod operations on $H^*(X,K)$ and $H^*(X,K')$. Moreover, for a morphism of topoi $f\colon X\to Y$, Steenrod operations are compatible with the isomorphism $H^*(X,K)\simeq H^*(Y,Rf_*K)$.

It is easy to check the following properties of Steenrod operations, where we write H^* for $H^*(X,K)$:

- For $x\in H^i$ (resp. $x\in H^{2i}$), $P^ix=x^\ell$ if $\ell=2$ (resp. $\ell>2$);
- If one defines
$$P_t\colon H^*\to H^*[t,t^{-1}]$$
by $P_t(x)=\sum_{i\in\mathbb{Z}}P^i(x)t^i$, then P_t is a ring homomorphism (Cartan's formula).

In the case where X has enough points and $K\in\mathrm{Mod}(X,\mathbb{F}_\ell)$, Epstein showed the following additional properties [16, Theorem 8.3.4]:

- $P_t\colon H^*\to H^*[t]$. In other words, $P^i=0$ for $i<0$.
- $P^0=\mathrm{id}$ for $K=\mathbb{F}_\ell$ (this depends on the choices of C and x above).

In particular, $P_t(x)=x+x^\ell t$ for $x\in H^1(X,\mathbb{F}_\ell)$, $\ell=2$ (resp. $x\in H^2(X,\mathbb{F}_\ell)$, $\ell>2$).

The above can be easily adapted to D^+_{cart} of Artin stacks. For a morphism of Artin stacks $f\colon\mathcal{X}\to\mathcal{Y}$ and a commutative ring $K\in D^+_{\mathrm{cart}}(\mathcal{X},\mathbb{F}_\ell)$, Steenrod operations are compatible with the isomorphism $H^*(X,K)\simeq H^*(Y,Rf_*K)$. Therefore, for $K'\in D^+_{\mathrm{cart}}(Y,\mathbb{F}_\ell)$, Steenrod operations are compatible with the restriction homomorphism $H^*(Y,K')\to H^*(X,f^*K')$.

For related results on the Chow rings of classifying spaces and much more, we refer the reader to Totaro's book [45] and the bibliography thereof.

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