Parity and symmetry in intersection and ordinary cohomology

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To the memory of Torsten Ekedahl

Abstract

In this paper, we show that the Galois representations provided by ℓ-adic cohomology of proper smooth varieties, and more generally by ℓ-adic intersection cohomology of proper varieties, over any field, are orthogonal or symplectic according to the degree. We deduce this from a preservation result of orthogonal and symplectic pure perverse sheaves by proper direct image. We show moreover that the subgroup of the Grothendieck group generated by orthogonal pure perverse sheaves of even weights and symplectic pure perverse sheaves of odd weights are preserved by Grothendieck's six operations. Over a finite field, we deduce parity and symmetry results for Jordan blocks appearing in the Frobenius action on intersection cohomology of proper varieties, and virtual parity results for the Frobenius action on ordinary cohomology of arbitrary varieties.

1 Introduction

The n-th cohomology of a compact Kähler manifold X is equipped with a pure Hodge structure of weight n:

$$\mathrm{H}^n(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=n} \mathrm{H}^{pq},$$

where $\mathrm{H}^{pq} \simeq \mathrm{H}^q(X, \Omega_X^p)$ satisfies $\overline{\mathrm{H}^{pq}} = \mathrm{H}^{qp}$. In particular, $\mathrm{H}^n(X, \mathbb{Q})$ is even-dimensional for n odd. Hodge decomposition and Hodge symmetry extend to proper smooth schemes over $\mathbb{C}$ [10, Proposition 5.3] by Chow's lemma and resolution of singularities. Thus, in this case, $\mathrm{H}^n(X(\mathbb{C}), \mathbb{Q})$ is also even-dimensional for n odd. Moreover, the pure Hodge structure of weight n on $\mathrm{H}^n(X(\mathbb{C}), \mathbb{Q})$ is polarizable, in the sense that there exists a morphism of Hodge structures

$$\mathrm{H}^n(X(\mathbb{C}), \mathbb{Q}) \otimes \mathrm{H}^n(X(\mathbb{C}), \mathbb{Q}) \to \mathbb{Q}(-n),$$

symmetric for n even and alternating for n odd, satisfying certain positivity conditions, which implies that the pairing is perfect.

Now let $\bar{k}$ be a separably closed field of characteristic $p \geq 0$ and let $\ell \neq p$ be a prime number. For a projective smooth scheme X of finite type over $\bar{k}$, hard Lefschetz theorem [11, Théorème 4.1.1] and Poincaré duality equip the n-th ℓ-adic cohomology $\mathrm{H}^n(X, \mathbb{Q}_\ell)$ of X with a nondegenerate bilinear form that is symmetric for n even and alternating for n odd. In particular, $\mathrm{H}^n(X, \mathbb{Q}_\ell)$ is even-dimensional for n odd. In a remark following [11, Corollaire 4.1.5], Deligne predicts that the evenness of the odd-degree Betti numbers should hold more generally for proper smooth schemes over $\bar{k}$. This is recently shown by Suh [45, Corollary 2.2.3] using crystalline cohomology.

The goal of this article is to study problems of parity and symmetry in more general settings, including symmetry of Galois actions on cohomology. Our approach is different from that of Suh as we do not use p-adic cohomology.

For a general scheme X of finite type over $\bar{k}$, $\mathrm{H}^n(X, \mathbb{Q}_\ell)$ is not "pure" and not necessarily even-dimensional for n odd. Before going into results for such mixed situations, let us first state our results in the pure case for intersection cohomology.

Theorem 1.1. *Let k be an arbitrary field of characteristic $p \geq 0$ and let $\bar{k}$ be its separable closure. Let X be a proper, equidimensional scheme over k. Then, for n even (resp. odd), the n-th ℓ-adic intersection cohomology group admits a $\mathrm{Gal}(\bar{k}/k)$-equivariant symmetric (resp. alternating) perfect pairing*

$$\mathrm{IH}^n(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes \mathrm{IH}^n(X_{\bar{k}}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(-n).$$

Here $\mathrm{Gal}(\bar{k}/k)$ denotes the Galois group of k and $X_{\bar{k}} = X \otimes_k \bar{k}$.

By definition, $\mathrm{IH}^n(X_{\bar{k}}, \mathbb{Q}_\ell) = \mathrm{H}^{n-d}(X_{\bar{k}}, \mathrm{IC}_X)$, where $\mathrm{IC}_X = j_{!*}(\mathbb{Q}_\ell[d])$, $d = \dim(X)$, $j\colon U \to X$ is an open dense immersion such that U_{red} is regular. For X proper smooth, we have $\mathrm{IH}^n(X_{\bar{k}}, \mathbb{Q}_\ell) = \mathrm{H}^n(X_{\bar{k}}, \mathbb{Q}_\ell)$, and the theorem takes the following form. The statement was suggested to us by Takeshi Saito. One may compare such pairings with polarizations of pure Hodge structures mentioned at the beginning of the introduction.

Corollary 1.2. *Let X be a proper smooth scheme over k. Then, for n even (resp. odd), the n-th ℓ-adic cohomology group admits a $\mathrm{Gal}(\bar{k}/k)$-equivariant symmetric (resp. alternating) perfect pairing*

$$\mathrm{H}^n(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes \mathrm{H}^n(X_{\bar{k}}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell(-n).$$

Ignoring Galois actions, we obtain the following corollary. For X proper smooth, this gives another proof of Suh's result mentioned earlier.

Corollary 1.3. *Let X be a proper, equidimensional scheme over k. Then $\mathrm{IH}^n(X_{\bar{k}}, \mathbb{Q}_\ell)$ is even-dimensional for n odd.*

To demonstrate the strength of Theorem 1.1, we give a reformulation in the case where $k = \mathbb{F}_q$ is a finite field. In this case, the Galois action is determined by Frobenius action. We let $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ denote the geometric Frobenius $x \mapsto x^{1/q}$. The eigenvalues of Frob_q acting on $\mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ are *q-Weil integers of weight n*, by which we mean algebraic integers λ such that for every embedding $\alpha\colon \mathbb{Q}(\lambda) \to \mathbb{C}$, we have $|\alpha(\lambda)|^2 = q^n$. We let $\mu_\lambda \in \mathbb{Z}_{\geq 0}$ denote the multiplicity of the eigenvalue λ for the action of Frob_q on $\mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$. In other words, we put

$$(1.3.1) \qquad \det(1 - T\mathrm{Frob}_q \mid \mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = \prod_\lambda (1 - \lambda T)^{\mu_\lambda}.$$

For $e \geq 1$, we let $\mu_{\lambda,e} \in \mathbb{Z}_{\geq 0}$ denote the number of $e \times e$ Jordan blocks with eigenvalue λ in the Jordan normal form of Frob_q acting on $\mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell)$. We have $\mu_\lambda = \sum_{e \geq 1} e\mu_{\lambda,e}$.

Corollary 1.4. *Let X be a proper, equidimensional scheme over $\mathbb{F}_q$. In the above notation, $\mu_{\lambda,e} = \mu_{q^n/\lambda,e}$. Moreover, $\mu_{\sqrt{q^n},e}$ and $\mu_{-\sqrt{q^n},e}$ are even for $n + e$ even. In particular, $\mu_\lambda = \mu_{q^n/\lambda}$ and, for n odd, $\mu_{\sqrt{q^n}}$ and $\mu_{-\sqrt{q^n}}$ are even.*

The last statement of Corollary 1.4 implies that for n odd,

$$\det(\mathrm{Frob}_q \mid \mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = q^{nb_n/2},$$

$$\det(1 - T\mathrm{Frob}_q \mid \mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)) = q^{nb_n/2}T^{b_n} \det(1 - q^{-n}T^{-1}\mathrm{Frob}_q \mid \mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)),$$

where $b_n = \dim \mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$. Moreover, it implies that $\dim \mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = \sum_\lambda \mu_\lambda$ is even.

Remark 1.5. Some special cases of the last statement of Corollary 1.4 were previously known.

(1) Gabber's theorem on independence of ℓ for intersection cohomology [16, Theorem 1] states that (1.3.1) belongs to $\mathbb{Z}[T]$ and is independent of ℓ. The fact that (1.3.1) belongs to $\mathbb{Q}[T]$ implies $\mu_\lambda = \mu_{\lambda'}$ for λ and λ' in the same $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit. In particular, $\mu_\lambda = \mu_{q^n/\lambda}$, and, if q is *not* a square and n is odd, $\mu_{\sqrt{q^n}} = \mu_{-\sqrt{q^n}}$, so that $\dim \mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) = \sum_\lambda \mu_\lambda$ is even in this case.

(2) For X proper smooth, the fact that $\mu_{\sqrt{q^n}}$ and $\mu_{-\sqrt{q^n}}$ are even for n odd follows from a theorem of Suh [45, Theorem 3.3.1].

Remark 1.6. The first two statements of Corollary 1.4 are consistent with the conjectural semisimplicity of the Frobenius action on $\mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ (namely, $\mu_{\lambda,e} = 0$ for $e \geq 2$), which would follow from the standard conjectures. To see this implication, let $X' \to X$ be a surjective generically finite morphism

such that X' is projective smooth over $\mathbb{F}_q$, which exists by de Jong's alterations [8, Theorem 4.1]. Then $\mathrm{IH}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ as a $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-module is a direct summand of $\mathrm{H}^n(X'_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$.[1] The semisimplicity of the Frobenius action on $\mathrm{H}^n(X'_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ would follow from the Lefschetz type standard conjecture for X' and the Hodge type standard conjecture for $X' \times X'$ [27, Theorem 5.6 (2)].

To prove Theorem 1.1, we may assume that k is finitely generated over its prime field. We will keep this assumption in the rest of the introduction. This includes notably the case of a number field. We deduce Theorem 1.1 from a relative result with coefficients. In the case where k is a finite field, the coefficients are pure perverse sheaves. In the general case, we apply the formalism of pure *horizontal* perverse sheaves of Annette Huber [21], as extended by Sophie Morel [34]. Since the proofs are the same in the two cases, we recommend readers not familiar with horizontal perverse sheaves to concentrate on the case of a finite field and to ignore the word "horizontal". Unless otherwise stated, we will only consider the middle perversity. We let $\overline{\mathbb{Q}}_\ell$ denote the algebraic closure of $\mathbb{Q}_\ell$.

Definition 1.7. Let X be a scheme of finite type over k and let $A \in \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ be a horizontal perverse sheaf on X, pure of weight w. We say that A is *orthogonal*, if there exists a symmetric perfect pairing $A \otimes A \to K_X(-w)$. We say that A is *symplectic*, if there exists an alternating perfect pairing $A \otimes A \to K_X(-w)$.

Here $K_X = \mathrm{R}a_X^! \overline{\mathbb{Q}}_\ell$ is the dualizing complex on X, where $a_X \colon X \to \mathrm{Spec}(k)$ is the structural morphism.

Theorem 1.8 (Special case of Theorem 5.1.5)**.** *Let $f \colon X \to Y$ be a proper morphism of schemes of finite type over k and let $A \in \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ be an orthogonal (resp. symplectic) pure horizontal perverse sheaf on X. Then*

$$(1.8.1) \qquad \mathrm{R}f_* A \simeq \bigoplus_n ({}^p\mathrm{R}^n f_* A)[-n]$$

and ${}^p\mathrm{R}^n f_* A$ is orthogonal (resp. symplectic) for n even and symplectic (resp. orthogonal) for n odd.

Recall that the Beilinson-Bernstein-Deligne-Gabber decomposition theorem [3, Théorème 5.4.5] implies that (1.8.1) holds after base change to the algebraic closure $\bar{k}$ of k.

Theorem 1.1 follows from Theorem 1.8 applied to the morphism $a_X \colon X \to \mathrm{Spec}(k)$. Even if one is only interested in Theorem 1.1, our proof leads one to consider the relative situation of Theorem 1.8.

Next we state results for operations that do not necessarily preserve pure (horizontal) complexes. For a scheme X of finite type over k, we let $\mathrm{K}_{\mathrm{orth}}(X, \overline{\mathbb{Q}}_\ell)$ denote the subgroup of the Grothendieck group $\mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$ of $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ generated by orthogonal pure horizontal perverse sheaves of even weights and symplectic pure horizontal perverse sheaves of odd weights.

Theorem 1.9 (Special case of Theorem 5.2.2)**.** *Grothendieck's six operations preserve $\mathrm{K}_{\mathrm{orth}}$.*

Note that the preservation of $\mathrm{K}_{\mathrm{orth}}$ by each of the six operations is nontrivial. The crucial case turns out to be the preservation by the extension by zero functor $j_!$ for certain open immersions j (Proposition 4.3.1).

As Michel Gros points out, one may compare these theorems to Morihiko Saito's theory of mixed Hodge modules [41]. By definition a mixed Hodge module admits a weight filtration for which the graded pieces are *polarizable* pure Hodge modules. One may compare Definition 1.7 to polarizable pure Hodge modules.

Let us state a consequence of Theorem 1.9 on Galois action on cohomology in the case where $k = \mathbb{F}_q$ is a finite field. Let X be a scheme of finite type over $\mathbb{F}_q$. The eigenvalues of Frob_q acting on $\mathrm{H}^*(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ are q-Weil integers[2] of integral weights. We let $m_\lambda \in \mathbb{Z}$ denote the multiplicity of the eigenvalue λ. In other words, we put

$$(1.9.1) \qquad \prod_n \det(1 - T\mathrm{Frob}_q \mid \mathrm{H}^n(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell))^{(-1)^n} = \prod_\lambda (1 - \lambda T)^{m_\lambda}.$$

Applying the theorem to $(a_X)_*$, where $a_X \colon X \to \mathrm{Spec}(\mathbb{F}_q)$, we obtain the following.

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[1] This argument is also used in Gabber's proof of the integrality of (1.3.1).

[2] The integrality is a special case of [51, Variante 5.1].

Corollary 1.10. *Let X be a scheme of finite type over $\mathbb{F}_q$. In the above notation, $m_\lambda = m_{q^w/\lambda}$ for every q-Weil integer λ of weight w, and for w odd, $m_{\sqrt{q^w}}$ and $m_{-\sqrt{q^w}}$ are even. In particular, for w odd, $\sum_\lambda m_\lambda$ (where λ runs through q-Weil integers of weight w), namely the dimension of the weight w part of $\mathrm{H}^*(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$, is even.*

Theorem 1.9 also implies analogues of Corollary 1.10 for compactly supported cohomology $\mathrm{H}^*_c(X, \mathbb{Q}_\ell)$, and, if X is equidimensional, intersection cohomology $\mathrm{IH}^*(X, \mathbb{Q}_\ell)$ and compactly supported intersection cohomology $\mathrm{IH}^*_c(X, \mathbb{Q}_\ell)$. In the case of $\mathrm{H}^*_c(X, \mathbb{Q}_\ell)$, the analogue of (1.9.1) is the inverse of the zeta function and the analogue of Corollary 1.10 was established by Suh [45, Theorem 3.3.1] using rigid cohomology.

Remark 1.11. Some special cases of Corollary 1.10 were previously known. By Gabber's theorem on independence of ℓ [16, Theorem 2], (1.9.1) belongs to $\mathbb{Q}(T)$ and is independent of ℓ. The fact that (1.9.1) belongs to $\mathbb{Q}(T)$ implies that $m_\lambda = m_{\lambda'}$ for λ and λ' in the same $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbit. In particular, $m_\lambda = m_{q^w/\lambda}$ for every q-Weil integer λ of weight w, and, if q is *not* a square and w is odd, $m_{\sqrt{q^w}} = m_{-\sqrt{q^w}}$ so that $\sum_\lambda m_\lambda$ (where λ runs through q-Weil integers of weight w) is even in this case.

One ingredient in the proof of Theorem 1.9 is de Jong's alterations. Note that even for a finite étale cover $f\colon X \to Y$, one cannot recover the parity of an object on Y from the parity of its pullback to X. More precisely, for an element A in the Grothendieck group of mixed horizontal perverse sheaves on Y such that $f^*A \in \mathrm{K}_{\mathrm{orth}}$, we do not have $A \in \mathrm{K}_{\mathrm{orth}}$ in general. We use equivariant alterations to compensate for this loss of information. In order to better deal with the equivariant situation, we will work systematically with Deligne-Mumford stacks in the main text. We note however that the proofs of Theorems 1.1 and 1.8 (and the corollaries to Theorem 1.1) do not depend on stacks, and readers only interested in these results may, in the corresponding portions of the text (Sections 2, 3 and Subsection 5.1), assume every stack to be a scheme.

The paper is organized as follows. In Section 2, we study symmetry of complexes and perverse sheaves over a general field. In Section 3, we study symmetry and decomposition of pure complexes over a finite field and prove Theorem 1.8 in this case. In Section 4, we study symmetry in Grothendieck groups over a finite field and prove Theorem 1.9 in this case. In Section 5, we study symmetry of horizontal complexes over a general field finitely generated over its prime field and finish the proof of the theorems. In Appendix A, we collect some general symmetry properties in categories with additional structures, which are used in the main body of the paper.

Conventions

Unless otherwise indicated, we let X, Y, etc. denote Deligne-Mumford stacks *of finite presentation* (i.e. of finite type and quasi-separated) over a base field k; this rules out stacks such as $B\mathbb{Z}$. We recall that for schemes, being of finite presentation over k is the same as being of finite type over k.

We let ℓ denote a prime number invertible in k, and let $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ denote the derived category of $\overline{\mathbb{Q}}_\ell$-complexes on X. We refer the reader to [53] for the construction of $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ and of Grothendieck's six operations. We denote by $a_X\colon X \to \mathrm{Spec}(k)$ the structural morphism, by $K_X \coloneqq \mathrm{R}a_X^! \overline{\mathbb{Q}}_\ell$ the dualizing complex on X, and by D_X the dualizing functor $D_{K_X} \coloneqq \mathrm{R}\mathscr{H}om(-, K_X)$.

As mentioned above, we will only consider the middle perversity unless otherwise stated. We let $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ denote the full subcategory of perverse $\overline{\mathbb{Q}}_\ell$-sheaves on X. For a separated quasi-finite morphism $f\colon X \to Y$, the middle extension functor $f_{!*}\colon \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(Y, \overline{\mathbb{Q}}_\ell)$ is the image of the support-forgetting morphism ${}^p\mathscr{H}^0 f_! \to {}^p\mathscr{H}^0 \mathrm{R}f_*$.

Throughout the article we let σ and σ' represent elements of $\{\pm 1\}$.

Acknowledgments

2 Symmetry of complexes and perverse sheaves

In this section, we study symmetry properties of $\overline{\mathbb{Q}}_\ell$-complexes, namely objects of $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$, over an arbitrary field k. In Subsection 2.1, we define σ-self-dual complexes and we study their behavior under operations that commute with duality. In Subsection 2.2, we analyze σ-self-dual semisimple perverse sheaves. In this generality none of the results is difficult, but they will be used quite often in the sequel.

2.1 Symmetry of complexes

Tensor product endows $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ with the structure of a closed symmetric monoidal structure. The definition below only makes use of the symmetry constraint $c_{AB}\colon A \otimes B \xrightarrow{\sim} B \otimes A$ and the internal mapping object $\mathrm{R}\mathscr{H}om$.

Definition 2.1.1 (σ-self-dual complexes)**.** Let $A, C \in \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$. We say that A is *1-self-dual with respect to C* (resp. *-1-self-dual with respect to C*) if there exists a pairing $A \otimes A \to C$ that is
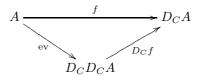
- *symmetric* (resp. *alternating*), in the sense that the following diagram commutes (resp. anticommutes)

$$(2.1.1) \qquad \begin{array}{ccc} A \otimes A & \xrightarrow{\;\;c_{AA}\;\;} & A \otimes A \\ & \searrow \qquad \swarrow & \\ & C, & \end{array}$$

- and *perfect*, in the sense that the pairing induces an isomorphism $A \xrightarrow{\sim} D_C A \coloneqq \mathrm{R}\mathscr{H}om(A, C)$.

We say that A is *self-dual with respect to C* if there exists an isomorphism $A \xrightarrow{\sim} D_C A$.

The symmetry of the pairing $A \otimes A \to C$ can also be expressed in terms of the induced morphism $f\colon A \to D_C A$. In fact, the diagram (2.1.1) σ-commutes if and only if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & D_C A \\ {\scriptstyle \mathrm{ev}} \searrow & & \nearrow {\scriptstyle D_C f} \\ & D_C D_C A & \end{array}$$

σ-commutes (Lemma A.4.2).

Remark 2.1.2. Similarly one can define self-dual and σ-self-dual E_λ-complexes, where E_λ is any algebraic extension of $\mathbb{Q}_\ell$. Note that for $A, C \in \mathrm{D}^b_c(X, E_\lambda)$, A is self-dual (resp. σ-self-dual) with respect to C if and only if $A \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$ satisfies the same property with respect to $C \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$. Indeed, the "only if" part is obvious. To see the "if" part, consider $U = \mathbf{Isom}(A_{\bar{k}}, D_C A_{\bar{k}})$ as in Lemma 2.1.3 below. Here $A_{\bar{k}}$ denotes the pullback of A to $X_{\bar{k}}$. Recall that rational points form a Zariski dense subset of any affine space over an infinite field. If $A \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$ is self-dual (resp. σ-self-dual) with respect to $C \otimes_{E_\lambda} \overline{\mathbb{Q}}_\ell$, then $U \cap V$ is nonempty, hence has a E_λ-point. Here $V \subseteq \mathbf{Hom}(A_{\bar{k}}, D_C A_{\bar{k}})$ is represented by the E_λ-vector subspace, image of morphisms (resp. σ-symmetric morphisms) $A \to D_C A$. For the above reason, we will work almost exclusively with $\overline{\mathbb{Q}}_\ell$-complexes.

Lemma 2.1.3. *Let $A, B \in \mathrm{D}^b_c(X, E_\lambda)$ such that $\dim_{E_\lambda} \mathrm{Hom}(A, B) < \infty$. Then there exists a Zariski open subscheme $U = \mathbf{Isom}(A, B)$ of the affine space $\mathbf{Hom}(A, B)$ over E_λ represented by the E_λ-vector space $\mathrm{Hom}(A, B)$ such that for any algebraic extension E'_λ of E_λ, $U(E'_\lambda)$ corresponds to the set of isomorphisms $A \otimes_{E_\lambda} E'_\lambda \xrightarrow{\sim} B \otimes_{E_\lambda} E'_\lambda$.*

Proof. Assume $A, B \in \mathrm{D}^{[a,b]}$. Choose a stratification of X by connected geometrically unibranch substacks such that the restrictions of $\mathscr{H}^n A$ and $\mathscr{H}^n B$ to each stratum are lisse sheaves. Choose a geometric point x in each stratum. Then a morphism $f\colon A \otimes_{E_\lambda} E'_\lambda \to B \otimes_{E_\lambda} E'_\lambda$ is an isomorphism if and only if $\mathscr{H}^n f_x$ is for every $n \in [a, b]$ and every x in the finite collection. Then $\mathbf{Isom}(A, B)$ is the intersection of the pullbacks of the open subsets $\mathbf{Isom}(\mathscr{H}^n A_x, \mathscr{H}^n B_x) \subseteq \mathbf{Hom}(\mathscr{H}^n A_x, \mathscr{H}^n B_x)$. $\qquad \square$

We will mostly be interested only in duality with respect to Tate twists $K_X(-w)$, $w \in \mathbb{Z}$ of the dualizing complex $K_X = \mathrm{R}a_X^! \overline{\mathbb{Q}}_\ell$. In this case, the evaluation morphism $A \to D_{K_X(-w)} D_{K_X(-w)} A$ is an isomorphism. The functor $D_{K_X(-w)}$ preserves perverse sheaves. We will sometimes write K for K_X when no confusion arises.

In the rest of this subsection, we study the behavior of σ-self-dual complexes under operations that commute with the dualizing functors (up to shift and twist). The results are mostly formal, but for completeness we provide a proof for each result, based on general facts on symmetry in categories collected in Appendix A. Readers willing to accept these results may skip the proofs.

Most of the proofs consist of showing that the natural isomorphism representing the commutation of the functor in question with duality is symmetric in the sense of Definition A.3.3. It then follows from Lemma A.3.9 that the functor in question preserves σ-self-dual objects.

Remark 2.1.4 (Preservation of σ-self-dual complexes). Let $f \colon X \to Y$ be a morphism. Let $w, w' \in \mathbb{Z}$.

(1) *For $n \in \mathbb{Z}$, Tate twist $A \mapsto A(n)$ carries σ-self-dual $\overline{\mathbb{Q}}_\ell$-complexes with respect to C to σ-self-dual $\overline{\mathbb{Q}}_\ell$-complexes with respect to $C(2n)$; the shift functor $A \mapsto A[n]$ carries $\overline{\mathbb{Q}}_\ell$-complexes σ-self-dual with respect to C to $(-1)^n \sigma$-self-dual $\overline{\mathbb{Q}}_\ell$-complexes with respect to $C[2n]$.*

This follows from Lemma A.5.6.

(2) *D_X carries σ-self-dual $\overline{\mathbb{Q}}_\ell$-complexes with respect to $K_X(-w)$ to σ-self-dual $\overline{\mathbb{Q}}_\ell$-complexes with respect to $K_X(w)$.*

Since $D_X A \simeq (D_{K_X(-w)} A)(w)$, the assertion follows from (1).

(3) *Assume that f is proper. Then $\mathrm{R}f_* \colon \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_c^b(Y, \overline{\mathbb{Q}}_\ell)$ preserves σ-self-dual objects with respect to $K(-w)$. In other words, $\mathrm{R}f_*$ carries $\overline{\mathbb{Q}}_\ell$-complexes σ-self-dual with respect to $K_X(-w)$ to $\overline{\mathbb{Q}}_\ell$-complexes σ-self-dual with respect to $K_Y(-w)$.*

Since $\mathrm{R}f_*$ is a right-lax symmetric functor (Definition A.1.5), the morphism $\mathrm{R}f_* D_X \to D_{\mathrm{R}f_* K_X} \mathrm{R}f_*$ is symmetric by Construction A.4.6. Composing with the adjunction map $\mathrm{R}f_* K_X \simeq \mathrm{R}f_! K_X \to K_Y$, we obtain a symmetric isomorphism $\mathrm{R}f_* D_X \xrightarrow{\sim} D_Y \mathrm{R}f_*$.

(4) *Assume that f is a closed immersion and let $A \in \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$. Then A is σ-self-dual with respect to $K_X(-w)$ if and only if $f_* A$ is σ-self-dual with respect to $K_Y(-w)$.*

Since the functor f_* is fully faithful in this case, the assertion follows from the proof of (3) above and Lemma A.3.9.

(5) *Assume that f is an open immersion and let $A \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ be a perverse sheaf. Then A is σ-self-dual with respect to $K_X(-w)$ if and only if $f_{!*} A$ is σ-self-dual with respect to $K_Y(-w)$.*

Since $f_!$ is a symmetric functor, the morphism $f_! D_X \to D_{f_! K_X} f_!$ is symmetric by Construction A.4.6. It follows that the composite map in the commutative square

$$
\begin{array}{ccc}
f_! D_X & \xrightarrow{\;\alpha\;} & \mathrm{R}f_* D_X \\
\downarrow & & \simeq \downarrow \beta \\
D_{f_! K_X} f_! & \longrightarrow & D_Y f_!
\end{array}
$$

is symmetric. Here the lower horizontal map is given by the adjunction map $f_! K_X \to K_Y$. Moreover, we have a commutative square

$$
\begin{array}{ccc}
f_{!*} D_X & \xrightarrow{\;\gamma\;}_{\sim} & D_Y f_{!*} \\
\downarrow & & \downarrow \\
\mathrm{R}\,{}^p\mathrm{R}^0 f_* D_X & \xrightarrow[\sim]{\;{}^p\mathscr{H}^0 \beta\;} & D_Y\, {}^p\mathrm{R}^0 f_!
\end{array}
$$

By Lemma A.3.10, γ is symmetric. Since the functor $f_{!*}$ is fully faithful by Lemma 2.1.5 below, it then suffices to apply Lemma A.3.9. The "if" part also follows from (8) below.

(6) *Let $A \in \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ be σ-self-dual with respect to $K_X(-w)$, and let $B \in \mathrm{D}_c^b(X', \overline{\mathbb{Q}}_\ell)$ be σ'-self-dual with respect to $K_{X'}(-w')$. Then the exterior tensor product $A \boxtimes B \in \mathrm{D}_c^b(X \times X', \overline{\mathbb{Q}}_\ell)$ is $\sigma\sigma'$-self-dual with respect to $K_{X \times X'}(-w - w')$.*

Since $-\boxtimes-$ is a symmetric monoidal functor, the Künneth isomorphism (cf. [20, (1.7.6), Proposition 2.3])

$$D_X(-) \boxtimes D_{X'}(-) \xrightarrow{\sim} D_{K_X \boxtimes K_{X'}}(- \boxtimes -) \simeq D_{X \times X'}(- \boxtimes -)$$

is symmetric by Construction A.4.6.

(7) *If f is smooth, purely of dimension d, then $f^*[d]$ carries $\overline{\mathbb{Q}}_\ell$-complexes σ-self-dual with respect to $K_Y(-w)$ to $\overline{\mathbb{Q}}_\ell$-complexes $(-1)^d \sigma$-self-dual with respect to $K_X(-d-w)$.*

Since f^* is a symmetric monoidal functor, the isomorphism

$$f^* T^d D_Y \xrightarrow{\sim} D_{f^* K_Y[2d]} f^* T^d \simeq D_{K_X(-d)} f^* T^d$$

is $(-1)^d$-symmetric by Construction A.4.6 and Remark A.5.10.

(8) *If X and Y are regular, purely of dimension d and e respectively, then $f^*[d-e] \colon \mathrm{D}^b_{\mathrm{lisse}}(Y, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}^b_{\mathrm{lisse}}(X, \overline{\mathbb{Q}}_\ell)$ carries $\overline{\mathbb{Q}}_\ell$-complexes σ-self-dual with respect to $K_Y(-w)$ to $\overline{\mathbb{Q}}_\ell$-complexes $(-1)^{d-e} \sigma$-self-dual with respect to $K_X(-(d-e)-w)$. Here $\mathrm{D}^b_{\mathrm{lisse}}$ denotes the full subcategory of D^b_c consisting of complexes with lisse cohomology sheaves.*

Let $r = d - e$. As in (7), the natural transformation

$$f^* T^r D_Y \to D_{f^* K_Y[2r]} f^* T^r \simeq D_{K_X(-r)} f^* T^r$$

is $(-1)^r$-symmetric. For $A \in \mathrm{D}^b_c(Y, \overline{\mathbb{Q}}_\ell)$, this natural transformation can be computed as

$$f^* T^r D_Y A \simeq f^* T^r D_{K_Y(-r)} A \otimes \mathrm{R}f^! \overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \mathrm{R}f^! D_{K_Y(-r)} T^r A \simeq D_{K_X(-r)} f^* T^r A.$$

For $A \in \mathrm{D}^b_{\mathrm{lisse}}$, α is an isomorphism.

Similar results hold for self-dual $\overline{\mathbb{Q}}_\ell$-complexes.

Lemma 2.1.5. *Let $j \colon U \to X$ be an immersion. Then the functor $j_{!*} \colon \mathrm{Perv}(U, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ is fully faithful.*

Proof. Let A and B be perverse $\overline{\mathbb{Q}}_\ell$-sheaves on U, and let $\alpha \colon \mathrm{Hom}(A, B) \to \mathrm{Hom}(j_{!*}A, j_{!*}B)$ be the map induced by $j_{!*}$. As the composite map

$$\mathrm{Hom}(A, B) \xrightarrow{\alpha} \mathrm{Hom}(j_{!*}A, j_{!*}B) \xrightarrow{\beta} \mathrm{Hom}(^p\mathscr{H}^0 j_! A, \, ^p\mathscr{H}^0 \mathrm{R}j_* B) \simeq \mathrm{Hom}(j_! A, \mathrm{R}j_* B)$$

is an isomorphism and β is an injection, α is an isomorphism. $\qquad\square$

Example 2.1.6. Assume that X is equidimensional. We define the *intersection complex* of X by $\mathrm{IC}_X = j_{!*}(\mathbb{Q}_\ell[d])$, where $j \colon U \to X$ is a dominant open immersion such that U_{red} is regular and $d = \dim X$. Then by Parts (5) and (7) of Remark 2.1.4, IC_X is $(-1)^d$-self-dual with respect to $K_X(-d)$.

Although we do not need it, let us mention the following stability of σ-self-dual complexes under nearby cycles.

Remark 2.1.7. *Let S be the spectrum of a Henselian discrete valuation ring, of generic point η and closed point s, on which ℓ is invertible. Let $\mathfrak{X}$ be a Deligne-Mumford stack of finite presentation over S. Then the nearby cycle functor $\mathrm{R}\Psi \colon \mathrm{D}^b_c(\mathfrak{X}_\eta, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}^b_c(\mathfrak{X}_s \times_s \eta, \overline{\mathbb{Q}}_\ell)$ preserves σ-self-dual objects with respect to $K(-w)$.*

Indeed, $\mathrm{R}\Psi$ is a right-lax symmetric monoidal functor. Hence, by Construction A.4.6, the composite $\mathrm{R}\Psi D_{\mathfrak{X}_\eta} \to D_{\mathrm{R}\Psi K_{\mathfrak{X}_\eta}} \mathrm{R}\Psi \to D_{\mathfrak{X}_s} \mathrm{R}\Psi$, which is a natural isomorphism (cf. [22, Théorème 4.2]), is symmetric.

Remark 2.1.4 (6) can be applied to the exterior tensor power functor $(-)^{\boxtimes m} \colon \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}^b_c(X^m, \overline{\mathbb{Q}}_\ell)$, $m \geq 0$. We now discuss a refinement

$$(2.1.2) \qquad \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}^b_c([X^m/\mathfrak{S}_m], \overline{\mathbb{Q}}_\ell), \quad A \mapsto A^{\boxtimes m}$$

given by permutation. Readers not interested in this refinement may skip this part as it will not be used in the proofs of the results mentioned in the introduction.

We briefly recall one way to define the symmetric product stack $[X^m/\mathfrak{S}_m]$. For every k-scheme S, $[X^m/\mathfrak{S}_m](S)$ is the groupoid of pairs (T, x), where T is a finite étale cover of S of degree m and x is an object of $X(T)$, with isomorphisms of pairs defined in the obvious way.

Remark 2.1.8. *The functor* (2.1.2) *carries σ-self-dual complexes with respect to $K_X(-w)$ to σ^m-self-dual complexes with respect to $K_{[X^m/\mathfrak{S}_m]}(-mw)$.*

Indeed, $(-)^{\boxtimes m}$ is a symmetric monoidal functor. Hence, the isomorphism

$$(D_X(-))^{\boxtimes m} \xrightarrow{\sim} D_{K_X^{\boxtimes m}}((-)^{\boxtimes m}) \simeq D_{[X^m/\mathfrak{S}_m]}((-)^{\boxtimes m})$$

is symmetric by Construction A.4.6.

2.2 Symmetry of perverse sheaves

In this subsection, we study σ-self-dual perverse sheaves. We first prove a two-out-of-three property, which will play an important role in later sections. We then discuss a trichotomy for indecomposable perverse $\overline{\mathbb{Q}}_\ell$-sheaves. From this we deduce a criterion for semisimple perverse $\overline{\mathbb{Q}}_\ell$-sheaves to be σ-self-dual in terms of multiplicities of simple factors.

Proposition 2.2.1 (Two-out-of-three). *Let A be a perverse $\overline{\mathbb{Q}}_\ell$-sheaf such that $A \simeq A' \oplus A''$. If A and A' are σ-self-dual with respect to $K_X(-w)$, then so is A''.*

Proof. We write D for $D_{K_X(-w)}$. Let $f \colon A \xrightarrow{\sim} DA$ and $g' \colon A' \xrightarrow{\sim} DA'$ be σ-symmetric isomorphisms. We let $f' \colon A' \to DA'$ denote the restriction of f to A', namely the composite

$$A' \xrightarrow{i} A \xrightarrow[\sim]{f} DA \xrightarrow{Di} DA',$$

where $i \colon A' \to A$ is the inclusion. Let $g \colon A \to DA$ be the direct sum of g' with the zero map $A'' \to DA''$. These are σ-symmetric morphisms. Consider linear combinations $h_{a,b} = af + bg$ and $h'_{a,b} = af' + bg'$, where $a, b \in \overline{\mathbb{Q}}_\ell$. By Lemma 2.1.3, there are only finitely many values of $(a : b)$ for which $h_{a,b}$ is not an isomorphism. The same holds for $h'_{a,b}$. Therefore, there exist $a, b \in \overline{\mathbb{Q}}_\ell$ such that $h_{a,b}$ and $h'_{a,b}$ are isomorphisms. Consider the orthogonal complement of A' in A with respect to $h_{a,b}$:

$$B = \mathrm{Ker}(A \xrightarrow[\sim]{h_{a,b}} DA \xrightarrow{Di} DA').$$

Then $h_{a,b}$ induces a σ-symmetric isomorphism $B \xrightarrow{\sim} DB$. Moreover, $A \simeq A' \oplus B$, so that $B \simeq A''$. Here we used the Krull-Schmidt theorem [1, Theorem 1] and the fact that perverse sheaves have finite lengths. $\qquad\square$

Remark 2.2.2. The two-out-of-three property also holds more trivially for self-dual complexes. In fact, if we have decompositions of perverse $\overline{\mathbb{Q}}_\ell$-sheaves $A \simeq A' \oplus A''$ and $B \simeq B' \oplus B''$ such that $A \simeq D_X B(-w)$ and $A' \simeq D_X B'(-w)$, then we have $A'' \simeq D_X B''(-w)$ by the Krull-Schmidt theorem.

Proposition 2.2.3 (Trichotomy). *Let A be an indecomposable perverse $\overline{\mathbb{Q}}_\ell$-sheaf on X. Then exactly one of the following occurs:*

- *A is 1-self-dual with respect to $K_X(-w)$;*

- *A is -1-self-dual with respect to $K_X(-w)$;*

- *A is not self-dual with respect to $K_X(-w)$.*

This follows from general facts (Lemma A.2.7 and Remark A.2.8) applied to the category of perverse $\overline{\mathbb{Q}}_\ell$-sheaves.

Remark 2.2.4. In the case of a simple perverse $\overline{\mathbb{Q}}_\ell$-sheaf, the proof can be somewhat simplified with the help of Schur's lemma. This case is analogous to a standard result on complex representations of finite or compact groups ([44, Section 13.2, Proposition 38], [4, Proposition II.6.5]).

Remark 2.2.5. An indecomposable perverse E_λ-sheaf on X, self-dual with respect to $K_X(-w)$, is either 1-self-dual or -1-self-dual, by Lemma A.2.7 and Remark A.2.8. Note that a simple perverse E_λ-sheaf can be 1-self-dual and -1-self-dual with respect to $K_X(-w)$ at the same time.

Corollary 2.2.6. *Let $A \simeq \bigoplus_B B^{n_B}$ be a semisimple perverse $\overline{\mathbb{Q}}_\ell$-sheaf on X, where B runs through isomorphism classes of simple perverse $\overline{\mathbb{Q}}_\ell$-sheaves on X. Then A is σ-self-dual with respect to $K_X(-w)$ if and only if the following conditions hold:*

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(1) $n_B = n_{(D_X B)(-w)}$ *for B not self-dual with respect to $K_X(-w)$;*

(2) n_B *is even for B that are $-\sigma$-self-dual with respect to $K_X(-w)$.*

Moreover, A is self-dual with respect to $K_X(-w)$ if and only if (1) holds.

In particular, if B and B' are respectively 1-self-dual and -1-self-dual simple perverse sheaves on X, then $B \oplus B'$ is self-dual but neither 1-self-dual nor -1-self-dual.

Proof. It is clear that (1) is equivalent to the condition that A is self-dual. If (1) and (2) hold, then A is σ-self-dual by Proposition 2.2.3 and the fact that $B \oplus (D_X B)(-w)$ is σ-self-dual with respect to $K_X(-w)$ for all B (Remark A.2.6 (2)). It remains to show that if A is σ-self-dual, then (2) holds. Let B be $-\sigma$-self-dual. As $B^{\oplus n_B} \simeq B \otimes V$ is σ-self-dual, where $V = \overline{\mathbb{Q}}_\ell^{\oplus n_B}$, the isomorphism

$$\mathrm{Hom}(B \otimes V, D_X(B \otimes V)(-w)) \simeq \mathrm{Hom}(B, (D_X B)(-w)) \otimes \mathrm{Hom}(V, V^*) \simeq \mathrm{Hom}(V, V^*)$$

provides a skew-symmetric $n_B \times n_B$ matrix with entries in $\overline{\mathbb{Q}}_\ell$, which implies that n_B is even. More formally, we can apply the second part of Lemma A.3.9 to the fully faithful functor $F \colon V \mapsto B \otimes V$ from the category of finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces to $\mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$. The natural isomorphism $FD \xrightarrow{\sim} D_{K_X(-w)}F$ is $-\sigma$-self-dual. $\qquad\square$

Remark 2.2.7. The semisimplification of a σ-self-dual perverse sheaf is σ-self-dual by Lemma A.2.9. The converse does not hold. See Example 3.1.5 below.

We will need to consider more generally geometrically semisimple perverse sheaves, namely perverse sheaves whose pullbacks to $X_{\bar{k}}$ are semisimple. Part (1) of the following lemma extends [3, Corollaire 5.3.11] for pure perverse sheaves.

Lemma 2.2.8. *Let A be a geometrically semisimple perverse $\overline{\mathbb{Q}}_\ell$-sheaf on X.*

(1) Let $i \colon Y \to X$ be a closed immersion with complementary open immersion $j \colon U \to X$. Then A admits a unique decomposition $A \simeq j_{!*}j^*A \oplus i_*B$, where B is a perverse sheaf on Y. Moreover, we have $B \simeq {}^p\mathscr{H}^0 i^*A \simeq {}^p\mathrm{R}^0 i^! A$.

(2) A admits a unique decomposition $A \simeq \bigoplus_V A_V$, where V runs through irreducible closed substacks of X, and the support of each indecomposable direct summand of the perverse sheaf A_V is V.

Assume moreover that A is indecomposable. Then, by Part (1) of the lemma, we have $j_{!*}j^*A \simeq A$ if U intersects with the support of A (and $j^*A = 0$ otherwise). Moreover, the support of A is irreducible, and A is isomorphic to $f_{!*}(\mathscr{F}[d])$ for some immersion $f \colon W \to X$ with W regular irreducible of dimension d, and some lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}$ on V.

Proof. (1) The proof is identical to that of [3, Corollaire 5.3.11]. The uniqueness of the decomposition is clear. For the existence, it suffices to check that

- the adjunction map ${}^p\mathscr{H}^0 j_! j^*A \to A$ factorizes through the quotient $j_{!*}j^*A$ of ${}^p\mathscr{H}^0 j_! j^*A$, and the adjunction map $A \to {}^p\mathrm{R}^0 j_* j^*A$ factorizes through the sub-object $j_{!*}j^*A$ of ${}^p\mathrm{R}^0 j_* j^*A$;

- the composite of the adjunction maps $i_* {}^p\mathrm{R}^0 i^! A \to A \to i_* {}^p\mathscr{H}^0 i^*A$ is an isomorphism;

and these maps provide a decomposition of A. These statements can be easily checked over $\bar{k}$.

(2) Again the uniqueness is clear. The existence follows from the fact that the support of every indecomposable direct summand of A is irreducible. $\qquad\square$

Remark 2.2.9. In Lemma 2.2.8, A is σ-self-dual with respect to $K_X(-w)$ if and only if each direct summand A_V in the support decomposition is σ-self-dual with respect to $K_X(-w)$.

As an application, we show that for geometrically semisimple perverse sheaves the property of being σ-self-dual is local for the Zariski topology. This Zariski local nature will be useful in Section 4.

Proposition 2.2.10. *Let $(X_\alpha)_{\alpha \in I}$ be a Zariski open covering of X, and let A and B be geometrically semisimple perverse $\overline{\mathbb{Q}}_\ell$-sheaves on X. Then $A \simeq (D_X B)(-w)$ if and only if $A \,|\, X_\alpha \simeq (D_{X_\alpha} B \,|\, X_\alpha)(-w)$ for every $\alpha \in I$. Moreover, A is σ-self-dual with respect to $K_X(-w)$ if and only if $A \,|\, X_\alpha$ is so with respect to $K_{X_\alpha}(-w)$ for every $\alpha \in I$.*

Proof. We prove the second assertion, the proof of the first assertion being simpler. It suffices to show the "if" part. Let $j_\alpha \colon X_\alpha \to X$. By Parts (5) and (8) of Remark 2.1.4, $j_{\alpha!*}j_\alpha^* A$ is σ-self-dual. Since $j_{\alpha!*}j_\alpha^* A \simeq \bigoplus_V A_V$, where V satisfies $V \cap X_\alpha \neq \emptyset$, we conclude that each A_V is σ-self-dual.

Alternatively we may apply Lemma 2.2.11 below. Indeed, by quasi-compactness, we may assume that I is finite. For $J \subseteq I$ nonempty, $j_{J!*}j_J^* A$ is σ-self-dual. Thus the same holds for $A \simeq j_{\emptyset!*}j_\emptyset^* A$ by the two-out-of-three property. $\square$

Lemma 2.2.11. *Let $(X_\alpha)_{\alpha \in I}$ be a finite Zariski open covering of X, and let A be a geometrically semisimple perverse $\overline{\mathbb{Q}}_\ell$-sheaf on X. Then*

$$\bigoplus_{\substack{J \subseteq I \\ \#J \text{ even}}} j_{J!*}j_J^* A \simeq \bigoplus_{\substack{J \subseteq I \\ \#J \text{ odd}}} j_{J!*}j_J^* A,$$

where $j_J \colon \bigcap_{\alpha \in J} X_\alpha \to X$ is the open immersion.

Proof. We may assume that A is indecomposable. Then both sides are direct sums of copies of A and the multiplicities are equal:

$$\sum_{\substack{0 \leq i \leq m \\ i \text{ even}}} \binom{m}{i} = \sum_{\substack{0 \leq i \leq m \\ i \text{ odd}}} \binom{m}{i}.$$

Here $m \geq 1$ is the number of indices $\alpha \in I$ such that the support of A intersects with X_α. $\square$

3 Symmetry and decomposition of pure complexes

In this section, we study symmetry of pure perverse sheaves and more generally of pure complexes that decompose into shifts of perverse sheaves. We first work over a finite field. In Subsection 3.1, we analyze σ-self-dual pure perverse sheaves and give a criterion in terms of multiplicities of factors. In Subsection 3.2, we study the behavior of such perverse sheaves under operations that preserve purity. The main result of this section is the preservation of certain class of complexes under derived proper direct image (Theorem 3.2.3), which implies the finite field case of Theorem 1.8. Such preservation results constitute the starting point of the analysis in Section 4 of the effects of more general operations in the mixed case. In Subsection 3.3, we work over a separably closed base field and we prove preservation results for certain semisimple complexes, by reducing to the finite field case.

3.1 Symmetry of pure perverse sheaves over a finite field

In this subsection and the next we work over a finite field $k = \mathbb{F}_q$. Recall that X, Y, etc. denote Deligne-Mumford stacks of finite presentation over $\mathbb{F}_q$. Let $\iota \colon \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ be an embedding.

In this subsection, we study σ-self-dual ι-pure perverse sheaves. We give a criterion for ι-pure perverse sheaves to be σ-self-dual in terms of multiplicities of factors.

For $n \geq 1$, let E_n be the sheaf on $\mathrm{Spec}(\mathbb{F}_q)$ of stalk $(\overline{\mathbb{Q}}_\ell)^n = \bigoplus_{i=1}^n \overline{\mathbb{Q}}_\ell e_i$ on which Frobenius $F = \mathrm{Frob}_q$ acts unipotently with one Jordan block: $Fe_1 = e_1$ and $Fe_i = e_i + e_{i-1}$ for $i > 1$. Recall that any indecomposable ι-pure perverse sheaf A on X is isomorphic to a perverse sheaf of the form $B \otimes a_X^* E_n$, where B is a simple perverse sheaf on X, $n \geq 1$, and $a_X \colon X \to \mathrm{Spec}(\mathbb{F}_q)$.

Proposition 3.1.1. *Let A be a perverse $\overline{\mathbb{Q}}_\ell$-sheaf on X, isomorphic to $\bigoplus_B (B \otimes a_X^* E_n)^{m_{B,n}}$, where B runs over simple perverse $\overline{\mathbb{Q}}_\ell$-sheaves on X, and $w \in \mathbb{Z}$. Then A is σ-self-dual with respect to $K_X(-w)$ if and only if the following conditions hold:*

(1) $m_{B,n} = m_{(D_X B)(-w),n}$ for B not self-dual with respect to $K_X(-w)$;

(2) $m_{B,n}$ is even for B σ-self-dual with respect to $K_X(-w)$ and n even;

(3) $m_{B,n}$ is even for B $-\sigma$-self-dual with respect to $K_X(-w)$ and n odd.

Moreover, A is self-dual with respect to $K_X(-w)$ if and only if (1) holds.

Proof. The equivalence between (1) and the condition that A is self-dual follows from the isomorphism $D_X(B \otimes a_X^* E_n) \simeq D_X B \otimes a_X^* E_n$. For the "if" part of the σ-self-dual case, note that $C \oplus D_X C(-w)$ is σ-self dual with respect to $K_X(-w)$, so that by the trichotomy in Proposition 2.2.3, it suffices to show

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that $B \otimes a_X^* E_n$ is σ'-self-dual (resp. $-\sigma'$-self-dual) for B σ'-self-dual and n odd (resp. even). For the "only if" part, we reduce to the case where $m_{B,n} = 0$ for all B except for one σ'-self-dual B. For both parts, consider the functor $F = B \otimes a_X^* -\colon \mathrm{Perv}(\mathrm{Spec}(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$, which is fully faithful by Lemma 3.1.2 below. The natural isomorphism $F D_{\mathrm{Spec}(\mathbb{F}_q)} \simeq D_{K_X(-w)} F$ is σ'-symmetric. By Lemma A.3.8, we are then reduced to the case where $X = \mathrm{Spec}(\mathbb{F}_q)$ and $B = (\overline{\mathbb{Q}}_\ell)_X$, which follows from Lemma 3.1.3 below. $\qquad\square$

Lemma 3.1.2. *Let B be a simple perverse $\overline{\mathbb{Q}}_\ell$-sheaf on X. Then the functor $B \otimes a_X^* -$ is fully faithful. In other words, for $\overline{\mathbb{Q}}_\ell$-sheaves E and E' on $\mathrm{Spec}(\mathbb{F}_q)$, the map $\alpha\colon \mathrm{Hom}(E, E') \to \mathrm{Hom}(B \otimes a_X^* E, B \otimes a_X^* E')$ is an isomorphism.*

Proof. We have $B \simeq j_{!*}(\mathscr{F}[d])$, where $j\colon U \to X$ is an immersion, with U_{red} connected regular purely of dimension d, and where $\mathscr{F}$ is a simple lisse $\overline{\mathbb{Q}}_\ell$-sheaf on U. We have $B \otimes a_X^* E \simeq j_{!*}((\mathscr{F} \otimes a_U^* E)[d])$. The map α is the composite

$$\mathrm{Hom}(E, E') \xrightarrow{\beta} \mathrm{Hom}(\mathscr{F} \otimes a_U^* E, \mathscr{F} \otimes a_U^* E') \xrightarrow{\gamma} \mathrm{Hom}(B \otimes a_X^* E, B \otimes a_X^* E'),$$

where γ is an isomorphism by Lemma 2.1.5. The map β is obviously injective. To show that β is an isomorphism, we may assume that $E = E_n$, $E' = E_m$. Since the socle of $\mathscr{F} \otimes a_U^* E_m$ is $\mathscr{F}$,

$$\mathrm{Hom}(\mathscr{F}, \mathscr{F}) \simeq \mathrm{Hom}(\mathscr{F}, \mathscr{F} \otimes a_U^* E_m)$$

is one-dimensional. Dually, since the cosocle of $\mathscr{F} \otimes a_U^* E_n$ is $\mathscr{F}$,

$$\mathrm{Hom}(\mathscr{F} \otimes a_U^* E_n, \mathscr{F}) \simeq \mathrm{Hom}(\mathscr{F}, \mathscr{F})$$

is one-dimensional. Thus

$$\dim \mathrm{Hom}(\mathscr{F} \otimes a_U^* E_n, \mathscr{F} \otimes a_U^* E_m) \leq \min\{n, m\} = \dim \mathrm{Hom}(E_n, E_m).$$

It follows that β is an isomorphism. $\qquad\square$

Lemma 3.1.3. *Let L be a field of characteristic $\neq 2$, and let $N_n \in \mathrm{M}_{n \times n}(L)$ be the matrix defined by $(N_n)_{i,j} = 1$ for $i = j - 1$ and $(N_n)_{i,j} = 0$ otherwise. Let $m_1, \cdots, m_l \geq 0$ be integers, and let*

$$N := N(m_1, \ldots, m_l) := \mathrm{diag}(N_1, \ldots, N_1, \ldots, N_l, \ldots, N_l),$$

where each N_n is repeated m_n times. Then there exists an invertible symmetric (resp. invertible skew-symmetric) matrix A such that $AN = -N^T A$ if and only if m_n is even for n even (resp. m_n is even for n odd).

For L of characteristic 0, the equality is equivalent to $\exp(N)^T A \exp(N) = A$.

Proof. We denote the entries of A by $a_{n,c,i}^{n',c',i'}$, where $1 \leq n, n' \leq l$, $1 \leq c \leq m_n$, $1 \leq c' \leq m_{n'}$, $1 \leq i \leq n$, $1 \leq i' \leq n'$. Then $AN = -N^T A$ if and only if $a_{n,c,i}^{n',c',i'} = -a_{n,c,i-1}^{n',c',i'+1}$ for $1 < i \leq n$, $1 \leq i' < n'$ and $a_{n,c,i}^{n',c',i'} = 0$ for $i + i' \leq \max\{n, n'\}$. Let A_n be the $n \times n$ matrix given by $(A_n)_{ij} = (-1)^i$ for $i + j = n + 1$ and $(A_n)_{ij} = 0$ otherwise. For $N = N_n$, n odd (resp. even), we can take $A = A_n$. For $N = \mathrm{diag}(N_n, N_n)$, n even (resp. odd), we can take $A = \begin{pmatrix} 0 & A_n \\ -A_n & 0 \end{pmatrix}$. The "if" part follows. The "only if" part follows from the case $\sigma = -1$ of Proposition A.6.8, because P_{1-n} is $(-1)^{1-n}$-self-dual (resp. $(-1)^n$-self-dual) and $\dim P_{1-n} = m_n$. Let us give a more elementary proof of the "only if" part by induction on l. For $l = 1$, the assertion is void (resp. A defines a nondegenerate alternating bilinear form on an m_1-dimensional vector space, which implies that m_1 is even). For $l \geq 2$, consider the $m_l \times m_l$ submatrices $B = (a_{l,c,1}^{l,c',l})_{c,c'}$, $C = (a_{l,c,l}^{l,c',1})_{c,c'}$ of A. Let A' be the matrix obtained from A by removing the rows and columns in A that contain entries of B or C. Note that for $i' < l$, we have $a_{l,c,1}^{n',c',i'} = 0$, and for $i < l$, we have $a_{n,c,i}^{l,c',1} = 0$. Thus, up to reordering the indices, we have

$$A = \begin{pmatrix} A' & 0 & B' \\ 0 & 0 & B \\ C' & C & D \end{pmatrix}.$$

It follows that $B^T = \sigma' C = (-1)^{l-1} \sigma' B$, where $\sigma' = 1$ (resp. $\sigma' = -1$), and B is invertible, so that m_l is even for l even (resp. l odd). Moreover, A' is invertible symmetric (resp. invertible skew-symmetric) and $A'N' = -N'^T A'$, where $N' = N(m_1, \ldots, m_{l-3}, m_{l-2} + m_l, m_{l-1})$ ($N' = N(m_1)$ for $l = 2$). The assertion then follows from induction hypothesis. $\qquad\square$

Example 3.1.4. Let A and A' be sheaves on $X = \operatorname{Spec}(\mathbb{F}_q)$ and let $w \in \mathbb{Z}$. For $n \geq 1$ and $\lambda \in \overline{\mathbb{Q}}_\ell^\times$, we let $\mu_{\lambda,n}$ and $\mu'_{\lambda,n}$ denote the number of $n \times n$ Jordan blocks of eigenvalue λ in the Jordan normal forms of the Frobenius Frob_q acting on $A_{\overline{\mathbb{F}}_q}$ and $A'_{\overline{\mathbb{F}}_q}$, respectively. Then

- $A \simeq (D_X A')(-w)$ if and only if $\mu_{\lambda,n} = \mu'_{q^w/\lambda,n}$ for all $n \geq 1$ and all λ. In particular, A is self-dual with respect to $\overline{\mathbb{Q}}_\ell(-w)$ if and only if $\mu_{\lambda,n} = \mu_{q^w/\lambda,n}$ for all $n \geq 1$ and all λ. Note that the last condition trivially holds for $\lambda = \pm q^{w/2}$.

- A is 1-self-dual (resp. -1-self-dual) with respect to $\overline{\mathbb{Q}}_\ell(-w)$ if and only if it is self-dual with respect to $\overline{\mathbb{Q}}_\ell(-w)$ and $\mu_{q^{w/2},n}$, $\mu_{-q^{w/2},n}$ are even for n even (resp. n odd).

Example 3.1.5. Let B be a simple perverse sheaf ι-pure of weight w, not self-dual with respect to $K_X(-w)$. Then $A = B^{\oplus 2} \oplus ((D_X B)(-w) \otimes a_X^* E_2)$ is not self-dual, but the semisimplification of A is both 1-self-dual and -1-self-dual.

Remark 3.1.6.

(1) An ι-pure complex self-dual with respect to $K_X(-w)$ is necessarily of weight w.

(2) Every simple perverse $\overline{\mathbb{Q}}_\ell$-sheaf is ι-pure by a theorem of Lafforgue [28, Corollaire VII.8] (with a gap filled by Deligne [13, Théorème 1.6]; see [46, Remark 2.8.1] for the case of stacks).

(3) The two-out-of-three property (Proposition 2.2.1) in the case of ι-pure perverse sheaves also follows from the criterion of Proposition 3.1.1.

(4) Since ι-pure perverse sheaves are geometrically semisimple, for such perverse sheaves the property of being self-dual (resp. σ-self-dual) is local for the Zariski topology by Proposition 2.2.10.

3.2 Symmetry and decomposition of pure complexes over a finite field

In this subsection, we study the behavior of σ-self-dual ι-pure perverse sheaves under operations that preserve purity. The main goal is to prove the finite field case of Theorem 1.8 on derived proper direct image of σ-self-dual ι-pure perverse sheaves. The behavior of σ-self-dual complexes has already been described in Subsection 2.1. The focus of this subsection is on decomposition and on the self-duality of individual perverse cohomology sheaves. To state our results, it is convenient to introduce the following terminology.

Definition 3.2.1 (Split complexes). We say that a complex of $\overline{\mathbb{Q}}_\ell$-sheaves A is *split* if it is a direct sum of shifts of perverse sheaves, or, in other words, $A \simeq \bigoplus_i ({}^p\mathscr{H}^i A)[-i]$.

Definition 3.2.2 ($\mathrm{D}_{\iota,\sigma}^w$). Let $w \in \mathbb{Z}$. We denote by $\mathrm{D}_{\iota,\sigma}^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{Ob}(\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell))$ (resp. $\mathrm{D}_{\iota,\mathrm{sd}}^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{Ob}(\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell))$) the subset consisting of split ι-pure complexes A of weight w such that ${}^p\mathscr{H}^i A$ is $(-1)^{w+i}\sigma$-self-dual (resp. self-dual) with respect to $K_X(-w-i)$ for all i. We denote by $\mathrm{D}_{\iota,\mathrm{d}}^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{Ob}\big(\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \times \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)\big)$ the subset consisting of pairs (A, B) of split ι-pure complexes of weight w such that ${}^p\mathscr{H}^i A$ is isomorphic to $(D_X \, {}^p\mathscr{H}^i B)(-w-i)$ for all i.

By definition, $\mathrm{D}_{\iota,\mathrm{sd}}^w(X, \overline{\mathbb{Q}}_\ell) = \Delta^{-1}(\mathrm{D}_{\iota,\mathrm{d}}^w(X, \overline{\mathbb{Q}}_\ell))$, where $\Delta \colon \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \times \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ is the diagonal embedding.

Since in this subsection we will only consider operations that preserve purity, the factor $(-1)^w$ in the definition above is fixed, hence not essential. We include this factor here to make the definition compatible with the mixed case studied in Section 4, where the factor is essential (see Definition 4.2.1).

The main result of this section is the following preservation result under proper direct image, which clearly implies the finite field case of Theorem 1.8.

Theorem 3.2.3. *Let $f \colon X \to Y$ be a proper morphism of Deligne-Mumford stacks of finite presentation over $\mathbb{F}_q$, where Y has finite inertia. Then $\mathrm{R}f_*$ preserves $\mathrm{D}_{\iota,\sigma}^w$ and $\mathrm{D}_{\iota,\mathrm{d}}^w$. In other words, for $A \in \mathrm{D}_{\iota,\sigma}^w(X, \overline{\mathbb{Q}}_\ell)$ we have $\mathrm{R}f_* A \in \mathrm{D}_{\iota,\sigma}^w(Y, \overline{\mathbb{Q}}_\ell)$, and for $(A, B) \in \mathrm{D}_{\iota,\mathrm{d}}^w(X, \overline{\mathbb{Q}}_\ell)$ we have $(\mathrm{R}f_* A, \mathrm{R}f_* B) \in \mathrm{D}_{\iota,\mathrm{d}}^w$.*

The preservation of $\mathrm{D}_{\iota,\mathrm{d}}^w$ has the following two consequences, obtained respectively by considering the diagonal embedding and the first factor.

Corollary 3.2.4. *Let $f \colon X \to Y$ be a proper morphism, where Y has finite inertia. Then $\mathrm{R}f_*$ preserves $\mathrm{D}_{\iota,\mathrm{sd}}^w$.*

Corollary 3.2.5. *Let $f\colon X \to Y$ be a proper morphism, where Y has finite inertia. Then $\mathrm{R}f_*$ preserves split ι-pure complexes of weight w. In other words, if A is a split ι-pure complex of weight w on X, then $\mathrm{R}f_*A$ is a split ι-pure complex of weight w on Y.*

Corollary 3.2.5 clearly extends to the case where $w \in \mathbb{R}$. Recall that the Beilinson-Bernstein-Deligne-Gabber decomposition theorem [3, Théorème 5.4.5] ([47, Theorem 1.2] for the case of stacks) implies that the pullback of $\mathrm{R}f_*A$ (or any ι-pure complex on Y) to $Y \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q$ is split.

Remark 3.2.6. For $w \in \mathbb{Z}$, let $\mathrm{D}^w_{\iota,\mathrm{spl}}(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ denote the full subcategory consisting of split ι-pure complexes of weight w. Consider the twisted dualizing functor $\overline{D}^w_{\iota,X}\colon \mathrm{D}^w_{\iota,\mathrm{spl}}(X, \overline{\mathbb{Q}}_\ell)^{\mathrm{op}} \to \mathrm{D}^w_{\iota,\mathrm{spl}}(X, \overline{\mathbb{Q}}_\ell)$ carrying A to $\bigoplus_i (D_X\, {}^p\mathscr{H}^i A)(-w - i)[-i]$. Then $\mathrm{D}^w_{\iota,\mathrm{d}}$ is the collection of pairs $(A, \overline{D}^w_{\iota,X}A)$. Thus the preservation of $\mathrm{D}^w_{\iota,\mathrm{d}}$ by $\mathrm{R}f_*$ is equivalent to the preservation of $\mathrm{D}^w_{\iota,\mathrm{spl}}$ and the existence of an isomorphism $\mathrm{R}f_*\overline{D}^w_{\iota,X}A \simeq \overline{D}^w_{\iota,X}\mathrm{R}f_*A$ for every object A of $\mathrm{D}^w_{\iota,\mathrm{spl}}(X, \overline{\mathbb{Q}}_\ell)$. Our proof of Theorem 3.2.3 relies on the two-out-of-three property and the resulting isomorphism is not necessarily functorial in A. Thus our proof does *not* provide a natural isomorphism between the functors $\mathrm{R}f_*\overline{D}^w_{\iota,X}$ and $\overline{D}^w_{\iota,X}\mathrm{R}f_*$.

Let us first recall that the following operations preserve ι-pure complexes [3, Stabilités 5.1.14, Corollaire 5.4.3] ([47] for the case of stacks). The proof makes use of the fact that these operations commute with duality (up to shift and twist).

Remark 3.2.7 (Preservation of ι-pure complexes)**.** Let $f\colon X \to Y$ be a morphism, and let $w, w' \in \mathbb{R}$.

(1) For $A \in \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ ι-pure of weight w, $A(n)$ is ι-pure of weight $w - 2n$ and $A[n]$ is ι-pure of weight $w + n$ for $n \in \mathbb{Z}$.

(2) For $A \in \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$, A is ι-pure of weight w if and only if $D_X A$ is ι-pure of weight $-w$.

(3) If f is an open immersion, the functor $f_{!*}$ preserves ι-pure perverse sheaves of weight w.

(4) Assume that f is smooth. Then f^* preserves ι-pure complexes of weight w. Moreover, if f is surjective, then $A \in \mathrm{D}^b_c(Y, \overline{\mathbb{Q}}_\ell)$ is ι-pure of weight w if and only if f^*A is so.

(5) Assume that X and Y are regular. Then f^* preserves ι-pure complexes of weight w in $\mathrm{D}^b_{\mathrm{lisse}}$. Moreover, if f is surjective, then $A \in \mathrm{D}^b_{\mathrm{lisse}}(Y, \overline{\mathbb{Q}}_\ell)$ is ι-pure of weight w if and only if f^*A is so.

(6) For $A \in \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ which is ι-pure of weight w, and $A' \in \mathrm{D}^b_c(X', \overline{\mathbb{Q}}_\ell)$ which is ι-pure of weight w', $A \boxtimes A' \in \mathrm{D}^b_c(X \times X', \overline{\mathbb{Q}}_\ell)$ is ι-pure of weight $w + w'$.

(7) For $A \in \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ and $m \geq 1$, A is ι-pure of weight w if and only if $A^{\boxtimes m} \in \mathrm{D}^b_c([X^m/\mathfrak{S}_m], \overline{\mathbb{Q}}_\ell)$ is ι-pure of weight mw.

(8) For a proper morphism f, $\mathrm{R}f_*$ preserves ι-pure complexes of weight w.

(9) Assume that f is a closed immersion and let $A \in \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$. Then A is ι-pure of weight w if and only if f_*A is so.

Recall that these operations also preserve σ-self-dual complexes (Remark 2.1.4). With the exception of derived proper direct image, the operations also preserve perversity (up to shift). Hence they also preserve $\mathrm{D}^w_{\iota,\sigma}$, up to modification of w and σ. The details are given below. The case of $(-)^{\boxtimes m}$ requires some additional arguments and will be given in Proposition 3.2.14 later.

Remark 3.2.8 (Preservation of $\mathrm{D}^w_{\iota,\sigma}$, easy part)**.** Let $f\colon X \to Y$ be a morphism, and let $w, w' \in \mathbb{Z}$.

(1) If $A \in \mathrm{D}^w_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$, then $A[n] \in \mathrm{D}^{w+n}_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$ and $A(n) \in \mathrm{D}^{w-2n}_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$ for $n \in \mathbb{Z}$.

(2) D_X carries $\mathrm{D}^w_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$ to $\mathrm{D}^{-w}_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$.

(3) Assume that X is regular and let $\mathscr{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X, punctually ι-pure of weight w. Then there exists a nondegenerate σ-symmetric pairing $\mathscr{F} \otimes \mathscr{F} \to \overline{\mathbb{Q}}_\ell(-w)$ if and only if $\mathscr{F}$ belongs to $\mathrm{D}^w_{\iota,(-1)^w\sigma}(X, \overline{\mathbb{Q}}_\ell)$.

(4) If f is smooth, then f^* preserves $\mathrm{D}^w_{\iota,\sigma}$.

(5) If X and Y are regular, then f^* preserves $\mathrm{D}^w_{\iota,\sigma} \cap \mathrm{D}^b_{\mathrm{lisse}}$.

(6) The functor $-\boxtimes -$ carries $\mathrm{D}^w_{\iota,\sigma}(X,\overline{\mathbb{Q}}_\ell) \times \mathrm{D}^{w'}_{\iota,\sigma'}(X',\overline{\mathbb{Q}}_\ell)$ to $\mathrm{D}^{w+w'}_{\iota,\sigma\sigma'}(X\times X',\overline{\mathbb{Q}}_\ell)$.

(7) Assume that f is a closed immersion and let $A\in \mathrm{D}^b_c(X,\overline{\mathbb{Q}}_\ell)$. Then $A\in \mathrm{D}^w_{\iota,\sigma}(X,\overline{\mathbb{Q}}_\ell)$ if and only if $f_*A\in \mathrm{D}^w_{\iota,\sigma}(Y,\overline{\mathbb{Q}}_\ell)$.

(8) Assume that f is an open immersion and let $A\in \mathrm{Perv}(X,\overline{\mathbb{Q}}_\ell)$. Then $A\in \mathrm{D}^w_{\iota,\sigma}(X,\overline{\mathbb{Q}}_\ell)$ if and only if $f_{!*}A\in \mathrm{D}^w_{\iota,\sigma}(Y,\overline{\mathbb{Q}}_\ell)$.

Similar properties hold for $\mathrm{D}^w_{\iota,\mathrm{d}}$.

Next we transcribe the two-out-of-three property (Proposition 2.2.1) established earlier in terms of $\mathrm{D}^w_{\iota,\sigma}$.

Remark 3.2.9. If $A, A', A''\in \mathrm{D}^b_c(X,\overline{\mathbb{Q}}_\ell)$ satisfy $A\simeq A'\oplus A''$, and two of the three complexes are in $\mathrm{D}^w_{\iota,\sigma}$, then so is the third one. A similar property holds for $\mathrm{D}_{\iota,\mathrm{d}}$.

Note that proper direct image $\mathrm{R}f_*$ does not preserve perversity and in general there seems to be no canonical way to produce pairings on the perverse cohomology sheaves ${}^p\mathrm{R}^i f_* A$ from pairing on A. In the case of projective direct image, the relative hard Lefschetz theorem provides such pairings. Let us first fix some terminology on projective morphisms of Deligne-Mumford stacks.

Definition 3.2.10. Let $f\colon X\to Y$ be a quasi-compact schematic morphism of Deligne-Mumford stacks. We say that an invertible sheaf $\mathscr{L}$ on X is f-*ample* if for one (or, equivalently, for every) étale surjective morphism $g\colon Y'\to Y$ where Y' is a scheme, $h^*\mathscr{L}$ is f'-ample [19, Définition 4.6.1]. Here h and f' are as shown in the following Cartesian square

$$
\begin{array}{ccc}
X' & \xrightarrow{\ h\ } & X \\
{\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
Y' & \xrightarrow{\ g\ } & Y.
\end{array}
$$

We say that a morphism $f\colon X\to Y$ of Deligne-Mumford stacks is *quasi-projective* if it is schematic, of finite presentation and if there exists an f-ample invertible sheaf on X. We say that f is *projective* if it is quasi-projective and proper.

The following is an immediate extension of the case of schemes [3, Théorème 5.4.10].

Proposition 3.2.11 (Relative hard Lefschetz). *Let $f\colon X\to Y$ be a projective morphism of Deligne-Mumford stacks of finite presentation over $\mathbb{F}_q$, and let $\eta\in \mathrm{H}^2(X,\overline{\mathbb{Q}}_\ell(1))$ be the first Chern class of an f-ample invertible sheaf on X. Let A be an ι-pure perverse sheaf on X. Then, for $i\geq 0$, the morphism*

$$
{}^p\mathrm{R}^{-i}f_*(\eta^i\otimes \mathrm{id}_A)\colon {}^p\mathrm{R}^{-i}f_*A\to {}^p\mathrm{R}^i f_*A(i)
$$

is an isomorphism.

By Deligne's decomposition theorem [12], the proposition implies that $\mathrm{R}f_*A$ is split.

Proposition 3.2.12. *Let $f\colon X\to Y$ be a projective morphism, and let $w\in \mathbb{Z}$. Then $\mathrm{R}f_*$ preserves $\mathrm{D}^w_{\iota,\sigma}$ and $\mathrm{D}^w_{\iota,\mathrm{d}}$.*

Proof. We prove the case of $\mathrm{D}^w_{\iota,\sigma}$, the case of $\mathrm{D}^w_{\iota,\mathrm{d}}$ being simpler. It suffices to show that for every $(-1)^w\sigma$-self-dual ι-pure perverse sheaf A of weight w, $\mathrm{R}f_*A$ belongs to $\mathrm{D}^w_{\iota,\sigma}$. Given a $(-1)^w\sigma$-symmetric isomorphism $A\xrightarrow{\sim}(D_X A)(-w)$, the isomorphism

$$
{}^p\mathrm{R}^{-i}f_*A\xrightarrow{\sim}{}^p\mathrm{R}^{-i}f_*(D_X A)(-w)\xrightarrow[\eta^i]{\sim}{}^p\mathrm{R}^i f_*(D_X A)(i-w)\xrightarrow{\sim}(D_Y\,{}^p\mathrm{R}^{-i}f_*A)(i-w)
$$

corresponding to the pairing obtained from

$$
\mathrm{R}f_*A[-i]\otimes \mathrm{R}f_*A[-i]\to \mathrm{R}f_*(A\otimes A)[-2i]\to \mathrm{R}f_*K_X(-w)[-2i]\xrightarrow{\eta^i}\mathrm{R}f_*K_X(i-w)\to K_Y(i-w)
$$

is $(-1)^{w+i}\sigma$-symmetric by Lemma A.5.11. $\qquad\square$

Proof of Theorem 3.2.3. We will prove the case of $\mathrm{D}_{\iota,\sigma}^w$. The case of $\mathrm{D}_{\iota,\mathrm{d}}^w$ is similar.

Consider the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f_1} Y \times_{\bar{Y}} \bar{X} \longrightarrow & Y \\
& \downarrow & \downarrow \\
& \bar{X} \xrightarrow{\ \bar{f}\ } & \bar{Y}
\end{array}
$$

where $\bar{f}$ is the morphism of coarse moduli spaces (they exist by the Keel-Mori theorem [26]) associated to f. Since f_1 is proper and quasi-finite, $\mathrm{R}f_{1*}$ is t-exact for the perverse t-structures, and by Remarks 3.2.7 (8) and 2.1.4 (3), we see that the theorem holds for f_1. Thus we may assume that f is representable. We proceed by induction on the dimension of X. Let $A \in \mathrm{D}_{\iota,\sigma}^w(X, \overline{\mathbb{Q}}_\ell)$; we may assume that A is perverse. Applying Chow's lemma [39, Corollaire I.5.7.13] to the proper morphism $\bar{f}$ of algebraic spaces, we obtain a projective birational morphism $\bar{g}\colon \bar{X}' \to \bar{X}$ such that $\bar{f}\bar{g}\colon \bar{X}' \to \bar{Y}$ is projective. Let $g\colon X' \to X$ be the base change of $\bar{g}$. Let U be a dense open substack of X such that g induces an isomorphism $g^{-1}(U) \xrightarrow{\sim} U$. Let j and j' be the open immersions, as shown in the commutative diagram

$$
\begin{array}{ccc}
& & X' \\
& \nearrow^{j'} & \downarrow g \\
U & \xrightarrow{\ j\ } & X.
\end{array}
$$

By [3, Corollaire 5.3.11] (cf. Lemma 2.2.8), we have

$$A \simeq j_{!*}j^*A \oplus B,$$

where $B \in \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ is supported on $X \backslash U$. By Remark 3.2.8 (4), (8), we have $j_{!*}j^*A \in \mathrm{D}_{\iota,\sigma}^w(X, \overline{\mathbb{Q}}_\ell)$. By the two-out-of-three property, we have $B \in \mathrm{D}_{\iota,\sigma}^w(X, \overline{\mathbb{Q}}_\ell)$. As g is projective, by Proposition 3.2.12 we have $\mathrm{R}g_*j'_{!*}j^*A \in \mathrm{D}_{\iota,\sigma}^w(X, \overline{\mathbb{Q}}_\ell)$, so

$$\mathrm{R}g_*j'_{!*}j^*A \simeq j_{!*}j^*A \oplus C,$$

where $C \in \mathrm{D}_{\iota,\sigma}^w(X, \overline{\mathbb{Q}}_\ell)$ is supported on $X \backslash U$. Applying $\mathrm{R}f_*$ to $A \oplus C \simeq \mathrm{R}g_*j'_{!*}j^*A \oplus B$, we obtain

$$\mathrm{R}f_*A \oplus \mathrm{R}f_*C \simeq \mathrm{R}f_*\mathrm{R}g_*j'_{!*}j^*A \oplus \mathrm{R}f_*B.$$

By induction hypothesis, $\mathrm{R}f_*B$ and $\mathrm{R}f_*C$ belong to $\mathrm{D}_{\iota,\sigma}^w(Y, \overline{\mathbb{Q}}_\ell)$. As fg is projective, by Proposition 3.2.12 we have $\mathrm{R}(fg)_*j'_{!*}j^*A \in \mathrm{D}_{\iota,\sigma}^w(Y, \overline{\mathbb{Q}}_\ell)$. It then follows from the two-out-of-three property that $\mathrm{R}f_*A$ belongs to $\mathrm{D}_{\iota,\sigma}^w(Y, \overline{\mathbb{Q}}_\ell)$. $\qquad\square$

Remark 3.2.13 (Gabber)**.** In the case $Y = \mathrm{Spec}(\mathbb{F}_q)$, the proof of Theorem 3.2.3 still makes use of the relative hard Lefschetz theorem (applied to the morphism g). With the help of a refined Chow's lemma, it is possible to prove this case of Theorem 3.2.3 using only the absolute hard Lefschetz theorem, at least in the case of schemes.

The following is a preservation result for the exterior tensor power functor $(-)^{\boxtimes m}$. Unlike the functors listed in Remark 3.2.8, $(-)^{\boxtimes m}$ is not additive and the reduction to the case of perverse sheaves is not trivial.

Proposition 3.2.14. *Let A be an ι-mixed $\overline{\mathbb{Q}}_\ell$-complex of integral weights on X such that for all $n, w \in \mathbb{Z}$, $\mathrm{gr}_w^W {}^p\mathscr{H}^n A$ is $(-1)^w\sigma$-self-dual with respect to $K_X(-w)$. Then $\mathrm{gr}_w^W {}^p\mathscr{H}^n(A^{\boxtimes m})$ is $(-1)^w\sigma^m$-self-dual with respect to $K_{[X^m/\mathfrak{S}_m]}(-w)$ for all $n, w \in \mathbb{Z}$. Here W denotes the ι-weight filtrations. In particular, the functor $(-)^{\boxtimes m}$ carries $\mathrm{D}_{\iota,\sigma}^w(X, \overline{\mathbb{Q}}_\ell)$ to $\mathrm{D}_{\iota,\sigma^m}^{mw}([X^m/\mathfrak{S}_m], \overline{\mathbb{Q}}_\ell)$.*

Similar results hold for $\mathrm{D}_{\iota,\mathrm{d}}$.

Proof. We write $A^n = {}^p\mathscr{H}^n A$. For $\tau \in \mathfrak{S}_m$, τ acts on X^m by $(x_1, \ldots, x_m) \mapsto (x_{\tau(1)}, \ldots, x_{\tau(m)})$. By the Künneth formula,

$$
{}^p\mathscr{H}^n(A^{\boxtimes m}) \simeq \bigoplus_{n_1 + \cdots + n_m = n} A^{n_1} \boxtimes \cdots \boxtimes A^{n_m},
$$

where τ acts on the right hand side by $\prod_{\substack{i < j \\ \tau(i) > \tau(j)}} (-1)^{n_i n_j}$ times the canonical isomorphism

$$
\tau^*(A^{n_{\tau(1)}} \boxtimes \cdots \boxtimes A^{n_{\tau(m)}}) \xrightarrow{\sim} A^{n_1} \boxtimes \cdots \boxtimes A^{n_m}.
$$

Note that $W_w \, {}^p\mathscr{H}^n(A^{\boxtimes m}) \subseteq {}^p\mathscr{H}^n(A^{\boxtimes m})$ is the perverse subsheaf given by

$$\sum_{\substack{n_1+\cdots+n_m=n \\ w_1+\cdots+w_m=w}} W_{w_1}A^{n_1} \boxtimes \cdots \boxtimes W_{w_m}A^{n_m}.$$

So

$$\mathrm{gr}^W_w \, {}^p\mathscr{H}^n(A^{\boxtimes m}) \simeq \bigoplus_{\substack{n_1+\cdots+n_m=n \\ w_1+\cdots+w_m=w}} \mathrm{gr}^W_{w_1}A^{n_1} \boxtimes \cdots \boxtimes \mathrm{gr}^W_{w_1}A^{n_m}.$$

Thus the $(-1)^{w_i}\sigma$-symmetric isomorphisms

$$\mathrm{gr}^W_{w_i}A^{n_i} \xrightarrow{\sim} (D_X \mathrm{gr}^W_{w_i}A^{n_i})(-w_i)$$

induce a $(-1)^w\sigma^m$-symmetric isomorphism

$$\mathrm{gr}^W_w \, {}^p\mathscr{H}^n(A^{\boxtimes m}) \xrightarrow{\sim} (D_{X^m}\mathrm{gr}^W_w \, {}^p\mathscr{H}^n(A^{\boxtimes m}))(-w),$$

compatible with the actions of $\mathfrak{S}_m$. $\qquad\square$

We conclude this subsection with a symmetry criterion in terms of traces of squares of Frobenius, analogue of the Frobenius-Schur indicator theorem in representation theory (cf. [44, Section 13.2, Proposition 39], [4, Proposition II.6.8]). This criterion will be used to show a result on the independence of ℓ of symmetry (Corollary 4.2.14). We refer the reader to [25, Theorem 1.9.6] for a related criterion on the symmetry of the geometric monodromy. For a groupoid $\mathscr{C}$, we let $|\mathscr{C}|$ denote the set of isomorphism classes of its objects.

Proposition 3.2.15. *Let X be a Deligne-Mumford stack of finite presentation over $\mathbb{F}_q$, connected and geometrically unibranch of dimension d. Let $\mathscr{F}$ be a semisimple lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X, punctually ι-pure of weight $w \in \mathbb{Z}$. Consider the series*

$$L^{(2)}(T) = \exp\left(\sum_{m \geq 1} \sum_{x \in |X(\mathbb{F}_{q^m})|} \frac{\iota\mathrm{tr}(\mathrm{Frob}_x^2 \mid \mathscr{F}_{\bar{x}})}{\#\mathrm{Aut}(x)}\frac{T^m}{m}\right),$$

where $\bar{x}$ denotes a geometric point above x, and $\mathrm{Frob}_x = \mathrm{Frob}_{q^m}$. Then the series $L^{(2)}(T)$ converges absolutely for $|T| < q^{-w-d}$ and extends to a rational function satisfying

$$-\mathrm{ord}_{T=q^{-w-d}}L^{(2)}(T) = \dim \mathrm{H}^0(X, (\mathrm{Sym}^2(\mathscr{F}^\vee))(-w)) - \dim \mathrm{H}^0(X, (\wedge^2(\mathscr{F}^\vee))(-w)).$$

In particular, if $\mathscr{F}$ is simple and σ-self-dual (resp. not self-dual) with respect to $\overline{\mathbb{Q}}_\ell(-w)$, then

$$-\mathrm{ord}_{T=q^{-w-d}}L^{(2)}(T) = \sigma \quad (\text{resp.} = 0).$$

Proof. For $x \in X(\mathbb{F}_{q^m})$,

$$\mathrm{tr}(\mathrm{Frob}_x^2 \mid \mathscr{F}_{\bar{x}}) = \mathrm{tr}(\mathrm{Frob}_x \mid \mathrm{Sym}^2 \mathscr{F}_{\bar{x}}) - \mathrm{tr}(\mathrm{Frob}_x \mid \wedge^2 \mathscr{F}_{\bar{x}}).$$

Thus $L^{(2)}(T) = L_\iota(X, \mathrm{Sym}^2\mathscr{F}, T)/L_\iota(X, \wedge^2\mathscr{F}, T)$ (see [46, Definition 4.1] for the definition of L-series $L_\iota(X, -, T)$). Note that $\mathscr{F} \otimes \mathscr{F}$ is lisse punctually ι-pure of weight $2w$, and semisimple by a theorem of Chevalley [6, IV, Proposition 5.2]. The same holds for $\mathrm{Sym}^2\mathscr{F}$ and $\wedge^2\mathscr{F}$. For $\mathscr{G}$ lisse punctually ι-pure of weight $2w$ on X, the series $L_\iota(X, \mathscr{G}, T)$ converges absolutely for $|T| < q^{-w-d}$ and extends to a rational function

$$\iota\prod_i \det(1 - T\mathrm{Frob}_q \mid \mathrm{H}^i_c(X_{\overline{\mathbb{F}}_q}, \mathscr{G}))^{(-1)^{i+1}}$$

by [46, Theorem 4.2]. Only the factor $i = 2d$ may contribute to poles on the circle $|T| = q^{-w-d}$, by [46, Theorem 1.4]. For every dense open substack U of X such that U_{red} is regular, we have

$$\mathrm{H}^{2d}_c(X_{\overline{\mathbb{F}}_q}, \mathscr{G}) \simeq \mathrm{H}^0(U_{\overline{\mathbb{F}}_q}, \mathscr{G}^\vee)^\vee(-d) \simeq \mathrm{H}^0(X_{\overline{\mathbb{F}}_q}, \mathscr{G}^\vee)^\vee(-d)$$

Here in the second assertion we used the hypothesis on X, which implies that the homomorphism $\pi_1(U) \to \pi_1(X)$ is surjective. Therefore,

$$-\mathrm{ord}_{T=q^{-w-d}}L_\iota(X, \mathscr{G}, T) = \dim \mathrm{H}^0(X, \mathscr{G}^\vee(-w))$$

for $\mathscr{G}$ semisimple. $\qquad\square$

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3.3 Variant: Semisimple complexes over a separably closed field

In this subsection let k be a separably closed field. We establish variants over k of results of Subsection 3.2. The main result is a preservation result under proper direct image (Theorem 3.3.7). We also include an example of Gabber (Remark 3.3.13) showcasing the difference in parity of Betti numbers between zero and positive characteristics.

One key point in Subsection 3.2 is the relative hard Lefschetz theorem for pure perverse sheaves. If k has characteristic zero, a conjecture of Kashiwara states that all semisimple perverse sheaves satisfy relative hard Lefschetz. Kashiwara's conjecture was proved by Drinfeld [14] assuming de Jong's conjecture for infinitely many primes ℓ, which was later proved by Gaitsgory for $\ell > 2$ [17].

Drinfeld's proof uses the techniques of reduction from k to finite fields in [3, Section 6]. The reduction holds in fact without restriction on the characteristic of k and provides a class of semisimple perverse sheaves over k for which the relative hard Lefschetz theorem holds. Let us briefly recall the reduction. Let X be a Deligne-Mumford stack of finite presentation over k. There exist a subring $R \subseteq k$ of finite type over $\mathbb{Z}[1/\ell]$ and a Deligne-Mumford stack X_R of finite presentation over $\mathrm{Spec}(R)$ such that $X \simeq X_R \otimes_R k$. For extra data $(\mathscr{T}, \mathscr{L})$ on X, we have an equivalence [3, 6.1.10]

$$(3.3.1) \qquad \mathrm{D}^b_{\mathscr{T},\mathscr{L}}(X, \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \mathrm{D}^b_{\mathscr{T},\mathscr{L}}(X_s, \overline{\mathbb{Q}}_\ell)$$

for every geometric point s above a closed point of $\mathrm{Spec}(R)$, provided that R is big enough (relative to the data $(\mathscr{T}, \mathscr{L})$). Here X_s denotes the base change of X_R by $s \to \mathrm{Spec}(R)$. Note that s is the spectrum of an algebraic closure of a finite field. Each $A \in \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ is contained in $\mathrm{D}^b_{\mathscr{T},\mathscr{L}}(X, \overline{\mathbb{Q}}_\ell)$ for some $(\mathscr{T}, \mathscr{L})$.

Definition 3.3.1 (Admissible semisimple complexes). Let A be a semisimple perverse $\overline{\mathbb{Q}}_\ell$-sheaf on X. If k is an algebraic closure of a finite field, we say that A is *admissible* if there exists a Deligne-Mumford stack X_0 of finite presentation over a finite subfield k_0 of k, an isomorphism $X \simeq X_0 \otimes_{k_0} k$, and a perverse $\overline{\mathbb{Q}}_\ell$-sheaf A_0 on X_0 such that A is isomorphic to the pullback of A_0. More generally, if k has characteristic > 0, we say that A is *admissible* if the images of A under the equivalences (3.3.1), for all geometric points s over a closed point of $\mathrm{Spec}(R)$, are admissible, for some R big enough. If k has characteristic zero, we adopt the convention that every semisimple perverse $\overline{\mathbb{Q}}_\ell$-sheaf is admissible.

We say that a complex $B \in \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ is *admissible semisimple* if $B \simeq \bigoplus_i ({}^p\mathscr{H}^i B)[-i]$ and, for each i, its i-th perverse cohomology sheaf ${}^p\mathscr{H}^i B$ is admissible semisimple.

Remark 3.3.2. In the case where k is the algebraic closure of a finite field k_0, for X_0 as above, the pullback of an ι-pure complex on X_0 to X is admissible semisimple by the decomposition theorems [3, Théorèmes 5.3.8, 5.4.5] ([47] for the case of stacks). Conversely, if a semisimple perverse $\overline{\mathbb{Q}}_\ell$-sheaf A on X is the pullback of A_0 on X_0 as above, then we may take A_0 to be pure (of weight 0 for example) by Lafforgue's theorem [28, Corollaire VII.8] mentioned in Remark 3.1.6 (2).

Remark 3.3.3. Following [3, 6.2.4], we say that a simple perverse $\overline{\mathbb{Q}}_\ell$-sheaf on X is *of geometric origin* if it belongs to the class of simple perverse $\overline{\mathbb{Q}}_\ell$-sheaves generated from the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\mathrm{Spec}(k)$ by taking composition factors of perverse cohomology sheaves under the six operations. By [3, Lemme 6.2.6] (suitably extended), simple perverse $\overline{\mathbb{Q}}_\ell$-sheaves of geometric origin are admissible.

The operations that preserve purity also preserve admissible semisimple complexes. The details are given below.

Remark 3.3.4 (Preservation of admissible semisimple complexes). Let $f: X \to Y$ be a morphism.

- The full subcategory of D^b_c consisting of objects A such that the composition factors of ${}^p\mathscr{H}^i A$ are admissible for all i is stable under the operations $\mathrm{R}f_*$, $\mathrm{R}f_!$, f^*, $\mathrm{R}f^!$, $\otimes$, $\mathrm{R}\mathscr{H}om$, and $(-)^{\boxtimes m}$.

- $D_X: \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)^{\mathrm{op}} \to \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$ preserves admissible semisimple complexes.

- If f is an open immersion, $f_{!*}$ preserves admissible semisimple perverse sheaves.

- Assume that f is a closed immersion and let $A \in \mathrm{D}^b_c(X, \overline{\mathbb{Q}}_\ell)$. Then A is admissible semisimple if and only if $f_* A$ is admissible semisimple.

- If f is smooth, f^* preserves admissible semisimple complexes.

- If X and Y are regular, f^* preserves admissible semisimple complexes in $\mathrm{D}^b_{\mathrm{lisse}}$.

- The functors

$$- \boxtimes - \colon \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \times \mathrm{D}_c^b(X', \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_c^b(X \times X', \overline{\mathbb{Q}}_\ell),$$
$$(-)^{\boxtimes m} \colon \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_c^b([X^m/\mathfrak{S}_m], \overline{\mathbb{Q}}_\ell), \quad m \geq 0$$

  preserve admissible semisimple complexes.

- For a proper morphism f, $\mathrm{R}f_*$ preserves admissible semisimple complexes.

These properties reduce to the corresponding properties for pure complexes over a finite field (Remark 3.2.7). Since the equivalences (3.3.1) are compatible with these operations, this reduction is clear in positive characteristic. The reduction in characteristic zero is more involved. The case of $\mathrm{R}f_*$ is done in [14] and the other cases can be done similarly.

Definition 3.3.5 (D_σ). We denote by $\mathrm{D}_\sigma(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{Ob}(\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell))$ (resp. $\mathrm{D}_{\mathrm{sd}}(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{Ob}(\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell))$) the subset consisting of admissible semisimple complexes A such that ${}^p\mathscr{H}^i A$ is $(-1)^i\sigma$-self-dual (resp. self-dual) with respect to K_X, for all i. We denote by $\mathrm{D}_{\mathrm{d}}(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{Ob}\big(\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \times \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)\big)$ the subset consisting of pairs (A, B) such that both A and B are admissible semisimple, and that ${}^p\mathscr{H}^i A$ is isomorphic to $D_X \, {}^p\mathscr{H}^i B$.

By definition, $\mathrm{D}_{\mathrm{sd}}(X, \overline{\mathbb{Q}}_\ell) = \Delta^{-1}(\mathrm{D}_{\mathrm{d}}(X, \overline{\mathbb{Q}}_\ell))$, where $\Delta \colon \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \times \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ is the diagonal embedding.

Example 3.3.6. For $X = \mathrm{Spec}(k)$, every object A of $\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ is admissible semisimple and belongs to $\mathrm{D}_{\mathrm{sd}}(X, \overline{\mathbb{Q}}_\ell)$. Let $d_i = \dim \mathscr{H}^i(A)$. Then

- A is 1-self-dual with respect to $\overline{\mathbb{Q}}_\ell$ if and only if it is self-dual with respect to $\overline{\mathbb{Q}}_\ell$, namely if $d_i = d_{-i}$ for all i. (Recall that in general self-dual objects are not necessarily 1-self-dual.)

- A is -1-self-dual with respect to $\overline{\mathbb{Q}}_\ell$ if and only if $d_i = d_{-i}$ for all i and d_0 is even.

- $A \in \mathrm{D}_1(X, \overline{\mathbb{Q}}_\ell)$ (resp. $\in \mathrm{D}_{-1}(X, \overline{\mathbb{Q}}_\ell)$) if and only if d_i is even for i odd (resp. even).

- For $A, B \in \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$, $(A, B) \in \mathrm{D}_{\mathrm{d}}(X, \overline{\mathbb{Q}}_\ell)$ if and only if $A \simeq B$.

The main result of this subsection is the following.

Theorem 3.3.7. *Let $f \colon X \to Y$ be a proper morphism of Deligne-Mumford stacks of finite presentation over k, where Y has finite inertia. Then $\mathrm{R}f_*$ preserves D_σ and D_{d}.*

Corollary 3.3.8. *Let $f \colon X \to Y$ be a proper morphism, where Y has finite inertia. Then $\mathrm{R}f_*$ preserves D_{sd}.*

The strategy for proving Theorem 3.3.7 is the same as Theorem 3.2.3. Let us recall that the operations listed in Remark 3.3.4 that preserve admissible semisimple complexes also preserve σ-self-dual complexes (Remark 2.1.4). With the exception of proper direct image $\mathrm{R}f_*$, they also preserve perversity, hence they preserve D_σ. The details are given below.

Remark 3.3.9 (Preservation of D_σ, easy part). Let $f \colon X \to Y$ be a morphism, and let $n \in \mathbb{Z}$.

(1) If $A \in \mathrm{D}_\sigma(X, \overline{\mathbb{Q}}_\ell)$, then $A[n] \in \mathrm{D}_{(-1)^n\sigma}(X, \overline{\mathbb{Q}}_\ell)$.

(2) D_X preserves $\mathrm{D}_\sigma(X, \overline{\mathbb{Q}}_\ell)$.

(3) Assume that X is regular and let $\mathscr{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X, admissible semisimple. Then there exists a nondegenerate σ-symmetric pairing $\mathscr{F} \otimes \mathscr{F} \to \overline{\mathbb{Q}}_\ell$ if and only if $\mathscr{F}$ belongs to $\mathrm{D}_\sigma(X, \overline{\mathbb{Q}}_\ell)$.

(4) If f is smooth, then f^* preserves D_σ.

(5) If X and Y are regular, then f^* preserves $\mathrm{D}_\sigma \cap \mathrm{D}_{\mathrm{lisse}}^b$.

(6) Assume that f is a closed immersion and let $A \in \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$. Then $A \in \mathrm{D}_\sigma(X, \overline{\mathbb{Q}}_\ell)$ if and only if $f_*A \in \mathrm{D}_\sigma(Y, \overline{\mathbb{Q}}_\ell)$.

(7) The exterior tensor product functors induce functors

$$- \boxtimes - \colon \mathrm{D}_\sigma(X, \overline{\mathbb{Q}}_\ell) \times \mathrm{D}_{\sigma'}(X', \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_{\sigma\sigma'}(X \times X', \overline{\mathbb{Q}}_\ell),$$
$$(-)^{\boxtimes m} \colon \mathrm{D}_\sigma(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_{\sigma^m}([X^m/\mathfrak{S}_m], \overline{\mathbb{Q}}_\ell), \quad m \geq 0.$$

For $(-)^{\boxtimes m}$, the reduction to perverse sheaves is nontrivial and is similar to Proposition 3.2.14.

Similar properties hold for D_d.

By Proposition 2.2.1, the two-out-of-three property holds for D_σ and D_d.

We state a relative hard Lefschetz theorem over an arbitrary field F in which ℓ is invertible.

Proposition 3.3.10 (Relative hard Lefschetz)**.** *Let $f \colon X \to Y$ be a projective morphism of Deligne-Mumford stacks of finite presentation over F, and let $\eta \in \mathrm{H}^2(X, \overline{\mathbb{Q}}_\ell(1))$ be the first Chern class of an f-ample invertible sheaf on X. Let A be a perverse $\overline{\mathbb{Q}}_\ell$-sheaf on X whose pullback to $X \otimes_F \bar{F}$ is admissible semisimple. Then, for $i \geq 0$, the morphism*

$$^p\mathscr{H}^{-i}(\eta^i \otimes \mathrm{id}_A) \colon {}^p\mathrm{R}^{-i}f_*A \to {}^p\mathrm{R}^i f_*A(i)$$

is an isomorphism.

That the morphism is an isomorphism can be checked on $X \otimes_F \bar{F}$. Thus we are reduced to the case where $F = k$ is separably closed. As mentioned earlier, the relative hard Lefschetz theorem in this case is obtained by reduction to the finite field case (Proposition 3.2.11).

Combining Proposition 3.3.10 with Lemma A.5.11, we obtain the following preservation result under projective direct image.

Proposition 3.3.11. *Let $f \colon X \to Y$ be a projective morphism (over k). Then $\mathrm{R}f_*$ preserves D_σ and D_d.*

The proof of Theorem 3.2.3 can now be repeated verbatim to prove Theorem 3.3.7.

Over an arbitrary field F in which ℓ is invertible, we may exploit the relative hard Lefschetz theorem to get analogues for split complexes that are geometrically admissible semisimple.

Proposition 3.3.12. *Let $f \colon X \to Y$ be a proper morphism of separated Deligne-Mumford stacks of finite type over F. Assume that X is regular. Let $\mathscr{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X, whose pullback to $X_{\bar{F}}$ is admissible semisimple. Then we have $\mathrm{R}f_*\mathscr{F} \simeq \bigoplus_i ({}^p\mathrm{R}^i f_*\mathscr{F})[-i]$.*

Proof. By [31, Théorème 16.6], there exists a finite surjective morphism $g_1 \colon X' \to X$ where X' is a scheme. Up to replacing X' by its normalization, we may assume that X' is normal. By de Jong's alterations [8, Theorem 4.1], there exists a proper surjective morphism $g_2 \colon X'' \to X'$, generically finite, such that X'' is regular and quasi-projective over F. Let $g = g_1 g_2 \colon X'' \to X$. By the relative hard Lefschetz theorem (Proposition 3.3.10) and Deligne's decomposition theorem [12], we have

$$\mathrm{R}(fg)_* g^* \mathscr{F} \simeq \bigoplus_i ({}^p\mathrm{R}^i (fg)_* g^* \mathscr{F})[-i].$$

Note that $g^*\mathscr{F} \simeq g^*\mathscr{F} \otimes \mathrm{R}g^!\overline{\mathbb{Q}}_\ell \simeq \mathrm{R}g^!\mathscr{F}$. Consider the composite

$$\alpha \colon \mathscr{F} \to \mathrm{R}g_* g^* \mathscr{F} \simeq \mathrm{R}g_! \mathrm{R}g^! \mathscr{F} \to \mathscr{F}$$

of the adjunction morphisms. Since α is generically multiplication by the degree of g, α is an isomorphism. It follows that $\mathscr{F}$ is a direct summand of $\mathrm{R}g_* g^* \mathscr{F}$, so that $\mathrm{R}f_*\mathscr{F}$ is a direct summand of $\mathrm{R}(fg)_* g^* \mathscr{F}$. $\quad\square$

Remark 3.3.13 (Gabber)**.** Let X be a proper smooth algebraic space over k and let $\mathscr{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X with finite monodromy, -1-self-dual with respect to $\overline{\mathbb{Q}}_\ell$. Then $\mathscr{F}$ is admissible semisimple (since each simple factor is of geometric origin), and $\mathscr{F}$ belongs to D_{-1}. By Theorem 3.3.7, $b_n(\mathscr{F}) :=
\dim \mathrm{H}^n(X, \mathscr{F})$ is even for n even.

If k has characteristic 0, then $b_n(\mathscr{F})$ is even for all n. To see this, we may assume $k = \mathbb{C}$, X connected, and $\mathscr{F}$ simple. Let G be the monodromy group of $\mathscr{F}$ and let $f \colon Y \to X$ be the corresponding Galois étale cover. Then

$$\mathrm{H}^n(X, \mathscr{F}) \simeq \mathrm{H}^n(Y, f^*\mathscr{F})^G \simeq (\mathrm{H}^n(Y, \overline{\mathbb{Q}}_\ell) \otimes_{\overline{\mathbb{Q}}_\ell} V)^G,$$

where V is the representation of G corresponding to $\mathscr{F}$. Thus $b_n(\mathscr{F})$ is the multiplicity of $V^\vee$ in the representation $\mathrm{H}^n(Y, \overline{\mathbb{Q}}_\ell)$ of G. Since the complex representation $\mathrm{H}^n(Y(\mathbb{C}), \mathbb{C})$ of G has a real structure $\mathrm{H}^n(Y(\mathbb{C}), \mathbb{R})$, it admits a G-invariant nondegenerate symmetric bilinear form. In other words, it is 1-self-dual. The same holds for $\mathrm{H}^n(Y, \overline{\mathbb{Q}}_\ell)$. Therefore, the multiplicity of $V^\vee$, which is -1-self-dual, is necessarily even (cf. [44, Section 13.2, Theorem 31], [4, Proposition II.6.6 (i), (ii), (iii)]).

By contrast, if k has characteristic 2 or 3, then $b_n(\mathscr{F})$ may be odd for n odd, as shown by the following counterexample. Let E be a supersingular elliptic curve over k and let G be its automorphism group. Let $X' \to X$ be a finite étale cover of connected projective smooth curves over k of Galois group G, which exists by [36, Theorem 7.4], as explained in [35, Section 3]. Let $f \colon Y = (X' \times E)/G \to X$ be the projection, where G acts diagonally on $X' \times E$. Then $\mathscr{F} = \mathrm{R}^1 f_* \overline{\mathbb{Q}}_\ell$ is a -1-self-dual simple lisse sheaf of rank 2 on X, of monodromy G. Note that $f^* \mathscr{F}$ is a -1-self-dual simple lisse sheaf on Y of monodromy G, since $\pi_1(Y)$ maps onto $\pi_1(X)$. We claim that $b_1(f^* \mathscr{F})$ and $b_1(\mathscr{F})$ are not of the same parity. Indeed, consider the Leray spectral sequence for $(f, f^* \mathscr{F})$:

$$E_2^{pq} = \mathrm{H}^p(X, R^q f_* f^* \mathscr{F}) \Rightarrow \mathrm{H}^{p+q}(Y, f^* \mathscr{F}).$$

By the projection formula,

$$f_* f^* \mathscr{F} \simeq f_* \overline{\mathbb{Q}}_\ell \otimes \mathscr{F} \simeq \mathscr{F}, \quad \mathrm{R}^1 f_* f^* \mathscr{F} \simeq \mathrm{R}^1 f_* \overline{\mathbb{Q}}_\ell \otimes \mathscr{F} = \mathscr{F} \otimes \mathscr{F},$$

so we have an exact sequence

$$0 \to \mathrm{H}^1(X, \mathscr{F}) \to \mathrm{H}^1(Y, f^* \mathscr{F}) \to \mathrm{H}^0(X, \mathscr{F} \otimes \mathscr{F}) \to E_2^{20} = 0.$$

Since $\dim \mathrm{H}^0(X, \mathscr{F} \otimes \mathscr{F}) = 1$, we get $b_1(f^* \mathscr{F}) = b_1(\mathscr{F}) + 1$. (That the Leray spectral sequence degenerates at E_2 also follows from a general theorem of Deligne [10].) By the Grothendieck-Ogg-Shafarevich formula, we have $b_1(\mathscr{F}) = (2g-2)\mathrm{rk}(\mathscr{F}) = 4g-4$ is even, where g is the genus of X. It follows that $b_1(f^* \mathscr{F}) = 4g-3$ is odd.

4 Symmetry in Grothendieck groups

In Section 3, we studied the behavior of σ-self-dual pure perverse sheaves under operations that preserve purity. Mixed Hodge theory suggests that one may expect results for more general operations in the mixed case. This section confirms such expectations in a weak sense, by working in Grothendieck groups. We work over a finite field $k = \mathbb{F}_q$. In Subsection 4.1, we review operations on Grothendieck groups. In Subsection 4.2, we define certain subgroups of the Grothendieck groups and state the main result of this section (Theorem 4.2.5), which says that these subgroups are preserved by Grothendieck's six operations, and contains the finite field case of Theorem 1.9. The proof is a bit involved and is given in Subsection 4.3.

4.1 Operations on Grothendieck groups

In this subsection, we review Grothendieck groups and operations on them. The six operations are easily defined. The action of the middle extension functor $f_{!*}$ on Grothendieck groups is more subtle, and we justify our definition with the help of purity.

Construction 4.1.1 (Six operations on Grothendieck groups). Let X be a Deligne-Mumford stack of finite presentation over a field. We let $\mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$ denote the Grothendieck group of $\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$, which is a free Abelian group generated by the isomorphism classes of simple perverse $\overline{\mathbb{Q}}_\ell$-sheaves. For an object A of $\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$, we let $[A]$ denote its class in $\mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$. The usual operations on derived categories induce maps between Grothendieck groups. More precisely, for a morphism $f \colon X \to Y$ of Deligne-Mumford stacks of finite presentation over a field, we have $\mathbb{Z}$-(bi)linear maps

$$- \otimes -, \ \mathscr{H}om(-, -) \colon \mathrm{K}(X, \overline{\mathbb{Q}}_\ell) \times \mathrm{K}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}(X, \overline{\mathbb{Q}}_\ell), \qquad D_X \colon \mathrm{K}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}(X, \overline{\mathbb{Q}}_\ell),$$
$$- \boxtimes - \colon \mathrm{K}(X, \overline{\mathbb{Q}}_\ell) \times \mathrm{K}(Y, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}(X \times Y, \overline{\mathbb{Q}}_\ell),$$
$$f^*, f^! \colon \mathrm{K}(Y, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}(X, \overline{\mathbb{Q}}_\ell), \qquad f_*, f_! \colon \mathrm{K}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}(Y, \overline{\mathbb{Q}}_\ell).$$

Tensor product $(- \otimes -)$ endows $\mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$ with a ring structure. The map f^* is a ring homomorphism.

The Grothendieck ring is equipped with the structure of a λ-ring as follows. Readers not interested in this structure may skip this part as it is not used in the proof of the theorems in the introduction. For $m \geq 0$, we have a map

$$(-)^{\boxtimes m} \colon \mathrm{K}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}([X^m/\mathfrak{S}_m], \overline{\mathbb{Q}}_\ell),$$

which preserves multiplication and satisfies $(n[A])^{\boxtimes m} = n^m [A]^{\boxtimes m}$ (with the convention $0^0 = 1$) for $n \geq 0$ and $(-[A])^{\boxtimes m} = (-1)^m [\mathscr{S}] \otimes [A]^{\boxtimes m}$, where $\mathscr{S}$ is the lisse sheaf of rank 1 on $[X^m/\mathfrak{S}_m]$ given by the sign character $\mathfrak{S}_m \to \overline{\mathbb{Q}}_\ell^\times$. The maps $\lambda^m \colon \mathrm{K}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$ given by $\lambda^m(x) = (-1)^m p_* \Delta^*(-x)^{\boxtimes m}$, where $\Delta \colon X \times B\mathfrak{S}_m \to [X^m/\mathfrak{S}_m]$ is the diagonal morphism and $p \colon X \times B\mathfrak{S}_m \to X$ is the projection, endow $\mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$ with the structure of a special λ-ring. The map f^* is a λ-ring homomorphism. We refer the reader to [18, Section 4] and [2] for the definitions of special λ-ring and λ-ring homomorphism.

Remark 4.1.2. For a separated quasi-finite morphism $f \colon X \to Y$, the functor $f_{!*} \colon \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(Y, \overline{\mathbb{Q}}_\ell)$ is *not exact* in general. There exists a unique homomorphism $f_{!*} \colon \mathrm{K}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}(Y, \overline{\mathbb{Q}}_\ell)$ such that $f_{!*}[A] = [f_{!*}A]$ for A perverse *semisimple*. As we shall see in Lemma 4.1.8, over a finite field this identity also holds for A pure perverse.

We note the following consequence of Lemma 2.2.11 (applied to semisimple perverse sheaves).

Lemma 4.1.3. *Let $(X_\alpha)_{\alpha \in I}$ be a finite Zariski open covering of X, and let $A \in \mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$. Then*

$$\sum_{\substack{J \subseteq I \\ \#J \text{ even}}} j_{J!*} j_J^* A = \sum_{\substack{J \subseteq I \\ \#J \text{ odd}}} j_{J!*} j_J^* A,$$

where $j_J \colon \bigcap_{\alpha \in J} X_\alpha \to X$ is the open immersion.

In the rest of this section we work over a finite field $k = \mathbb{F}_q$. We first recall the following injectivity, which will be used in the proof of Corollary 4.2.10.

Lemma 4.1.4. *The homomorphism $\mathrm{K}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Map}(\coprod_{m \geq 1} |X(\mathbb{F}_{q^m})|, \overline{\mathbb{Q}}_\ell)$ sending A to $x \mapsto \mathrm{tr}(\mathrm{Frob}_x \mid A_{\bar{x}})$ is injective.*

As in [30, Théorème 1.1.2], this injectivity follows from Chebotarev's density theorem [43, Theorem 7], which extends to the case of Deligne-Mumford stacks as follows.

Lemma 4.1.5. *Let $Y \to X$ be a Galois étale cover of Galois group G of irreducible Deligne-Mumford stacks of dimension d of finite presentation over $\mathbb{F}_q$. Let $R \subseteq G$ be a subset stable under conjugation. Then*

$$\lim_{T \to (q^{-d})^-} \sum_{m \geq 1} \sum_x \frac{1}{\#\mathrm{Aut}(x)} \frac{T^m}{m} \Big/ \log \frac{1}{T - q^{-d}} = \#R/\#G,$$

where x runs through isomorphism classes of $X(\mathbb{F}_{q^m})$ such that the image F_x of Frob_x in G (well-defined up to conjugation) lies in R.

Proof. For a character $\chi \colon G \to \overline{\mathbb{Q}}_\ell$ of a $\overline{\mathbb{Q}}_\ell$-representation of G, consider the L-series

$$L(X, \iota\chi, T) = L_\iota(X, \mathscr{F}_\chi, T) = \exp\left(\sum_{m \geq 1} \sum_{x \in |X(\mathbb{F}_{q^m})|} \frac{\iota\chi(F_x)}{\#\mathrm{Aut}(x)} \frac{T^m}{m} \right)$$

associated to the corresponding lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}_\chi$ on X [46, Definition 4.1]. The series $L(X, \iota\chi, T)$ converges absolutely for $|T| < q^{-d}$ and extends to a rational function

$$\iota \prod_i \det(1 - T\mathrm{Frob}_q \mid \mathrm{H}_c^i(X_{\overline{\mathbb{F}}_q}, \mathscr{F}_\chi))^{(-1)^{i+1}}$$

by [46, Theorem 4.2]. As $\mathrm{H}_c^{2d}(X_{\overline{\mathbb{F}}_q}, \mathscr{F}_\chi) \simeq \mathrm{H}^0(U_{\overline{\mathbb{F}}_q}, \mathscr{F}_\chi^\vee)^\vee(-d)$ for a dense open substack U of X such that U_{red} is regular, $-\mathrm{ord}_{T=q^{-d}} L(X, \iota\chi, T) = \dim \mathrm{H}^0(U, \mathscr{F}_\chi^\vee)$ is the multiplicity of the identity character in χ, so that

$$\lim_{T \to (q^{-d})^-} \sum_{m \geq 1} \sum_{x \in |X(\mathbb{F}_{q^m})|} \frac{\iota\chi(F_x)}{\#\mathrm{Aut}(x)} \frac{T^m}{m} \Big/ \log \frac{1}{T - q^{-d}} = \sum_{g \in G} \iota\chi(g)/\#G.$$

This equality extends to an arbitrary class function $\chi \colon G \to \overline{\mathbb{Q}}_\ell$. It then suffices to take χ to be the characteristic function of R. $\qquad\square$

Next we discuss purity.

Notation 4.1.6. For $w \in \mathbb{R}$, we let $\mathrm{K}_\iota^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$ denote the subgroup generated by perverse sheaves ι-pure of weight w on X. We put $\mathrm{K}_\iota^{\mathbb{Z}}(X, \overline{\mathbb{Q}}_\ell) := \bigoplus_{w \in \mathbb{Z}} \mathrm{K}_\iota^w(X, \overline{\mathbb{Q}}_\ell)$.

The group $\mathrm{K}_\iota^w(X, \overline{\mathbb{Q}}_\ell)$ is a free Abelian group generated by the isomorphism classes of simple perverse sheaves ι-pure of weight w on X. We have $\bigoplus_{w \in \mathbb{R}} \mathrm{K}_\iota^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$ and the λ-subring $\mathrm{K}_\iota^{\mathbb{Z}}(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$ is stable under Grothendieck's six operations and duality. For $w \in \mathbb{Z}$, the group $\mathrm{K}^w(X, \overline{\mathbb{Q}}_\ell) := \bigcap_\iota \mathrm{K}_\iota^w(X, \overline{\mathbb{Q}}_\ell)$ is a free Abelian group generated by the isomorphism classes of perverse sheaves pure of weight w on X.

Remark 4.1.7. In fact, we have $\bigoplus_{w \in \mathbb{R}} \mathrm{K}_\iota^w(X, \overline{\mathbb{Q}}_\ell) = \mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$, as every $\overline{\mathbb{Q}}_\ell$-sheaf on X is ι-mixed by Lafforgue's theorem [28, Corollaire VII.8] mentioned in Remark 3.1.6 (2).

For a subset $I \subseteq \mathbb{R}$, we let $\mathrm{Perv}_\iota^I(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ denote the full subcategory of perverse sheaves ι-mixed of weights contained in I. Lemmas 4.1.8 and 4.1.9 below, which justify the definition of the map $f_{!*}$ in Remark 4.1.2, are taken from [50, Lemme 2.9, Corollaire 2.10].

Lemma 4.1.8. *Let $f \colon X \to Y$ be a separated quasi-finite morphism. For $w \in \mathbb{R}$, the functor*

$$f_{!*} \colon \mathrm{Perv}_\iota^{\{w, w+1\}}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}_\iota^{\{w, w+1\}}(Y, \overline{\mathbb{Q}}_\ell)$$

is exact. In particular, $f_{!*}[A] = [f_{!*}A]$ for A in $\mathrm{Perv}_\iota^{\{w, w+1\}}(X, \overline{\mathbb{Q}}_\ell)$.

Proof. As the assertion is local for the étale topology on Y and trivial for f proper quasi-finite, we may assume that f is an open immersion. Let $i \colon Z \to Y$ be the closed immersion complementary to f. We proceed by induction on the dimension d of Z. Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be a short exact sequence in $\mathrm{Perv}_\iota^{\{w, w+1\}}(X, \overline{\mathbb{Q}}_\ell)$. As in Gabber's proof of his theorem on independence on ℓ for middle extensions [16, Theorem 3], up to shrinking Z, we may assume that Z is smooth equidimensional and that $\mathscr{H}^n i^* \mathrm{R} f_* A_j$ is lisse for every j and every n. It follows that the distinguished triangle

$$i_* \mathrm{R} i^! f_{!*} A_j \to f_{!*} A_j \to \mathrm{R} f_* A_j \to$$

induces isomorphisms $f_{!*} A_j \xrightarrow{\sim} {}^P\tau^{\le -d-1} \mathrm{R} f_* A_j$ and ${}^P\tau^{\ge -d} \mathrm{R} f_* A_j \xrightarrow{\sim} i_* \mathrm{R} i^! f_{!*} A_j[1]$ for every j. Here P denotes the t-structure obtained by gluing $(\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell), 0)$ and the canonical t-structure on $\mathrm{D}_c^b(Z, \overline{\mathbb{Q}}_\ell)$. Thus ${}^P\mathrm{R}^{-d-1} f_* A_j \simeq i_* \mathscr{H}^{-d-1} i^* f_{!*} A_j$ has punctual ι-weights $\le w - d$, while ${}^P\mathrm{R}^{-d} f_* A_j \simeq i_* \mathscr{H}^{-d+1} \mathrm{R} i^! f_{!*} A_j$ has punctual ι-weights $\ge w - d + 1$. Therefore, the morphism ${}^P\mathrm{R}^{-d-1} f_* A_3 \to {}^P\mathrm{R}^{-d} f_* A_1$ is zero. Applying Lemma 4.1.9 below, we get a distinguished triangle

$${}^P\tau^{\le -d-1} \mathrm{R} f_* A_1 \to {}^P\tau^{\le -d-1} \mathrm{R} f_* A_2 \to {}^P\tau^{\le -d-1} \mathrm{R} f_* A_3 \to .$$

Taking perverse cohomology sheaves, we get the exactness of the sequence

$$0 \to f_{!*} A_1 \to f_{!*} A_2 \to f_{!*} A_3 \to 0.$$

$\square$

Lemma 4.1.9. *Let P be a t-structure on a triangulated category $\mathscr{D}$ and let $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1]$ be a distinguished triangle such that ${}^P\mathrm{H}^0 c \colon {}^P\mathrm{H}^0 C \to {}^P\mathrm{H}^1 A$ is zero. Then there exists a unique nine-diagram of the form*

(4.1.1)

$$
\begin{array}{ccccccc}
{}^P\tau^{\le 0}A & \xrightarrow{{}^P\tau^{\le 0}a} & {}^P\tau^{\le 0}B & \xrightarrow{{}^P\tau^{\le 0}b} & {}^P\tau^{\le 0}C & \xrightarrow{c_0} & ({}^P\tau^{\le 0}A)[1] \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle u} \quad (*) & & \downarrow \\
A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & A[1] \\
\downarrow & & \downarrow & & \downarrow \quad (**) & & \downarrow{\scriptstyle v} \\
{}^P\tau^{\ge 1}A & \xrightarrow{{}^P\tau^{\ge 1}a} & {}^P\tau^{\ge 1}B & \xrightarrow{{}^P\tau^{\ge 1}b} & {}^P\tau^{\ge 1}C & \xrightarrow{c_1} & ({}^P\tau^{\ge 1}A)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
({}^P\tau^{\le 0}A)[1] & \xrightarrow{({}^P\tau^{\le 0}a)[1]} & ({}^P\tau^{\le 0}B)[1] & \xrightarrow{({}^P\tau^{\le 0}b)[1]} & ({}^P\tau^{\le 0}C)[1] & \xrightarrow{c_0[1]} & ({}^P\tau^{\le 0}A)[2],
\end{array}
$$

where the columns are the canonical distinguished triangles.

By a *nine-diagram* in a triangulated category (cf. [3, Proposition 1.1.11]), we mean a diagram

$$
\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\
\downarrow & & \downarrow & & \downarrow & & \vdots \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \\
\downarrow & & \downarrow & & \downarrow & & \vdots \\
A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & A''[1] \\
\downarrow & & \downarrow & & \downarrow & {\scriptstyle -} & \vdots \\
A[1] \dashrightarrow & & B[1] \dashrightarrow & & C[1] \dashrightarrow & & A[2],
\end{array}
$$

in which the square marked with "$-$" is anticommutative and all other squares are commutative, the dashed arrows are induced from the solid ones by translation, and the rows and columns in solid arrows are distinguished triangles.

Proof. First note that vcu is the image of ${}^P\mathrm{H}^0 c$ under the isomorphism

$$
\mathrm{Hom}({}^P\mathrm{H}^0 C, {}^P\mathrm{H}^1 A) \xrightarrow{\sim} \mathrm{Hom}({}^P\tau^{\leq 0}C, ({}^P\tau^{\geq 1}A)[1]).
$$

Hence $vcu = 0$. Moreover, $\mathrm{Hom}({}^P\tau^{\leq 0}C, {}^P\tau^{\geq 1}A) = 0$. Thus by [3, Proposition 1.1.9], there exist a unique c_0 making $(*)$ commutative and a unique c_1 making $(**)$ commutative. This proves the uniqueness of (4.1.1). It remains to show that (4.1.1) thus constructed is a nine-diagram. To do this, we extend the upper left square of (4.1.1) into a nine-diagram

$$
(4.1.2) \quad
\begin{array}{ccccccc}
{}^P\tau^{\leq 0}A & \xrightarrow{{}^P\tau^{\leq 0}a} & {}^P\tau^{\leq 0}B & \longrightarrow & C_0 & \longrightarrow & ({}^P\tau^{\leq 0}A)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & A[1] \\
\downarrow & & \downarrow & & \downarrow & {\scriptstyle (***)} & \downarrow \\
{}^P\tau^{\geq 1}A & \xrightarrow{{}^P\tau^{\geq 1}a} & {}^P\tau^{\geq 1}B & \longrightarrow & C_1 & \longrightarrow & ({}^P\tau^{\geq 1}A)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
({}^P\tau^{\leq 0}A)[1] & \xrightarrow{({}^P\tau^{\leq 0}a)[1]} & ({}^P\tau^{\leq 0}B)[1] & \longrightarrow & C_0[1] & \longrightarrow & ({}^P\tau^{\leq 0}A)[2].
\end{array}
$$

By the first and third rows of (4.1.2), $C_0 \in {}^P\mathscr{D}^{\leq 0}$ and $C_1 \in {}^P\mathscr{D}^{\geq 0}$. Taking ${}^P\mathrm{H}^0$ of $(***)$, we obtain a commutative diagram

$$
\begin{array}{ccc}
{}^P\mathrm{H}^0 C & \xrightarrow{0} & {}^P\mathrm{H}^1 A \\
{\scriptstyle e}\downarrow & & \| \\
{}^P\mathrm{H}^0 C_1 & \xrightarrow{d} & {}^P\mathrm{H}^1 A,
\end{array}
$$

where e is an epimorphism and d is a monomorphism. Thus ${}^P\mathrm{H}^0 C_1 = 0$, so that $C_1 \in {}^P\mathscr{D}^{\geq 1}$. Further applying [3, Proposition 1.1.9], we may identify (4.1.2) with (4.1.1). □

4.2 Statement and consequences of main result

In this subsection, we define a subgroup $\mathrm{K}_{\iota,\sigma}$ of the Grothendieck group and state its preservation by Grothendieck's six operations (Theorem 4.2.5), which contains the finite field case of Theorem 1.9. We then give a number of consequences and discuss the relationship with independence of ℓ and Laumon's theorem on Euler characteristics.

Definition 4.2.1 ($\mathrm{K}_{\iota,\sigma}$)**.** We define $\mathrm{K}_{\iota,\sigma}^w(X,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_\iota^w(X,\overline{\mathbb{Q}}_\ell)$ (resp. $\mathrm{K}_{\iota,\mathrm{sd}}^w(X,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_\iota^w(X,\overline{\mathbb{Q}}_\ell)$), for $w \in \mathbb{Z}$, to be the subgroup generated by $[B]$, for B perverse, ι-pure of weight w, and $(-1)^w\sigma$-self-dual

(resp. self-dual) with respect to $K_X(-w)$. We put

$$\mathrm{K}_{\iota,\sigma}(X,\overline{\mathbb{Q}}_\ell) = \bigoplus_{w \in \mathbb{Z}} \mathrm{K}_{\iota,\sigma}^w(X,\overline{\mathbb{Q}}_\ell) \quad (\text{resp. } \mathrm{K}_{\iota,\mathrm{sd}}(X,\overline{\mathbb{Q}}_\ell) = \bigoplus_{w \in \mathbb{Z}} \mathrm{K}_{\iota,\mathrm{sd}}^w(X,\overline{\mathbb{Q}}_\ell)).$$

We define the *twisted dualizing map*

$$\overline{D}_{\iota,X} \colon \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell) \to \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell)$$

to be the direct sum of the group automorphisms $\overline{D}_{\iota,X}^w \colon \mathrm{K}_\iota^w(X,\overline{\mathbb{Q}}_\ell) \to \mathrm{K}_\iota^w(X,\overline{\mathbb{Q}}_\ell)$ sending $[A]$ to $[(D_X A)(-w)]$. We let $\mathrm{K}_{\iota,\mathrm{d}}^w(X,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_\iota^w(X,\overline{\mathbb{Q}}_\ell)^2$ denote the graph of $\overline{D}_{\iota,X}^w$. We put

$$\mathrm{K}_{\iota,\mathrm{d}}(X,\overline{\mathbb{Q}}_\ell) = \bigoplus_{w \in \mathbb{Z}} \mathrm{K}_{\iota,\mathrm{d}}^w(X,\overline{\mathbb{Q}}_\ell).$$

We note that $\overline{D}_{\iota,X}\overline{D}_{\iota,X} = \mathrm{id}$, and $\mathrm{K}_{\iota,\mathrm{d}}(X,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell)^2$ is the graph of $\overline{D}_{\iota,X}$.

Example 4.2.2. For $X = \mathrm{Spec}(\mathbb{F}_q)$, an element $A \in \mathrm{K}(X,\overline{\mathbb{Q}}_\ell)$ is determined by the determinant

$$P(A,T) := \det(1 - T\mathrm{Frob}_q \mid A_{\overline{\mathbb{F}}_q}) \in \overline{\mathbb{Q}}_\ell(T).$$

Assume $A \in \mathrm{K}_\iota^w(X,\overline{\mathbb{Q}}_\ell)$, $w \in \mathbb{Z}$. For $\lambda \in \overline{\mathbb{Q}}_\ell$ satisfying $|\iota(\lambda)| = q^{w/2}$, we let m_λ and m'_λ denote the order at $T = 1/\lambda$ of $P(A,T)$ and $P(\overline{D}_{\iota,X}A,T)$, respectively. We then have $m_\lambda = m'_{q^w/\lambda}$; in other words,

$$(4.2.1) \qquad\qquad\qquad \iota P(A,T) = \bar\iota P(\overline{D}_{\iota,X}A,T).$$

We have $\mathrm{K}_{\iota,(-1)^{w+1}}^w(X,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_{\iota,(-1)^w}^w(X,\overline{\mathbb{Q}}_\ell) = \mathrm{K}_{\iota,\mathrm{sd}}^w(X,\overline{\mathbb{Q}}_\ell)$. The following conditions are equivalent:

(1) $A \in \mathrm{K}_{\iota,(-1)^w}^w(X,\overline{\mathbb{Q}}_\ell) = \mathrm{K}_{\iota,\mathrm{sd}}^w(X,\overline{\mathbb{Q}}_\ell)$;

(2) $m_\lambda = m_{q^w/\lambda}$ for all λ;

(3) $\iota P(A,T) \in \mathbb{R}(T)$.

Furthermore, the following conditions are equivalent:

(1) $A \in \mathrm{K}_{\iota,(-1)^{w+1}}^w(X,\overline{\mathbb{Q}}_\ell)$;

(2) $m_\lambda = m_{q^w/\lambda}$ for all λ and $m_{q^{w/2}}$, $m_{-q^{w/2}}$ are even;

(3) $\iota P(A,T) \in \mathbb{R}(T)$, the rank $b = \sum_\lambda m_\lambda \in \mathbb{Z}$ of A is even, and $\det(\mathrm{Frob}_q \mid A_{\overline{\mathbb{F}}_q}) = q^{wb/2}$.

Remark 4.2.3. Let $w \in \mathbb{Z}$.

(1) By definition, $\mathrm{K}_{\iota,\sigma}(X,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}(X,\overline{\mathbb{Q}}_\ell)$ (resp. $\mathrm{K}_{\iota,\mathrm{sd}}(X,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}(X,\overline{\mathbb{Q}}_\ell)$) is generated by the image of $\mathrm{D}_{\iota,\sigma}^w(X,\overline{\mathbb{Q}}_\ell)$ (resp. $\mathrm{D}_{\iota,\mathrm{sd}}^w(X,\overline{\mathbb{Q}}_\ell)$), as defined in Definition 3.2.2. Moreover, $\mathrm{K}_{\iota,\mathrm{d}}(X,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}(X,\overline{\mathbb{Q}}_\ell)^2$ is generated by the image of $\mathrm{D}_{\iota,\mathrm{d}}^w(X,\overline{\mathbb{Q}}_\ell)$.

(2) By Remark 2.2.7, in the definition of $\mathrm{K}_{\iota,\sigma}^w$, one may restrict to semisimple perverse sheaves. This also holds for $\mathrm{K}_{\iota,\mathrm{sd}}^w$. Thus $\mathrm{K}_{\iota,\sigma}^w(X,\overline{\mathbb{Q}}_\ell)$ (resp. $\mathrm{K}_{\iota,\mathrm{sd}}^w(X,\overline{\mathbb{Q}}_\ell)$) is generated by $[A] + [(D_X A)(-w)]$ for A simple perverse ι-pure of weight w, and $[B]$ for B simple perverse ι-pure of weight w and $(-1)^w\sigma$-self-dual (resp. self-dual) with respect to $K_X(-w)$.

(3) By Proposition 2.2.3, we have $\mathrm{K}_{\iota,\mathrm{sd}}^w(X,\overline{\mathbb{Q}}_\ell) = \mathrm{K}_{\iota,1}^w(X,\overline{\mathbb{Q}}_\ell) + \mathrm{K}_{\iota,-1}^w(X,\overline{\mathbb{Q}}_\ell)$.

(4) $\mathrm{K}_{\iota,\mathrm{d}}(X,\overline{\mathbb{Q}}_\ell)$ is generated by $([B], [(D_X A)(-w)])$ for B simple perverse ι-pure of weight w. Thus $\mathrm{K}_{\iota,\mathrm{sd}}(X,\overline{\mathbb{Q}}_\ell) = \Delta^{-1}(\mathrm{K}_{\iota,\mathrm{d}}(X,\overline{\mathbb{Q}}_\ell))$, where $\Delta \colon \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell) \to \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell)^2$ is the diagonal embedding. In other words, for $A \in \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell)$, A belongs to $\mathrm{K}_{\iota,\mathrm{sd}}(X,\overline{\mathbb{Q}}_\ell)$ if and only if $A = \overline{D}_{\iota,X}A$.

(5) For $A \in \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell)$ and $n \in \mathbb{Z}$, we have $\overline{D}_{\iota,X}(A(n)) = (\overline{D}_{\iota,X}A)(n)$. For $A \in \mathrm{K}_{\iota,\sigma}^w(X,\overline{\mathbb{Q}}_\ell)$, we have $A(n) \in \mathrm{K}_{\iota,\sigma}^{w-2n}(X,\overline{\mathbb{Q}}_\ell)$.

(6) Let A be a perverse sheaf on X, ι-pure of weight w. By Corollary 2.2.6, $[A] \in \mathrm{K}_{\iota,\sigma}^w(X,\overline{\mathbb{Q}}_\ell)$ if and only if the semisimplification of A is $(-1)^w\sigma$-self-dual with respect to $K_X(-w)$. Similar results hold for $\mathrm{K}_{\iota,\mathrm{sd}}$ and $\mathrm{K}_{\iota,\mathrm{d}}$.

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Remark 4.2.4. Although we do not need it in the sequel, let us give two more descriptions of $\mathrm{K}_{\iota,\sigma}^w$. In our definition of $\mathrm{K}_{\iota,\sigma}^w$, we consider self-dual perverse sheaves B only and do not take the bilinear form $B \simeq D_X B(-w)$ as part of the data. Alternatively, we can also include the form and consider the Grothendieck group GS of symmetric spaces in $\mathrm{Perv}_\iota^{\{w\}}$ (equipped with the duality $D_X(-w)$ and the evaluation map modified by a factor of $(-1)^w \sigma$). The Grothendieck-Witt group GW is a quotient of GS, equipped with a homomorphism $\mathrm{GW} \to \mathrm{K}_\iota^w$. We refer the reader to [37, page 280], [42, Section 2.2] for the definition of the Grothendieck-Witt group of an Abelian category with duality (generalizing Quillen's definition [38, (5.1)] for representations). In our situation, the canonical maps

$$\mathrm{GS} \to \mathrm{GW} \to \mathrm{K}_{\iota,\sigma}^w$$

are isomorphisms. In fact, by definition, $\mathrm{K}_{\iota,\sigma}^w$ is the image of GW. Moreover, since we work over the algebraically closed field $\overline{\mathbb{Q}}_\ell$, symmetric spaces with isomorphic underlying objects are isometric [37, 3.4 (3)].

We now consider preservation of $\mathrm{K}_{\iota,\sigma}$ and $\mathrm{K}_{\iota,\mathrm{d}}$ by cohomological operations. The preservation of $\mathrm{K}_{\iota,\mathrm{d}}$ is equivalent to the commutation with the twisted dualizing map $\overline{D}_\iota$. The main result of this section is the following generalization of Theorem 1.9.

Theorem 4.2.5. *Let $f\colon X \to Y$ be a morphism between Deligne-Mumford stacks of finite inertia and finite presentation over $\mathbb{F}_q$. Then Grothendieck's six operations induce maps*

$$- \otimes -, \ \mathscr{H}om(-,-)\colon \mathrm{K}_{\iota,\sigma}(X,\overline{\mathbb{Q}}_\ell) \times \mathrm{K}_{\iota,\sigma'}(X,\overline{\mathbb{Q}}_\ell) \to \mathrm{K}_{\iota,\sigma\sigma'}(X,\overline{\mathbb{Q}}_\ell),$$

$$f^*, f^!\colon \mathrm{K}_{\iota,\sigma}(Y,\overline{\mathbb{Q}}_\ell) \to \mathrm{K}_{\iota,\sigma}(X,\overline{\mathbb{Q}}_\ell), \qquad f_*, f_!\colon \mathrm{K}_{\iota,\sigma}(X,\overline{\mathbb{Q}}_\ell) \to \mathrm{K}_{\iota,\sigma}(Y,\overline{\mathbb{Q}}_\ell).$$

Moreover, Grothendieck's six operations on $\mathrm{K}_\iota^{\mathbb{Z}}$ commute with the twisted dualizing map $\overline{D}_\iota$.

The proof will be given in the next section. We now make a list of pure cases in which the preservation has already been established. Most items of the list below follow from Remark 3.2.8.

Remark 4.2.6 (Preservation of $\mathrm{K}_{\iota,\sigma}$, pure cases)**.** Let $f\colon X \to Y$ be a morphism, $w,w' \in \mathbb{Z}$.

(1) D_X carries $\mathrm{K}_{\iota,\sigma}^w(X,\overline{\mathbb{Q}}_\ell)$ to $\mathrm{K}_{\iota,\sigma}^{-w}(X,\overline{\mathbb{Q}}_\ell)$.

(2) If f is smooth, then f^* preserves $\mathrm{K}_{\iota,\sigma}^w$.

(3) If f is an open immersion, then $f_{!*}$ preserves $\mathrm{K}_{\iota,\sigma}^w$.

(4) The functor $- \boxtimes -$ carries $\mathrm{K}_{\iota,\sigma}^w(X,\overline{\mathbb{Q}}_\ell) \times \mathrm{K}_{\iota,\sigma'}^{w'}(X',\overline{\mathbb{Q}}_\ell)$ to $\mathrm{K}_{\iota,\sigma\sigma'}^{w+w'}(X \times X',\overline{\mathbb{Q}}_\ell)$, and the functor $(-)^{\boxtimes m}$, $m \geq 0$, carries $\mathrm{K}_{\iota,\sigma}^w(X,\overline{\mathbb{Q}}_\ell)$ to $\mathrm{K}_{\iota,\sigma^m}^{mw}([X^m/\mathfrak{S}_m],\overline{\mathbb{Q}}_\ell)$. For the latter we used Proposition 3.2.14.

(5) Assume that f is a closed immersion and let $A \in \mathrm{K}(X,\overline{\mathbb{Q}}_\ell)$. Then $A \in \mathrm{K}_{\iota,\sigma}^w(X,\overline{\mathbb{Q}}_\ell)$ if and only if $f_* A \in \mathrm{K}_{\iota,\sigma}^w(Y,\overline{\mathbb{Q}}_\ell)$.

(6) Assume that f is proper. If f is projective or Y has finite inertia, then f_* preserves $\mathrm{K}_{\iota,\sigma}^w$, by Proposition 3.2.12 and Theorem 3.2.3.

Similar properties hold for $\mathrm{K}_{\iota,\mathrm{d}}$.

The following Zariski local nature of $\mathrm{K}_{\iota,\sigma}^w$ will be used in the proof of Theorem 4.2.5. It follows from the Zariski local nature of σ-self-dual perverse sheaves (Proposition 2.2.1). It also follows from Remark 4.2.6 (3) and Lemma 4.1.3.

Remark 4.2.7 (Zariski local nature)**.** Let $(X_\alpha)_{\alpha \in I}$ be a Zariski open covering of X and let $A \in \mathrm{K}(X,\overline{\mathbb{Q}}_\ell)$. Then $A \in \mathrm{K}_{\iota,\sigma}^w(X,\overline{\mathbb{Q}}_\ell)$ if and only if $A \mid X_\alpha \in \mathrm{K}_{\iota,\sigma}^w(X_\alpha,\overline{\mathbb{Q}}_\ell)$ for every α. The same holds for $\mathrm{K}_{\iota,\mathrm{d}}$.

We now turn to consequences of Theorem 4.2.5. The ring part of the following two corollaries follow from the two assertions of Theorem 4.2.5 applied to a_X^* (recall $a_X\colon X \to \mathrm{Spec}(\mathbb{F}_q)$) and $- \otimes -$. For the λ-ring part, we apply Remark 4.2.6 (4) to the map $(-)^{\boxtimes m}$ and Theorem 4.2.5 to the maps Δ^* and p_* in the definition of λ^m in Construction 4.1.1.

Corollary 4.2.8. *Assume that X has finite inertia. Then $\mathrm{K}_{\iota,1}(X,\overline{\mathbb{Q}}_\ell)$ is a λ-subring of $\mathrm{K}(X,\overline{\mathbb{Q}}_\ell)$. In particular, $\mathrm{K}_{\iota,1}(X,\overline{\mathbb{Q}}_\ell)$ contains the class $[\overline{\mathbb{Q}}_\ell]$ of the constant sheaf $\overline{\mathbb{Q}}_\ell$ on X.*

Corollary 4.2.9. *Assume that X has finite inertia. Then $\overline{D}_{\iota,X}\colon \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell) \to \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell)$ is a λ-ring homomorphism. In particular, $\overline{D}_{\iota,X}[\overline{\mathbb{Q}}_\ell] = [\overline{\mathbb{Q}}_\ell]$.*

Another consequence of Theorem 4.2.5 is the following pointwise characterization of $\mathrm{K}_{\iota,\mathrm{d}}$ and $\mathrm{K}_{\iota,\mathrm{sd}}$. We let $\mathrm{K}_{\iota,\bar\iota}(X,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}(X,\overline{\mathbb{Q}}_\ell)^2$ (resp. $\mathrm{K}_{\iota,\mathbb{R}}(X,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}(X,\overline{\mathbb{Q}}_\ell)$) denote the subgroup consisting of elements (A,A') (resp. A) such that for every morphism $x\colon \mathrm{Spec}(\mathbb{F}_{q^m}) \to X$ and every geometric point $\bar x$ above x, we have

$$\iota\mathrm{tr}(\mathrm{Frob}_x, A_{\bar x}) = \bar\iota\mathrm{tr}(\mathrm{Frob}_x, A'_{\bar x}) \quad (\text{resp. } \iota\mathrm{tr}(\mathrm{Frob}_x, A_{\bar x}) \in \mathbb{R}).$$

The notation $\mathrm{K}_{\iota,\bar\iota}$ and $\mathrm{K}_{\iota,\mathbb{R}}$ will only be used in Corollary 4.2.10 and Remark 4.2.11 below.

Corollary 4.2.10. *Assume that X has finite inertia. Let $A \in \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell)$. Then for every $m \geq 1$, every morphism $x\colon \mathrm{Spec}(\mathbb{F}_{q^m}) \to X$ and every geometric point $\bar x$ above x, we have*

$$(4.2.2) \qquad\qquad \iota\mathrm{tr}(\mathrm{Frob}_x, A_{\bar x}) = \bar\iota\mathrm{tr}(\mathrm{Frob}_x, (\overline{D}_{\iota,X}A)_{\bar x}).$$

Moreover, $\mathrm{K}_{\iota,\mathrm{d}}(X,\overline{\mathbb{Q}}_\ell) = \mathrm{K}_{\iota,\bar\iota}(X,\overline{\mathbb{Q}}_\ell) \cap \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell)^2$. In particular, $\mathrm{K}_{\iota,\mathrm{sd}}(X,\overline{\mathbb{Q}}_\ell) = \mathrm{K}_{\iota,\mathbb{R}} \cap \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell)$.

Proof. By the second assertion of Theorem 4.2.5 applied to x^*, we have $\overline{D}_{\iota,\mathrm{Spec}(\mathbb{F}_{q^m})}x^*A = x^*\overline{D}_{\iota,X}A$. Thus (4.2.1) in Example 4.2.2 implies (4.2.2). It follows that $\mathrm{K}_{\iota,\mathrm{d}}(X,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_{\iota,\bar\iota}(X,\overline{\mathbb{Q}}_\ell) \cap \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell)^2$. The inclusion in the other direction follows from the injectivity of the homomorphism $\mathrm{K}(X,\overline{\mathbb{Q}}_\ell) \to \mathrm{Map}(\coprod_{m\geq 1}|X(\mathbb{F}_{q^m})|, \overline{\mathbb{Q}}_\ell)$ (Lemma 4.1.4). The last assertion of Corollary 4.2.10 follows from the second one. $\qquad\square$

Remark 4.2.11. Corollary 4.2.10 also follows from [25, Lemma 1.8.1 1), 4)], which in turn follows from Gabber's theorem on independence of ℓ for middle extensions [16, Theorem 3]. By Gabber's theorem on independence of ℓ for Grothendieck's six operations [16, Theorem 2] (see [52, 3.2] for a different proof and [52, Proposition 5.8] for the case of stacks), $\mathrm{K}_{\iota,\bar\iota}$ and $\mathrm{K}_{\iota,\mathbb{R}}$ in Corollary 4.2.10 are stable under the six operations. Thus the second assertion of Theorem 4.2.5 follows from Gabber's theorems on independence of ℓ. We will not use Gabber's theorems on independence of ℓ in our proof of Theorem 4.2.5.

Remark 4.2.12. The pointwise characterization of $\mathrm{K}_{\iota,\mathrm{sd}}$ in Corollary 4.2.10 does not extend to $\mathrm{K}_{\iota,\sigma}$. For instance, if X is regular and geometrically connected and if $f\colon E \to X$ is a family of elliptic curves with nonconstant j-invariant, then $\mathscr{F} = \mathrm{R}^1 f_*\overline{\mathbb{Q}}_\ell$ is a geometrically simple lisse $\overline{\mathbb{Q}}_\ell$-sheaf on X by [11, Lemme 3.5.5], so that $[\mathscr{F}] \in \mathrm{K}_{\iota,1}^1(X,\overline{\mathbb{Q}}_\ell)\backslash\mathrm{K}_{\iota,-1}^1(X,\overline{\mathbb{Q}}_\ell)$, but for every closed point x of X, we have $[\mathscr{F}_x] \in \mathrm{K}_{\iota,1}^1(x,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_{\iota,-1}^1(x,\overline{\mathbb{Q}}_\ell)$.

Remark 4.2.13. Let $f\colon X \to Y$ be as in Theorem 4.2.5 and let $\mathrm{I}(Y,\overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}(Y,\overline{\mathbb{Q}}_\ell)$ be the ideal generated by $[\overline{\mathbb{Q}}_\ell(1)]-[\overline{\mathbb{Q}}_\ell]$. A theorem of Laumon [29] ([23, Theorem 3.2] for the case of Deligne-Mumford stacks) states that $f_* \equiv f_!$ modulo $\mathrm{I}(Y,\overline{\mathbb{Q}}_\ell)$. This is equivalent to the congruence $D_Y f_* \equiv f_* D_X$ (and to $D_Y f_! \equiv f_! D_X$) modulo $\mathrm{I}(Y,\overline{\mathbb{Q}}_\ell)$. Thus the second assertion of Theorem 4.2.5 can be seen as a refinement of Laumon's theorem.

In the case of $\mathrm{K}_{\iota,\sigma}$, we have the following result on independence of (ℓ,ι). Let $\ell' \neq q$ be a prime number and let $\iota'\colon \overline{\mathbb{Q}}_{\ell'} \to \mathscr{C}$ be an embedding.

Corollary 4.2.14. *Assume that X has finite inertia. Let $A \in \mathrm{K}_\iota^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_\ell)$, $A' \in \mathrm{K}_{\iota'}^{\mathbb{Z}}(X,\overline{\mathbb{Q}}_{\ell'})$. Assume that A and A' are compatible in the sense that for every morphism $x\colon \mathrm{Spec}(\mathbb{F}_{q^m}) \to X$ and every geometric point $\bar x$ above x, we have*

$$\iota\mathrm{tr}(\mathrm{Frob}_x, A_{\bar x}) = \iota'\mathrm{tr}(\mathrm{Frob}_x, A'_{\bar x}).$$

Then A belongs to $\mathrm{K}_{\iota,\sigma}(X,\overline{\mathbb{Q}}_\ell)$ if and only if A' belongs to $\mathrm{K}_{\iota',\sigma}(X,\overline{\mathbb{Q}}_{\ell'})$.

Proof. Let $(X_\alpha)_{\alpha\in I}$ be a stratification of X. Then $A = \sum_{\alpha\in I} j_{\alpha!}j_\alpha^*A$ and $A' = \sum_{\alpha\in I} j_{\alpha!}j_\alpha^*A'$, where $j_\alpha\colon X_\alpha \to X$ is the immersion. Thus, by Theorem 4.2.5, up to replacing X by a stratum, we may assume that X is regular and A belongs to the subgroup generated by lisse $\overline{\mathbb{Q}}_\ell$-sheaves, that is, $A = \sum_{\mathscr{F}} n_{\mathscr{F}}[\mathscr{F}]$, where $\mathscr{F}$ runs over isomorphism classes of simple lisse $\overline{\mathbb{Q}}_\ell$-sheaves. For each $\mathscr{F}$ appearing in the decomposition, let $\mathscr{F}'$ be the companion of $\mathscr{F}$ [15] ([54] for the case of stacks), namely the simple lisse Weil $\overline{\mathbb{Q}}_{\ell'}$-sheaf such that

$$\iota\mathrm{tr}(\mathrm{Frob}_x, \mathscr{F}_{\bar x}) = \iota'\mathrm{tr}(\mathrm{Frob}_x, \mathscr{F}'_{\bar x})$$

for all x and $\bar x$ as above. Since $A' = \sum_{\mathscr{F}} n_{\mathscr{F}}[\mathscr{F}']$, each $\mathscr{F}'$ is an honest $\overline{\mathbb{Q}}_{\ell'}$-sheaf. By Corollary 4.2.10, we have $(\overline{D}\mathscr{F})' = \overline{D}(\mathscr{F}')$. Therefore, we may assume that $A = [\mathscr{F}]$ and $A' = [\mathscr{F}']$. In this case, the assertion follows from the symmetry criterion in terms of squares of Frobenius (Proposition 3.2.15). $\qquad\square$

4.3 Proof of main result

The situation of Theorem 4.2.5 is quite different from that of Gabber's theorem on independence of ℓ [16, Theorem 2]. In Gabber's theorem, the preservation by $-\otimes-$ and f^* is trivial and the preservation by $f_!$ follows from the Grothendieck trace formula. The key point of Gabber's theorem is thus the preservation by D_X. The preservation by middle extensions [16, Theorem 3] follows from the preservation by the six operations. In Theorem 4.2.5, the stability under each of the six operations is nontrivial, but the preservation by D_X and middle extensions is easy. To prove Theorem 4.2.5, we will first deduce that $f_!$ preserves $\mathrm{K}_{\iota,\sigma}$ and $\mathrm{K}_{\iota,\mathrm{d}}$ in an important special case from the preservation by middle extensions.
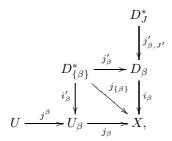
Proposition 4.3.1. *Let X be a regular Deligne-Mumford stack of finite presentation over $\mathbb{F}_q$ and let $D = \sum_{\alpha\in I} D_\alpha$ be a strict normal crossing divisor, with D_α regular. Assume that there exists a finite étale morphism $f\colon Y \to X$ such that $f^{-1}(D_\alpha)$ is defined globally by $t_\alpha \in \Gamma(Y, \mathscr{O}_Y)$ for all $\alpha \in I$. Let $\mathscr{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $U = X - D$, tamely ramified along D. Assume that $[\mathscr{F}] \in \mathrm{K}_\iota^{\mathbb{Z}}(U, \overline{\mathbb{Q}}_\ell)$. Then $\overline{D}_{\iota,X}[j_!\mathscr{F}] = j_!\overline{D}_{\iota,U}[\mathscr{F}]$ and, if $[\mathscr{F}] \in \mathrm{K}_{\iota,\sigma}(U, \overline{\mathbb{Q}}_\ell)$, $[j_!\mathscr{F}]$ belongs to $\mathrm{K}_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$, where $j\colon U \to X$ is the open immersion.*

[Added after publication: the assumption on the existence of Y is superfluous. The proof has been corrected accordingly.]

Proof. We will prove the case of $\mathrm{K}_{\iota,\sigma}$. The case of $\mathrm{K}_{\iota,\mathrm{d}}$ is similar.

We apply the construction of [11, 1.7.9, 1.7.10] to our setting as follows. For $J \subseteq I$, let $U_J = X - \bigcup_{\beta\in I-J} D_\beta$ and let $D_J^* = \bigcap_{\beta\in J} D_\beta \cap U_J$. Let $p_\beta\colon T_\beta \to D_{\{\beta\}}^*$ be the normal bundle of $D_{\{\beta\}}^*$ in $U_\beta = U_{\{\beta\}}$. If D_β is defined by $t_\beta = 0$, then for each locally constant constructible sheaf of sets $\mathscr{G}$ on U, tamely ramified along $D_{\{\beta\}}^*$, there exists an integer n invertible in $\mathbb{F}_q$ such that $\mathscr{G}$ extends to $\mathscr{G}'$ on the cover $U_\beta[t_\beta^{1/n}]$ of U_β, and we let $\mathscr{G}[D_{\{\beta\}}^*]$ denote the restriction of $\mathscr{G}'$ to $D_{\{\beta\}}^*$, which is locally constant constructible and equipped with an action of μ_n. In general we have a locally constant constructible sheaf $\mathscr{G}[D_{\{\beta\}}^*]'$ on T_β, obtained by applying the above construction to the deformation to the normal bundle and the pullback of $\mathscr{G}$. Extending this construction to $\overline{\mathbb{Q}}_\ell$-sheaves by taking limits, we obtain a lisse $\overline{\mathbb{Q}}_\ell$-sheaf $\mathscr{F}[D_{\{\beta\}}^*]'$ on T_β endowed with an action of $G_{\{\beta\}} = \hat{\mathbb{Z}}_L(1)$. Here L denotes the set of primes invertible in $\mathbb{F}_q$.

Let us first show that for all $J \subseteq I$, $[j_J^* \mathrm{R} j_* \mathscr{F}] \in \mathrm{K}_{\iota,\sigma}(D_J^*, \overline{\mathbb{Q}}_\ell)$, where $j_J\colon D_J^* \to X$ is the immersion. We proceed by induction on $\#J$. The assertion is trivial for J empty, as $j_\emptyset = j$. For J nonempty, choose $\beta \in J$. Consider the diagram with Cartesian square

$$
\begin{array}{ccc}
 & & D_J^* \\
 & & \downarrow{\scriptstyle j'_{\beta,J'}} \\
D_{\{\beta\}}^* & \xrightarrow{\;j'_\beta\;} & D_\beta \\
{\scriptstyle i'_\beta}\downarrow & {\scriptstyle j_{\{\beta\}}}\searrow\;\;{\scriptstyle i_\beta}\downarrow & \\
U \xrightarrow{\;j^\beta\;} U_\beta & \xrightarrow{\;j_\beta\;} & X,
\end{array}
$$

where $U_\beta = X - \bigcup_{\alpha\in I-\{\beta\}} D_\alpha$, $D_{\{\beta\}} = D_\beta \cap U_\beta$, and $J' = J - \{\beta\}$. By [51, Lemme 3.7] (or by direct computation using [11, 1.7.9]), the base change morphism

$$
i_\beta^* \mathrm{R} j_* \mathscr{F} \to \mathrm{R} j'_{\beta*} i_\beta'^* \mathrm{R} j_*^\beta \mathscr{F}
$$

is an isomorphism, so that

$$
j_J^* \mathrm{R} j_* \mathscr{F} \simeq (j'_{\beta,J'})^* i_\beta^* \mathrm{R} j_* \mathscr{F} \simeq (j'_{\beta,J'})^* \mathrm{R} j'_{\beta*} i_\beta'^* \mathrm{R} j_*^\beta \mathscr{F}.
$$

Since $\#J' < \#J$, by induction hypothesis applied to j'_β and J', it suffices to show that the class of $i_\beta'^* \mathrm{R} j_* \mathscr{F} \simeq j_{\{\beta\}}^* \mathrm{R} j_* \mathscr{F}$ is in $\mathrm{K}_{\iota,\sigma}(D_{\{\beta\}}^*, \overline{\mathbb{Q}}_\ell)$. For this, we may assume $\mathscr{F}$ ι-pure of weight $w \in \mathbb{Z}$. Let $H_\beta <= \hat{\mathbb{Z}}_L(1) = G_{\{\beta\}}$ be an open subgroup whose action on $V = \mathscr{F}[D_{\{\beta\}}^*]'$ is unipotent. Let $N\colon V(1) \to V$ be the logarithm of this action and let M be the monodromy filtration on V. We have

$$j_{\{\beta\}}^* \mathrm{R} j_* \mathscr{F} \in \mathrm{D}^{[0,1]},$$

$$p_\beta^* j_{\{\beta\}}^* j_* \mathscr{F} \simeq (\mathrm{Ker}(N)(-1))^{G_{\{\beta\}}/H_\beta}, \qquad p_\beta^* j_{\{\beta\}}^* \mathrm{R}^1 j_* \mathscr{F} \simeq (\mathrm{Coker}(N))^{G_{\{\beta\}}/H_\beta}(-1),$$

$$\mathrm{gr}_i^M(\mathrm{Ker}(N)(-1)) \simeq \begin{cases} P_i(V,N) & i \le 0, \\ 0 & i > 0, \end{cases} \qquad \mathrm{gr}_i^M(\mathrm{Coker}(N)) \simeq \begin{cases} P_{-i}(V,N)(-i) & i \ge 0, \\ 0 & i < 0. \end{cases}$$

By [11, Corollaire 1.8.7, Remarque 1.8.8], $P_i(V,N)$ is pure of weight $w + i$ for $i \le 0$. Moreover, $P_i(V,N)$ is $(-1)^{w+i}\sigma$-self-dual by Proposition A.6.8. It follows that $[j_{\{\beta\}}^* j_* \mathscr{F}], [j_{\{\beta\}}^* \mathrm{R}^1 j_* \mathscr{F}] \in \mathrm{K}_{\iota,\sigma}$.

Next we show that, if $\mathscr{F}$ is ι-pure of weight $w \in \mathbb{Z}$, then for all $n \ge 0$ and $J \subseteq I$, $[j_J^* \mathrm{R} j_{n*} j_{!*}^n \mathscr{F}] \in \mathrm{K}_{\iota,\sigma}(D_J^*, \overline{\mathbb{Q}}_\ell)$. Here $U \xrightarrow{j^n} U_n \xrightarrow{j_n} X$ are immersions, $U_n = X - \bigcup_{K \subseteq I, \#K \ge n} D_K^*$, $j_{!*}^n \mathscr{F} := (j_{!*}^n(\mathscr{F}[d]))[-d]$, $d = \dim(X)$ (a function on $\pi_0(X)$). The proof is similar to Gabber's proof of independence of ℓ for middle extensions [16, Theorem 3]. We proceed by induction on n. For $n = 0$, $U_0 = U$ and the assertion is shown in the preceding paragraph. For $n \ge 1$, consider the immersions $U_n - U_{n-1} \xrightarrow{i_n} U_n \xleftarrow{j_{n-1}^n} U_{n-1}$ and the distinguished triangle

$$i_{n*} \mathrm{R} i_n^! j_{!*}^n \mathscr{F} \to j_{!*}^n \mathscr{F} \to \mathrm{R}(j_{n-1}^n)_* j_{!*}^{n-1} \mathscr{F} \to .$$

The second and third arrows of the triangle induce isomorphisms $j_{!*}^n \mathscr{F} \xrightarrow{\sim} \tau^{\le n-1} \mathrm{R}(j_{n-1}^n)_* j_{!*}^{n-1} \mathscr{F}$ and $\tau^{\ge n} \mathrm{R}(j_{n-1}^n)_* j_{!*}^{n-1} \mathscr{F} \xrightarrow{\sim} i_{n*} \mathrm{R} i_n^! j_{!*}^n \mathscr{F}[1]$. By induction hypothesis, the left hand side of

$$[i_n^* \mathrm{R}(j_{n-1}^n)_* j_{!*}^{n-1} \mathscr{F}] = [i_n^* j_{!*}^n \mathscr{F}] - [\mathrm{R} i_n^! j_{!*}^n \mathscr{F}]$$

belongs to $\mathrm{K}_{\iota,\sigma}$. Moreover, the first term of the right hand side belongs to $\bigoplus_{w' \le w+n-1} \mathrm{K}_\iota^{w'}$ and the second term belongs to $\bigoplus_{w' \ge w+n+1} \mathrm{K}_\iota^{w'}$. It follows that both terms belong to $\mathrm{K}_{\iota,\sigma}$. Thus, by the preceding paragraph, $[j_J^* R(j_n i_n)_* \mathrm{R} i_n^! j_{!*}^n \mathscr{F}] \in \mathrm{K}_{\iota,\sigma}$. Moreover, by induction hypothesis, $[j_J^* \mathrm{R}(j_{n-1})_* j_{!*}^{n-1} \mathscr{F}] \in \mathrm{K}_{\iota,\sigma}$. Therefore, $[j_J^* \mathrm{R} j_{n*} j_{!*}^n \mathscr{F}] = [j_J^* R(j_n i_n)_* \mathrm{R} i_n^! j_{!*}^n \mathscr{F}] + [j_J^* \mathrm{R}(j_{n-1})_* j_{!*}^{n-1} \mathscr{F}] \in \mathrm{K}_{\iota,\sigma}$.

Taking $n = 1 + \#I$ so that $U_n = X$ in the preceding paragraph, we get that for $\mathscr{F}$ ι-pure and $J \subseteq I$, $[j_J^* j_{!*} \mathscr{F}] \in \mathrm{K}_{\iota,\sigma}(D_J^*, \overline{\mathbb{Q}}_\ell)$. Here $j_{!*} \mathscr{F} := (j_{!*}(\mathscr{F}[d]))[-d]$.

Finally, we show the proposition by induction on $\#I$. The assertion is trivial for I empty. For I nonempty, we may assume $\mathscr{F}$ ι-pure. We have

$$[j_{!*} \mathscr{F}] = [j_! \mathscr{F}] + \sum_{\emptyset \ne J \subseteq I} [j_{J!} j_J^* j_{!*} \mathscr{F}].$$

By the preceding paragraph and induction hypothesis, for $\emptyset \ne J \subseteq I$, we have $[j_{J!} j_J^* j_{!*} \mathscr{F}] \in \mathrm{K}_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$. Moreover, $[j_{!*} \mathscr{F}] \in \mathrm{K}_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$. It follows that $[j_! \mathscr{F}] \in \mathrm{K}_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$. $\qquad \square$

Lemma 4.3.2. *Let X be a Noetherian Deligne-Mumford stack of separated diagonal. Then there exist a finite group G and a G-equivariant dominant open immersion $V \to W$ of schemes, such that the induced morphism $[V/G] \to [W/G]$ fits into a commutative diagram*

where j is an open immersion and f is quasi-finite, proper, and surjective.

Proof. By [31, 16.6.3], there exists a finite group G acting on a scheme V fitting into a Cartesian square
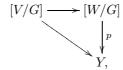

where Z is a scheme, g is finite surjective, and j is a dense open immersion. It then suffices to take W to be the schematic closure of V in $(Z/X)^G$ (fiber product over X of copies of Z indexed by G) endowed with the action of G by permutation of factors. $\qquad \square$

Proof of Theorem 4.2.5. We will prove the preservation of $K_{\iota,\sigma}$. The commutation with $\overline{D}_\iota$ is similar.

(1) Let us first show the case of $f_!$ for an open immersion f. Since Y has finite inertia, there exists a Zariski open covering (Y_α) of Y with Y_α separated. By the Zariski local nature of $K_{\iota,\sigma}$ (Remark 4.2.7), we may assume Y separated. We may assume f dominant.

We proceed by induction on $d = \dim X$. For $d < 0$ (i.e. $X = \emptyset$), the assertion is trivial. For $d \geq 0$, let $A \in K_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$. Note that if $A' \in K(X, \overline{\mathbb{Q}}_\ell)$ is such that the support of $A - A'$ has dimension $< d$, then, by induction hypothesis, to show $f_! A \in K_{\iota,\sigma}(Y, \overline{\mathbb{Q}}_\ell)$, it suffices to show $A' \in K_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$ and $f_! A' \in K_{\iota,\sigma}(Y, \overline{\mathbb{Q}}_\ell)$. This applies in particular to $A' = j_! j^* A$, where $j \colon U \to X$ is a dominant open immersion. In this case $f_! A' = (fj)_! j^* A$. This allows us to shrink X.

Applying Lemma 4.3.2 to Y, we obtain a finite group G, a G-equivariant dominant open immersion of schemes $V \to W$, and a commutative diagram

$$
\begin{array}{ccc}
[V/G] & \longrightarrow & [W/G] \\
 & \searrow & \downarrow{\scriptstyle p} \\
 & & Y,
\end{array}
$$

where p is proper quasi-finite surjective, and the oblique arrow is an open immersion. Let $j \colon U \to X$, where $U = X \cap [V/G] = [V'/G]$. By the remark above, it suffices to show that $j_!$ and $(fj)_!$ preserve $K_{\iota,\sigma}$. In the case of $j_!$, up to replacing Y by X and p by its restriction to X, we are reduced to the case of $(fj)_!$. Since $(fj)_! = f'_! p_*$, where $f' \colon U \to [W/G]$, we are reduced to the case of $f'_!$. Thus, changing notation, we are reduced to the case of $f_!$, where $f \colon X = [V/G] \to [W/G] = Y$ is given by a G-equivariant open immersion of schemes $V \to W$.

The reduction of this case to the case where V is the complement of a G-strict normal crossing divisor of W is similar to parts of [52, Section 3]. We may assume V reduced. Shrinking V, we may assume V normal and $A = [\mathscr{F}]$, where $\mathscr{F} = \mathscr{F}_{\mathscr{O}} \otimes_{\mathscr{O}} \overline{\mathbb{Q}}_\ell$, where $\mathscr{F}_{\mathscr{O}}$ is a lisse $\mathscr{O}$-sheaf, $\mathscr{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_\ell$. Applying [52, Lemme 3.5], we obtain a G-stable dense open subscheme U of V and an equivariant morphism $(u, \alpha) \colon (U', G') \to (U, G)$, where α is surjective and u is a Galois étale cover of group $\mathrm{Ker}(\alpha)$ trivializing $\mathscr{F}_{\mathscr{O}}/\mathfrak{m}\mathscr{F}_{\mathscr{O}}$, where $\mathfrak{m}$ is the maximal ideal of $\mathscr{O}$. By Nagata compactification, this can be completed into a commutative diagram

$$
\begin{array}{ccc}
(U', G') & \xrightarrow{(f',\mathrm{id})} & (W', G') \\
{\scriptstyle (u,\alpha)}\downarrow & & \downarrow{\scriptstyle (w,\alpha)} \\
(U, G) & \longrightarrow & (W, G),
\end{array}
$$

where w is proper and f' is an open immersion. Since $[u/\alpha]$ is an isomorphism and Remark 4.2.6 (6) applies to $[w/\alpha]_*$, shrinking X and changing notation, we are reduced to the case where $\mathscr{F}_{\mathscr{O}}/\mathfrak{m}\mathscr{F}_{\mathscr{O}}$ is constant on every connected component. We may assume W reduced. Let k' be a finite extension of k such that the irreducible components of $W \otimes_k k'$ are geometrically irreducible. Up to replacing W by $W \otimes_k k'$ and G by $G \times \mathrm{Gal}(k'/k)$, we may assume that the irreducible components of W are geometrically irreducible and there exists a G-equivariant morphism $W \to \mathrm{Spec}(k')$. Shrinking V, we may assume V regular. Moreover, we may assume that G acts transitively on $\pi_0(V)$. Let V_0 be an irreducible component of V and let G_0 be the decomposition group. Then f can be decomposed as $X \simeq [V_0/G_0] \to [W/G_0] \xrightarrow{g} [W/G] = Y$, where g is finite. Changing notation, we may assume V irreducible. Up to replacing W by the closure of V, we may assume W irreducible, thus geometrically irreducible.

Applying Gabber's refinement [52, Lemme 3.8] (see also [49, 4.4]) of de Jong's alterations [9], we obtain a diagram with Cartesian square

$$
\begin{array}{ccccc}
(U', G') & \longrightarrow & (V', G') & \longrightarrow & (W', G') \\
 & & {\scriptstyle (v,\alpha)}\downarrow & & \downarrow{\scriptstyle (w,\alpha)} \\
 & & (V, G) & \longrightarrow & (W, G),
\end{array}
$$

where (w, α) is a Galois alteration, W' is regular, quasi-projective over k, and U' is the complement of a G'-strict normal crossing divisor of W'. As $\mathscr{F}$ is lisse and $[V/G]$, $[V'/G']$ are regular, $A' = [v/\alpha]^* A$ belongs to $K_{\iota,\sigma}([V'/G'], \overline{\mathbb{Q}}_\ell)$, so that $[v/\alpha]_* A$ belongs to $K_{\iota,\sigma}([V/G], \overline{\mathbb{Q}}_\ell)$. Moreover, the support of $A - [v/\alpha]_* A'$ has dimension $< d$. Thus it suffices to show that $f_! [v/\alpha]_* A' = [w/\alpha]_* f'_! A'$ belongs to $K_{\iota,\sigma}(Y, \overline{\mathbb{Q}}_\ell)$, where

$f' \colon [W'/G'] \to [V'/G']$. Let $j' \colon [U'/G'] \to [V'/G']$. It suffices to show that $j'_! j'^* A'$ and $(f'j')_! j'^* A'$ belong to $\mathrm{K}_{\iota,\sigma}$. Changing notation, we are reduced to showing $f_![\mathscr{F}] \in \mathrm{K}_{\iota,\sigma}([W/G], \overline{\mathbb{Q}}_\ell)$ for $f \colon [V/G] \to [W/G]$, where V is the complement of a G-strict normal crossing divisor D of a regular quasi-projective scheme W over k and $\mathscr{F}$ is a lisse sheaf on $[V/G]$ tame along D such that $[\mathscr{F}] \in \mathrm{K}_{\iota,\sigma}([V/G], \overline{\mathbb{Q}}_\ell)$.

Note that W admits a Zariski open covering by G-stable affine schemes. Thus, by the Zariski local nature of $\mathrm{K}_{\iota,\sigma}$ (Remark 4.2.7), we may assume W affine. In this case, the assertion is a special case of Proposition 4.3.1. This finishes the proof of the case of $f_!$ for f an open immersion.

(2) Next we establish the general case of $f_!$. Let $(X_\alpha)_{\alpha \in I}$ be a Zariski open covering of X with X_α separated. For $J \subseteq I$, let $j_J \colon \bigcap_{\beta \in J} X_\beta \to X$ be the open immersion. Let $A \in \mathrm{K}_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$. Then $A = \sum_{\emptyset \neq J \subseteq I} (-1)^{1+\#J} j_{J!} j_J^* A$ by Lemma 4.1.3. Thus we may assume X separated. Applying Nagata compactification [7] to the morphism $\bar{X} \to \bar{Y}$ of coarse spaces, we obtain a diagram with Cartesian squares

$$
\begin{array}{ccccccc}
X & \xrightarrow{f_1} & \bar{X} \times_{\bar{Y}} Y & \xrightarrow{f_2} & \bar{Z} \times_{\bar{Y}} Y & \xrightarrow{f_3} & Y \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \bar{X} & \xrightarrow{g_2} & \bar{Z} & \xrightarrow{g_3} & \bar{Y},
\end{array}
$$

where f_1 is proper and quasi-finite, g_2 is an open immersion, and g_3 is proper. Thus $f_! = f_{3*} f_{2!} f_{1*}$ preserves $\mathrm{K}_{\iota,\sigma}$.

The case of $f_* = D_Y f_! D_X$ follows immediately.

(3) Next we establish the case of f^*. The argument is similar to the deduction of the congruence $f^* \equiv f^!$ modulo $\mathrm{I}(X, \overline{\mathbb{Q}}_\ell)$ [53, Corollary 9.5] from Laumon's theorem mentioned in Remark 4.2.13. Let $B \in \mathrm{K}_{\iota,\sigma}(Y, \overline{\mathbb{Q}}_\ell)$. If f is a closed immersion, then $B = j_! j^* B + f_* f^* B$, where j is the complementary open immersion. It follows that $f_* f^* B \in \mathrm{K}_{\iota,\sigma}(Y, \overline{\mathbb{Q}}_\ell)$, so that $f^* B \in \mathrm{K}_{\iota,\sigma}$. In the general case, let $(Y_\alpha)_{\alpha \in I}$ be a stratification of Y such that each Y_α is the quotient stack of an affine scheme by a finite group action. For each α, form the Cartesian square

$$
\begin{array}{ccc}
X_\alpha & \xrightarrow{j'_\alpha} & X \\
f_\alpha \downarrow & & \downarrow f \\
Y_\alpha & \xrightarrow{j_\alpha} & Y.
\end{array}
$$

Then $f^* B = \sum_{\alpha \in I} f^* j_{\alpha!} j_\alpha^* B = \sum_{\alpha \in I} j'_{\alpha!} f_\alpha^* j_\alpha^* B$. Thus we may assume $Y = [Y'/H]$, where Y' is an affine scheme endowed with an action of a finite group H. Similarly, we may assume $X = [X'/G]$, where X' is an affine scheme endowed with an action of a finite group G. Up to changing X' and G, we may further assume that $f = [f'/\gamma]$, for $(f, \gamma) \colon (X', G) \to (Y', H)$, by [52, Proposition 5.1]. In this case f' can be decomposed into G-equivariant morphisms $X' \xrightarrow{i} Z' \xrightarrow{p} Y'$ where i is a closed immersion and p is an affine space. Thus $f^* \simeq [i/\mathrm{id}]^* [p/\gamma]^*$ preserves $\mathrm{K}_{\iota,\sigma}$.

The assertions for the other operations follow immediately: $f^! = D_X f^* D_Y$, $- \otimes - = \Delta_X^* (- \boxtimes -)$, $\mathscr{H}om(-, -) = D(- \otimes D-)$. $\qquad\square$

5 Variant: Horizontal complexes

In this section, let k be a field finitely generated over its prime field. This includes notably the case of a number field. Many results in previous sections over finite fields can be generalized to Annette Huber's horizontal complexes [21], as extended by Sophie Morel [34], over k. In Subsection 5.1, after briefly reviewing horizontal complexes, we discuss symmetry and decomposition of pure horizontal complexes, and prove analogues of results of Subsection 3.2. In Subsection 5.2, we discuss symmetry in Grothendieck groups of horizontal complexes, and give analogues of results of Section 4. This section stems from a suggestion of Takeshi Saito.

5.1 Symmetry and decomposition of pure horizontal complexes

Let X be a Deligne-Mumford stack of finite presentation over k. Huber [21] (see also Morel [34]) defines a triangulated category $\mathrm{D}_h^b(X, \overline{\mathbb{Q}}_\ell)$ of horizontal complexes. For a finite extension Λ of $\mathbb{Z}_\ell$, $\mathrm{D}_h^b(X, \Lambda)$ is the 2-colimit of the categories $\mathrm{D}_c^b(X_R, \Lambda)$, indexed by triples (R, X_R, u), where $R \subseteq k$ is a subring of finite type over $\mathbb{Z}[1/\ell]$ such that $k = \mathrm{Frac}(R)$, X_R is a Deligne-Mumford stack of finite presentation over

Spec(R), and $u\colon X \to X_R \otimes_R k$ is an isomorphism. We may restrict to R regular and X_R flat over Spec(R). We have Grothendieck's six operations on $\mathrm{D}_h^b(X, \Lambda)$.

The triangulated category $\mathrm{D}_h^b(X, \overline{\mathbb{Q}}_\ell)$ is equipped with a canonical t-structure and a perverse t-structure. We let $\mathrm{Sh}_h(X, \overline{\mathbb{Q}}_\ell)$ and $\mathrm{Perv}_h(X, \overline{\mathbb{Q}}_\ell)$ denote the respective hearts. The pullback functors via $X \to X_R$ induce a conservative functor $\eta^*\colon \mathrm{D}_h^b(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell)$ t-exact for the canonical t-structures and the perverse t-structures, and compatible with the six operations. Moreover, η^* induces *fully faithful* exact functors $\mathrm{Sh}_h(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Sh}(X, \overline{\mathbb{Q}}_\ell)$ and $\mathrm{Perv}_h(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ [34, Propositions 2.3, 2.5]. Every object of $\mathrm{Perv}_h(X, \overline{\mathbb{Q}}_\ell)$ has finite length.

Remark 5.1.1. The functor $\eta^*\colon \mathrm{Perv}_h(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ preserves indecomposable objects. By the description of simple objects, the functor also preserves simple objects. Thus, via the functor, $\mathrm{Perv}_h(X, \overline{\mathbb{Q}}_\ell)$ can be identified with a full subcategory of $\mathrm{Perv}(X, \overline{\mathbb{Q}}_\ell)$ stable under subquotients. The subcategory is *not* stable under extensions in general.

By restricting to closed points of $\mathrm{Spec}(R[1/\ell])$, we get a theory of weights for horizontal complexes. Weight filtration does not always exist, but this will not be a problem for us. The analogue of Remark 3.2.7 for the preservation of pure complexes holds. Moreover, the analogues of [3, Théorèmes 5.3.8, 5.4.5] hold for the decomposition of the pullbacks of pure horizontal complexes to $\bar{k}$. In other words, the functor

$$\bar{\eta}^*\colon \mathrm{D}_h^b(X, \overline{\mathbb{Q}}_\ell) \xrightarrow{\eta^*} \mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{D}_c^b(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell)$$

obtained by composing η^* with the pullback functor carries pure complexes to admissible semisimple complexes (Definition 3.3.1). Indeed, both theorems follow from [3, Proposition 5.1.15 (iii)], which has the following analogue, despite the fact that the analogue of [3, Proposition 5.1.15 (ii)] does not hold in general.

Proposition 5.1.2. *Let $K, L \in \mathrm{D}_h^b(X, \overline{\mathbb{Q}}_\ell)$, with K mixed of weights $\leq w$ and L mixed of weights $\geq w$. Then $\mathrm{Ext}^i(\bar{\eta}^* K, \bar{\eta}^* L)^{\mathrm{Gal}(\bar{k}/k)} = 0$ for $i > 0$. In particular, the map $\mathrm{Ext}^i(\eta^* K, \eta^* L) \to \mathrm{Ext}^i(\bar{\eta}^* K, \bar{\eta}^* L)$ is zero.*

Proof. The second assertion follows from the first one, as the map factors through $E^i := \mathrm{Ext}^i(\bar{\eta}^* K, \bar{\eta}^* L)^{\mathrm{Gal}(\bar{k}/k)}$. For the first assertion, consider the horizontal complex $\mathscr{E} = \mathrm{R}a_{X*}\mathrm{R}\mathscr{H}om(K, L)$ on $\mathrm{Spec}(k)$, which has weight ≥ 0. Thus $E^i \simeq \Gamma(\mathrm{Spec}(k), \mathscr{H}^i\mathscr{E}) = 0$ for $i > 0$. $\square$

Since pure horizontal perverse sheaves are geometrically semisimple, Lemma 2.2.8 on the support decomposition applies (cf. [3, Corollaire 5.3.11]).

General preservation properties of σ-self-dual complexes listed in Remark 2.1.4 still hold for D_h^b. The two-out-of-three property (Proposition 2.2.1) holds for σ-self-dual horizontal perverse sheaves. The trichotomy for indecomposable horizontal perverse sheaves (Proposition 2.2.3) also holds.

We say that a horizontal complex of $\overline{\mathbb{Q}}_\ell$-sheaves A is *split* if it is a direct sum of shifts of horizontal perverse sheaves, or, in other words, $A \simeq \bigoplus_i (^p\mathscr{H}^i A)[-i]$. Definition 3.2.2 can be repeated as follows.

Definition 5.1.3 ($\mathrm{D}_{h,\sigma}^w$)**.** Let $w \in \mathbb{Z}$. We denote by $\mathrm{D}_{h,\sigma}^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{Ob}(\mathrm{D}_h^b(X, \overline{\mathbb{Q}}_\ell))$ (resp. $\mathrm{D}_{h,\mathrm{sd}}^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{Ob}(\mathrm{D}_c^b(X, \overline{\mathbb{Q}}_\ell))$) the subset consisting of split pure horizontal complexes A of weight w such that $^p\mathscr{H}^i A$ is $(-1)^{w+i}\sigma$-self-dual (resp. self-dual) with respect to $K_X(-w-i)$ for all i. We denote by $\mathrm{D}_{h,\mathrm{d}}^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{Ob}\big(\mathrm{D}_h^b(X, \overline{\mathbb{Q}}_\ell) \times \mathrm{D}_h^b(X, \overline{\mathbb{Q}}_\ell)\big)$ the subset consisting of pairs (A, B) of split pure horizontal complexes of weight w such that $^p\mathscr{H}^i A$ is isomorphic to $(D_X \, {}^p\mathscr{H}^i B)(-w-i)$ for all i.

The analogue of Remark 3.2.8 holds for the preservation of $\mathrm{D}_{h,\sigma}^w$ and $\mathrm{D}_{h,\mathrm{d}}^w$. The two-out-of-three property, analogue of Remark 3.2.9, also holds for $\mathrm{D}_{h,\sigma}^w$ and $\mathrm{D}_{h,\mathrm{d}}^w$. We have the following analogue of Proposition 3.2.14, which holds with the same proof as before, and a similar result for $\mathrm{D}_{h,\mathrm{d}}^w$.

Proposition 5.1.4. *Let $m \geq 0$. Let A be a mixed horizontal complex on X such that for all $n \in \mathbb{Z}$, $^p\mathscr{H}^n A$ admits a weight filtration W, and that $\mathrm{gr}_w^W {}^p\mathscr{H}^n A$ is $(-1)^w\sigma$-self-dual with respect to $K_X(-w)$, for all $w \in \mathbb{Z}$. Then for all $n \in \mathbb{Z}$, $^p\mathscr{H}^n(A^{\boxtimes m})$ admits a weight filtration W, and $\mathrm{gr}_w^W {}^p\mathscr{H}^n(A^{\boxtimes m})$ is $(-1)^w\sigma^m$-self-dual with respect to $K_{[X^m/\mathfrak{S}_m]}(-mw)$ for all $w \in \mathbb{Z}$. Moreover, the functor $(-)^{\boxtimes m}$ carries $\mathrm{D}_{h,\sigma}^w(X, \overline{\mathbb{Q}}_\ell)$ to $\mathrm{D}_{h,\sigma^m}^{mw}([X^m/\mathfrak{S}_m], \overline{\mathbb{Q}}_\ell)$.*

We have the following analogues of Theorems 3.2.3 and Corollary 3.2.5, which hold with the same proofs as before.

Theorem 5.1.5. *Let $f\colon X \to Y$ be a proper morphism of Deligne-Mumford stacks of finite presentation over k. Assume that Y has finite inertia. Then Rf_* preserves $D_{h,\sigma}^w$ and $D_{h,\mathrm{d}}^w$.*

Corollary 5.1.6. *Assume that Y has finite inertia. Then Rf_* preserves split pure complexes of weight w.*

The analogue of Corollary 3.2.4 also holds for $D_{h,\mathrm{sd}}^b$.

Theorem 1.8 is a special case of Theorem 5.1.5. Applying it to a_X, we obtain Theorem 1.1. Indeed, as we remarked in the introduction, in Theorem 1.1 we may assume that k is finitely generated over its prime field. The horizontal perverse sheaf IC_X, pure of weight d, is $(-1)^d$-self-dual with respect to $K_X(-d)$ by Example 2.1.6, so $\mathrm{IC}_X[-d] \in D_{h,1}^0$, hence $Ra_{X*}\mathrm{IC}_X[-d] \in D_{h,1}^0$ by Theorem 1.8, which proves Theorem 1.1.

5.2 Symmetry in Grothendieck groups of horizontal complexes

Let X be a Deligne-Mumford stack of finite presentation over k. We let $\mathrm{K}_h(X, \overline{\mathbb{Q}}_\ell)$ denote the Grothendieck group of $D_h^b(X, \overline{\mathbb{Q}}_\ell)$, which is a free Abelian group generated by the isomorphism classes of simple horizontal perverse $\overline{\mathbb{Q}}_\ell$-sheaves. The functor η^* induces an injection $\mathrm{K}_h(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$, which identifies $\mathrm{K}_h(X, \overline{\mathbb{Q}}_\ell)$ with a λ-subring of $\mathrm{K}(X, \overline{\mathbb{Q}}_\ell)$. The operations on Grothendieck groups in Construction 4.1.1 and Remark 4.1.2 induce operations on K_h.

For $w \in \mathbb{Z}$, we let $\mathrm{K}_h^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_h(X, \overline{\mathbb{Q}}_\ell)$ denote the subgroup generated by pure horizontal perverse sheaves of weight w on X. Let $\mathrm{K}_h^{\mathbb{Z}}(X, \overline{\mathbb{Q}}_\ell) = \bigoplus_{w \in \mathbb{Z}} \mathrm{K}_h^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_h(X, \overline{\mathbb{Q}}_\ell)$. The analogue of Lemma 4.1.8 holds, which further justifies the definition of the map $f_{!*}$ in Remark 4.1.2.

We repeat Definition 4.2.1 as follows.

Definition 5.2.1 ($\mathrm{K}_{h,\sigma}$). We define $\mathrm{K}_{h,\sigma}^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_h^w(X, \overline{\mathbb{Q}}_\ell)$ (resp. $\mathrm{K}_{h,\mathrm{sd}}^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_h^w(X, \overline{\mathbb{Q}}_\ell)$), for $w \in \mathbb{Z}$, to be the subgroup generated by $[B]$, for B perverse, ι-pure of weight w, and $(-1)^w \sigma$-self-dual (resp. self-dual) with respect to $K_X(-w)$. We put

$$\mathrm{K}_{h,\sigma}(X, \overline{\mathbb{Q}}_\ell) = \bigoplus_{w \in \mathbb{Z}} \mathrm{K}_{h,\sigma}^w(X, \overline{\mathbb{Q}}_\ell) \quad (\text{resp. } \mathrm{K}_{\iota,\mathrm{sd}}(X, \overline{\mathbb{Q}}_\ell) = \bigoplus_{w \in \mathbb{Z}} \mathrm{K}_{h,\mathrm{sd}}^w(X, \overline{\mathbb{Q}}_\ell)).$$

We define the *twisted dualizing map*

$$\overline{D}_{h,X}\colon \mathrm{K}_h^{\mathbb{Z}}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}_h^{\mathbb{Z}}(X, \overline{\mathbb{Q}}_\ell)$$

to be the direct sum of the group automorphisms $\overline{D}_{h,X}^w\colon \mathrm{K}_h^w(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}_h^w(X, \overline{\mathbb{Q}}_\ell)$ sending $[A]$ to $[(D_X A)(-w)]$. We let $\mathrm{K}_{h,\mathrm{d}}^w(X, \overline{\mathbb{Q}}_\ell) \subseteq \mathrm{K}_h^w(X, \overline{\mathbb{Q}}_\ell)^2$ denote the graph of $\overline{D}_{h,X}^w$. We put

$$\mathrm{K}_{h,\mathrm{d}}(X, \overline{\mathbb{Q}}_\ell) = \bigoplus_{n \in Z} \mathrm{K}_{h,\mathrm{d}}^w(X, \overline{\mathbb{Q}}_\ell).$$

The subgroup $\mathrm{K}_{\mathrm{orth}}(X, \overline{\mathbb{Q}}_\ell)$ in the introduction is $\mathrm{K}_{h,1}(X, \overline{\mathbb{Q}}_\ell)$. If k is a finite field, $\mathrm{K}_{h,\sigma}(X, \overline{\mathbb{Q}}_\ell)$ is the intersection $\bigcap_\iota \mathrm{K}_{\iota,\sigma}(X, \overline{\mathbb{Q}}_\ell)$ of the subgroups $\mathrm{K}_{\iota,\sigma}$ of Definition 4.2.1, where ι runs over embeddings $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$.

The analogue of Remark 4.2.3 holds for the definition of $\mathrm{K}_{h,\sigma}$, $\mathrm{K}_{h,\mathrm{sd}}$, and $\mathrm{K}_{h,\mathrm{d}}$. The analogues of Remarks 4.2.6 and 4.2.7 hold for the preservation and Zariski local nature of $\mathrm{K}_{h,\sigma}$ and $\mathrm{K}_{h,\mathrm{d}}$ with the same proofs.

The following analogue of Theorem 4.2.5 holds with the same proof. In particular, the analogue of Proposition 4.3.1 holds.

Theorem 5.2.2. *Let X and Y be Deligne-Mumford stacks of finite inertia and finite presentation over k and let $f\colon X \to Y$ be a morphism. Then Grothendieck's six operations induce maps*

$$- \otimes -, \quad \mathscr{H}om(-, -)\colon \mathrm{K}_{h,\sigma}(X, \overline{\mathbb{Q}}_\ell) \times \mathrm{K}_{h,\sigma'}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}_{h,\sigma\sigma'}(X, \overline{\mathbb{Q}}_\ell),$$

$$f^*, f^!\colon \mathrm{K}_{h,\sigma}(Y, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}_{h,\sigma}(X, \overline{\mathbb{Q}}_\ell), \qquad f_*, f_!\colon \mathrm{K}_{h,\sigma}(X, \overline{\mathbb{Q}}_\ell) \to \mathrm{K}_{h,\sigma}(Y, \overline{\mathbb{Q}}_\ell).$$

Moreover, Grothendieck's six operations on $\mathrm{K}_h^{\mathbb{Z}}$ commute with the twisted dualizing map $\overline{D}_h$.

The analogues of Corollaries 4.2.8 and 4.2.9 hold. The relationship with Laumon's theorem (Remark 4.2.13) also holds.

Theorem 1.9 is a special case of Theorem 5.2.2.

A   Appendix. Symmetry and duality in categories

In the appendix, we collect some general symmetry properties in categories with additional structures. Tensor product equips the derived category of ℓ-adic sheaves with a symmetric structure. We discuss symmetry of pairings in symmetric categories in Subsection A.1. The category of perverse sheaves is not stable under tensor product, but is equipped with a duality functor. We study symmetry in categories with duality in Subsections A.2 and A.3. We discuss the relation of the two points of view in Subsection A.4. We then study effects of translation on symmetry in Subsection A.5, which is applied in the main text to the Lefschetz pairing. In Subsection A.6, we study symmetry of primitive parts under a nilpotent operator, which is applied in the main text to the monodromy operator. The results of the appendix are formal but are used in the main text. The presentation here is influenced by [37], [40, Section 12], and [42]. Recall $\sigma, \sigma' \in \{\pm 1\}$.

A.1   Symmetric categories

In this subsection, we discuss symmetry of pairings in symmetric categories.

Definition A.1.1 (Symmetric category)**.** A *symmetric category* is a category $\mathscr{C}$ endowed with a bifunctor $-\otimes-\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and a natural isomorphism (called the *symmetry constraint*) $c_{AB}\colon A \otimes B \to B \otimes A$, for objects A and B of $\mathscr{C}$, satisfying $c_{AB}^{-1} = c_{BA}$. We say that the symmetric category $\mathscr{C}$ is *closed* if for every object A of $\mathscr{D}$, the functor $-\otimes A\colon \mathscr{C} \to \mathscr{C}$ admits a right adjoint, which we denote by $\mathscr{H}om(A, -)$.

In our applications, we mostly encounter symmetric *monoidal* categories (see, for example, [33, Section VII.7] for the definition), but the associativity and unital constraints are mostly irrelevant to the results of this article.

To deal with signs, we need the following additive variant of Definition A.1.1.

Definition A.1.2. A *symmetric additive category* is a symmetric category $(\mathscr{D}, \otimes)$ such that $\mathscr{D}$ is an additive category and $-\otimes-\colon \mathscr{D} \times \mathscr{D} \to \mathscr{D}$ is an additive bifunctor (namely, a bifunctor additive in each variable). A *closed symmetric additive category* is a closed symmetric category $(\mathscr{D}, \otimes)$ such that $\mathscr{D}$ is an additive category.

A closed symmetric additive category is necessarily a symmetric additive category and the internal Hom functor $\mathscr{H}om(-, -)\colon \mathscr{D}^{\mathrm{op}} \times \mathscr{D} \to \mathscr{D}$ is an additive bifunctor.

Definition A.1.3. Let $(\mathscr{C}, \otimes, c)$ be a symmetric category. Assume that $\mathscr{C}$ is an additive category if $\sigma = -1$. Let A, B, K be objects of $\mathscr{C}$.

(1) We define the *transpose* of a pairing $g\colon B \otimes A \to K$ to be the composite
$$g^T\colon A \otimes B \xrightarrow{c} B \otimes A \xrightarrow{g} K.$$
We call σg^T the *σ-transpose* of g.

(2) We say that a pairing $f\colon A \otimes A \to K$ is *σ-symmetric* if $f = \sigma f^T$.

We have $(g^T)^T = g$. We will often say "symmetric" instead of "1-symmetric".

Note that for a pair of pairings $f\colon A \otimes B \to K$ and $g\colon B \otimes A \to K$ in a symmetric additive category, $(2f, 2g)$ is a sum of a pair of 1-transposes and a pair of -1-transposes:
$$(2f, 2g) = (f + g^T, g + f^T) + (f - g^T, g - f^T).$$

Remark A.1.4. Let $(\mathscr{C}, \otimes, c)$ be a symmetric category such that $\mathscr{C}$ is an additive category. Then $(\mathscr{C}, \otimes, -c)$ is another symmetric category. The -1-transpose in $(\mathscr{C}, \otimes, c)$ of a pairing $g\colon A \otimes B \to K$ is the transpose in $(\mathscr{C}, \otimes, -c)$ of g.

Next we consider effects of functors on symmetry.

Definition A.1.5. Let $\mathscr{C}$ and $\mathscr{D}$ be symmetric categories. A *right-lax symmetric functor* (resp. *symmetric functor*) from $\mathscr{C}$ to $\mathscr{D}$ is a functor $G\colon \mathscr{C} \to \mathscr{D}$ endowed with a natural transformation (resp. natural isomorphism) of functors $\mathscr{C} \times \mathscr{C} \to \mathscr{D}$ given by morphisms $G(A) \otimes G(B) \to G(A \otimes B)$ in $\mathscr{D}$ for objects A, B of $\mathscr{C}$, such that the following diagram commutes

$$
\begin{array}{ccc}
G(A) \otimes G(B) & \longrightarrow & G(A \otimes B) \\
{\scriptstyle c_{GA,GB}}\downarrow & & \downarrow{\scriptstyle G(c_{A,B})} \\
G(B) \otimes G(A) & \longrightarrow & G(B \otimes A).
\end{array}
$$

Between symmetric monoidal categories, one has the notions of symmetric monoidal functors and lax symmetric monoidal functors, which are compatible with the associativity constraints and unital constraints. In our applications we will need to consider symmetric functors between symmetric monoidal categories that are *not* symmetric monoidal functors. For example, if f is an open immersion, then $f_!$ is a symmetric functor compatible with the associativity constraint, but not compatible with the unital constraints except in trivial cases. Again we emphasize that the compatibility with the associativity and unital constraints is irrelevant to the results in this article.

Example A.1.6. Let $\mathscr{C}$ and $\mathscr{D}$ be symmetric categories. Let $F\colon \mathscr{C} \to \mathscr{D}$ be a functor admitting a right adjoint $G\colon \mathscr{D} \to \mathscr{C}$. Then every symmetric structure on F induces a right-lax symmetric structure on G, given by the morphism $G(A) \otimes G(B) \to G(A \otimes B)$ adjoint to

$$F(G(A) \otimes G(B)) \xrightarrow{\sim} F(G(A)) \otimes F(G(B)) \to A \otimes B.$$

This construction extends to *left-lax* symmetric structures on F and provides a bijection between left-lax symmetric structures on F and right-lax symmetric structures on G. Since we do not need this extension, we omit the details.

Example A.1.7. Let $\mathscr{C}$ be a symmetric *monoidal* category. Then $\mathscr{C} \times \mathscr{C}$ is a symmetric monoidal category and the functor $- \otimes -\colon \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is a symmetric monoidal functor and, in particular, a symmetric functor. The symmetric structure of the functor is given by the isomorphisms $(A \otimes A') \otimes (B \otimes B') \xrightarrow{\sim} (A \otimes B) \otimes (A' \otimes B')$ for objects A, A', B, B' of $\mathscr{C}$.

Construction A.1.8. Let $\mathscr{C}$ and $\mathscr{D}$ be symmetric categories and let $G\colon \mathscr{C} \to \mathscr{D}$ be a right-lax symmetric functor. Let A, B, K be objects of $\mathscr{C}$. A pairing $A \otimes B \to K$ induces a pairing $G(A) \otimes G(B) \to G(A \otimes B) \to G(K)$.

The following lemma follows immediately from the definitions.

Lemma A.1.9. *Let $\mathscr{C}$ and $\mathscr{D}$ be symmetric categories and let $G\colon \mathscr{C} \to \mathscr{D}$ be a right-lax symmetric functor. Let A, B, K be objects of $\mathscr{C}$. Let $A \otimes B \to K$ and $B \otimes A \to K$ be transposes of each other. Then the induced pairings $GA \otimes GB \to GK$ and $GB \otimes GA \to GK$ are transposes of each other.*

A.2 Categories with duality

In this subsection, we study symmetry in categories with duality.

Definition A.2.1 (Duality)**.** Let $\mathscr{C}$ be a category. A *duality* on $\mathscr{C}$ is a functor $D\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{D}$ endowed with a natural transformation $\mathrm{ev}\colon \mathrm{id}_{\mathscr{C}} \to DD$ such that the composite $D \xrightarrow{\mathrm{ev}D} DDD \xrightarrow{D\mathrm{ev}} D$ is equal to id_D. The duality (D, ev) is said to be *strong* if ev is a natural isomorphism.

We are mostly interested in strong dualities in the main text. However, for the proofs of many results on strong dualities, it is necessary to consider general dualities (for example, the duality $D_{\mathrm{R}f_*K_X}$ in the proof of Remark 2.1.4 (3) is not strong in general). Our terminology here is consistent with [42, Definition 3.1]. Some authors refer to a strong duality simply as "duality" [37].

The underlying functor of a strong duality is an equivalence of categories. If $\mathscr{C}$ is an additive category, we say that a duality on $\mathscr{C}$ is *additive* if the underlying functor is additive. By an *additive category with duality*, we mean an additive category equipped with an additive duality.

A basic example of duality is provided by the internal Hom functor in a closed symmetric category. We will discuss this in detail in Subsection A.4. By analogy with this case, we sometimes refer to morphisms $B \to DA$ in a category with duality as *forms*. We have the following notion of symmetry for forms.

Definition A.2.2 (Symmetry of forms)**.** Let $(\mathscr{C}, D, \mathrm{ev})$ be a category with duality for $\sigma = 1$ (resp. additive category with duality for $\sigma = -1$) and let A, B be objects of $\mathscr{D}$.

(1) We define the *transpose* of a morphism $g\colon B \to DA$ to be the composite

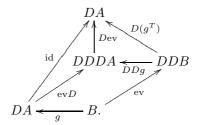$$A \xrightarrow{\mathrm{ev}} DDA \xrightarrow{Dg} DB.$$

We call σg^T the *σ-transpose* of g.

(2) We say that a morphism $f\colon A \to DA$ is *σ-symmetric* if $f = \sigma f^T$.

Again we will often say "symmetric" instead of "1-symmetric". The terminology above is justified by the following lemma.

Lemma A.2.3. *We have $(g^T)^T = g$. Moreover, the map $\mathrm{Hom}_{\mathscr{C}}(B, DA) \to \mathrm{Hom}_{\mathscr{C}}(A, DB)$ carrying g to g^T is a bijection.*

Proof. The first assertion follows from the commutativity of the following diagram

$$
\begin{array}{ccc}
 & DA & \\
 & \uparrow\uparrow\nwarrow{\scriptstyle D(g^T)} & \\
{\scriptstyle\mathrm{id}}\nearrow & {\scriptstyle D\mathrm{ev}}\big\uparrow & \\
DDDA & \xleftarrow{DDg} & DDB \\
{\scriptstyle\mathrm{ev}D}\nearrow & & \nearrow{\scriptstyle\mathrm{ev}} \\
DA & \xleftarrow{\ g\ } & B.
\end{array}
$$

For the second assertion, note that the map carrying $h\colon A \to DB$ to h^T is the inverse of the map $g \mapsto g^T$, by the first assertion. $\square$

Remark A.2.4. Let $(\mathscr{C}, D, \mathrm{ev})$ be an additive category with duality. Then $(\mathscr{C}, D, -\mathrm{ev})$ is another additive category with duality. The -1-transpose in $(\mathscr{C}, D, \mathrm{ev})$ of a morphism $g\colon B \to DA$ is the transpose in $(\mathscr{C}, D, -\mathrm{ev})$ of g. This allows us in the sequel to omit the -1-symmetric case in many results without loss of generality.

We will be especially interested in objects A that admit isomorphisms $A \xrightarrow{\sim} DA$.

Definition A.2.5. Let $(\mathscr{C}, D, \mathrm{ev})$ be a category with duality and let A be an object of $\mathscr{C}$.

(1) We say that A is *self-dual* if there exists an isomorphism $A \xrightarrow{\sim} DA$.

(2) Assume that $(\mathscr{C}, D, \mathrm{ev})$ is an additive category with duality if $\sigma = -1$. We say that A is *σ-self-dual* if there exists a σ-symmetric isomorphism $A \xrightarrow{\sim} DA$.

We warn the reader that being 1-self-dual is more restrictive than being self-dual. If A is 1-self-dual or -1-self-dual, then $\mathrm{ev}\colon A \to DDA$ is an isomorphism.

Remark A.2.6. Let $(\mathscr{C}, D, \mathrm{ev})$ be an additive category with duality.

(1) The classes of self-dual objects and σ-self-dual objects of $\mathscr{D}$ are stable under finite products.

(2) If A is an object of $\mathscr{D}$ such that $\mathrm{ev}\colon A \to DDA$ is an isomorphism, then $A \oplus DA$ is 1-self-dual and -1-self-dual. In fact, the isomorphism $A \oplus DA \xrightarrow{\sigma\mathrm{ev}\oplus\mathrm{id}} DDA \oplus DA \simeq D(A \oplus DA)$ is σ-symmetric.

(3) There are self-dual objects that are neither 1-self-dual nor -1-self-dual (Corollary 2.2.6).

We close this subsection with a couple of lemmas on σ-self-dual objects. They are used in Subsection 2.2 but not in the rest of this appendix.

A σ-symmetric isomorphism $f\colon A \xrightarrow{\sim} DA$ induces an involution on $\mathrm{End}(A)$ carrying $g \in \mathrm{End}(A)$ to $f^{-1}(Dg)f$. If $\mathscr{C}$ is a k-linear category and D is a k-linear functor, then the involution is k-linear.

Lemma A.2.7. *Let $(\mathscr{C}, D, \mathrm{ev})$ be an additive category with duality. Let A be an object of $\mathscr{D}$ such that $R = \mathrm{End}(A)$ is a local ring and that 2 is invertible in R.*

(1) If A is self-dual with respect to D, then A is 1-self-dual or -1-self-dual with respect to D.

(2) If A is both 1-self-dual and -1-self-dual with respect to D, then every symmetric (resp. -1-symmetric) isomorphism $f\colon A \to DA$ induces a nontrivial involution on the residue division ring of R.

This is essentially [37, Proposition 2.5]. We recall the proof in our notation. It will be apparent from the proof that the additional assumption in [37] that D is a strong duality is not used.

Proof. (1) Since $A \simeq DA$, we have $\mathrm{End}(A) \simeq \mathrm{Hom}(A, DA)$. The image $M \subseteq \mathrm{Hom}(A, DA)$ of the maximal ideal of $R = \mathrm{End}(A)$ is the complement of the set of isomorphisms. For any $f \in \mathrm{Hom}(A, DA)$, we have $2f = (f + f^T) + (f - f^T)$, where $f + f^T$ is symmetric and $f - f^T$ is -1-symmetric. If f is an isomorphism, then $2f$ is an isomorphism, so that either $f + f^T$ or $f - f^T$ is an isomorphism.

(2) Let $g\colon A \to DA$ be a -1-symmetric (resp. symmetric) isomorphism. Then $h = f^{-1}g$ is a unit of R whose image under the involution induced by f is $-h$. Thus the involution is nontrivial on the residue field of R. $\square$

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Remark A.2.8.

(1) If $\mathscr{C}$ is an Abelian category and A is an indecomposable object of finite length, then $\mathrm{End}(A)$ is a local ring [1, Lemma 7].

(2) Let k be a separably closed field of characteristic $\neq 2$. Assume that $\mathscr{C}$ is a k-linear category, D is a k-linear functor, and $R = \mathrm{End}(A)$ is a finite k-algebra. Then any k-linear involution on R is trivial on the residue field. It follows then from Lemma A.2.7 that exactly one of the following holds: A is 1-self-dual; A is -1-self-dual; A is not self-dual.

Lemma A.2.9. *Let (D, ev) be a strong duality on an Abelian category $\mathscr{C}$. Let A be a σ-self-dual object of finite length. Then the semisimplification A^{ss} of A is σ-self-dual.*

Note that by assumption D is an equivalence of categories, hence an exact functor.

Proof. We fix a σ-symmetric isomorphism $f \colon A \to DA$. For any sub-object N of A, we let $N^{\perp}$ denote the kernel of the morphism $A \xrightarrow[\sim]{f} DA \to DN$. Then we have $A/N^{\perp} \simeq DN$, so that $N^{\mathrm{ss}} \oplus (A/N^{\perp})^{\mathrm{ss}}$ is σ-self-dual by Remark A.2.6. If N is totally isotropic, namely $N \subseteq N^{\perp}$, then f induces a σ-symmetric isomorphism $N^{\perp}/N \xrightarrow{\sim} D(N^{\perp}/N)$ (cf. [37, Lemma 5.2]). Now let N be a maximal totally isotropic sub-object of A. By [37, Theorem 6.12], $N^{\perp}/N$ is semisimple. Therefore, $A^{\mathrm{ss}} \simeq N^{\mathrm{ss}} \oplus (A/N^{\perp})^{\mathrm{ss}} \oplus N^{\perp}/N$ is σ-self-dual. $\qquad\square$

A.3 Duality and functors

In this subsection, we study symmetry of functors between categories with duality.

Given categories with duality $(\mathscr{C}, D_{\mathscr{C}}, \mathrm{ev})$ and $(\mathscr{D}, D_{\mathscr{D}}, \mathrm{ev})$, and functors $F, G \colon \mathscr{C} \to \mathscr{D}$, we sometimes refer to natural transformations $G D_{\mathscr{C}} \to D_{\mathscr{D}} F$ as *form transformations*. If $\mathscr{C}$ is the category with one object $*$ and one morphism id, and if we identify functors $\{*\} \to \mathscr{D}$ with objects of $\mathscr{D}$, then a form transformation is simply a form in $\mathscr{D}$. Form transformations are composed as follows.

Construction A.3.1. Let $(\mathscr{B}, D_{\mathscr{B}}, \mathrm{ev})$, $(\mathscr{C}, D_{\mathscr{C}}, \mathrm{ev})$, and $(\mathscr{D}, D_{\mathscr{D}}, \mathrm{ev})$ be categories with duality. Let $F, G \colon \mathscr{C} \to \mathscr{D}$ and $F', G' \colon \mathscr{B} \to \mathscr{C}$ be functors. Let $\alpha \colon F D_{\mathscr{C}} \to D_{\mathscr{D}} G$ and $\alpha' \colon F' D_{\mathscr{B}} \to D_{\mathscr{C}} G'$ be natural transformations. We define the *composite* of α and α' to be

$$\alpha\alpha' \colon FF'D_{\mathscr{B}} \xrightarrow{F\alpha'} FD_{\mathscr{C}}G' \xrightarrow{\alpha G'} D_{\mathscr{D}}GG'.$$

As the name suggests, form transformations act on forms. This can be seen as the case $\mathscr{B} = \{*\}$ of the preceding construction, as follows.

Construction A.3.2. Let $(\mathscr{C}, D_{\mathscr{C}}, \mathrm{ev})$ and $(\mathscr{D}, D_{\mathscr{D}}, \mathrm{ev})$ be categories with duality. Let $F, G \colon \mathscr{C} \to \mathscr{D}$ be functors and let $\alpha \colon F D_{\mathscr{C}} \to D_{\mathscr{D}} G$ be a natural transformation. Let A, B be objects of $\mathscr{C}$ and let $f \colon A \to D_{\mathscr{C}} B$ be a morphism in $\mathscr{C}$. The action of α on f is the composite

$$\alpha f \colon FA \xrightarrow{Ff} FD_{\mathscr{C}}B \xrightarrow{\alpha_B} D_{\mathscr{D}}GB.$$

We have the following notion of symmetry for form transformations.

Definition A.3.3 (Symmetry of form transformations)**.** Let $(\mathscr{C}, D_{\mathscr{C}}, \mathrm{ev})$ and $(\mathscr{D}, D_{\mathscr{D}}, \mathrm{ev})$ be categories with duality. Let $F, G \colon \mathscr{C} \to \mathscr{D}$ be functors. Assume that $(\mathscr{D}, D_{\mathscr{D}}, \mathrm{ev})$ is an additive category with duality if $\sigma = -1$.

(1) We define the *transpose* of a natural transformation $\beta \colon G D_{\mathscr{C}} \to D_{\mathscr{D}} F$ to be the composite

$$\beta^T \colon FD_{\mathscr{C}} \xrightarrow{\mathrm{ev}FD_{\mathscr{C}}} D_{\mathscr{D}}D_{\mathscr{D}}FD_{\mathscr{C}} \xrightarrow{D_{\mathscr{D}}\beta D_{\mathscr{C}}} D_{\mathscr{D}}GD_{\mathscr{C}}D_{\mathscr{C}} \xrightarrow{D_{\mathscr{D}}G\mathrm{ev}} D_{\mathscr{D}}G.$$

We call $\sigma\beta^T$ the σ-*transpose* of β.

(2) We say that a natural transformation $\alpha \colon F D_{\mathscr{C}} \to D_{\mathscr{D}} F$ is σ-*symmetric* if $\alpha = \sigma\alpha^T$.

Again we will often say "symmetric" instead of "1-symmetric". The terminology above is justified by the following easy lemma.

Lemma A.3.4. *We have $(\beta^T)^T = \beta$. Moreover, the map $\mathrm{Nat}(GD_{\mathscr{C}}, D_{\mathscr{D}}F) \to \mathrm{Nat}(FD_{\mathscr{C}}, D_{\mathscr{D}}G)$ carrying β to β^T is a bijection.*

The transpose $\alpha = \beta^T$ is uniquely characterized by the commutativity of the diagram

(A.3.1)
$$
\begin{array}{ccc}
F & \xrightarrow{\ F\mathrm{ev}\ } & FD_{\mathscr{C}}D_{\mathscr{C}} \\
{\scriptstyle \mathrm{ev}F}\downarrow & & \downarrow{\scriptstyle \alpha D_{\mathscr{C}}} \\
D_{\mathscr{D}}D_{\mathscr{D}}F & \xrightarrow{\ D_{\mathscr{D}}\beta\ } & D_{\mathscr{D}}GD_{\mathscr{C}}.
\end{array}
$$

If $\mathscr{C} = \{*\}$, Definition A.3.3 reduces to Definition A.2.2.

Remark A.3.5. A more direct analogue of Definition A.2.2 (1) for functors is as follows. Let $\mathscr{C}$ be a category and let $(\mathscr{D}, D_{\mathscr{D}}, \mathrm{ev})$ be a category with duality. Let $G \colon \mathscr{C} \to \mathscr{D}$ and $H \colon \mathscr{C} \to \mathscr{D}^{\mathrm{op}}$ be functors. Then the map $\mathrm{Nat}(G, D_{\mathscr{D}}H) \to \mathrm{Nat}(H, D_{\mathscr{D}}G)$ carrying $\gamma \colon G \to D_{\mathscr{D}}H$ to $\gamma^* \colon H \xrightarrow{\mathrm{ev}H} D_{\mathscr{D}}D_{\mathscr{D}}H \xrightarrow{D_{\mathscr{D}}\gamma} D_{\mathscr{D}}G$ is a bijection. Indeed, γ is a collection $(\gamma_A \colon GA \to D_{\mathscr{D}}HA)_A$ of forms in $\mathscr{D}$ and γ^* is characterized by $(\gamma^*)_A = (\gamma_A)^T$ for all objects A of $\mathscr{C}$, so that $(\gamma^*)^* = \gamma$. As one of the referee points out, this operation does not lead to a notion of symmetry, since G and H do not have the same variance.

Remark A.3.6. In the situation of Definition A.3.3, we have a bijection

(A.3.2)
$$
\mathrm{Nat}(GD_{\mathscr{C}}, D_{\mathscr{D}}F) \xrightarrow{\ \sim\ } \mathrm{Nat}(F, D_{\mathscr{D}}GD_{\mathscr{C}})
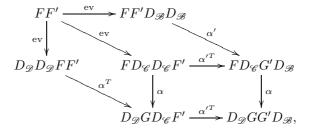$$

carrying β to β^*. Note that β^* is the composite $F \xrightarrow{F\mathrm{ev}} FD_{\mathscr{C}}D_{\mathscr{C}} \xrightarrow{\beta^T D_{\mathscr{C}}} D_{\mathscr{D}}GD_{\mathscr{C}}$, and β^T is the composite $FD_{\mathscr{C}} \xrightarrow{\beta^* D_{\mathscr{C}}} D_{\mathscr{D}}GD_{\mathscr{C}}D_{\mathscr{C}} \xrightarrow{D_{\mathscr{D}}G\mathrm{ev}} D_{\mathscr{D}}G$.

If equip the functor category $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ with the duality carrying G to $D_{\mathscr{D}}GD_{\mathscr{C}}$ and evaluation transformation given by $G \xrightarrow{\mathrm{ev}G\mathrm{ev}} D_{\mathscr{D}}D_{\mathscr{D}}GD_{\mathscr{C}}D_{\mathscr{C}}$, then natural transformations $F \to D_{\mathscr{D}}GD_{\mathscr{C}}$ are forms in this category with duality and Definition A.3.3 of transposes of forms applies. Definition A.3.3 is compatible with Definition A.2.2 via the bijection (A.3.2) in the sense that we have $(\beta^*)^T = (\beta^T)^*$.

Composition of form transformations is compatible with transposition.

Lemma A.3.7. *Let $(\mathscr{B}, D_{\mathscr{B}}, \mathrm{ev})$, $(\mathscr{C}, D_{\mathscr{C}}, \mathrm{ev})$, $(\mathscr{D}, D_{\mathscr{D}}, \mathrm{ev})$ be categories with duality. Let $F, G \colon \mathscr{C} \to \mathscr{D}$ and $F', G' \colon \mathscr{B} \to \mathscr{C}$ be functors. Let $\alpha \colon FD_{\mathscr{C}} \to D_{\mathscr{D}}G$ and $\alpha' \colon F'D_{\mathscr{B}} \to D_{\mathscr{C}}G'$ be natural transformations and let $\alpha\alpha' \colon FF'D_{\mathscr{B}} \to D_{\mathscr{D}}GG'$ be the composite. Then $(\alpha\alpha')^T = \alpha^T\alpha'^T$.*

Proof. In the diagram

all inner cells commute. It follows that the outer hexagon commutes. $\qquad\square$

Taking $\mathscr{B} = \{*\}$, we obtain the following compatibility of transposition with action of form transformations.

Lemma A.3.8. *Let $(\mathscr{C}, D_{\mathscr{C}}, \mathrm{ev})$ and $(\mathscr{D}, D_{\mathscr{D}}, \mathrm{ev})$ be categories with duality. Let $F, G \colon \mathscr{C} \to \mathscr{D}$ be functors equipped with a natural transformation $\alpha \colon FD_{\mathscr{C}} \to D_{\mathscr{D}}G$. Let $f \colon A \to D_{\mathscr{C}}B$ be a morphism in $\mathscr{C}$. Then $(\alpha f)^T = \alpha^T f^T$.*

The following consequence of Lemma A.3.8 is used many times in Section 2.

Lemma A.3.9. *Let $(\mathscr{C}, D_{\mathscr{C}}, \mathrm{ev})$ and $(\mathscr{D}, D_{\mathscr{D}}, \mathrm{ev})$ be categories with duality. Let $F \colon \mathscr{C} \to \mathscr{D}$ be a functor endowed with a symmetric natural isomorphism $\alpha \colon FD_{\mathscr{C}} \xrightarrow{\sim} D_{\mathscr{D}}F$.*

(1) F carries 1-self-dual objects of $\mathscr{C}$ to 1-self-dual objects of $\mathscr{D}$.

(2) If F is fully faithful, then the converse holds: any object A of $\mathscr{C}$ such that FA is 1-self-dual is 1-self-dual.

Proof. For (1), let $f\colon A \xrightarrow{\sim} D_{\mathscr{C}}A$ be a symmetric isomorphism. By Lemma A.3.8, αf is symmetric. The assertion follows from the fact that αf is an isomorphism. For (2), let $g\colon FA \xrightarrow{\sim} D_{\mathscr{D}}FA$ be a symmetric isomorphism. Since F is fully faithful, there exists a unique morphism $f\colon A \to D_{\mathscr{C}}A$ such that $\alpha f = g$. Note that f is an isomorphism. Since $\alpha f^T = g$, we have $f^T = f$. $\qquad\square$

The following lemma is used in Subsection 2.1 to show the symmetry of the middle extension functor.

Lemma A.3.10. *Let $(\mathscr{C}, D_{\mathscr{C}}, \mathrm{ev})$ and $(\mathscr{D}, D_{\mathscr{D}}, \mathrm{ev})$ be categories with duality. Assume that $\mathscr{D}$ is an Abelian category and $D_{\mathscr{D}}$ carries epimorphisms in $\mathscr{D}$ to monomorphisms. Let $E, G\colon \mathscr{C} \to \mathscr{D}$ be functors endowed with natural transformations $\alpha\colon E \to G$ and $\beta\colon GD_{\mathscr{C}} \to D_{\mathscr{D}}E$ such that the composite $ED_{\mathscr{C}} \xrightarrow{\alpha D_{\mathscr{C}}} GD_{\mathscr{C}} \xrightarrow{\beta} D_{\mathscr{D}}E$ is symmetric and such that the image functor $F\colon \mathscr{C} \to \mathscr{D}$ of α fits into a commutative diagram*
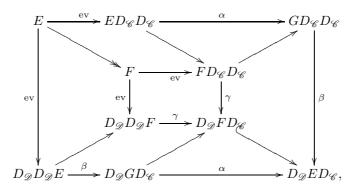
$$
\begin{array}{ccc}
FD_{\mathscr{C}} & \xrightarrow{\ \gamma\ } & D_{\mathscr{D}}F \\
\downarrow & & \downarrow \\
GD_{\mathscr{C}} & \xrightarrow{\ \beta\ } & D_{\mathscr{D}}E.
\end{array}
$$

Then the natural transformation $\gamma\colon FD_{\mathscr{C}} \to D_{\mathscr{D}}F$ is symmetric.

Proof. In fact, in the diagram

the outer square commutes by the symmetry of $\beta\alpha$ and all inner cells except the inner square commute. It follows that the inner square commutes. $\qquad\square$

We conclude this subsection with another example of form transformation, which will be used to handle the sign of the Lefschetz pairing (see Lemma A.5.11). We refer to [24, Remark 10.1.10 (ii)] for the convention on distinguished triangles in the opposite category of a triangulated category.

Lemma A.3.11. *Let $\mathscr{D}$ be a triangulated category equipped with a t-structure P. Let $(D, \mathrm{ev})\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{D}$ be a duality on the underlying category of $\mathscr{D}$. Assume that D underlies a right t-exact triangulated functor. We consider $\tau = {}^P\tau^{\geq a}$ and $\tau' = {}^P\tau^{\leq -a}$ as functors $\mathscr{D} \to \mathscr{D}$. Then the form transformations $\tau D \to D\tau'$ and $\tau'D \to D\tau$ induced by the diagrams*

$$(A.3.3) \qquad\qquad \tau D \to \tau D\tau' \xleftarrow{\sim} D\tau',$$

$$(A.3.4) \qquad\qquad \tau'D \xleftarrow{\sim} \tau'D\tau \to D\tau$$

are transposes of each other.

The second arrow in (A.3.3) is an isomorphism by the assumption that D carries ${}^P\tau^{\leq -a}$ to ${}^P\tau^{\geq -a}$. To see that the first arrow in (A.3.4) is an isomorphism, consider, for any object A of $\mathscr{D}$, the distinguished triangle

$$D\,{}^P\tau^{\geq a}A \xrightarrow{f} DA \to D\,{}^P\tau^{\leq a-1}A \to .$$

By assumption, $D\,{}^P\tau^{\leq a-1}A$ is in ${}^P\mathscr{D}^{\geq 1-a}$. Thus, by Lemma 4.1.9, ${}^P\tau^{\leq -a}f$ is an isomorphism.

Proof. The commutativity of (A.3.1) follows from the commutativity of the diagram

$$
\begin{array}{ccccc}
\tau & \longrightarrow & \tau DD & \longrightarrow & \tau D\tau' D & \overset{\sim}{\longleftarrow} & D\tau' D \\
\downarrow & & & & \downarrow & & \\
DD\tau & & & & DD\tau D\tau' D & \overset{\sim}{\longleftarrow} & DDD\tau' D \\
\downarrow & & & & \downarrow & & \\
D\tau' D\tau D & \longrightarrow & D\tau' D\tau DD & \longrightarrow & D\tau' D\tau D\tau' D & & D\tau' D \\
\simeq \uparrow & & \simeq \uparrow & & \nwarrow \simeq & & \downarrow \simeq \\
D\tau' D & \longrightarrow & D\tau' DDD & \longrightarrow & D\tau' DD\tau' D & & \\
& & & & & & D\tau' D & \overset{\sim}{\longrightarrow} & D\tau'\tau' D.
\end{array}
$$

$\square$

Remark A.3.12. For any truncation functor $\tau = {}^P\tau^{[a,b]}$ with dual truncation functor $\tau' = {}^P\tau^{[-b,-a]}$, combining the two form transformations in the lemma, we obtain a form transformation $\gamma_\tau \colon \tau D \to D\tau'$ whose transpose is γ'_τ. The form transformation γ_τ is an isomorphism if D is *t-exact*.

A.4 Duality in closed symmetric categories

In this subsection, we study dualities given by internal Hom functors in closed symmetric categories. Let $(\mathscr{C}, \otimes, c)$ be a closed symmetric category (Definition A.1.1).

Construction A.4.1. Let K be an object of $\mathscr{C}$. We let D_K denote the functor $\mathscr{H}om(-, K) \colon \mathscr{D}^{op} \to \mathscr{D}$. For an object A, the composite

$$
A \otimes D_K A \overset{c}{\to} D_K A \otimes A \overset{\mathrm{adj}}{\longrightarrow} K,
$$

where adj denotes the adjunction morphism, corresponds by adjunction to a morphism $A \to D_K D_K A$. This defines a natural transformation $\mathrm{ev} \colon \mathrm{id}_{\mathscr{D}} \to D_K D_K$, which makes D_K a duality on $\mathscr{D}$. The latter follows by adjunction from the commutativity of the diagram

$$
\begin{array}{ccc}
D_K A \otimes D_K D_K A & \overset{c}{\longrightarrow} & D_K D_K A \otimes D_K A \\
\mathrm{id}\otimes\mathrm{ev} \uparrow & & \mathrm{ev}\otimes\mathrm{id} \uparrow \quad \searrow {\scriptstyle \mathrm{adj}} \\
D_K A \otimes A & \overset{c}{\longrightarrow} & A \otimes D_K A \\
& \searrow {\scriptstyle \mathrm{adj}} & \downarrow \\
& & K.
\end{array}
$$

We defined transposes of pairings in symmetric categories (Definition A.2.2) and in categories with duality (Definition A.1.3). The two definitions are compatible via the above construction, by the following lemma.

Lemma A.4.2. *Let A, B, K be objects of $\mathscr{C}$. We put $D = D_K$. Then the following diagram commutes*

$$
\begin{array}{ccc}
\mathrm{Hom}(B \otimes A, K) & \overset{-\circ c}{\longrightarrow} & \mathrm{Hom}(A \otimes B, K) \\
\simeq \downarrow & & \downarrow \simeq \\
\mathrm{Hom}(B, DA) & \overset{D}{\longrightarrow} \mathrm{Hom}(DDA, DB) \overset{-\circ\mathrm{ev}(A)}{\longrightarrow} & \mathrm{Hom}(A, DB).
\end{array}
$$

Proof. Let $f \in \mathrm{Hom}(B, DA)$. The two images of f in $\mathrm{Hom}(A \otimes B, K)$ are the two composite morphisms in the commutative diagram

$$
\begin{array}{ccc}
A \otimes B & \overset{c}{\longrightarrow} & B \otimes A \\
\mathrm{id}\otimes f \downarrow & & \downarrow f\otimes\mathrm{id} \\
A \otimes DA & \overset{c}{\longrightarrow} DA \otimes A & \overset{\mathrm{adj}}{\longrightarrow} K.
\end{array}
$$

$\square$

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Following Definition A.2.5, we say A is *self-dual* with respect to K if $A \simeq D_K A$. We say A is σ-*self-dual* with respect to K if there exists a σ-symmetric isomorphism $A \xrightarrow{\sim} D_K A$, or, in other words, if there exists a σ-symmetric pairing $A \otimes A \to K$ that is *perfect* in the sense that it induces an isomorphism $A \xrightarrow{\sim} D_K A$.

Definition A.4.3. A *dualizing object* of $\mathscr{C}$ is an object K of $\mathscr{C}$ such that the evaluation transformation $\mathrm{ev} \colon \mathrm{id}_{\mathscr{C}} \to D_K D_K$ is a natural isomorphism, or, in other words, that (D_K, ev) is a strong duality.

Remark A.4.4. Let $\mathscr{B}$ be a closed symmetric *monoidal* category and let K be an object of $\mathscr{B}$. The associativity constraint induces an isomorphism $\mathscr{H}om(A, D_K B) \simeq D_K(A \otimes B)$ for objects A, B of $\mathscr{D}$. In particular, if K is a dualizing object, then $\mathscr{H}om(A, B) \simeq \mathscr{H}om(A, D_K D_K B) \simeq D_K(A \otimes D_K B)$.

We close this subsection with the construction of two symmetric form transformations.

Construction A.4.5. For a morphism $f \colon K \to L$ of $\mathscr{C}$, the natural transformation $D_f \colon \mathrm{id}_{\mathscr{C}} D_K \to D_L \mathrm{id}_{\mathscr{C}}$ is symmetric. This follows from the commutativity of the diagram

$$
\begin{array}{ccccc}
A \otimes D_K A & \xrightarrow{c} & D_K A \otimes A & \xrightarrow{\mathrm{adj}} & K \\
{\scriptstyle \mathrm{id} \otimes D_f} \downarrow & & {\scriptstyle D_f \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle f} \\
A \otimes D_L A & \xrightarrow{c} & D_L A \otimes A & \xrightarrow{\mathrm{adj}} & L.
\end{array}
$$

The action of D_f on forms (Construction A.3.2) carries $A \otimes B \to K$ to the composite $A \otimes B \to K \xrightarrow{f} L$.

Construction A.4.6. Let $\mathscr{C}$ and $\mathscr{D}$ be closed symmetric categories and let $G \colon \mathscr{C} \to \mathscr{D}$ be a right-lax symmetric functor (Definition A.1.5). For objects A, K of $\mathscr{C}$, consider the morphism

$$
G\mathscr{H}om(A, K) \xrightarrow{\mathrm{adj}} \mathscr{H}om(GA, G\mathscr{H}om(A, K) \otimes GA) \to \mathscr{H}om(GA, G(\mathscr{H}om(A, K) \otimes A))
$$
$$
\xrightarrow{\mathrm{adj}} \mathscr{H}om(GA, GK).
$$

This induces a symmetric natural transformation $GD_K \to D_{GK}G$ (cf. [40, Théorème 12.2.5]), whose action on forms carries $A \otimes B \to K$ to the pairing $GA \otimes GB \to GK$ of Construction A.1.8.

A.5 Symmetry and translation

The derived category of ℓ-adic sheaves is equipped with a shift functor $A \mapsto A[1]$ and the Tate twist functor $A \mapsto A(1)$. In this subsection, we study the effects of such translation functors on symmetry. Lemma A.5.11 is used in the main text to handle the symmetry of the Lefschetz pairing.

Recall that a *category with translation* [24, Definition 10.1.1 (i)] is a category $\mathscr{D}$ equipped with an equivalence of categories $T \colon \mathscr{D} \to \mathscr{D}$. We let $T^{-1} \colon \mathscr{D} \to \mathscr{D}$ denote a quasi-inverse of T. For an integer n, we will often write $[n]$ for T^n. Recall that a *functor of categories of translation* [24, Definition 10.1.1 (ii)] $(\mathscr{D}, T) \to (\mathscr{D}', T')$ is a functor $F \colon \mathscr{D} \to \mathscr{D}'$ endowed with a natural isomorphism $\eta \colon FT \xrightarrow{\sim} T'F$. Recall that a *morphism of functors of categories with translation* $(F, \eta) \to (G, \xi)$ is a natural transformation $\alpha \colon F \to G$ of functors such that the following diagram commutes

$$
\begin{array}{ccc}
FT & \xrightarrow[\sim]{\eta} & T'F \\
{\scriptstyle \alpha T} \downarrow & & \downarrow {\scriptstyle T'\alpha} \\
GT & \xrightarrow[\sim]{\xi} & T'G.
\end{array}
$$

Our first goal is to define duality on categories with translation, variant of Definition A.2.1. We endow $\mathscr{D}^{\mathrm{op}}$ with the translation functor $(T^{\mathrm{op}})^{-1} \colon \mathscr{D}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$. We endow $F^{\mathrm{op}} \colon \mathscr{A}^{\mathrm{op}} \to \mathscr{A}'^{\mathrm{op}}$ with the isomorphism $F^{\mathrm{op}}(T^{\mathrm{op}})^{-1} \xrightarrow{\sim} (T'^{\mathrm{op}})^{-1}F^{\mathrm{op}}$ induced by

$$
\eta^{\mathrm{op}} \colon T'^{\mathrm{op}}F^{\mathrm{op}} \xrightarrow{\sim} F^{\mathrm{op}}T^{\mathrm{op}}.
$$

Definition A.5.1 (Duality on a category with translation)**.** Let $(\mathscr{D}, T)$ be a category with translation. A *duality on* $(\mathscr{D}, T)$ is a functor of categories with translation $(D, \eta) \colon (\mathscr{D}^{\mathrm{op}}, (T^{\mathrm{op}})^{-1}) \to (\mathscr{D}, T)$ endowed

with a structure of duality on the underlying functor $D\colon \mathscr{D}^{\mathrm{op}} \to \mathscr{D}$ such that $\mathrm{ev}\colon \mathrm{id}_{\mathscr{D}} \to DD^{\mathrm{op}}$ is a morphism of functors of categories with translation. This means that the diagram

$$
\begin{array}{ccc}
T & \xrightarrow{\ \ \mathrm{ev}\ \ } & DD^{\mathrm{op}}T \\
{\scriptstyle \mathrm{ev}}\big\downarrow & & \big\downarrow {\scriptstyle D(T^{\mathrm{op}})^{-1}\eta^{\mathrm{op}}T} \\
TDD^{\mathrm{op}} & \xleftarrow{\ \eta D^{\mathrm{op}}\ } & D(T^{\mathrm{op}})^{-1}D^{\mathrm{op}}
\end{array}
$$

commutes. In other words, the isomorphisms $\eta^{-1}\colon TD \xrightarrow{\sim} D(T^{\mathrm{op}})^{-1}$ and $T^{-1}\eta T^{\mathrm{op}}\colon T^{-1}D \xrightarrow{\sim} DT^{\mathrm{op}}$ are *transposes* of each other in the sense of Definition A.3.3.

The above definitions have obvious additive variants. An *additive category with translation* is defined to be a category with translation whose underlying category is additive. For additive categories with translation $\mathscr{D}$ and $\mathscr{D}'$, a *functor of additive categories with translation* $\mathscr{D} \to \mathscr{D}'$ is defined to be a functor of categories with translation whose underlying functor is additive. An *additive duality* on an additive category with translation is a duality on the category with translation such that the underlying functor is additive.

As in the case without translation, a basic example of dualities on categories with translation is provided by closed symmetric categories with translation (see Construction A.5.9 below). Our next goal is to define symmetric categories with translation, variant of Definition A.1.1. Note that in the example of ℓ-adic sheaves, the shift and twist functors differ in signs with regard to tensor products. To deal with the two cases simultaneously, we let $\epsilon = \pm 1$. The case $\epsilon = -1$ of the following definition corresponds to [24, Definition 10.1.1 (v)]. For a more general notion, see [48, Définition I.1.4.4].

Definition A.5.2. Let $\mathscr{D}, \mathscr{D}', \mathscr{D}''$ be additive categories with translation. An ϵ-*bifunctor of additive categories with translation* $F\colon \mathscr{D} \times \mathscr{D}' \to \mathscr{D}''$ is an additive bifunctor endowed with functorial isomorphisms $F(A[1], B) \simeq F(A, B)[1]$ and $F(A, B[1]) \simeq F(A, B)[1]$ for objects A of $\mathscr{D}$ and B of $\mathscr{D}'$, such that the following diagram ϵ-commutes

$$
\begin{array}{ccc}
F(A[1], B[1]) & \xrightarrow{\ \sim\ } & F(A, B[1])[1] \\
{\scriptstyle \simeq}\big\downarrow & & \big\downarrow {\scriptstyle \simeq} \\
F(A[1], B)[1] & \xrightarrow{\ \sim\ } & F(A, B)[2].
\end{array}
$$

It then follows that the following diagram ϵ^{mn}-commutes

$$
\begin{array}{ccc}
F(X[m], Y[n]) & \xrightarrow{\ \sim\ } & F(X, Y[n])[m] \\
{\scriptstyle \simeq}\big\downarrow & & \big\downarrow {\scriptstyle \simeq} \\
F(X[m], Y)[n] & \xrightarrow{\ \sim\ } & F(X, Y)[m + n].
\end{array}
$$

Definition A.5.3 (Symmetric category with translation)**.** An ϵ-*symmetric additive category with translation* is an additive category with translation $\mathscr{D}$ endowed with a symmetric structure $\otimes$ and a structure of additive ϵ-bifunctor of categories with translation on $- \otimes -\colon \mathscr{D} \times \mathscr{D} \to \mathscr{D}$, such that the symmetry constraint, when restricted to each variable, is a morphism of functors of categories with translation $\mathscr{D} \to \mathscr{D}$. We say that an ϵ-symmetric additive category with translation is *closed* if its underlying symmetric category is closed.

For $\epsilon = 1$, Definitions A.5.2 and A.5.3 make sense without assuming that the categories in question are additive.

Example A.5.4. Let $\mathscr{C}$ be a symmetric *monoidal* category and let X be a *dualizable* object of $\mathscr{C}$, that is, there exists an object B of $\mathscr{C}$ such that $A \otimes B \simeq \mathbf{1}$. Then $- \otimes A$ endows $\mathscr{C}$ with the structure of a 1-symmetric category with translation. This applies in particular to the Tate twist functor on the Abelian category of perverse $\overline{\mathbb{Q}}_{\ell}$-sheaves.

Example A.5.5. The derived category of any commutatively ringed topos is a closed -1-symmetric additive category with translation. Similarly, the derived category of $\overline{\mathbb{Q}}_{\ell}$-sheaves is a closed -1-symmetric additive category with translation.

Let $\mathscr{D}$ be an ϵ-symmetric additive category with translation.

Lemma A.5.6. *The diagram*

$$A[m] \otimes B[n] \xrightarrow{\sim} (A \otimes B[n])[m] \xrightarrow{\sim} (A \otimes B)[m+n]$$

$$c \downarrow \simeq \qquad\qquad\qquad\qquad \simeq \downarrow c[m+n]$$

$$B[n] \otimes A[m] \xrightarrow{\sim} (B \otimes A[m])[n] \xrightarrow{\sim} (B \otimes A)[m+n]$$

ϵ^{mn}-*commutes for all objects* A, B *of* $\mathscr{D}$ *and all integers* m, n*. Here* c *denotes the symmetry constraint.*

Proof. In the diagram

$$A[m] \otimes B[n] \xrightarrow{\sim} (A \otimes B[n])[m] \xrightarrow{\sim} (A \otimes B)[m+n]$$

$$c \downarrow \qquad\qquad c[m] \downarrow \qquad\qquad c[m+n] \downarrow$$

$$B[n] \otimes A[m] \xrightarrow{\sim} (B[n] \otimes A)[m] \xrightarrow{\sim} (B \otimes A)[m+n]$$

$$(B \otimes A[m])[n],$$

the upper squares commute by functoriality, and the lower triangle ϵ^{mn}-commutes by the definition of ϵ-bifunctor of additive categories with translation. $\qquad\square$

Construction A.5.7. Let A, B, K be objects of $\mathscr{D}$. A pairing $A \otimes B \to K$ induces a pairing

$$(A[m]) \otimes (B[n]) \simeq (A \otimes B[n])[m] \simeq (A \otimes B)[m+n] \to K[m+n].$$

Lemma A.5.6 implies the following.

Lemma A.5.8. *Let* A, B, K *be objects of* $\mathscr{D}$*. Let* $A \otimes B \to K$ *and* $B \otimes A \to K$ *be pairings that are* σ*-transposes of each other. Then the induced pairings* $(A[m]) \otimes (B[n]) \to K[m+n]$ *and* $(B[n]) \otimes (A[m]) \to K[m+n]$ *are* $\epsilon^{mn}\sigma$*-transposes of each other.*

Let $\mathscr{D}$ be a *closed* ϵ-symmetric additive category with translation.

Construction A.5.9. Consider the isomorphisms

$$\alpha_n \colon \mathscr{H}om(A[-n], B) \xrightarrow{\sim} \mathscr{H}om(A, B)[n],$$
$$\beta_n \colon \mathscr{H}om(A, B[n]) \xrightarrow{\sim} \mathscr{H}om(A, B)[n]$$

given by the isomorphisms

$$\mathrm{Hom}(C, \mathscr{H}om(A[-n], B)) \simeq \mathrm{Hom}(C \otimes A[-n], B) \simeq \mathrm{Hom}((C \otimes A)[-n], B),$$
$$\mathrm{Hom}(C, \mathscr{H}om(A, B[n])) \simeq \mathrm{Hom}(C \otimes A, B[n]) \simeq \mathrm{Hom}((C \otimes A)[-n], B),$$
$$\mathrm{Hom}((C \otimes A)[-n], B) \simeq \mathrm{Hom}(C[-n] \otimes A, B)$$
$$\simeq \mathrm{Hom}(C[-n], \mathscr{H}om(A, B)) \simeq \mathrm{Hom}(C, \mathscr{H}om(A, B)[n])$$

for objects A, B, C of $\mathscr{D}$. We have $\alpha_m \alpha_n = \epsilon^{mn} \alpha_{m+n}$, $\beta_m \beta_n = \beta_{mn}$, $\alpha_m \beta_n = \epsilon^{mn} \beta_n \alpha_m$. We endow $\mathscr{H}om(-, -) \colon \mathscr{D}^{\mathrm{op}} \times \mathscr{D} \to \mathscr{D}$ with the structure of ϵ-bifunctor of additive categories with translation given by $\epsilon\alpha_1$ and β_1.[3] Let $\tilde{\alpha}_n = (\epsilon\alpha_1)^n = \epsilon^{n(n+1)/2}\alpha_n$.

In particular, $D_A \colon \mathscr{D}^{\mathrm{op}} \to \mathscr{D}$ is endowed with the structure of functor of additive categories with translation, which, together with $\mathrm{ev} \colon \mathrm{id}_\mathscr{D} \to D_A D_A^{\mathrm{op}}$, defines an additive duality on the additive category with translation (cf. [5, Proposition 3.2.1]).

Remark A.5.10. Construction A.5.7 corresponds to the construction sending $f \colon A \to D_K B$ to $\epsilon^{n(n-1)/2}$ times the morphism

$$A[m] \xrightarrow{f[m]} (D_K B)[m] \xrightarrow[\sim]{\tilde{\alpha}_{-n}} D_K(B[n])[m+n] \xrightarrow{\beta_{m+n}^{-1}} D_{K[m+n]}(B[n]),$$

---

[3]The sign convention is adopted here only to fix ideas. Our results do not depend on the convention.

where $\tilde{\alpha}$ and β are as in Construction A.5.9. In fact, the following diagram $\epsilon^{n(n-1)/2}$-commutes

$$
\begin{array}{ccc}
\mathrm{Hom}(A[m] \otimes B[n], K[m+n]) & \xrightarrow{\sim} & \mathrm{Hom}((A \otimes B[n])[m], K[m+n]) \\
\simeq \downarrow & & \downarrow \simeq \\
\mathrm{Hom}(A[m], D_{K[m+n]}(B[n])) & & \mathrm{Hom}((A \otimes B)[m+n], K[m+n]) \xrightarrow{\sim} \mathrm{Hom}(A \otimes B, K) \\
\simeq \downarrow \tilde{\alpha}_{-n} & & \downarrow \simeq \\
\mathrm{Hom}(A[m], D_K(B[n])[m+n]) & \xrightarrow[\sim]{\beta_{m+n}^{-1}} & \mathrm{Hom}(A[m], D_K B[m]) \xrightarrow{\quad\sim\quad} \mathrm{Hom}(A, D_K B).
\end{array}
$$

Thus, Construction A.5.7 corresponds to the form transformation $\gamma_{m,n} \colon T^m D_K \xrightarrow{\sim} D_{K[m+n]} T^n$, defined to be $\epsilon^{n(n-1)/2}$ times the isomorphism

$$
T^m D_K \xrightarrow[\sim]{\tilde{\alpha}_{-n}} T^{m+n} D_K T^n \xrightarrow[\sim]{\beta_{m+n}^{-1}} D_{K[m+n]} T^n
$$

given by Construction A.5.9. By the above, the ϵ^{mn}-transpose of $\gamma_{m,n}$ is $\gamma_{n,m}$.

We combine the above discussion on translation with our previous discussion on truncation into the following, which is applied in the proof of Proposition 3.2.12 to the Lefschetz pairing.

Lemma A.5.11. *Let $\mathscr{D}$ be a closed -1-symmetric additive category with translation. Assume that the underlying category with translation is further equipped with a triangulated structure and a t-structure P. Let K and L be objects of $\mathscr{D}$ such that D_L is a right t-exact triangulated functor. For any σ-symmetric pairing $A \otimes A \to K$ and any morphism $\xi \colon K[2n] \to L$, the pairing ${}^P\mathrm{H}^n A \otimes {}^P\mathrm{H}^n A \to L$ induced by*

$$
A[n] \otimes A[n] \xrightarrow{\sim} (A \otimes A)[2n] \to K[2n] \xrightarrow{\xi} L
$$

is $(-1)^n \sigma$-symmetric.

In fact, the form transformation ${}^P\mathrm{H}^n D_K \to D_L {}^P\mathrm{H}^n$ given by

$$
\tau^{[0,0]} T^n D_K \xrightarrow{\gamma_{n,n}} \tau^{[0,0]} D_{K[2n]} T^n \xrightarrow{D_\xi} \tau^{[0,0]} D_L T^n \xrightarrow{\gamma_\tau} D_L \tau^{[0,0]} T^n
$$

is $(-1)^n$-symmetric. Here $\gamma_{n,n}$ and γ_τ are as in Remarks A.5.10 and A.3.12.

Remark A.5.12. Let us mention in passing that Lurie's theory of stable ∞-categories [32, Chapter 1] provides a nicer framework for symmetric monoidal structures in derived categories. If $(\mathscr{D}, \otimes)$ is a closed symmetric monoidal ∞-category such that the underlying ∞-category $\mathscr{D}$ is stable, then $- \otimes -$ and $\mathscr{H}om(-, -)$ are automatically exact in each variable and the homotopy category of $\mathscr{D}$ is a closed -1-symmetric additive category with translation.

A.6 Duality and nilpotence

In this subsection, we study symmetry of primitive parts under a (twisted) nilpotent operator. We formulate the problem in the language of duality on category with translation introduced in Definition A.5.1. The main result of this subsection is Proposition A.6.8. This is applied in the main text to the logarithm of the monodromy operator associated to a normal crossing divisor to show that Grothendieck's six operations preserve $\mathrm{K}_{\mathrm{orth}}$ (see the proof of Proposition 4.3.1).

Let $(\mathscr{A}, T)$ be an additive category with translation. In this subsection, we denote $T^n A$ by $A(n)$ instead of $A[n]$. Our first goal is to define a category of objects with nilpotent operators and record its relation with duality.

Construction A.6.1. Consider the additive category $\mathrm{Nil}(\mathscr{A}, T)$ of pairs (A, N) of an object A of $\mathscr{A}$ and a morphism $N \colon A(1) \to A$ which is nilpotent in the sense that there exists an integer $d \geq 0$ such that $N^d := N \circ N(1) \circ \cdots \circ N(d-1) \colon A(d) \to A$ is the zero morphism. A morphism $(A, N) \to (A', N')$ is a morphism $f \colon A \to A'$ of $\mathscr{A}$ satisfying $N' f(1) = f N$.

There are two ways to identify $\mathrm{Nil}(\mathscr{A}, T)^{\mathrm{op}}$ and $\mathrm{Nil}(\mathscr{A}^{\mathrm{op}}, (T^{\mathrm{op}})^{-1})$, which differ by a sign. We fix $\sigma = \pm 1$ and consider the isomorphism of categories

$$
E_{\mathscr{A}} = E_{(\mathscr{A}, T), \sigma} \colon \mathrm{Nil}(\mathscr{A}, T)^{\mathrm{op}} \to \mathrm{Nil}(\mathscr{A}^{\mathrm{op}}, (T^{\mathrm{op}})^{-1})
$$

sending $(A, N\colon A(1) \to A)$ to $(A, \sigma N(-1)\colon A \to A(-1))$. The composite

$$\mathrm{Nil}(\mathscr{A}, T) \xrightarrow{E_{\mathscr{A}}^{\mathrm{op}}} \mathrm{Nil}(\mathscr{A}^{\mathrm{op}}, (T^{\mathrm{op}})^{-1})^{\mathrm{op}} \xrightarrow{E_{\mathscr{A}^{\mathrm{op}}}} \mathrm{Nil}(\mathscr{A}, T)$$

equals the identity. The duality we put on $\mathrm{Nil}(\mathscr{A}, T)$ will depend on the choice of σ. In the main text we take $\sigma = -1$.

Let $F\colon (\mathscr{A}, T) \to (\mathscr{A}', T')$ be a functor of additive categories with translation. Then F induces an additive functor $\mathrm{Nil}_F\colon \mathrm{Nil}(\mathscr{A}, T) \to \mathrm{Nil}(\mathscr{A}', T')$ carrying $(A, N\colon TA \to A)$ to $(FA, T'FA \simeq FTA \xrightarrow{FN} FA)$ and $f\colon (A, N) \to (A', N')$ to Ff. Let $\gamma\colon F \to F'$ be a morphism of functors of categories with translation. Then γ induces a natural transformation $\mathrm{Nil}_\gamma\colon \mathrm{Nil}_F \to \mathrm{Nil}_{F'}$, which is a natural isomorphism if γ is an isomorphism.

The following diagrams commute

$$
\begin{array}{ccc}
\mathrm{Nil}(\mathscr{A}, T)^{\mathrm{op}} & \xrightarrow{\mathrm{Nil}_F^{\mathrm{op}}} & \mathrm{Nil}(\mathscr{A}', T')^{\mathrm{op}} \\
{\scriptstyle E_{\mathscr{A}}} \downarrow & & \downarrow {\scriptstyle E_{\mathscr{A}'}} \\
\mathrm{Nil}(\mathscr{A}^{\mathrm{op}}, (T^{\mathrm{op}})^{-1}) & \xrightarrow{\mathrm{Nil}_{F^{\mathrm{op}}}} & \mathrm{Nil}(\mathscr{A}'^{\mathrm{op}}, (T'^{\mathrm{op}})^{-1})
\end{array}
\qquad
\begin{array}{ccc}
E_{\mathscr{A}'}\mathrm{Nil}_{F'}^{\mathrm{op}} & \xrightarrow{\mathrm{Nil}_\gamma^{\mathrm{op}}} & E_{\mathscr{A}'}\mathrm{Nil}_F^{\mathrm{op}} \\
\| & & \| \\
\mathrm{Nil}_{F^{\mathrm{op}}}E_{\mathscr{A}'} & \xrightarrow{\mathrm{Nil}_{\gamma^{\mathrm{op}}}} & \mathrm{Nil}_{F^{\mathrm{op}}}E_{\mathscr{A}}.
\end{array}
$$

Construction A.6.2. Let $D\colon (\mathscr{A}^{\mathrm{op}}, (T^{\mathrm{op}})^{-1}) \to (\mathscr{A}, T)$ be an additive duality on the additive category with translation (see the comment following Definition A.5.1). Consider the functor $D_{\mathrm{Nil}(\mathscr{A}, T)}$, composite of

$$\mathrm{Nil}(\mathscr{A}, T)^{\mathrm{op}} \xrightarrow[\sim]{E_{\mathscr{A}}} \mathrm{Nil}(\mathscr{A}^{\mathrm{op}}, (T^{\mathrm{op}})^{-1}) \xrightarrow{\mathrm{Nil}_D} \mathrm{Nil}(\mathscr{A}, T),$$

and the natural transformation

$$\mathrm{id}_{\mathrm{Nil}(\mathscr{A}, T)} \xrightarrow{\mathrm{Nil}_{\mathrm{ev}}} \mathrm{Nil}_D\mathrm{Nil}_{D^{\mathrm{op}}} = \mathrm{Nil}_D\mathrm{Nil}_{D^{\mathrm{op}}}E_{\mathscr{A}^{\mathrm{op}}}E_{\mathscr{A}}^{\mathrm{op}} = \mathrm{Nil}_D E_{\mathscr{A}}\mathrm{Nil}_D^{\mathrm{op}}E_{\mathscr{A}}^{\mathrm{op}} = D_{\mathrm{Nil}(\mathscr{A}, T)}D_{\mathrm{Nil}(\mathscr{A}, T)}^{\mathrm{op}}.$$

These define an additive duality on the additive category $\mathrm{Nil}(\mathscr{A}, T)$, which is strong if D is strong on $\mathscr{A}$.

In the rest of this section, let $(\mathscr{A}, T)$ be an *Abelian category with translation*, namely an additive category with translation whose underlying category $\mathscr{A}$ is Abelian. Our next goal is to review the decomposition into primitive parts. The following is a variant of [11, Proposition 1.6.1, 1.6.14], with essentially the same proof.

Lemma A.6.3. *Let* (A, N) *be an object of* $\mathrm{Nil}(\mathscr{A}, T)$. *Then there exists a unique finite increasing filtration* M *of* A *satisfying* $NM_j(1) \subseteq M_{j-2}$ *and such that for* $k \geq 0$, N^k *induces an isomorphism* $\mathrm{gr}_k^M A(k) \xrightarrow{\sim} \mathrm{gr}_{-k}^M A$.

Proof. Let $d \geq 0$ be an integer such that $N^{d+1} = 0$. We proceed by induction on d. We have $M_d = A$ and $M_{-d-1} = 0$. For $d > 0$, $M_{d-1} = \mathrm{Ker}(N^d)(-d)$ and $M_{-d} = \mathrm{Im}(N^d)$. We have $N^d = 0$ on $\mathrm{Ker}(N^d)(-d)/\mathrm{Im}(N^d)$ and let M' be the corresponding filtration given by induction hypothesis. For $-d \leq i \leq d - 1$, M_i is the inverse image in $\mathrm{Ker}(N^d)(-d)$ of $M_i' \subseteq \mathrm{Ker}(N^d)(-d)/\mathrm{Im}(N^d)$. $\qquad\square$

The following is an immediate consequence of the construction of the filtration M.

Lemma A.6.4. *Let* $f\colon (A, N) \to (A', N')$ *be a morphism of* $\mathrm{Nil}(\mathscr{A}, T)$. *Then* f *is compatible with the corresponding filtrations. More precisely, if* M *and* M' *denote the corresponding filtrations, then* $f(M_j) \subseteq M_j'$.

For $i \leq 0$, let $P_i(A, N) = \mathrm{Ker}(N\colon \mathrm{gr}_i^M A(1) \to \mathrm{gr}_{i-2}^M A)(-1)$. The inclusion $\mathrm{Ker}(N)(-1) \subseteq A$ induces an isomorphism $\mathrm{gr}_i^M(\mathrm{Ker}(N)(-1)) \xrightarrow{\sim} P_i(A, N)$. We thus obtain functors

$$P_i = P_{i, \mathscr{A}}\colon \mathrm{Nil}(\mathscr{A}, T) \to \mathscr{A}.$$

For all j, we have

$$\mathrm{gr}_j^M A \simeq \bigoplus_{\substack{k \geq |j| \\ k \equiv j \pmod 2}} P_{-k}(A, N)\left(-\tfrac{j+k}{2}\right).$$

We now proceed to define form transformations on the primitive part functors. Let (A, N) be an object of $\mathrm{Nil}(\mathscr{A}, T)$. If $M = M(A, N)$ and $M^* = M(E_{\mathscr{A}}(A, N))$, then we have the following short exact sequence in $\mathscr{A}$

$$0 \to M_{-j-1} \to A \to M_j^* \to 0,$$

so that $\operatorname{gr}^M_{-j} A$ can be identified with $\operatorname{gr}^{M^*}_j A$. Moreover, $N^{-i} \colon \operatorname{gr}^M_{-i} A(-i) \xrightarrow{\sim} \operatorname{gr}^M_i A$ induces an isomorphism in $\mathscr{A}$

$$\alpha_{\mathscr{A}}(A, N) \colon P_{i,\mathscr{A}^{\mathrm{op}}}(E_{\mathscr{A}}(A, N))(-i) \simeq \operatorname{gr}^M_{-i}(\operatorname{Coker}(N))(-i) \xrightarrow{\sim} \operatorname{gr}^M_i(\operatorname{Ker}(N)(-1)) \simeq P_{i,\mathscr{A}}(A, N).$$

This defines a natural isomorphism of functors $\alpha_{\mathscr{A}} \colon P^{\mathrm{op}}_{i,\mathscr{A}} \xrightarrow{\sim} (T^{\mathrm{op}})^{-i} P_{i,\mathscr{A}^{\mathrm{op}}} E_{\mathscr{A}}$. By definition, we have the following.

Lemma A.6.5. *The isomorphism* $T^{-i} P^{\mathrm{op}}_{i,\mathscr{A}^{\mathrm{op}}} E^{\mathrm{op}}_{\mathscr{A}} \xrightarrow[\sim]{\alpha^{\mathrm{op}}_{\mathscr{A}}} P_{i,\mathscr{A}} E_{\mathscr{A}^{\mathrm{op}}} E^{\mathrm{op}}_{\mathscr{A}} = P_{i,\mathscr{A}}$ *is* $\sigma^i \alpha^{\mathrm{op}}_{\mathscr{A}}$.

Let $(\mathscr{A}, T)$ and $(\mathscr{A}', T')$ be Abelian categories with translation and let $F \colon (\mathscr{A}, T) \to (\mathscr{A}', T')$ be a functor of categories with translation such that the underlying functor $\mathscr{A} \to \mathscr{A}'$ is *exact*. Let (A, N) be an object of $(\mathscr{A}, T)$ and let $M = M(A, N)$, $M' = M(\operatorname{Nil}_F(A, N))$. The exactness of F allows us to identify $F(M_j A)$ as a sub-object of FA, and under this identification we have $F(M_j A) = M'_j(FA)$. We have an obvious natural isomorphism $\beta_F \colon P_{i,\mathscr{A}'} \operatorname{Nil}_F \xrightarrow{\sim} F P_{i,\mathscr{A}}$. The following functoriality of β is obvious.

Lemma A.6.6. *Let* $F, F' \colon (\mathscr{A}, T) \to (\mathscr{A}', T')$ *be functors of categories with translation such that the underlying functors are exact and let* $\gamma \colon F \to F'$ *be a morphism of functors of categories with translation. Then the following diagram commutes*

$$
\begin{array}{ccc}
P_{i,\mathscr{A}'} \operatorname{Nil}_F & \xrightarrow[\sim]{\beta_F} & F P_{i,\mathscr{A}} \\
{\scriptstyle \operatorname{Nil}_\gamma} \downarrow & & \downarrow {\scriptstyle \gamma} \\
P_{i,\mathscr{A}'} \operatorname{Nil}_{F'} & \xrightarrow[\sim]{\beta_{F'}} & F' P_{i,\mathscr{A}}.
\end{array}
$$

By construction, the isomorphisms α and β have the following compatibility.

Lemma A.6.7. *The following diagram commutes*

$$
\begin{array}{ccccc}
(T'^{\mathrm{op}})^{-i} F^{\mathrm{op}} P_{i,\mathscr{A}^{\mathrm{op}}} E_{\mathscr{A}} & \xleftarrow{\;\sim\;} & F^{\mathrm{op}} (T^{\mathrm{op}})^{-i} P_{i,\mathscr{A}^{\mathrm{op}}} E_{\mathscr{A}} & \xleftarrow[\sim]{\alpha_{\mathscr{A}}} & F^{\mathrm{op}} P^{\mathrm{op}}_{i,\mathscr{A}} \\
{\scriptstyle \beta_{F^{\mathrm{op}}}} \uparrow {\scriptstyle \simeq} & & & & {\scriptstyle \simeq} \downarrow {\scriptstyle \beta^{\mathrm{op}}_F} \\
(T'^{\mathrm{op}})^{-i} P_{i,\mathscr{A}'^{\mathrm{op}}} \operatorname{Nil}_{F^{\mathrm{op}}} E_{\mathscr{A}} & = \!\!\!= & (T'^{\mathrm{op}})^{-i} P_{i,\mathscr{A}'^{\mathrm{op}}} E_{\mathscr{A}'} \operatorname{Nil}^{\mathrm{op}}_F & \xleftarrow[\sim]{\alpha_{\mathscr{A}'}} & P^{\mathrm{op}}_{\mathscr{A}'} \operatorname{Nil}^{\mathrm{op}}_F.
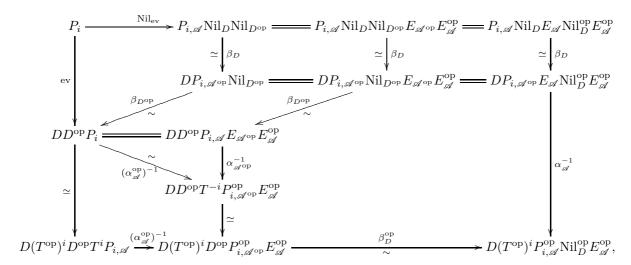\end{array}
$$

The following is the main result of this subsection.

Proposition A.6.8. *Let* $(\mathscr{A}, T)$ *be an Abelian category with translation and let* $D \colon (\mathscr{A}^{\mathrm{op}}, (T^{\mathrm{op}})^{-1}) \to (\mathscr{A}, T)$ *be a duality such that the underlying functor* $\mathscr{A}^{\mathrm{op}} \to \mathscr{A}$ *is exact. Then, for* $i \le 0$, *the composite isomorphism*

$$P_{i,\mathscr{A}} D_{\operatorname{Nil}(\mathscr{A}, T)} = P_{i,\mathscr{A}} \operatorname{Nil}_D E_{\mathscr{A}} \xrightarrow[\sim]{\beta_D} D P_{i,\mathscr{A}^{\mathrm{op}}} E_{\mathscr{A}} \xrightarrow[\sim]{\alpha^{-1}_{\mathscr{A}}} (D(T^{\mathrm{op}})^i) P^{\mathrm{op}}_{i,\mathscr{A}}$$

is σ^i*-symmetric.*

Note that $T^{-i} D \simeq D(T^{\mathrm{op}})^i \colon \mathscr{A}^{\mathrm{op}} \to \mathscr{A}$ endowed with the natural transformation $\operatorname{id}_{\mathscr{A}} \xrightarrow{\mathrm{ev}} DD^{op} \simeq (D(T^{\mathrm{op}})^i)(T^{-i} D)^{\mathrm{op}}$ is a duality on $\mathscr{A}$. By the proposition, $P_{i,\mathscr{A}}$ carries σ'-self-dual objects of $\operatorname{Nil}(\mathscr{A}, T)$ to $\sigma^i \sigma'$-self-dual objects of $\mathscr{A}$.

Proof. In the diagram

$$
\begin{array}{ccccccc}
P_i & \xrightarrow{\mathrm{Nil_{ev}}} & P_{i,\mathscr{A}}\mathrm{Nil}_D\mathrm{Nil}_{D^{\mathrm{op}}} & = & P_{i,\mathscr{A}}\mathrm{Nil}_D\mathrm{Nil}_{D^{\mathrm{op}}}E_{\mathscr{A}^{\mathrm{op}}}E_{\mathscr{A}}^{\mathrm{op}} & = & P_{i,\mathscr{A}}\mathrm{Nil}_D E_{\mathscr{A}}\mathrm{Nil}_D^{\mathrm{op}}E_{\mathscr{A}}^{\mathrm{op}}
\end{array}
$$

the triangle σ^i-commutes by Lemma A.6.5, the upper-left inner cell commutes by Lemma A.6.6, the lower-right inner cell commutes by Lemma A.6.7, and the other inner cells trivially commute. $\square$

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