Companions on Artin stacks

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Abstract

Deligne's conjecture that ℓ-adic sheaves on normal schemes over a finite field admit ℓ'-companions was proved by L. Lafforgue in the case of curves and by Drinfeld in the case of smooth schemes. In this paper, we extend Drinfeld's theorem to smooth Artin stacks and deduce Deligne's conjecture for coarse moduli spaces of smooth Artin stacks. In contrast to the case of smooth schemes, our proof relies on Gabber's theorem on the preservation of companionship. We also extend related theorems on Frobenius eigenvalues and traces.

Let $\mathbb{F}_q$ be a finite field and let ℓ and ℓ' be prime numbers not dividing q. We let $\overline{\mathbb{Q}_\ell}$ denote an algebraic closure of $\mathbb{Q}_\ell$. Deligne conjectured [8, Conjecture 1.2.10] that every lisse $\overline{\mathbb{Q}_\ell}$-sheaf on a normal scheme separated of finite type over $\mathbb{F}_q$ admits a lisse $\overline{\mathbb{Q}_{\ell'}}$-companion. Drinfeld [10, Theorem 1.1] proved this conjecture for smooth schemes. The goal of this paper is to extend Drinfeld's theorem to smooth Artin stacks. We deduce that Deligne's conjecture holds for coarse moduli spaces of smooth Artin stacks. We also extend related theorems on Frobenius eigenvalues and traces.

For an Artin stack X of finite presentation over $\mathbb{F}_q$ and a Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X, we let $E(\mathcal{F})$ denote the subfield of $\overline{\mathbb{Q}_\ell}$ generated by the local Frobenius traces $\mathrm{tr}(\mathrm{Frob}_x, \mathcal{F}_{\bar{x}})$, where $x \in X(\mathbb{F}_{q^n})$ and $n \geq 1$. Here $\mathrm{Frob}_x = \mathrm{Frob}_{q^n}$ denotes the geometric Frobenius, and $\bar{x}$ denotes a geometric point above x. Let $E_{\lambda'}$ be an algebraic extension of $\mathbb{Q}_{\ell'}$ and let $\sigma\colon E(\mathcal{F}) \to E_{\lambda'}$ be a field embedding, *not* necessarily continuous. We say that a Weil $E_{\lambda'}$-sheaf $\mathcal{F}'$ is a *σ-companion* of $\mathcal{F}$ if for all $x \in X(\mathbb{F}_{q^n})$ with $n \geq 1$, we have $\mathrm{tr}(\mathrm{Frob}_x, \mathcal{F}'_{\bar{x}}) = \sigma\mathrm{tr}(\mathrm{Frob}_x, \mathcal{F}_{\bar{x}})$.

Our main results on Frobenius eigenvalues and traces are as follows.

Theorem 0.1. *Let X be a geometrically unibranch Artin stack of finite presentation over $\mathbb{F}_q$ and let $\mathcal{F}$ be a simple lisse $\overline{\mathbb{Q}_\ell}$-sheaf of rank r on X such that* $\det(\mathcal{F})$ *has finite order.*

(1) (Frobenius eigenvalues) Let $x \in X(\mathbb{F}_{q^n})$ and let α be an eigenvalue of Frob_x *acting on $\mathcal{F}_{\bar{x}}$. Then α is a q-Weil number of weight* 0.[1] *Moreover, for every valuation v on $\mathbb{Q}(\alpha)$ such that $v(q^n) = 1$, we have $|v(\alpha)| \leq \frac{r-1}{2}$.*

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[1]We adopt the convention that a *q-Weil number of weight* 0 is an algebraic number α such that for every place (finite or Archimedean) λ of $\mathbb{Q}(\alpha)$ not dividing q, we have $|\alpha|_\lambda = 1$.

(2) *(Frobenius traces) The field $E(\mathcal{F})$ is a number field (namely, a finite extension of $\mathbb{Q}$).*

The statement of Theorem 0.1, with a slightly weaker bound for the p-adic valuations, is conjectured to hold for normal schemes separated of finite type over $\mathbb{F}_q$ by Deligne [8, Conjecture 1.2.10 (i)–(iv)]. In the case of curves, the theorem with a weaker bound is a consequence of the Langlands conjecture for $\mathrm{GL}(n)$ over function fields proved by L. Lafforgue [15, Théorème VII.6]. The improvement of the bound is due to V. Lafforgue [15, Corollaire 2.2]. The extension from curves to schemes is stated by L. Lafforgue [15, Proposition VII.7] for part (1), and due to Deligne [9, Théorème 3.1] for part (2).

Recently Drinfeld and Kedlaya [11, Theorem 1.3.3] proved a refinement of V. Lafforgue's bound for Newton polygons, which can be thought of as an analogue of Griffiths transversality. We also extend this result from smooth schemes to normal stacks (Theorem 2.5).

The following is our main result on companions.

Theorem 0.2 (Companions on smooth stacks)**.** *Let X be a smooth Artin stack over $\mathbb{F}_q$ of finite presentation and separated diagonal. Let $\mathcal{F}$ be a lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on X. Then, for every embedding $\sigma\colon E(\mathcal{F}) \to \overline{\mathbb{Q}_{\ell'}}$, $\mathcal{F}$ admits a lisse σ-companion $\mathcal{F}'$. Moreover, if $E(\mathcal{F})$ is a number field, then there exists a finite extension E of $E(\mathcal{F})$ such that for every finite place λ' of E not dividing q, $\mathcal{F}$ admits a lisse $\sigma_{\lambda'}$-companion. Here $\sigma_{\lambda'}\colon E(\mathcal{F}) \to E \to E_{\lambda'}$, and $E_{\lambda'}$ denotes the completion of E at λ'.*

The statements of Theorem 0.2 are conjectured to hold for normal schemes separated of finite type over $\mathbb{F}_q$ by Deligne [8, Conjecture 1.2.10 (v)]. In the case of curves, the first assertion of the theorem is due to L. Lafforgue [15, Théorème VII.6], and the second to Chin [6]. The extension from curves to smooth schemes is due to Drinfeld [10, Theorem 1.1].

As an application of Theorem 0.2, we deduce that Deligne's conjecture holds for coarse moduli spaces of smooth Artin stacks.

Corollary 0.3 (Companions on coarse moduli spaces)**.** *Let X be a scheme or algebraic space that is Zariski locally the coarse moduli space of a smooth Artin stack of finite inertia and finite presentation over $\mathbb{F}_q$ (e.g. X has quotient singularities). Then the statements of Theorem 0.2 hold for X.*

The general normal case seems difficult. Drinfeld deduces his result from an equivalence between lisse sheaves on a regular scheme X and compatible systems of lisse sheaves on curves on X [10, Theorem 2.5]. This equivalence fails for X normal in general [10, Section 6].

Gabber has shown that companionship is preserved by operations on the Grothendieck groups [13]. Combining this with Theorem 0.2, one obtains the existence of perverse companions. In the case of schemes, Gabber's theorem is not used in the proof of Theorem 0.2. By contrast, it turns out to be a key ingredient in reducing Theorem 0.2 to the case of schemes. In fact, for both Theorems 0.1 and 0.2, we reduce to the case of schemes and to showing that it suffices to check the assertions on any dense open substack.

The paper is organized as follows. In Section 1, we establish some preliminary results on Weil sheaves. In Section 2, we prove Theorems 0.1 (1) on Frobenius eigenvalues and Theorem 2.5 on Newton polygons. We deduce from Theorem 0.1 (1) that the bounded derived category $D^b(X, \overline{\mathbb{Q}_\ell})$ is a direct sum of twists of the derived category of weakly motivic complexes for any Artin stack X of finite presentation over $\mathbb{F}_q$. In Section 3, we prove Theorem 0.1 (2) on Frobenius traces. In Section 4, we prove Theorem 0.2 and Corollary 0.3 on lisse companions. We deduce results on perverse companions and companions in Grothendieck groups on Artin stacks of finite presentation and separated diagonal over $\mathbb{F}_q$. In an appendix (Section 5), we prove that pure perverse sheaves on X are geometrically semisimple, without assuming that the stabilizers are affine, extending a result of Sun [23, Theorem 3.11].

Unless otherwise stated, all stacks are assumed to be Artin stacks of finite presentation over $\mathbb{F}_q$, not necessarily of separated diagonal, and sheaves are assumed to be constructible. We write $D(X,\overline{\mathbb{Q}_\ell})$ for $D_c(X,\overline{\mathbb{Q}_\ell})$. We will only consider the middle perversity.

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1 Weil sheaves

For problems concerning companions, it is convenient to work with Weil sheaves. In this section, we establish some preliminary results on Weil sheaves. The main result is Proposition 1.7 on the determinant of lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaves on geometrically unibranch stacks. We deduce that the category of Weil $\overline{\mathbb{Q}_\ell}$-sheaves is a direct sum of the twists of the category of $\overline{\mathbb{Q}_\ell}$-sheaves (Proposition 1.12).

Let E_λ be an algebraic extension of $\mathbb{Q}_\ell$. A *Weil E_λ-sheaf* on a stack X is an E_λ-sheaf $\mathcal{F}$ on $X\otimes_{\mathbb{F}_q}\overline{\mathbb{F}_q}$ equipped with an action of the Weil group $W(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ lifting the action of $W(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on $X\otimes_{\mathbb{F}_q}\overline{\mathbb{F}_q}$. A morphism of Weil E_λ-sheaves on X is a morphism of the underlying E_λ-sheaves on $X\otimes_{\mathbb{F}_q}\overline{\mathbb{F}_q}$ compatible with the action of $W(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. More formally, the category $\mathrm{Shv}^W(X,E_\lambda)$ of Weil E_λ-sheaves on X is the 2-limit of the diagram (i.e. pseudofunctor) $B\mathbb{Z}\to\mathcal{C}at$ given by the action of $W(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on the category $\mathrm{Shv}(X\otimes_{\mathbb{F}_q}\overline{\mathbb{F}_q},E_\lambda)$, where $B\mathbb{Z}$ is the groupoid associated to the group $\mathbb{Z}$ and $\mathcal{C}at$ is the 2-category of categories. If we let $\mathcal{C}at^{B\mathbb{Z}}$ denote the 2-category of diagrams $B\mathbb{Z}\to\mathcal{C}at$, the forgetful 2-functor $\mathcal{C}at^{B\mathbb{Z}}\to\mathcal{C}at$ and the limit 2-functor $\lim\colon\mathcal{C}at^{B\mathbb{Z}}\to\mathcal{C}at$ preserve limits (up to equivalences).

Remark 1.1. The functor $\mathrm{Shv}(X,E_\lambda)\to\mathrm{Shv}^W(X,E_\lambda)$ carrying $\mathcal{F}$ to $(\mathcal{F}_{\overline{\mathbb{F}_q}},\phi)$, where $\mathcal{F}_{\overline{\mathbb{F}_q}}$ is the pullback of $\mathcal{F}$ to $X\otimes_{\mathbb{F}_q}\overline{\mathbb{F}_q}$ and ϕ is the restriction of the action of $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ to $W(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, is fully faithful. Moreover, its essential image is stable under extension by the following general facts on extensions (cf. [4, Proposition 5.1.2]).

(1) Let $(\mathcal{A},F)$ be an Abelian category with $\mathbb{Z}$-action (i.e. a pseudofunctor from $B\mathbb{Z}$ to the 2-category of Abelian categories). For objects (A,ϕ) and (B,ψ) of the limit category $\mathcal{A}^F$, we have a short exact sequence of Abelian groups (cf. [4, page 124])

$$0\to\mathrm{Hom}_{\mathcal{A}}(A,B)_{\mathbb{Z}}\to\mathrm{Ext}^1_{\mathcal{A}^F}((A,\phi),(B,\psi))\to\mathrm{Ext}^1_{\mathcal{A}}(A,B)^{\mathbb{Z}}\to 0.$$

(2) Let $\mathcal{D}$ be a triangulated category equipped with a t-structure. Note that $\mathcal{D}$ is not necessarily equivalent to the derived category of its heart $\mathcal{A}$. Nonetheless, for A and B in $\mathcal{A}$, we have an isomorphism $\mathrm{Hom}_{\mathcal{D}}(A, B[1]) \simeq \mathrm{Ext}^1_{\mathcal{A}}(A, B)$ carrying f to the extension given by completing f into a distinguished triangle (cf. [4, Remarque 3.1.17 (ii)]).

Lemma 1.2. *Let $f\colon X \to Y$ be a universally submersive morphism of stacks. Then f is of effective descent for Weil E_λ-sheaves and for E_λ-sheaves.*

The statement for Weil E_λ-sheaves means that f^* induces an equivalence of categories from $\mathrm{Shv}^W(Y, E_\lambda)$ to the category of descent data, namely the 2-limit of the diagram

$$\mathrm{Shv}^W(X, E_\lambda) \rightrightarrows \mathrm{Shv}^W(X \times_Y X, E_\lambda) \rightrightarrows\!\!\!\!\!\to \mathrm{Shv}^W(X \times_Y X \times_Y X, E_\lambda).$$

induced by inverse image functors. An object of the category of descent data is a Weil E_λ-sheaf $\mathcal{F}$ on X endowed with an isomorphism $p_1^*\mathcal{F} \simeq p_2^*\mathcal{F}$ satisfying the cocycle condition. Here $p_1, p_2\colon X \times_Y X \to X$ are the two projections.

Proof. By general properties of 2-limits, the case of Weil E_λ-sheaves reduces to the case of E_λ-sheaves (on stacks over $\overline{\mathbb{F}_q}$). That f is of descent then follows from the case of $\mathcal{O}$-sheaves (cf. [14, Proposition 2.4]), where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbb{Q}_\ell$ contained in E_λ. For the effectiveness, we argue directly as follows. By general properties of descent, we are reduced to the case where f is either smooth representable or a morphism of schemes. If f is smooth, then the result follows from smooth base change (cf. [1, VIII 9.4.1]). If f is a morphism of schemes, by a theorem of Voevodsky [26, Theorem 3.1.9], we further reduce to the following cases: (a) f is proper surjective; (b) f is faithfully flat. Case (a) follows from proper base change as in [1, VIII 9.4.2]. In case (b), we may repeat the argument of [1, VIII 9.4.3]. □

A Weil E_λ-sheaf $\mathcal{F}$ on a stack X is called *lisse* if there exists a smooth presentation $f\colon Y \to X$ such that the pullback of $\mathcal{F}$ to $Y \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ is isomorphic to $\mathcal{G} \otimes_{\mathcal{O}} E_\lambda$ for a lisse $\mathcal{O}$-sheaf $\mathcal{G}$. Lemma 1.2 also holds for lisse Weil E_λ-sheaves (and lisse E_λ-sheaves). This follows from the lemma and the following fact.

Lemma 1.3. *Let $f\colon X \to Y$ be a universally submersive morphism of stacks and let $\mathcal{F}$ be a Weil E_λ-sheaf on Y. Then $\mathcal{F}$ is lisse if and only if $f^*\mathcal{F}$ is lisse.*

Proof. The "only if" part is trivial. To show the "if" part, by taking presentations, we are reduced to the case of schemes (over $\overline{\mathbb{F}_q}$) and $\mathcal{O}$-sheaves. In this case, the assertion follows from the fact that f is of effective descent for étale morphisms [20, Theorem 5.17]. □

Remark 1.4. Let X be a geometrically unibranch stack. Every lisse Weil E_λ-sheaf $\mathcal{F}$ satisfies $j_*j^*\mathcal{F} \simeq \mathcal{F} \otimes j_*E_\lambda \simeq \mathcal{F}$ for every dominant open immersion $j\colon U \to X$. It follows that the pullback of $\mathcal{F}$ to $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ comes from a lisse $\mathcal{O}$-sheaf. If X is connected, then for any geometric point $\bar{x} \to X$, the functor $\mathrm{Shv}^W_{\mathrm{lisse}}(X, E_\lambda) \to \mathrm{Vect}(E_\lambda)$ carrying $\mathcal{F}$ to its stalk $\mathcal{F}_{\bar{x}}$ at $\bar{x}$ is conservative. Moreover, if X is connected, then Weil E_λ-sheaves on X correspond to E_λ-representations of the Weil group of X.

Remark 1.5. Let X be a stack. For Weil E_λ-sheaves $\mathcal{F}$ and $\mathcal{G}$ on X, we have $\mathcal{F} \simeq \mathcal{G}$ if and only if $\mathcal{F} \otimes_{E_\lambda} \overline{\mathbb{Q}_\ell} \simeq \mathcal{G} \otimes_{E_\lambda} \overline{\mathbb{Q}_\ell}$. Similarly, for A and B in the bounded derived category $D^b(X, E_\lambda)$ of E_λ-sheaves, $A \simeq B$ if and only if $A \otimes_{E_\lambda} \overline{\mathbb{Q}_\ell} \simeq B \otimes_{E_\lambda} \overline{\mathbb{Q}_\ell}$. This follows from

Lemma 1.6 below and the fact that rational points form a Zariski dense subset of any affine space over an infinite field (here E_λ).

Moreover, a Weil E_λ-sheaf $\mathcal{F}$ on X is a E_λ sheaf if and only if the Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F} \otimes_{E_\lambda} \overline{\mathbb{Q}_\ell}$ is a $\overline{\mathbb{Q}_\ell}$-sheaf. Indeed, we reduce to the case of schemes by Lemma 1.2 and then to lisse sheaves on irreducible geometrically unibranch schemes by Remark 1.1. In this case, the assertion is clear, as the Weil group is dense in the fundamental group.

For these reasons, we will work mostly with Weil $\overline{\mathbb{Q}_\ell}$-sheaves rather than Weil E_λ-sheaves.

The following is a variant of [25, Lemma 2.1.3].

Lemma 1.6. *Let X be a stack and let A and B be Weil E_λ-sheaves on X (resp. $A, B \in D^b(X, E_\lambda)$). Then there exists a Zariski open subscheme $U = \mathbf{Isom}(A,B)$ of the affine space $\mathbf{Hom}(A,B)$ over E_λ represented by the E_λ-vector space* $\mathrm{Hom}(A,B)$ *such that for any algebraic extension E'_λ of E_λ, the set $U(E'_\lambda)$ is the set of isomorphisms $A \otimes_{E_\lambda} E'_\lambda \xrightarrow{\sim} B \otimes_{E_\lambda} E'_\lambda$.*

Note that $\mathrm{Hom}(A,B)$ is finite-dimensional. Indeed, we have $\mathrm{Hom}(A,B) = \mathrm{Hom}(A_{\overline{\mathbb{F}_q}}, B_{\overline{\mathbb{F}_q}})^{W(\overline{\mathbb{F}_q}/\mathbb{F}_q)}$ in the case of Weil sheaves, and a short exact sequence

$$0 \to \mathrm{Hom}(A_{\overline{\mathbb{F}_q}}, B_{\overline{\mathbb{F}_q}}[-1])_{\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)} \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(A_{\overline{\mathbb{F}_q}}, B_{\overline{\mathbb{F}_q}})^{\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)} \to 0$$

in the case of $D^b(X, E_\lambda)$. Here $A_{\overline{\mathbb{F}_q}}$ and $B_{\overline{\mathbb{F}_q}}$ denote the pullback of A and B to $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. The proof of the lemma is the same as [25, Lemma 2.1.3], by taking a finite number of stalk functors (cf. Remark 1.4).

Following [8, 1.2.7], for $a \in \overline{\mathbb{Q}_\ell}^\times$, we let $\overline{\mathbb{Q}_\ell}^{(a)}$ denote the Weil sheaf on $\mathrm{Spec}(\mathbb{F}_q)$ of rank one such that the geometric Frobenius $\mathrm{Frob}_q \in \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ acts by multiplication by a. For a stack X, we still denote $\pi_X^* \overline{\mathbb{Q}_\ell}^{(a)}$ by $\overline{\mathbb{Q}_\ell}^{(a)}$, where $\pi_X \colon X \to \mathrm{Spec}(\mathbb{F}_q)$ is the projection. We put $\mathcal{F}^{(a)} := \mathcal{F} \otimes \overline{\mathbb{Q}_\ell}^{(a)}$.

The following is an extension to stacks of Deligne's result on determinants [8, Propositions 1.3.4 (i), 1.3.14] (cf. [9, 0.4]).

Proposition 1.7. *Let X be a connected geometrically unibranch stack. Then, for every lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X, there exists $a \in \overline{\mathbb{Q}_\ell}^\times$ such that* $\det(\mathcal{F}^{(a)})$ *has finite order. Moreover, every* simple *lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X such that* $\det(\mathcal{F})$ *has finite order is a $\overline{\mathbb{Q}_\ell}$-sheaf.*

Note that a is unique up to multiplication by roots of unity. It follows from the proposition that every simple lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf is the twist of some $\overline{\mathbb{Q}_\ell}$-sheaf.

Even if we restrict our attention to $\overline{\mathbb{Q}_\ell}$-sheaves, the first part of the proposition is still necessary for the following sections. Our proof of the proposition relies on the following lemma, which will be used in later sections as well.

Lemma 1.8. *Let $\mathcal{F}$ be a lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on a stack X. Then there exists a dominant open immersion $j \colon U \to X$ and a gerbe $f \colon U \to Y$, where Y is a Deligne-Mumford stack, such that the adjunction map $f^*\mathcal{G} \to j^*\mathcal{F}$, where $\mathcal{G} := f_* j^* \mathcal{F}$, is an isomorphism. Moreover, if X is connected and geometrically unibranch, then* $\det(\mathcal{F})$ *and* $\det(\mathcal{G})$ *have the same (possibly infinite) order.*

Note that $f^*\mathcal{G} \simeq j^*\mathcal{F}$ implies that $\mathcal{G}$ is a lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf, which is simple if $j^*\mathcal{F}$ is simple.

Proof. By Behrend's dévissage for stacks [3, Propositions 5.1.11, 5.1.14], there exists a dominant open immersion $j\colon U \to X$ and a gerbe $f\colon U \to Y$, where Y is a Deligne-Mumford stack, such that the diagonal of f has connected geometric fibers. By generic base change [14, Proposition 2.11], up to shrinking Y (and U), we may assume that $f_*\mathcal{F}$ commutes with base change. For the first assertion, it then suffices to check that the adjunction $f_y^* f_{y*}(\mathcal{F}\,|\,U_y) \to \mathcal{F}\,|\,U_y$ is an isomorphism for every geometric fiber $f_y\colon U_y \to y$ of f. Since U_y is the classifying stack of a connected group scheme over y, any sheaf on U_y is constant and the assertion is trivial.

Next we show the second assertion. If $\det(\mathcal{G})^{\otimes n} \simeq \overline{\mathbb{Q}_\ell}$, then $j^* \det(\mathcal{F})^{\otimes n} \simeq \overline{\mathbb{Q}_\ell}$, so that $\det(\mathcal{F})^{\otimes n} \simeq \overline{\mathbb{Q}_\ell}$. It remains to show that $\det(\mathcal{F})^{\otimes n} \simeq \overline{\mathbb{Q}_\ell}$ implies $\det(\mathcal{G})^{\otimes n} \simeq \overline{\mathbb{Q}_\ell}$. By generic base change, up to shrinking Y, we may assume that $f_*\overline{\mathbb{Q}_\ell}$ commutes with base change. Then the adjunction $\det(\mathcal{G})^{\otimes n} \to f_*f^* \det(\mathcal{G})^{\otimes n} \simeq \overline{\mathbb{Q}_\ell}$ is an isomorphism. □

Remark 1.9. Since any gerbe over a finite field is neutral [3, Corollary 6.4.2], any point $y \in Y(\mathbb{F}_{q^n})$ lifts to a point of $U(\mathbb{F}_{q^n})$. In particular, $E(\mathcal{G}) = E(j^*\mathcal{F})$.

Proof of Proposition 1.7. Applying Lemma 1.8, we are reduced to the case where X is a Deligne-Mumford stack. Here we have used the fact that $\mathcal{F} \simeq j_*j^*\mathcal{F}$ (and the same holds for tensor powers of $\det(\mathcal{F}^{(a)})$). Up to shrinking X, we may assume $X = [Y/G]$, where Y is an irreducible scheme and G is a finite group acting on Y. Let $g\colon Y \to X$. Then $g^*\mathcal{F}$ corresponds to the restriction to the open normal subgroup $\pi_1(Y) \lhd \pi_1(X)$ of quotient G. By the case of schemes of the first assertion [8, Proposition 1.3.4 (i)], there exists $a \in \overline{\mathbb{Q}_\ell}^\times$ such that $\det(g^*\mathcal{F}^{(a)})^{\otimes n} \simeq \overline{\mathbb{Q}_\ell}$ for some $n \geq 1$. Then $\det(\mathcal{F}^{(a)})^{\otimes n\#G}\overline{\mathbb{Q}_\ell}$. This finishes the proof of the first part of Proposition 1.7. Now assume that $\mathcal{F}$ is simple and $\det(\mathcal{F})$ has finite order. By Lemma 1.10 below and the case of schemes of the second assertion of Proposition 1.7 [8, Proposition 1.3.14], $g^*\mathcal{F}$ is a $\overline{\mathbb{Q}_\ell}$-sheaf, so that the same holds for $\mathcal{F}$ by Lemma 1.2. □

Lemma 1.10. *Let $f\colon X \to Y$ be a finite étale morphism of geometrically unibranch Deligne-Mumford stacks and let $\mathcal{F}$ be a simple lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on Y such that* $\det(\mathcal{F})$ *has finite order. Then $f^*\mathcal{F} \simeq \bigoplus_i \mathcal{F}_i$ is semisimple with simple factors $\mathcal{F}_i$ such that each* $\det(\mathcal{F}_i)$ *has finite order.*

Proof. We may assume that X and Y are irreducible. Since $\pi_1(X) < \pi_1(Y)$ is an open subgroup finite index, $f^*\mathcal{F} \simeq \bigoplus_i \mathcal{F}_i$ is semisimple. For the assertion on simple factors, we may assume that f is a Galois cover of group G. Fix an i_0. By the first assertion of Proposition 1.7, there exists $a \in \overline{\mathbb{Q}_\ell}^\times$ such that $\det(\mathcal{F}_{i_0}^{(a)})$ has finite order. Since the simple factors are permuted by G, $\det(\mathcal{F}_i^{(a)})$ has finite order for each i. It follows that $\det(\mathcal{F}^{(a)}) \simeq \bigotimes_i \det(\mathcal{F}_i^{(a)})$ has finite order. This implies that a is a root of unity. Therefore, each $\det(\mathcal{F}_i)$ has finite order. □

Remark 1.11. (1) In the above proof of Proposition 1.7, we first reduce to the case of Deligne-Mumford stacks by Lemma 1.8, and then reduce to the case of schemes. The same strategy will be used for Theorems 0.1 and 0.2. Another approach is to use the generic existence of a presentation with geometrically connected fibers [18, Théorème 6.5] to reduce directly to the case of schemes, which works for Proposition 1.7 and Theorem 0.1 (1), but fails for Theorems 0.1 (2) and 0.2.

(2) The reduction from Deligne-Mumford stacks to schemes here and in Theorem 0.1 (1) uses Lemma 1.10. We may replace this by Lemma 2.8 below, making the proofs closer to those of Theorem 0.1 (2) and Theorem 0.2.

(3) We can also prove Proposition 1.7 directly by imitating the proof of the case of schemes. Indeed, as in [8, Proposition 1.3.4, Variante], the first assertion follows from the case of curves by joining by curves (see the proof of Proposition 2.2) and Cebotarev's density theorem (Proposition 4.2). As in [8, Theorem 1.3.8], this implies Grothendieck's theorem on the identity component G^{00} of the geometric monodromy group: the radical of G^{00} is unipotent. Finally, as in [8, Proposition 1.3.14], the second assertion follows from the theorem and the first assertion.

Proposition 1.7 has the following consequence on the structure of Weil $\overline{\mathbb{Q}_\ell}$-sheaves. For a stack X, we let $\mathrm{Shv}(X,\overline{\mathbb{Q}_\ell})^{(a)}\subseteq \mathrm{Shv}^W(X,\overline{\mathbb{Q}_\ell})$ denote the full subcategory spanned by Weil $\overline{\mathbb{Q}_\ell}$-sheaves of the form $\mathcal{F}^{(a)}$ with $\mathcal{F}\in\mathrm{Shv}(X,\overline{\mathbb{Q}_\ell})$. The subcategory only depends on the class of a in $\overline{\mathbb{Q}_\ell}^\times/\overline{\mathbb{Z}_\ell}^\times$, where $\overline{\mathbb{Z}_\ell}$ denotes the ring of integers of $\overline{\mathbb{Q}_\ell}$.

Proposition 1.12. *Let X be a stack. We have a canonical decomposition:*

$$\mathrm{Shv}^W(X,\overline{\mathbb{Q}_\ell})\simeq\bigoplus_{a\in\overline{\mathbb{Q}_\ell}^\times/\overline{\mathbb{Z}_\ell}^\times}\mathrm{Shv}(X,\overline{\mathbb{Q}_\ell})^{(a)}.$$

Proof of Proposition 1.12. It suffices to show the following:

- (generation) Every object of $\mathrm{Shv}^W(X,\overline{\mathbb{Q}_\ell})$ is a successive extension of objects of $\mathrm{Shv}(X,\overline{\mathbb{Q}_\ell})^{(a)}$;
- (orthogonality) $\mathrm{Ext}^i(A^{(a)},B^{(b)})=0$ for $A,B\in\mathrm{Shv}(X,\overline{\mathbb{Q}_\ell})$, $a/b\notin\overline{\mathbb{Z}_\ell}^\times$ and $i=0,1$.

The first point follows from Proposition 1.7. Let us show the orthogonality. We have $\mathrm{Hom}(A^{(a)},B^{(b)})=\mathrm{Hom}(A^{(a)}_{\overline{\mathbb{F}}_q},B^{(b)}_{\overline{\mathbb{F}}_q})^{W(\overline{\mathbb{F}}_q/\mathbb{F}_q)}$ and a short exact sequence (Remark 1.1 (1), (2))

$$0\to\mathrm{Hom}(A^{(a)}_{\overline{\mathbb{F}}_q},B^{(b)}_{\overline{\mathbb{F}}_q})_{W(\overline{\mathbb{F}}_q/\mathbb{F}_q)}\to\mathrm{Ext}^1(A^{(a)},B^{(b)})\to\mathrm{Hom}(A^{(a)}_{\overline{\mathbb{F}}_q},B^{(b)}_{\overline{\mathbb{F}}_q}[1])^{W(\overline{\mathbb{F}}_q/\mathbb{F}_q)}.$$

The $\overline{\mathbb{Q}_\ell}$-vector space $\mathrm{Hom}(A^{(a)}_{\overline{\mathbb{F}}_q},B^{(b)}_{\overline{\mathbb{F}}_q}[i])$ with $W(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-action can be identified with the Weil $\overline{\mathbb{Q}_\ell}$-sheaf $H^i(R\pi_{X*}R\mathcal{H}om(A,B))^{(b/a)}$ on $\mathrm{Spec}(\mathbb{F}_q)$, where $\pi_X\colon X\to\mathrm{Spec}(\mathbb{F}_q)$. The eigenvalues of Frob_q are all in the class of b/a, so the action has no nonzero invariants or coinvariants. Therefore, $\mathrm{Ext}^i(A^{(a)},B^{(b)})=0$ for $i=0,1$. □

Corollary 1.13. *Let X be a stack. A Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ is a $\overline{\mathbb{Q}_\ell}$-sheaf if and only if for every $x\in X(\mathbb{F}_{q^n})$, $n\geq 1$, the eigenvalues of Frob_x acting on $\mathcal{F}_{\bar{x}}$ are ℓ-adic units.*

By reducing to curves (see Proposition 2.2 below), we see that for lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaves, it suffices to check the condition in the corollary for one given x in each connected component of X.

2 Frobenius eigenvalues

In this section, we prove Theorem 0.1 (1) on Frobenius eigenvalues. We then extend the theorem of Drinfeld and Kedlaya on Newton polygons from smooth schemes to normal stacks (Theorem 2.5). Finally, following Drinfeld [10, Appendix B], we study the category of weakly motivic complexes, whose cohomology sheaves have "motivic" Frobenius eigenvalues (Theorems 2.12 and 2.14).

Notation 2.1. We let M_0 denote the group of algebraic numbers $\alpha \in \overline{\mathbb{Q}}^\times$ of weight 0 relative to q, namely, such that for every Archimedean place λ of $\mathbb{Q}(\alpha)$, we have $|\alpha|_\lambda = 1$. We let $W_0(q) \subseteq M_0$ denote the subgroup of *q-Weil numbers of weight* 0, namely algebraic numbers α such that for every place λ of $\mathbb{Q}(\alpha)$ not dividing q (finite or Archimedean), we have $|\alpha|_\lambda = 1$. For a subset $S \subseteq \mathbb{Q}$ of slopes, we let $W_0^S(q) \subseteq W_0(q)$ denote the subgroup of $\alpha \in W_0(q)$ of slope in S, namely such that for every valuation v on $\mathbb{Q}(\alpha)$ such that $v(q) = 1$, we have $v(\alpha) \in S$.

Note that $W_0(q)$ only depends on the characteristic p of $\mathbb{F}_q$ and that $W_0^{\{0\}}(q)$ is simply the set of roots of unity.

In the notation above, Theorem 0.1 (1) says that the eigenvalues of Frob_x acting on $\mathcal{F}_{\bar{x}}$ belong to $W_0^{[-\frac{r-1}{2},\frac{r-1}{2}]\cap\mathbb{Q}}(q^n)$ for all $x \in X(\mathbb{F}_{q^n})$ and all $n \geq 1$. As mentioned earlier, we prove this by reducing to the case of schemes. For the reduction to work, we need to show that the statement can be checked on any dense open substack. We start by reviewing Deligne's argument of joining by curves [9, Proposition 1.9] and extending it to stacks.

Proposition 2.2. *Let X be a connected stack. Then there exists an integer $M \geq 1$ such that, for every lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X of rank $r \geq 1$, and for all $m, n \geq 1$, $x \in X(\mathbb{F}_{q^m})$, $y \in X(\mathbb{F}_{q^n})$, if we let $\alpha_1, \dots, \alpha_r$ (resp. $\beta_1, \dots, \beta_r$) denote the eigenvalues of* Frob_x *(resp.* Frob_y*) acting on $\mathcal{F}_{\bar{x}}$ (resp. $\mathcal{F}_{\bar{y}}$), then, up to reordering, we have $\beta_i^{1/n}/\alpha_i^{1/m} \in W_0^{[-M(r-1),M(r-1)]\cap\mathbb{Q}}(q)$ for $1 \leq i \leq r$. Moreover, we may take $M-1$ to be the diameter of the intersection graph of the irreducible components of a flat presentation of X by a separated algebraic space.*

In the situation of the proposition, if $\alpha_1, \dots, \alpha_r \in W_0(q)$, then $\beta_1, \dots, \beta_r \in W_0(q)$.

Proof. Let $Y \to X$ be a flat presentation with Y separated. Consider the intersection graph of the irreducible components of Y over X. The vertices are the irreducible components of Y. There is an edge between two vertices v and w if and only if the corresponding components Y_v and Y_w are such that $Y_v \times_X Y_w$ is nonempty. We may assume that the graph is connected. For each edge $e = (v, w)$ of the graph, choose a closed point x_e of $Y_v \times_X Y_w$. We take $M-1$ to be the diameter of the graph. For x and y as in the statement of the proposition, let v and w be vertices such that $\bar{x}$ lifts to Y_v and $\bar{y}$ lifts to Y_w. Let $v = v_1 \xrightarrow{e_1} v_2 \cdots v_{N-1} \xrightarrow{e_{N-1}} v_N = w$ be a path of length $N-1 \leq M-1$. Note that any pair of closed points on an irreducible algebraic space of finite presentation over a field are in the image of a connected regular curve (cf. [19, Section 6]). Thus there exists a diagram above X

$$x_0 \to C_1 \leftarrow x_1 \to \cdots \leftarrow x_{N-1} \to C_N \leftarrow x_N,$$

where C_j, $1 \leq j \leq N$ are connected smooth curves over $\mathbb{F}_q$ above Y_{v_j}, and $x_j = \mathrm{Spec}(\mathbb{F}_{q^{m_j}})$, $0 \leq j \leq N$ such that x_0 is above x, x_j is above x_{e_j} for $1 \leq j \leq N-1$, and x_N is above y. We apply the proof of L. Lafforgue's theorem [15, Théorème VII.6] (or V. Lafforgue's improvement of the bound [16, Corollaire 2.2]) to C_j, $1 \leq j \leq N$, and to the simple subquotients of the pullback of $\mathcal{F}$ to C_j. If $\alpha_1^{(j)}, \dots, \alpha_r^{(j)}$ denote the eigenvalues of Frob_{x_j}, then up to reordering the r values, we have $(\alpha_i^{(j)})^{1/n_j}/(\alpha_i^{(j-1)})^{1/n_{j-1}} \in W_0^{[-(r-1),(r-1)]}(q)$. □

Proposition 2.3. *Let R be an integrally closed subring of $\overline{\mathbb{Q}_\ell}$. Let $j\colon U \to X$ be a dominant open immersion of stacks and let $\mathcal{F}$ be a lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on X such that $j^*\mathcal{F}$ is R-integral (resp. inverse R-integral). Then $\mathcal{F}$ is R-integral (resp. inverse R-integral).*

Following [27, Variante 5.13, Définition 6.1], we say that a Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X is *R-integral* (resp. *inverse R-integral*) if for all $n \geq 1$ and all $x \in X(\mathbb{F}_{q^n})$, the eigenvalues (resp. inverse eigenvalues) of Frob_x acting on $\mathcal{F}_{\bar{x}}$ are in R.

Proof. Up to replacing X by a presentation, we may assume that X is a scheme. Up to replacing X by its normalization, we may further assume that X is normal. In this case, $\mathcal{F} \simeq j_* j^* \mathcal{F}$ is R-integral (resp. inverse R-integral) by [27, Théorème 2.5, Variante 5.1]. □

Remark 2.4. Let $I = [a, b] \cap \mathbb{Q}$ be an interval with $a, b \in \mathbb{Q}$. It follows from the propositions that if $\mathcal{F}$ is a lisse $\overline{\mathbb{Q}_\ell}$-sheaf on a stack X such that the eigenvalues of Frob_x acting on $\mathcal{F}_{\bar{x}}$ belong to $W_0^I(q^n)$ for all $n \geq 1$ and all $x \in U(\mathbb{F}_{q^n})$, where U is some dense open substack of X, then the same holds for all $x \in X(\mathbb{F}_{q^n})$. Indeed, the eigenvalues belong to $W_0(q)$ by Proposition 2.2, and $\mathcal{F}^{(q^{-a})}$ is $\overline{\mathbb{Z}}$-integral and $\mathcal{F}^{(q^{-b})}$ is inverse $\overline{\mathbb{Z}}$-integral (for all representatives of q^{-a} and q^{-b}) by Proposition 2.3. Here $\overline{\mathbb{Z}}$ denotes the ring of algebraic integers.

Proof of Theorem 0.1 (1). By Remark 2.4, we may shrink X. Thus, by Lemma 1.8, we are reduced to the case of Deligne-Mumford stacks. Up to shrinking X, we may assume that there exists a finite étale cover $f\colon Y \to X$, where Y is a scheme. By Lemma 1.10, $f^*\mathcal{F} \simeq \bigoplus_i \mathcal{F}_i$, with $\mathcal{F}_i$ simple and $\det(\mathcal{F}_i)$ of finite order. We are thus reduced to the case where X is a scheme. This case was stated in [15, Proposition VII.7], and the gap in the proof has been fixed by Deligne [9, Théorème 1.6] and others. Indeed, by Remark 2.4 again, we may assume that X is a smooth separated scheme. By a consequence of Hilbert irreducibility ([10, Proposition 2.17] or [12, Proposition B.1]), for any closed point x of X, there exists a smooth curve C over $\mathbb{F}_q$ and a morphism $g\colon C \to X$ such that x is in the image of g and $g^*\mathcal{F}$ is simple. It then suffices to apply L. Lafforgue's theorem for curves [15, Théorème VII.6] and V. Lafforgue's improvement of the bound [16, Corollaire 2.2]. □

More generally V. Lafforgue proved an inequality for the Newton polygon in the case of curves. Recently Drinfeld and Kedlaya [11, Theorem 1.3.3] gave a refinement for the lowest Newton polygon in the case of smooth schemes. These results extend to normal stacks as follows.

For a stack X, we let $|X|$ denote the set of isomorphism classes of the groupoid $X(\overline{\mathbb{F}_q})$. If X is a Deligne-Mumford stack, then $|X|$ can be identified with the set of closed points of X. In general, following [11, Lemma 5.3.4], we equip $|X|$ with the following topology T: a subset $U \subseteq |X|$ is T-open if and only if for every morphism $C \to X$ from a smooth curve C to X, the inverse image of U under the map $|C| \to |X|$ is open for the Zariski topology on $|C|$.

We fix a valuation v on $\overline{\mathbb{Q}}$ such that $v(q) = 1$. For a $\overline{\mathbb{Q}}$-integral Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X and $x \in X(\mathbb{F}_{q^n})$, we let $s_1^x(\mathcal{F}) \leq \cdots \leq s_r^x(\mathcal{F})$ denote the images under v/n of the eigenvalues of Frob_x acting on $\mathcal{F}_{\bar{x}}$. These rational numbers are called the *slopes* of $\mathcal{F}$ at x and depend on x only through the image of x in $|X|$.

Theorem 2.5. *Let X be an irreducible geometrically unibranch stack. Let $\mathcal{F}$ be a $\overline{\mathbb{Q}}$-integral lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf of rank r on X. Then*

(1) *There exist rational numbers $s_1(\mathcal{F}) \leq \cdots \leq s_r(\mathcal{F})$ such that $\sum_{j=1}^i s_j(\mathcal{F}) \leq \sum_{j=1}^i s_j^x(\mathcal{F})$ for all x and all i and the set $Y \subseteq |X|$ of y satisfying $s_i^y(\mathcal{F}) = s_i(\mathcal{F})$ for all i is nonempty and T-open.*

(2) If $\mathcal{F}$ is simple and $\det(\mathcal{F})$ *has finite order, then* $s_{i+1}(\mathcal{F}) \leq s_i(\mathcal{F}) + 1$ *for all* $1 \leq i \leq r-1$*, so that* $s_i^x(\mathcal{F}) \geq -i(r-i)/2$ *for all x and all i.*

The numbers $s_i(\mathcal{F})$ are the slopes of the lowest Newton polygon. To prove the theorem, we need a few lemmas. Lemmas 2.6 and 2.7 below extend [11, Lemmas 5.3.1, 5.3.4].

Lemma 2.6. *Let X be a stack and let $\mathcal{F}$ be a $\overline{\mathbb{Q}}$-integral lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on X. For all i, the function $x \mapsto \sum_{j=1}^{i} s_j^x(\mathcal{F})$ on $|X|$ is upper semi-continuous for the topology T, bounded, and takes values in $N^{-1}\mathbb{Z}$ for some N.*

Proof. By the definition of the T-topology, for the semi-continuity we may assume that X is a smooth curve. We reduce then to the case $\mathcal{F}$ simple, and then to the case where $\det(\mathcal{F})$ has finite order. In this case, the assertion follows from Abe's theorem on crystalline companions [2, Theorem 4.3.1] and the corresponding statement for overconvergent F-isocrystals. The boundedness follows from Proposition 2.2. The last assertion follows from the fact that $E(\mathcal{F})$ is a number field (Theorem 3.1) by the proof of [11, Lemma 5.3.1]. □

Lemma 2.7. *Let X be an irreducible scheme or an irreducible geometrically unibranch stack. Then $|X|$ is irreducible for the topology T.*

Proof. If X is a separated algebraic space, then the assertion follows from the fact that each pair of closed points can be joined by a smooth curve, as recalled in the proof of Proposition 2.2. Next note that if X admits a Zariski open cover (X_i) such that $|X_i|$ is T-irreducible for all i, then X is T-irreducible. Indeed, for any nonempty T-open subset $U \subseteq |X|$, there exists i such that $U \cap |X_i|$ is nonempty, and it follows that $U \cap |X_j|$ is nonempty for all j. Thus the case of schemes reduces to the case of separated schemes. Assume that X is a geometrically unibranch stack. Let $f\colon Y \to X$ be a presentation with Y a scheme. The images of the connected components Y_i of Y form a Zariski open cover of X. Thus we are reduced to the case where Y is nonempty and connected, hence irreducible. Since $|Y|$ is T-irreducible and $|f|$ is a surjection, $|X|$ is T-irreducible. □

Lemma 2.8. *Let $X = [Y/G]$ be a separated geometrically unibranch quotient stack of an algebraic space Y by an affine group scheme G. Then there exists a surjective morphism $f\colon X' \to X$ with X' a separated smooth scheme such that for every simple lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on X, $f^*\mathcal{F}$ is simple.*

Proof. We may assume X integral. Note that the last assertion holds for morphisms between irreducible geometrically unibranch stacks that either admit sections or are dominant open immersions. Choose an embedding $G \to \mathrm{GL}_m$. Consider the embedding $\mathrm{GL}_m \to \mathbb{A}^{m^2}$ into the affine space of $m \times m$ matrices, which is equivariant under the action of GL_m. The projection $Z = [Y \times \mathbb{A}^{m^2}/G] \to [Y/G]$ admits a section. Thus we may replace X by the dense open subspace $Y \wedge^G \mathrm{GL}_m = (Y \times \mathrm{GL}_m)/G \subseteq Z$. By Chow's lemma, we may further assume that X is a quasi-projective scheme. Applying Gabber's refinement [28, Lemme 3.8] of de Jong's alterations [7], we find a Galois alteration $(Y', H) \to (X, \{1\})$ with Y' smooth and quasi-projective. We are now reduced to the case $X = [Y'/H]$. It then suffices to perform the reduction step again and take $X' = Y' \wedge^H \mathrm{GL}_n$ for some embedding $H \to \mathrm{GL}_n$. □

Proof of Theorem 2.5. (1) By Lemma 2.6, the function $a_i\colon x \mapsto \sum_{j=1}^{i} s_j^x(\mathcal{F})$ on $|X|$ attains a minimum. We define $s_1 \leq \cdots \leq s_r$ so that the minimum of a_i is $\sum_{j=1}^{i} s_j$. Moreover, the locus $Y_i \subseteq |X|$ on which a_i attains the minimum is T-open. Therefore, $Y = \bigcap_{i=1}^{r-1} Y_i$ is nonempty and T-open by Lemma 2.7.

(2) The second inequality follows from the first one. To show the first inequality, we may shrink X by Lemma 2.7. Thus, by Lemma 1.8, we may assume that X is a Deligne-Mumford stack. Further shrinking X, we may assume that X is a separated quotient stack $[Y/G]$ by a finite group G. By Lemma 2.8, we reduce to the case where X is a separated smooth scheme, which is [11, Theorem 1.3.3]. □

Let $\iota\colon \overline{\mathbb{Q}_\ell} \to \mathbb{C}$ be an embedding. Following [22, 2.4.3], we say that a Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on a stack X is *punctually ι-pure of weight $w \in \mathbb{R}$* if for every $x \in X(\mathbb{F}_{q^n})$, $n \geq 1$ and every eigenvalue α of Frob_x acting on $\mathcal{F}_{\bar{x}}$, we have $|\iota\alpha| = q^{w/2}$. The results of Sun in [22] and [23] extend to stacks not necessarily of separated diagonal, using the fact that group algebraic spaces of finite presentation over fields are group schemes [21, Tags 08BH, 0B8G]. For $w \in \mathbb{Z}$, we say that $\mathcal{F}$ is *punctually pure of weight w* if it is punctually ι-pure of weight w for all ι.

Remark 2.9. It follows from Theorem 0.1 (1) and Proposition 1.7 that every simple lisse Weil sheaves on a geometrically unibranch stack is punctually ι-pure. It follows that every Weil $\overline{\mathbb{Q}_\ell}$-sheaf on a stack is ι-mixed, namely, a successive extension of ι-pure sheaves) (cf. [22, Remark 2.8.1]). Similarly, a Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on a stack is mixed, namely, a successive extension of pure sheaves (of integral weights), if and only if for every $x \in X(\mathbb{F}_{q^n})$, $n \geq 1$, the eigenvalues of Frob_x acting on $\mathcal{F}_{\bar{x}}$ belong to $M(q^n)$. Here $M(q) := \bigcup_{w\in\mathbb{Z}} q^{w/2} M_0$. (Recall that M_0 is the group of algebraic numbers of weight 0.)

The structure of punctually ι-pure Weil $\overline{\mathbb{Q}_\ell}$-sheaves can be described as follows. We let $\mathcal{E}_n$ denote the $\overline{\mathbb{Q}_\ell}$-sheaf on $\mathrm{Spec}(\mathbb{F}_q)$ of stalk $\overline{\mathbb{Q}_\ell}^n$ on which Frob_q acts unipotently with one Jordan block.

Proposition 2.10. *Let X be a geometrically unibranch stack. Then indecomposable punctually ι-pure lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaves are of the form $\mathcal{F} \otimes \pi_X^* \mathcal{E}_n$ with $\mathcal{F}$ simple, where $\pi_X\colon X \to \mathrm{Spec}(\mathbb{F}_q)$.*

In the appendix we will prove an analogue for pure perverse sheaves. The proposition still holds with $\overline{\mathbb{Q}_\ell}$ replaced by an algebraic extension of $\mathbb{Q}_\ell$.

Proof. As in the case of schemes [4, Proposition 5.3.9 (i)], this follows from the geometric semisimplicity of ι-pure lisse $\overline{\mathbb{Q}_\ell}$-sheaves [23, Theorem 2.1 (iii)]. □

Let $W(q) = \bigcup_{w\in\mathbb{Z}} q^{w/2} W_0(q)$ be the group of q-Weil numbers (of integral weights). We say that $K \in D^b(X, \overline{\mathbb{Q}_\ell})$ is *weakly motivic* if for all $n \geq 1$, $x \in X(\mathbb{F}_{q^n})$, and $i \in \mathbb{Z}$, the eigenvalues of Frob_x acting on $H^i K_{\bar{x}}$ belong to $W(q^n)$. We let $D_{\mathrm{mot}}(X, \overline{\mathbb{Q}_\ell}) \subset D(X, \overline{\mathbb{Q}_\ell})$ denote the full subcategory spanned by weakly motivic complexes, which is a thick subcategory. For $* \in \{+, -, b\}$, we put $D^*_{\mathrm{mot}} = D^* \cap D_{\mathrm{mot}}$. By definition, $D_{\mathrm{mot}}(X, \overline{\mathbb{Q}_\ell})^{(a)}$ only depends on the class of a in $\overline{\mathbb{Z}_\ell}^\times / W(q)$.

Remark 2.11. By Proposition 2.2, for a lisse $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on a connected stack X and a fixed $x \in X(\mathbb{F}_{q^n})$, $\mathcal{F}$ is weakly motivic if and only if the eigenvalues of Frob_x acting on $\mathcal{F}_{\bar{x}}$ are in $W(q^n)$.

The following result generalizes [10, Theorems B.3, B.4].

Theorem 2.12. *Let f be a morphism of stacks. The six operations and Verdier duality induce*

$$\begin{gathered}\otimes\colon D^-_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})\times D^-_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})\to D^-_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell}),\\ R\mathcal{H}om\colon D^-_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})^{\mathrm{op}}\times D^+_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})\to D^+_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell}),\\ D\colon D_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})^{\mathrm{op}}\to D_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell}),\quad f^*,f^!\colon D_{\mathrm{mot}}(Y,\overline{\mathbb{Q}_\ell})\to D_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell}),\\ f_*\colon D^+_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})\to D^+_{\mathrm{mot}}(Y,\overline{\mathbb{Q}_\ell}),\quad f_!\colon D^-_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})\to D^-_{\mathrm{mot}}(Y,\overline{\mathbb{Q}_\ell}).\end{gathered}$$

Proof. Note that $W(q)=R(q)^\times\cap M(q)$, where $R(q)$ is the algebraic closure of $\mathbb{Z}[1/q]$. By [27, Variante 5.13, Section 6] (which extends easily to stacks not necessarily of separated diagonals), complexes with R-integral (resp. inverse R-integral) cohomology sheaves are preserved by the operations. By Remark 2.9, having Frobenius eigenvalues in $M(q^n)$ is equivalent to being mixed, and complexes with mixed cohomology sheaves are preserved by the operations by [22, Remark 2.12]. □

As in [4, Stabilités 5.1.7], the theorem has the following consequence.

Corollary 2.13. *The perverse truncation functors on $D(X,\overline{\mathbb{Q}_\ell})$ preserve $D_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})$ and induce a t-structure on $D_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})$.*

Theorems 0.1 (1) and 2.12 imply the following.

Theorem 2.14. *For any stack X, we have a canonical decomposition for the bounded derived category of $\overline{\mathbb{Q}_\ell}$-sheaves:*

$$D^b(X,\overline{\mathbb{Q}_\ell})\simeq\bigoplus_{a\in\overline{\mathbb{Z}_\ell}^\times/W(q)}D^b_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})^{(a)}.$$

The case of schemes is [10, Theorem B.7].

Proof. The proof is very similar to the case of schemes and parallel to the proof of Proposition 1.12. It suffices to show the following:

- (generation) Every object of $D^b(X,\overline{\mathbb{Q}_\ell})$ is a successive extension of objects of $D^b_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})^{(a)}$;
- (orthogonality) $\mathrm{Hom}(A^{(a)},B^{(b)})=0$ for $A,B\in D^b_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})$, $a/b\notin W(q)$.

The first point follows from Proposition 1.7 and Theorem 0.1 (1). For the orthogonality, note that

$$\mathrm{Hom}(A^{(a)},B^{(b)})\simeq H^0(\mathrm{Spec}(\mathbb{F}_q),R\pi_{X*}R\mathcal{H}om(A,B)^{(b/a)})=0,$$

where $\pi_X\colon X\to\mathrm{Spec}(\mathbb{F}_q)$. Here we used the fact that $R\pi_{X*}R\mathcal{H}om(A,B)$ is in $D^+_{\mathrm{mot}}(\mathrm{Spec}(\mathbb{F}_q),\overline{\mathbb{Q}_\ell})$ by Theorem 2.12. □

Remark 2.15. The same decomposition holds for categories of $\overline{\mathbb{Q}_\ell}$-sheaves and perverse $\overline{\mathbb{Q}_\ell}$-sheaves. In particular, the subcategory of weakly motivic perverse sheaves $\mathrm{Perv}_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})\subseteq\mathrm{Perv}(X,\overline{\mathbb{Q}_\ell})$ is stable under subquotient.

3 Frobenius traces

Theorem 0.1 (2) follows immediately from Theorem 0.1 (1) and the following.

Theorem 3.1. *Let X be a stack and let $\mathcal{F}$ be a Weil $\overline{\mathbb{Q}_\ell}$-sheaf on X. Then $E(\mathcal{F})$ is a finitely generated extension of $\mathbb{Q}$. In particular, $E(\mathcal{F})$ is a number field if and only if for all $n \geq 1$ and all $x \in X(\mathbb{F}_{q^n})$, the eigenvalues of* Frob_x *acting on $\mathcal{F}_{\bar{x}}$ are algebraic numbers.*

The case of schemes is a theorem of Deligne [9, Théorème 3.1, Remarque 3.9].

Proof. To show the first assertion, by induction, we may replace X by a dense open substack. In particular, we may assume that $\mathcal{F}$ is lisse. Moreover, by Lemma 1.8 and Remark 1.9 (or the fact that any sub-extension of a finitely generated field extension is finitely generated [5, page V.113, Corollaire 3]), we may assume that X is a Deligne-Mumford stack. We may further assume that $X \simeq [Y/G]$ for a finite group G acting on an affine scheme Y. Choose an embedding $G \to \mathrm{GL}_m$ and take $Z = Y \wedge^G \mathrm{GL}_m = (Y \times \mathrm{GL}_m)/G$. Then $f\colon Z \to X$ is a GL_m-torsor. We have $E(f^*\mathcal{F}) = E(\mathcal{F})$. Indeed, any point $x \in X(\mathbb{F}_{q^n})$ lifts to a point of $Z(\mathbb{F}_{q^n})$ by Hilbert's Theorem 90. We then apply the case of schemes [9, Théorème 3.1, Remarque 3.9] to $f^*\mathcal{F}$ on Z. For the second assertion, it suffices to note that the Frobenius eigenvalues are algebraic numbers if and only if the Frobenius traces are algebraic numbers. □

Corollary 3.2. *Let $\mathcal{F}$ be a Weil $\overline{\mathbb{Q}_\ell}$-sheaf on a stack X and let $\mathcal{G}$ be a subquotient of $\mathcal{F}$. Then $E(\mathcal{G})$ is contained in a finite extension of $E(\mathcal{F})$.*

Proof. For each $x \in X(\mathbb{F}_{q^n})$, the eigenvalues of Frob_x on $\mathcal{F}_{\bar{x}}$ are contained in a finite extension of $E(x^*\mathcal{F}) \subseteq E(\mathcal{F})$. The assertion then follows from the theorem, which says that $E(\mathcal{G})$ is generated by the traces $\mathrm{tr}(\mathrm{Frob}_x, \mathcal{G}_{\bar{x}})$ at a finite number of points x with varying n. □

For any morphism $f\colon X \to Y$ of stacks and any Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{G}$ on Y, we have $E(f^*\mathcal{G}) \subseteq E(\mathcal{G})$.

Corollary 3.3. *Let $f\colon X \to Y$ be a morphism of stacks with X nonempty and Y connected. For any* lisse *Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{G}$ on Y, the field $E(\mathcal{G})$ is a finite extension of $E(f^*\mathcal{G})$. In particular, for any $n \geq 1$ and any $y \in Y(\mathbb{F}_{q^n})$, the field $E(\mathcal{G})$ is a finite extension of $E(y^*\mathcal{G})$ (the field generated by $\mathrm{tr}(\mathrm{Frob}_y^m, \mathcal{F}_{\bar{y}})$, $m \geq 1$).*

Proof. The first assertion follows from the second one. For any $y' \in X(\mathbb{F}_{q^{n'}})$, the eigenvalues of $\mathrm{Frob}_{y'}$ on $\mathcal{F}_{\overline{y'}}$ is contained in a finite extension of $E(y^*\mathcal{G})$ by Proposition 2.2. The first assertion then follows from the theorem, which says that $E(\mathcal{G})$ is generated by the traces $\mathrm{tr}(\mathrm{Frob}_{y'}, \mathcal{G}_{\overline{y'}})$ at a finite number of points y' with varying n'. □

Corollary 3.4. *Let $f\colon X \to Y$ be a surjective morphism of stacks. Then, for any Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{G}$ on Y, the field $E(\mathcal{G})$ is a finite extension of $E(f^*\mathcal{G})$.*

Proof. This follows from Corollary 3.3 by taking a stratification of Y by connected strata such that the restriction of $\mathcal{G}$ to each stratum is lisse. □

Under additional assumptions, Corollaries 3.3 and 3.4 admit Propositions 3.5 and 3.7 below as refinements. We say that a morphism of stacks is *irreducible* if the geometric fibers are irreducible.

Proposition 3.5. *Let $f\colon X\to Y$ be an irreducible morphism of stacks. Let $\mathcal{G}$ be a Weil $\overline{\mathbb{Q}_\ell}$-sheaf on Y. Then $E(f^*\mathcal{G})=E(\mathcal{G})$. Moreover, if $\mathcal{G}'$ is a Weil $\overline{\mathbb{Q}_{\ell'}}$-sheaf on Y such that $f^*\mathcal{G}'$ is a σ-companion of $f^*\mathcal{G}$ for some embedding $\sigma\colon E(\mathcal{G})\to\overline{\mathbb{Q}_{\ell'}}$, then $\mathcal{G}'$ is a σ-companion of $\mathcal{G}$.*

Proof. Since $\mathrm{tr}(F^n)$ for all integers $n\geq m$ for some fixed m determine the characteristic polynomial of F, the proposition follows from the following lemma. □

Lemma 3.6. *Let X be a geometrically irreducible stack over $\mathbb{F}_q$. Then there exists an integer m such that X admits a $\mathbb{F}_{q^n}$-point for every $n\geq m$.*

Proof. Let d be the dimension of X. Consider $H_c^i=H_c^i(X\otimes_{\mathbb{F}_q}\overline{\mathbb{F}_q},\overline{\mathbb{Q}_\ell})$. Then $H_c^{2d}\simeq\overline{\mathbb{Q}_\ell}(-d)$, and for $j>0$, $H_c^{2d+j}=0$ and H_c^{2d-j} has weights $\leq 2d-\frac{j}{2}$ [22, Theorem 1.4]. Let $\iota\colon\overline{\mathbb{Q}_\ell}\to\mathbb{C}$ be an embedding. By [22, Theorem 4.3], $M_n=\sum_\alpha|\iota\alpha|^n<\infty$, where α runs through the multiset of eigenvalues of H_c^{2d-j}, $j>0$. Since $M_n\leq q^{(n-1)(d-\frac{1}{4})}M_1$, we have $M_n<q^{dn}$ for $n\gg 0$. By the trace formula, we then have

$$\sum_{x\in X(\mathbb{F}_q^n)}\frac{1}{\#\mathrm{Aut}(x)}=q^{dn}+\sum_\alpha(\pm\alpha^n)>0.$$

□

The following is a consequence of [28, Proposition 3.10], which in turn is a consequence of Gabber's theorem on the preservation of companionship.

Proposition 3.7. *Let $f\colon X\to Y$ be a dominant open immersion of smooth stacks, Y having separated diagonal, and let $\mathcal{G}$ be a lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on Y. Then $E(f^*\mathcal{G})=E(\mathcal{G})$. Moreover, if $\mathcal{G}'$ is a lisse Weil $\overline{\mathbb{Q}_{\ell'}}$-sheaf on X such that $f^*\mathcal{G}'$ is a σ-companion of $f^*\mathcal{G}$ for some embedding $\sigma\colon E(\mathcal{G})\to\overline{\mathbb{Q}_{\ell'}}$, then $\mathcal{G}'$ is a σ-companion of $\mathcal{G}$.*

Proof. By the existence of smooth neighborhood [18, Théorème 6.3] (here we used the assumption that Y has separated diagonal), any point $x\in X(\mathbb{F}_{q^n})$ factorizes through a smooth morphism $Y'\to Y$, where Y' is a scheme. We are thus reduced to the case of schemes, which follows from [28, Proposition 3.10]. □

4 Companions

In this section, we prove Theorem 0.2 on the existence of lisse companions on smooth stacks of separated diagonal and Corollary 0.3 on the existence of lisse companions on coarse moduli spaces. We then deduce the existence of perverse companions on stacks of separated diagonal (Theorem 4.10). We also deduce that companionship induces isomorphisms among the Grothendieck groups of Weil $\overline{\mathbb{Q}_\ell}$-sheaves for varying ℓ (Corollary 4.12).

To apply the reduction steps to Theorem 0.2, again we need to show that we may shrink X. This is done by combining Gabber's theorem with Drinfeld's theorem. Let $E_{\lambda'}$ be an algebraic extension of $\mathbb{Q}_{\ell'}$.

Proposition 4.1. *Let $j\colon U\to X$ be a dominant open immersion of smooth stacks, X having separated diagonal. Let $\mathcal{F}$ be a lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on X and let $\sigma\colon E(\mathcal{F})\to E_{\lambda'}$ and $\iota'\colon E_{\lambda'}\to\mathbb{C}$ be embeddings. Assume that $j^*\mathcal{F}$ admits a lisse (punctually) ι'-pure σ-companion $\mathcal{G}'$. Then $j_*\mathcal{G}'$ is a lisse σ-companion of $\mathcal{F}$.*

Proof. If $j_*\mathcal{G}'$ is lisse, then $j_*\mathcal{G}'$ is a σ-companion of $\mathcal{F}$ by Proposition 3.7. It remains to show that $j_*\mathcal{G}'$ is lisse. For this we may assume $E_{\lambda'} = \overline{E_{\lambda'}}$. Since any pullback of $\mathcal{G}'$ is punctually ι'-pure, we may assume that X is a scheme. By [4, Proposition 5.3.9 (i)] (see Proposition 2.10), $j_*\mathcal{G}'$ is lisse if and only if $j_*\mathcal{G}'^{\mathrm{ss}}$ is lisse, where $\mathcal{G}'^{\mathrm{ss}}$ is the semisimplification of $\mathcal{G}'$. Indeed, we have $j_*(\mathcal{G}_0 \otimes \pi_U^*\mathcal{E}_n) \simeq (j_*\mathcal{G}_0) \otimes \pi_X^*\mathcal{E}_n$ for any sheaf $\mathcal{G}_0$ on U. By Drinfeld's theorem, $\mathcal{F}$ admits a lisse σ-companion $\mathcal{F}'$, which we may assume semisimple. By Chebotarev's density theorem, we have $\mathcal{G}'^{\mathrm{ss}} \simeq j^*\mathcal{F}'$. Therefore, $j_*\mathcal{G}'^{\mathrm{ss}} \simeq j_*j^*\mathcal{F}' \simeq \mathcal{F}'$ is lisse. □

Assuming Theorem 0.2 on the existence of lisse companions, we have the following consequence of Chebotarev's density theorem.

Proposition 4.2. *Let $\mathcal{F}$ be a lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on a geometrically unibranch stack X and let $\sigma\colon E(\mathcal{F}) \to E_{\lambda'}$ be an embedding. Then lisse σ-companions of $\mathcal{F}$ are unique up to semisimplification. Moreover, if $\mathcal{F}$ is simple, then, up to isomorphism, there exists at most one lisse σ-companion $\mathcal{G}$ of $\mathcal{F}$, and $\mathcal{G}$ is simple if it exists.*

It is convenient to slightly extend terminology as follows. Given a Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{F}$ on a stack X and an embedding $\sigma\colon E' \to E_{\lambda'}$ where E' is an extension of $E(\mathcal{F})$, we will refer to $(\sigma \mid E(\mathcal{F}))$-companions of $\mathcal{F}$ simply as σ-companions.

Proof. The first assertion follows from Chebotarev's density theorem (see Proposition 4.5 below). In the second assertion, it suffices to show the simplicity, as the uniqueness then follows from the first assertion. Up to replacing X by a dense substack, we may assume that X is smooth and of separated diagonal. We may further assume that $E_{\lambda'} = \overline{E_{\lambda'}}$. Extend σ to an isomorphism $\overline{\mathbb{Q}_\ell} \to E_{\lambda'}$, which we still denote by σ. Let $\mathcal{F}'$ be a lisse σ-companion of $\mathcal{F}$. Up to replacing $\mathcal{F}'$ by its semisimplification, we may assume $\mathcal{F}' = \bigoplus_i \mathcal{F}'_i$, with each $\mathcal{F}'_i$ simple. By Theorem 0.2, there exists a σ^{-1}-companion (i.e. $(\sigma^{-1} \mid E(\mathcal{G}_i))$-companion) $\mathcal{F}_i$ of $\mathcal{F}'_i$. Then $\mathcal{F}$ is the semisimplification of $\bigoplus_i \mathcal{F}_i$, since they are both σ^{-1}-companions of $\mathcal{F}'$. Thus $\mathcal{F}_i = 0$ for all but one i, and the same holds for $\mathcal{F}'_i$. Therefore, $\mathcal{F}'$ is simple. □

Proof of Theorem 0.2. By Corollary 3.2, we may assume that $\mathcal{F}$ is simple. In this case, we show in addition to the statements of the theorem, that any lisse σ'-companion $\mathcal{F}'$ is ι'-pure for any embedding $\iota'\colon \overline{\mathbb{Q}_{\ell'}} \to \mathbb{C}$. By Proposition 4.1 and Proposition 3.7 (or Corollary 3.3), we may shrink X. Applying Lemma 1.8 and Remark 1.9 (or Corollary 3.4), we reduce to the case where X is a Deligne-Mumford stack. Up to shrinking X, we may further assume that $X = [Y/G]$, where G is a finite group acting on an affine scheme Y. Choose an embedding $G \to \mathrm{GL}_m$. Consider the embedding $\mathrm{GL}_m \to \mathbb{A}^{m^2}$, which is equivariant under the action of GL_m. Let $p\colon Z = [Y \times \mathbb{A}^{m^2}/G] \to [Y/G]$ be the projection and let s be the zero section. Since $s^*p^*\mathcal{F} \simeq \mathcal{F}$, $p^*\mathcal{F}$ is simple. It suffices to show the assertions for $(Z, p^*\mathcal{F})$. Indeed, if $\mathcal{G}'$ is a lisse σ-companion of $p^*\mathcal{F}$, then $\mathcal{F}' = s^*\mathcal{G}'$ is a σ-companion of $s^*p^*\mathcal{F} \simeq \mathcal{F}$. Applying Propositions 4.1 and 3.7 to the dense open subscheme $[Y \times \mathrm{GL}_m/G]$ of Z, we are reduced to the case of schemes. In this case, the existence of lisse companions is Drinfeld's theorem [10, Theorem 1.1]. Moreover, if $\mathcal{F}'$ is a lisse σ-companion of $\mathcal{F}$, then $\mathcal{F}'$ is simple by the proof of Proposition 4.2, hence ι'-pure. □

Definition 4.3. Let $f\colon X \to Y$ be a morphism of stacks. We say that f *creates lisse companions* if for every lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{G}$ on Y and every embedding $\sigma\colon E(\mathcal{G}) \to E_\lambda$ such that $f^*\mathcal{G}$ admits a σ-companion, then $\mathcal{G}$ admits a σ-companion.

Note that we do not ask for the existence of a companion $\mathcal{G}'$ such that $f^*\mathcal{G}'$ is isomorphic to a given companion of $f^*\mathcal{G}$. Morphisms creating lisse companions are stable under composition. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a sequence of morphisms of stacks, and if gf creates lisse companions, then g creates lisse companions.

Corollary 0.3 follows from Theorem 0.2 and the following.

Proposition 4.4. *Let $f\colon X \to Y$ be a morphism of stacks. Then f creates lisse companions if it satisfies* any *of the following conditions:*

(1) f is a proper universal homeomorphism.

(2) $f\colon X = \coprod_i X_i \to Y$ is a Zariski open covering, and Y is geometrically unibranch.

Proof. Let $\mathcal{G}$ be a lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on Y, let $\sigma\colon E(\mathcal{G}) \to E_\lambda$ be an embedding, and let $\mathcal{F}'$ be a lisse σ-companion of $f^*\mathcal{G}$.

(1) The reduced geometric fibers of f are classifying stacks BG of finite group schemes G. By proper base change, the adjunction map $\mathcal{G} \to f_*f^*\mathcal{G}$ is an isomorphism. Moreover, $f_*\mathcal{F}'$ is a σ-companion of $f_*f^*\mathcal{G} \simeq \mathcal{G}$. It remains to show that $f_*\mathcal{F}'$ is lisse. Consider the adjunction map $a\colon f^*f_*\mathcal{F}' \to \mathcal{F}'$. The restriction of a to the reduced geometric fibers BG of f can be identified with the inclusion of the G-invariants. Thus a is a monomorphism. Both $f^*f_*\mathcal{F}'$ and $\mathcal{F}'$ are σ-companions of $f^*\mathcal{G}$, and the rank of the stalk at any $\overline{\mathbb{F}_q}$-point of X is determined by the Frobenius traces. Thus, a is an isomorphism. By Lemma 1.3, $f_*\mathcal{F}'$ is lisse.

(2) We may assume Y irreducible and each X_i nonempty. Then $U = \bigcap_i X_i$ is nonempty. Let $j\colon U \to Y$ and let $\mathcal{G}'_U$ be a semisimple lisse σ-companion of $j^*\mathcal{G}$. Let $\mathcal{F}'_i = \mathcal{F}' \mid X_i$. Then $\mathcal{G}'_U \simeq \mathcal{F}'^{\mathrm{ss}}_i \mid U$, so that $j_*\mathcal{G}'_U \mid X_i \simeq \mathcal{F}'^{\mathrm{ss}}_i$. Thus $j_*\mathcal{G}'_U$ is a lisse σ-companion of $\mathcal{G}$. □

In the rest of the section, we discuss companions of perverse sheaves and in Grothendieck groups. For this, it is convenient to introduce perverse Weil sheaves. Let E_λ be an algebraic extension of $\mathbb{Q}_\ell$. A *perverse Weil E_λ-sheaf* on a stack X is a perverse E_λ-sheaf $\mathcal{P}$ on $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ equipped with an action of the Weil group $W(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ lifting the action of $W(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$. A morphism of perverse Weil E_λ-sheaves on X is a morphism of the underlying perverse E_λ-sheaves on $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ compatible with the action of $W(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. As in the case of schemes [4, Proposition 5.1.2] or E_λ-sheaves (Remark 1.1), we have a fully faithful functor $\mathrm{Perv}^W(X, E_\lambda) \to \mathrm{Perv}(X, E_\lambda)$ and the essential image is stable under extension. Remark 1.5 on extending scalars to $\overline{\mathbb{Q}_\ell}$ still holds. The analogue of Proposition 1.12 holds with the same proof:

$$\mathrm{Perv}^W(X, \overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{a \in \overline{\mathbb{Q}_\ell}^\times/\overline{\mathbb{Z}_\ell}^\times} \mathrm{Perv}(X, \overline{\mathbb{Q}_\ell})^{(a)}.$$

We let $K^W_{\mathrm{lisse}}(X, E_\lambda)$ denote the Grothendieck group of $\mathrm{Shv}^W_{\mathrm{lisse}}(X, E_\lambda)$, which is a free Abelian group generated by the isomorphism classes of simple lisse Weil E_λ-sheaves on X. We let $K^W(X, E_\lambda)$ denote the Grothendieck group of $\mathrm{Shv}^W(X, E_\lambda)$, which is also the Grothendieck group of $\mathrm{Perv}^W(X, E_\lambda)$, and is a free Abelian group generated by the isomorphism classes of simple perverse Weil E_λ-sheaves on X. For a sheaf or perverse sheaf $\mathcal{F}$, we let $[\mathcal{F}]$ denote its class in the Grothendieck group. We have a commutative diagram

$$\begin{array}{ccc} K^W_{\mathrm{lisse}}(X, E_\lambda) & \longrightarrow & K^W(X, E_\lambda) \\ \big\downarrow & & \big\downarrow \\ K^W_{\mathrm{lisse}}(X, \overline{\mathbb{Q}_\ell}) & \longrightarrow & K^W(X, \overline{\mathbb{Q}_\ell}) \end{array}$$

of groups. The horizontal arrows are also injections if X is geometrically unibranch.

We have the following Chebotarev's density theorem.

Proposition 4.5. *Let X be a stack.*

(1) The homomorphism $t_X\colon K^W(X,\overline{\mathbb{Q}_\ell}) \to \overline{\mathbb{Q}_\ell}^{\coprod_{n\geq 1}|X(\mathbb{F}_{q^n})|}$ sending A to $t_X(A)\colon x \mapsto \mathrm{tr}(\mathrm{Frob}_x, A_{\bar{x}})$ is injective. Here $|X(\mathbb{F}_{q^n})|$ is the set of isomorphism classes of the groupoid $X(\mathbb{F}_{q^n})$.

(2) For X irreducible and geometrically unibranch, the conjugates of the images of Frob_x in the fundamental group $\pi_1(X)$ form a dense subset.

Proof. The two assertions are equivalent and as in [17, Théorème 1.1.2], we reduce to showing that the map $t_X\colon K^W_{\mathrm{lisse}}(X,\overline{\mathbb{Q}_\ell}) \to \overline{\mathbb{Q}_\ell}^{\coprod_{m\geq 1}|X(\mathbb{F}_{q^m})|}$ is injective for X smooth. We reduce to the case of Deligne-Mumford stacks as follows. Let $[\mathcal{F}]-[\mathcal{G}]$ be an element in the kernel of t_X with $\mathcal{F}$ and $\mathcal{G}$ lisse. We apply Lemma 1.8 to $\mathcal{F}\oplus\mathcal{G}$ to find, up to shrinking X, a gerbe $f\colon X \to Y$ where Y is a Deligne-Mumford stack such that $f^*f_*\mathcal{F} \simeq \mathcal{F}$ and $f^*f_*\mathcal{G} \simeq \mathcal{G}$. By Remark 1.9, $t_Y([f_*\mathcal{F}]-[f_*\mathcal{G}]) = 0$. Thus, by Chebotarev's density theorem for Deligne-Mumford stacks [25, Lemma 4.1.4], we have $[f_*\mathcal{F}] = [f_*\mathcal{G}]$. Therefore, $[\mathcal{F}] = [\mathcal{G}]$. □

The definitions of $E(\mathcal{F})$ and σ-companions at the beginning of the paper extend to perverse Weil sheaves and elements of Grothendieck groups, with trace defined as alternating sums of traces. Proposition 4.5 implies that σ-companions in Grothendieck groups are unique and perverse σ-companions are unique up to semisimplification.

Let $K_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})$ denote the Grothendieck group of $\mathrm{Perv}_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})$ (see Remark 2.15) in $K(X,\overline{\mathbb{Q}_\ell})$. Then we have

$$K^W(X,\overline{\mathbb{Q}_\ell}) \simeq \bigoplus_{a\in\overline{\mathbb{Q}_\ell}^\times/W(q)} K_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})^{(a)}.$$

Proposition 4.6. *Let X be a stack and let $A \in K^W(X,\overline{\mathbb{Q}_\ell})$. The following conditions are equivalent:*

(1) $E(A)$ is a number field.

(2) $\mathrm{tr}(\mathrm{Frob}_x, A_{\bar{x}})$ is an algebraic number for all $n \geq 1$ and all $x \in X(\mathbb{F}_{q^n})$.

(3) A belongs to $K_{\mathrm{alg}}(X,\overline{\mathbb{Q}_\ell}) := \bigoplus_{a\in\overline{\mathbb{Q}}^\times/W(q)} K_{\mathrm{mot}}(X,\overline{\mathbb{Q}_\ell})^{(a)}$.

Thus, if we identify $K^W(X,\overline{\mathbb{Q}_\ell})$ with its image under t_X, then

$$K_{\mathrm{alg}}(X,\overline{\mathbb{Q}_\ell}) = \overline{\mathbb{Q}}^{\coprod_{n\geq 1}|X(\mathbb{F}_{q^n})|} \cap K^W(X,\overline{\mathbb{Q}_\ell}).$$

Proof. It is clear that (1) implies (2). By Theorem 3.1, (3) implies (1). Now assume that (2) holds. Let $A = B + C$, where B is the projection of A in $K_{\mathrm{alg}}(X,\overline{\mathbb{Q}_\ell})$. Since $\det(1-t\mathrm{Frob}_x, A_{\bar{x}}) \in \overline{\mathbb{Q}}(t)$, we have $\det(1-t\mathrm{Frob}_x, C_{\bar{x}}) = 1$, so that $\mathrm{tr}(\mathrm{Frob}_x^m, C_{\bar{x}}) = 0$ for $m \geq 1$. Thus $C = 0$ by Proposition 4.5. □

Remark 4.7. Let $f\colon X \to Y$ be a morphism of stacks, we have (bi)linear maps

$$-\otimes-,\ \mathcal{H}om(-,-)\colon K^W(X,\overline{\mathbb{Q}_\ell})\times K^W(X,\overline{\mathbb{Q}_\ell})\to K^W(X,\overline{\mathbb{Q}_\ell}),$$
$$D_X\colon K^W(X,\overline{\mathbb{Q}_\ell})\to K^W(X,\overline{\mathbb{Q}_\ell}),$$
$$f^*, f^!\colon K^W(Y,\overline{\mathbb{Q}_\ell})\to K^W(X,\overline{\mathbb{Q}_\ell}).$$

If f is relatively Deligne-Mumford, then we have linear maps

$$f_*, f_!\colon K^W(X,\overline{\mathbb{Q}_\ell})\to K^W(Y,\overline{\mathbb{Q}_\ell}).$$

For an immersion of stacks $f\colon X\to Y$, the middle extension functor $f_{!*}\colon \mathrm{Perv}^W(X,\overline{\mathbb{Q}_\ell})\to \mathrm{Perv}^W(Y,\overline{\mathbb{Q}_\ell})$ is not exact in general. We define a linear map

$$f_{!*}\colon K^W(X,\overline{\mathbb{Q}_\ell})\to K^W(Y,\overline{\mathbb{Q}_\ell})$$

such that $f_{!*}[\mathcal{P}]=[f_{!*}\mathcal{P}]$ for $\mathcal{P}\in \mathrm{Perv}^W(X,\overline{\mathbb{Q}_\ell})$ semisimple.

The definition of $f_{!*}$ on Grothendieck groups is further justified by the following fact. Let $\iota\colon \overline{\mathbb{Q}_\ell}\to\mathbb{C}$ be an embedding and let $w\in\mathbb{R}$. We let $\mathrm{Perv}^W_{\iota,\{w,w+1\}}(X,\overline{\mathbb{Q}_\ell})$ denote the category of perverse Weil $\overline{\mathbb{Q}_\ell}$-sheaves on X, ι-mixed of weights w and $w+1$. Then $f_{!*}[\mathcal{P}]=[f_{!*}\mathcal{P}]$ for $\mathcal{P}\in \mathrm{Perv}^W_{\iota,\{w,w+1\}}(X,\overline{\mathbb{Q}_\ell})$ by the following immediate extension from the case of schemes [25, Lemma 4.2.5].

Lemma 4.8. *Let $f\colon X\to Y$ be an immersion of stacks. The functor*

$$f_{!*}\colon \mathrm{Perv}^W_{\iota,\{w,w+1\}}(X,\overline{\mathbb{Q}_\ell})\to \mathrm{Perv}^W_{\iota,\{w,w+1\}}(Y,\overline{\mathbb{Q}_\ell})$$

is exact.

Remark 4.9. Let X and Y be stacks with separated diagonal. Then the operations in Remark 4.7 preserve σ-companions. Moreover the operations preserve E in the sense that $E(FA)\subseteq E(A)$ for any operation F in the list. It follows that $E(D_XA)=E(A)$. In the case of schemes, these assertions are theorems of Gabber [13, Theorems 2, 3]. The case of stacks follow easily by the existence of smooth neighborhoods. See [28, Propositions 5.7, 5.8].

Due to cancellation in the alternating sum, the analogue of Corollary 3.4 does not hold: $E(A)$ is not necessarily a finite extension of $E(f^*A)$ for f surjective. For example, for $f\colon \mathrm{Spec}(\mathbb{F}_{q^2})\to\mathrm{Spec}(\mathbb{F}_q)$ and $A=\left[\overline{\mathbb{Q}_\ell}^{(a)}\right]-\left[\overline{\mathbb{Q}_\ell}^{(-a)}\right]$, we have $E(A)=\mathbb{Q}(a)$ but $E(f^*A)=\mathbb{Q}$.

The analogues of Propositions 3.5 and 3.7 hold in Grothendieck groups. In fact, in each of the two cases, the first assertion follows from the second assertion, and the second assertion in Grothendieck groups reduces to the case of sheaves.

Combining Theorem 0.2 with the fact that $f_{!*}$ on Grothendieck groups preserves companions (Remark 4.9), we obtain the existence of perverse companions.

Theorem 4.10. *Let X be a stack of separated diagonal. Let $\mathcal{P}$ be a perverse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on X. Then, for every embedding $\sigma\colon E(\mathcal{P})\to\overline{\mathbb{Q}_{\ell'}}$, $\mathcal{P}$ admits a perverse σ-companion $\mathcal{F}'$, unique up to semisimplification. Moreover, if $E(\mathcal{P})$ is a number field, then there exists a finite extension E of $E(\mathcal{P})$ such that for every finite place λ' of E not dividing q, $\mathcal{P}$ admits a perverse $\sigma_{\lambda'}$-companion. Here $\sigma_{\lambda'}\colon E(\mathcal{P})\to E\to E_{\lambda'}$, and $E_{\lambda'}$ denotes the completion of E at λ.*

Corollary 4.11. *Let X be a stack of separated diagonal. Let $\mathcal{P}$ be a simple perverse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on X. Then, for every embedding $\sigma\colon E(\mathcal{P}) \to \overline{\mathbb{Q}_{\ell'}}$, there exists a unique perverse σ-companion $\mathcal{P}'$. Moreover, $\mathcal{P}'$ is simple.*

Proof. It suffices to show the simplicity. The proof is the same as the end of the proof of Proposition 4.2. We extend σ to an isomorphism $\overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \overline{\mathbb{Q}_{\ell'}}$. Assume $\mathcal{P}' \simeq \bigoplus_i \mathcal{P}'_i$ with $\mathcal{P}'_i$ simple. Then there exists $\mathcal{P}_i$ such that $\mathcal{P}'_i$ is the σ-companion of $\mathcal{P}_i$. It follows that $\mathcal{P} \simeq \bigoplus_i \mathcal{P}_i$, so that $\mathcal{P}_i = 0$ for all but one i, and the same holds for $\mathcal{P}'_i$. □

Corollary 4.12. *Let X be a stack of separated diagonal. Let $\sigma\colon \overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \overline{\mathbb{Q}_{\ell'}}$ be an isomorphism. For any $A \in K(X, \overline{\mathbb{Q}_\ell})$, there exists a unique σ-companion A'. The map $K(X, \overline{\mathbb{Q}_\ell}) \to K(X, \overline{\mathbb{Q}_{\ell'}})$ sending A to its σ-companion A' is an isomorphism. Moreover, if $E(A)$ is a number field, then there exists a finite extension E of $E(A)$ such that for every finite place λ' of E not dividing q, A admits a perverse $\sigma_{\lambda'}$-companion, where $\sigma_{\lambda'}$ is as in Theorem 4.10.*

Note that if $A = \sum_{\mathcal{P}} n_{\mathcal{P}}[\mathcal{P}]$, where $\mathcal{P}$ runs through isomorphism classes of simple perverse $\overline{\mathbb{Q}_\ell}$-sheaves, then $A' = \sum_{\mathcal{P}} n_{\mathcal{P}}[\mathcal{P}']$, where $\mathcal{P}'$ is the perverse σ-companion of $\mathcal{P}$, is the σ-companion of A.

Proof. The existence of σ-companion follows from Theorem 4.10 or 0.2. For the second assertion, note that sending A' to its σ^{-1}-companion defines an inverse of the map. For the last assertion, note that $A = [\mathcal{P}] - [\mathcal{Q}]$ with $E(\mathcal{P})$ and $E(\mathcal{Q})$ being number fields by Proposition 4.6, so that it suffices to apply the last assertion of the theorem. □

Remark 4.13. Let X be a stack of separated diagonal. The group of functions $K(X, \mathbb{C}) \subseteq \mathbb{C}^{\coprod_{n\geq 1}|X(\mathbb{F}_{q^n})|}$ of the form $\iota \circ t_X(A)$, where A belongs to $K^W(X, \overline{\mathbb{Q}_\ell})$ and $\iota\colon \overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \mathbb{C}$ is an isomorphism, does not depend on the choice of ℓ and ι by Corollary 4.12. Similarly, the subgroups $K_{\mathrm{mot}}(X, \overline{\mathbb{Q}}) \subseteq K_{\mathrm{alg}}(X, \overline{\mathbb{Q}}) \subseteq \overline{\mathbb{Q}}^{\coprod_{n\geq 1}|X(\mathbb{F}_{q^n})|}$, inverse images via an embedding $i\colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_\ell}$ of the corresponding subgroups of $t_X(K^W(X, \overline{\mathbb{Q}_\ell}))$, do not depend on the choice of ℓ and i (cf. [10, Corollary 1.6]). We have

$$K(X, \mathbb{C}) \simeq \bigoplus_{a \in \mathbb{C}^\times/\overline{\mathbb{Q}}^\times} K_{\mathrm{alg}}(X, \overline{\mathbb{Q}})^{(a)}, \quad K_{\mathrm{alg}}(X, \overline{\mathbb{Q}}) \simeq \bigoplus_{a \in \overline{\mathbb{Q}}^\times/W(q)} K_{\mathrm{mot}}(X, \overline{\mathbb{Q}})^{(a)}.$$

Remark 4.14. The support of a simple perverse Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{P}$ on a stack X equals to the maximal reduced closed substack Y of X such that $\mathrm{tr}(\mathrm{Frob}_x, \mathcal{P}_{\bar{x}}) = 0$ for all $n \geq 1$ and all $x \in (X - Y)(\mathbb{F}_{q^n})$ by Proposition 4.5. Assume that X has separated diagonal. Then the perverse σ-companion $\mathcal{P}'$ of $\mathcal{P}$ has the same support as $\mathcal{P}$. Sun [24] defines the open support of $\mathcal{P}$ to be the maximal smooth Zariski open of Y on which $\mathcal{P}$ is the shift of a lisse Weil $\overline{\mathbb{Q}_\ell}$-sheaf. As he observed, $\mathcal{P}$ and $\mathcal{P}'$ have the same open support by Theorem 0.2.

5 Appendix: Structure of pure perverse sheaves

The goal of this appendix is to prove the following geometric semisimplicity theorem. Let $\iota\colon \overline{\mathbb{Q}_\ell} \to \mathbb{C}$ be an embedding.

Theorem 5.1. *Let X be a stack and let $\mathcal{P}$ be a ι-pure perverse Weil $\overline{\mathbb{Q}_\ell}$-sheaf on X. Then the pullback of $\mathcal{P}$ to $X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ is semisimple.*

The case of affine stabilizers is a theorem of Sun [23, Theorem 3.11], extending the case of schemes [4, Théorème 5.3.8]. Note that the decomposition theorem of pure complexes [23, Theorem 3.12] does *not* extend to general stacks, as shown in [23, Section 1].

As in the case of schemes [4, Proposition 5.3.9], Theorem 5.1 has the following consequence on the structure of pure perverse sheaves. As before we let $\mathcal{E}_n$ denote the $\overline{\mathbb{Q}_\ell}$-sheaf on $\mathrm{Spec}(\mathbb{F}_q)$ of stalk $\overline{\mathbb{Q}_\ell}^n$ on which Frob_q acts unipotently with one Jordan block.

Corollary 5.2. *Let X be a stack. The indecomposable ι-pure perverse Weil $\overline{\mathbb{Q}_\ell}$-sheaves on X are of the form $\mathcal{P}\otimes\pi_X^*\mathcal{E}_n$ with $\mathcal{P}$ simple, where $\pi_X\colon X\to\mathrm{Spec}(\mathbb{F}_q)$. Moreover, for every simple perverse Weil $\overline{\mathbb{Q}_\ell}$-sheaf $\mathcal{P}$, there exists a unique $m\geq 1$ such that $\mathcal{P}\simeq p_*\mathcal{Q}$, where $p\colon X\otimes_{\mathbb{F}_q}\mathbb{F}_{q^m}\to X$ is the projection, $\mathcal{Q}$ is geometrically simple (i.e. the pullback of $\mathcal{Q}$ to $X\otimes_{\mathbb{F}_{q^m}}\overline{\mathbb{F}_q}$ is simple) and not isomorphic to any of its conjugates under $\mathrm{Gal}(\mathbb{F}_{q^m}/\mathbb{F}_q)$.*

The first assertion of the corollary still holds with $\overline{\mathbb{Q}_\ell}$ replaced by an algebraic extension of $\mathbb{Q}_\ell$.

The key to the proof of Theorem 5.1 is a weight estimate. Let $w\in\mathbb{R}$ and let $K\in D_c(X,\overline{\mathbb{Q}_\ell})$. We say that K has *ι-weights $\leq w$* if the i-th cohomology sheaf of K has punctually ι-weights $\leq w+i$ for all i, and K has *ι-weights $\geq w$* if DK has ι-weights $\leq -w$.

Proposition 5.3. *Let X be a stack and let $\pi\colon X\to\mathrm{Spec}(\mathbb{F}_q)$ be the projection. let $K\in D^{\geq 0}(X,\overline{\mathbb{Q}_\ell})$ be a complex of ι-weights $\geq w$ and vanishing i-th cohomology for $i<0$. Then for all $i\geq 0$, $R^i\pi_*K$ has ι-weights $\geq w+\lceil\frac{i}{2}\rceil$. Moreover $H^i(X\otimes_{\mathbb{F}_q}\overline{\mathbb{F}_q},K)^{\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)}=0$ for $i>0$ if $w\geq 0$, and $R\Gamma(X,K)=0$ if $w>0$.*

The estimate is optimal. Indeed, for $X=\mathbb{G}_\mathrm{m}\times BA$, where A is an Abelian variety, $R^i\pi_*\overline{\mathbb{Q}_\ell}$ is pure of weight $\lceil\frac{i}{2}\rceil$. Unlike the case of schemes or stacks with affine stabilizers, $R\pi_*K$ is *not* of ι-weights $\geq w$ in general.

Proof. The second assertion follows from the first one and the short exact sequence

$$0\to H^{i-1}(X\otimes_{\mathbb{F}_q}\overline{\mathbb{F}_q},K)_{\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)}\to H^i(X,K)\to H^i(X\otimes_{\mathbb{F}_q}\overline{\mathbb{F}_q},K)^{\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)}\to 0.$$

Note that for any stratification of X into locally closed substacks $(j_\alpha\colon X_\alpha\to X)_\alpha$ such that the closure of every stratum is a union of strata, K is a successive extension of $j_{\alpha*}Rj_\alpha^!K$. Thus we may assume that X is smooth of dimension d and K has lisse cohomology sheaves. We may further assume $K=\mathcal{F}[-n]$, with $\mathcal{F}$ lisse of weight $w+n$ and $n\geq 0$. Then the ι-weights of $(R^i\pi_*K)^\vee\simeq(R^{2d-(i+n)}\pi_!\mathcal{F}^\vee)(d)$ are at most

$$d+\frac{2d-(i+n)}{2}-(w+n)-2d=-w-\frac{i+n}{2}\leq -w-\frac{i}{2}$$

by [22, Theorem 1.4]. We conclude by the fact that the ι-weights are in $w+\mathbb{Z}$. □

Corollary 5.4. *Let X be a stack and let $\mathcal{P}$ and $\mathcal{Q}$ be perverse $\overline{\mathbb{Q}_\ell}$-sheaves on X, with $\mathcal{P}$ of ι-weights $\leq w$, and $\mathcal{Q}$ of ι-weights $\geq w$. Then for $i>0$, $\mathrm{Hom}^i(\mathcal{P}_{\overline{\mathbb{F}_q}},\mathcal{Q}_{\overline{\mathbb{F}_q}})^{\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)}=0$, so that the canonical map $\mathrm{Hom}^i(\mathcal{P},\mathcal{Q})\to\mathrm{Hom}^i(\mathcal{P}_{\overline{\mathbb{F}_q}},\mathcal{Q}_{\overline{\mathbb{F}_q}})$ is zero. Moreover, if $\mathcal{Q}$ has ι-weights $>w$, then $R\mathrm{Hom}(\mathcal{P},\mathcal{Q})=0$.*

For perverse Weil $\overline{\mathbb{Q}_\ell}$-sheaves and $i=1$, the first assertion holds with Hom^1 replaced by Ext^1 and $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ replaced by $W(\overline{\mathbb{F}_q}/F_q)$.

Proof. We apply the proposition to $K = R\mathcal{H}om(\mathcal{P}, \mathcal{Q}) \in D^{\geq 0}(X, \overline{\mathbb{Q}_\ell})$, which has ι-weights ≥ 0. If $\mathcal{Q}$ has ι-weights $> w$, then K has ι-weights > 0. □

The proof of Theorem 5.1 is identical to the proof of [4, Théorème 5.3.8], with [4, Proposition 5.1.15] replaced by Corollary 5.4.

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