# Duality and nearby cycles over general bases 

Qing Lu* Weizhe Zheng ${ }^{\dagger \ddagger}$

October 7, 2019


#### Abstract

This paper studies the sliced nearby cycle functor and its commutation with duality. Over a Henselian discrete valuation ring, we show that this commutation holds, confirming a prediction of Deligne. As an application we give a new proof of Beilinson's theorem that the vanishing cycle functor commutes with duality up to twist. Over an excellent base scheme, we show that the sliced nearby cycle functor commutes with duality up to modification of the base. We deduce that duality preserves universal local acyclicity over an excellent regular base. We also present Gabber's theorem that local acyclicity implies universal local acyclicity over a Noetherian base.


## Contents

1 Fiber products of topoi ..... 3
2 The functor $L C o$ ..... 17
3 Nearby and vanishing cycles over Henselian discrete valuation rings ..... 24
4 Sliced nearby cycles over general bases ..... 26
5 Nearby cycles and duality ..... 39
6 Local acyclicity (after Gabber) ..... 46

## Introduction

### 0.1 Over a Henselian discrete valuation ring

Let $S$ be the spectrum of a Henselian discrete valuation ring, of closed point $s$ and generic point $\eta$. Let $a: X \rightarrow S$ be a morphism of schemes. Let $\Lambda$ be a Noetherian commutative ring such that $m \Lambda=0$ for some integer $m$ invertible on $S$. We have the classical nearby cycle functor [SGA7II, XIII 2.1.1]

$$
R \Psi_{\eta}^{s}: D\left(X_{\eta}, \Lambda\right) \rightarrow D\left(X_{s} \bar{x}_{s} \eta, \Lambda\right),
$$

where $X_{\eta}:=X \times_{S} \eta$ and $X_{s}:=X \times_{S} s$. The vanishing cycle functor $\Phi^{s}$ is a composition

$$
D(X, \Lambda) \xrightarrow{R \Psi^{s}} D\left(X_{s} \bar{x}_{s} S, \Lambda\right) \xrightarrow{L \mathrm{Co}} D\left(X_{s} \bar{x}_{s} \eta, \Lambda\right),
$$

[^0]where $L \operatorname{Co}(M)$ computes the cone of the specialization map $M_{s} \rightarrow M_{\eta}$ (2.17). We call $R \Psi^{s}$ the sliced nearby cycle functor. Here $\bar{x}_{s}$ denotes fiber products of étale topoi. (In the case where $\eta$ is a scheme over $s, X \bar{x}_{s} \eta$ is typically not the étale topos of $X \times_{s} \eta$.) Unless otherwise indicated, we work in the unbounded derived categories.

Assume $X$ separated and of finite type over $S$. Gabber proved that the classical nearby cycle functor $R \Psi_{\eta}^{s}$, when restricted to $D_{c \mathrm{ft}}$ (the full subcategory of $D_{c}^{b}$ spanned by complexes of finite toramplitude, where $D_{c}^{*}$ denotes the full subcategory of $D^{*}$ spanned by complexes with constructible cohomology sheaves), commutes with duality [I1, Théorème 4.2]. Our first result confirms Deligne's prediction [D2] that the same holds for the sliced nearby cycle functor $R \Psi^{s}$. Let $D_{X}=R \mathcal{H} o m\left(-, R a^{!} \Lambda_{S}\right)$, $D_{X_{s} \bar{x}_{s} S}=R \mathcal{H o m}\left(-, R\left(a_{s} \bar{x}_{s} \operatorname{id}_{S}\right)^{!} \Lambda_{S}\right)$.

## Theorem 0.1.

(1) The canonical map $R \Psi^{s} D_{X} L \rightarrow D_{X_{s} \bar{x}_{s} S} R \Psi^{s} L$ (3.4) is an isomorphism for $L \in D_{c}^{-}(X, \Lambda)$.
(2) We have a natural isomorphism $D_{X_{s} \bar{x}_{s} \eta} L \mathrm{Co} \simeq \tau L \operatorname{Co} D_{X_{s} \bar{x}_{s} S}$ (2.19).

Here $\tau$ (Definition [2.8) is a functor that modifies only the tame part by what we call, following Illusie, the Iwasawa twist. We immediately deduce that the vanishing cycle functor commutes with duality up to the Iwasawa twist, which is a theorem of Beilinson (at least when $s$ is separably closed) [B1, 2.3].

Corollary 0.2 (Beilinson). We have an isomorphism $D_{X_{s} \bar{x}_{s} \eta} \Phi^{s} L \simeq \tau \Phi^{s} D_{X} L$, functorial in $L \in$ $D_{c}^{-}(X, \Lambda)$.

Though the functors in Theorem 0.1 (1) involve only the fiber product $X_{s} \bar{x}_{s} S$, the construction of the map and the proof uses the oriented product $X \overleftarrow{x}_{S} S$. Theorem 0.1 (2) follows from quasi-periodic adjunctions for the functor $L$ Co (2.8). Beilinson's proof of Corollary 0.2 was rather different, using his maximal extension functor $\Xi$.

### 0.2 Over general bases

For an arbitrary base scheme $S$ and any scheme $X$ over $S$, Deligne defined the vanishing topos $X \overleftarrow{\times}_{S} S$ and the nearby cycle functor ( (L), [ILO, XI Section 4])

$$
R \Psi: D(X, \Lambda) \rightarrow D\left(X \overleftarrow{\times}_{S} S, \Lambda\right)
$$

Recently Illusie proved a Künneth formula for $R \Psi$ [13, Theorems 2.3, A.3], generalizing Gabber's theorem over a Henselian discrete valuation ring ([II, Théorème 4.7], [BB, Lemma 5.1.1]). It was used in Saito's work on characteristic cycles, notably in his proof of the global index formula ([S1], [S2]).

Following a suggestion of Illusie, we study the commutation with duality of nearby cycles over general bases. While there is typically no good duality on the vanishing topos $X \overleftarrow{\times}_{S} S$ as a whole (Remark (3.6), there is a good duality on the slice $X_{s} \overleftarrow{×}_{S} S \simeq X_{s} \bar{x}_{s} S_{(s)}$ for every point $s$ of $S$, under usual assumptions, where $S_{(s)}$ denotes the Henselization of $S$ at $s$. We define the sliced nearby cycle functor $R \Psi^{s}$ to be the composition

$$
D(X, \Lambda) \xrightarrow{R \Psi} D\left(X \overleftarrow{\times}_{S} S, \Lambda\right) \rightarrow D\left(X_{s} \bar{x}_{s} S_{(s)}, \Lambda\right)
$$

of $R \Psi$ with the restriction functor. Our main result is that the sliced nearby cycle functor commutes with duality up to modification of the base, generalizing the excellent case of Theorem 0.1 (1). More generally, we have the following result, where $K_{S}$ is not assumed to be a dualizing complex.
Theorem 0.3. Let $S$ be an excellent scheme and let $K_{S} \in D_{c}^{b}(S, \Lambda)$. Let $a: X \rightarrow S$ be a separated morphism of schemes of finite type and let $L \in D_{c}^{b}(X, \Lambda)$ such that $R \mathcal{H} o m\left(L, R a!K_{S}\right) \in D_{c}^{b}(X, \Lambda)$. Then there exists a modification $S^{\prime} \rightarrow S$ such that for every morphism $T \rightarrow S^{\prime}$ separated of finite type, and for every point $t \in T$, the canonical map (5.1)

$$
R \Psi^{t} D_{X_{T}}\left(\left.L\right|_{X_{T}}\right) \rightarrow D_{X_{t} \bar{x}_{t} T_{(t)}} R \Psi^{t}\left(\left.L\right|_{X_{T}}\right)
$$

is an isomorphism. Here $X_{T}:=X \times_{S} T, X_{t}:=X \times_{S} t, D_{X_{T}}:=R \mathcal{H o m}\left(-, K_{X_{T}}\right), D_{X_{t} \bar{x}_{t} T_{(t)}}:=$ $R \mathcal{H o m}\left(-, K_{X_{t} \bar{x}_{t} T_{(t)}}\right), K_{X_{T}}$ and $K_{X_{t} \bar{x}_{t} T_{(t)}}$ are !-pullbacks of $K_{S}$.

Here a modification means a proper birational morphism. Excellent schemes are assumed to be Noetherian.

Further restrictions of the nearby cycle functor to shreds and local sections were previously studied by Orgogozo [0, Section 6] and Illusie [13, Section 1], but these restrictions carry too little information on the base and fibers, respectively, for an analogue of Theorem 0.3 to hold (Remark 5.10).

One ingredient of the proof of Theorem 0.3 is Orgogozo's theorem that the nearby cycle functor commutes with base change after modification of the base [0, Théorème 2.1]. Since duality swaps pullback and !-pullback, we are also lead to study the commutation of the sliced nearby cycle functor with !-pullback. The proof of Theorem 0.3] relies on both Orgogozo's theorem and an analogue (Theorem 4.32) thereof for !-pullback.

As an application of Theorem 0.3, we show that universal local acyclicity over a regular excellent base is preserved by duality (Corollary 5.13), answering a question of Illusie. Gabber gave a different proof of the case of finite tor-amplitude of this corollary. He also showed that over a Noetherian base, local acyclicity implies universal local acyclicity (Corollary 6.6), answering a question of M. Artin. We deduce that weak singular support over a regular excellent base is compatible with duality (Corollary 5.14).

The results of this paper have applications to the perversity of nearby cycles, which we hope to explore in a future article.

Organization In Section we define and study duality and other operations on fiber products of topoi, on which the sliced nearby cycles live. In Section 2, after preliminaries on the Iwasawa twist, we study adjunctions and duality for the functor $L$ Co and prove Theorem 0.1 (2). In Section 3 we prove Theorem 0.1 (1) on duality and the sliced nearby cycle functor over Henselian discrete valuation rings. In Section [4 we study the sliced nearby cycle functor over general bases and prove Theorem 4.32, the analogue of Orgogozo's theorem for !-pullback, which is a key step toward the proof of Theorem 0.3. In Section 5, we prove Theorem 0.3 and the applications to local acyclicity and singular support. In Section [6] we present Gabber's results on local acyclicity.

Section 5 depends on Section 4. Sections 2 through 5 depend on Section 1 . For those interested only in the case over a Henselian discrete valuation ring (Sections 2 and 3), we recommend consulting Section $\square$ only when necessary.

Acknowledgments This paper grew out of suggestions from Luc Illusie, and would not have existed without his insights. We thank him for hundreds of comments on drafts of this paper. We thank Ofer Gabber for kindly allowing us to present his results on local acyclicity in Section 6 and for many crucial suggestions, including the removal of a finite dimensionality assumption on the base (Remark 4.35). We thank Takeshi Saito and the referee for valuable comments. We thank Lei Fu and Haoyu Hu for useful discussions. Part of this work was done during visits to l'Institut des Hautes Études Scientifiques, the University of Hong Kong, and Princeton University. We thank these institutions for hospitality and support.

## 1 Fiber products of topoi

The sliced nearby cycles live on fiber products of topoi over the étale topos of a point. The goal of this section is to define and study various operations on fiber products of topoi, including the dualizing functor in our main theorems.

### 1.1 Over general bases

In this subsection, we prove a proper base change theorem for fiber products of topoi over general bases (Proposition 1.6) and use it to construct various operations on such fiber products. The construction provides operations on slices of the vanishing topos used in our main theorems. It also applies to the vanishing topos itself (Construction 4.9).

Let $\Lambda$ be a commutative ring. For any topos $S$, we write $\operatorname{Shv}(S, \Lambda)$ for the category of sheaves of $\Lambda$-modules on $S$ and we write $D(S, \Lambda)$ for its derived category.

Let $X \rightarrow S$ and $Y \rightarrow S$ be morphisms of topoi. We refer to [ILO, XI] for the constructions of the oriented product $X \overleftarrow{\times}_{S} Y$ and the fiber product $X \overline{\times}_{S} Y$. We adopt the notation $\overline{\times}$ for products and fiber products of topoi to avoid confusion with fiber products and products of schemes.

The following base change results for oriented products of topoi will be used in Proposition 1.17 and Lemma 4.13 ,

Proposition 1.1. Let $X \xrightarrow{a} S \leftarrow Y \stackrel{g}{\leftarrow} Y^{\prime}$ be coherent morphisms [SGA4, VI Définition 3.1] of coherent topoi [SGA4, VI Définition 2.3]. Consider $\operatorname{id}_{X} \overleftarrow{×}_{S} g: X \overleftarrow{\times}_{S} Y^{\prime} \rightarrow X \overleftarrow{×}_{S} Y$. For $L \in D^{+}\left(X \overleftarrow{\times}_{S} Y^{\prime}, \Lambda\right)$, the stalk of $R\left(\operatorname{id}_{X} \overleftarrow{\times}_{S} g\right)_{*} L$ at any point $(x, y, \phi)$ of $X \overleftarrow{\times}_{S} Y$ is isomorphic to $R \Gamma\left(Y_{(y)}^{\prime}, c^{*} L\right)$, where $Y_{(y)}^{\prime}=Y^{\prime} \bar{×}_{Y} Y_{(y)}, Y_{(y)}$ is the localization of $Y$ at $y$, c is the composite $Y_{(y)}^{\prime} \rightarrow Y^{\prime} \xrightarrow{\sigma} X \overleftarrow{\times}_{S} Y^{\prime}$, and $\sigma$ is the canonical section induced by $x$ [ILO, XI 2.2].

By [ILO, XI Lemme 2.5], $\operatorname{id}_{X} \overleftarrow{\times}_{S} g: X \overleftarrow{\times}_{S} Y^{\prime} \rightarrow X \overleftarrow{\times}_{S} Y$ is a coherent morphism of coherent topoi.
Proof. By limit arguments, the stalk in question can be identified with $R \Gamma\left(T,\left.L\right|_{T}\right)$, where $T=$ $X_{(x)} \overleftarrow{\times}_{S_{(a(x))}} Y_{(y)}^{\prime}$. We have $R \Gamma\left(T,\left.L\right|_{T}\right) \simeq R \Gamma\left(Y_{(y)}^{\prime}, R p_{2 *}\left(\left.L\right|_{T}\right)\right)$, where $p_{2}: T \rightarrow Y_{(y)}^{\prime}$ is the projection and $R p_{2 *}\left(\left.L\right|_{T}\right) \simeq c^{*} L$ by [ILO, XI Proposition 2.3].

Corollary 1.2. Let $X^{\prime} \xrightarrow{f} X \xrightarrow{a} S \stackrel{b}{\leftarrow} Y \stackrel{g}{\leftarrow} Y^{\prime}$ be morphisms of coherent topoi with a, af, b, g coherent. Then the base change map

$$
\alpha:\left(f \overleftarrow{×}_{S} \operatorname{id}_{Y}\right)^{*} R\left(\operatorname{id}_{X} \overleftarrow{×}_{S} g\right)_{*} \rightarrow R\left(\operatorname{id}_{X^{\prime}} \overleftarrow{×}_{S} g\right)_{*}\left(f \overleftarrow{×}_{S} \operatorname{id}_{Y^{\prime}}\right)^{*}
$$

associated to the square (Cartesian by ILO, XI Proposition 4.2])

$$
\begin{aligned}
& X^{\prime} \overleftarrow{ธ}_{S} Y^{\mathrm{jd}_{X^{\prime}} \overleftarrow{x}_{S} g} X^{\prime} \overleftarrow{ธ}_{S} Y
\end{aligned}
$$

is an isomorphism on $D^{+}\left(X \overleftarrow{×}_{S} Y^{\prime}, \Lambda\right)$.
Here $\Leftrightarrow$ denotes an isomorphism of morphisms of topoi.
Recall that any locally coherent topos [SGA4, VI Définition 2.3] has enough points by Deligne's theorem [SGA4, VI Théorème 9.0]. The stalk of $\alpha$ at every point $\left(x^{\prime}, y, \phi\right)$ of $X^{\prime} \overleftarrow{\times}_{S} Y$ is an isomorphism by Proposition 1.1.

Let $f: X \rightarrow S$ and $g: Y \rightarrow S$ be morphisms of topoi. For sheaves $X^{\prime}, Y^{\prime}, S^{\prime}$ on $X, Y, S$, equipped with morphisms $X^{\prime} \rightarrow S^{\prime}$ and $Y^{\prime} \rightarrow S^{\prime}$ above $f$ and $g$, we write $X^{\prime} \overline{\times}_{S^{\prime}} Y^{\prime}$ for the object $p_{1}^{*} X^{\prime} \times_{h^{*} S^{\prime}} p_{2}^{*} Y^{\prime}$ of $X \overline{\times}_{S} Y$, where $p_{1}: X \overline{\times}_{S} Y \rightarrow X$ and $p_{2}: X \bar{×}_{S} Y \rightarrow Y$ are the projections and $h$ denotes $f p_{1} \simeq g p_{2}$.

The following is an analogue of [ILO, XI Lemme 2.5], with essentially the same proof.
Lemma 1.3. Let $X \rightarrow S$ and $Y \rightarrow S$ be coherent morphisms of coherent topoi. Then $X \bar{x}_{S} Y$ is coherent and the projections $p_{1}$ and $p_{2}$ are coherent.

Lemma 1.3 implies that objects of the form $X^{\prime} \bar{x}_{S^{\prime}} Y^{\prime}$ with $X^{\prime}, Y^{\prime}, S^{\prime}$ coherent form a generating family.

The following is an analogue of [ILO, XI Corollaire 2.3.2].
Lemma 1.4. Let $f: X \rightarrow S$ and $g: Y \rightarrow S$ be local morphisms of local topoi. Then the fiber product $X \bar{×}_{S} Y$ is a local topos of center $z=(x, y, \phi)$, where $x$ and $y$ are the centers of $X$ and $Y$, respectively, and $\phi: f(x) \simeq g(y)$ is the unique isomorphism.

Proof. It suffices to show that for every sheaf $\mathcal{F}$ on $X \bar{x}_{S} Y$, any element of the stalk $\mathcal{F}_{z}$ lifts uniquely to a section. For this we may assume that $\mathcal{F}=X^{\prime} \overline{\times}_{S^{\prime}} Y^{\prime}$ for sheaves $X^{\prime}, Y^{\prime}, S^{\prime}$ on $X, Y, S$, equipped with morphisms $X^{\prime} \rightarrow S^{\prime}, Y^{\prime} \rightarrow S^{\prime}$ above $f$ and $g$. Any element of $\mathcal{F}_{z}$ corresponds to compatible elements of $X_{x}^{\prime}, Y_{y}^{\prime}, S_{s}^{\prime}$, which correspond in turn to sections of $X^{\prime}, Y^{\prime}, S^{\prime}$, providing a section of $\mathcal{F}$. Here $s$ denotes the center of $S$.

Remark 1.5. As observed in [ILO, XI Exemples 3.4 (2)], given morphisms of schemes $X \rightarrow S \leftarrow Y$, the morphism of topoi

$$
\left(X \times_{S} Y\right)_{\mathrm{et}} \rightarrow X_{\mathrm{et}} \overline{\times}_{S_{\mathrm{et}}} Y_{\mathrm{et}}
$$

is not an equivalence in general, even if $X, Y$, and $S$ are spectra of fields 1 Indeed, if $S=\operatorname{Spec}(k)$ with $k$ separably closed, and $X=\operatorname{Spec}\left(k_{1}\right), Y=\operatorname{Spec}\left(k_{2}\right)$ with $k_{1} / k$ and $k_{2} / k$ transcendental, then $X_{\text {et }} \overline{\times}_{S_{\text {et }}} Y_{\text {et }}$ has only one isomorphism class of points, while $\left(X \times_{S} Y\right)_{\text {et }}$ has infinitely many.

Let $\Lambda$ be a torsion commutative ring. The following is a generalization of [O, Lemme 10.1].
Proposition 1.6. Let $f: X \rightarrow S$ be a proper morphism of schemes. Let $g: Y \rightarrow S_{\text {et }}$ be a locally coherent morphism [SGA4, VI Définition 3.7] of locally coherent topoi. Then for any point y of $Y$, any geometric point $s$ of $S$, and any isomorphism $\phi: s_{\mathrm{et}} \xrightarrow{\sim} g(y)$, the base change map $\alpha:\left(R p_{2 *} L\right)_{y} \rightarrow$ $R \Gamma\left(X_{s}, c^{*} L\right)$ associated to the square

is an isomorphism for $L \in D\left(X_{\mathrm{et}} \overline{\times}_{S_{\mathrm{et}}} Y, \Lambda\right)$. Here $X_{s}=X \times_{S} s$, and $c$ is induced by the diagram

where the triangle is given by $\phi$.
Proof. The proof is similar to that of [O, Lemme 10.1]. We may assume that $S$ is strictly local of center $s$ and $Y$ is local of center $y$. There exists an integer $d$ such that the dimensions of the fibers of $f$ are $\leq d$. It suffices to show that $\alpha$ is an isomorphism for $L \in D^{+}$. Indeed, this implies that $R p_{2 *}$ has cohomological dimension $\leq 2 d$, and the general case follows.

Consider the diagram of topoi


[^1]and the base change map $\beta: i^{*} R p_{1 *} L \rightarrow c^{*} L$. Then $\alpha$ can be identified with the composition
$$
R \Gamma\left(X, R p_{1 *} L\right) \xrightarrow{\sim} R \Gamma\left(X_{s}, i^{*} R p_{1 *} L\right) \xrightarrow{R \Gamma\left(X_{s}, \beta\right)} R \Gamma\left(X_{s}, c^{*} L\right),
$$
where the first arrow is proper base change. For any geometric point $x$ of $X_{s}, \beta_{x}$ can be identified with the map $R \Gamma\left(\left(X_{(x)}\right)_{\text {et }} \overline{\times}_{S_{\mathrm{et}}} Y, L\right) \rightarrow L_{(x, y, \phi)}$, which is an isomorphism by Lemma 1.4. It follows that $\beta$ and $\alpha$ are isomorphisms.

## Remark 1.7.

(1) The square (1.1) is not Cartesian in general. Indeed, in the case where $S=\operatorname{Spec}(k)$ with $k$ a separably closed field, $\operatorname{dim}(X) \geq 1, Y=\mathrm{pt}$, and $s=\operatorname{Spec}\left(k^{\prime}\right)$ with $k^{\prime}$ a transcendental (separably closed) extension of $k$, the morphism $\left(X_{s}\right)_{\text {et }} \rightarrow X_{\text {et }}$ is not an equivalence.
(2) For locally coherent morphisms of locally coherent topoi $S_{\text {et }} \rightarrow B \stackrel{h}{\leftarrow} B^{\prime}$, we can take $g: S_{\text {et }} \bar{x}_{B}$ $B^{\prime} \rightarrow S_{\text {et }}$ to be the first projection, which is a locally coherent morphism of locally coherent topoi by Lemma 1.3. In this case, $p_{2}$ can be identified with $f_{\text {et }} \bar{x}_{B} B^{\prime}: X_{\text {et }} \bar{x}_{B} B^{\prime} \rightarrow S_{\text {et }} \bar{x}_{B} B^{\prime}$.
(3) For locally coherent morphisms of locally coherent topoi $S_{\mathrm{et}} \rightarrow B \leftarrow B^{\prime}$, we can take $g: Y=$ $S_{\text {et }} \overleftarrow{\times}_{B} B^{\prime} \rightarrow S_{\text {et }}$ to be the first projection, which is a locally coherent morphism of locally coherent topoi by [ILO, XI Lemme 2.5]. In this case, $p_{2}$ can be identified with $f \overleftarrow{\times}_{B} B^{\prime}: X_{\text {et }} \overleftarrow{\times}_{B} B^{\prime} \rightarrow$ $S_{\text {et }} \overleftarrow{×}_{B} B^{\prime}$ by [ILO, XI Proposition 4.2], and we recover [O] Lemme 10.1]. This can be identified with the special case of (2) with $h$ being the first projection $B \overleftarrow{\times}_{B} B^{\prime} \rightarrow B$.
(4) For $f$ integral, Proposition 1.6 holds without the assumption that $\Lambda$ is torsion. Indeed, in the proof above, it suffices to replace proper base change by integral base change [SGA4, VIII Corollaire 5.6].
Recall that for any morphism of topoi $f: X \rightarrow Y$, we have a projection formula map

$$
R f_{*} L \otimes^{L} M \rightarrow R f_{*}\left(L \otimes^{L} f^{*} M\right)
$$

adjoint to the composition

$$
f^{*}\left(R f_{*} L \otimes^{L} M\right) \simeq f^{*} R f_{*} L \otimes^{L} f^{*} M \rightarrow L \otimes^{L} f^{*} M
$$

Construction 1.8. Applying Nagata's compactification theorem [C Theorem 4.1] and Deligne's gluing formalism [SGA4, XVII 3.3], we define for $f: X^{\prime} \rightarrow X$ a separated morphism of finite type of coherent schemes equipped with locally coherent morphisms of locally coherent topoi $X_{\text {et }} \rightarrow S \leftarrow Y$, a functor

$$
R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)_{!}: D\left(X_{\mathrm{et}}^{\prime} \bar{x}_{S} Y, \Lambda\right) \rightarrow D\left(X_{\mathrm{et}} \bar{x}_{S} Y, \Lambda\right),
$$

isomorphic to $R\left(f_{\text {et }} \bar{x}_{S} \text { id }_{Y}\right)_{*}$ for $f$ proper and left adjoint to $\left(f_{\text {et }} \bar{x}_{S} \text { id }{ }_{Y}\right)^{*}$ for $f$ an open immersion, and compatible with composition.

Given a Cartesian square

of coherent schemes with $f$ proper and $j$ an open immersion, as in [SGA4, XVII Lemme 5.1.6] we need to check that the morphism

$$
\begin{equation*}
\left(j_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)_{!}\left(f_{U \mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)_{*} \rightarrow\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)_{*}\left(j_{\mathrm{et}}^{\prime} \bar{x}_{S} \operatorname{id}_{Y}\right)_{!} \tag{1.2}
\end{equation*}
$$

induced by the inverse of the isomorphism

$$
\left(j_{\mathrm{et}} \bar{x}_{S} \mathrm{id}_{Y}\right)^{*}\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)_{*} \xrightarrow{\sim}\left(f_{U \mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)_{*}\left(j_{\mathrm{et}}^{\prime} \bar{x}_{S} \operatorname{id}_{Y}\right)^{*}
$$

is an isomorphism. The restriction of (1.2) to $U_{\text {et }} \bar{x}_{S} Y$ is trivially an isomorphism. Let $X_{1}=X-U$ be a complement of $U$. The restriction of the left-hand side of (1.2) to $\left(X_{1}\right)_{\text {et }} \bar{x}_{S} Y$ is zero, and the restriction of the right-hand side of (1.2) to $\left(X_{1}\right)_{\text {et }} \bar{x}_{S} Y$ is zero by Proposition 1.6 and Remark 1.7 (2).

We have the following isomorphisms.
(1) (base change on $X$ ) For any Cartesian square of coherent schemes

we have

$$
\left(h_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)^{*} R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{idd}_{Y}\right)!\simeq R\left(f_{\mathrm{et}}^{\prime} \bar{x}_{S} \operatorname{id}_{Y}\right)_{!}\left(h_{\mathrm{et}}^{\prime} \bar{x}_{S} \mathrm{id}_{Y}\right)^{*} .
$$

(2) (base change on $Y$ ) For any locally coherent morphism $g: Y^{\prime} \rightarrow Y$ of locally coherent topoi, we have

$$
\left(\operatorname{id}_{X_{\mathrm{et}}} \bar{x}_{S} g\right)^{*} R\left(f_{\mathrm{et}} \bar{x}_{S} \mathrm{id}_{Y}\right)!\simeq R\left(f_{\mathrm{et}} \overline{\times}_{S} \operatorname{id}_{Y^{\prime}}\right)!\left(\mathrm{id}_{X_{\mathrm{et}}^{\prime}} \bar{x}_{S} g\right)^{*}
$$

(3) (projection formula) For $L \in D\left(X_{\mathrm{et}}^{\prime} \bar{x}_{S} Y, \Lambda\right), M \in D\left(X_{\mathrm{et}} \bar{x}_{S} Y, \Lambda\right)$, we have

$$
R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)!L \otimes^{L} M \simeq R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)!\left(L \otimes^{L}\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)^{*} M\right) .
$$

For $f$ proper, these maps are given by adjunction. In this case the maps in (1) and (2) are isomorphisms by Proposition 1.6 and Remark 1.7 (2). To show that the map in (3) is an isomorphism in this case, we reduce by Proposition 1.6 and Remark 1.7 (2) to the classical case where $Y=S=\mathrm{pt}$. For $f$ an open immersion, the inverses of these isomorphisms are standard [Z1, Constructions 2.6, 2.7].
Construction 1.9. The functor $R\left(f_{\text {et }} \bar{x}_{S}\right.$ id $\left._{Y}\right)$ ! has cohomological dimension $\leq 2 d$, where $d$ is the maximum of the dimensions of the fibers of $f$. Thus the functor admits a right adjoint

$$
\begin{equation*}
R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)^{!}: D\left(X_{\mathrm{et}} \overline{\times}_{S} Y, \Lambda\right) \rightarrow D\left(X_{\mathrm{et}}^{\prime} \bar{x}_{S} Y, \Lambda\right) \tag{1.3}
\end{equation*}
$$

by Lemma 1.10 below applied to the proper case.
Assume $m \Lambda=0$ with $m$ invertible on $X$ and $f$ flat with fibers of dimension $\leq d$. Then the trace $\operatorname{map} \operatorname{Tr}_{f_{\mathrm{et}}}(\Lambda): R f_{\mathrm{et}!} \Lambda_{X_{\mathrm{et}}^{\prime}} \rightarrow \Lambda_{X_{\mathrm{et}}}(-d)[-2 d]$ for $f_{\mathrm{et}}$ induces a trace map

$$
\operatorname{Tr}_{f_{\mathrm{ett}} \bar{\chi}_{S} \mathrm{id}_{Y}}(\Lambda): R\left(f_{\mathrm{et}} \overline{\times}_{S} \operatorname{id}_{Y}\right)!\Lambda \simeq p_{1}^{*} R f_{\mathrm{et}!} \Lambda \rightarrow \Lambda(-d)[-2 d]
$$

for $f_{\text {et }} \bar{X}_{S} \operatorname{id}_{Y}$, where $p_{1}: X_{\text {et }} \bar{X}_{S} Y \rightarrow X_{\text {et }}$ denotes the projection. Here we used base change (2) on $Y$. By the projection formula (3), this induces a natural transformation
which induces, by adjunction, a natural transformation

$$
\begin{equation*}
\operatorname{tr}_{f_{\text {et }} \bar{x}_{S} \mathrm{id}_{Y}}:\left(f_{\mathrm{et}} \bar{x}_{S} \mathrm{id}_{Y}\right)^{*}(d)[2 d] \rightarrow R\left(f_{\mathrm{et}} \bar{x}_{S} \mathrm{id}_{Y}\right)^{!} \tag{1.4}
\end{equation*}
$$

Lemma 1.10. Let $h: Z^{\prime} \rightarrow Z$ be a coherent morphism of locally coherent topoi such that there exists a covering family of objects $Y_{\alpha}$ of $Z$ with $h^{*} Y_{\alpha}$ algebraic [SGA4, VI Définition 2.3]. Then for all $q$, $R^{q} h_{*}$ commutes with filtered colimits. If, moreover, $R h_{*}: D\left(Z^{\prime}, \Lambda\right) \rightarrow D(Z, \Lambda)$ has finite cohomological dimension, then $R h_{*}$ admits a right adjoint.

Proof. For the first assertion we may assume $Z$ coherent and the assertion becomes [SGA4, VI Théorème 5.1]. The second assertion follows by the Brown representability theorem. More precisely, one applies [KS, Corollary 14.3.7] to the collection of $h_{*}$-acyclic sheaves, which is stable under small direct sums by the first assertion.

Construction 1.11. Consider functors

and adjunctions $L \dashv R$ and $L^{\prime} \dashv R^{\prime}$. Then natural transformations $\alpha: L F \rightarrow G L^{\prime}$ correspond by adjunction to natural transformations $\beta: F R^{\prime} \rightarrow R G$. For $\alpha$ given, $\beta$ is the composition

$$
F R^{\prime} \rightarrow R L F R^{\prime} \xrightarrow{\alpha} R G L^{\prime} R^{\prime} \rightarrow R G .
$$

The same holds for adjunctions in 2-categories [A, Proposition 1.1.9].
Lemma 1.12. Let $Y^{\prime}$ be an object of $Y$ and let $g: Y^{\prime} \rightarrow Y$. The map

$$
\begin{equation*}
\left(\operatorname{id}_{X_{\mathrm{et}}^{\prime}} \bar{x}_{S} g\right)^{*} R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)^{!} \rightarrow R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y^{\prime}}\right)^{!}\left(\operatorname{id}_{X_{\mathrm{et}}} \bar{x}_{S} g\right)^{*} \tag{1.5}
\end{equation*}
$$

induced by Constructions 1.8 (2) and 1.11 is a natural isomorphism.
Proof. We may assume $f$ proper. It suffices to show that the map $\left(\operatorname{id}_{X} \bar{x}_{S} g\right)!R\left(f_{\mathrm{et}} \bar{x}_{S} \mathrm{id}_{Y^{\prime}}\right)_{*} \rightarrow$ $R\left(f_{\text {et }} \bar{x}_{S} \operatorname{id}_{Y}\right)_{*}\left(\operatorname{id}_{X^{\prime}} \bar{x}_{S} g\right)$ !, left adjoint of (1.5), is an isomorphism. For this, we may assume that $Y$ is a point. In this case, $Y^{\prime}$ is a small set $I$ and (id $\bar{X}_{S} g$ )! can be identified with $\bigoplus_{I}$. It then suffices to note that $R\left(f_{\mathrm{et}} \bar{x}_{S} \mathrm{id}_{Y}\right)_{*}$ commutes with small direct sums.

By adjunction, the projection formula induces the following isomorphisms (see for example [Z1, Proposition 2.24]).

Corollary 1.13. We have isomorphisms, natural in $L, M \in D\left(X_{\mathrm{et}} \bar{x}_{S} Y, \Lambda\right)$ and $L^{\prime} \in D\left(X_{\mathrm{et}}^{\prime} \bar{x}_{S} Y, \Lambda\right)$,

$$
\begin{align*}
R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)^{!} R \mathcal{H o m}(L, M) & \simeq R \mathcal{H o m}\left(\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)^{*} L, R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)^{!} M\right)  \tag{1.6}\\
R \mathcal{H o m}\left(R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)!L^{\prime}, M\right) & \simeq R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)_{*} R \mathcal{H o m}\left(L, R\left(f_{\mathrm{et}} \bar{x}_{S} \operatorname{id}_{Y}\right)^{!} M\right), \tag{1.7}
\end{align*}
$$

For $L \in D(X, \Lambda), M \in D(Y, \Lambda)$, we write $L \boxtimes^{L} M$ for $p_{1}^{*} L \otimes^{L} p_{2}^{*} M$, where $p_{1}: X \bar{x}_{S} Y \rightarrow X$ and $p_{2}: X \bar{x}_{S} Y \rightarrow Y$ are the projections. The base change isomorphisms and projection formula induce the following Künneth formula.

Corollary 1.14. Let $f: X^{\prime} \rightarrow X$ and $g: Y^{\prime} \rightarrow Y$ be morphisms separated of finite type of coherent schemes, and let $X_{\mathrm{et}} \rightarrow S$ and $Y_{\mathrm{et}} \rightarrow S$ be locally coherent morphisms of locally coherent topoi. Let $L \in D\left(X_{\mathrm{et}}^{\prime}, \Lambda\right)$ and $M \in D\left(Y_{\mathrm{et}}^{\prime}, \Lambda\right)$. Then we have an isomorphism

$$
R f_{!} L \boxtimes^{L} R g!M \xrightarrow{\sim} R\left(f \overline{\times}_{S} g\right)!\left(L \boxtimes^{L} M\right) .
$$

Here $R\left(f \bar{x}_{S} g\right)$ ! denotes the functor $R\left(f \overline{\times}_{S} \operatorname{id}_{X}\right)!R\left(\operatorname{id}_{Y^{\prime}} \bar{x}_{S} g\right)!\simeq R\left(\operatorname{id}_{X} \overline{\times}_{S} g\right)!R\left(f \bar{x}_{S} \operatorname{id}_{Y^{\prime}}\right)!$.
Here we have omitted the subscript "et" when no confusion arises.

### 1.2 Over a classifying topos

In this subsection, we study operations on fiber products of topoi over the classifying topos of a profinite group, which include slices of vanishing topoi. Among other things we prove Künneth formulas and biduality, which will be used in Subsection 2.4 and in many places in Sections 4 and 5

Let $s=B G$ be the classifying topos of a profinite group $G$, consisting of discrete sets equipped with a continuous action of $G$. In our applications $s$ will be the étale topos of a spectrum of a field. Let $X$ and $Y$ be topoi over $s$. For $G=\{1\}, X \overline{\times}_{s} Y$ is the product topos $X \overline{\times} Y$.

Remark 1.15. For any topos $Z$, the category of morphisms of topoi $Z \rightarrow s$ is a groupoid. In other words, if $p_{1}$ and $p_{2}$ are morphisms $Z \rightarrow s$, then any natural transformation $\alpha: p_{1}^{*} \rightarrow p_{2}^{*}$ is an isomorphism. Indeed, since $p_{i}^{*}$ carries the final object to the final object and preserves coproducts, $\alpha$ is an isomorphism on constant sheaves. It follows that $\alpha$ is an isomorphism on locally constant sheaves. It then suffices to note that every sheaf on $s$ is a coproduct of locally constant sheaves.

It follows that $X \bar{x}_{s} Y$ is also the oriented product $X \overleftarrow{×}_{s} Y$.
Remark 1.16. Given a generating family $\{U\}$ of $X$ and a generating family $\{V\}$ of $Y,\left\{\Lambda_{U \bar{x}_{s} V, X \bar{x}_{s} Y}\right\}$ is a generating family of $\operatorname{Shv}\left(X \bar{x}_{s} Y, \Lambda\right)$. Here, $\Lambda_{U \bar{x}_{s} V, X \bar{x}_{s} Y}$ is the free sheaf of $\Lambda$-modules on $X \bar{x}_{s} Y$ generated by $U \bar{x}_{s} V$ [SGA4, IV Proposition 11.3.3]. Our notation $U \bar{x}_{s} V$ is consistent with [ILO, XI 1.10 (b)] (cf. AGT, Proposition VI.3.15]), using the same notation for objects of a topos and the corresponding localized topoi.

Indeed, by the construction of $X \bar{x}_{s} Y$, objects of the form $U \times_{a, B H, b} V, H$ running through open normal subgroups of $G$, form a generating family of $X \bar{x}_{s} Y$. Here $a: U \rightarrow B H$ and $b: V \rightarrow B H$ are morphisms of topoi. We have

$$
\begin{equation*}
B H \bar{x}_{s} B H \simeq \coprod_{g \in G / H} B H . \tag{1.8}
\end{equation*}
$$

The two projections on the component indexed by $g H, g \in G$ are given respectively by the identity and by the morphism $c_{g}: B H \rightarrow B H$ induced by conjugation by $g$. The isomorphism class of $c_{g}$ depends only on $g H$. It follows that $U \bar{×}_{s} V \simeq \amalg_{g \in G / H} U \bar{×}_{a, B H, c_{g} b} V$.

We restrict our attention to coherent topoi equipped with morphisms to $s$, which we call coherent topoi over $s$. Note that $s$ is a coherent topos and any morphism $X \rightarrow s$ with $X$ coherent is coherent. Indeed, $B H, H$ running through open normal subgroups of $G$, form a generating family of coherent objects of $s$, and by SGA4, VI Proposition 3.2], it suffices to show that $B H \bar{x}_{s} X$ is coherent. By (1.8) the pullback of the projection $p: B H \bar{x}_{s} X \rightarrow X$ by itself is coherent, so that $p$ is coherent by [SGA4, VI Proposition 1.10 (ii)].

Let $\mathrm{pt}=\bar{s} \rightarrow s$ be the point of $s$ with inverse image given by the functor forgetting the action of $G$. By Remark 1.15, all points of $s$ are isomorphic.

Corollary 1.2 takes the following form.
Proposition 1.17. Let $f: X^{\prime} \rightarrow X$ and $g: Y^{\prime} \rightarrow Y$ be morphisms of coherent topoi over s. Assume that $f$ is coherent. Then the base change map

$$
\mathrm{BC}_{g}:\left(\operatorname{id}_{X} \bar{x}_{s} g\right)^{*} R\left(f \bar{x}_{s} \operatorname{id}_{Y}\right)_{*} \rightarrow R\left(f \bar{x}_{s} \operatorname{id}_{Y^{\prime}}\right)_{*}\left(\operatorname{id}_{X^{\prime}} \bar{x}_{s} g\right)^{*}
$$

associated to the (Cartesian) square of topoi
is a natural isomorphism on $D^{+}\left(X^{\prime} \bar{x}_{s} Y, \Lambda\right)$.
Remark 1.18. Note that $R f_{*}$ and $R f_{\bar{s} *}$ have the same cohomological dimension. Moreover, $R\left(f \bar{x}_{s} \mathrm{id}\right)_{*}$ has the same cohomological dimension by the proposition applied to $Y^{\prime}=\mathrm{pt}$. If these cohomological dimensions are finite, then the proposition holds more generally on the unbounded derived category $D\left(X^{\prime} \bar{x}_{s} Y, \Lambda\right)$.
Remark 1.19. The analogue of Proposition 1.17 does not hold for fiber products of schemes. For example, let $s=\operatorname{Spec}(k)$, where $k$ is a field, $X=Y=\mathbb{P}_{k}^{1}$, and let $f: X^{\prime}=\mathbb{A}_{k}^{1} \rightarrow X$ and $g: Y^{\prime}=$ $\{\infty\} \rightarrow Y$ be the immersions. Let $\Gamma: X^{\prime} \rightarrow X^{\prime} \times_{s} Y$ be the graph of $f$. Then $\left(\operatorname{id}_{X^{\prime}} \times_{s} g\right)^{*}\left(\Gamma_{*} \Lambda\right)=0$ while $\left(\operatorname{id}_{X} \times_{s} g\right)^{*}\left(f \times_{s} \operatorname{id}_{Y}\right)_{*}\left(\Gamma_{*} \Lambda\right)$ can be identified with $g_{*} \Lambda$.

Corollary 1.20. Let $f$ and $Y$ be as above, $L \in D\left(X^{\prime} \overline{\times}_{s} Y, \Lambda\right), M \in D(Y, \Lambda)$. Assume that one of the following holds:
(a) $R f_{*}$ has finite cohomological dimension; or
(b) $L, M \in D^{+}$, and $\Lambda$ has finite weak dimension; or
(c) $L \in D^{+}$and $M$ has finite tor-amplitude; or
(d) $f=\left(f_{0}\right)_{\text {et }}$, where $f_{0}: X_{0}^{\prime} \rightarrow X_{0}$ is a morphism of finite type of Noetherian schemes, $m \Lambda=0$ for an integer $m$ invertible on $X_{0}$, and $L \in D^{+}, L \otimes^{L} p_{2}^{*} M \in D^{+}$.
Then the projection formula map

$$
R\left(f \overline{\times}_{s} \operatorname{id}_{Y}\right)_{*} L \otimes^{L} p_{2}^{*} M \rightarrow R\left(f \overline{\times}_{s} \operatorname{id}_{Y}\right)_{*}\left(L \otimes^{L} p_{2}^{*} M\right)
$$

is an isomorphism. Here $p_{2}: X \overline{\times}_{s} Y \rightarrow X$ and $p_{2}^{\prime}: X^{\prime} \overline{\times}_{s} Y \rightarrow X^{\prime}$ denote the projections.
Recall that the weak dimension of $\Lambda$ is the supremum of integers $n$ such that $\operatorname{Tor}_{n}^{\Lambda}(A, B) \neq 0$ for some $\Lambda$-modules $A$ and $B$.

Proof. We may assume that $s$ is a point. By Proposition 1.17, we may further assume that $Y=s$. This case is standard. For (a), see [I3, Lemma A.8]. For (b), we reduce to the case $M \in D^{b}$, which is a special case of (c). For (c), we reduce to the case where $M$ is a flat $\Lambda$-module, and then to the trivial case where $M$ is a finite free $\Lambda$-module. For (d), we reduce by localization at a point to the case where $X_{0}$ is finite-dimensional, which is a special case of (a) by Gabber's theorem on the finiteness of cohomological dimension [ILO, XVIII ${ }_{\mathrm{A}}$ Corollary 1.4].

Remark 1.21. In case (d), the projection formula implies that $R\left(f \bar{x}_{s} \mathrm{id}_{Y}\right)_{*}$ preserves complexes of tor-amplitude $\geq n$.

Combining the corollary with the proposition, we obtain the following Künneth formula for $R\left(f \bar{x}_{s}\right.$ $g)_{*}$. The analogue for fiber products of schemes over a field is [SGA5, III (1.6.4)].

Corollary 1.22. Let $f: X^{\prime} \rightarrow X$ and $g: Y^{\prime} \rightarrow Y$ be coherent morphisms of coherent topoi over $s$. Let $L \in D\left(X^{\prime}, \Lambda\right), M \in D\left(Y^{\prime}, \Lambda\right)$. Assume that either one of the following holds:
(a) $R f_{*}$ and $R g_{*}$ have finite cohomological dimensions;
(b) $L, M \in D^{+}$, and $\Lambda$ has finite weak dimension.

Then the Künneth formula map

$$
R f_{*} L \boxtimes^{L} R g_{*} M \rightarrow R\left(f \overline{\times}_{s} g\right)_{*}\left(L \boxtimes^{L} M\right)
$$

is an isomorphism.
By a scheme over $s$ we mean a scheme $Z$ equipped with a morphism of topoi $Z_{\text {et }} \rightarrow s$. Let $X$ be a coherent scheme over $s$. For any open subgroup $H<G$, we have $X_{\text {et }} \bar{X}_{s} t \simeq\left(X_{t}\right)_{\text {et }}$, where $t=B H$ and $X_{t} \rightarrow X$ is a finite étale cover. Taking limit, we get $X_{\mathrm{et}} \bar{X}_{s} \bar{s} \simeq\left(X_{\bar{s}}\right)_{\text {et }}$, where $X_{\bar{s}}=\lim X_{t}$.

Let $Y$ be a coherent topos over $s$. Assume $\Lambda$ of torsion. By Constructions 1.8 and 1.9, for $f: X^{\prime} \rightarrow X$ a morphism of schemes separated of finite type, the functors $\left(f_{\text {et }} \bar{x}_{s} \operatorname{id}_{Y}\right)!$ and $\left(f_{\text {et }} \overline{\times}_{s} \operatorname{id}_{Y}\right)$ ! are defined. In fact, in this case, the proper base change used in the construction of $\left(f_{\text {et }} \bar{x}_{s} \text { id }\right)_{Y}$ ! does not require Proposition 1.6 and can be reduced by Proposition 1.17 to classical proper base change.

Assume $m \Lambda=0$ for some $m$ invertible on $X$.
Proposition 1.23. For $f$ smooth of dimension $d$, the trace map (1.4)

$$
\operatorname{tr}_{f_{\mathrm{et}} \bar{x}_{s} \mathrm{id}_{Y}}:\left(f_{\mathrm{et}} \bar{x}_{s} \operatorname{id}_{Y}\right)^{*}(d)[2 d] \rightarrow R\left(f_{\mathrm{et}} \overline{\mathrm{X}}_{s} \mathrm{id}_{Y}\right)^{!}
$$

is an isomorphism.

Proof. Let us first show that $\operatorname{tr}(L)$ is an isomorphism for $L \in D^{+}\left(X_{\text {et }} \bar{x}_{s} Y, \Lambda\right)$. For this, it suffices to check that $R \Gamma\left(U_{\text {et }}^{\prime} \bar{x}_{s} V,\left.\operatorname{tr}(L)\right|_{U_{\text {et }}^{\prime} \bar{x}_{s} V}\right)$ is an isomorphism for $u: U^{\prime} \rightarrow X^{\prime}$ an étale morphism of coherent schemes and $V$ a coherent object of $Y$. By Lemma 1.12, $\left.\operatorname{tr}(L)\right|_{X_{\mathrm{et}}^{\prime} \bar{x}_{s} V} \simeq \operatorname{tr}\left(\left.L\right|_{X_{\mathrm{ett}} \bar{x}_{s} V}\right)$. Changing notation, it suffices to check that

$$
R \Gamma\left(U_{\mathrm{et}}^{\prime} \bar{x}_{s} Y,\left(u \bar{x}_{s} \operatorname{id}_{Y}\right)^{*} \operatorname{tr}(L)\right) \simeq R \Gamma\left(U_{\mathrm{et}}^{\prime}, R p_{1 *}^{\prime \prime}\left(u \bar{x}_{s} \operatorname{id}_{Y}\right)^{*} \operatorname{tr}(L)\right) \simeq R \Gamma\left(U_{\mathrm{et}}^{\prime}, u^{*} R p_{1 *}^{\prime} \operatorname{tr}(L)\right)
$$

is an isomorphism, where $p_{1}^{\prime}: X_{\text {et }}^{\prime} \bar{x}_{s} Y \rightarrow X_{\text {et }}^{\prime}$ and $p_{1}^{\prime \prime}: U_{\mathrm{et}}^{\prime} \bar{x}_{s} Y \rightarrow U_{\mathrm{et}}^{\prime}$ are the projections. We have a commutative diagram

$$
\begin{gathered}
f_{\mathrm{et}}^{*} R p_{1 *} L(d)[2 d] \xrightarrow{\sim} \xrightarrow{\operatorname{tr}\left(R p_{1 * L}\right)} R f_{\mathrm{et}}^{!} R p_{1 *} L \\
\simeq \downarrow p_{1 *}\left(f_{\mathrm{et}} \bar{x}_{s} \mathrm{id}_{Y}\right)^{*} L(d)[2 d] \xrightarrow{R p_{1 *}^{\prime} \operatorname{tr}(L)} R p_{1 *}^{\prime} R\left(f_{\mathrm{et}} \overline{\mathrm{x}}_{s} \mathrm{id}_{Y}\right)^{!} L .
\end{gathered}
$$

The left vertical isomorphism is given by Proposition 1.17 applied to the pair $\left(g, f_{\mathrm{et}}\right)$, where $g: Y \rightarrow s$. The right vertical isomorphism is given by adjunction from Construction 1.8 (2) applied to the pair $(f, g)$. The upper horizontal isomorphism is [SGA4, XVIII Théorème 3.2.5]. It follows that $R p_{1 *}^{\prime} \operatorname{tr}(L)$ is an isomorphism.

As in the proof of [SGA4, XVIII Théorème 3.2.5], the above result extends to unbounded $L$ by a more explicit construction of $R\left(f_{\mathrm{et}} \bar{x}_{s} \mathrm{id}_{Y}\right)^{!}$. For any coherent topos $E, \operatorname{Shv}(E, \Lambda)$ is equivalent to the Ind-category of the category of finitely presented sheaves of $\Lambda$-modules on $E$ by [SGA4, VI Corollaire 1.24.2, I Corollaire 8.7.7]. In fact, for each finitely presented object $U$ of $E, u_{!} \Lambda_{U}$ is a finitely presented object of $\operatorname{Shv}(E, \Lambda)$, where $u: U \rightarrow E$. Imitating [SGA4, XVIII Section 3.1], we define the modified Godement resolution for a sheaf $\mathcal{F}=\operatorname{colim}_{i} \mathcal{F}_{i}$ of $\Lambda$-modules on $E$ by $M^{\bullet}(\mathcal{F})=\operatorname{colim}_{i} G^{\bullet}\left(\mathcal{F}_{i}\right)$, where $G^{\bullet}$ denotes the Godement resolution associated to a conservative family of points of $E$ (which exists by Deligne's theorem [SGA4, VI Théorème 9.0]), and $\mathcal{F}_{i}$ are finitely presented. For a compactification $f=\bar{f} j$, we define $\left(f_{\text {et }} \bar{x}_{s} \mathrm{id}\right)_{!}^{\bullet} \mathcal{F}:=\left(\bar{f}_{\text {et }} \bar{x}_{s} \mathrm{id}\right)_{*} \tau^{\leq 2 d} M^{\bullet}\left(j_{\text {et }} \bar{x}_{s} \mathrm{id}\right)_{!} \mathcal{F}$, where $d$ is the maximum of the dimensions of the fibers of $f$. Then $R\left(f_{\mathrm{et}} \bar{x}_{s} \mathrm{id}\right)^{!}$is the derived functor of the complex of functors $\left(f_{\text {et }} \bar{x}_{s}\right.$ id)! , right adjoint to $\left(f_{\text {et }} \bar{x}_{s}\right.$ id)! . By [SGA4, XVII Proposition 1.2.10] (variant for a bounded complex of functors), $R\left(f_{\mathrm{et}} \bar{x}_{s} \mathrm{id}\right)^{!}$has finite cohomological amplitude on the unbounded derived category.

Proposition 1.24. Assume $f$ separated of finite presentation. The map

$$
\left(\operatorname{id}_{X_{\mathrm{et}}^{\prime}} \bar{x}_{s} g\right)^{*} R\left(f_{\mathrm{et}} \overline{\mathrm{x}}_{s} \operatorname{idd}_{Y}\right)^{!} \rightarrow R\left(f_{\mathrm{et}} \overline{\mathrm{x}}_{s} \operatorname{id}_{Y^{\prime}}\right)^{!}\left(\operatorname{id}_{X_{\mathrm{et}}} \overline{\mathrm{x}}_{s} g\right)^{*}
$$

induced by Constructions 1.8 (2) and 1.11 is a natural isomorphism on $D^{+}$.
Proof. We reduce to the following cases: (a) $f$ is smooth; (b) $f$ is a closed immersion. For (a), we apply the preceding proposition. For (b), we apply Proposition 1.17 to $R j_{*}$, where $j$ is the complementary open immersion.

Similarly, we have the following projection formula.
Proposition 1.25. Assume $X$ Noetherian finite-dimensional. Then the map

$$
R\left(f_{\mathrm{et}} \overline{\mathrm{x}}_{s} \mathrm{id}_{Y}\right)^{!} L \otimes^{L} p_{2}^{\prime *} M \rightarrow R\left(f_{\mathrm{et}} \bar{x}_{s} \operatorname{id}_{Y}\right)^{!}\left(L \otimes^{L} p_{2}^{*} M\right)
$$

adjoint to the composition

$$
R\left(f_{\mathrm{et}} \bar{x}_{s} \operatorname{id}_{Y}\right)_{!}\left(R\left(f_{\mathrm{et}} \bar{x}_{s} \operatorname{id}_{Y}\right)^{!} L \otimes^{L} p_{2}^{\prime *} M\right) \simeq R\left(f_{\mathrm{et}} \bar{x}_{s} \operatorname{id}_{Y}\right)_{!} R\left(f_{\mathrm{et}} \bar{x}_{s} \operatorname{id}_{Y}\right)^{!} L \otimes^{L} p_{2}^{*} M \rightarrow L \otimes^{L} p_{2}^{*} M
$$

of Construction 1.8 (3) and the adjunction map, is an isomorphism for $L \in D\left(X_{\mathrm{et}} \overline{\mathrm{x}}_{s} Y, \Lambda\right)$ and $M \in D(Y, \Lambda)$.

Proof. We proceed as in the proof of Proposition 1.24, For (a) we apply Proposition 1.23, For (b) we apply Corollary 1.20 (a) to $R j_{*}$. That $R j_{*}$ has finite cohomological dimension is a theorem of Gabber [ILO, XVIII $_{\text {A }}$ Corollary 1.4].

Let $X$ and $Y$ be coherent schemes over $s$. Again we omit the index "et" when no confusion arises. For separated morphisms of finite type of schemes $f: X^{\prime} \rightarrow X$ and $g: Y^{\prime} \rightarrow Y$, and $\Lambda$ of torsion, the functors $R\left(f \bar{x}_{s} g\right)$ ! and $R\left(f \bar{x}_{s} g\right)$ ! are defined. The base change isomorphism and projection formula above induce the following Künneth formula. The analogue for fiber products of schemes over a field is [SGA5, III (1.7.3)].

Corollary 1.26. Assume $X$ and $Y$ are Noetherian finite-dimensional and $m \Lambda=0$ for some $m$ invertible on $X$ and $Y$. Then for $L \in D(X, \Lambda), M \in D(Y, \Lambda)$, we have

$$
R f^{!} L \boxtimes^{L} R g^{\prime} M \xrightarrow{\sim} R\left(f \bar{x}_{s} g\right)^{!}\left(L \boxtimes^{L} M\right) .
$$

Let $\Lambda$ be a Noetherian commutative ring. Imitating [O, Section 9.1], we say that a sheaf of $\Lambda$ modules $\mathcal{F}$ on $X \bar{x}_{s} Y$ is constructible if every stalk is finitely generated and there exist finite partitions $X=\bigcup_{i} X_{i}, Y=\bigcup_{j} Y_{j}$ into disjoint constructible locally closed subsets such that the restriction $\mathcal{F}$ to each $X_{i} \bar{X}_{s} Y_{j}$ is locally constant. We write $\operatorname{Shv}_{c}(-, \Lambda)$ for the full subcategory of $\operatorname{Shv}(-, \Lambda)$ spanned by constructible sheaves. The subcategory $\operatorname{Shv}_{c}\left(X \bar{x}_{s} Y, \Lambda\right) \subseteq \operatorname{Shv}\left(X \bar{x}_{s} Y, \Lambda\right)$ is stable under kernels, cokernels, and extensions, but in general not stable under subobjects or quotients. Even for $X$ and $Y$ Noetherian, constructible sheaves are in general very different from Noetherian sheaves. For example, in the case where $X$ and $Y$ are spectra of separably closed fields, pt $\bar{x}_{s}$ pt can be identified with the topos associated to the underlying topological space of $G$, on which a sheaf is Noetherian if and only if the support is finite and the stalks are finitely generated. See however Lemma 1.28 below.

Lemma 1.27. For $\mathcal{F}$ constructible, there exist étale morphisms $u: U \rightarrow X$ and $v: V \rightarrow Y$ of schemes with $U$ and $V$ affine and an epimorphism $\left(u \bar{x}_{s} v\right)!\Lambda_{U \bar{x}_{s} V} \rightarrow \mathcal{F}$.

Proof. The proof is similar to that of [SGA4, IX Proposition 2.7]. Given $\alpha:\left(u_{\alpha} \bar{x}_{s} v_{\alpha}\right)!\Lambda_{U_{\alpha}} \bar{x}_{s} V_{\alpha} \rightarrow \mathcal{F}$, the set $E_{\alpha}$ of points $(x, y) \in X^{\text {cons }} \times Y^{\text {cons }}$ such that $\alpha$ is an epimorphism on $x \bar{x}_{s} y$ is open. Here $X^{\text {cons }}$ and $Y^{\text {cons }}$ denote respectively $X$ and $Y$ equipped with the constructible topology. By Remark 1.16. $\left(E_{\alpha}\right)$ form an open cover of $X^{\text {cons }} \times Y^{\text {cons }}$, and thus admits a finite subcover $\left(E_{i}\right)$ by the quasicompactness of $X^{\text {cons }} \times Y^{\text {cons. }}$. It suffices to take $U=\coprod_{i} U_{i}$ and $V=\coprod_{i} V_{i}$.

The following lemma is not needed in the rest of this paper.
Lemma 1.28. Assume that $X$ and $Y$ are Noetherian schemes. Assume that for each pair of points $x \in X, y \in Y$, the double coset $G_{x} \backslash G / G_{y}$ is finite. Here $G_{x}$ and $G_{y}$ denote respectively the images of $\operatorname{Gal}(\bar{x} / x)$ and $\operatorname{Gal}(\bar{y} / y)$ in $G$, well-defined up to conjugation. Then $\mathcal{F} \in \operatorname{Shv}\left(X \bar{x}_{s} Y, \Lambda\right)$ is constructible if and only if it is Noetherian.

Proof. The proof is similar to those of [SGA4, IX Proposition 2.9] and [O, Lemme 9.3]. Since $\operatorname{Shv}\left(X \bar{x}_{s}\right.$ $Y, \Lambda)$ admits a family of constructible generators, it suffices to show that every constructible sheaf $\mathcal{F}$ is Noetherian. We show that every filtered family $\left(\mathcal{F}_{\alpha}\right)$ of subsheaves of $\mathcal{F}$ admits a maximal element. Since each $\mathcal{F}_{\alpha}$ is a filtered colimit of constructible sheaves, and every morphism of constructible sheaves has constructible image, each $\mathcal{F}_{\alpha}$ is a filtered union of constructible subsheaves. Thus we may assume that each $\mathcal{F}_{\alpha}$ is constructible. The maximal points of $X \bar{x}_{s} Y$ are of the form $p=(\bar{x}, \bar{y}, \phi)$, where $\bar{x} \rightarrow X, \bar{y} \rightarrow Y$ are geometric generic points. For every point $q$ of $X \bar{x}_{s} Y$, there exists a specialization from a maximal point $p$ to $q$. The assumptions imply that there are only finitely many isomorphism classes of maximal points. Let $\alpha$ be such that $\left(\mathcal{F}_{\alpha}\right)_{p}=\left(\mathcal{F}_{\beta}\right)_{p}$ for every maximal point $p$ and every $\beta \geq \alpha$. Let $U \subseteq X$ and $V \subseteq Y$ be nonempty open subsets such that $\left.\mathcal{F}_{\alpha}\right|_{U \bar{x}_{s} V}$ and $\left.\mathcal{F}\right|_{U \bar{x}_{s} V}$ are locally constant. Then $\left.\mathcal{F}_{\alpha}\right|_{U \bar{x}_{s} V}=\left.\mathcal{F}_{\beta}\right|_{U \bar{x}_{s} V}$ for all $\beta \geq \alpha$ and we conclude by Noetherian induction.

Lemma 1.29. Let $f: X_{0} \rightarrow X$ be a morphism of coherent schemes. Assume either (a) $f$ weakly étale [BS, Definition 1.2] or (b) $(f, \mathbb{Z} / m \mathbb{Z})$ universally locally acyclic for some integer $m>0$ such that $m \Lambda=0$. For $L \in D_{c}^{-}\left(X \bar{x}_{s} Y, \Lambda\right)$ and $M \in D^{+}\left(X \bar{x}_{s} Y, \Lambda\right)$, the canonical map

$$
\left(f \bar{x}_{s} \operatorname{id}_{Y}\right)^{*} R \mathcal{H o m}(L, M) \rightarrow R \mathcal{H} \operatorname{com}\left(\left(f \bar{x}_{s} \operatorname{id}_{Y}\right)^{*} L,\left(f \bar{x}_{s} \operatorname{id}_{Y}\right)^{*} M\right)
$$

is an isomorphism.
Proof. In case (a), we may assume that $X_{0}$ and $X$ are affine and that $f$ is ind-étale, by [BS, Theorem 1.3 (3)]. By Lemma 1.27, we may assume that $L=\left(u \bar{x}_{s} v\right)_{!} \Lambda$. In this case, the map can be identified with the base change map $\left(f \bar{x}_{s} \operatorname{id}_{Y}\right)^{*} R\left(u \bar{x}_{s} v\right)_{*} M^{\prime} \rightarrow R\left(u_{X_{0}} \bar{x}_{s} v\right)_{*}\left(f_{U} \bar{x}_{s} \operatorname{id}_{Y}\right)^{*} M^{\prime}$, which is an isomorphism by limit arguments for $f$ ind-étale and by [SGA4, XVI Théorème 1.1] in case (b). Here $M^{\prime}=\left(u \bar{x}_{s} v\right)^{*} M$.

The following lemma will be used later in Subsection 4.5,
Lemma 1.30. Let $f: X_{0} \rightarrow X$ be a weakly étale morphism of coherent schemes. Let $g: X^{\prime} \rightarrow X$ be a finite morphism of finite presentation. Form a Cartesian square


Then the map $f^{\prime *} R g^{\prime} \rightarrow R g_{0}^{\prime} f^{*}$ adjoint to the inverse of the finite base change isomorphism $f^{*} g_{*} \xrightarrow{\sim}$ $g_{0 *} f^{\prime *}$ via Construction 1.11 is an isomorphism on $D^{+}(X, \Lambda)$.
Proof. For $M \in D(X, \Lambda)$, we have a commutative diagram


For $M \in D^{+}$, since $g_{*} \Lambda$ is constructible, the left vertical arrow is an isomorphism by Lemma 1.29 (case $Y=s$ ). It follows that the same holds for the right vertical arrow. We conclude by the fact that $g_{0 *}$ is conservative.

Lemma 1.31. Let $X$ be a Noetherian scheme with $m \Lambda=0$ for some $m$ invertible on $X$. Let $L \in$ $D_{\text {cft }}^{\leq a}(X, \Lambda)$ and $M \in D^{+}(X, \Lambda)$. Assume that $M$ has tor-amplitude $\geq b$. Then $R \mathcal{H} o m(L, M)$ has tor-amplitude $\geq b-a$.

Proof. The proof is similar to D1, Th. finitude, Remarque 1.7]. The case where $L$ has locally constant cohomology sheaves is trivial. For the general case, we may assume that $L=u_{!} \mathcal{F}$ for an immersion $u: U \rightarrow X$ and $\mathcal{F} \in D_{\text {cft }}$ with locally constant cohomology sheaves. Then $R \mathcal{H} o m(L, M) \simeq$ $R u_{*} R \mathcal{H o m}\left(\mathcal{F}, R u^{!} M\right)$. It remains to show that $R u_{*}$ and $R u^{!}$preserve objects of tor-amplitude $\geq n$. For $R u_{*}$ this follows from Remark 1.21 (with $Y=s$ ). For $R u^{!}$we may assume that $u$ is a closed immersion, and it suffices to apply the assertion for $R j_{*}$, where $j$ is the complementary open immersion.

Proposition 1.32. Assume $X$ and $Y$ are both Noetherian schemes, and $m \Lambda=0$ for some $m$ invertible on $X$ and $Y$. Let $L \in D_{c}^{-}(X, \Lambda), M \in D_{c f t}(Y, \Lambda), L^{\prime} \in D^{+}(X, \Lambda), M^{\prime} \in D^{+}(Y, \Lambda)$. Assume that either of the following holds:
(a) $L \in D_{c \mathrm{ft}}(X, \Lambda)$ and $L^{\prime} \boxtimes^{L} M^{\prime} \in D^{+}$; or
(b) $M^{\prime}$ has tor-amplitude $\geq n$ for some integer $n$.

$$
\begin{aligned}
& R \mathcal{H o m}\left(L, L^{\prime}\right) \boxtimes^{L} R \mathcal{H} o m\left(M, M^{\prime}\right) \rightarrow R \mathcal{H o m}\left(p_{1}^{*} L, p_{1}^{*} L^{\prime}\right) \otimes^{L} R \mathcal{H} o m\left(p_{2}^{*} M, p_{2}^{*} M^{\prime}\right) \\
& \rightarrow R \mathcal{H o m}\left(L \boxtimes^{L} M, L^{\prime} \boxtimes^{L} M^{\prime}\right)
\end{aligned}
$$

is an isomorphism in $D\left(X \overline{\times}_{s} Y, \Lambda\right)$.
Proof. By Lemma 1.29, we may replace $X$ and $Y$ by strict localizations at geometric points, which are finite-dimensional. In case (b), by Lemma [1.31, $R \mathcal{H} o m\left(M, M^{\prime}\right)$ has tor-amplitude bounded from below, so that we may assume $L=f_{!} \Lambda$ for $f: X^{\prime} \rightarrow X$ étale with $X^{\prime}$ affine, which is a special case of (a). Let us prove case (a). We may assume $L=u!\mathcal{F}, M=v!\mathcal{G}$ for immersions $u: U \rightarrow X, v: V \rightarrow Y$ and $\mathcal{F}, \mathcal{G}$ in $D_{\text {cft }}$ of locally constant cohomology sheaves. In this case, $L \boxtimes^{L} M \simeq\left(u \overline{\times}_{s} v\right)!\left(\mathcal{F} \boxtimes^{L} \mathcal{G}\right)$ and the map in question is the composition

$$
\begin{aligned}
& R u_{*} R \mathcal{H o m}\left(\mathcal{F}, R u^{!} L^{\prime}\right) \boxtimes^{L} R v_{*} R \mathcal{H o m}\left(\mathcal{G}, R v^{!} M^{\prime}\right) \\
\simeq & R\left(u \bar{x}_{s} v\right)_{*}\left(R \mathcal{H o m}\left(\mathcal{F}, R u^{!} L^{\prime}\right) \boxtimes^{L} R \mathcal{H o m}\left(\mathcal{G}, R v^{!} M^{\prime}\right)\right) \\
\rightarrow & R\left(u \bar{x}_{s} v\right)_{*} R \mathcal{H o m}\left(\mathcal{F} \boxtimes^{L} \mathcal{G}, R u^{!} L^{\prime} \boxtimes^{L} R v^{!} M^{\prime}\right) \\
\simeq & R\left(u \bar{x}_{s} v\right)_{*} R \mathcal{H o m}\left(\mathcal{F} \boxtimes^{L} \mathcal{G}, R\left(u \bar{x}_{s} v\right)^{!}\left(L^{\prime} \boxtimes^{L} M^{\prime}\right)\right) .
\end{aligned}
$$

The first and last isomorphisms are Corollaries 1.22 (a) and 1.26. Here again we used Gabber's theorem that $R u_{*}$ and $R v_{*}$ have finite cohomological dimension [ILO, XVIII ${ }_{\mathrm{A}}$ Corollary 1.4]. Thus we are reduced to showing the proposition under the additional assumption that the cohomology sheaves of $L$ and $M$ are locally constant. This case is obvious by taking stalks.

Proposition 1.33. Assume $\Lambda$ of torsion. Let $f: X^{\prime} \rightarrow X$ and $g: Y^{\prime} \rightarrow Y$ be separated morphisms of finite presentation. Then $R\left(f \bar{x}_{s} g\right)$ ! preserves $D_{c}^{b}$ and $D_{\mathrm{ft}}$.

Proof. It suffices to show $R\left(f \bar{x}_{s} g\right)!\mathcal{F} \in D_{c}^{b}$ for $\mathcal{F}$ a constructible sheaf. By Lemma 1.27, we may assume $\mathcal{F}=\Lambda$. In this case, we apply Künneth formula $R\left(f \bar{x}_{s} g\right)_{!} \Lambda \simeq R f_{!} \Lambda \boxtimes^{L} R g_{!} \Lambda$ (Corollary 1.14) and the classical theorem that $R f_{!} \Lambda$ and $R g!\Lambda$ are in $D_{c}^{b}$ [SGA4, XVII Théorème 5.3.6]. It follows from projection formula that $R\left(f \overline{\times}_{s} g\right)$ ! preserves complexes of finite tor-dimension.
Lemma 1.34. Assume $\ell \Lambda=0$ for a prime number $\ell$. Let $\mathcal{F} \in \operatorname{Shv}_{c}\left(X \bar{x}_{s} Y, \Lambda\right)$ be a sheaf that becomes constant on $U \bar{x}_{s} V$ for finite étale covers $U \rightarrow X$ and $V \rightarrow Y$. Then there exist finite étale covers $p: X^{\prime} \rightarrow X$ and $q: Y^{\prime} \rightarrow Y$ such that $\mathcal{F}$ is a direct summand of a sheaf $\mathcal{F}^{\prime}$ equipped with a finite filtration with graded pieces of the form $\left(p \bar{x}_{s} q\right)_{*} C_{n}$ with $C_{n}$ constant.

Proof. The proof is similar to that of [SGA4, IX Proposition 5.5]. We may assume that $U \rightarrow X$ and $V \rightarrow Y$ are Galois of groups $G$ and $H$, respectively. Choose $\ell$-Sylow subgroups $G^{\prime}$ of $G$ and $H^{\prime}$ of $H$. Let $X^{\prime}=U / G^{\prime}, Y^{\prime}=V / H^{\prime}$. Take $\mathcal{F}^{\prime}=\left(p \bar{x}_{s} q\right)_{*}\left(p \bar{x}_{s} q\right)^{*} \mathcal{F}$. The composition $\mathcal{F} \rightarrow \mathcal{F}^{\prime} \xrightarrow{\operatorname{Tr}} \mathcal{F}$ of the adjunction and trace maps equals $\left[G: G^{\prime}\right]\left[H: H^{\prime}\right]$ id, which is an isomorphism. Thus $\mathcal{F}$ is a direct summand of $\mathcal{F}^{\prime}$. Moreover, $\left(p \bar{x}_{s} q\right)^{*} \mathcal{F}$ is a $\Lambda[L]$-module $M$, where $L=G^{\prime} \times H^{\prime}$. Since $(g-1)^{\# L}=0$ in $\Lambda[L]$ for each $g \in L$, the augmentation ideal $I$ of $\Lambda[L]$ satisfies $I^{(\# L)^{2}}=0$. Thus the filtration $\left(I^{n} M\right)_{n \geq 0}$ of $M$ is finite and provides the desired filtration of $\mathcal{F}^{\prime}$.

Assume $m \Lambda=0$ with $m$ invertible on $X$ and on $Y$.
Proposition 1.35. Assume that $X$ is (a) quasi-excellent or (b) of finite type over a Noetherian regular scheme of dimension 1. Assume that $Y$ is (a) or (b). Let $f: X^{\prime} \rightarrow X$ and $g: Y^{\prime} \rightarrow Y$ be morphisms of finite type. Then $R\left(f \bar{x}_{s} g\right)_{*}$ and $R\left(f \bar{x}_{s} g\right)^{!}$(the latter for $f$ and $g$ separated) preserve $D_{c}^{b}$ and $D_{c \mathrm{ct}}$. Moreover, RHom $X_{X} \bar{x}_{s}$ induces $\left(D_{c}^{-}\right)^{\mathrm{op}} \times D_{c}^{+} \rightarrow D_{c}^{+}$.

Proof. Let us first show that $R\left(f \bar{x}_{s} g\right)_{*}$ preserves $D_{c}^{b}$. The preservation of $D_{c \mathrm{ft}}$ follows from this and Remark 1.21. We may assume $\ell \Lambda=0$ for some prime $\ell \mid m$. By Noetherian induction on $X^{\prime}$ and $Y^{\prime}$, it suffices to show that for $\mathcal{F} \in \operatorname{Shv}_{c}\left(X^{\prime} \bar{x}_{s} Y^{\prime}, \Lambda\right)$, there exist open immersions $u: U^{\prime} \rightarrow X^{\prime}$, $v: V^{\prime} \rightarrow Y^{\prime}$ with $U^{\prime}$ and $V^{\prime}$ nonempty such that $R\left(u \bar{x}_{s} v\right)_{*} \mathcal{F}^{\prime} \in D_{c}^{+}$and $R\left(f u \bar{x}_{s} g v\right)_{*} \mathcal{F}^{\prime} \in D_{c}^{+}$, where $\mathcal{F}^{\prime}=\left.\mathcal{F}\right|_{U^{\prime} \bar{x}_{s} V^{\prime}}$. For this it suffices to show that for some open subgroup $H<G$ and $t=B H$, we have $R\left(u_{t} \bar{x}_{t} v_{t}\right)_{*} \mathcal{F}_{t}^{\prime} \in D_{c}^{+}$and $R\left(f_{t} u_{t} \bar{x}_{t} g_{t} v_{t}\right)_{*} \mathcal{F}_{t}^{\prime} \in D_{c}^{+}$, where $\mathcal{F}_{t}^{\prime}=\left.\mathcal{F}^{\prime}\right|_{U_{t}^{\prime} \bar{x}_{t} V_{t}^{\prime}}$. There exist $U^{\prime}, V^{\prime}$, and $t$ such that $\mathcal{F}_{t}^{\prime}$ is locally constant and trivialized by finite étale covers on each pair of connected components of $U_{t}^{\prime}$ and $V_{t}^{\prime}$. Changing notation (replacing $\mathcal{F}$ by $\mathcal{F}_{t}^{\prime}$ ), it suffices to show that for $\mathcal{F}$ locally constant and trivialized by finite étale covers, we have $R\left(f \bar{x}_{s} g\right)_{*} \mathcal{F} \in D_{c}^{b}$. By Lemma 1.34, we may assume that $\mathcal{F}=\left(p \overline{\times}_{s} q\right)_{*} C$ for finite étale covers $p: X^{\prime \prime} \rightarrow X^{\prime}$ and $q: Y^{\prime \prime} \rightarrow Y^{\prime}$ and $C$ constant. Changing notation again, we may assume $\mathcal{F}$ constant. By projection formula, we may assume $\Lambda=\mathbb{Z} / \ell \mathbb{Z}$ and $\mathcal{F}=\Lambda$. This case follows from the Künneth formula (Corollary 1.22 (b)) and finiteness theorems for $R f_{*} \Lambda$ and $R g_{*} \Lambda$ of Deligne [D1, Th. finitude, Théorème 1.1] and Gabber [ILO, Introduction, Théorème 1].

For $R\left(f \bar{x}_{s} g\right)^{\text {! }}$, we reduce to the case of smooth morphisms, which follows from Proposition 1.23 , and the case of closed immersions, which follows from the first assertion applied to the complementary open immersions.

To show $R \mathcal{H o m}(\mathcal{F}, L) \in D_{c}^{+}$for $\mathcal{F}$ constructible and $L \in D_{c}^{+}$, we may assume $\mathcal{F}=\left(u \bar{x}_{s} v\right)!\Lambda$ by Lemma 1.27. Then $\operatorname{RHom}(\mathcal{F}, L) \simeq R\left(u \overline{\times}_{s} v\right)_{*}\left(u \overline{\times}_{s} v\right)^{*} L$.

Assume that $X$ is (a) excellent admitting a dimension function or (b) of finite type over a regular Noetherian scheme of dimension one. Then $X$ is equipped with a dualizing complex $K_{X}$ for $D_{c f t}(X, \Lambda)$, unique up to tensor product with invertible objects. For $f: X^{\prime} \rightarrow X$ separated of finite type, $R f^{!} K_{X}$ is a dualizing complex for $D_{c f t}\left(X^{\prime}, \Lambda\right)$. If $\Lambda$ is Gorenstein, then $K_{X}$ is also a dualizing complex for $D_{c}^{b}(X, \Lambda)$ ([D1, Th. finitude, 4.7], [ILO, XVII Théorèmes 6.1.1, 7.1.2, 7.1.3]). Let $\Lambda=\prod_{i} \Lambda_{i}$ with each $\operatorname{Spec}\left(\Lambda_{i}\right)$ connected. For each $i, K_{X}$ determines a dimension function $\delta_{X, i}$ : for each $x \in X$, $R \Gamma_{x}(K) \otimes_{\Lambda} \Lambda_{i}$ is concentrated in degree $-2 \delta_{X, i}(x)$.

Assume $Y$ is (a) or (b). Then $X$ and $Y$ are equipped with dualizing complexes $K_{X}$ and $K_{Y}$. Let $K_{X \bar{x}_{s} Y}=K_{X} \boxtimes^{L} K_{Y}$ and $D_{X \bar{x}_{s} Y}=\operatorname{RHom}\left(-, K_{X \bar{x}_{s} Y}\right)$. Proposition 1.32 (b) takes the following form.

Corollary 1.36. For $L \in D_{c}^{-}(X, \Lambda)$ and $M \in D_{c \mathrm{ft}}(Y, \Lambda)$, the canonical map

$$
D_{X} L \boxtimes^{L} D_{Y} M \rightarrow D_{X \bar{x}_{s} Y}\left(L \boxtimes^{L} M\right)
$$

is an isomorphism.

## Proposition 1.37.

(1) $D_{X \bar{x}_{s} Y}$ preserves $D_{c \mathrm{ft}}\left(X \bar{x}_{s} Y, \Lambda\right)$. The evaluation map $\operatorname{ev}_{L}: L \rightarrow D_{X \bar{x}_{s} Y} D_{X \bar{x}_{s} Y} L$ is an isomorphism for $L \in D_{c f t}\left(X \overline{\times}_{s} Y, \Lambda\right)$.
(2) Assume $\Lambda$ Gorenstein. $D_{X \bar{x}_{s} Y}$ preserves $D_{c}^{b}\left(X \bar{x}_{s} Y, \Lambda\right)$. The evaluation map $\operatorname{ev}_{L}: L \rightarrow D_{X \bar{x}_{s} Y} D_{X \bar{x}_{s} Y} L$ is an isomorphism for $L \in D_{c}^{b}\left(X \overline{\times}_{s} Y, \Lambda\right)$. The quasi-injective dimension of $K_{X \bar{x}_{s} Y}$ (namely, the cohomological dimension of $D_{X \bar{x}_{s} Y}$ when restricted to $\left.D_{c}^{b}\left(X \overline{\times}_{s} Y, \Lambda\right)\right)$ is

$$
d=\sup _{i}\left(\operatorname{dim} \Lambda_{i}-2 \inf \delta_{X, i}-2 \inf \delta_{Y, i}\right)
$$

In particular, in (2) $K_{X \bar{x}_{s} Y}$ has finite quasi-injective dimension if and only if $\Lambda, X$ and $Y$ are finite-dimensional.

Proof. Let us first show the preservation of $D_{c \mathrm{ft}}$ and $D_{c}^{b}$ (the latter assuming $\Lambda$ Gorenstein) and biduality. This is trivial if $X$ and $Y$ are regular and we restrict to complexes with locally constant cohomology sheaves. We reduce the general case to the case of $L=\left(u \bar{x}_{s} v\right)!\mathcal{F}$, where $u: U \rightarrow X$ and $v: V \rightarrow Y$ are immersions with $U$ and $V$ regular and $\mathcal{F}$ has locally constant cohomology sheaves.

Then $D_{X \bar{x}_{s} Y} L \simeq R\left(u \bar{x}_{s} v\right)_{*} D_{U \bar{x}_{s} V} \mathcal{F}$. Note that $R\left(u \overline{\times}_{s} v\right)_{*}$ preserves $D_{c}^{b}$ and $D_{c \mathrm{ft}}$ by Proposition 1.35. The map $\operatorname{ev}_{L}$ is the composition

$$
\begin{aligned}
&\left(u \bar{x}_{s} v\right)_{!} \mathcal{F} \xrightarrow[\sim]{\operatorname{ev} \mathcal{F}_{\mathcal{F}}}\left(u \overline{\times}_{s} v\right)!D_{U \bar{x}_{s} V} D_{U \bar{x}_{s} V} \mathcal{F} \\
& \xrightarrow[\sim]{\sim} D_{X \bar{x}_{s} Y} R\left(u \overline{\times}_{s} v\right)_{*} D_{U \bar{x}_{s} V} \mathcal{F} \simeq D_{X \bar{x}_{s} Y} D_{X \bar{x}_{s} Y}\left(u \overline{\times}_{s} v\right)!\mathcal{F}
\end{aligned}
$$

where we used Proposition 1.38 below.
It remains to show that for $\Lambda$ Gorenstein, the quasi-injective dimension $c$ of $K_{X} \bar{x}_{s} Y$ is $d$. This is similar to the proof of [ILO, XVII Proposition 6.2.4.1]. For each maximal ideal $\mathfrak{m}$ of $\Lambda_{i}$, taking $L$ to be the constant sheaf $\Lambda / \mathfrak{m}$ on $\overline{\{x\}} \overline{\times}_{s} \overline{\{y\}}$, extended to $X \overline{\times}_{s} Y$, we get $c \geq \operatorname{dim} \Lambda_{\mathfrak{m}}-2 \delta_{X, i}(x)-2 \delta_{Y, i}(y)$. It follows that $c \geq d$. To show $c \leq d$, we reduce to the case $L=\left(u \bar{x}_{s} v\right)!\mathcal{F}$ as above with $\mathcal{F}$ a sheaf. We may further assume $U$ and $V$ are connected, with generic points $x$ and $y$ respectively. Then $D_{U \bar{x}_{s} V} \mathcal{F}$ has cohomological degrees $\leq \sup _{i}\left(\operatorname{dim} \Lambda_{i}-\delta_{X, i}(x)-\delta_{Y, i}(y)\right)$. Moreover, by [ILO, XVIII ${ }_{\text {A }}$ Theorem 1.1] and Remark [1.18, $R\left(u \overline{\times}_{s} v\right)_{*}$ has cohomological dimension $\leq 2 \operatorname{dim} \bar{U}+2 \operatorname{dim} \bar{V}$. It follows that $c \leq d$.

Let $f: X^{\prime} \rightarrow X$ and $g: Y^{\prime} \rightarrow Y$ be separated morphisms of finite type. We have a natural transformation

$$
\begin{align*}
R\left(f \bar{x}_{s} g\right)_{!} D_{X^{\prime}} \bar{x}_{s} Y^{\prime} \xrightarrow{\mathrm{ev}} & D_{X} \bar{x}_{s} Y  \tag{1.9}\\
& D_{X}^{\mathrm{op}} \bar{x}_{s} Y
\end{align*} R\left(f \bar{x}_{s} g\right)!D_{X^{\prime} \bar{x}_{s} Y^{\prime}} .
$$

where ev denotes the evaluation maps.
Proposition 1.38. Let $L \in D_{c}^{-}\left(X^{\prime} \bar{x}_{s} Y^{\prime}, \Lambda\right)$. Assume either $L \in D_{c}^{b}$ or $X$ and $Y$ are finitedimensional. Then $R\left(f \overline{\times}_{s} g\right)_{!} D_{X^{\prime}} \bar{x}_{s} Y^{\prime} L \rightarrow D_{X} \bar{x}_{s} Y\left(f \overline{\times}_{s} g\right)_{*} L$ is an isomorphism.
Proof. For $L \in D_{c}^{b}$, by Lemma [1.29, we may assume that $X$ and $Y$ are strictly local. Thus we may assume $X$ and $Y$ finite-dimensional. Since $R\left(f \bar{x}_{s} g\right)_{*}$ has finite cohomological dimension, we may assume $L \in \operatorname{Shv}_{c}$. By Lemma 1.27 , we may further assume $L=\left(u \overline{\times}_{s} v\right)!\Lambda \simeq M \boxtimes^{L} N$ for $u$ and $v$ affine and étale, where $M=u_{!} \Lambda$ and $N \in v_{!} \Lambda$ in $D_{c \mathrm{ft}}$. Via Künneth formula for $R\left(f \bar{x}_{s} g\right)_{!}, R\left(f \overline{\times}_{s} g\right)_{*}$, and $D$ (Corollaries $1.14, ~\left(1.22, ~(1.36),(1.9) ~ c a n ~ b e ~ i d e n t i f i e d ~ w i t h ~ t h e ~ \boxtimes^{L}\right.$ of the isomorphisms

$$
R f_{!} D_{X^{\prime}} M \xrightarrow{\sim} D_{X} R f_{*} M, \quad R g_{!} D_{Y^{\prime}} N \xrightarrow{\sim} D_{Y} R g_{*} N
$$

Remark 1.39. The natural transformation (1.9) can be interpreted formally as follows. We write $\alpha: F \dashv G$ for an adjunction $\alpha_{X, Y}: \operatorname{Hom}(F X, Y) \simeq \operatorname{Hom}(X, G Y)$. Then $\alpha^{\mathrm{op}}: G^{\mathrm{op}} \dashv F^{\mathrm{op}}$, where $\alpha_{Y, X}^{\mathrm{op}}=\alpha_{X, Y}^{-1}$. By a duality on a category $\mathcal{C}$ we mean a functor $D: \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{C}$ equipped with an adjunction $\alpha: D^{\mathrm{op}} \dashv D$ such that $\alpha^{\mathrm{op}}=\alpha$. Here $D^{\mathrm{op}}:\left(\mathcal{C}^{\mathrm{op}}\right)^{\mathrm{op}} \rightarrow \mathcal{C}^{\mathrm{op}}$ and we have identified $\left(\mathcal{C}^{\mathrm{op}}\right)^{\mathrm{op}}$ with $\mathcal{C}$. Similarly $\alpha^{\mathrm{op}}: D^{\mathrm{op}} \dashv\left(D^{\mathrm{op}}\right)^{\mathrm{op}}$ and we have identified $\left(D^{\mathrm{op}}\right)^{\mathrm{op}}$ with $D$. Equivalently, a duality on $\mathcal{C}$ is a functor $D: \mathcal{C}^{\mathrm{op}} \rightarrow \mathcal{C}$ equipped with a natural transformation ev: $\mathrm{id}_{\mathcal{C}} \rightarrow D D^{\mathrm{op}}$ such that the composite $D \xrightarrow{\text { ev } D} D D^{\mathrm{op}} D \xrightarrow{D \mathrm{ev}^{\mathrm{op}}} D$ is the identity. Note that we do not require ev to be a natural isomorphism. If $(\mathcal{C}, \otimes)$ is a closed symmetric category and $K$ is an object of $\mathcal{C}$, then $D_{K}=\mathcal{H o m}(-, K)$, where $\mathcal{H o m}$ is the internal Hom functor, is a duality on $\mathcal{C}$ [SZ, Construction A.4.1]. The adjunction $D_{K}^{\mathrm{op}} \dashv D_{K}$ is given by

$$
\operatorname{Hom}\left(L, D_{K} M\right) \simeq \operatorname{Hom}(L \otimes M, K) \simeq \operatorname{Hom}(M \otimes L, K) \simeq \operatorname{Hom}\left(M, D_{K} L\right) \simeq \operatorname{Hom}_{\mathcal{C}^{\mathrm{op}}}\left(D_{K}^{\mathrm{op}} L, M\right)
$$

Here the second isomorphism is induced by the symmetric constraint $L \otimes M \simeq M \otimes L$.
The natural transformation (1.9) is obtained from Construction 1.11applied to the opposite of the canonical natural isomorphism

$$
\begin{equation*}
R\left(u \overline{\times}_{s} v\right)_{*} D_{U \bar{x}_{s} V} \simeq D_{X \bar{x}_{s} Y} R\left(u \overline{\times}_{s} v\right)_{!}^{\mathrm{op}} . \tag{1.10}
\end{equation*}
$$

In the terminology of [SZ, Definition A.3.3], (1.9) and (1.10) are form transformations, transposes of each other.

Example 1.40. Let $S$ be a Henselian local scheme and let $i: s \rightarrow S$ be the inclusion of the closed point. The functor $i^{-1}: \operatorname{Et}(S) \rightarrow \mathrm{Et}(s)$ between étale sites admits a left adjoint $\pi^{-1}: \operatorname{Et}(s) \rightarrow \operatorname{Et}(S)$, which commutes with finite limits and extends the equivalence between finite étale sites. The functor $\pi^{-1}$ is continuous and induces a morphism of topoi $\pi: S_{\text {et }} \rightarrow s_{\text {et }}$, left adjoint to $i_{\text {et }}$ (which means $\pi_{*} \simeq i^{*}$ here). For any scheme $Y$ over $s$, we have $Y \overline{\times}_{s} S \simeq Y \overleftarrow{\times}_{s} S \simeq Y \overleftarrow{×}_{S} S$ by Remark 1.15 and Lemma 1.41 below.

In the special case where $S$ is the spectrum of a Henselian discrete valuation ring, we recover the topos $Y \bar{x}_{s} S$ of [SGA7II, XIII Section 1.2]. A sheaf $\mathcal{F}$ on $Y \bar{x}_{s} S$ can be identified with a triple $\left(\mathcal{F}_{s}, \mathcal{F}_{\eta}, \phi\right)$, where $\mathcal{F}_{s}$ is a sheaf on $Y, Y_{\eta}$ is a sheaf on $Y \overline{\times}_{s} \eta \simeq Y \overleftarrow{\times}_{s} \eta \simeq Y \overleftarrow{\times}_{S} \eta$, and $\phi: p^{*} \mathcal{F}_{s} \rightarrow \mathcal{F}_{\eta}$ is a morphism of sheaves. Here $p: Y \bar{x}_{s} \eta \rightarrow Y$ denotes the projection. The functor $R\left(f \bar{x}_{s} \mathrm{id}_{S}\right)_{\text {! in }}$ this case reduces to [SGA7II, XIII 2.1.6 c)].

Lemma 1.41. Consider morphisms $X \xrightarrow{a} s \underset{\pi}{\frac{i}{T}} S \stackrel{b}{\longleftrightarrow} Y$ in a 2-category with $\pi$ left adjoint to i. Then $X \overleftarrow{\times}_{a, s, \pi b} Y \simeq X \overleftarrow{\times}_{i a, S, b} Y$.

Proof. The two oriented products satisfy the same universal property. Indeed, for $X \stackrel{x}{\leftarrow} T \xrightarrow{y} Y$, giving $a x \Leftarrow \pi b y$ is equivalent to giving $i a x \Leftarrow \pi b$.

## 2 The functor $L C$ o

After preliminaries on the Iwasawa twist (Sections 2.1 and 2.2), we study adjunctions and duality for the functor $L C$. The main results of this section are Theorems 2.12 and 2.20 ,

### 2.1 Iwasawa twist

Let $\Lambda$ be a commutative ring with $m \Lambda=0$ for some integer $m>0$. Let $G$ be a profinite group. Recall that the completed group ring

$$
R=\Lambda[[G]]=\lim _{H} \Lambda[G / H]
$$

where $H$ runs through open normal subgroups of $G$, can be identified with the ring of $\Lambda$-valued measures on $G$ with convolution product. For any discrete $\Lambda$ - $G$-module $M$ and any normal subgroup $P \triangleleft G$, we write $M^{P}$ and $M_{P}$ for the $\Lambda$ - $G$-modules of $G$-invariants and $G$-coinvariants, respectively. Note that if the (supernatural) order of $P$ is prime to $m$, then the canonical map $M^{P} \rightarrow M_{P}$ is an isomorphism, with inverse carrying the class of $x \in M$ to $\int p x d \mu(p)$, where $\mu$ is the Haar measure on $P$ of mass 1. In this case we write ${ }^{P} M=\operatorname{Coker}\left(M^{P} \rightarrow M\right)$ and we have a canonical isomorphism

$$
M \simeq M^{P} \oplus^{P} M
$$

Let $A$ be a set of prime numbers containing all prime divisors of $m$ and let $\mathbb{Z}_{A}=\prod_{\ell \in A} \mathbb{Z}_{\ell}$. Let $G$ be a profinite group fitting into a short exact sequence

$$
\begin{equation*}
1 \rightarrow Z_{A}(1) \rightarrow G \xrightarrow{\pi} G_{s} \rightarrow 1 \tag{2.1}
\end{equation*}
$$

where $Z_{A}(1)$ is isomorphic to $\mathbb{Z}_{A}$. The conjugation action of $G_{s}$ on $Z_{A}(1)$ provides a character $\chi: G_{s} \rightarrow \mathbb{Z}_{A}^{\times}$. Let $\sigma$ be a generator of $Z_{A}(1)$ and let $t=\sigma-1 \in \Lambda\left[\left[Z_{A}(1)\right]\right] \subseteq \Lambda[[G]]$. Recall that the classical Iwasawa algebra $\Lambda\left[\left[Z_{A}(1)\right]\right]$ is simply $\Lambda[[t]]$.

Lemma 2.1. $t$ is a non-zero-divisor in $R$ and $t R=R t$ does not depend on the choice of $\sigma$.

Proof. Let $r \in R$ such that $\operatorname{tr}=0$. We regard $r$ as a measure on $G$ with values in $\Lambda$. For any open and closed subset $X \subseteq G$, we have $r(\sigma X)=r(X)$. Let $H$ be an open normal subgroup of $G$ such that $H \cap Z_{A}(1)=m n Z_{A}(1)$, and let $H^{\prime}=n Z_{A}(1) \cdot H$. Then $H^{\prime}=\coprod_{i=1}^{m} \sigma^{i n} H$, so that $r\left(H^{\prime} g\right)=m r(H g)=0$ for any $g \in G$. Since any open subgroup of $G$ contains such an $H^{\prime}$, we have $r=0$. Similarly, $r t=0$ implies $r=0$.

That $t R=R t$ follows from the identity

$$
\begin{equation*}
g t g^{-1}=(1+t)^{\chi(g)}-1=\sum_{n \geq 1}\binom{\chi(g)}{n} t^{n} \tag{2.2}
\end{equation*}
$$

in $R$ for $g \in G$. Here we have denoted $\chi(\bar{g})$ by $\chi(g)$, where $\bar{g}$ is the image of $g$ in $G_{s}$. That $t R$ does not depend on the choice of $\sigma$ follows from the standard fact that $t \Lambda\left[\left[Z_{A}(1)\right]\right]$ does not depend on the choice of $\sigma$, which follows in turn from the expansion of $(1+t)^{s}-1$ for $s \in \mathbb{Z}_{A}^{\times}$.
Remark 2.2. Although we do not need this, let us show that the canonical homomorphism $\Lambda[[\pi]]: \Lambda[[G]] \rightarrow$ $\Lambda\left[\left[G_{s}\right]\right]$ induces an isomorphism $R / t R \xrightarrow{\sim} \Lambda\left[\left[G_{s}\right]\right]$. Clearly the image of $t$ in $\Lambda\left[\left[G_{s}\right]\right]$ is zero. Moreover, any continuous section of $\pi$ (not necessarily a homomorphism) induces a section of $\Lambda[[\pi]]$. Now let $s \in R$ be a measure on $G$ such that $\Lambda[[\pi]](s)=0$. We need to find a measure $r \in R$ such that $s=t r$. For $H$ and $H^{\prime}$ as in the proof of Lemma 2.1 and $g \in G$, we take $r\left(H^{\prime} g\right)=(\tau(n) s)(H g)$, where

$$
\tau(n):=\sum_{i=1}^{m} \sum_{j=0}^{i n-1} \sigma^{j}=-\sum_{j=0}^{m n-1}\left\lfloor\frac{j}{n}\right\rfloor \sigma^{j} \in R .
$$

Note that the definition depends only on $H^{\prime} g$. Indeed, we have $\sigma^{n} \tau(n)-\tau(n)=\sum_{j=0}^{m n-1} \sigma^{j}$, so that $\left(\sigma^{n} \tau(n) s-\tau(n) s\right)(H g)=s\left(Z_{A}(1) H g\right)=0$ by the assumption on $s$. Let $H_{0}<H$ be an open normal subgroup of $G$ with $Z_{A}(1) \cap H_{0}=a m n Z_{A}(1)$ and let $H_{0}^{\prime}=a n Z_{A}(1) \cdot H_{0}$. We have $H=$ $\amalg_{h \in S} \amalg_{k=0}^{a-1} \sigma^{-k m n} H_{0} h$ for a subset $S \subseteq H$ and $H^{\prime}=\amalg_{h \in S} \amalg_{k=0}^{a-1} H_{0}^{\prime} \sigma^{-k n} h$. Applying the identity $b=\sum_{k=0}^{a-1}\left\lfloor\frac{b+k}{a}\right\rfloor$ to $b=\lfloor j / n\rfloor$, we get

$$
\sum_{k=0}^{a-1} \sigma^{k n} \tau(a n)=-\sum_{j=0}^{a m n-1}\left\lfloor\frac{j}{n}\right\rfloor \sigma^{j}=\sum_{k=0}^{a-1} \sigma^{k m n} \tau(n),
$$

so that $r\left(H^{\prime} g\right)=\sum_{h \in S} \sum_{k=0}^{a-1} r\left(H_{0}^{\prime} \sigma^{-k n} h g\right)$. It follows that $r$ is a well-defined measure on $G$. Finally, $t \tau(n)=\sum_{i=1}^{m} \sigma^{i n}$ so that $(\operatorname{tr})\left(H^{\prime} g\right)=s\left(H^{\prime} g\right)$.

The following definition is due to Beilinson, at least for $G=\mathbb{Z}_{\ell}(1)$.
Definition 2.3 (Iwasawa twist). Consider the $R$-bimodule $R(1)^{\tau}=t R$. For an $R$-module $M$, we consider the $R$-modules

$$
M(1)^{\tau}=R(1)^{\tau} \otimes_{R} M, \quad M(-1)^{\tau}=\operatorname{Hom}_{R}\left(R(1)^{\tau}, M\right)
$$

By Lemma [2.1, we have isomorphisms of $R$-modules $M \xrightarrow{\sim} M(1)^{\tau}(-1)^{\tau}$ and $M(-1)^{\tau}(1)^{\tau} \xrightarrow{\sim}$ $M$. The $R$-bimodule $R(1)^{\tau}$ is invertible in the sense that we have an isomorphism of $R$-bimodules $R(1)^{\tau} \otimes_{R} R(-1)^{\tau} \simeq R$.

The $R$-module $M(1)^{\tau}$ can be described more explicitly as follows. For each $\sigma$, we define a ring isomorphism $\rho_{\sigma}: R \rightarrow R$ by

$$
\rho_{\sigma}(g)=\sum_{n \geq 1}\binom{\chi(g)}{n} t^{n-1} g
$$

for $g \in G$. That $\rho_{\sigma}$ defines a ring homomorphism follows from the identity $g t=t \rho_{\sigma}(g)$ (2.2) and the fact that $t$ is a non-zero-divisor in $R$. We have

$$
\rho_{\sigma}^{-1}(g)=g \sum_{n \geq 1}\binom{\chi\left(g^{-1}\right)}{n} t^{n-1} .
$$

Consider the isomorphism of $\Lambda$-modules $u: M \rightarrow M(1)^{\tau}$ carrying $x$ to $t \otimes x$. For $g \in G$, we have $g(t \otimes x)=t \otimes \rho_{\sigma}(g) x$, namely $u\left(\rho_{\sigma}(g) x\right)=g u(x)$. Thus if $M=\left(M, \alpha: R \rightarrow \operatorname{End}_{\Lambda}(M)\right)$, then $u:\left(M, \alpha \circ \rho_{\sigma}\right) \xrightarrow{\sim} M(1)^{\tau}$ is an isomorphism of $R$-modules.
Remark 2.4. The constructions preserve discrete $\Lambda$ - $G$-modules. For $M$ a discrete $\Lambda$ - $G$-module, $M(-1)^{\tau}$ can be described as follows. Composition with the crossed homomorphism $Z_{A}(1) \rightarrow R(1)^{\tau}$ carrying $\xi$ to $\xi-1$ gives an isomorphism of $\Lambda$ - $G$-modules $M(-1)^{\tau} \xrightarrow{\sim} Z_{\text {cont }}^{1}\left(Z_{A}(1), M\right)$, where $Z_{\text {cont }}^{1}\left(Z_{A}(1), M\right)$ is the $\Lambda$-module of continuous crossed homomorphisms $f: Z_{A}(1) \rightarrow M$, with $G$-action given by $(g f)(\xi)=$ $g\left(f\left(g^{-1} \xi g\right)\right)$ for $g \in G$.
Remark 2.5. If $t^{n} M=0$ for some integer $n$ such that all primes $\ell \leq n$ are invertible in $\Lambda$, then we have an isomorphism of $R$-modules $M(1):=Z_{A}(1) \otimes_{\mathbb{Z}_{A}} M \xrightarrow{\sim} M(1)^{\tau}$ carrying $\sigma \otimes x$ to $t \otimes$ $\sum_{i=1}^{n}(-1)^{i-1} t^{i-1} x / i$.
Remark 2.6. Let $A^{\prime}$ be a subset of $A$ containing all prime factors of $m$ and let $Z_{A^{\prime}}(1)$ be the maximal pro- $A^{\prime}$ quotient of $Z_{A}(1)$. Let $K=\operatorname{Ker}\left(Z_{A}(1) \rightarrow Z_{A^{\prime}}(1)\right)$. For a discrete $G$-module $M$, we have $M(1)^{\tau} \simeq M^{K}(1)^{\tau} \oplus{ }^{K} M$. Indeed, for $M$ such that $M^{K}=0$, since $M^{Z_{A}(1)}=M_{Z_{A}(1)}=0, t$ acts bijectively on $M$.
Remark 2.7. In our applications (2.1) splits so that $G \simeq Z_{A}(1) \rtimes G_{s}$. In this case an $R$-module gives rise to a $G_{s}-R_{0}$-module, where $R_{0}=\Lambda\left[\left[Z_{A}(1)\right]\right]$ is a $G_{s}$-ring. One can define Iwasawa twist for $G_{s}-R_{0}$-module by $t R_{0} \otimes_{R_{0}}$-, which is compatible with Definition 2.3.

### 2.2 Iwasawa twist on fiber products

Let $S$ be the spectrum of a Henselian discrete valuation ring of residue characteristic $p \nmid m$. Let $s$ and $\eta$ be the closed point and the generic point of $S$, respectively. Let $\bar{\eta} \rightarrow \eta$ be an algebraic geometric point and let $\bar{s}$ be the closed point of the normalization of $S$ in $\bar{\eta}$. Let $G_{\eta}=\operatorname{Gal}(\bar{\eta} / \eta)$. Assume that $p \notin A$. Let $I<G$ be the inertia group and $P<I$ be the inertia group prime to $A$. Then $G=G_{\eta} / P$ fits into a split short exact sequence (2.1), with $Z_{A}(1)$ being the Tate twist $\mathbb{Z}_{A}(1)$ of $\mathbb{Z}_{A}, G_{s}=\operatorname{Gal}(\bar{s} / s)$, and $\chi$ the cyclotomic character. For a sheaf $M$ of $\Lambda$-modules on $\eta$, we define $M( \pm 1)^{\tau}=M^{P}( \pm 1)^{\tau} \oplus^{P} M$. By Remark [2.6, this does not depend on the choice of $A$. In particular, we can take $A$ to be either maximal (the set of all primes except $p$ ) or minimal (the set of prime divisors of $m$ ).

Now let $Y$ be a scheme over $s$. A sheaf on $Y \bar{x}_{s} B G$ is a pair $(M, \alpha)$ where $M$ is a sheaf on $Y_{\bar{s}}$ and $\alpha$ is a continuous action of $G$ on $M$, compatible with the action of $G$ on $Y_{\bar{s}}$ via $G_{s}$. The construction $(M, \alpha) \mapsto\left(M, \alpha \circ \rho_{\sigma}\right)$ extends to the topos $Y \bar{x}_{s} B G$ and does not depend on the choice of $\sigma$ up to isomorphism. Indeed, for $s \in \mathbb{Z}_{A}^{\times}$, we have $\alpha(r):\left(M, \alpha \circ \rho_{\sigma^{s}}\right) \xrightarrow{\sim}\left(M, \alpha \circ \rho_{\sigma}\right)$, where $r=\sum_{n \geq 1}\binom{s}{n} t^{n-1} \in R^{\times}$. We define $M(1)^{\tau}=\left(M, \alpha \circ \rho_{\sigma}\right)$ and $M(-1)^{\tau}=\left(M, \alpha \circ \rho_{\sigma}^{-1}\right)$. The latter has a more canonical description by Remark [2.4; For $U$ a quasi-compact object of the étale site of $Y_{\bar{s}}$, we have

$$
\begin{equation*}
M(-1)^{\tau}(U)=Z_{\mathrm{cont}}^{1}\left(\mathbb{Z}_{A}(1), M(U)\right) . \tag{2.3}
\end{equation*}
$$

We have a canonical map

$$
M \rightarrow M(-1)^{\tau}
$$

carrying $x \in M(U)$ to the crossed homomorphism $\sigma \mapsto \sigma x-x$.
Definition 2.8. For a sheaf $M$ of $\Lambda$-modules on $Y \bar{x}_{s} \eta$, we define

$$
\begin{equation*}
M( \pm 1)^{\tau}=M^{P}( \pm 1)^{\tau} \oplus^{P} M \tag{2.4}
\end{equation*}
$$

We will sometimes write $\tau M$ for $M(1)^{\tau}$. We get an auto-equivalence

$$
\tau: \operatorname{Shv}\left(Y \bar{x}_{s} \eta, \Lambda\right) \rightarrow \operatorname{Shv}\left(Y \bar{x}_{s} \eta, \Lambda\right)
$$

We consider the map

$$
\begin{equation*}
\iota: M \rightarrow M(-1)^{\tau} \tag{2.5}
\end{equation*}
$$

given by the canonical map on $M^{P}$ and the identity on ${ }^{P} M$.

The kernel of (2.5) is $M^{I}$ and the cokernel is $M_{I}(-1)$. Here we used the fact that if the action of $I$ on $M$ is trivial, then $M( \pm 1)^{\tau} \simeq M( \pm 1)$ (Remark (2.5).

Let $p: Y \bar{x}_{s} \eta \rightarrow Y$ be the projection. The functor $p^{*}$ on categories of sheaves is fully faithful. For a sheaf $M$ on $Y \bar{x}_{s} \eta, p_{*} M$ can be identified with $M^{I}$.

Lemma 2.9. For $M \in D\left(Y \bar{x}_{s} \eta, \Lambda\right), p^{*} R p_{*} M$ is computed by $\operatorname{Cone}\left(M \xrightarrow{\iota} M(-1)^{\tau}\right)[-1]$, with the adjunction $p^{*} R p_{*} M \rightarrow M$ given by the totalization of the map of bicomplexes


Proof. Since the functor $M \mapsto M(-1)^{\tau}$ is exact, it suffices to show that for $M$ an injective sheaf, the sequence $0 \rightarrow M^{I} \rightarrow M \xrightarrow{\iota} M(-1)^{\tau} \rightarrow 0$ is exact. We have already seen that $\operatorname{Ker}(\iota)=M^{I}$. It remains to show that $M^{P} \rightarrow M^{P}(-1)^{\tau}$ is an epimorphism. This can be identified with $C_{\text {cont }}^{0}\left(K, M^{P}\right) \rightarrow$ $Z_{\text {cont }}^{1}\left(K, M^{P}\right)$, where $K=I / P \simeq \mathbb{Z}_{A}(1)$, which is an epimorphism by the following lemma applied to the injective sheaf $M^{P}$ on $Y \bar{x}_{s} B G$.

Lemma 2.10. Let $H \rightarrow G_{s}$ be an epimorphism of profinite groups of kernel $K$ and let $q: Y \bar{\chi}_{s} B H \rightarrow Y$ be the projection. Then for $M \in \operatorname{Shv}\left(Y \bar{×}_{s} B H, \Lambda\right), q^{*} R q_{*} M$ is computed by the complex $C_{\text {cont }}^{*}(K, M)$. Here $C_{\text {cont }}^{i}(K, M)$ is the sheaf on $Y \bar{x}_{s} B H$ such that for any quasi-compact object $U$ of the étale site of $Y_{\bar{s}}, C_{\text {cont }}^{i}(K, M)(U)=C_{\text {cont }}^{i}(K, M(U))$ is the $\Lambda$-module of continuous $i$-cochains, with $H$-action given by $(g f)\left(\xi_{0}, \ldots, \xi_{i}\right)=g\left(f\left(g^{-1} \xi_{0} g, \ldots, g^{-1} \xi_{i} g\right)\right)$ for $g \in H$.

Proof. We have $M=\operatorname{colim}_{J} M^{J}$, where $J$ runs through open subgroups of $K$, normal in $H$, and $C_{\text {cont }}^{i}(K, M)=\operatorname{colim}_{J} C_{\mathrm{cont}}^{i}\left(K / J, M^{J}\right)$. Since the functors $C_{\text {cont }}^{i}(K,-)$ are exact, it suffices to show that $H^{i} C_{\text {cont }}^{*}(K, M)=0$ for $i>0$ and $M$ injective. In this case, $M^{J}$ is injective as sheaf on $Y \bar{x}_{s}$ $B(H / J)$. Thus we may assume $K$ finite. In this case $C_{\text {cont }}^{*}(K, M)=\mathcal{H o m}(L, M)$, where $L$ is the standard resolution of $\mathbb{Z}$ by $\mathbb{Z}[K]$-modules, regarded as sheaves on $X \bar{x}_{s} B K$.

Remark 2.11. By Lemma 2.9, $R p_{*}$ has cohomological dimension $\leq 1$, and $R^{1} p_{*} M \simeq M_{I}(-1)$ for $M \in \operatorname{Shv}\left(Y \bar{x}_{s} \eta, \Lambda\right)$.

### 2.3 Some quasi-periodic adjunctions

Let $j: Y \bar{x}_{s} \eta \rightarrow Y \bar{x}_{s} S$ be the inclusion of the open subtopos. Between categories of abelian sheaves, we have a sequence of adjoint functors

$$
\begin{equation*}
\mathrm{Co} \dashv j!\dashv j^{*} \dashv j_{*}, \tag{2.7}
\end{equation*}
$$

where $\operatorname{Co\mathcal {F}}=\operatorname{Coker}\left(\phi: p^{*} \mathcal{F}_{s} \rightarrow \mathcal{F}_{\eta}\right)$ for $\mathcal{F}=\left(\mathcal{F}_{s}, \mathcal{F}_{\eta}, \phi\right)$ as in Example 1.40. This sequence cannot be extended, as neither Co nor $j_{*}$ is exact (unless $Y$ is empty). We will see however that the derived adjunction sequence can be extended into a loop up to twist.

Note that Co admits a left derived functor

$$
L \mathrm{Co}: D\left(Y \bar{x}_{s} S, \Lambda\right) \rightarrow D\left(Y \bar{x}_{s} \eta, \Lambda\right) .
$$

For each $M \in D\left(Y \bar{x}_{s} S, \Lambda\right), L \mathrm{Co} M$ is computed by $\mathrm{Co}^{\prime}$, where $M^{\prime} \rightarrow M$ is a quasi-isomorphism and $M^{\prime}$ is a complex of sheaves of the form $\left(\mathcal{F}_{s}, \mathcal{F}_{\eta}, \phi\right)$ for which $\phi$ is a monomorphism. In fact, $L C$ o is the functor $\Phi$ of [SGA7II, XIII 1.4.2]. The adjunctions (2.7) induce adjunctions of derived functors $L \mathrm{Co} \dashv j!\dashv j^{*} \dashv R j_{*}$ (see for example [KS, Theorem 14.4.5]).
Theorem 2.12. Between the derived categories $D\left(Y \bar{x}_{s} \eta, \Lambda\right)$ and $D\left(Y \bar{x}_{s} S, \Lambda\right)$, we have a canonical adjunction $R j_{*} \dashv \tau L$ Co.

Therefore, we have a quasi-periodic sequence of adjoint functors

$$
\begin{equation*}
L \mathrm{Co} \dashv j_{!} \dashv j^{*} \dashv R j_{*} \dashv \tau L \mathrm{Co} . \tag{2.8}
\end{equation*}
$$

We start by constructing an adjunction for the projection $p: Y \bar{x}_{s} \eta \rightarrow Y$.
Proposition 2.13. Between derived categories $D\left(Y \bar{x}_{s} \eta, \Lambda\right)$ and $D(Y, \Lambda)$, we have a canonical adjunction $R p_{*} \dashv p^{*}(1)[1]$.

Thus we have a quasi-periodic sequence of adjoint functors

$$
\begin{equation*}
p^{*} \dashv R p_{*} \dashv p^{*}(1)[1] . \tag{2.9}
\end{equation*}
$$

Proof. The isomorphism $R^{1} p_{*} \Lambda \simeq \Lambda(-1)$ in Remark 2.11 induces $\operatorname{Tr}_{\Lambda}: R p_{*} p^{*} \Lambda(1)[1] \rightarrow \Lambda$. We define the co-unit of the adjunction to be the trace map

$$
\operatorname{Tr}_{L}: R p_{*} p^{*} L(1)[1] \stackrel{\sim}{\leftarrow} p_{*} \Lambda(1)[1] \otimes^{L} L \xrightarrow{L \otimes^{L} \operatorname{Tr}_{\Lambda}} L
$$

for $L \in D(Y, \Lambda)$, where the first arrow is the projection formula map, which is an isomorphism by Corollary 1.20 (a) or by Lemma 2.9. By construction, $p^{*} \operatorname{Tr}_{L}$ is given by the totalization of the morphism of double complexes


Indeed, this is clear in the case $L=\Lambda$ and the general case follows. We define the unit $\epsilon_{M}: M \rightarrow$ $p^{*} R p_{*} M(1)[1]$ for $M \in D\left(Y \bar{x}_{s} \eta, \Lambda\right)$ to be the totalization of the morphism of double complexes


It follows that $p^{*} \operatorname{Tr}_{L} \circ \epsilon_{p^{*} L}=$ id and $\operatorname{Tr}_{R p_{*} M(1)[1]} \circ R p_{*} \epsilon_{M}(1)[1]=$ id. For the latter we reduce to the easy case where $M$ is an injective sheaf.

Remark 2.14. The proposition is a form of the Poincaré-Verdier duality for the inertia group $I$, and can be compared with other Poincaré-Verdier dualities. For $f$ a proper topological submersion of locally compact spaces (resp. proper smooth morphism of schemes) of relative dimension $d$, we have adjunctions

$$
f^{*} \dashv R f_{*} \dashv R f^{!}
$$

where $R f^{!} \simeq\left(f^{*}-\otimes o_{f}\right)[d]$ (resp. $\left.R f^{!} \simeq f^{*}(d)[2 d]\right)$. Here $o_{f}$ is the orientation sheaf.
To construct the adjunction $R j_{*} \dashv \tau L C$, we need (part (1) of) the following.
Lemma 2.15. Let $j: U \rightarrow X$ be an open subtopos and let $i: V \rightarrow X$ be the complementary closed subtopos. Let $\mathcal{C}$ be a category. Let $F: \mathcal{C} \rightarrow D(X, \Lambda)$ and $G: D(X, \Lambda) \rightarrow \mathcal{C}$ be functors.
(1) A natural transformation id $\rightarrow G F$ is an adjunction if and only if the compositions id $\rightarrow G F \rightarrow$ $\left(G i_{*}\right)\left(i^{*} F\right)$ and $\mathrm{id} \rightarrow G F \rightarrow\left(G R j_{*}\right)\left(j^{*} F\right)$ are adjunctions.
(2) A natural transformation $G F \rightarrow$ id is an adjunction if and only if the compositions $\left(G i_{*}\right)\left(R i^{!} F\right) \rightarrow$ $G F \rightarrow \mathrm{id}$ and $\left(G j_{!}\right)\left(j^{*} F\right) \rightarrow G F \rightarrow \mathrm{id}$ are adjunctions.

That $\epsilon:$ id $\rightarrow G F$ is the unit of an adjunction (or, in short, is an adjunction) means that the composite

$$
\begin{equation*}
\operatorname{Hom}(F A, B) \rightarrow \operatorname{Hom}(G F A, G B) \xrightarrow{\epsilon_{A}} \operatorname{Hom}(A, G B) \tag{2.10}
\end{equation*}
$$

is a bijection for all $A \in \mathcal{C}$ and $B \in D(X, \Lambda)$. That $\eta: G F \rightarrow$ id is the co-unit of an adjunction (or, in short, is an adjunction) means that the composite

$$
\begin{equation*}
\operatorname{Hom}(B, F A) \rightarrow \operatorname{Hom}(G B, G F A) \xrightarrow{\eta_{A}} \operatorname{Hom}(G B, A) \tag{2.11}
\end{equation*}
$$

is a bijection for all $A \in \mathcal{C}$ and $B \in D(X, \Lambda)$.
Proof. The "only if" parts are trivial. For the "if" part of (1), the assumption implies that (2.10) is an isomorphism for $B=R j_{*} M$ and for $B=i_{*} L$, which implies that the same holds for all $B$. For the "if" part of (2), the assumption implies that (2.11) is an isomorphism for $B=j_{!} M$ and for $B=i_{*} L$, which implies that the same holds for all $B$.

Let $i: Y \simeq Y \overline{\times}_{s} s \rightarrow Y \overline{\times}_{s} S$ be the inclusion. Then $i^{*} j_{*} \simeq p_{*}$.
Proof of Theorem [2.12. Consider the topos $Y \bar{x}_{s} \eta^{[1]}$ of morphisms of $Y \bar{x}_{s} \eta$ and the morphism of topoi $\lambda: Y \bar{x}_{s} \eta^{[1]} \rightarrow Y \bar{x}_{s} S$ with $\lambda^{*}$ carrying $\left(\mathcal{F}_{s}, \mathcal{F}_{\eta}, \phi\right)$ to $\left(p^{*} \mathcal{F}_{s}, \mathcal{F}_{\eta}, \phi\right)$. We have $L$ Co $\simeq L$ Coker $\circ \lambda^{*}$. By Lemma 2.9, for $M \in D\left(Y \bar{x}_{s} \eta, \Lambda\right), \lambda^{*} R j_{*} M$ is computed by the totalization of the diagram (2.6) considered as a double complex in $\operatorname{Shv}\left(Y \bar{x}_{s} \eta^{[1]}, \Lambda\right)$, with the vertical arrows representing $\phi$. Thus $L \mathrm{Co} R j_{*} M$ is computed by the diagram (2.6) considered as a triple complex in $\operatorname{Shv}\left(Y \bar{x}_{s} \eta\right.$, $\Lambda$ ), with the rows of the diagram numbered -1 and 0 . This gives an isomorphism id $\xrightarrow{\sim} \tau L \mathrm{Co} R j_{*}$. It is easy to check that the composition

$$
\mathrm{id} \xrightarrow{\sim} \tau L \mathrm{Co} R j_{*} \rightarrow\left(\tau L \mathrm{Co} i_{*}\right)\left(i^{*} R j_{*}\right) \simeq\left(p^{*}(1)[1]\right) R p_{*}
$$

is the adjunction constructed in Proposition 2.13 and

$$
\mathrm{id} \xrightarrow{\sim} \tau L \mathrm{Co} R j_{*} \rightarrow\left(\tau L \mathrm{Co} R j_{*}\right)\left(j^{*} R j_{*}\right) \simeq \mathrm{id} \circ \mathrm{id}
$$

is the trivial adjunction. Thus, by the above lemma, we get $R j_{*} \dashv \tau L$ Co.
Remark 2.16. Between $\operatorname{Shv}(Y, \Lambda)$ and $\operatorname{Shv}\left(Y \bar{×}_{s} S, \Lambda\right)$, we have a sequence of adjoint functors

$$
\pi_{?} \dashv \pi^{*} \dashv \pi_{*}=i^{*} \dashv i_{*} \dashv i^{!}
$$

Here $\pi: Y \bar{x}_{s} S \rightarrow Y$ is the projection and $\pi_{?}=R^{2} i^{!}(1)=R^{1} p_{*} j^{*}(1)$. Between $D(Y, \Lambda)$ and $D\left(Y \overline{\times}_{s}\right.$ $S, \Lambda$ ), we have adjunctions

$$
\begin{equation*}
R i^{!}(1)[2] \dashv \pi^{*} \dashv \pi_{*}=i^{*} \dashv i_{*} \dashv R i^{!} . \tag{2.12}
\end{equation*}
$$

Since $p=\pi j$, compositions of the corresponding functors of the sequences (2.12) and (2.8) collapse into (2.9).

To construct the adjunction $\operatorname{Ri}^{!}(1)[2] \dashv \pi^{*}$, we apply the last assertion of Lemma 2.15, The co-unit $\left(R i^{!}(1)[2]\right) \pi^{*} \xrightarrow{\sim}$ id is given by the cycle class map. The compositions

$$
\begin{gathered}
(\mathrm{id}(1)[2])(\mathrm{id}(-1)[-2]) \simeq\left(\left(R i^{!}(1)[2]\right) i_{*}\right)\left(R i^{!} \pi^{*}\right) \xrightarrow{\sim}\left(R i^{!}(1)[2]\right) \pi^{*} \xrightarrow{\sim} \mathrm{id} \\
\quad\left(R p_{*}(1)[1]\right)\left(p^{*}\right) \simeq\left(\left(R i^{!}(1)[2]\right) j_{!}\right)\left(j^{*} \pi^{*}\right) \rightarrow\left(R i^{!}(1)[2]\right) \pi^{*} \xrightarrow{\sim} \mathrm{id}
\end{gathered}
$$

are adjunctions.

Remark 2.17. Let $M \in D\left(Y \bar{x}_{s} S, \Lambda\right)$. We have distinguished triangles

$$
\begin{gather*}
\pi^{*} \pi_{*} M \rightarrow M \rightarrow j!L \mathrm{Co} M \rightarrow \pi^{*} \pi_{*} M[1],  \tag{2.13}\\
j!j^{*} M \rightarrow M \rightarrow i_{*} i^{*} M \rightarrow j!j^{*} M[1],  \tag{2.14}\\
i_{*} R i^{!} M \rightarrow M \rightarrow R j_{*} j^{*} M \rightarrow i^{*} R i^{!} M[1],  \tag{2.15}\\
R j_{*} \tau L \mathrm{Co} M \rightarrow M \rightarrow \pi^{*} R i^{!} M(1)[2] \rightarrow R j_{*} \tau L \mathrm{Co} M[1] . \tag{2.16}
\end{gather*}
$$

Each triangle above is right adjoint of the preceding one (the first triangle being right adjoint to the last one). Applying $j^{*}$ to (2.13) and (2.16), we obtain distinguished triangles

$$
\begin{gather*}
p^{*} M_{s} \rightarrow M_{\eta} \xrightarrow{\text { can }} L \mathrm{CoM} M p^{*} M_{s}[1],  \tag{2.17}\\
L \mathrm{Co} M \xrightarrow{\text { var }} M_{\eta}(-1)^{\tau} \rightarrow p^{*} R i^{!} M[2] \rightarrow L \mathrm{Co} M[1] . \tag{2.18}
\end{gather*}
$$

Here var is the variation map. One can check that var o can is the canonical map $\iota: M_{\eta} \rightarrow M_{\eta}(-1)^{\tau}$.
For $M$ tame (namely, $M^{P} \simeq M$ ), the morphism var corresponds via (2.3) to morphisms $\operatorname{var}(\xi):(L \mathrm{CoM})_{\bar{\eta}} \rightarrow$ $M_{\bar{\eta}}$ for $\xi \in I$, which are the classical variation maps [SGA7II, XIII (1.4.3.1)].

### 2.4 Duality and $L C o$

Construction 2.18. Let $\mathcal{C}$ and $\mathcal{D}$ be categories equipped with dualities $D_{\mathcal{C}}: \mathcal{C}^{\text {op }} \rightarrow \mathcal{C}$ and $D_{\mathcal{D}}: \mathcal{D}^{\mathrm{op}} \rightarrow$ $\mathcal{D}$ (Remark 1.39). Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ and $F^{\prime}, G^{\prime}: \mathcal{D} \rightarrow \mathcal{C}$ be functors equipped with adjunctions $F^{\prime} \dashv F$ and $G \dashv G^{\prime}$. A natural transformation $F D_{\mathcal{C}} \rightarrow D_{\mathcal{D}} G^{\mathrm{op}}$ corresponds by adjunction to $G^{\prime \text { op }} D_{\mathcal{D}}^{\mathrm{op}} \rightarrow$ $D_{\mathcal{C}}^{\mathrm{op}} F^{\prime}$, which corresponds by taking opposites to a natural transformation $D_{\mathcal{C}} F^{\prime \mathrm{op}} \rightarrow G^{\prime} D_{\mathcal{D}}$.

Let $f: Y \rightarrow s$ be a separated morphism of finite type. We let $K_{Y \bar{x}_{s} \eta}=p^{*} R f^{!} \Lambda_{s} \simeq R\left(f \bar{x}_{s} \mathrm{id}_{\eta}\right)^{!} \Lambda_{\eta}$ and $K_{Y \bar{x}_{s} S}=\pi^{*} R f^{!} \Lambda_{s} \simeq R\left(f \bar{x}_{s} \mathrm{id}_{S}\right)^{!} \Lambda_{S}$ (by Corollary (1.26).
Construction 2.19. We construct a natural transformation

$$
\begin{equation*}
D_{Y \bar{x}_{s} \eta}(L \mathrm{Co})^{\mathrm{op}} \rightarrow \tau L \mathrm{Co} D_{Y \bar{x}_{s} S} \tag{2.19}
\end{equation*}
$$

of functors $D\left(Y \bar{x}_{s} S, \Lambda\right)^{\text {op }} \rightarrow D\left(Y \bar{x}_{s} \eta, \Lambda\right)$, by applying Construction 2.18 and Theorem 2.12 to the natural transformation (1.9)

$$
\begin{equation*}
\gamma: j_{!} D_{Y \bar{x}_{s} \eta} \rightarrow D_{Y \bar{x}_{s} S}\left(R j_{*}\right)^{\mathrm{op}} \tag{2.20}
\end{equation*}
$$

adjoint to the canonical isomorphism $D_{Y \bar{x}_{s} \eta} \simeq j^{*} D_{Y \bar{x}_{s} S}\left(R j_{*}\right)^{\text {op }}$.
By construction (2.19) is given by

$$
\begin{aligned}
\operatorname{Hom}(L, D(L \mathrm{Co}) M) & \simeq \operatorname{Hom}((L \mathrm{Co}) M, D L) \\
& \simeq \operatorname{Hom}\left(M, j_{!} D L\right) \\
\xrightarrow{\gamma_{L}} \operatorname{Hom}\left(M, D R j_{*} L\right) & \simeq \operatorname{Hom}\left(R j_{*} L, D M\right) \simeq \operatorname{Hom}(L, \tau(L \mathrm{Co}) D M) .
\end{aligned}
$$

Theorem 2.20 (= $0.1(2))$. The natural transformations (2.19) and (2.20) are isomorphisms.
Proof. It suffices to show that $\gamma_{L}$ is an isomorphism for every $L \in D\left(Y \bar{x}_{s} \eta, \Lambda\right)$. Since the source and target of $\gamma$ both carry coproducts to products, we may assume $L \in D^{-}$. Since $R j_{*}$ has cohomological amplitude contained in $[0,1]$ and both $D_{Y \bar{x}_{s} S}$ and $D_{Y \bar{x}_{s} \eta}$ have cohomological amplitude $\geq-2 \operatorname{dim}(Y)$, we may assume $L$ is a sheaf. We may further assume that $L$ is a coproduct of sheaves of the form $u_{!} \Lambda$ with $u$ étale and affine. The case of $u_{!} \Lambda$ is a special case of Proposition 1.38,
Remark 2.21. We let $K_{Y}=R f^{!} \Lambda_{s}(-1)[-2]$. Similarly to the above, we have natural isomorphisms

$$
\begin{gather*}
i^{*} D_{Y \bar{x}_{s} S} \xrightarrow{\sim} D_{Y}\left(R i^{\prime}\right)^{\mathrm{op}},  \tag{2.21}\\
D_{Y \bar{x}_{s} S}\left(\pi^{*}\right)^{\mathrm{op}} \xrightarrow{\sim} \pi^{*}(1)[2] D_{Y} \tag{2.22}
\end{gather*}
$$

on the unbounded derived categories, adjoint to each other. Moreover, (2.22) is the inverse of the Künneth formula map. Via the isomorphisms (2.20) and (2.21), the distinguished triangles (2.14) and (2.15) are compatible. Similarly, via the isomorphisms (2.19) through (2.22), the distinguished triangles (2.13) and (2.16) are compatible.

## 3 Nearby and vanishing cycles over Henselian discrete valuation rings

In this section we assume as in Section 2.2 that $S$ is the spectrum of a Henselian discrete valuation ring, of generic point $\eta$ and closed point $s$. Let $\Lambda$ be a Noetherian ring satisfying $m \Lambda=0$ for some integer $m$ invertible on $S$.

Let $X \rightarrow S$ be a morphism of schemes. Consider the morphisms of topoi

$$
X \xrightarrow{\Psi} X \overleftarrow{\times}_{S} S \stackrel{\overleftarrow{i}}{\leftarrow} X_{s} \overline{\times}_{s} S
$$

We call $\Psi^{s}=\overleftarrow{i}^{*} \Psi_{*}$ the sliced nearby cycle functor. As explained in [I3, (1.2.9)], this is the functor $\Psi$ of [SGA7II, XIII 1.3.3]. Between derived categories equipped with the symmetric monoidal structures given by derived tensor products, the functor $\overleftarrow{i}^{*}$ is a symmetric monoidal functor, and $R \Psi_{*}$ is a right-lax symmetric monoidal functor (see for example [IZ, Construction 3.7]). The composite $R \Psi^{s}$ is a right-lax symmetric monoidal functor. By adjunction (see for example ILO, XVII Définition 12.2.3]), we get a morphism natural in $A, K \in D(X, \Lambda)$

$$
\begin{equation*}
R \Psi^{s} R \mathcal{H o m}(A, K) \rightarrow R \mathcal{H o m}\left(R \Psi^{s} A, R \Psi^{s} K\right) \tag{3.1}
\end{equation*}
$$

Let $f: X \rightarrow Y$ be a morphism of schemes over $S, f$ separated of finite type. Via Construction 1.11, the natural transformation [SGA7II, XIII (2.1.7.3)]

$$
\begin{equation*}
R\left(f_{s} \overline{\times}_{s} \operatorname{id}_{S}\right)!R \Psi_{X}^{s} \rightarrow R \Psi_{Y}^{s} R f_{!} \tag{3.2}
\end{equation*}
$$

corresponds to a natural transformation

$$
\begin{equation*}
R \Psi_{X}^{s} R f^{!} \rightarrow R\left(f_{s} \bar{x}_{s} \operatorname{id}_{S}\right)^{!} R \Psi_{Y}^{s} \tag{3.3}
\end{equation*}
$$

See (4.8) and (4.9) for generalizations.
We fix $K_{S} \in D_{c}^{b}(S, \Lambda)$, not necessarily dualizing. For $a: X \rightarrow S$ separated of finite type, we define $K_{X}:=R a^{!} K_{S}$ and $K_{X_{s} \bar{x}_{s} S}:=R\left(f_{s} \bar{x}_{s} \mathrm{id}_{S}\right)^{!} K_{S}$. Applying (3.3) to $K_{S}$, we get $R \Psi^{s} K_{X} \rightarrow K_{X_{s} \bar{x}_{s} S}$. Composing with (3.1), we get a natural transformation

$$
\begin{equation*}
R \Psi^{s} D_{X} \rightarrow D_{X_{s} \bar{x}_{s} S} R \Psi^{s} \tag{3.4}
\end{equation*}
$$

Theorem 3.1. The canonical map $R \Psi^{s} D_{X} L \rightarrow D_{X_{s} \bar{x}_{s} S} R \Psi^{s} L$ (3.4) is an isomorphism for $L \in$ $D_{c}^{-}(X, \Lambda)$.

This confirms a prediction of Deligne D2]. Since $j^{*} R \Psi^{s} \simeq R \Psi_{\eta}^{s} j_{0}^{*}$, where $j_{0}: X_{\eta} \rightarrow X$ and $j: X_{s} \overline{\times}_{s} \eta \rightarrow X_{s} \overline{\times}_{s} S$ are the inclusions, the theorem extends Gabber's theorem on duality for $R \Psi_{\eta}^{s}$.

Corollary 3.2 (Gabber). The canonical map $R \Psi_{\eta}^{s} D_{X_{\eta}} L \rightarrow D_{X_{s} \bar{x}_{s} \eta} R \Psi_{\eta}^{s} L$ is an isomorphism for $L \in D_{c}^{-}\left(X_{\eta}, \Lambda\right)$. Here $K_{X_{s} \bar{x}_{s} \eta}:=R\left(f_{s} \overline{\times}_{s} \mathrm{id}_{\eta}\right)^{!} K_{\eta}$.

Remark 3.3. In [11, Théorème 4.2], Corollary 3.2 is proved for $K_{\eta}=\Lambda_{\eta}$ and $L \in D_{c \mathrm{ft}}\left(X_{\eta}, \Lambda\right)$. The case $K_{\eta}=\Lambda_{\eta}$ and $L \in D_{c}^{-}\left(X_{\eta}, \Lambda\right)$ follows. Indeed, since $R \Psi_{\eta}^{s}$ has cohomological amplitude contained in $\left[0, \operatorname{dim}\left(X_{s}\right)\right], D_{X_{\eta}}$ has cohomological amplitude $\geq-2 \operatorname{dim}\left(X_{\eta}\right)$, and $D_{X_{s} \bar{x}_{s} \eta}$ has cohomological amplitude $\geq-2 \operatorname{dim}\left(X_{s}\right)$, we may assume $L \in \operatorname{Shv}_{c}$. We then reduce to the case $L=j_{!} \Lambda$ for $j$ étale of finite type. In this case $L \in D_{c \mathrm{ft}}\left(X_{\eta}, \Lambda\right)$.

Combining Theorem 3.1 with Theorem 2.20, we obtain duality for $\Phi^{s}=L \mathrm{Co} \circ R \Psi^{s}$ (Corollary (0.2), which is (at least for $S$ strictly local) a theorem of Beilinson [B1, 2.3]. Theorem 3.1] thus encodes duality for both nearby cycles and vanishing cycles and provides a new proof of Beilinson's theorem.

In the rest of this section, we show that Theorem 3.1 follows from our results over general bases. The case where $S$ is excellent of Theorem 3.1 (and Corollary 3.2 ) follows from Theorem 5.1] by Example
$4.26(2)$, which extends to the unbounded derived category since $R \Psi$ has finite cohomological dimension in our case. The general case of Corollary 3.2 follows from the excellent case by Lemma 1.29 (b) and the universal local acyclicity of $\hat{\eta} \rightarrow \eta$, because $R \Psi_{\eta}^{s}$ commutes with the base change $\hat{S} \rightarrow S$. Here $\hat{S}$ denotes the completion of $S$ at $s$ and $\hat{\eta}$ denotes the generic point of $\hat{S}$. We will deduce Theorem 3.1] from Corollary [3.2, after some preliminaries on the vanishing topos $X \overleftarrow{\times}_{S} S$.

Let $X$ be a scheme over $S$. The topos $X \overleftarrow{\times}_{S} S$ is glued from the pieces $X_{s}, X_{s} \overline{\times}_{s} \eta$, and $X_{\eta}$ Consider the diagram of topoi


Here $\Psi_{\eta}, \overleftarrow{j}, j_{0}, j$ are open embeddings, and $i, \overleftarrow{i}, \overleftarrow{i}_{\eta}, i_{0}$ are closed embeddings. We have $\Psi^{s}=\overleftarrow{i}^{*} \Psi_{*}$ and $\Psi_{\eta}^{s}=\overleftarrow{i}_{\eta}^{*} \Psi_{\eta *}\left(\right.$ denoted respectively by $\Psi$ and $\Psi_{\eta}$ in [SGA7II, XIII 1.3]).

Lemma 3.4. We have

$$
\begin{align*}
R \Psi^{s} i_{0 *} & \stackrel{\sim}{\rightarrow} i_{*},  \tag{3.6}\\
R \Psi^{s} j_{0!} & \sim  \tag{3.7}\\
R & j_{!} R \Psi_{\eta}^{s},  \tag{3.8}\\
R \Psi^{s} R j_{0 *} & \sim \sim \\
\sim & j_{*} \\
& \Psi_{\eta}^{s} .
\end{align*}
$$

Here the functors are between unbounded derived categories.
Proof. This follows from

$$
\begin{align*}
& R \Psi i_{0 *} \simeq(\overleftarrow{i} i)_{*}, \quad R \Psi j_{0!}  \tag{3.9}\\
& \leftarrow \leftarrow \overleftarrow{j} R \Psi_{\eta}, \quad R \Psi R j_{0 *} \simeq R \overleftarrow{j}_{*} R \Psi_{\eta}  \tag{3.10}\\
& \overleftarrow{i} * \overleftarrow{i}_{*} \xrightarrow{\sim} \mathrm{id}, \quad \overleftarrow{i} * \overleftarrow{j}!\sim \overleftarrow{j}_{!}^{*} \overleftarrow{i}_{\eta}^{*}, \quad \overleftarrow{i}_{*} \overleftarrow{j}_{\rightarrow}^{\sim} R j_{*} \overleftarrow{i}_{\eta}^{*}
\end{align*}
$$

To show that the middle arrow in (3.9) is an isomorphism, we apply the functors $\overleftarrow{j}^{*}$ and $\left(\overleftarrow{i}_{i}\right)^{*}$ and use the fact that $\Psi^{*} R \Psi_{*} \xrightarrow{\sim}$ id (Remark 4.3 below). All the other isomorphisms except the last one of (3.10) are trivial. The last isomorphism of (3.10) follows from Corollary 1.2, but we give a direct proof here. The last arrow of (3.10) being trivially an isomorphism on $\overleftarrow{i}_{\eta *} M$, it remains to show

$$
\begin{equation*}
\overleftarrow{i}^{*} R \overleftarrow{j}_{*} \Psi_{\eta!}=0 \tag{3.11}
\end{equation*}
$$

Let $y \rightarrow X_{s}$ be a geometric point. Note that

$$
\begin{equation*}
\left(R \overleftarrow{j}_{*} L\right)_{y} \simeq R \Gamma\left(X_{(y)} \overleftarrow{\times}_{S^{\prime}} \eta^{\prime}, L_{(y)}\right) \simeq R \Gamma\left(\eta^{\prime}, L_{y}\right) \tag{3.12}
\end{equation*}
$$

for $L \in D^{+}\left(X \overleftarrow{\times}_{S} \eta, \Lambda\right)$, where $S^{\prime}=S_{(s)}$ is the strict Henselization of $S, \eta^{\prime}$ is the generic point of $S^{\prime}$, and $L_{(y)}$ and $L_{y}$ are the restrictions of $L$ to $X_{(y)} \overleftarrow{×}_{S^{\prime}} \eta^{\prime}$ and $y \overleftarrow{\times}_{S^{\prime}} \eta^{\prime} \simeq \eta^{\prime}$, respectively. Indeed, if $p: X_{(y)} \overleftarrow{×}_{S^{\prime}} \eta^{\prime} \rightarrow \eta^{\prime}$ denotes the projection, then $p_{*}$ is isomorphic to the restriction by [ILO, XI Corollaire 2.3.1]. It follows that $\overleftarrow{j}_{*}$ has cohomological dimension $\leq 1$ and consequently (3.12) holds for $L$ unbounded. (3.11) follows.

Proof of Theorem 3.1. For $L=i_{0 *} M$, the map can be identified with the isomorphism

$$
R \Psi^{s} D_{X} i_{0 *} M \simeq R \Psi^{s} i_{0 *} D_{X_{s}} M \stackrel{(3.6)}{\sim} i_{*} D_{X_{s}} M \simeq D_{X_{s} \bar{x}_{s} S} i_{*} M \stackrel{(3.6)}{\sim} D_{X_{s} \bar{x}_{s} S} R \Psi^{s} i_{0 *} M
$$

For $L=j_{0!} N$, the map can be identified with

$$
\begin{aligned}
R \Psi^{s} D_{X} j_{0!} N \simeq R \Psi^{s} R j_{0 *} D_{X_{\eta}} N & \stackrel{(3.8)}{\sim} R j_{*} R \Psi_{\eta}^{s} D_{X_{\eta}} N \\
& \stackrel{a}{\sim} R j_{*} D_{X_{s} \bar{x}_{s} \eta} R \Psi_{\eta}^{s} N \xrightarrow[\sim]{\sim} D_{X_{s} \bar{x}_{s} S} j_{!} R \Psi_{\eta}^{s} N \xrightarrow[\sim]{\sim} D_{X_{s} \bar{x}_{s} S} R \Psi^{s} j_{0!} N
\end{aligned}
$$

where $a$ is an isomorphism by Corollary 3.2.
Remark 3.5. Let $X$ be a scheme over $S$. The map $R i_{0}^{!} \rightarrow R i^{!} R \Psi^{s}$ induced from (3.6) is an isomorphism. Indeed, by the lemma, this holds on $i_{0 *} M$ and $R j_{0 *} N$. Thus, for $L \in D(X, \Lambda)$, the distinguished triangles (2.17) and (2.18) applied to $R \Psi^{s} L$ are

$$
\begin{gather*}
p^{*} L_{s} \rightarrow R \Psi^{s}\left(L_{\eta}\right) \xrightarrow{\text { can }} \Phi^{s}(L) \rightarrow p^{*} L_{s}[1]  \tag{3.13}\\
\Phi^{s}(L) \xrightarrow{\mathrm{var}} R \Psi^{s}\left(L_{\eta}\right)(-1)^{\tau} \rightarrow p^{*} R i_{0}^{!} L[2] \rightarrow \Phi^{s}(L)[1] . \tag{3.14}
\end{gather*}
$$

A special case of (3.14) was given in [12, Théorème 3.3].
By Theorem 3.1) and Remark 2.21, when $X$ is separated of finite type over $S$, (3.13) and (3.14) are compatible with each other via duality.
Remark 3.6. There is no good duality on $X \overleftarrow{\star}_{S} S$ unless $X=X_{\eta} \amalg X_{s}$ as topological spaces or $\Lambda=0$. Indeed, by (3.11), we have $\left(\overleftarrow{j} \Psi_{\eta}\right)!\xrightarrow{\sim} R \overleftarrow{j}_{*} \Psi_{\eta!}$. Now if there are dualities that swap J! and $R J_{*}$ for the open immersions $J=\overleftarrow{j}$ and $J=\Psi_{\eta}$, then $R\left(\overleftarrow{j} \Psi_{\eta}\right)_{*} \simeq \overleftarrow{j}!R \Psi_{\eta *}$, which implies $i_{0}^{*} R j_{0 *} \simeq i_{0}^{*} \Psi^{*} R \Psi_{*} R j_{0 *} \simeq\left(\Psi i_{0}\right)^{*} R\left(\overleftarrow{j} \Psi_{\eta}\right)_{*}=0$

## 4 Sliced nearby cycles over general bases

In this section, we study the sliced nearby cycle functor over general bases and base change. The main result is Theorem 4.32, the analogue of Orgogozo's theorem for !-pullback. The proof of Theorem 0.3 will be completed in Section 5 .

### 4.1 Definition

In this subsection we define the sliced nearby cycle functor and discuss some basic properties.
Let $f: X \rightarrow S$ be a morphism of schemes. For a point $s \rightarrow S$ with values in a field, we consider the slice $X_{s} \overleftarrow{\times}_{S} S$ of the vanishing topos, where $X_{s}=X \times_{S} s$, and the morphisms of topoi

$$
X \xrightarrow{\Psi_{f}} X \overleftarrow{\times}_{S} S \stackrel{i \overleftarrow{×}_{S} \mathrm{id}_{S}}{\rightleftarrows} X_{s} \overleftarrow{×}_{S} S
$$

Let $\Lambda$ be a commutative ring. We define the sliced nearby cycle functor

$$
R \Psi^{s}=R \Psi_{f}^{s}: D(X, \Lambda) \rightarrow D\left(X_{s} \overleftarrow{×}_{S} S, \Lambda\right)
$$

to be $\left(i \overleftarrow{×}_{S} \operatorname{id}_{S}\right)^{*} R \Psi_{f}$. The most essential case is when $s$ is a point of $S$. However, we will also need the case where $s$ is a geometric point.

Note that $R \Psi_{f}^{s}$ is the right derived functor of $\Psi_{f}^{s}=R^{0} \Psi_{f}^{s}$, with $R \Psi_{f}^{s} L$ computed by $\Psi_{f}^{s} L^{\prime}$, where $L^{\prime} \rightarrow L$ is a quasi-isomorphism with $L^{\prime}$ homotopically injective. The functor $\left(i \overleftarrow{\times}_{S} \operatorname{id}_{S}\right)^{*}$ is a symmetric monoidal functor, and $R \Psi_{f}$ is a right-lax symmetric monoidal functor (see for example
[IZ, Construction 3.7]). The composite $R \Psi_{f}^{s}$ is a right-lax symmetric monoidal functor. By adjunction (see for example [ILO, XVII Définition 12.2.3]), we get a morphism natural in $L, K \in D(X, \Lambda)$

$$
\begin{equation*}
R \Psi_{f}^{s} R \mathcal{H o m}(L, K) \rightarrow R \mathcal{H} o m\left(R \Psi_{f}^{s} L, R \Psi_{f}^{s} K\right) \tag{4.1}
\end{equation*}
$$

We show that the slice is in fact a fiber product of topoi. Let $S_{(s)}$ be the Henselization of $S$ at $s$ and let $i: s_{0} \rightarrow S_{(s)}$ be the inclusion of the closed point (so that $s_{0}$ is the spectrum of the separable closure in $s$ of the residue field of $S$ at the image of $s$ ). The morphism of topoi $i_{\text {et }}$ admits a left adjoint $\pi:\left(S_{(s)}\right)_{\text {et }} \rightarrow\left(s_{0}\right)_{\text {et }}$ (Example 1.40).

Lemma 4.1. We have equivalences of topoi $X_{s} \bar{x}_{s_{0}} S_{(s)} \simeq X_{s} \overleftarrow{\times}_{S_{(s)}} S_{(s)} \xrightarrow{\sim} X_{s} \overleftarrow{×}_{S} S$.
Proof. The first equivalence follows from Example 1.40. The second equivalence follows from ILO, XI Proposition 1.11].

In the sequel we will often identify the equivalent topoi in the lemma. For $L \in D^{+}(X, \Lambda), R \Psi_{f}^{s} L$ can be identified with $R \Psi_{f_{(s)}}^{s}\left(\left.L\right|_{X_{(s)}}\right)$, where $f_{(s)}: X_{(s)} \rightarrow S_{(s)}$ is the base change of $f$.
Remark 4.2. Assume $s$ is a geometric point and $L \in D^{+}(X, \Lambda)$. One can further restrict the sliced nearby cycles $R \Psi^{s} L$ on $X_{s} \overline{\times} S_{(s)}$ in two ways. (a) For any geometric point $t \rightarrow S_{(s)}$, the restriction of $R \Psi^{s} L$ to the shred $X_{s} \overline{\times} t$ (which is called "slice" in [O, Section 6] and [13, Section 1.3]) is computed by Orgogozo's shredded nearby cycle functor

$$
R \Psi_{t}^{s}=\left(i^{s}\right)^{*} R\left(j_{t}^{s}\right)_{*}: D^{+}\left(X_{(t)}, \Lambda\right) \rightarrow D^{+}\left(X_{s}, \Lambda\right)
$$

Here $i^{s}: X_{s} \rightarrow X_{(s),}, j_{t}^{s}: X_{(t)} \rightarrow X_{(s)}$. (b) For any geometric point $x \rightarrow X_{s}$, the restriction of $R \Psi^{s} L$ to the local section $x \times S_{(s)}$ is computed by the localized pushforward $R f_{(x) *}\left(\left.L\right|_{X_{(x)}}\right)$ [33, (1.12.6)], where $f_{(x)}: X_{(x)} \rightarrow S_{(s)}$ is the strict localization of $f$ at $x$.
Remark 4.3. The morphism of topoi $\Psi_{f}: X \rightarrow X \overleftarrow{\times}_{S} S$ is an embedding. In other words the adjunction map $\Psi_{f}^{*} \Psi_{f *} \rightarrow$ id is an isomorphism. This follows from the identification of $\Psi_{f}^{*}$ with $p_{1 *}$ ILO, XI Proposition 4.4] or the computation by shreds: $\Psi_{s}^{s}=\left(i^{s}\right)^{*}$.

It follows that the adjunction map $\Psi_{f}^{*} R \Psi_{f} \rightarrow \mathrm{id}$ is an isomorphism. For $L, K \in D(X, \Lambda)$, we have an isomorphism

$$
A_{f}: R \Psi_{f} R \mathcal{H} o m(L, K) \xrightarrow{\sim} R \mathcal{H o m}\left(R \Psi_{f} L, R \Psi_{f} K\right)
$$

given by

$$
\operatorname{Hom}\left(\Psi_{f}^{*} M \otimes^{L} L, K\right) \xrightarrow{\sim} \operatorname{Hom}\left(\Psi_{f}^{*}\left(M \otimes^{L} R \Psi_{f} L\right), K\right) \simeq \operatorname{Hom}\left(M \otimes^{L} R \Psi_{f} L, R \Psi_{f} K\right)
$$

for $M \in D\left(X \overleftarrow{\times}_{S} S, \Lambda\right)$. The map (4.1) is the composite

$$
R \Psi_{f}^{s} R \mathcal{H} o m(L, K) \xrightarrow[\sim]{A_{f}}\left(i \overleftarrow{\times}_{S} \operatorname{id}_{S}\right)^{*} R \mathcal{H} o m\left(R \Psi_{f} L, R \Psi_{f} K\right) \rightarrow R \mathcal{H o m}\left(R \Psi_{f}^{s} L, R \Psi_{f}^{s} K\right)
$$

Lemma 4.4. Let $X \xrightarrow{f} T \xrightarrow{g} S$ be morphisms of schemes. For any geometric point $t \rightarrow T$, we have $a$ natural isomorphism

$$
\begin{equation*}
\left.R\left(\Psi_{g f}^{t} L\right)\right|_{X_{t} \overline{\times} S_{(t)}} \xrightarrow{\sim} R\left(\operatorname{id} \overline{\times} g_{(t)}\right)_{*} R \Psi_{f}^{t} L \tag{4.2}
\end{equation*}
$$

in $L \in D^{+}(X, \Lambda)$, where $X_{t}=X \times_{T} t, g_{(t)}: T_{(t)} \rightarrow S_{(t)}$ is the strict localization of $g$ at $t$, and $t$ in $\Psi_{g f}^{t}$ denotes the composition $t \rightarrow T \xrightarrow{g} S$.

Consider the diagram of topoi


Via Lemma 4.1, (4.2) is

$$
\left(i \overleftarrow{\times} \mathrm{id}_{S}\right)^{*} R \Psi_{g f} \simeq\left(i \overleftarrow{\times} \mathrm{id}_{S}\right)^{*} R\left(\mathrm{id}_{X} \overleftarrow{\times} g\right)_{*} R \Psi_{f} \xrightarrow{\sim} R\left(\mathrm{id}_{X_{t}} \overleftarrow{\times} g\right)_{*}\left(i \overleftarrow{\times} \mathrm{id}_{T}\right)^{*} R \Psi_{f}
$$

where the second isomorphism is trivial base change [13, Lemma A.9].

### 4.2 Review of local acyclicity

Definition 4.5. Let $f: X \rightarrow S$ be a morphism of schemes and let $L \in D(X, \Lambda)$.
(1) Following [D1, Th. finitude, Définition 2.12], we say that $(f, L)$ is locally acyclic if the canonical map $\alpha_{L}: L_{x} \rightarrow R \Gamma\left(X_{(x) t}, L\right)$ is an isomorphism for every geometric point $x \rightarrow X$ and every algebraic geometric point $t \rightarrow S_{(x)}$. Here $X_{(x) t}:=X_{(x)} \times{ }_{S_{(x)}} t$ denotes the Milnor fiber.
(2) Following [D1, Th. finitude, App., 2.9], we say that $(f, L)$ is strongly locally acyclic if $L \otimes^{L} M$ is locally acyclic for every $\Lambda$-module $M$.
For ( $*$ ) denoting one of the above properties, we say that $(f, L)$ is universally $(*)$ if $(*)$ holds after every base change $g: T \rightarrow S$.

Remark 4.6. Assume $L \in D^{+}$. Then $(f, L)$ is locally acyclic if and only if $\Phi_{f} L=0$ (namely, the canonical map $p_{1}^{*} L \rightarrow R \Psi_{f} L$ is an isomorphism) and $R \Psi_{f} L$ commutes with locally quasi-finite base change ([S1, Proposition 2.7], [13, Example 1.7 (b)], see also [SGA4, XV Corollaire 1.17]).

The following extends [ [, Proposition 7.6.2, Theorem 7.6.9 (i) $\Leftrightarrow($ iii $)$ ].
Lemma 4.7. Assume $(f, L)$ locally acyclic and that either of the following conditions holds:
(1) The Milnor fibers $X_{(x) t}$ have finite cohomological dimension.
(2) L has tor-amplitude $\geq n$ for some integer $n$ and every $\Lambda$-module of finite presentation can be embedded into a free $\Lambda$-module.
Then $(f, L)$ is strongly locally acyclic. Moreover, under condition (1), for any $N \in D(X, \Lambda)$ of locally constant cohomology sheaves, $\left(f, L \otimes^{L} N\right)$ is locally acyclic.

Condition (1) is satisfied if $\Lambda$ is torsion and $f$ is locally of finite type, by limit arguments ( F , Corollary 7.5.7].

Proof. Note that $\alpha_{L \otimes \otimes^{L} M}$ is the composition

$$
L_{x} \otimes^{L} M \xrightarrow{\alpha_{L} \otimes^{L} M} R \Gamma\left(X_{(x) t}, L\right) \otimes^{L} M \xrightarrow{\beta} R \Gamma\left(X_{(x) t}, L \otimes^{L} M\right) .
$$

In case (1), the projection formula map $\beta$ is an isomorphism [13, Lemma A.8]. The same formula implies the last statement of the lemma, as $\left.N\right|_{X_{(x)}} \simeq \pi^{*} N_{x}$, where $\pi:\left(X_{(x)}\right)_{\text {et }} \rightarrow$ pt. In case (2), we show by induction on $m$ that $\operatorname{Cone}\left(\alpha_{L \otimes^{L} M}\right) \in D^{\geq m}$. This is trivial for $m=n-1$. For the general case, we may assume $M$ of finite presentation. Choose a short exact sequence $0 \rightarrow M \rightarrow F \rightarrow M^{\prime} \rightarrow 0$ with $F$ free. Since $\alpha_{L \otimes^{L} F}$ is an isomorphism, $\operatorname{Cone}\left(\alpha_{L \otimes^{L} M}\right) \simeq \operatorname{Cone}\left(\alpha_{L \otimes^{L} M^{\prime}}\right)[-1]$ and it suffices to apply the induction hypothesis to $M^{\prime}$.

### 4.3 Functoriality in $X$

In the sequel we will often omit the notation $R$ for right derived functors.
In this subsection, we study the functoriality of the sliced nearby cycle functor in $X$ by giving analogues of the natural transformations [SGA7II, XIII (2.1.7.1)-(2.1.7.4)] over general bases. These will be used many times in the sequel, including in the definition of the map (5.1) in Theorem 0.3

Let $X \xrightarrow{f} Y \xrightarrow{b} S$ be morphisms of schemes. Consider the diagram of topoi
where the square on the left is Cartesian. The base change maps

$$
\begin{gather*}
\left(f \overleftarrow{\times} \mathrm{id}_{S}\right)^{*} \Psi_{b} \rightarrow \Psi_{b f} f^{*}  \tag{4.4}\\
i_{Y}^{*}\left(f \overleftarrow{\times} \mathrm{id}_{S}\right)_{*} \rightarrow\left(f_{s} \overleftarrow{\times} \mathrm{id}_{S}\right)_{*} i_{X}^{*} \tag{4.5}
\end{gather*}
$$

induce

$$
\begin{align*}
& \left(f_{s} \overline{\times}_{s_{0}} \operatorname{id}_{S_{(s)}}\right)^{*} \Psi_{b}^{s} \rightarrow \Psi_{b f}^{s} f^{*},  \tag{4.6}\\
& \Psi_{b}^{s} f_{*} \rightarrow\left(f_{s} \bar{x}_{s 0} \operatorname{id}_{\left.S_{(s)}\right)}\right)_{*} \Psi_{b f}^{s} . \tag{4.7}
\end{align*}
$$

Remark 4.8.
(1) If $f$ is étale, or if $\Lambda$ is torsion and $f$ is locally acyclic locally of finite type, then (4.4) and (4.6) are isomorphisms, by [13, Remark 1.20] (which extends to the unbounded derived category by the finiteness of the cohomological dimension of $\Psi$ (O, Proposition 3.1]).
(2) If $f$ is integral, or if $\Lambda$ is torsion and $f$ is proper, then (4.5) and (4.7) are isomorphisms by Proposition 1.6 and Remark 1.7 .
Assume $\Lambda$ of torsion. Assume $X$ and $Y$ coherent and $f$ separated and of finite type. We will define maps

$$
\begin{align*}
& \left(f_{s} \overline{\times}_{s_{0}} \operatorname{id}_{S_{(s)}}\right)!\Psi_{b f}^{s} \rightarrow \Psi_{b}^{s} f_{!},  \tag{4.8}\\
& \Psi_{b f}^{s} f^{!} \rightarrow\left(f_{s} \bar{x}_{s_{0}} \operatorname{id}_{\left.S_{(s)}\right)}\right) \Psi_{b}^{s}, \tag{4.9}
\end{align*}
$$

adjoint to each other via Construction 1.11, The precise definitions will be given in (4.15) and (4.16). For $f$ proper, (4.8) is the inverse of (4.7). For $f$ an open immersion, (4.9) is the inverse of (4.6). For general $f$, we take a compactification of $f$. The task of checking that the construction does not depend on the choice of the compactification can be conveniently divided into two with the help of oriented !-pushforward and !-pullback functors, that we will now discuss.

Construction 4.9. We construct, for morphisms of schemes $X \xrightarrow{f} Y \xrightarrow{b} S$ with $X$ and $Y$ coherent and $f$ separated of finite type, a functor

$$
R\left(f \overleftarrow{\times}_{S} \operatorname{id}_{S}\right)!: D\left(X \overleftarrow{\times}_{S} S, \Lambda\right) \rightarrow D\left(Y \overleftarrow{\times}_{S} S, \Lambda\right)
$$

isomorphic to $R\left(f \overleftarrow{\times}_{S} \text { id }_{S}\right)_{*}$ for $f$ proper and left adjoint to $\left(f \overleftarrow{×}_{S} \operatorname{id}_{S}\right)^{*}$ for $f$ an open immersion, and compatible with composition. For this, we apply Construction 1.8 to $f_{\text {et }}$ and to the diagram $Y_{\text {et }} \xrightarrow{b_{\text {et }}} S_{\text {et }} \stackrel{p_{1}}{\stackrel{ }{\sim}} S \overleftarrow{\times}_{S} S$. By [ILO, XI Lemme 2.5], $p_{1}$ is a locally coherent morphism of locally coherent topoi. Moreover, the source and target are of the desired form by [ILO, XI Proposition 4.2] (cf. Remark 1.7 (3)).

Consider a Cartesian square

of coherent schemes over $S$ with $f$ separated of finite type. We apply Constructions 1.8 (1), (2) (to $g=\Psi: S_{\text {et }} \rightarrow S \overleftarrow{×}_{S} S$ ), (3), and obtain isomorphisms

$$
\begin{align*}
& \left(h \overleftarrow{×}_{S} \operatorname{id}_{S}\right)^{*} R\left(f \overleftarrow{\times}_{S} \mathrm{id}_{S}\right)!\simeq R\left(f^{\prime} \overleftarrow{\times}_{S} \mathrm{id}_{S}\right)!\left(h^{\prime} \overleftarrow{\times}_{S} \mathrm{id}_{S}\right)^{*},  \tag{4.10}\\
& \Psi_{b}^{*} R\left(f \overleftarrow{\times}_{S} \mathrm{id}_{S}\right)!\simeq R f_{!} \Psi_{b f}^{*},  \tag{4.11}\\
& R\left(f \overleftarrow{×}_{S} \operatorname{id}_{S}\right)!L \otimes^{L} M \simeq R\left(f \overleftarrow{\times}_{S} \operatorname{id}_{S}\right)!\left(L \otimes^{L}\left(f \overleftarrow{×}_{S} \operatorname{id}_{S}\right)^{*} M\right) \tag{4.12}
\end{align*}
$$

for $L \in D\left(X \overleftarrow{\times}_{S} S, \Lambda\right)$ and $M \in D\left(Y \overleftarrow{\times}_{S} S, \Lambda\right)$.
Next we apply Construction 1.9. The functor $R\left(f \overleftarrow{\times}_{S} \mathrm{id}_{S}\right)$ ! has cohomological dimension $\leq 2 d$, where $d=\max _{y \in Y} \operatorname{dim}\left(X_{y}\right)$, and admits a right adjoint

$$
R\left(f \overleftarrow{\times}_{S} \operatorname{id}_{S}\right)^{!}: D\left(Y \overleftarrow{\times}_{S} S, \Lambda\right) \rightarrow D\left(X \overleftarrow{×}_{S} S, \Lambda\right)
$$

For $m \Lambda=0$ with $m$ invertible on $Y$ and $f$ flat with fibers of dimension $\leq d$, we have a trace map (1.4)

$$
\operatorname{tr}_{f \overleftarrow{×}_{S i \mathrm{id}_{S}}}:\left(f \overleftarrow{\times}_{S} \mathrm{id}_{S}\right)^{*}(d)[2 d] \rightarrow R\left(f \overleftarrow{×}_{S} \mathrm{id}_{S}\right)^{!}
$$

Consider

$$
\begin{gather*}
\alpha^{\prime}:\left(f \overleftarrow{×}_{S} \operatorname{id}_{S}\right)!\Psi_{b f} \rightarrow \Psi_{b} f_{!}, \quad \alpha: \Psi_{b f} f^{!} \xrightarrow{\sim}\left(f \overleftarrow{×}_{S} \operatorname{id}_{S}\right)^{!} \Psi_{b},  \tag{4.13}\\
\beta^{\prime}:\left(f_{s} \overleftarrow{×}_{S} \operatorname{id}_{S}\right)!i_{X}^{*} \xrightarrow{\sim} i_{Y}^{*}\left(f \overleftarrow{×}_{S} \operatorname{id}_{S}\right)!, \quad \beta: i_{X}^{*}\left(f \overleftarrow{×}_{S} \operatorname{id}_{S}\right)^{!} \rightarrow\left(f_{s} \overleftarrow{×}_{S} \operatorname{id}_{S}\right)^{!} i_{Y}^{*}, \tag{4.14}
\end{gather*}
$$

where the two maps in each line are adjoint to each other via Construction 1.11, $\alpha$ is right adjoint to (4.11) and $\beta^{\prime}$ is the inverse of (4.10). The map (4.8) is the composite

$$
\begin{equation*}
\left(f_{s} \overleftarrow{×}_{S} \operatorname{id}_{S}\right)!i_{X}^{*} \Psi_{b f} \xrightarrow[\sim]{\beta^{\prime} \Psi_{b f}} i_{Y}^{*}\left(f \overleftarrow{\times}_{S} \operatorname{id}_{S}\right)!\Psi_{b f} \xrightarrow{i_{Y}^{*} \alpha^{\prime}} i_{Y}^{*} \Psi_{b} f_{!}, \tag{4.15}
\end{equation*}
$$

and (4.9) is the composite

$$
\begin{equation*}
i_{X}^{*} \Psi_{b f} f^{!} \xrightarrow[\sim]{i_{X}^{*} \alpha} i_{X}^{*}\left(f \overleftarrow{×}_{S} \operatorname{id}_{S}\right)^{!} \Psi_{b} \xrightarrow{\beta \Psi_{b}}\left(f_{s} \overleftarrow{×}_{S} \mathrm{id}_{S}\right)^{!} i_{Y}^{*} \Psi_{b} \tag{4.16}
\end{equation*}
$$

For $f$ proper, $\alpha^{\prime}$ is the obvious isomorphism.
Lemma 4.10. Assume $m \Lambda=0$ with $m$ invertible on $Y$ and $f$ flat with fibers of dimension $\leq d$. Then the following square commutes


If, moreover, $f$ is smooth of dimension $d$, then (4.9) is an isomorphism.

Proof. The square (4.17) decomposes into

$$
\begin{aligned}
& \left(f_{s} \overleftarrow{\times}_{S} \mathrm{id}\right)^{*} i_{Y}^{*} \Psi_{b}(d)[2 d] \underset{\sim}{\sim} i_{X}^{*}\left(f \overleftarrow{\times}_{S} \text { id }\right)^{*} \Psi_{b}(d)[2 d] \xrightarrow{[4.4]} i_{X}^{*} \Psi_{b f} f^{*}(d)[2 d]
\end{aligned}
$$

$$
\begin{aligned}
& \left(f_{s} \overleftarrow{\times}_{S} \mathrm{id}\right)^{\prime} i_{Y}^{*} \Psi_{b} \longleftarrow{ }^{\beta \Psi_{b}} i_{X}^{*}\left(f \overleftarrow{\times}_{S} \text { id }\right)^{\prime} \Psi_{b} \underset{\sim}{i_{X}^{*} \alpha} i_{X}^{*} \Psi_{b f} f^{!} .
\end{aligned}
$$

The inner cells commute by construction. For $f$ smooth of dimension $d$, (4.9) is an isomorphism, because the other three sides of the square (4.17) are isomorphisms by Proposition 1.23 and Remark 4.8 (1).

Remark 4.11. One can show that if $m \Lambda=0$ with $m$ invertible on $Y, f$ is smooth of dimension $d$, $S$ is Noetherian and either $S$ is excellent or $b$ is of finite type, then $\operatorname{tr}_{f \overleftarrow{\times}{ }_{S} \mathrm{id} S}:\left(f \overleftarrow{\times}_{S} \mathrm{id}_{S}\right)^{*}(d)[2 d] \rightarrow$ $R\left(f \overleftarrow{\times}_{S} \operatorname{id}_{S}\right)!$ is an isomorphism. In this case, $\beta$ is an isomorphism.

### 4.4 Base change and sliced !-base change

Theorem 0.3 states that the sliced nearby cycle functor commutes with duality after pullback by a modification of the base, and remains so after further pullback. As duality swaps pullback and !pullback, in order to prove Theorem 0.3 we need to consider the commutation of the sliced nearby cycle functor with both pullback and !-pullback. In this subsection, we study such commutation.

Consider a Cartesian square

of schemes. Let $t \rightarrow T$ be a geometric point and let $X_{t}=X \times_{S} t$. Consider the diagram of topoi

Let $\Lambda$ be a commutative ring. The base change maps

$$
\begin{array}{r}
\mathrm{BC}_{f, g}:\left(g_{X} \overleftarrow{×}_{g} g\right)^{*} \Psi_{f} \rightarrow \Psi_{f_{T}} g_{X}^{*} \\
i^{*}\left(g_{X} \overleftarrow{×}_{g} g\right)_{*} \rightarrow\left(\operatorname{id}_{X_{t}} \overleftarrow{×}_{g} g\right)_{*} i_{T, t}^{*} \tag{4.21}
\end{array}
$$

induce

$$
\begin{gather*}
\mathrm{BC}_{f, g}^{t}:\left(\mathrm{id} \overline{\times} g_{(t)}\right)^{*} \Psi_{f}^{t} \rightarrow \Psi_{f_{T}}^{t} g_{X}^{*},  \tag{4.22}\\
\quad \Psi_{f}^{t} g_{X *} \rightarrow\left(\mathrm{id} \overline{\times} g_{(t)}\right)_{*} \Psi_{f_{T}}^{t} \tag{4.23}
\end{gather*}
$$

Here $t$ in $\Psi_{f}^{t}$ denotes the composition $t \rightarrow T \xrightarrow{g} S$.
For $L \in D(X, \Lambda)$, we say that $\Psi_{f} L$ commutes with base change by $g$ if $\mathrm{BC}_{f, g}(L)$ is an isomorphism. Note that this holds if and only if $\mathrm{BC}_{f, g}^{t}(L)$ is an isomorphism for all geometric points $t$ of $T$. For $g$ étale, $\mathrm{BC}_{f, g}(L)$ is an isomorphism for all $f$ and $L$. For cases where $\mathrm{BC}_{f, g}(L)$ is an isomorphism for all $g$, see Section 4.5.

Lemma 4.12. Consider a diagram of schemes with Cartesian squares


Let $\alpha:\left(r_{X} \overleftarrow{\times}_{r} r\right)^{*}\left(\operatorname{id}_{X} \overleftarrow{×}_{b} b\right)_{*} \rightarrow\left(\operatorname{id}_{X^{\prime}} \overleftarrow{\times}_{b^{\prime}} b^{\prime}\right)_{*}\left(r_{X} \overleftarrow{×}_{r_{Y}} r_{Y}\right)^{*}$ be the base change map. Then the diagrams of base change maps


commute. For every geometric point $s^{\prime} \rightarrow S^{\prime}$, the diagram

$$
\begin{align*}
& \left(\mathrm{id} \overline{\times} r_{\left(s^{\prime}\right)}\right)^{*} \Psi_{b}^{s^{\prime}} f_{*} \xrightarrow{\text { (4.77) }}\left(\mathrm{id} \overline{\times} r_{\left(s^{\prime}\right)}\right)^{*}\left(f_{s^{\prime}} \overline{\times} \mathrm{id}\right)_{*} \Psi_{b f}^{s^{\prime}} \longrightarrow\left(f_{s^{\prime}} \overline{\times} \mathrm{id}\right)_{*}\left(\mathrm{id} \overline{\times} r_{\left(s^{\prime}\right)}\right)^{*} \Psi_{b f}^{s^{\prime}}  \tag{4.26}\\
& \left.\begin{array}{c}
\mathrm{BC}_{b, r}^{s^{\prime}} f_{*} \\
\Psi_{b^{\prime}}^{s^{\prime}} r_{X}^{*} f_{*} \longrightarrow \Psi_{b^{\prime}}^{s^{\prime}} f_{r^{\prime}}^{\prime} r_{Y}^{*} \longrightarrow \text { (4.7) }\left(f_{s^{\prime}} \overline{\times} \mathrm{id}\right)_{*} \mathrm{BC}_{b, r}^{s^{\prime}}
\end{array} f_{s^{\prime}} \overline{\times} \mathrm{id}\right)_{*} \Psi_{b^{\prime} f^{\prime}}^{s^{\prime}} r_{Y}^{*}
\end{align*}
$$

commutes. Moreover,
(1) Assume $b$ finite. Then the map $\alpha$ is an isomorphism. In particular, $\Psi_{b f} L$ commutes with base change by $r$ if and only if $\Psi_{f} L$ commutes with base change by $r_{Y}$.
(2) Assume $f$ and $f^{\prime}$ are locally acyclic locally of finite type and $\Lambda$ of torsion. Then the vertical arrows of (4.25) are isomorphisms. In particular, if $\Psi_{b} M$ commutes with base change by $r$, then $\Psi_{b f}\left(f^{*} M\right)$ commutes with base change by $r$.
(3) Assume either $f$ integral or $\Lambda$ of torsion and $f$ proper. Then the horizontal arrows of (4.26) are isomorphisms. In particular, if $\Psi_{b f} L$ commutes with base change by $r$, then $\Psi_{b}\left(f_{*} L\right)$ commutes with base change by $r$.
(4) Assume $f$ finite. Then $\Psi_{b f} L$ commutes with base change by $r$ if and only if $\Psi_{b}\left(f_{*} L\right)$ commutes with base change by $r$.

Proof. The commutativity of the squares follows from the definition. Then (2) follows from Remark 4.8 (1). Moreover, (3) follows from Remark 4.8 (2), integral base change, and proper base change, and (4) follows from (3) and the fact that $\left(f_{s^{\prime}} \overline{\times} \mathrm{id}\right)_{*}$ is conservative. For (1), note that by [I3, Proposition 1.13], for $L \in D^{+}\left(X \overleftarrow{\times}_{Y} Y, \Lambda\right)$ and every geometric point $x^{\prime} \rightarrow X^{\prime}$, the restriction of $\alpha(L)$ to $x^{\prime} \overleftarrow{\times}_{S^{\prime}} S^{\prime}$ is the base change map $r_{\left(s^{\prime}\right)}^{*} b_{(y) *}\left(\left.L\right|_{x \overleftarrow{×}_{Y} Y}\right) \rightarrow b_{\left(y^{\prime}\right) *}^{\prime}\left(r_{Y}\right)_{\left(y^{\prime}\right)}^{*}\left(\left.L\right|_{x \overleftarrow{x}_{Y} Y}\right)$ associated to the commutative square of strict localizations

$$
\begin{aligned}
Y_{\left(y^{\prime}\right)}^{\prime} & \xrightarrow{b_{\left(y^{\prime}\right)}^{\prime}} S_{\left(s^{\prime}\right)}^{\prime} \\
\left(r_{Y}\right)_{\left(y^{\prime}\right)} \mid & \|_{\left(s^{\prime}\right)} \\
Y_{(y)} & \xrightarrow{b_{(y)}} \|_{(s)},
\end{aligned}
$$

where the geometric points are images of $x^{\prime}$. For $b$ finite, the square is Cartesian and the functors are exact, so that $\alpha$ is an isomorphism for $L \in D\left(X \overleftarrow{\times}_{Y} Y, \Lambda\right)$. Moreover, in this case, the functors $b_{\left(y^{\prime}\right) *}^{\prime}$ are conservative, so that $\left(\operatorname{id}_{X^{\prime}} \overleftarrow{×}_{b^{\prime}} b^{\prime}\right)_{*}$ is conservative.

We will construct an analogue of $\mathrm{BC}_{f, g}^{t}$ for !-pullback. We start by the finite case.
Lemma 4.13. Assume $g$ finite. Let $s \rightarrow S$ be a geometric point. Then (4.23) induces an isomorphism

$$
\begin{equation*}
\Psi_{f}^{s} g_{X *} \xrightarrow{\sim} \prod_{t}\left(\mathrm{id} \overline{\times} g_{(t)}\right)_{*} \Psi_{f_{T}}^{t}, \tag{4.27}
\end{equation*}
$$

where $t$ runs through points of $T \times{ }_{S}$ s. Moreover, for each $t$, the map

$$
\begin{equation*}
\Psi_{f_{T}}^{t} g_{X}^{\prime} \rightarrow\left(\operatorname{id} \overline{\times} g_{(t)}\right)^{!} \Psi_{f}^{s} \tag{4.28}
\end{equation*}
$$

adjoint via Construction [1.11 to the map $\left(\operatorname{id} \overline{\times} g_{(t)}\right)_{*} \Psi_{f_{T}}^{t} \rightarrow \Psi_{f}^{s} g_{X *}$ induced by the inverse of (4.27) is an isomorphism on $D^{+}$if $g$ is of finite presentation.

Proof. By [O, Lemme 9.1], the morphisms $g_{X} \overleftarrow{×}_{g} \mathrm{id}_{T}: X_{T} \overleftarrow{×}_{T} T \rightarrow X \overleftarrow{\times}_{S} T$ and $\amalg_{t} X_{t} \overleftarrow{×}_{T} T \rightarrow X_{s} \overleftarrow{×}_{S} T$ are equivalences. Thus (4.19) is deduced from the diagram

The map (4.27) is the composite

$$
i^{*} \Psi_{f} g_{X *} \xrightarrow[\sim]{i^{*} \alpha} i^{*}\left(\operatorname{id}_{X} \overleftarrow{×}_{S} g\right)_{*} \Psi_{f_{T}} \xrightarrow[\sim]{\beta \Psi_{f_{T}}}\left(\operatorname{id}_{X_{s}} \overleftarrow{×}_{S} g\right)_{*} i_{T}^{*} \Psi_{f_{T}}
$$

where $\alpha: \Psi_{f} g_{X *} \xrightarrow{\sim}\left(\operatorname{id}_{X} \overleftarrow{\times}_{S} g\right)_{*} \Psi_{f_{T}}$, and $\beta: i^{*}\left(\operatorname{id}_{X} \overleftarrow{\times}_{S} g\right)_{*} \rightarrow\left(\operatorname{id}_{X_{s}} \overleftarrow{x}_{S} g\right)_{*} i_{T}^{*}$ is the base change map. By Proposition 1.1 and Corollary 1.2, the functors in $\beta$ are exact and $\beta$ is an isomorphism.

Note that $S_{(s)} \times{ }_{S} T$ is the disjoint union of $T_{(t)}$, and for each $t$ and each geometric point $x \rightarrow X_{s}$, the square

is Cartesian. By Remark 4.2, on $D^{+}$, the restriction of (4.27) to $x \overline{\times} S_{(s)}$ can be identified with $\prod_{t}$ of $f_{(x) *}\left(g_{X}\right)_{(x, t) *} \simeq g_{(t) *}\left(f_{T}\right)_{(x, t) *}$. Moreover, for $g$ of finite presentation, on $D^{+}$, the restriction of (4.28) to $x \overline{\times} S_{(s)}$ can be identified with the isomorphism $\left(f_{T}\right)_{(x, t) *}\left(g_{X}\right)_{(x, t)}^{!} \xrightarrow{\sim} g_{(t)}^{\prime} f_{(x) *}$ of Lemma 1.30, adjoint to finite base change.

Remark 4.14. Assume $g$ finite. By Lemma 1.10, $\left(\operatorname{id}_{X} \overleftarrow{×}_{S} g\right)_{*}$ admits a right adjoint $R\left(\operatorname{id}_{X} \overleftarrow{\times}_{S} g\right)^{!}$. The map (4.28) is the restriction of the composite

$$
i_{T}^{*} \Psi_{f_{T}} g_{X}^{!} \xrightarrow[\sim]{i_{T}^{*} \alpha^{\prime}} i_{T}^{*}\left(\mathrm{id}_{X} \overleftarrow{\times}_{S} g\right)^{!} \Psi_{f} \xrightarrow{\beta^{\prime} \Psi_{f}}\left(\mathrm{id}_{X_{s}} \overleftarrow{\times}_{S} g\right)^{!} i^{*} \Psi_{f}
$$

where $\alpha^{\prime}: \Psi_{f_{T}} g_{X}^{\prime} \xrightarrow{\sim}\left(\operatorname{id}_{X} \overleftarrow{\times}_{S} g\right)^{!} \Psi_{f}$ and $\beta^{\prime}: i_{T}^{*}\left(\operatorname{id}_{X} \overleftarrow{\times}_{S} g\right)^{!} \rightarrow\left(\operatorname{id}_{X_{s}} \overleftarrow{\triangle}_{S} g\right)^{!} i^{*}$ are adjoint to $\alpha^{-1}$ and $\beta^{-1}$, respectively, via Construction 1.11, Note that $\alpha^{\prime}$ is right adjoint to the base change map $\Psi_{f}^{*}\left(\operatorname{id}_{X} \overleftarrow{\times}_{S}\right.$ $g)_{*} \rightarrow g_{X *} \Psi_{f_{T}}^{*}$, which is an isomorphism by Proposition 1.1. If $g$ is a closed immersion of finite presentation, then $\beta^{\prime}$ is an isomorphism on $D^{+}$by Corollary 1.2 applied to the complement of $g$.

Next we construct ( $\mathrm{id} \times g_{(t)}$ )! in the case where $g$ is not necessarily finite. Assume $m \Lambda=0$ for $m$ invertible on $S$.

Construction 4.15. Let $X$ be a coherent topos. Consider the following subcategory $\mathcal{C}$ of the category of schemes. The objects are strictly local schemes (namely, spectra of strictly Henselian rings). A morphism $f: T \rightarrow S$ belongs to $\mathcal{C}$ if it is the strict localization of a morphism of schemes $f_{0}: T_{0} \rightarrow S_{0}$ of finite presentation at a geometric point of $T_{0}$. We define for such $f$ a functor

$$
(\operatorname{id} \overline{\times} f)^{!}: D^{+}(X \overline{\times} S, \Lambda) \rightarrow D^{+}(X \overline{\times} T, \Lambda),
$$

isomorphic to $(\operatorname{id} \overline{\times} f)^{*}(d)[2 d]$ for $f$ a strict localization of a smooth morphism of dimension $d$, and right adjoint to $(\operatorname{id} \times f)_{*}$ for $f$ finite, and compatible with composition. Note that each $f$ in $\mathcal{C}$ admits a decomposition $f=f_{2} f_{1}$ in $\mathcal{C}$, with $f_{1}$ a closed immersion and $f_{2}$ a strict localization of a smooth morphism.

We apply Ayoub's gluing formalism [A, Théorème 1.3.1] (see [Z2, Theorem 1.1] for a common generalization of this and Deligne's gluing formalism used earlier) to the category $\mathcal{C}_{F /}$ of objects of $\mathcal{C}$ under a fixed object $F$. This category has the advantage of admitting fiber products, the fiber product of $T$ and $S^{\prime}$ over $S$ being the strict localization of $T \times{ }_{S} S^{\prime}$ at the image of the closed point of $F$. The construction of $(\operatorname{id} \overline{\times} f)$ ! does not depend on the choice of $F \rightarrow T$. Alternatively we may apply [Z2, Theorem 5.9] directly to $\mathcal{C}$.

Given a commutative square $h g=f i$ in $\mathcal{C}$ with $h, i$ finite, $f$ (resp. $g$ ) a strict localization of a smooth morphism of dimension $d$ (resp. e), we need to construct an isomorphism

$$
\begin{equation*}
(\operatorname{id} \overline{\times} g)^{!}(\operatorname{id} \bar{x} h)^{!} \simeq(\operatorname{id} \overline{\times} i)^{!}(\operatorname{id} \overline{\times} f)^{!} \tag{4.29}
\end{equation*}
$$

We decompose the square into a commutative diagram in $\mathcal{C}$ with Cartesian inner square


For the inner square, the map

$$
\begin{equation*}
\left(\operatorname{id} \overline{\times} f^{\prime}\right)^{!}(\operatorname{id} \overline{\times} h)^{!} \rightarrow\left(\operatorname{id} \overline{\times} h^{\prime}\right)^{!}(\operatorname{id} \overline{\times} f)^{!} \tag{4.30}
\end{equation*}
$$

adjoint to the inverse of the base change isomorphism $(\operatorname{id} \overline{\times} f)^{*}(\operatorname{id} \overline{\times} h)_{*} \simeq\left(\operatorname{id} \overline{\times} h^{\prime}\right)_{*}\left(\operatorname{id} \bar{x} f^{\prime}\right)^{*}$ via Construction 1.11 is an isomorphism. Indeed, if $f=p f_{0}$ with $f_{0}$ smooth and $p$ a strict localization, the analogous assertion for $f_{0}$ is clear and the one for $p$ follows from Proposition 1.24 and Lemma 1.30. Note that $k$ is a local complete intersection of virtual relative dimension $e-d$. The cycle class map $\Lambda \rightarrow k^{!} \Lambda(d-e)[2(d-e)]$ of $k$ [ILO, XVI Définition 2.5.11] induces

$$
\begin{equation*}
(\mathrm{id} \overline{\times} g)^{!} \rightarrow(\mathrm{id} \overline{\times} k)^{!}\left(\mathrm{id} \overline{\times} f^{\prime}\right)^{!}, \tag{4.31}
\end{equation*}
$$

which is an isomorphism. Indeed, $k$ is the strict localization of a finite morphism $R_{0} \rightarrow T_{0}^{\prime}$, with $R_{0}$ and $T_{0}^{\prime}$ affine and smooth over $S^{\prime}$ of relative dimension $e$ and $d$, respectively. Combining (4.30) and (4.31), we get (4.29).

Remark 4.16. For $f: T \rightarrow S$ as above and $g: X \rightarrow Y$ a morphism of coherent topos, we have an isomorphism

$$
(g \overline{\times} \mathrm{id})^{*}(\mathrm{id} \overline{\times} f)^{!} \simeq(\mathrm{id} \overline{\times} f)^{!}(g \overline{\times} \mathrm{id})^{*}
$$

by Proposition 1.24.

Remark 4.17. Let $g: T \rightarrow S$ be a separated morphism of finite presentation of coherent schemes. Let $t \rightarrow T$ be a geometric point. Consider the commutative square


Note that $g_{(t)}$ is a morphism of $\mathcal{C}$. We have an isomorphism

$$
(\mathrm{id} \overline{\times} j)^{*}(\mathrm{id} \overline{\times} g)^{!} \xrightarrow{\sim}\left(\mathrm{id} \overline{\times} g_{(t)}\right)^{!}(\mathrm{id} \overline{\times} k)^{*} .
$$

Indeed, we may assume that $g$ admits a factorization $T \xrightarrow{i} R \xrightarrow{p} S$ with $p$ smooth and $i$ a closed immersion. The assertion for $p$ is trivial and the assertion for $i$ follows from Proposition 1.24 and Lemma 1.30 (or from limit arguments applied to the complement of $i$ ).
Construction 4.18. We construct, for each morphism $f: T \rightarrow S$ of $\mathcal{C}$ and $L \in D^{-}(X \overline{\times} S, \Lambda)$, $M \in D^{+}(X \overline{\times} S, \Lambda)$, a map

$$
\begin{equation*}
(\operatorname{id} \overline{\times} f)^{!} R \mathcal{H} o m(L, M) \rightarrow R \mathcal{H o m}\left((\operatorname{id} \overline{\times} f)^{*} L,(\operatorname{id} \overline{\times} f)^{!} M\right) \tag{4.32}
\end{equation*}
$$

For $f$ a strict localization of a smooth morphism, we take the restriction map. For $f$ finite, we take (1.6). In general we take a decomposition. One checks that the resulting map does not depend on choices and is compatible with composition.
Lemma 4.19. Let $X$ be a coherent scheme and let $f: T \rightarrow S$ be a morphism of $\mathcal{C}$. For $L \in D_{c}^{-}(X \bar{x}$ $S, \Lambda), M \in D^{+}(X \overline{\times} S, \Lambda)$, (4.32) is an isomorphism.
Proof. We decompose $f$ as $g j$, with $g$ separated of finite presentation and $j$ a strict localization. By Remark 4.17, we have $(\operatorname{id} \overline{\times} f)^{!} \simeq(\operatorname{id} \overline{\times} j)^{*}(\operatorname{id} \overline{\times} g)^{!}$. The lemma follows from Lemma 1.29 applied to $j$ and (1.6) applied to $g$.

Construction 4.20. Consider a Cartesian square (4.18) of coherent schemes with $g$ separated of finite presentation. Let $t$ be a geometric point of $T$. We construct the sliced !-base change map

$$
\begin{equation*}
\mathrm{BC}_{f, g}^{\prime t}:\left(\operatorname{id} \overline{\times} g_{(t)}\right)^{!} \Psi_{f}^{t} \rightarrow \Psi_{f_{T}}^{t} g_{X}^{\prime} \tag{4.33}
\end{equation*}
$$

as follows. For $g$ smooth, we take the map induced by $\mathrm{BC}_{f, g}^{t}$ (4.22). For $g$ a finite morphism, we take the inverse of (4.28), which is an isomorphism. In general, the restriction of $g$ to an open subscheme $U \subseteq T$ containing $t$ can be decomposed into $U \xrightarrow{g^{\prime}} S^{\prime} \xrightarrow{g^{\prime \prime}} S$ with $g^{\prime}$ finite (one may even take $g^{\prime}$ to be a closed immersion) and $g^{\prime \prime}$ smooth. We take the composition

$$
\left(\operatorname{id} \overline{\times} g_{(t)}^{\prime}\right)^{\prime}\left(\operatorname{id} \overline{\times} g_{(t)}^{\prime \prime}\right) \Psi_{f}^{t} \Psi_{f, g^{\prime \prime}}^{\mathrm{BC}_{f}^{\prime t}}\left(\operatorname{id} \overline{\times} f_{(t)}^{\prime}\right)^{\prime} \Psi_{f_{S}}^{t} g_{X}^{\prime \prime \prime} \xrightarrow{\mathrm{BC}_{f_{S^{\prime}}, g}^{\prime t}} \Psi_{f_{U}}^{t} g_{X}^{\prime \prime} g_{X}^{\prime \prime \prime}
$$

One checks by restricting to local sections that the resulting base change map does not depend on choices and is compatible with composition. (4.33) is an isomorphism for $g$ quasi-finite by Zariski's main theorem.
Remark 4.21. Assume that $g$ is smooth or a closed immersion. With the notation of (4.19), $\mathrm{BC}_{f, g}^{\prime t}$ is the composite

$$
\left.\left(\operatorname{id}_{X_{t}} \overline{\times} g_{(t)}\right)\right)^{!} i^{*} \Psi_{f} \xrightarrow[\sim]{\gamma \Psi_{f}} i_{T, t}^{*}\left(g_{X} \overleftarrow{×}_{g} g\right)^{!} \Psi_{f} \xrightarrow{i_{T, t}^{*} \mathrm{BC}_{f, g}^{\prime}} i_{T, t}^{*} \Psi_{f_{T}} g_{X}^{\prime}
$$

where $\gamma:\left(\operatorname{id}_{X_{t}} \overline{\times} g_{(t)}\right)^{!} \cdot i^{*} \xrightarrow{\sim} i_{T, t}^{*}\left(g_{X} \overleftarrow{\times}_{g} g\right)^{!}$and $\mathrm{BC}_{f, g}^{!}:\left(g_{X} \overleftarrow{\times}_{g} g\right)^{!} \Psi_{f} \rightarrow \Psi_{f_{T}} g_{X}^{\prime}$. For $g$ a closed immersion, these maps are given by $\beta^{\prime-1}$ and $\alpha^{\prime-1}$, where $\alpha^{\prime}$ and $\beta^{\prime}$ are as in Remark 4.14. For $g$ smooth of dimension $d$, we define $\left(g_{X} \overleftarrow{\times}_{g} g\right)^{!}:=\left(g_{X} \overleftarrow{\times}_{g} g\right)^{*}(d)[2 d], \gamma$ is the obvious isomorphism, and $\mathrm{BC}_{f, g}^{!}$is induced by $\mathrm{BC}_{f, g}$.

It would be nice to extend this interpretation of $\mathrm{BC}_{f, g}^{\prime t}$ to more general $g$.

In the setting of Construction 4.20 , for $L \in D^{+}(X, \Lambda)$, we say that $\Psi_{f} L$ commutes with sliced !-base change by $g$ if $\mathrm{BC}_{f, g}^{!t}$ is an isomorphism for all geometric points $t$ of $T$. We say that $\Psi_{f} L$ commutes with sliced !-base change if it commutes with sliced !-base change by $g$ for all $g$ separated and of finite presentation. The following lemma is immediate from the construction of $\mathrm{BC}_{f, g}!$.
Lemma 4.22. $\Psi_{f} L$ commutes with smooth base change if and only if $\Psi_{f} L$ commutes with sliced !-base change.

## Lemma 4.23.

(1) If $\Psi_{f} L$ commutes with smooth base change, then for every $g: T \rightarrow S$ separated of finite presentation, $\Psi_{f_{T}} g_{X}^{!} L$ commutes with smooth base change.
(2) If for some $g$ finite surjective of finite presentation, $\Psi_{f_{T}} g_{X} L$ commutes with smooth base change, then $\Psi_{f} L$ commutes with smooth base change.

Part (2) is a dual version of [O, Lemme 3.3].
Proof. By Lemma 4.22, we may replace smooth base change by sliced !-base change. Then (1) follows from the compatibility of the sliced !-base change map with composition. For (2), let $h: S^{\prime} \rightarrow S$ be a separated morphism of finite presentation and form the Cartesian square


By the compatibility of the sliced !-base change map with composition, for every geometric point $t^{\prime} \rightarrow T^{\prime}$, we have

$$
\left(\mathrm{BC}_{f_{S^{\prime}, g^{\prime}}^{!} h_{X}^{\prime}}^{!}\right)\left(\left(\mathrm{id} \overline{\times} g_{\left(t^{\prime}\right)}^{\prime}\right) \mathrm{BC}_{f, h}^{!s^{\prime}}\right)=\left(\mathrm{BC}_{f_{T}, h_{T}}^{!t^{\prime}} g_{X}^{!}\right)\left(\left(\mathrm{id} \overline{\times}\left(h_{T}\right)_{\left(t^{\prime}\right)}\right) \mathrm{BC}_{f, g}^{!t}\right)
$$

where $s^{\prime} \rightarrow S^{\prime}$ and $t \rightarrow T$ denote the images of $t^{\prime} \rightarrow T^{\prime}$. Since $\mathrm{BC}_{f_{S^{\prime}}, g^{\prime}}^{!t^{\prime}}, \mathrm{BC}_{f, g}^{!t}, \mathrm{BC}_{f_{T}, h_{T}}^{!t^{\prime}}\left(g_{X}^{!} L\right)$ are isomorphisms, and the family of functors $\left(\mathrm{id} \overline{\times} g_{\left(t^{\prime}\right)}^{\prime}\right)^{!}, t^{\prime}$ above a fixed $s^{\prime}$, is conservative by Lemma 4.24 (1) below, $\mathrm{BC}_{f, h}^{!s^{\prime}} L$ is an isomorphism.

Lemma 4.24. Let $X$ be a locally coherent topos and $g: T \rightarrow S$ a separated surjective morphism of finite presentation of schemes. Let $\Lambda$ be a commutative ring. Assume either $g$ quasi-finite or $\Lambda$ of torsion.
(1) For any $L \in D(X \overline{\times} S, \Lambda)$, $\left(\mathrm{id}_{X} \overline{\times} g\right)^{!} L \in D^{\geq a}$ implies $L \in D^{\geq a}$. In particular, $\left(\mathrm{id}_{X} \overline{\times} g\right)^{!}: D(X \overline{\times}$ $S, \Lambda) \rightarrow D(X \overline{\times} T, \Lambda)$ is conservative.
(2) Assume moreover that $g$ is proper and $m \Lambda=0$ for $m$ invertible on $S$. Then for every geometric point $s \rightarrow S$, the family of functors $\left(\mathrm{id}_{X} \overline{\times} g_{(t)}\right)!: D\left(X \overline{\times} S_{(s)}, \Lambda\right) \rightarrow D\left(X \overline{\times} T_{(t)}, \Lambda\right)$ is conservative. Here $t$ runs through geometric points of $T$ above $s \rightarrow S$.
Proof. For (1), let $S^{\prime} \subseteq S$ be the smallest closed subset such that $\tau^{\leq a-1} L$ is supported on $X \overline{\times} S^{\prime}$. Assume $S^{\prime}$ nonempty. By [EGAIV, Proposition 17.16.4], the base change of $g$ to $S^{\prime}$ admits generically a quasi-section: there exist a nonempty open subscheme $U \subseteq S^{\prime}$ and a commutative diagram

where $g^{\prime}$ is finite surjective of finite presentation. Let $L^{\prime}=\left.L\right|_{X \overline{\times} U}$. Since $i_{T} h$ is quasi-finite, we have $\left(\operatorname{id}_{X} \overline{\times} g^{\prime}\right)^{!} L^{\prime} \simeq\left(\operatorname{id}_{X} \overline{\times} i_{T} h\right)^{!}\left(\operatorname{id}_{X} \overline{\times} g\right)^{!} L \in D^{\geq a}$, so that

$$
R \mathcal{H o m}\left(\left(\operatorname{id}_{X} \overline{\times} g^{\prime}\right)_{*} \Lambda, L^{\prime}\right) \simeq\left(\operatorname{id}_{X} \overline{\times} g^{\prime}\right)_{*}\left(\operatorname{id}_{X} \overline{\times} g^{\prime}\right)^{!} L^{\prime} \in D^{\geq a}
$$

Moreover, $\left(\operatorname{id}_{X} \overline{\times} g^{\prime}\right)_{*} \Lambda \simeq p_{2}^{*} g_{*}^{\prime} \Lambda$, where $p_{2}: X \overline{\times} S^{\prime} \rightarrow S^{\prime}$ is the projection. The stalks of $g_{*}^{\prime} \Lambda$ are nonzero finite free $\Lambda$-modules. Up to shrinking $U$, we may assume that $g_{*}^{\prime} \Lambda$ is locally constant. It follows that $\left.L\right|_{X \bar{x} U} \in D^{\geq a}$, contradicting the minimality of $S^{\prime}$. Therefore, $S^{\prime}$ is empty and $L \in D^{\geq a}$.

For (2), we may assume that $S$ is strictly local of closed point $s$. Then, by Remark4.17, (id $\left.\overline{\times} g_{(t)}\right)^{!}=$ $\left(\operatorname{id} \overline{\times} j_{t}\right)^{*}(\operatorname{id} \overline{\times} g)^{!}$, where $j_{t}: T_{(t)} \rightarrow T$ is the strict localization. It suffices to show that the family $\left(j_{t}\right)$ is surjective. Let $y \in T$ be a point. Since $g$ is a closed map, $y$ specializes to a point $z \in g^{-1}(s)$. Let $t \rightarrow T$ be a geometric point above $z$. Since $j_{t}$ is flat, hence generizing by [EGAIV, Proposition 2.3.4], $y$ belongs to the image of $j_{t}$.

Remark 4.25. If $S$ is Noetherian finite-dimensional, Constructions 4.15 and 4.20 extend to the unbounded derived categories, as the functors have finite cohomological dimensions [ILO, XVIII A Corol- $^{\text {Cl }}$ lary 1.4].

## 4.5 $\Psi$-goodness and weak $\Psi$-goodness

Let $f: X \rightarrow S$ be a morphism of schemes and let $L \in D(X, \Lambda)$. Following [I3, Appendix] we say that $(f, L)$ is $\Psi$-good if $\Psi_{f} L$ commutes with base change. Examples (3) through (5) below are not needed in this paper.

## Example 4.26.

(1) For $L \in D^{+}(X, \Lambda),(f, L)$ is universally locally acyclic if and only if $(f, L)$ is $\Psi$-good and $R \Phi_{f} L=0$ (Remark 4.6).
(2) Assume $m \Lambda=0$ for some $m$ invertible on $S$. Assume that $S$ has only finitely many irreducible components and for every modification $r: S^{\prime} \rightarrow S$, there exists a finite surjective morphism $T \rightarrow S$ that factors through $r$. Then all pairs $(f, L)$ with $L \in D^{+}(X, \Lambda)$ are $\Psi$-good. In particular, if $(f, L)$ is locally acyclic, then $(f, L)$ is universally locally acyclic (compare with Corollary 6.6).
The condition on $S$ holds if (a) $S$ is the spectrum of a valuation ring, or (b) $S$ is Noetherian of dimension $\leq 1$. In case (a), $r$ admits a section by valuative criterion. In case (b), $r$ is quasi-finite by dimension formula [EGAIV, (5.6.5.1)], hence finite.
To show the $\Psi$-goodness, we reduce by standard limit arguments to the case $f$ of finite presentation, $\Lambda=\mathbb{Z} / m \mathbb{Z}$, and $L \in \operatorname{Shv}_{c}$. We conclude by Orgogozo's theorem (Theorem 4.27 (1) below), the assumption on $S$, and [O, Lemme 3.3].
(3) Assume $f$ of finite type and $\Lambda$ of torsion. Assume that there exists an open immersion $j: U \rightarrow X$ with complement $Y=X-U$ quasi-finite over $S$ such that $\left(f j,\left.L\right|_{U}\right)$ is $\Psi$-good. Then $(f, L)$ is $\Psi$ good. This extends [0, Proposition 6.1] and the proof is similar. Let us show more generally that $\left.\mathrm{BC}_{f, g}^{t}(L)\right|_{y \overline{\times} T_{(t)}}$ is an isomorphism for every isolated point $y$ of $Y_{t}$ (without assuming $Y$ quasifinite over $S$. For this, we may assume $f$ proper. By Lemma $4.12(3), R\left(f_{t} \overline{\times} \mathrm{id}\right)_{*} \mathrm{BC}_{\underline{f}, g}(L)$ is an isomorphism. It then suffices to note that the cone of $\mathrm{BC}_{f, g}^{t}(L)$ is supported on $Y_{t} \overline{\times} T_{(t)}$ by assumption.
(4) Assume $\Lambda$ of torsion. Let $f_{i}: X_{i} \rightarrow S, i=1,2$ be morphisms locally of finite type. Then for $\left(f_{i}, L_{i}\right) \Psi$-good with $L_{i} \in D^{-}\left(X_{i}, \Lambda\right),\left(f_{1} \times_{S} f_{2}, L_{1} \boxtimes^{L} L_{2}\right)$ is $\Psi$-good by the Künneth formula for $R \Psi$ [I3, Theorem A.3].
(5) Assume $\Lambda$ of torsion and $f$ locally of finite type. Then for $(f, L) \Psi$-good with $L \in D^{-}(X, \Lambda)$, $\left(f, L \otimes^{L} f^{*} M\right)$ is $\Psi$-good for all $M \in D(S, \Lambda)$. This follows from the projection formula for $R \Psi$ [I3, Proposition A.6].

Let us recall the following form of Orgogozo's theorem (O).
Theorem 4.27 (Orgogozo). Assume $S$ has only finitely many irreducible components. Assume $f$ of finite presentation, and $\Lambda$ Noetherian such that $m \Lambda=0$ for some $m$ invertible on $S$.
(1) For $L \in D_{c}^{b}(X, \Lambda)$, there exists a modification $g: T \rightarrow S$ such that $\left(f_{T}, g_{X}^{*} L\right)$ is $\Psi$-good.
(2) For $L \in D_{c}(X, \Lambda)$, if $(f, L)$ is $\Psi$-good, then $R \Psi_{f} L \in D_{c}\left(X \overleftarrow{\times}_{S} S, \Lambda\right)$.

A sheaf of $\Lambda$-modules $\mathcal{F}$ on $X \overleftarrow{\times}_{S} S$ is said to be constructible if every stalk is finitely generated and there exist finite partitions $X=\bigcup_{i} X_{i}, S=\bigcup_{j} S_{j}$ into disjoint constructible locally closed subsets such that the restriction of $\mathcal{F}$ to each $X_{i} \overleftarrow{\times}_{S} S_{j}$ is locally constant.

Proof. The results of [O] are stated only for the case $\Lambda=\mathbb{Z} / m \mathbb{Z}$, but as remarked in [I3, Theorem 1.6.1], the proofs can be easily adapted to the case of general $\Lambda$ as above. Indeed, in the last paragraph of [O, Section 4.4], we reduce to the case where $\mathcal{F}$ is constant instead of $\mathcal{F}=\Lambda$, and at the end of the proofs [O, Sections $5.1,10.2,10.3]$ we work with constant constructible sheaves instead of $\Lambda$.
(1) is [O, Théorème 2.1] (for general $\Lambda$ ). For (2), by ©, Lemme 10.5], it suffices to show that for each $i$, there exists a proper surjective morphism $T \rightarrow S$ such that $\left.\left(R^{i} \Psi_{f} L\right)\right|_{X_{T}} \overleftarrow{\times}_{T} T$ is constructible For $(f, L) \Psi$-good, we have $\left.\left(R^{i} \Psi_{f} L\right)\right|_{X_{T} \overleftarrow{×}_{T} T} \simeq R^{i} \Psi_{f_{T}}\left(\left.L\right|_{X_{T}}\right)$. By [O, Proposition 3.1], there exists a truncation $L^{\prime} \in D_{c}^{b}(X, \Lambda)$ of $L$ such that $R^{i} \Psi_{f_{T}}\left(\left.L^{\prime}\right|_{X_{T}}\right) \simeq R^{i} \Psi_{f_{T}}\left(\left.L\right|_{X_{T}}\right)$ for all $T \rightarrow S$. Then it suffices to apply ©, Théorème 8.1] to $L^{\prime}$.

Remark 4.28. It follows from Theorem4.27 (1) that for $L \in D_{c}^{b}(X, \Lambda)$, if $R \Psi_{f} L$ commutes with base change by modifications, then it commutes with any base change.

Assume $S$ Noetherian, $f$ of finite type, and $\Lambda$ Noetherian such that $m \Lambda=0$ for some $m$ invertible on $S$.

Corollary 4.29. For $L \in D_{c}^{b}(X, \Lambda)$, there exists an open subscheme $U \subseteq S$ of complement of codimension $\geq 2$ such that $\left(f_{U},\left.L\right|_{X_{U}}\right)$ is $\Psi$-good.

Proof. There exists an open subscheme $U \subseteq S$ of complement of codimension $\geq 2$ such that the restriction of the modification $g$ in Theorem4.27(1) to $U$ is quasi-finite, by Chevalley's semicontinuity theorem and dimension formula [EGAIV, (5.6.5.1)], hence finite. We then conclude by [O, Lemme 3.3].

Definition 4.30. For $L \in D_{c}(X, \Lambda)$, we say that $(f, L)$ is weakly $\Psi$-good if $\Psi_{f}^{s}(L)$ belongs to $D_{c}\left(X_{s} \overline{\times}\right.$ $\left.S_{(s)}, \Lambda\right)$ for every geometric point $s \rightarrow S$ and $\Psi_{f}(L)$ commutes with smooth base change.

By Theorem $4.27(2)$, for $L \in D_{c}$, if $(f, L)$ is $\Psi$-good, then $(f, L)$ is weakly $\Psi$-good. We do not know whether the converse holds.
Remark 4.31. Consider a Cartesian square (4.18) with $g$ finite. Let $M \in D_{c}\left(X_{T}, \Lambda\right)$. If $\left(f_{T}, M\right)$ is weakly $\Psi$-good, then $\left(f, g_{X *} M\right)$ is weakly $\Psi$-good. Indeed, $\Psi_{g f_{T}}(M)$ commutes with smooth base change by Lemma 4.12 (1) and $\Psi_{f}\left(g_{X *} M\right)$ commutes with smooth base change by Lemma 4.12 (4), and $\Psi_{f}^{s}\left(g_{X *} M\right)$ is in $D_{c}$ by Lemma 4.13.

The following is an analogue of Orgogozo's theorem for !-pullback. The quasi-excellence of $S$ (not needed in Orgogozo's theorem) ensures that $p_{X}^{!} L \in D_{c}^{b}$.

Theorem 4.32. Assume $S$ quasi-excellent. Let $f: X \rightarrow S$ be a morphism of finite type and let $L \in D_{c}^{b}(X, \Lambda)$. There exists a modification $p: S^{\prime} \rightarrow S$ such that $\left(f_{S^{\prime}}, p_{X}^{!} L\right)$ is weakly $\Psi$-good.

Our proof of the theorem relies on the following preliminary results on !-pullback.
Lemma 4.33. Assume $S$ quasi-excellent. Let $f: X \rightarrow S$ be a morphism of finite type and let $L \in$ $D_{c}^{+}(X, \Lambda)$.
(1) If $(f, L)$ is weakly $\Psi$-good, then for every $g: T \rightarrow S$ separated of finite type, $\left(f_{T}, g_{X}^{!} L\right)$ is weakly $\Psi$-good.
(2) If for some $g: T \rightarrow S$ finite surjective, $\left(f_{T}, g_{X}^{!} L\right)$ is weakly $\Psi$-good, then $(f, L)$ is weakly $\Psi$-good.

Proof. The commutation with smooth base change follows from Lemma 4.23, In (1) and (2), for every geometric point $t \rightarrow T$ above $s \rightarrow S$, we have $\Psi_{f_{T}}^{t}\left(g_{X}^{!} L\right) \simeq\left(\operatorname{id} \overline{\times} g_{(t)}\right)^{!} \Psi_{f}^{s} L$. Then (1) follows from the fact that $\left(\operatorname{id} \overline{\times} g_{(t)}\right)$ ! preserves $D_{c}^{+}$(Proposition 1.35), and (2) follows from the following lemma.

Lemma 4.34. Let $X$ be a quasi-excellent scheme. Let $g: T \rightarrow S$ be a separated surjective morphism of finite type between quasi-excellent schemes. Let $\Lambda$ be a Noetherian commutative ring such that $m \Lambda=0$ for some $m$ invertible on $X$ and on $S$. Let $L \in D^{+}(X \overline{\times} S, \Lambda)$ such that $R\left(\operatorname{id}_{X} \overline{\times} g\right)^{!} L \in D_{c}^{+}(X \overline{\times} T, \Lambda)$. Then $L \in D_{c}^{+}$.

Proof. The proof is similar to that of Lemma 4.24 (1). Let $S^{\prime} \subseteq S$ be the smallest closed subset such that $\left.L\right|_{X \overline{\times}\left(S-S^{\prime}\right)}$ belongs to $D_{c}^{+}$. Assume $S^{\prime}$ nonempty. Consider the diagram (4.34) and let $L^{\prime}=R\left(\operatorname{id}_{X} \overline{\times} i j\right)!L$. We have $R \mathcal{H}$ om $\left(\left(\operatorname{id}_{X} \overline{\times} g^{\prime}\right)_{*} \Lambda, L^{\prime}\right) \simeq\left(\operatorname{id}_{X} \overline{\times} g^{\prime}\right)_{*} R\left(\mathrm{id}_{X} \overline{\times} g^{\prime}\right)^{!} L^{\prime} \simeq\left(\operatorname{id}_{X} \overline{\times} g^{\prime}\right)_{*} R\left(\operatorname{id}_{X} \overline{\times}\right.$ $\left.i_{T} h\right)^{!} R\left(\operatorname{id}_{X} \overline{\times} g\right)^{!} L$, which belongs to $D_{c}^{+}$by the assumptions. Recall that $g^{\prime}$ is finite and surjective. The stalks of $g_{*}^{\prime} \Lambda$ are nonzero finite free $\Lambda$-modules. Up to shrinking $U$, we may assume $g_{*}^{\prime} \Lambda$ locally constant. Then $L^{\prime} \in D_{c}^{+}$. Let $S^{\prime \prime}=S^{\prime}-U$ and let $u: U \rightarrow S-S^{\prime \prime}$ and $v: S-S^{\prime} \rightarrow S-S^{\prime \prime}$ be the inclusions. The distinguished triangle

$$
\left.\left.\left(\operatorname{id}_{X} \overline{\times} u\right)_{*} L^{\prime} \rightarrow L\right|_{X \overline{\times}\left(S-S^{\prime \prime}\right)} \rightarrow R\left(\operatorname{id}_{X} \overline{\times} v\right)_{*} L\right|_{X \overline{\times}\left(S-S^{\prime}\right)} \rightarrow\left(\operatorname{idd}_{X} \overline{\times} u\right)_{*} L^{\prime}[1]
$$

implies $\left.L\right|_{X \bar{x}\left(S-S^{\prime \prime}\right)} \in D_{c}^{+}$, contradicting the minimality of $S^{\prime}$. Thus $S^{\prime}$ is empty and $L \in D_{c}^{+}$.
Proof of Theorem 4.32. We prove first the case $L=i_{X *} M$, where $i: F \subseteq S$ is a closed immersion with $F$ reduced and there exists a modification $m: F^{\prime} \rightarrow F$ such that $\left(f_{F^{\prime}}, m_{X}^{\prime} M\right)$ is weakly $\Psi$-good. The proof is similar to [O, Section 4.2]. We construct a commutative diagram of schemes

where $p$ and $\pi_{\text {red }}: T \rightarrow S_{\text {red }}$ are modifications, $q$ and $r$ are finite surjective, and the square in the middle is Cartesian, as follows. Applying [O, Lemme 4.3] to $i_{\text {red }} m$, we get the left and middle squares. Applying [O, Lemme 3.2] to $\pi$, we get the right square. By Lemma 4.33 (1), ( $\left.f_{G}, q_{X}^{\prime} \pi_{X_{F}}^{!} M\right)$ is weakly $\Psi$-good. By Lemma $4.33(2),\left(f_{F_{T}}, \pi_{X_{F}}^{!} M\right)$ is weakly $\Psi$-good. By Remark 4.31, $\left(f_{T}, \pi_{X}^{!} L\right)$ is weakly $\Psi$-good, as $\pi_{X}^{!} L \simeq i_{X_{T} * p_{X_{F}}^{\prime}}^{\prime} M$ by base change. By Lemma 4.33(1), $\left(f_{T^{\prime}}, r_{X}^{!} p_{X}^{\prime} L\right)$ is weakly $\Psi$-good. By Lemma 4.33 (2), $\left(f_{S^{\prime}}, p_{X} L\right)$ is weakly $\Psi$-good.

For the general case, by Orgogozo's theorem (Theorem4.27), there exists a modification $g: T \rightarrow S$ such that $\left(f_{T}, g_{X}^{*} L\right)$ is $\Psi$-good, hence weakly $\Psi$-good. Let $j: U \rightarrow T$ be the complement of the exceptional locus $S_{1} \subseteq T$. Then Cone $\left(j_{!}\left(\left.L\right|_{U}\right) \rightarrow g_{X}^{*} L\right)$ and Cone $\left(j_{!}\left(\left.L\right|_{U}\right) \rightarrow g_{X}^{\prime} L\right)$ are supported on $X \times_{S} S_{1}$. By the special case above, we are reduced to proving the theorem for $S_{1}$.

Repeating this process, we obtain a sequence $S=S_{0} \leftarrow S_{1} \leftarrow S_{2} \leftarrow \cdots$, each $S_{i+1}$ being the exceptional locus of a modification of $S_{i}$. It remains to show that $S_{n}$ is empty for $n \gg 0$. Assume the contrary. Up to replacing $S$ by an étale cover, we may assume that $S$ admits a dimension function $\delta_{0}$ [ILO, XIV Théorème 2.3.1]. We equip $S_{n}$ with the induced dimension function $\delta_{n}$ IILO, XIV Corollaire 2.5.2]. There exists a sequence of generic points $\eta_{i}$ of $S_{i}$ such that $\eta_{i}$ specializes to the image of $\eta_{i+1}$ in $S_{i}$. Then $\delta_{i+1}\left(\eta_{i+1}\right)<\delta_{i}\left(\eta_{i}\right)$, so that $\delta_{i}\left(\eta_{i}\right) \rightarrow-\infty$ as $i \rightarrow \infty$. Let $\eta_{i}^{\prime}$ denote the image of $\eta_{i}$ in $S$. Then $\delta_{0}\left(\eta_{i}^{\prime}\right) \leq \delta_{i}\left(\eta_{i}\right)$. Thus $\delta\left(\eta_{i}^{\prime}\right) \rightarrow-\infty$. Each $\eta_{i+1}^{\prime}$ is a specialization of $\eta_{i}^{\prime}$. This contradicts the assumption that $S$ is Noetherian.

Remark 4.35. We originally proved Theorem 4.32 under the additional assumption that $S$ is finitedimensional. The argument of dimension function above which allows to remove this assumption is due to Gabber.

## 5 Nearby cycles and duality

In this section, we prove Theorem 5.1 on the commutation of the sliced nearby cycle functor with duality, and deduce Theorem [0.3, We then give applications to local acyclicity (Corollary 5.13) and singular support (Corollary 5.14).

### 5.1 Duality

Let $S$ be a coherent scheme and let $\Lambda$ be a torsion commutative ring. We fix $K_{S} \in D(S, \Lambda)$, not necessarily dualizing. For $a: X \rightarrow S$ separated of finite type, we take $K_{X}=R a!K_{S}$. For a point $s$
 $K_{S}$. Note that $\Psi_{\mathrm{id}_{S}}^{s} K_{S} \simeq K_{S_{(s)}}$. Applying (4.9) to $b=\mathrm{id}_{S}$ and $K_{S}$, we obtain $\Psi_{a}^{s} K_{X} \rightarrow K_{X_{s} \bar{x}_{s_{0}} S_{(s)}}$. Composing with (4.1), we get a natural transformation

$$
\begin{equation*}
A_{a}^{s}: \Psi_{a}^{s} D_{X} \rightarrow D_{X_{s} \bar{x}_{s_{0}} S_{(s)}} \Psi_{a}^{s} \tag{5.1}
\end{equation*}
$$

where $D_{X}=R \mathcal{H o m}\left(-, K_{X}\right), D_{X_{s} \bar{x}_{s_{0}} S_{(s)}}=\operatorname{RHom}\left(-, K_{X_{s} \bar{x}_{s_{0}} S_{(s)}}\right)$.
By Remark 4.3, $A_{a}^{s}$ is the composition of $i^{*} A_{a}$ with

$$
\begin{equation*}
i^{*} D_{X \overleftarrow{×}_{S} S} \rightarrow D_{X_{s} \bar{x}_{s_{0}} S_{(s)}} i^{*} \tag{5.2}
\end{equation*}
$$

where $i: X_{s} \overline{\times}_{s_{0}} S_{(s)} \rightarrow X \overleftarrow{×}_{S} S$ and

$$
A_{a}: \Psi_{a} D_{X} \xrightarrow{\sim} D_{X \overleftarrow{x}_{S} S} \Psi_{a}
$$

is the trivial duality. Here $D_{X \overleftarrow{×}_{S} S}:=R \mathcal{H o m}\left(-, K_{X \overleftarrow{×}_{S} S}\right), K_{X \overleftarrow{x}_{S} S}:=\Psi_{a} K_{X}$.
In the rest of Subsection 5.1, let $S$ be a Noetherian scheme and let $\Lambda$ be a Noetherian commutative ring with $m \Lambda=0$ for some $m$ invertible on $S$. In this case, $K_{X_{s} \bar{x}_{s_{0}} S_{(s)}} \simeq\left(R a_{s}^{!} \Lambda\right) \boxtimes{ }^{L} K_{S_{(s)}}$ by Corollary 1.26 .

Theorem 5.1. Assume $S$ excellent. Let $a: X \rightarrow S$ be a separated morphism of finite type and let $K \in D_{c}^{b}(S, \Lambda)$. Let $L \in D_{c}^{-}(X, \Lambda)$ such that $(a, L)$ is $\Psi$-good and $\Psi_{a}\left(D_{X} L\right)$ commutes with smooth base change. Then, for every point $s$ of $S$ with values in a field, the map $A_{a}^{s}(L)$ is an isomorphism.

We refer to Example 4.26 for examples of $\Psi$-good pairs. As $D_{X}=R \mathcal{H} o m\left(-, a^{!} K_{S}\right)$, the theorem can be seen as a dual of the projection formula for $R \Psi$ [I3, Proposition A.6]: if $\Psi_{a} L$ commutes with finite base change, then

$$
R \Psi_{a} L \otimes^{L} p_{2}^{*} M \xrightarrow{\sim} R \Psi_{a}\left(L \otimes^{L} a^{*} M\right)
$$

Remark 5.2.
(1) If $(a, L)$ satisfies the assumptions of the theorem, then for any morphism $g: T \rightarrow S$ separated and of finite type, the same holds for $\left(a_{T}, g_{X}^{*} L\right)$. Indeed, $R \Psi_{a_{T}}\left(D_{X_{T}} g_{X}^{*} L\right)$ commutes with smooth base change by Lemma 4.23 (1), since $D_{X_{T}} g_{X}^{*} L \simeq g_{X}^{!} D_{X} L$.
(2) If $X \xrightarrow{f} Y \xrightarrow{b} S$ are separated morphisms of finite type with $f$ proper and if ( $b f, L$ ) satisfies the assumptions of the theorem, the same holds for $\left(b, f_{*} L\right)$ by Lemma 4.12 (3).

For the proof of Theorem 5.1, we need the following compatibilities between $A_{a}^{s}$ and the constructions of Subsections 4.3 and 4.5.

Lemma 5.3. For any separated morphism $g: T \rightarrow S$ of finite type and any geometric point $t \rightarrow T$, the diagram
commutes. Here $\delta:\left(\mathrm{id} \overline{\times} g_{(t)}\right)^{!} D_{X_{t} \overline{\times} S_{(t)}} \rightarrow D_{X_{t} \overline{\times} T_{(t)}}\left(\mathrm{id} \overline{\times} g_{(t)}\right)^{*}$ is (4.32).

Proof. We may assume that $g$ is smooth or a closed immersion. By Remark 4.21, with the notation of (4.19), the diagram decomposes into

$$
\begin{aligned}
&\left(\mathrm{id} \overline{\times} g_{(t)}\right)!i^{*} \Psi_{a} D_{X} \xrightarrow{\gamma} i_{T, t}^{*}\left(g_{X} \overleftarrow{\times}_{g} g\right)!\Psi_{a} D_{X} \xrightarrow{\mathrm{BC}_{a, g}^{!}} i_{T, t}^{*} \Psi_{a_{T}} g_{X}^{!} D_{X} \longrightarrow i_{T, t}^{*} \Psi_{a_{T}} D_{X_{T}} g_{X}^{*} \\
& \simeq \downarrow A_{a} \simeq \downarrow A_{a} \\
& \simeq \downarrow A_{a_{T}}
\end{aligned}
$$


(5.2)
$\left(\mathrm{id} \overline{\times} g_{(t)}\right)^{!} D_{X_{t} \overline{\times} S_{(t)}} i^{*} \Psi_{a} \stackrel{\delta}{\longrightarrow} D_{X_{t} \overline{\times} T_{(t)}}\left(\mathrm{id} \overline{\times} g_{(t)}\right)^{*} i^{*} \Psi_{a} \simeq D_{X_{t} \overline{\times} T_{(t)}} i_{T, t}^{*}\left(g_{X} \overleftarrow{×}_{g} g\right)^{*} \Psi_{a}^{\mathrm{BC} \mathrm{BC}_{a, g}} D_{X_{t} \overline{\times} T_{(t)}} i_{T, t}^{*} \Psi_{a_{T}} g_{X}^{*}$,
where $\epsilon:\left(g_{X} \overleftarrow{\times}_{g} g\right)^{!} D_{X \overleftarrow{\times}_{S} S} \rightarrow D_{X_{T} \overleftarrow{\times}_{T} T}\left(g_{X} \overleftarrow{\times}_{g} g\right)^{*}$ is given by

$$
\begin{aligned}
\left(g_{X} \overleftarrow{×}_{g} g\right)^{!} R \mathcal{H} \operatorname{Hom}\left(M, \Psi_{a} K_{X}\right) \stackrel{\zeta}{\rightarrow} R \mathcal{H o m}\left(\left(g_{X} \overleftarrow{×}_{g} g\right)^{*} M,\left(g_{X} \overleftarrow{\times}_{g} g\right)^{!} \Psi_{a} K_{X}\right) \\
\xrightarrow{\mathrm{BC}_{a, g}^{!}\left(K_{X}\right)} \operatorname{RHom}\left(\left(g_{X} \overleftarrow{\times}_{g} g\right)^{*} M, \Psi_{a_{T}} g_{X}^{!} K_{X}\right)
\end{aligned}
$$

Here $\zeta$ is the restriction map for $g$ smooth and the canonical isomorphism for $g$ a closed immersion (in this case, as in the proof of Lemma 4.13, $g_{X} \overleftarrow{\times}_{g} g$ can be identified with the closed immersion $\operatorname{id}_{X} \overleftarrow{×}_{S} g$ ). The inner squares commute by construction.

Remark 5.4. We have $R\left(\operatorname{id} \overline{\times} g_{(t)}\right)!K_{X_{t} \overline{\times} S_{(t)}} \simeq K_{X_{t} \overline{\times} T_{(t)}}$ by Remark 4.17. Thus, by Lemma 4.19, $\delta$ is an isomorphism on $D_{c}^{-}$. It follows that for $L$ as in Theorem [5.1, the horizontal arrows of (5.3) are isomorphisms.
Lemma 5.5. Let $X \xrightarrow{f} Y \xrightarrow{b} S$ be morphisms of schemes with $Y$ coherent and $f$ separated of finite type. For any geometric point $s \rightarrow S$, the following diagrams commute:


For $f$ proper (resp. smooth), the horizontal arrows of (5.4) (resp. (5.5)) are isomorphisms by Remark 4.8 (2) (resp. Remark 4.8 (1) and Lemma 4.10).

Proof. By (4.15), with the notation of (4.3), the diagram (5.4) decomposes into

where $\gamma: D_{Y \overleftarrow{×}_{S} S}\left(f \overleftarrow{\times}_{S} \mathrm{id}\right)!\simeq\left(f \overleftarrow{\times}_{S} \mathrm{id}\right)_{*} D_{X} \overleftarrow{\times}_{S} S$ is given by

$$
\begin{aligned}
& R \mathcal{H o m}\left(\left(f \overleftarrow{\times}_{S} \mathrm{id}\right)!N, \Psi_{b} K_{Y}\right) \stackrel{(1.7)}{\sim}\left(f \overleftarrow{\times}_{S} \mathrm{id}\right)_{*} R \mathcal{H o m}\left(N,\left(f \overleftarrow{\times}_{S} \mathrm{id}\right)^{!} \Psi_{b} K_{Y}\right) \\
& \xrightarrow[\sim]{\alpha^{-1}\left(K_{Y}\right)}\left(f \overleftarrow{\times}_{S} \mathrm{id}\right)_{*} R \mathcal{H o m}\left(N, \Psi_{b f} f^{!} K_{Y}\right)
\end{aligned}
$$

Here $\alpha$ is defined in (4.13). The inner squares commute by construction.
The commutativity of (5.5) can be proved similarly, using (4.16).
Proof of Theorem 5.1. Parts of the proof are similar to [O, Sections 4.4, 5.1], but the third induction step below is in the opposite direction. By Lemma 1.29, up to replacing $S$ by the strict localization at a geometric point above $s$, we may assume that $S$ is finite-dimensional and $s$ is a geometric point. By Lemmas 4.24 (2), 5.3 and Remark 5.4, if the theorem holds for $\left(a_{T}, g_{X}^{*} L\right)$ for a proper surjective morphism $g: T \rightarrow S$, then it holds for $(a, L)$. In particular, we may alter $S$.

We proceed by a triple induction. First we proceed by induction on the dimension $d_{S}$ of $S$. Up to replacing $S$ by an irreducible component, we may assume $S$ integral, of generic point $\eta$. For $S$ empty the assertion is trivial. For each $d_{S} \geq 0$, we proceed by induction on the dimension $d_{X_{\eta}}$ of the generic fiber $X_{\eta}$ of $a$. Note that the assertion holds if $X_{\eta}$ is empty. Indeed, if $T \subseteq S$ denotes the schematic image of $a$, then for $s \rightarrow S-T$ the source and target of $A_{a}^{s}(L)$ are both zero. For $t \rightarrow T$, the source and target of $A_{a}^{t}(L)$ are supported on $X_{t} \overline{\times} T_{(t)}$. By Lemma 5.3, the restriction of $A_{a}^{t}(L)$ to $X_{t} \overline{\times} T_{(t)}$ can be identified with $A_{a_{T}}^{t}(L)$, which is an isomorphism by the induction hypothesis on $d_{S}$.

Assume $d_{X_{\eta}} \geq 0$. Note that if $L$ is supported on $Y \subseteq X$ such that $Y_{\eta} \subseteq X_{\eta}$ is nowhere dense, then the assertion holds. Indeed, if $L=i_{*} M$, where $i: Y \rightarrow X$ is the inclusion, then $A_{a}^{s}(L)$ can be identified with $\left(i_{s} \overline{\times} \mathrm{id}\right)_{*} A_{a i}^{s}(M)$ by Lemma [5.5, and the induction hypothesis on $d_{X_{\eta}}$ applies to ai. This applies in particular if $Y \subseteq X$ is nowhere dense.

For any alteration $g: T \rightarrow S, g_{!}$has cohomological amplitude $\leq 2 d_{g}$, where $d_{g}$ denotes the maximum dimension of the fibers of $g$, so that $g$ ! has cohomological amplitude $\geq-2 d_{g} \geq-2 d_{S}$ by dimension formula EGAIV, (5.6.5.1)]. Thus, up to replacing $K$ by a shift, we may assume that for every alteration $g: T \rightarrow S, g^{!} K_{S} \in D^{\geq 0}$. We proceed by induction on $n$ to show that for every $L \in D_{c}^{\leq 0}(X, \Lambda)$ satisfying the assumptions of the theorem, the cone of $A_{a}^{s}(L)$ belongs to $D^{\geq n}$, and the same holds with $S$ replaced by an alteration. As above, $a^{!}$has cohomological amplitude $\geq-2 d_{a}$, where $d_{a}$ denotes the maximum dimension of the fibers of $a$. Thus the source and target of $A_{a}^{s}(L)$ both belong to $D^{\geq-2 d_{a}}$, and the assertion is trivial for $n=-2 d_{a}-1$.

Recall that every $C \in \operatorname{Shv}_{c}(X, \Lambda)$ is Noetherian. Thus, for any epimorphism $M \rightarrow C$, there exists $u_{!} \Lambda \rightarrow M$ with $u: U \rightarrow X$ étale, separated and of finite type, such that the composition $u_{!} \Lambda \rightarrow C$ is an epimorphism. By [KS, Lemma 13.2.1 (b)], we may assume that $L^{q}=0$ for $q>0$ and $L^{0}$ has the form $u_{!} \Lambda$. Note that $D_{X}\left(u_{!} \Lambda\right) \simeq u_{*} K_{U} \in D_{c}^{b}$. By Orgogozo's theorem (Theorem 4.27) and Theorem 4.32, up to modifying $S$, we may assume that $\left(a, u_{!} \Lambda\right)$ satisfies the assumptions of the theorem. It follows that $\left(a, \sigma^{\leq-1} L\right)$ satisfies the assumptions of the theorem, where $\sigma^{\leq-1} L$ denotes the naive truncation of $L$. By induction hypothesis on $n$, we have $\operatorname{Cone}\left(A_{a}^{s}\left(\sigma^{\leq-1} L\right)\right) \in D^{\geq n}$. Thus it suffices to show that $A_{a}^{s}(u!\Lambda)$ is an isomorphism.

Choose a compactification $X \xrightarrow{f} Y \xrightarrow{b} S$ of $a$ with $\operatorname{dim} Y_{\eta}=\operatorname{dim} X_{\eta}$. By Zariski's main theorem, $f u$ admits a factorization $U \xrightarrow{j} V \xrightarrow{v} Y$ with $v$ finite and $j$ a dominant open immersion. Let $V^{\prime}$ be the disjoint union of the irreducible components of $V$ and let $v^{\prime}: V^{\prime} \rightarrow X$. Then the composition $(f u)_{!} \Lambda \rightarrow v_{*} \Lambda \rightarrow v_{*}^{\prime} \Lambda$ is a monomorphism and the cokernel is supported on a nowhere dense closed subset. Up to modifying $S$, we may assume that $(b,(f u)!\Lambda),\left(b v^{\prime}, \Lambda\right)$ and (consequently) $\left(b, v_{*}^{\prime} \Lambda\right)$ satisfy the assumptions of the theorem. By Lemma 5.5, $A_{a}^{s}\left(u_{!} \Lambda\right)$ can be identified with $\left(f_{s} \overline{\times} \mathrm{id}\right)^{*} A_{b}^{s}((f u)!L)$. Thus it suffices to show that $A_{b}^{s}\left(v_{*}^{\prime} \Lambda\right)$ is an isomorphism. By Lemma 5.5, $A_{b}^{s}\left(v_{*}^{\prime} \Lambda\right)$ can be identified with $\left(v_{s}^{\prime} \times \mathrm{id}\right)_{*} A_{b v^{\prime}}^{s}(\Lambda)$. Changing notation, we are reduced to showing that $A_{a}^{s}(\Lambda)$ is an isomorphism for $a$ proper, $X$ integral, and $(a, \Lambda)$ satisfying the assumptions of the theorem.

By [O, Lemme 4.7], up to altering $S$, there exists an alteration $X^{\prime} \amalg X^{\prime \prime} \rightarrow X$ such that $X^{\prime} \rightarrow S$ is a plurinodal morphism and $X^{\prime \prime} \rightarrow S$ is non-dominant. Let $p: X^{\prime} \rightarrow X$. There exists an open immersion $j: U \rightarrow X$ with $j_{\eta}$ dominant such that $p_{U}$ is finite flat and surjective. Consider the maps


The horizontal map is induced by the trace map and is surjective. The cones of the vertical maps are supported on closed subsets of $X$ having nowhere dense intersections with $X_{\eta}$. Up to modifying $S$, we may assume that the pairs $(a, j!\Lambda),\left(a, j!p_{U *} \Lambda\right),(a p, \Lambda)$ and (consequently, by Remark 5.2 (2)) $\left(a, R p_{*} \Lambda\right)$ satisfy the assumptions of the theorem. Thus it suffices to show that $A_{a}^{s}\left(R p_{*} \Lambda\right)$ is an isomorphism. By Lemma 5.5, $A_{a}^{s}\left(R p_{*} \Lambda\right)$ can be identified with $\left(p_{s} \times \mathrm{id}\right)_{*} A_{a p}^{s}(\Lambda)$. Changing notation, we are reduced to showing that $A_{a}^{s}(\Lambda)$ is an isomorphism for $a$ plurinodal and $(a, \Lambda)$ satisfying the assumptions of the theorem.

If $d_{X_{\eta}}=0$, then $a$ is an isomorphism and $\Psi_{a}^{s}$ is the restriction from $S$ to $S_{(s)}$, so that $A_{a}^{s}(\Lambda)$ can be identified with the identity on $K_{S_{(s)}}$. Assume $d_{X_{\eta}} \geq 1$. Then $a$ decomposes into $X \xrightarrow{f} Y \xrightarrow{b} S$, where $\operatorname{dim}\left(Y_{\eta}\right)=d_{X_{\eta}}-1$ and $f$ is a projective flat curve with geometric fibers having at most ordinary quadratic singularities. Up to modifying $S$, we may assume that $(b, \Lambda)$ satisfies the assumptions of the theorem. By Lemma [5.5, $\left(f_{s} \overline{\times} \mathrm{id}\right)_{*} A_{a}^{s}(\Lambda)$ can be identified with $A_{b}^{s}\left(f_{*} \Lambda\right)$. Since $\left(b, f_{*} \Lambda\right)$ satisfies the assumption of the theorem, $A_{b}^{s}\left(f_{*} \Lambda\right)$ is an isomorphism by the induction hypothesis on $d_{X_{\eta}}$, so that it suffices to show that $\operatorname{Cone}\left(A_{a}^{s}(\Lambda)\right)$ is supported on $(X-U)_{s} \overline{\times} S_{(s)}$, where $U$ is the smooth locus of $f$ (with $X-U$ finite over $Y$ ). By Lemma [5.5, on $U_{s} \overline{\times} S_{(s)}, A_{a}^{s}(\Lambda)$ coincides with $\left(f_{s} \overline{\times} \mathrm{id}\right)^{*} A_{b}^{s}(\Lambda)(1)[2]$, which is an isomorphism by the induction hypothesis on $d_{X_{\eta}}$.

Proof of Theorem 0.3. By Orgogozo's theorem (Theorem4.27) and Theorem4.32, there exists a modification $g: S^{\prime} \rightarrow S$ such that $\left(a_{S^{\prime}},\left.L\right|_{X_{S^{\prime}}}\right)$ is $\Psi$-good and $\left(a_{S^{\prime}}, g_{X}^{!} D_{X} L\right)$ is weakly $\Psi$-good. Since $D_{X_{S^{\prime}}}\left(\left.L\right|_{X_{S^{\prime}}}\right) \simeq g_{X}^{!} D_{X} L$, Theorem 5.1] applies to $\left(a_{T},\left.L\right|_{X_{T}}\right)$ for any $T \rightarrow S^{\prime}$ separated of finite type by Remark 5.2 (1).

Corollary 5.6. Under the assumptions of Theorem 0.3, there exists an open subscheme $U \subseteq S$ of complement of codimension $\geq 2$, such that for every morphism $T \rightarrow U$ separated of finite type, and for every point $t \in T$, the map $A_{a_{T}}^{t}\left(\left.L\right|_{X_{T}}\right)$ is an isomorphism.

Proof. By Corollary 4.29, there exists an open subscheme $U \subseteq S$ of complement of codimension $\geq 2$, such that $\left(a_{U},\left.L\right|_{X_{U}}\right)$ and $\left(a_{U},\left.\left(D_{X} L\right)\right|_{X_{U}}\right)$ are $\Psi$-good and weakly $\Psi$-good. Since $D_{X_{U}}\left(\left.L\right|_{X_{U}}\right) \simeq$ $\left.\left(D_{X} L\right)\right|_{X_{U}}$, we then conclude by Theorem 5.1 and Remark 5.2 (1).

In the case where $K_{S}$ is a dualizing complex, Theorem 5.1 has the following dual.
Corollary 5.7. Assume $S$ excellent equipped with a dimension function and let $K_{S}$ be a dualizing complex for $D_{c \mathrm{ft}}(S, \Lambda)$. Let $a: X \rightarrow S$ be a separated morphism of finite type and let $L \in D_{c}^{b}(X, \Lambda)$ such that $(a, L)$ is weakly $\Psi$-good and $\left(a, D_{X} L\right)$ is $\Psi$-good. Assume either $L \in D_{c \mathrm{ft}}$ or $\Lambda$ Gorenstein. Then for every point $s$ of $S$ with values in a field, $A_{a}^{s}(L)$ is an isomorphism.

Proof. The assumption that $L \in D_{\text {cft }}$ or $\Lambda$ Gorenstein implies that $L \rightarrow D_{X} D_{X} L$ is an isomorphism and $M \rightarrow D_{X_{s}} \bar{x}_{s_{0}} S_{(s)} D_{X_{s}} \bar{x}_{s_{0}} S_{(s)} M$ is an isomorphism for $M=\Psi_{a}^{s} D_{X} L$ by Proposition 1.37, (For $L \in D_{c \mathrm{ft}}, D_{X} L \in D_{c \mathrm{ft}}$, hence $M \in D_{c \mathrm{ft}}$ by [O, Remarque 8.3].) Thus $A_{a}^{s}\left(D_{X} L\right)$ is an isomorphism Theorem 5.1, The corollary follows from the following formal result (see for example [SZ, Constructions A.4.5, A.4.6]).

Lemma 5.8. The square

where the vertical arrow are induced by the evaluation maps id $\rightarrow D D$, is commutative.
In the case where $K_{S}$ is a dualizing complex, Theorem 0.3 has the following dual.
Corollary 5.9. Assume $S$ excellent equipped with a dimension function and let $K_{S}$ be a dualizing complex for $D_{\text {cft }}(S, \Lambda)$. Let $a: X \rightarrow S$ be a separated morphism of finite type and let $L \in D_{c}^{b}(X, \Lambda)$. Assume either $L \in D_{\text {cft }}$ or $\Lambda$ Gorenstein. Then there exists a modification $g: S^{\prime} \rightarrow S$ such that for every morphism $T \rightarrow S^{\prime}$ separated of finite type, and for every point $t$ of $T, A_{a_{T}}^{t}\left(h_{X}^{!} L\right)$ is an isomorphism. Here $h$ denotes the composition $T \rightarrow S^{\prime} \xrightarrow{g} S$.

Proof. By Orgogozo's theorem (Theorem4.27) and Theorem4.32, there exists $g$ such that $\left(a_{S^{\prime}}, g_{X}^{\prime} L\right)$ is weakly $\Psi$-good and $\left(a_{S^{\prime}}, g_{X}^{*} D_{X} L\right)$ is $\Psi$-good. Then $\left(a_{T}, h_{X}^{!} L\right)$ is weakly $\Psi$-good and $\left(a_{T}, D_{X_{T}} h_{X}^{!} L\right)$ is $\Psi$-good, as $D_{X_{T}} h_{X}^{!} L \simeq h_{X}^{*} D_{X} L$. We conclude by Corollary 5.7.

Remark 5.10. The analogue of Theorem 0.3 does not hold for the restriction to shreds or local sections (Remark 4.2). In fact, the shredded nearby cycle functor $\Psi_{s}^{s}=\left(i^{s}\right)^{*}$ typically does not commute with duality, even if $X=S$. Moreover, for $x \rightarrow X$ a geometric point above a geometric generic point $s \rightarrow S$, the local section $x \overline{\times} S_{(s)}$ is a point and the restriction $\left.\left(\Psi^{s} L\right)\right|_{x \overline{\times} S_{(s)}}$ can be identified with $L_{x}$, and $L \mapsto L_{x}$ typically does not commute with duality, even if $S$ is a point.

### 5.2 Vanishing cycles and local acyclicity

The functor $L$ Co studied in Section 2 admits the following generalization.
Remark 5.11. Let $X$ be a topos glued from an open subtopos $U$ and a closed subtopos $Y$. Let $j: U \rightarrow X$ and $i: Y \rightarrow X$ be the embeddings. Then $i$ admits a left adjoint $\pi: X \rightarrow Y$ if and only if the gluing functor $p_{*}=i^{*} j_{*}: U \rightarrow Y$ admits an exact left adjoint $p^{*}$. For the "only if" part we take $p=\pi j$. For the "if" part, we take $\pi^{*} \mathcal{G}$ to be $\left(\mathcal{G}, p^{*} \mathcal{G}, \varphi\right)$, where $\varphi: \mathcal{G} \rightarrow p_{*} p^{*} \mathcal{G}$ is the adjunction.

Assume that the above conditions hold. Sheaves $\mathcal{F}$ on $X$ are triples $\left(\mathcal{F}_{Y}, \mathcal{F}_{U}, \phi\right)$, where $\phi: p^{*} \mathcal{F}_{Y} \rightarrow$ $\mathcal{F}_{U}$ is a morphism. Between $\operatorname{Shv}(-, \Lambda)$, we have adjoint functors

$$
\mathrm{Co} \dashv j_{!} \dashv j^{*} \dashv j_{*}, \quad \pi^{*} \dashv \pi_{*}=i^{*} \dashv i_{*} \dashv i^{!},
$$

where $\operatorname{Co} \mathcal{F}=\operatorname{Coker}\left(\phi: p^{*} \mathcal{F}_{Y} \rightarrow \mathcal{F}_{U}\right)$. Between derived categories $D(-, \Lambda)$, we have adjoint functors

$$
L \mathrm{Co} \dashv j_{!} \dashv j^{*} \dashv R j_{*}, \quad .
$$

For $M \in D(X, \Lambda), L C o M$ is computed by $\operatorname{Co} M^{\prime}$, where $M^{\prime} \rightarrow M$ is a quasi-isomorphism and $M^{\prime q}=\left(M_{Y}^{\prime q}, M_{U}^{\prime q}, \phi^{q}\right)$ with $\phi^{q}$ a monomorphism for every $q$. In fact, we can take $M^{\prime}$ to be $\operatorname{Ker}(M \oplus$ Cone $\left(\mathrm{id}_{\pi^{*} \pi_{*} M}\right) \rightarrow$ Cone $\left(\operatorname{id}_{i_{*} i^{*} M}\right)$ ), where the map is induced by the adjunctions $\pi^{*} \pi_{*} M \rightarrow M \rightarrow$ $i_{*} i^{*} M$, and take $M^{\prime} \rightarrow M$ to be the map induced by projection.

Generalizing (2.13), we have a distinguished triangle

$$
\pi^{*} \pi_{*} M \rightarrow M \rightarrow j!L \mathrm{Co} M \rightarrow \pi^{*} \pi_{*} M[1]
$$

Let $(S, s)$ be a Henselian local pair. Let $Y$ be a locally coherent topos over $s$. Let $S^{\circ}:=S-\{s\}$. Remark 5.11 applies to the inclusions $j: Y \bar{x}_{s} S^{\circ} \rightarrow Y \times_{s} S$ and $i: Y \simeq s \overline{\times}_{s} Y \rightarrow Y \bar{x}_{s} S$, with $\pi=p_{1}: Y \overline{\times}_{s} S \rightarrow Y$ given by the first projection. In particular, we have the functor

$$
L \mathrm{Co}: D\left(Y \bar{x}_{s} S\right) \rightarrow D\left(Y \overline{\times}_{s} S^{\circ}\right)
$$

The following remark is not needed in this paper.

Remark 5.12. If $S$ is Noetherian and $m \Lambda=0$ with $m$ invertible on $S$, then $R j_{*}$ has finite-cohomological dimension and admits a right adjoint $\mathrm{Co}^{\vee}$. If, moreover, $S$ is excellent equipped with a dimension function, $K_{S}$ is a dualizing complex for $D_{c f t}(S, \Lambda)$, and $Y$ is a scheme separated of finite type over $s$, then we have $D_{Y \bar{\chi} S^{\circ}}(L \mathrm{Co}) \simeq \mathrm{Co}^{\vee} D_{Y \bar{\chi} S}$, similarly to Theorem [2.20. However, unlike the case of Section 2, $\mathrm{Co}^{\vee}$ is very different from $L \mathrm{Co}$ in general. For example, if $S$ is regular, $Y=s, v: T \rightarrow S$ is the inclusion of a nonempty regular closed subscheme of codimension $c$, then $\mathrm{Co}^{\vee}\left(v_{*} \Lambda\right) \simeq R u_{*} \Lambda(c)[2 c-$ 1] while $L \operatorname{Co}\left(v_{*} \Lambda\right) \simeq u!\Lambda[1]$, where $u: S-T \rightarrow S^{\circ}$.

Let $S$ be an arbitrary scheme. For a morphism of schemes $a: X \rightarrow S$ and a point $s$ of $S$ with values in a field, we define the sliced vanishing cycle functor $\Phi_{a}^{s}$ to be the composition

$$
D(X, \Lambda) \xrightarrow{R \Psi_{a}^{s}} D\left(X_{s} \bar{x}_{s_{0}} S_{(s)}, \Lambda\right) \xrightarrow{L \mathrm{Co}} D\left(X_{s} \bar{x}_{s_{0}} S_{(s)}^{\circ}, \Lambda\right),
$$

where $s_{0}$ is the closed point of $S_{(s)}$.
Corollary 5.13. Let $a: X \rightarrow S$ be a morphism of finite type of excellent schemes, with $S$ regular. Let $\Lambda$ be a Noetherian commutative ring such that $m \Lambda=0$ for some $m$ invertible on $S$. Let $K_{X}$ be a dualizing complex for $D_{\text {cft }}(X, \Lambda)$ and let $L \in D_{c}^{b}(X, \Lambda)$. Assume either $L \in D_{\text {cft }}$ or $\Lambda$ Gorenstein. Assume ( $a, L$ ) is universally locally acyclic. Then $\left(a, D_{X} L\right)$ is universally locally acyclic. Here $D_{X}=R \mathcal{H}$ om $\left(-, K_{X}\right)$.

This gives an affirmative answer to a question of Illusie. Note that since $S$ is regular, $\Lambda_{S}$ is a dualizing complex for $D_{c f t}(S, \Lambda)$. See also [BG, B. 62 )] for $S$ smooth over a field.

Proof. We may assume $a$ separated. We have $K_{X} \simeq a^{!} \Lambda_{S} \otimes^{L} M$ for some invertible object $M$ of $D_{c f t}(X, \Lambda)$. Since $D_{X} L \simeq \operatorname{RHom}\left(L \otimes^{L} M, a^{!} \Lambda_{S}\right)$ with $L \otimes^{L} M$ universally locally acyclic, we may assume $K_{X}=a^{!} \Lambda_{S}$. By Corollary [5.9, there exists a modification $S_{0} \rightarrow S$ such that for $T \rightarrow S_{0}$ separated of finite type, and for every geometric point $t \rightarrow T$, we have

$$
\Psi^{t}\left(\left.\left(D_{X} L\right)\right|_{X_{T}}\right) \simeq \Psi^{t} D_{X_{T}} g_{X}^{\prime} L \xrightarrow[\sim]{A_{a_{T}}^{t}\left(g_{X}^{\prime} L\right)} D_{X_{t} \overline{\bar{x}} T_{(t)}} \Psi^{t} g_{X}^{\prime} L,
$$

where $g: T \rightarrow S$. Let $L_{t}=\left.L\right|_{X_{t}}$. By Example 4.26 (1), (a, L) is $\Psi$-good and $\Psi^{t} L=p_{1}^{*} L_{t} \in D_{c}^{b}$. Thus, by Lemmas 4.22 and 4.19 and biduality (Proposition (1.37),

$$
\begin{aligned}
D_{X_{t} \overline{\times} T_{(t)}} \Psi^{t} g_{X}^{\prime} L & \simeq D_{X_{t} \overline{\times} T_{(t)}}\left(\mathrm{id} \overline{\times} g_{(t)}\right)^{\prime} \Psi^{t} L \simeq D_{X_{t} \overline{\times} T_{(t)}}\left(\mathrm{id} \overline{\times} g_{(t)}\right)^{!} D_{X_{t} \overline{\times} S_{(t)}} D_{X_{t} \overline{\times} S_{(t)}} p_{1}^{*} L_{t} \\
& \stackrel{\sim(4.32)}{\sim} D_{X_{t} \overline{\times} T_{(t)}} D_{X_{t} \overline{\times} T_{(t)}}\left(\mathrm{id} \overline{\times} g_{(t)}\right)^{*} D_{X_{t} \overline{\bar{x}} S_{(t)}} p_{1}^{*} L_{t} \simeq\left(\mathrm{id} \overline{\times} g_{(t)}\right)^{*} D_{X_{t} \overline{\times} S_{(t)}} p_{1}^{*} L_{t} .
\end{aligned}
$$

By Künneth formula (Corollary 1.36), $D_{X_{t} \bar{x} S_{(t)}} p_{1}^{*} L_{t} \simeq p_{1}^{*} D_{X_{t}} L_{t}$. Thus $\Psi^{t}\left(\left.\left(D_{X} L\right)\right|_{X_{T}}\right)$ has the form $p_{1}^{*} L^{\prime}$. It follows that $\Phi_{a_{T}}\left(\left.\left(D_{X} L\right)\right|_{X_{T}}\right)=0$. Thus, for any morphism $S^{\prime} \rightarrow S$ separated and of finite type, $\Phi_{a_{S^{\prime}}}\left(\left.\left(D_{X} L\right)\right|_{X_{S^{\prime}}}\right)=0$ by Lemma 6.1 applied to the Čech nerve of $S_{0} \times{ }_{S} S^{\prime} \rightarrow S^{\prime}$. We conclude by Remark 4.28,

Gabber showed that universal local acyclicity in Corollary 5.13 is equivalent to local acyclicity. He also gave a different proof of the case $L \in D_{\text {cft }}$ of Corollary 5.13, independent of Corollary 5.9, We present Gabber's results in Section 6.

Generalizing constructions of Beilinson [B2], Hu and Yang [HY] recently defined relative versions of singular support and weak singular support over a Noetherian base scheme $S$, which exist and are equal on a dense open subscheme of $S$. For $X \rightarrow S$ smooth of finite type and $L \in D_{c}^{b}(X, \Lambda)$, the weak singular support $\mathrm{SS}^{w}(L, X / S)$ is defined to be the smallest element of $\mathcal{C}^{w}(L, X / S)$ (if it exists), where $\mathcal{C}^{w}(L, X / S)$ denotes the set of closed conical subsets $C$ of the cotangent bundle $T^{*}(X / S)$ such that $L$ is weakly micro-supported on $C$ relatively to $S$. $\mathrm{SS}^{w}(L, X / S)$ exists if $(f, L)$ is universally locally acyclic [HY, 4.3, Proposition 4.5].

The preservation of local acyclicity by duality implies the following compatibility of weak singular support with duality.

Corollary 5.14. Let $X \rightarrow S$ be a smooth morphism of regular excellent schemes. Let $\Lambda$ and $K_{X}$ be as in Corollary 5.13 and let $L \in D_{c}^{b}(X, \Lambda)$. Assume either $L \in D_{c \mathrm{ft}}$ or $\Lambda$ Gorenstein. Then

$$
\mathcal{C}^{w}(L, X / S)=\mathcal{C}^{w}\left(D_{X} L, X / S\right), \quad \mathrm{SS}^{w}(L, X / S)=\mathrm{SS}^{w}\left(D_{X} L, X / S\right)
$$

(The second equality means that $\mathrm{SS}^{w}(L, X / S)$ exists if and only if $\mathrm{SS}^{w}\left(D_{X} L, X / S\right)$ exists and the two are equal when they exist.)

In the case where $S$ is the spectrum of a field, one recovers [S1, Corollary 4.9].
Proof. The second equality follows from the first equality. For the first equality, we show more generally that for every test pair $X \stackrel{h}{\leftarrow} U \xrightarrow{g} Y$ of smooth schemes of finite type over $S$ with $h$ smooth of relative dimension $d,\left(g, h^{*} L\right)$ is locally acyclic if and only if $\left(g, h^{*} D_{X} L\right)$ is locally acyclic. Since $\left(h^{*} D_{X} L\right)(d)[2 d] \simeq D_{U} h^{*} L$, this follows from Corollaries 5.13 and 6.6.

## 6 Local acyclicity (after Gabber)

The results of this section are due to Ofer Gabber.
Lemma 6.1. Let $a: X \rightarrow S$ be a morphism of schemes and let $L \in D^{+}(X, \Lambda)$. Assume that there exists a hypercovering $g_{\bullet}: S \bullet \rightarrow S$ for the h-topology such that $\Phi_{a_{n}}\left(\left.L\right|_{X_{n}}\right)=0$ for all $n \geq 0$, where $a_{n}: X_{n} \rightarrow S_{n}$ denotes the base change of a by $g_{n}$. Then $\Phi_{a}(L)=0$.

Proof. Let $g_{X} \bullet$ be the base change of $g_{\bullet}$ to $X$ and let $\vec{g} \bullet_{\bullet}=g_{X} \bullet \overleftarrow{×}_{\bullet} g_{\bullet}$. We have a commutative diagram


By cohomological descent and oriented cohomological descent [ILO, XII ${ }_{\mathrm{A}}$ Théorème 2.2.3], the horizontal arrows are isomorphisms on $D^{+}$. The right vertical arrow is an isomorphism on $L$ by assumption. Thus the left vertical arrow is an isomorphism on $L$.

Lemma 6.2. Let $S$ be a Noetherian scheme and let $U \subseteq S$ be a dense open subset. For any $s \in S-U$, there exists an immediate Zariski generization $t$ of $s$ in $S$ with $t \in U$.

This follows from [M, Section 31, Lemma 1]. We include a proof for completeness.
Proof. We may assume $S$ local of center $s$. We proceed by induction on the dimension $d$ of $S$. For $d=1$, any $t \in U$ works. For $d>1$, there are infinitely many codimension 1 points of $S$. Indeed, for $S=\operatorname{Spec}(R)$, by Krull's principal ideal theorem, each proper principal ideal of $R$ is contained in a height 1 prime ideal, so that the maximal ideal $\mathfrak{m}$ of $R$ is the union of all height 1 prime ideals. On the other hand, by the prime avoidance lemma, $\mathfrak{m}$ cannot be the union of finitely many such primes, as $\mathfrak{m}$ has height $>1$. Let $x$ be a codimension 1 point of $S$ that is not a maximal point of $S-U$. Then $x \in U$. We conclude by induction hypothesis applied to the closure of $x$ in $S$.

Lemma 6.3. Let $f: X \rightarrow S$ be a morphism of finite type of Noetherian schemes, $T$ a subscheme of $S$, $(\bar{x}, \bar{t})$ a point of $P=X \overleftarrow{\times}_{S} T$ ( $\bar{x}$ being a point of $X_{\text {et }}, \bar{t}$ being a point of $T_{\text {et }}$, and $\bar{t} \rightsquigarrow f(\bar{x})$ being a morphism of points of $S_{\text {et }}$ ). Then $(\bar{x}, \bar{t})$ specializes to a point $\left(\bar{x}^{\prime}, \bar{t}^{\prime}\right)$ of $P$ such that the image $x^{\prime} \in X$ of $\bar{x}^{\prime}$ is closed and $\bar{t}^{\prime} \rightsquigarrow f\left(\bar{x}^{\prime}\right)$ is an (étale) specialization of codimension $\leq 1$.

Proof. The image $x \in X$ of $\bar{x}$ specializes to a closed point $x^{\prime} \in X$ and $\bar{x}$ specializes to a point $\bar{x}^{\prime}$ of $X_{\text {et }}$ above $x^{\prime}$. Let $t$ be the image of $\bar{t} \rightarrow S_{(f(\bar{x}))} \rightarrow S_{\left(f\left(\bar{x}^{\prime}\right)\right)}$ and let $U=T \times_{S} \overline{\{t\}}$, which is locally closed in $\overline{\{t\}}$ and contains $t$, hence open dense in $\overline{\{t\}}$. If $U=\overline{\{t\}}$, it suffices to take $\bar{t}^{\prime}=f\left(\bar{x}^{\prime}\right)$. Otherwise $f\left(\bar{x}^{\prime}\right) \notin U$ admits an immediate Zariski generization $t^{\prime} \in U$ by Lemma 6.2 and it suffices to take $\bar{t}^{\prime}$ to be a geometric point above $t^{\prime}$.

Lemma 6.4. Let $f: X \rightarrow S$ be a morphism of finite type of Noetherian schemes. The points $(\bar{x}, \bar{t})$ of $X \overleftarrow{×}_{S} S$ such that the image $x \in X$ of $\bar{x}$ is locally closed and $\bar{t} \rightsquigarrow f(\bar{x})$ is an (étale) specialization of codimension $\leq 1$, form a conservative family $\mathcal{P}$ for the category of constructible sheaves.

Note that by Chevalley's constructibility theorem, $x \in X$ locally closed implies $f(x) \in S$ locally closed. Thus $x \in X$ is locally closed if and only if $s=f(x) \in S$ is locally closed and $x$ is closed in the fiber $X_{s}$.

Proof. Let $\alpha: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of constructible sheaves such that $\alpha_{(\bar{x}, \bar{t})}$ is an isomorphism for all $(\bar{x}, \bar{t})$ in $\mathcal{P}$. There exist partitions $X=\bigcup_{i} X_{i}$ and $S=\bigcup_{j} S_{j}$ into locally closed subsets such that the restrictions of $\mathcal{F}$ and $\mathcal{G}$ to $P_{i, j}=X_{i} \overleftarrow{×}_{S} S_{j}$ are locally constant. By Lemma 6.3, each point of $P_{i, j}$ specializes to a point of $P_{i, j}$ in $\mathcal{P}$. Thus $\left.\alpha\right|_{P_{i, j}}$ is an isomorphism.
Theorem 6.5. Let $f: X \rightarrow S$ be a morphism of finite type of Noetherian schemes. Let $\Lambda$ be a Noetherian commutative ring such that $m \Lambda=0$ for some $m$ invertible on $S$ and let $L \in D_{c}^{b}(X, \Lambda)$. Assume that for every geometric point $\bar{s}$ of $S$ above $s \in S$ locally closed and every strictly local test curve $C \rightarrow S_{(\bar{s})},\left(f_{C},\left.L\right|_{X_{C}}\right)$ is locally acyclic. Then $(f, L)$ is universally locally acyclic.

By a strictly local test curve we mean a finite morphism $C \rightarrow S_{(\bar{s})}$ such that $C$ is integral and one-dimensional. Note that $C$ is strictly local.

Proof. By Orgogozo's theorem (Theorem 4.27), there exists a modification $g: T \rightarrow S$ such that $R \Psi_{f_{T}}\left(\left.L\right|_{X_{T}}\right)$ is constructible and commutes with base change. We claim $\Phi_{f_{T}}\left(\left.L\right|_{X_{T}}\right)=0$. Assuming the claim, we have, for any morphism $S^{\prime} \rightarrow S$ separated and of finite type, $\Phi_{f_{S^{\prime}}}\left(\left.L\right|_{X_{S^{\prime}}}\right)=0$ by Lemma 6.1 applied to the Čech nerve of the base change of $g$ by $S^{\prime} \rightarrow S$. We conclude by Remark 4.28.

To prove the claim, let $(\bar{x}, \bar{t})$ be a point of $X_{T} \overleftarrow{×}_{T} T$ as in Lemma 6.4. Let $C$ be the closure of the image of $\bar{t}$ in $T_{\left(f_{T}(\bar{x})\right)}$ (equipped with the reduced scheme structure). It suffices to show $\Phi_{f_{C}}\left(\left.L\right|_{X_{C}}\right)=0$. Let $\bar{s}=g\left(f_{T}(\bar{x})\right) \rightarrow S$. If the specialization $g(\bar{t}) \rightsquigarrow \bar{s}$ is an isomorphism, this follows from Deligne's theorem [D1, Th. finitude, Corollaire 2.16] that $\left(f_{\bar{s}},\left.L\right|_{X_{\bar{s}}}\right)$ is universally locally acyclic. Otherwise $C \rightarrow S_{(\bar{s})}$ is a strictly local test curve and $\Phi_{f_{C}}\left(\left.L\right|_{X_{C}}\right)=0$ by assumption. Indeed, $C$ is clearly integral and one-dimensional. Moreover, $T_{\left(f_{T}(\bar{x})\right)}$ is the strict localization of $T_{(\bar{s})}:=T \times{ }_{S} S_{(\bar{s})}$ at a closed point of the fiber $T_{\bar{s}}$. Thus $C$ is the limit of a system of affine schemes quasi-finite over $T_{(\bar{s})}$ with étale transition maps, hence a strict localization of $U$ at a closed point $u \in h^{-1}(\bar{s})$, with $h: U \rightarrow S_{(\bar{s})}$ of finite type. The localization $\operatorname{Spec}\left(\mathcal{O}_{U, u}\right)$ is irreducible of dimension 1 and the image in $S_{(\bar{s})}$ is not a point. It follows that $u$ is an isolated point $h^{-1}(\bar{s})$. By Chevalley's semicontinuity theorem, $h$ is quasi-finite at $u$. Thus $C$ is finite over $S_{(\bar{s})}$.

Since local acyclicity is stable under quasi-finite base change, we immediately deduce the following.
Corollary 6.6. Let $f$ and $\Lambda$ be as in Theorem 6.5. Let $L \in D_{c}^{b}(X, \Lambda)$ such that $(f, L)$ is locally acyclic. Then $(f, L)$ is universally locally acyclic.

In the case $L=\Lambda$, this answers a question of M. Artin [SGA4, XV Remarque 1.8 b )] (local variant) under the above assumptions. Compare with Example 4.26 (2).
Remark 6.7.
(1) We may replace the assumption in Theorem 6.5 by the existence, for every strictly local test curve $C \rightarrow S_{(\bar{s})}$ with $s \in S$ locally closed, of a surjective morphism of integral schemes $g: C^{\prime} \rightarrow C$ such that $\left(f_{C^{\prime}},\left.L\right|_{X_{C^{\prime}}}\right)$ is locally acyclic. Indeed, $\left(f_{C},\left.L\right|_{X_{C}}\right)$ is $\Psi$-good (by Example 4.26 (2) or by the proof of Theorem 6.5) and $\left(g_{X} \overleftarrow{\times}_{g} g\right)^{*}$ is conservative. For $S$ universally Japanese, taking $C^{\prime}$ to be the normalization of $C$, it thus suffices in Theorem 6.5 to assume the local acyclicity of ( $f_{C},\left.L\right|_{X_{C}}$ ) for strictly local test curves $C \rightarrow S_{(\bar{s})}$ with the additional hypothesis that $C$ is regular. Note that such a $C$ is the spectrum of a strictly Henselian discrete valuation ring.
(2) Assume that $S$ is of finite type over a field or over $\operatorname{Spec}(\mathbb{Z})$. Then it suffices in Theorem 6.5 to assume the local acyclicity of $\left(f_{T},\left.L\right|_{X_{T}}\right)$ for $T \rightarrow S$ quasi-finite with $T$ regular of dimension 1 (cf. [BG, B. 65 )]). Indeed, any $C \rightarrow S$ as above with $C$ regular factorizes through some $T \rightarrow S$.
Gabber's proof of the case $L \in D_{c f t}$ of Corollary 5.13 relies on Remark (6.7) (1) and the following consequence of the absolute purity theorem, which is a variant of [S1, Corollary 8.10].
Theorem 6.8. Let $g: T \rightarrow S$ be an immersion of Noetherian regular schemes, of codimension $c$. Let $\Lambda$ be a Noetherian commutative ring with $m \Lambda=0$ for some $m$ invertible on $S$. Let $f: X \rightarrow S$ be a morphism of schemes and let $L \in D^{b}(X, \Lambda)$ of finite tor-amplitude such that $(f, L)$ is strongly locally acyclic (Definition 4.5). Then the Gysin map a: $g_{X}^{*} L(-c)[-2 c] \rightarrow R g_{X}^{\prime} L$ is an isomorphism.

For $f=\operatorname{id}_{S}$ and $L=\Lambda$, we recover the absolute purity theorem.
Proof. We may assume that $g$ is a closed immersion. Let $j: U \rightarrow S$ denote the complement of $g$. We have a morphism of distinguished triangles

where $c$ is an isomorphism by [D1, Th. finitude, App., Proposition 2.10] (see [F, Lemma 7.6.7 (b)] for a more detailed proof). It follows that $b$ is an isomorphism. We have a commutative square

where the upper horizontal isomorphism is the absolute purity theorem. Thus $g_{X *} a$ is an isomorphism, and so is $a$.

Corollary 6.9. Let $g: T \rightarrow S$ be a morphism of finite type of Noetherian regular schemes admitting ample invertible sheaves. Let $(f, L)$ be as in Theorem 6.8 with $(f, L)$ universally strongly locally acyclic. Then the Gysin map $g_{X}^{*} L(d)[2 d] \rightarrow R g_{X}^{\prime} L$ is an isomorphism, where $d$ is the virtual relative dimension of $g$.
Proof. We factorize $g$ into an immersion followed by a smooth morphism. The case of a smooth morphism is obvious. The case of an immersion follows from Theorem 6.8.

Here is Gabber's proof of the case $L \in D_{\text {cft }}$ of Corollary 5.13. We may assume $K_{X}=a^{!} \Lambda_{S}$. By Remark $6.7(1)$, it suffices to show that $\left(a_{C}, g_{X}^{*}\left(D_{X} L\right)_{(\bar{s})}\right)$ is locally acyclic for strictly local test curves $g: C \rightarrow S_{(\bar{s})}$ with $C$ the spectrum of a strictly Henselian discrete valuation ring. By Corollary 6.6 and Lemma $4.7(1),(f, L)$ is universally strongly locally acyclic. Thus $g_{X}^{*}\left(D_{X} L\right)_{(\bar{s})} \simeq D_{X_{C}} R g_{X}^{\prime} L_{(\bar{s})} \simeq$ $D_{X_{C}}\left(g_{X}^{*} L_{(\bar{s})}(d)[2 d]\right)$ by Corollary [6.9, Thus it suffices to show that $D_{X_{C}}$ preserves local acyclicity over $C$. Changing notation, it suffices to show $\Phi_{f} D_{X} L=0$ for $S$ the spectrum of a strictly Henselian discrete valuation ring. This follows from Beilinson's theorem (Corollary (0.2), or from Theorem 3.1: $\Psi_{f}^{s} D_{X} L \simeq D_{X_{s} \overline{\times} S} \Psi_{f}^{s} L \simeq D_{X_{s} \overline{\times} S} p_{1}^{*}\left(\left.L\right|_{X_{s}}\right) \simeq p_{1}^{*} D_{X_{s}}\left(\left.L\right|_{X_{s}}\right)$, where the last isomorphism is Künneth formula (Corollary 1.36).

## References

［EGAIV］A．Grothendieck，Éléments de géométrie algébrique（avec la collaboration de J．Dieudonné）．IV．Étude locale des schémas et des morphismes de schémas，Inst．Hautes Études Sci．Publ．Math．20，24，28， 32 （1964） （French）．MR0173675，MR0199181，MR0217086，MR0238860 136，37，38，42
［SGA4］Théorie des topos et cohomologie étale des schémas，Lecture Notes in Mathematics，Vol．269，270，305，Springer－ Verlag，Berlin，1972－1973．Séminaire de Géométrie Algébrique du Bois－Marie 1963－1964（SGA 4）；Dirigé par M．Artin，A．Grothendieck，et J．L．Verdier．Avec la collaboration de N．Bourbaki，P．Deligne et B．Saint－Donat． MR0354652（50 \＃7130），MR0354653（50 \＃7131），MR0354654（50 \＃7132）146 67611121314484
［SGA5］Cohomologie l－adique et fonctions L，Lecture Notes in Mathematics，vol．589，Springer－Verlag，Berlin， 1977 （French）．Séminaire de Géometrie Algébrique du Bois－Marie 1965－1966（SGA 5）．Dirigé par A．Grothendieck， avec la collaboration de I．Bucur，C．Houzel，L．Illusie，J．－P．Jouanolou et J．－P．Serre．MR0491704（58 \＃10907） 110.12
［SGA7II］P．Deligne and N．Katz，Groupes de monodromie en géométrie algébrique．II，Lecture Notes in Mathematics， Vol．340，Springer－Verlag，Berlin－New York， 1973 （French）．Séminaire de Géométrie Algébrique du Bois－Marie 1967－1969（SGA 7 II）．MR0354657 囵［17，20，23，24，25， 29
［AGT］A．Abbes，M．Gros，and T．Tsuji，The p－adic Simpson correspondence，Annals of Mathematics Studies，vol．193， Princeton University Press，Princeton，NJ，2016．MR3444777 10
［A］J．Ayoub，Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique． I，Astérisque 314 （2007），x＋466 pp．（2008）（French，with English and French summaries）．MR2423375 8
［B1］A．A．Bellinson，How to glue perverse sheaves，K－theory，arithmetic and geometry（Moscow，1984），Lecture Notes in Math．，vol．1289，Springer，Berlin，1987，pp．42－51，DOI 10．1007／BFb0078366．MR923134 12 24
［B2］A．Beilinson，Constructible sheaves are holonomic，Selecta Math．（N．S．） 22 （2016），no．4，1797－1819． MR3573946 $\uparrow 45$
［BB］A．Beîlinson and J．Bernstein，A proof of Jantzen conjectures，I．M．Gel＇fand Seminar，Adv．Soviet Math．， vol．16，Amer．Math．Soc．，Providence，RI，1993，pp．1－50．MR1237825 12
［BS］B．Bhatt and P．Scholze，The pro－étale topology for schemes，Astérisque 369 （2015），99－201（English，with English and French summaries）．MR3379634 113
［BG］A．Braverman and D．Gaitsgory，Geometric Eisenstein series，Invent．Math． 150 （2002），no．2，287－384，DOI 10．1007／s00222－002－0237－8．MR1933587 455 48
［C］B．Conrad，Deligne＇s notes on Nagata compactifications，J．Ramanujan Math．Soc． 22 （2007），no．3，205－257． MR2356346（2009d：14002）蕾
［D1］P．Deligne，Cohomologie étale，Lecture Notes in Mathematics，Vol．569，Springer－Verlag，Berlin，1977．Séminaire de Géométrie Algébrique du Bois－Marie SGA 4 $\frac{1}{2}$ ．Avec la collaboration de J．F．Boutot，A．Grothendieck，L． Illusie et J．L．Verdier．MR0463174（57 \＃3132）1［13］［15］［28］ 47$][8$
［D2］，Lettre à L．Illusie， 19 mars 1999．12 24］
［F］L．Fu，Etale cohomology theory，Revised edition，Nankai Tracts in Mathematics，vol．14，World Scientific Publishing Co．Pte．Ltd．，Hackensack，NJ，2015．MR3380806 128， 48
［HY］H．Hu and E．Yang，Relative singular support and the semi－continuity of characteristic cycles for étale sheaves， Selecta Math．（N．S．） 24 （2018），no．3，2235－2273，DOI 10．1007／s00029－017－0355－1．MR3816504 145
［I1］L．Illusie，Autour du théorème de monodromie locale，Astérisque 223 （1994），9－57（French）．Périodes $p$－adiques （Bures－sur－Yvette，1988）．MR1293970 怆， 24
［I2］＿，Perversité et variation，Manuscripta Math． 112 （2003），no．3，271－295（French，with English sum－ mary）．MR2067039 126
［I3］＿＿，Around the Thom－Sebastiani theorem，with an appendix by Weizhe Zheng，Manuscripta Math． 152 （2017），no．1－2，61－125，DOI 10．1007／s00229－016－0852－0．MR3595371 12 3 ， 10 24，27，28，29， $32,37,3840$
［ILO］L．Illusie，Y．Laszlo，and F．Orgogozo（eds．），Travaux de Gabber sur l＇uniformisation locale et la cohomologie étale des schémas quasi－excellents，Société Mathématique de France，Paris， 2014 （French）．Séminaire à l＇École Polytechnique 2006－2008．［Seminar of the Polytechnic School 2006－2008］；With the collaboration of Frédéric Déglise，Alban Moreau，Vincent Pilloni，Michel Raynaud，Joël Riou，Benoît Stroh，Michael Temkin and Weizhe
 （37） 3946
［IZ］L．Illusie and W．Zheng，Quotient stacks and equivariant étale cohomology algebras：Quillen＇s theory revisited， J．Algebraic Geom． 25 （2016），no．2，289－400．MR3466353 124， 27
［KS］M．Kashiwara and P．Schapira，Categories and sheaves，Grundlehren der Mathematischen Wissenschaften ［Fundamental Principles of Mathematical Sciences］，vol．332，Springer－Verlag，Berlin，2006．MR2182076 （2006k：18001）价 2042
[L] G. Laumon, Vanishing cycles over a base of dimension $\geq 1$, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 143-150. MR726426 12
[M] H. Matsumura, Commutative ring theory, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. Translated from the Japanese by M. Reid. MR1011461 46
[O] F. Orgogozo, Modifications et cycles proches sur une base générale, Int. Math. Res. Not., posted on 2006, Art. ID 25315, 38, DOI 10.1155/IMRN/2006/25315 (French). MR2249998 13 5, 6, 12, 27, 29, 33, 36, 37, 38, 39, 42,43
[S1] T. Saito, The characteristic cycle and the singular support of a constructible sheaf, Invent. Math. 207 (2017), no. 2, 597-695. MR3595935 坆 28 46 48
[S2] _, Characteristic cycle of the external product of constructible sheaves, Manuscripta Math. 154 (2017), no. 1-2, 1-12. MR3682201 12
[SZ] S. Sun and W. Zheng, Parity and symmetry in intersection and ordinary cohomology, Algebra Number Theory 10 (2016), no. 2, 235-307, DOI 10.2140/ant.2016.10.235. MR3477743 16 1743
[Z1] W. Zheng, Six operations and Lefschetz-Verdier formula for Deligne-Mumford stacks, Sci. China Math. 58 (2015), no. 3, 565-632, DOI 10.1007/s11425-015-4970-z. MR3319927 17 8
[Z2] _, Gluing pseudo functors via n-fold categories, J. Homotopy Relat. Struct. 12 (2017), no. 1, 189-271. MR3613026 † ${ }^{34}$


[^0]:    *School of Mathematical Sciences, Beijing Normal University, Beijing 100875, China; School of Mathematical Sciences, University of the Chinese Academy of Sciences, Beijing 100049, China; email: qlu@bnu.edu. cn. Partially supported by National Natural Science Foundation of China Grants 11371043, 11501541; China Scholarship Council; Fundamental Research Funds for Central Universities of China.
    ${ }^{\dagger}$ Morningside Center of Mathematics and Hua Loo-Keng Key Laboratory of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; University of the Chinese Academy of Sciences, Beijing 100049, China; email: wzheng@math.ac.cn. Partially supported by National Natural Science Foundation of China Grants 11621061, 11688101, 11822110; National Center for Mathematics and Interdisciplinary Sciences, Chinese Academy of Sciences.
    ${ }^{\ddagger}$ Mathematics Subject Classification 2010: 14F20 (Primary); 18F10, 32S30 (Secondary).

[^1]:    ${ }^{1}$ The claim in [ILO, XI Exemples $\left.3.4(2)\right]$ that $G \mapsto B G$ preserves fiber products is false (cf. (1.8)).

