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**Gwo Dong Lin, Xiaoling Dou & Satoshi
Kuriki**

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The Bivariate Lack-of-Memory Distributions

Gwo Dong Lin

Academia Sinica, Taipei, Taiwan

Xiaoling Dou

Waseda University, Tokyo, Japan

Satoshi Kuriki

The Institute of Statistical Mathematics, Tokyo, Japan

Abstract

We treat all the bivariate lack-of-memory (BLM) distributions in a unified approach and develop some new general properties of the BLM distributions, including joint moment generating function, product moments, and dependence structure. Necessary and sufficient conditions for the survival functions of BLM distributions to be totally positive of order two are given. Some previous results about specific BLM distributions are improved. In particular, we show that both the Marshall–Olkin survival copula and survival function are totally positive of all orders, regardless of parameters. Besides, we point out that Slepian's inequality also holds true for BLM distributions.

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1 Introduction

The classical univariate lack-of-memory (LM) property is a remarkable characterization of the exponential distribution which plays a prominent role in reliability theory, queuing theory, and other applied fields (Feller, 1965; Fortet, 1977; Galambos and Kotz, 1978). The recent bivariate LM property is, however, shared by the famous Marshall and Olkin's, Block and Basu's as well as Freund's bivariate exponential distributions, among many others; see, e.g., Chapter 10 of Balakrishnan and Lai (2009), Chapter 47 of Kotz et al. (2000) and Kulkarni (2006). These bivariate distributions have been well investigated individually in the literature. Our main purpose in this paper

is, however, to develop in a unified approach some new general properties of the bivariate lack-of-memory (BLM) distributions which share the same bivariate LM property.

In Section 2, we first review the univariate and bivariate LM properties, and then summarize the important known properties of the BLM distributions. We derive in Section 3 some new general properties of the BLM distributions, including joint moment generating function, product moments, and stochastic inequalities. The dependence structures of the BLM distributions are investigated in Section 4. We find necessary and sufficient conditions for the survival functions (and the densities if they exist) of BLM distributions to be totally positive of order two. Some previous results about specific BLM distributions are improved. In particular, we show that both the Marshall–Olkin survival copula and survival function are totally positive of all orders, regardless of parameters. In Section 5, we study the stochastic comparisons in the family of all BLM distributions and point out that Slepian's lemma/inequality for bivariate normal distributions also holds true for BLM distributions.

2 Lack-of-Memory Property

We first review the well-known univariate lack-of-memory property. Let X be a nonnegative random variable with distribution function F . Then F satisfies (multiplicative) Cauchy's functional equation

$$\bar{F}(x + y) = \bar{F}(x)\bar{F}(y), \quad x \geq 0, y \geq 0, \quad (1)$$

where $\bar{F}(x) = 1 - F(x) = \Pr(X > x)$, if and only if $F(0) = 1$ (X degenerates at 0) or $F(x) = 1 - \exp(-\lambda x)$, $x \geq 0$, for some constant $\lambda > 0$, denoted by $X \sim \text{Exp}(\lambda)$ (X has an exponential distribution with positive parameter λ). If X is the lifetime of a system with positive survival function \bar{F} , then Eq. (1) is equivalent to

$$\Pr(X > x + y | X > y) = \Pr(X > x), \quad x \geq 0, y \geq 0. \quad (2)$$

This means that the conditional probability of a system surviving to time $x + y$ given surviving to time y is equal to the unconditional probability of the system surviving to time x . Namely, the failure performance of the system does not depend on the past, given its present condition. In such a case (2), we say that the distribution F lacks memory at each point y . So Eq. (1) is called the LM property or memoryless property of F .

For simplicity, we consider only *positive* random variable $X \sim F$ from now on. Then, the LM property (1) holds true iff $X \sim \text{Exp}(\lambda)$ for some $\lambda > 0$.

We next consider the bivariate LM property. Let the positive random variables X and Y have joint distribution H with marginals F and G . Namely, $(X, Y) \sim H$, $X \sim F$, $Y \sim G$. Moreover, denote the survival function of H by

$$\bar{H}(x, y) \equiv \Pr(X > x, Y > y) = 1 - F(x) - G(y) + H(x, y), \quad x, y \geq 0.$$

An intuitive extension of the LM property (2) to the bivariate case is the strict BLM property:

$$\Pr(X > x + s, Y > y + t \mid X > s, Y > t) = \Pr(X > x, Y > y), \quad x, y, s, t \geq 0$$

(H lacks memory at each pair (s, t)), which is equivalent to

$$\bar{H}(x + s, y + t) = \bar{H}(x, y)\bar{H}(s, t), \quad \forall x, y, s, t \geq 0, \tag{3}$$

if the survival function \bar{H} is positive. In a two-component system, this means as before that the conditional probability of two components surviving to times $(x + s, y + t)$ given surviving to times (s, t) is equal to the unconditional probability of these two components surviving to times (x, y) . But Eq. (3) has only one solution (Marshall and Olkin 1967, p. 33), namely, the independent bivariate exponential distribution with survival function

$$\bar{H}(x, y) = \exp[-(\lambda x + \delta y)], \quad x, y \geq 0,$$

for some constants $\lambda, \delta > 0$; in other words, X and Y are independent random variables and $X \sim \text{Exp}(\lambda)$, $Y \sim \text{Exp}(\delta)$ for some positive parameters λ, δ .

In their pioneering paper, Marshall and Olkin (1967) considered instead the weaker BLM property (with $s = t$)

$$\Pr(X > x + t, Y > y + t \mid X > t, Y > t) = \Pr(X > x, Y > y), \quad x, y, t \geq 0$$

(H lacks memory at each *equal* pair (t, t)), and solved the functional equation

$$\bar{H}(x + t, y + t) = \bar{H}(x, y)\bar{H}(t, t), \quad \forall x, y, t \geq 0. \tag{4}$$

It turns out that for given $(X, Y) \sim H$ with marginals F, G on $(0, \infty)$, H satisfies the BLM property (4) iff its survival function is of the form

$$\bar{H}(x, y) = \begin{cases} e^{-\theta y} \bar{F}(x - y), & x \geq y \geq 0 \\ e^{-\theta x} \bar{G}(y - x), & y \geq x \geq 0, \end{cases} \tag{5}$$

where θ is a positive constant (see also Barlow and Proschan 1981, p. 130).

For convenience, denote by $BLM(F, G, \theta)$ the BLM distribution H with marginals F, G , parameter $\theta > 0$ and survival function \bar{H} in (5), and denote by \mathcal{BLM} the family of all BLM distributions, namely,

$$\mathcal{BLM} = \{H : H = BLM(F, G, \theta), \text{ where } \theta > 0, \text{ and } F, G \text{ are marginal distributions}\}.$$

Theorem 1 below summarizes some important known properties of the BLM distributions; for more details, see Marshall and Olkin (1967), Block and Basu (1974), Block (1977), and Ghurye and Marshall (1984). For convenience, denote $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

THEOREM 1. *Let $(X, Y) \sim H = BLM(F, G, \theta) \in \mathcal{BLM}$. Then the following statements are true.*

- (i) *The marginals F, G have densities f, g , respectively. Moreover, the right-hand derivatives $f(x) = \lim_{\varepsilon \rightarrow 0^+} [F(x + \varepsilon) - F(x)]/\varepsilon$ and $g(x) = \lim_{\varepsilon \rightarrow 0^+} [G(x + \varepsilon) - G(x)]/\varepsilon$ exist for all $x \geq 0$, which are right-continuous and are of bounded variation on $[0, \infty)$.*
- (ii) *$\Pr(X - Y > t) = \bar{F}(t) - f(t)/\theta$ and $\Pr(Y - X > t) = \bar{G}(t) - g(t)/\theta$ for all $t \geq 0$.*
- (iii) *Both $e^{\theta x} f(x)$ and $e^{\theta x} g(x)$ are increasing (nondecreasing) in $x \geq 0$.*
- (iv) *$F(x) + G(x) \geq 1 - \exp(-\theta x)$, $x \geq 0$.*
- (v) *$X \wedge Y \sim Exp(\theta)$ and is independent of $X - Y$.*
- (vi) *$f(0) \vee g(0) \leq \theta \leq f(0) + g(0)$.*
- (vii) *$f'(x) + \theta f(x) \geq 0$, $g'(x) + \theta g(x) \geq 0$, $x \geq 0$, if f and g are differentiable.*

REMARK 1. Some of the above necessary conditions (i)–(vii) also play as sufficient conditions for (X, Y) to obey a BLM distribution. For example, in addition to the above conditions (vi) and (vii), assume that the marginal densities are absolutely continuous, then the \bar{H} in (5) is a *bona fide* survival function. This is a slight modification of Theorem 5.1 of Marshall and Olkin (1967) who required (vi') $[f(0) + g(0)]/2 \leq \theta \leq f(0) + g(0)$ instead of (vi) above. Note that conditions (vi) and (vi') are different unless $f(0) = g(0)$, and that (vi) is a consequence of (iii) and (iv) (see Corollary 2(i) below and Ghurye and Marshall 1984, p. 792). On the other hand, the condition (v) together with continuous marginals F, G also implies that (X, Y) has a BLM

distribution (Block 1977, p. 810). It is interesting to recall that for *independent* nondegenerate random variables X and Y , the above independence of $X \wedge Y$ and $X - Y$ is a characterization of the exponential/geometric distributions under suitable conditions (see Ferguson 1964, 1965, Crawford 1966, and Rao and Shanbhag 1994). Namely, in general, the BLM distributions share the same independence property of $X \wedge Y$ and $X - Y$ with independent exponential/geometric random variables.

REMARK 2. There are some more observations: (a) $\Pr(X = Y) = [f(0) + g(0)]/\theta - 1$ by the above (ii), (b) at least one of $f(0)$ and $g(0)$ is positive, (c) the survival function \bar{H} in (5) is purely singular (i.e., $X = Y$ almost surely) iff $\theta = [f(0) + g(0)]/2$ iff $f(0) = g(0) = \theta$ (because $f(0) \neq g(0)$ implies $\theta > [f(0) + g(0)]/2$ by (vi)), and (d) \bar{H} is absolutely continuous (i.e., $X \neq Y$ almost surely) iff the marginal densities together satisfy $f(0) + g(0) = \theta$ (see Ghurye and Marshall 1984, p. 792). In view of the above results, the survival function (5) of $H = \text{BLM}(F, G, \theta)$ can be rewritten as the convex combination of two extreme ones:

$$\bar{H}(x, y) = \left(2 - \frac{f(0) + g(0)}{\theta}\right) \bar{H}_a(x, y) + \left(\frac{f(0) + g(0)}{\theta} - 1\right) \bar{H}_s(x, y), \quad x, y \geq 0,$$

where H_a is absolutely continuous and H_s is purely singular with survival function $\bar{H}_s(x, y) = \exp[-\theta \max\{x, y\}]$, $x, y \geq 0$. Clearly, the parameter θ regulates H between H_a and H_s . On the other hand, Ghurye and Marshall (1984, Section 3) gave an interesting random decomposition of $(X, Y) \sim H \in \mathcal{BLM}$ and represented \bar{H} as a Laplace–Stieltjes integral by another bivariate survival function. See also Ghurye (1987) and Marshall and Olkin (2015) for further generalizations of the BLM distributions.

REMARK 3. Kulkarni (2006) proposed an interesting and useful approach to construct some BLM distributions by starting with marginal failure rate functions. First, choose two real-valued functions r_1, r_2 and a constant θ satisfying the following (modified) conditions:

- (a) The functions $r_i, i = 1, 2$, are *absolutely continuous* on $[0, \infty)$ and $\theta > 0$.
- (b) $0 \leq r_i(x) \leq \theta, x \geq 0, i = 1, 2$.
- (c) $\int_0^\infty r_i(x) dx = \infty, i = 1, 2$.
- (d) $r_i(x)(\theta - r_i(x)) + r_i'(x) \geq 0, x \geq 0, i = 1, 2$.
- (e) $r_1(0) + r_2(0) \geq \theta$.

Then set $\bar{F}(x) = \exp(-\int_0^x r_1(t)dt)$, $x \geq 0$, and $\bar{G}(y) = \exp(-\int_0^y r_2(t)dt)$, $y \geq 0$. In this way, the H defined through (5) is a *bona fide* BLM distribution because the above conditions (a)–(e) together imply that conditions (vi) and (vii) in Theorem 1 hold true (see Remark 1). Conversely, under the above smoothness conditions on r_i and the setting of F and G , if H in (5) is a BLM distribution, then its marginal failure rate functions r_i should satisfy conditions (b)–(e) from which some properties in Theorem 1 follow immediately (Kulkarni 2006, Proposition 1).

We now recall three important BLM distributions in the literature. For more details, see, e.g., Chapter 10 of Balakrishnan and Lai (2009).

Example 1. *Marshall and Olkin's (1967) bivariate exponential distribution (BVE)*

If both marginals F and G are exponential, then the $BLM(F, G, \theta) \in \mathcal{BLM}$ defined in (5) reduces to the Marshall–Olkin BVE with survival function of the form

$$\begin{aligned} \bar{H}(x, y) &= \exp[-\lambda_1 x - \lambda_2 y - \lambda_{12} \max\{x, y\}] & (6) \\ &\equiv \frac{\lambda_1 + \lambda_2}{\lambda} \bar{H}_a(x, y) + \frac{\lambda_{12}}{\lambda} \bar{H}_s(x, y), \quad x, y \geq 0, & (7) \end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_{12}$ are positive constants, $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$, and H_a, H_s (written explicitly below) are absolutely continuous and singular bivariate distributions, respectively.

In practice, the Marshall–Olkin BVE arises from a shock model for a two-component system. Formally, the lifetimes of two components are $(X, Y) = (X_1 \wedge X_3, X_2 \wedge X_3)$, where $X_1 \sim \text{Exp}(\lambda_1)$, $X_2 \sim \text{Exp}(\lambda_2)$ and $X_3 \sim \text{Exp}(\lambda_{12})$ are independent. So they have a joint survival function \bar{H} defined in (6). The singular part in (7) is identified by the conditional probability: $\bar{H}_s(x, y) = \Pr(X > x, Y > y | X_3 \leq X_1 \wedge X_2) = \exp[-\lambda \max\{x, y\}]$, while the absolutely continuous part \bar{H}_a is calculated from \bar{H} and \bar{H}_s via (7) (see the next example).

Example 2. *Block and Basu's (1974) bivariate exponential distribution*

The Block–Basu BVE is actually the absolute continuous part H_a of Marshall–Olkin BVE in (7) and has a joint density of the form

$$h(x, y) = \begin{cases} \frac{\lambda_2 \lambda (\lambda_1 + \lambda_{12})}{\lambda_1 + \lambda_2} \exp[-(\lambda_1 + \lambda_{12})x - \lambda_2 y], & x \geq y > 0 \\ \frac{\lambda_1 \lambda (\lambda_2 + \lambda_{12})}{\lambda_1 + \lambda_2} \exp[-\lambda_1 x - (\lambda_2 + \lambda_{12})y], & y > x > 0, \end{cases} \quad (8)$$

where $\lambda_1, \lambda_2, \lambda_{12} > 0$, and $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$. Its survival function is equal to

$$\begin{aligned} \overline{H}(x, y) &= \overline{H}_a(x, y) \\ &= \frac{\lambda}{\lambda_1 + \lambda_2} \exp[-\lambda_1 x - \lambda_2 y - \lambda_{12} \max\{x, y\}] \\ &\quad - \frac{\lambda_{12}}{\lambda_1 + \lambda_2} \exp[-\lambda \max\{x, y\}], \quad x, y \geq 0. \end{aligned}$$

Note that in this case, the marginals are not exponential but rather negative mixtures of two exponentials. Specifically, $\overline{F}(x) = \frac{\lambda}{\lambda_1 + \lambda_2} \exp[-(\lambda_1 + \lambda_{12})x] - \frac{\lambda_{12}}{\lambda_1 + \lambda_2} \exp(-\lambda x)$, $x \geq 0$, and $\overline{G}(y) = \frac{\lambda}{\lambda_1 + \lambda_2} \exp[-(\lambda_2 + \lambda_{12})y] - \frac{\lambda_{12}}{\lambda_1 + \lambda_2} \exp(-\lambda y)$, $y \geq 0$.

Example 3. Freund's (1961) bivariate exponential distribution

The Freund BVE has a joint density of the form

$$h(x, y) = \begin{cases} \alpha' \beta \exp[-(\alpha + \beta - \alpha')y - \alpha'x], & x \geq y > 0 \\ \alpha \beta' \exp[-(\alpha + \beta - \beta')x - \beta'y], & y > x > 0, \end{cases} \quad (9)$$

where $\alpha, \alpha', \beta, \beta' > 0$. If $\alpha + \beta > \alpha' \vee \beta'$, its survival function is equal to

$$\overline{H}(x, y) = \begin{cases} \frac{\beta}{\alpha + \beta - \alpha'} \exp[-(\alpha + \beta - \alpha')y - \alpha'x] + \frac{\alpha - \alpha'}{\alpha + \beta - \alpha'} \exp[-(\alpha + \beta)x], & x \geq y \geq 0 \\ \frac{\alpha}{\alpha + \beta - \beta'} \exp[-(\alpha + \beta - \beta')x - \beta'y] + \frac{\beta - \beta'}{\alpha + \beta - \beta'} \exp[-(\alpha + \beta)y], & y \geq x \geq 0. \end{cases}$$

It worths noting that by choosing $\alpha = \frac{\lambda_1 \lambda}{\lambda_1 + \lambda_2}$, $\beta = \frac{\lambda_2 \lambda}{\lambda_1 + \lambda_2}$, $\alpha' = \lambda_1 + \lambda_{12}$ and $\beta' = \lambda_2 + \lambda_{12}$, Freund's BVE (9) reduces to Block and Basu's BVE (8).

3 New General Properties of BLM Distributions

Let $(X, Y) \sim H = BLM(F, G, \theta) \in \mathcal{BLM}$ with marginals F and G on $(0, \infty)$, parameter $\theta > 0$ and survival function (5). Denote the Laplace-Stieltjes transform of X (Y , resp.) by L_X (L_Y , resp.), and that of (X, Y) by \mathcal{L} . Then we have

THEOREM 2. The Laplace-Stieltjes transform of $(X, Y) \sim H = BLM(F, G, \theta) \in \mathcal{BLM}$ is

$$\begin{aligned} \mathcal{L}(s, t) &\equiv E[e^{-sX - tY}] \\ &= \frac{1}{\theta + s + t} [(\theta + s)L_X(s) + (\theta + t)L_Y(t)] - \frac{\theta}{\theta + s + t}, \quad s, t > 0. \end{aligned}$$

To prove Theorem 2, we need the following lemma due to Lin et al. (2016).

Lemma 1. *Let $(X, Y) \sim H$ defined on $\mathbb{R}_+^2 = [0, \infty) \times [0, \infty)$. Then the Laplace-Stieltjes transform of (X, Y) is equal to*

$$\mathcal{L}(s, t) = st \int_0^\infty \int_0^\infty \bar{H}(x, y) e^{-sx-ty} dx dy - 1 + L_X(s) + L_Y(t), \quad s, t \geq 0.$$

PROOF OF THEOREM 2. We have to calculate the double integral

$$\int_0^\infty \int_0^\infty \bar{H}(x, y) e^{-sx-ty} dx dy = \iint_{x \geq y} + \iint_{y \geq x} \equiv A_1 + A_2,$$

where, by changing variables and by integration by parts,

$$\begin{aligned} A_1 &= \int_0^\infty e^{-(\theta+t)y} \int_y^\infty e^{-sx} \bar{F}(x-y) dx dy = \int_0^\infty e^{-(\theta+s+t)y} \int_0^\infty e^{-sz} \bar{F}(z) dz dy \\ &= \frac{1}{\theta + s + t} \left[-\frac{1}{s} \int_0^\infty \bar{F}(z) d e^{-sz} \right] = \frac{1}{\theta + s + t} \left[\frac{1}{s} (1 - L_X(s)) \right], \end{aligned}$$

and similarly,

$$A_2 = \frac{1}{\theta + s + t} \left[\frac{1}{t} (1 - L_Y(t)) \right].$$

Lemma 1 together with the above A_1 and A_2 completes the proof.

Denote the moment generating function (mgf) of X (Y , resp.) by M_X (M_Y , resp.), and that of (X, Y) by \mathcal{M} . Then we have the following general result.

THEOREM 3. *Let $(X, Y) \sim H = BLM(F, G, \theta) \in \mathcal{BLM}$ and let r, s be real numbers such that $s + t < \theta$. Then the mgf of (X, Y) is*

$$\mathcal{M}(s, t) \equiv E [e^{sX+tY}] = \frac{1}{\theta - s - t} [(\theta - s)M_X(s) + (\theta - t)M_Y(t)] - \frac{\theta}{\theta - s - t},$$

provided the expectations (mgfs) exist.

To prove Theorem 3, we need instead the following lemma due to Lin et al. (2014).

Lemma 2. *Let $(X, Y) \sim H$ defined on \mathbb{R}_+^2 . Let α and β be two increasing and left-continuous functions on \mathbb{R}_+ . Then the expectation of the product $\alpha(X)\beta(Y)$ is equal to*

$$\begin{aligned} E[\alpha(X)\beta(Y)] &= \int_0^\infty \int_0^\infty \bar{H}(x, y) d\alpha(x) d\beta(y) - \alpha(0)\beta(0) + \alpha(0)E[\beta(Y)] \\ &\quad + \beta(0)E[\alpha(X)], \end{aligned}$$

provided the expectations exist.

PROOF OF THEOREM 3. Case (i): $s, t \geq 0$. Let $\alpha(x) = e^{sx}$ and $\beta(y) = e^{ty}$ in Lemma 2, then

$$\mathcal{M}(s, t) = st \int_0^\infty \int_0^\infty \bar{H}(x, y)e^{sx+ty} dx dy - 1 + M_X(s) + M_Y(t).$$

We have to calculate the double integral

$$\int_0^\infty \int_0^\infty \bar{H}(x, y)e^{sx+ty} dx dy = \int \int_{x \geq y} + \int \int_{y \geq x} \equiv B_1 + B_2,$$

where, as before,

$$B_1 = \frac{1}{\theta - s - t} \left[\frac{1}{s}(-1 + M_X(s)) \right] \quad \text{and} \quad B_2 = \frac{1}{\theta - s - t} \left[\frac{1}{t}(-1 + M_Y(t)) \right].$$

Lemma 2 together with the above B_1 and B_2 completes the proof of Case (i). Case (ii): $s \geq 0, t < 0$. To apply Lemma 2, set $\alpha(x) = e^{sx}$ and $\beta(y) = 1 - e^{ty}$. Then both α and β are increasing functions on \mathbb{R}_+ and $E[\alpha(X)\beta(Y)] = M_X(s) - \mathcal{M}(s, t)$. Therefore,

$$\begin{aligned} \mathcal{M}(s, t) &= M_X(s) - E[\alpha(X)\beta(Y)] \\ &= M_X(s) - 1 + M_Y(t) + st \int_0^\infty \int_0^\infty \bar{H}(x, y)e^{sx+ty} dx dy. \end{aligned}$$

As before, we carry out the above double integral and complete the proof of Case (ii). Case (iii): $s < 0, t \geq 0$. Set $\alpha(x) = 1 - e^{sx}$ and $\beta(y) = e^{ty}$ in Lemma 2. The remaining proof is similar to that of Case (ii) and is omitted. Case (iv): $s, t < 0$. This case was treated in Theorem 1. The proof is completed.

Next, we consider the product moments of BLM distributions.

THEOREM 4. For positive integers i and j , the product moment $E[X^i Y^j]$ of $(X, Y) \sim H = BLM(F, G, \theta) \in \mathcal{BLM}$ is of the form

$$\begin{aligned} E[X^i Y^j] &= ij \sum_{k=0}^{i-1} \frac{1}{i-k} \binom{i-1}{k} \frac{\Gamma(j+k)}{\theta^{j+k}} E[X^{i-k}] \\ &\quad + ij \sum_{k=0}^{j-1} \frac{1}{j-k} \binom{j-1}{k} \frac{\Gamma(i+k)}{\theta^{i+k}} E[Y^{j-k}], \end{aligned}$$

provided the expectations exist.

The first product moment has a neat representation in terms of marginal means and the parameter θ , from which we can calculate Pearson's correlation of BLM distributions.

Corollary 1. $E[XY] = \frac{1}{\theta}(E[X] + E[Y])$ provided the expectations exist.

To prove Theorem 4 above, we will apply the following lemma due to Lin et al. (2014).

Lemma 3. Let $(X, Y) \sim H$ defined on \mathbb{R}_+^2 , and let the expectations $E[X^r Y^s]$, $E[X^r]$ and $E[Y^s]$ be finite for some positive real numbers r and s . Then the product moment

$$E[X^r Y^s] = rs \int_0^\infty \int_0^\infty \bar{H}(x, y) x^{r-1} y^{s-1} dx dy.$$

PROOF THEOREM 4. We have to calculate the double integral

$$\int_0^\infty \int_0^\infty \bar{H}(x, y) x^{i-1} y^{j-1} dx dy = \int \int_{x \geq y} + \int \int_{y \geq x} \equiv C_1 + C_2,$$

where, by changing variables and by integration by parts,

$$\begin{aligned} C_1 &= \int_0^\infty e^{-\theta y} y^{j-1} \int_y^\infty x^{i-1} \bar{F}(x-y) dx dy \\ &= \int_0^\infty e^{-\theta y} y^{j-1} \int_0^\infty (y+z)^{i-1} \bar{F}(z) dz dy \\ &= \sum_{k=0}^{i-1} \binom{i-1}{k} \int_0^\infty y^{j-1+k} e^{-\theta y} \int_0^\infty z^{i-1-k} \bar{F}(z) dz dy \\ &= \sum_{k=0}^{i-1} \binom{i-1}{k} \frac{\Gamma(j+k)}{\theta^{j+k}} \left[\frac{1}{i-k} \int_0^\infty \bar{F}(z) dz^{i-k} \right] \\ &= \sum_{k=0}^{i-1} \frac{1}{i-k} \binom{i-1}{k} \frac{\Gamma(j+k)}{\theta^{j+k}} E[X^{i-k}], \end{aligned}$$

and similarly,

$$C_2 = \sum_{k=0}^{j-1} \frac{1}{j-k} \binom{j-1}{k} \frac{\Gamma(i+k)}{\theta^{i+k}} E[Y^{j-k}].$$

Finally, Lemma 3 together with the above C_1 and C_2 completes the proof.

For moment generating functions of some *specific* BLM distributions, see Chapter 47 of Kotz et al. (2000), while for product moments of such distributions, see Nadarajah (2006).

For the next and later results, we need some notations in reliability theory. For random variables $X \sim F$ and $Y \sim G$, we say that X is smaller than

Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x , that X is smaller than Y in the hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x , and that X is smaller than Y in the reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $G(x)/F(x)$ is increasing in x . Suppose F and G have densities f and g , respectively. Then we say that X is smaller than Y in the likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in x . For more definitions of the related stochastic orders, see, e.g., Müller and Stoyan (2002), Shaked and Shanthikumar (2007), Lai and Xie (2006) as well as Kayid et al. (2016). The latter studied stochastic comparisons of the age replacement models.

On the other hand, for a distribution F itself we define the notions of increasing failure rate (IFR), decreasing failure rate (DFR), increasing failure rate in average (IFRA), and decreasing failure rate in average (DFRA) as follows. We say that

- (a) F is IFR (DFR, resp.) if $-\log \bar{F}(x)$ is convex (concave, resp.) in $x \geq 0$, and
- (b) F is IFRA (DFRA, resp.) if $-(1/x)\log \bar{F}(x)$ is increasing (decreasing, resp.) in $x > 0$, or, equivalently, $\bar{F}^\alpha(x) \leq (\geq, \text{ resp.}) \bar{F}(\alpha x)$ for all $\alpha \in (0, 1)$ and $x \geq 0$. (See Barlow and Proschan 1981, Chapters 3 and 4.)

The bivariate IFRA and bivariate DFRA distributions H can be defined similarly: H is bivariate IFRA (DFRA, resp.) if $\bar{H}^\alpha(x, y) \leq (\geq, \text{ resp.}) \bar{H}(\alpha x, \alpha y)$ for all $\alpha \in (0, 1)$ and $x, y \geq 0$ (see Block and Savits 1976, 1980). It worths mentioning that there are some other definitions of bivariate IFRA distributions that all extend the univariate case (see, e.g., Esary and Marshall 1979 or Shaked and Shanthikumar 1988).

Using reliability language, we have the following useful results. Especially, Theorem 5(iii) means that in the \mathcal{BLM} family, positive bivariate aging plays in favor of positive univariate aging in the sense of IFRA, and vice versa. This is in general not true even under the condition of positive dependence for lifetimes; see Bassan and Spizzichino (2005, Remark 6.8), which analyzed the relations among univariate and bivariate agings and dependence.

THEOREM 5. *Let $(X, Y) \sim H = BLM(F, G, \theta) \in \mathcal{BLM}$ and $Z \sim Exp(\theta)$. Then*

- (i) $Z \leq_{lr} X$ and $Z \leq_{lr} Y$;
- (ii) $Z \leq_{st} X$ and $Z \leq_{st} Y$; $Z \leq_{hr} X$ and $Z \leq_{hr} Y$; $Z \leq_{rh} X$ and $Z \leq_{rh} Y$;

- (iii) (X, Y) has a bivariate IFRA distribution iff both marginals F and G are IFRA;
- (iv) (X, Y) has a bivariate DFRA distribution iff both marginals F and G are DFRA.

PROOF. Part (i) follows immediately from Theorem 1(iii) (see Ghurye and Marshall 1984), while part (ii) follows from the fact that the likelihood ratio order is stronger than the usual stochastic order, hazard rate order, and reversed hazard rate order (Müller and Stoyan 2002, pp. 12–13). Part (iii) holds true by verifying that $\overline{H}(\alpha x, \alpha y) \geq \overline{H}^\alpha(x, y) \forall \alpha \in (0, 1), x, y \geq 0$, if, and only if, (a) $\overline{F}(\alpha x) \geq \overline{F}^\alpha(x) \forall \alpha \in (0, 1), x \geq 0$, and (b) $\overline{G}(\alpha y) \geq \overline{G}^\alpha(y) \forall \alpha \in (0, 1), y \geq 0$. The proof of part (iv) is similar.

Applying the above stochastic inequalities, we can simplify the proof of some previous known results. For example, we have

Corollary 2. *Let $(X, Y) \sim H = BLM(F, G, \theta) \in \mathcal{BLM}$. Then the following statements are true.*

- (i) Both hazard rates of marginals F, G are bounded by θ and hence $\theta \geq f(0) \vee g(0)$.
- (ii) Both the functions $F(-\frac{1}{\theta} \log(1 - t))$ and $G(-\frac{1}{\theta} \log(1 - t))$ are convex in $t \in [0, 1)$, and hence $f'(x) + \theta f(x) \geq 0, g'(x) + \theta g(x) \geq 0, x \geq 0$, if f and g are differentiable.
- (iii) Let S_F, S_G be the supports of marginals F, G with densities f, g , respectively. Then $S_F = [a_F, \infty), S_G = [a_G, \infty)$ for some nonnegative constants a_F, a_G with $a_F a_G = 0$, and f, g are positive on $(a_F, \infty), (a_G, \infty)$, respectively.
- (iv) If H is not absolutely continuous, then $f(0) > 0, g(0) > 0$, and hence $a_F = a_G = 0$.

PROOF. Part (i) follows from the facts $Z \leq_{hr} X$ and $Z \leq_{hr} Y$, where $Z \sim Exp(\theta)$, while part (ii) is due to the probability-probability plot characterization for $Z \leq_{lr} X$ and $Z \leq_{lr} Y$ (see Theorem 1.4.3 of Müller and Stoyan 2002). Part (iii) follows from the facts $Z \leq_{lr} X, Z \leq_{lr} Y$ and Theorem 1(iv), because the latter implies that at least one of the left extremities a_F and a_G of marginal distributions should be zero. Finally, to prove part (iv), we note that $\Pr(X - Y > 0) = 1 - f(0)/\theta$ and $\Pr(Y - X > 0) = 1 - g(0)/\theta$ by Theorem 1(ii) (see Ghurye and Marshall 1984, p. 789). So if H is not absolutely continuous, $\Pr(X = Y) > 0$, and hence $f(0) = \theta \Pr(X \leq Y) > 0$ and $g(0) = \theta \Pr(Y \leq X) > 0$. The proof is complete.

4 Dependence Structures of BLM Distributions

Recall that a bivariate distribution H with marginals F and G is positively quadrant dependent (PQD) if

$$H(x, y) \geq F(x)G(y) \quad \forall x, y \geq 0, \text{ or, equivalently, } \overline{H}(x, y) \geq \overline{F}(x)\overline{G}(y) \quad \forall x, y \geq 0,$$

which implies that H has a nonnegative covariance by Hoeffding representation for covariance (see, e.g., Lin et al. 2014, p. 2). A stronger (positive dependence) property than the PQD is the total positivity defined below. For a nonnegative function K on the rectangle $(a, b) \times (c, d)$ (or on the product of two subsets of \mathbb{R}), we say that $K(x, y)$ is totally positive of order r ($TP_r, r \geq 2$) in x and y if for each fixed $s \in \{2, 3, \dots, r\}$ and for all $a < x_1 < x_2 < \dots < x_s < b$ and $c < y_1 < y_2 < \dots < y_s < d$, the determinant of the $s \times s$ matrix $(K(x_i, y_j))$ is nonnegative. The function K is said to be TP_∞ if it is TP_r for any order $r \geq 2$ (Karlin, 1968).

The total positivity plays an important role on various concepts of bivariate dependence (see, e.g., Shaked 1977 and Lee 1985). Moreover, applying total positivity of the bivariate distribution or its survival function, we can derive some useful probability inequalities, among many applications to applied fields including statistics, reliability, and economics (see, e.g., Gross and Richards 1988, 2004, and Karlin and Proschan 1960). Especially, the latter studied the totally positive kernels that arise from convolutions of Pólya type distributions.

We now characterize the TP_2 property of the survival functions of BLM distributions.

THEOREM 6. *Let $(X, Y) \sim H = BLM(F, G, \theta) \in \mathcal{BLM}$. Then the survival function \overline{H} is TP_2 iff the marginal distributions F and G are IFR and together satisfy $\overline{F}(x)\overline{G}(x) \leq \exp(-\theta x), x \geq 0$.*

PROOF. We define the cross-product ratio of \overline{H} :

$$r \equiv r(x_1, x_2; y_1, y_2) = \frac{\overline{H}(x_1, y_1)\overline{H}(x_2, y_2)}{\overline{H}(x_1, y_2)\overline{H}(x_2, y_1)}, \quad 0 < x_1 < x_2, \quad 0 < y_1 < y_2.$$

Then, by definition, \overline{H} is TP_2 iff $r(x_1, x_2; y_1, y_2) \geq 1$ for all $0 < x_1 < x_2, 0 < y_1 < y_2$.

(Necessity) Suppose that \overline{H} is TP_2 . Then for all $0 < x_1 = y_1 < x_2 = y_2$, we have

$$r = r(x_1, x_2; x_1, x_2) = \frac{\overline{H}(x_1, x_1)\overline{H}(x_2, x_2)}{\overline{H}(x_1, x_2)\overline{H}(x_2, x_1)} = \frac{\exp(-\theta(x_2 - x_1))}{\overline{F}(x_2 - x_1)\overline{G}(x_2 - x_1)} \geq 1.$$

This implies that $\overline{F}(x)\overline{G}(x) \leq \exp(-\theta x)$, $x \geq 0$. Next, we prove that the marginal distribution G is IFR. Note that the following statements are equivalent:

- (i) $g(y)/\overline{G}(y)$ is increasing in $y \geq 0$,
- (ii) $\frac{\overline{G}(y+t)}{\overline{G}(t)}$ is decreasing in $t \in (0, \infty)$ for each $y \geq 0$ (Barlow and Proschan 1981, p. 54),
- (iii) $\frac{\overline{G}(t)}{\overline{G}(y+t)}$ is increasing in $t \in (0, \infty)$ for each $y \geq 0$,
- (iv) $\frac{\overline{G}(y-x_2)}{\overline{G}(y-x_1)}$ is increasing in $y > x_2$ for any fixed $0 < x_1 < x_2$,
- (v) the ratio $r_G^* \equiv \frac{\overline{G}(y_1-x_1)\overline{G}(y_2-x_2)}{\overline{G}(y_2-x_1)\overline{G}(y_1-x_2)} \geq 1$ for all $0 < x_1 < x_2 < y_1 < y_2$.

The latter is true because in this case $r_G^* = r(x_1, x_2; y_1, y_2) \geq 1$ by (5) and the assumption. Similarly, we can prove that F is IFR because the ratio

$$r_F^* \equiv \frac{\overline{F}(x_1 - y_1)\overline{F}(x_2 - y_2)}{\overline{F}(x_2 - y_1)\overline{F}(x_1 - y_2)} \geq 1 \text{ for all } 0 < y_1 < y_2 < x_1 < x_2.$$

(Sufficiency) Suppose that the marginal distributions F and G are IFR and together satisfy $\overline{F}(x)\overline{G}(x) \leq \exp(-\theta x)$, $x \geq 0$. Then we want to prove that \overline{H} is TP₂, that is, for all $0 < x_1 < x_2$, $0 < y_1 < y_2$, the cross-product ratio $r = r(x_1, x_2; y_1, y_2) \geq 1$. Without loss of generality, we consider only three possible cases below,

- (a) $0 < x_1 \leq x_2 \leq y_1 \leq y_2$, (b) $0 < x_1 \leq y_1 \leq x_2 \leq y_2$, (c) $0 < x_1 \leq y_1 \leq y_2 \leq x_2$,

because the remaining cases can be proved by exchanging the roles of F and G .

For case (a), we have $r \geq 1$ by the equivalence relations shown in the necessity part and by the continuity of H when $x_2 = y_1$. For case (b), the cross-product ratio

$$r = \frac{\exp(-\theta x_2)\overline{G}(y_1 - x_1)\overline{G}(y_2 - x_2)}{\exp(-\theta y_1)\overline{G}(y_2 - x_1)\overline{F}(x_2 - y_1)} \geq \frac{\overline{G}(y_1 - x_1)\overline{G}(y_2 - x_2)\overline{G}(x_2 - y_1)}{\overline{G}(y_2 - x_1)},$$

because $\overline{F}(x_2 - y_1)\overline{G}(x_2 - y_1) \leq \exp(-\theta(x_2 - y_1))$ by the assumption. Recall that any IFR distribution is new better than used (Barlow and Proschan

1981, p. 159). Therefore, $\overline{G}(x + y) \leq \overline{G}(x)\overline{G}(y)$ for all $x, y \geq 0$, and hence the last $r \geq 1$. Similarly, for case (c),

$$\begin{aligned} r &= \frac{\exp(-\theta y_2)\overline{G}(y_1 - x_1)\overline{F}(x_2 - y_2)}{\exp(-\theta y_1)\overline{G}(y_2 - x_1)\overline{F}(x_2 - y_1)} \\ &\geq \frac{\overline{G}(y_1 - x_1)\overline{G}(y_2 - y_1)}{\overline{G}(y_2 - x_1)} \times \frac{\overline{F}(y_2 - y_1)\overline{F}(x_2 - y_2)}{\overline{F}(x_2 - y_1)} \geq 1, \end{aligned}$$

by the assumptions. This completes the proof.

Recall also that for any bivariate distribution H with marginals F and G , there exist a copula C (a bivariate distribution with uniform marginals on $[0, 1]$) and a survival copula \hat{C} such that $H(x, y) = C(F(x), G(y))$ and $\overline{H}(x, y) = \hat{C}(\overline{F}(x), \overline{G}(y))$ for all $x, y \in \mathbb{R} \equiv (-\infty, \infty)$. Namely, C links H and (F, G) , while \hat{C} links \overline{H} and $(\overline{F}, \overline{G})$.

Corollary 3. *Let $(X, Y) \sim H = BLM(F, G, \theta) \in \mathcal{BLM}$. Then the survival copula \hat{C} of H is TP_2 iff the marginal distributions F and G are IFR and together satisfy $\overline{F}(x)\overline{G}(x) \leq \exp(-\theta x)$, $x \geq 0$.*

PROOF. Since the marginal F is absolutely continuous on the support $[a_F, \infty)$ with positive density f on (a_F, ∞) (see Corollary 2(iii) above), F is strictly increasing and continuous on (a_F, ∞) . Similarly, the marginal G is strictly increasing and continuous on (a_G, ∞) . By Theorem 6, it suffices to prove that \overline{H} is TP_2 on $(a_F, \infty) \times (a_G, \infty)$ iff its survival copula \hat{C} is TP_2 on $(0, 1)^2$. Recall the facts (i) $\overline{H}(x, y) = \hat{C}(\overline{F}(x), \overline{G}(y))$, $(x, y) \in (a_F, \infty) \times (a_G, \infty)$, (ii) $\hat{C}(u, v) = \overline{H}(\overline{F}^{-1}(u), \overline{G}^{-1}(v))$, $u, v \in (0, 1)$, where $\overline{F}^{-1}, \overline{G}^{-1}$ are inverse functions of $\overline{F}, \overline{G}$, respectively, and (iii) all the functions $\overline{F}, \overline{G}, \overline{F}^{-1}$ and \overline{G}^{-1} are decreasing. The required result then follows immediately (see, e.g., Lemma 5 (ii) below).

The counterpart of TP_2 property is the reverse regular of order two (RR_2). For a nonnegative function K on $(a, b) \times (c, d)$, we say that K is RR_2 if the determinant of the 2×2 matrix $(K(x_i, y_j))$ is non-positive for all $a < x_1 < x_2 < b$ and $c < y_1 < y_2 < d$ (see, e.g., Esna-Ashari and Asadi 2016 for examples of RR_2 joint densities and survival functions). Mimicking the proof of Theorem 6, we conclude that for $H = BLM(F, G, \theta) \in \mathcal{BLM}$, the survival function \overline{H} is RR_2 iff the survival copula \hat{C} of H is RR_2 iff the marginal distributions F and G are DFR and satisfy $\overline{F}(x)\overline{G}(x) \geq \exp(-\theta x)$, $x \geq 0$. To construct such a BLM distribution with RR_2 survival function, we first consider the Pareto Type II distribution (or Lomax distribution) F with density function $f(x) = (\alpha/\beta)(1 + x/\beta)^{-(\alpha+1)}$, $x \geq 0$, and survival function

$\bar{F}(x) = (1 + x/\beta)^{-\alpha}$, $x \geq 0$, where $\alpha, \beta > 0$. Then choose the parameters: $\alpha \geq 1$, $\beta > 0$ and $\theta = (\alpha + 1)/\beta$. It can be checked that the \bar{H} defined in (5) with $G = F$ is a *bona fide* survival function, and is RR_2 if $\alpha = 1$.

It is seen that all the conditions in Theorem 6 are satisfied by the Marshall–Olkin BVE. Therefore, the survival function and survival copula of the Marshall–Olkin BVE are both TP_2 , regardless of parameters; a more general result will be given in Theorem 8 below. We next characterize, by a different approach, the TP_2 property of some joint densities of absolutely continuous BLM distributions.

THEOREM 7. *Let $H = \text{BLM}(F, G, \theta) \in \mathcal{BLM}$ be absolutely continuous and have joint density function h . Suppose that the marginal density functions f and g are three times differentiable on $(0, \infty)$ and that $\theta f(0^+) + f'(0^+) = \theta g(0^+) + g'(0^+)$ is finite. Assume further the functions*

$$h_1(x|\theta) \equiv \theta f(x) + f'(x) > 0, \quad x > 0, \quad \text{and} \quad h_2(y|\theta) \equiv \theta g(y) + g'(y) > 0, \quad y > 0.$$

Then the joint density function h is TP_2 iff the marginal densities satisfy (i) $(h'_i(x|\theta))^2 \geq h''_i(x|\theta)h_i(x|\theta)$, $x > 0$, $i = 1, 2$, and (ii) $h_1(x|\theta)h_2(x|\theta) \leq h_1^2(0^+|\theta) \exp(-\theta x)$, $x > 0$.

To prove this theorem, we need the concept of local dependence function and the following lemma, in which part (ii) is essentially due to Holland and Wang (1987, p. 872). An alternative (complete) proof of part (ii) is provided below. In their proof, Holland and Wang (1987) assumed implicitly the *integrability* of the local dependence function, while Kemperman (1977, p. 329) gave without proof the same result under *continuity* (smoothness) condition (see also Newman 1984). Wang (1993) proved that a positive continuous bivariate density on a Cartesian product $(a, b) \times (c, d)$ is uniquely determined by its marginal densities and local dependence function when the latter exists and is integrable. On the other hand, Jones (1996, 1998) investigated the bivariate distributions with constant local dependence.

Lemma 4. *Let K be a positive function on $D = (a, b) \times (c, d)$. Then we have*

(i) *K is TP_2 on D iff $\log K$ is 2-increasing;*

(ii) *K is TP_2 on D iff the local dependence function $\gamma_K(x, y) \equiv \frac{\partial^2}{\partial x \partial y} \log K(x, y) \geq 0$ on D , provided the second-order partial derivatives exist.*

PROOF. Part (i) is trivial by the definition of 2-increasing functions (see Nelsen 2006, p. 8), and part (ii) follows from part (i) and the fact

that under the smoothness assumption, $\log K$ is 2-increasing iff the local dependence function $\gamma_K(x, y) \geq 0$. To prove part (ii) directly, note that the following statements are equivalent: (a) $\frac{\partial^2}{\partial x \partial y} \log K(x, y) \geq 0$ on D , (b) $\frac{\partial}{\partial y} \log[K(x_2, y)/K(x_1, y)] \geq 0$ for all y and for all $x_1 < x_2$, (c) $\log[K(x_2, y)/K(x_1, y)]$ is increasing in y for all $x_1 < x_2$, (d) $K(x_2, y)/K(x_1, y)$ is increasing in y for all $x_1 < x_2$, (e) $K(x_2, y_2)/K(x_1, y_2) \geq K(x_2, y_1)/K(x_1, y_1)$ for all $y_1 < y_2$, $x_1 < x_2$, (f) the cross-product ratio of K satisfies: $K(x_1, y_1)K(x_2, y_2)/[K(x_1, y_2)K(x_2, y_1)] \geq 1$ for all $x_1 < x_2$, $y_1 < y_2$, and (g) the function K is TP₂ on D . The proof is complete.

PROOF OF THEOREM 7. By the assumptions, the joint density function of H is of the form

$$h(x, y) = \begin{cases} e^{-\theta y} h_1(x - y|\theta), & x \geq y \\ e^{-\theta x} h_2(y - x|\theta), & x \leq y, \end{cases}$$

where $h_i(0|\theta) \equiv h_i(0^+|\theta)$, $i = 1, 2$. For $x \neq y$, the local dependence function of h is

$$\gamma_h(x, y) = \frac{\partial^2}{\partial x \partial y} \log h(x, y) = \begin{cases} \frac{[h'_1(x-y|\theta)]^2 - h''_1(x-y|\theta)h_1(x-y|\theta)}{h_1^2(x-y|\theta)}, & x > y \\ \frac{[h'_2(y-x|\theta)]^2 - h''_2(y-x|\theta)h_2(y-x|\theta)}{h_2^2(y-x|\theta)}, & x < y. \end{cases}$$

Therefore, $\gamma_h(x, y) \geq 0$ for all (x, y) with $x \neq y$ iff the property (i) holds true.

(Necessity) If h is TP₂ on $(0, \infty)^2$, then it is also TP₂ on each rectangle (rectangular area) in the region $\mathcal{A}_1 = \{(x, y) : x > y > 0\}$ or in $\mathcal{A}_2 = \{(x, y) : y > x > 0\}$, and hence the property (i) holds true by Lemma 4 and the above observation. Next, the property (ii) follows from the fact that for all $0 < x_1 = y_1 < x_2 = y_2$, the cross-product ratio r_h of h satisfies

$$1 \leq r_h \equiv r_h(x_1, x_2; y_1, y_2) = \frac{h(x_1, y_1)h(x_2, y_2)}{h(x_1, y_2)h(x_2, y_1)} = \frac{\exp(-\theta(x_2 - x_1))h_1(0|\theta)h_2(0|\theta)}{h_1(x_2 - x_1|\theta)h_2(x_2 - x_1|\theta)}.$$

This completes the proof of the necessity part.

(Sufficiency) Suppose $0 < x_1 < x_2$ and $0 < y_1 < y_2$, then we want to prove the cross-product ratio $r_h \geq 1$ under the assumptions (i) and (ii). If the rectangle with four vertices $P_i, i = 1, 2, 3, 4$, where $P_1 = (x_1, y_1), P_2 = (x_2, y_1), P_3 = (x_2, y_2), P_4 = (x_1, y_2)$, lies entirely in the region \mathcal{A}_1 or \mathcal{A}_2 , then $r_h \geq 1$ by the assumption (i) and Lemma 4. If $0 < x_1 = y_1 < x_2 = y_2$, then the assumption (ii) implies $r_h \geq 1$. For the remaining cases, we apply the technique of factorization of the cross-product ratio if necessary. For

example, if $P_* = (x_1, y_*) \in \overline{P_1P_4}$ and $P^* = (x^*, y_2) \in \overline{P_4P_3}$ denote the intersection of the diagonal line $x = y$ and boundary of the rectangle, where $x_1 < x^* < x_2$ and $y_1 < y_* < y_2$, then we split the original rectangle into four sub-rectangles by adding the new point (x^*, y_*) and calculate the ratio

$$\begin{aligned} r_h(x_1, x_2; y_1, y_2) &= r_h(x_1, x^*; y_1, y_*) r_h(x_1, x^*; y_*, y_2) r_h(x^*, x_2; y_1, y_*) r_h(x^*, x_2; y_*, y_2) \geq 1, \end{aligned}$$

each factor being greater than or equal to one by the previous results. The proof is complete.

It is known that the Marshall–Olkin BVE (6) is PQD, so are its copula C and survival copula \hat{C} (see Barlow and Proschan 1981, p. 129). Moreover, \overline{H} and \hat{C} are TP_2 due to Theorem 6 and its corollary (see also Nelsen 2006, p. 163, for a direct proof) and both are even TP_∞ if $\lambda_1 = \lambda_2$ (Lin et al. 2016). We are now able to extend these results to the following.

THEOREM 8. *The Marshall–Olkin survival function \overline{H} and survival copula \hat{C} are both TP_∞ , regardless of parameters.*

To prove this theorem, we need two more useful lemmas. Lemma 5 is well-known (see, e.g., Marshall et al. 2011, p. 758), while Lemma 6 is essentially due to Gantmacher and Krein (2002), pp. 78–79 (see also Karlin 1968, p. 112, for an alternative version).

Lemma 5. *Let $r \geq 2$ be an integer.*

- (i) *If $k(x, y)$ is TP_r in x and y , and if both u and v are nonnegative functions, then the product function $K(x, y) = u(x)v(y)k(x, y)$ is TP_r in x and y .*
- (ii) *If $k(x, y)$ is TP_r in x and y , and if u and v are both increasing, or both decreasing, then the composition function $K(x, y) = k(u(x), v(y))$ is TP_r in x and y .*

Lemma 6. *Let ϕ and ψ be two positive functions on (a, b) . Define the symmetric function*

$$K_s(x, y) = \begin{cases} \psi(x)\phi(y), & a < y \leq x < b \\ \phi(x)\psi(y), & a < x \leq y < b. \end{cases}$$

If $\phi(x)/\psi(x)$ is nondecreasing in $x \in (a, b)$, then the function $K_s(x, y)$ is TP_∞ in x and y .

PROOF OF THEOREM 8. We prove first that the Marshall–Olkin survival function \bar{H} is TP_∞ . Rewrite the survival function (6) as

$$\begin{aligned} \bar{H}(x, y) &= \begin{cases} \exp[-(\lambda_1 + \lambda_{12})x - \lambda_2 y], & x \geq y \\ \exp[-(\lambda_2 + \lambda_{12})y - \lambda_1 x], & x \leq y \end{cases} \\ &= \exp[-\lambda_1 x - \lambda_2 y] K_s(x, y), \end{aligned}$$

where the symmetric function

$$K_s(x, y) = \begin{cases} \exp(-\lambda_{12}x), & x \geq y \\ \exp(-\lambda_{12}y), & x \leq y. \end{cases} \tag{10}$$

Let $\phi(x) = 1$ and $\psi(y) = \exp(-\lambda_{12}y)$. Then by Lemma 6, we see that the function K_s in (10) is TP_∞ , so is \bar{H} by Lemma 5(i). Next, recall that the Marshall–Olkin survival copula

$$\hat{C}(u, v) = \bar{H}(\bar{F}^{-1}(u), \bar{G}^{-1}(v)), \quad u, v \in (0, 1),$$

where \bar{F}^{-1} and \bar{G}^{-1} are the inverse (decreasing) functions of $\bar{F}(x) = \exp[-(\lambda_1 + \lambda_{12})x]$ and $\bar{G}(y) = \exp[-(\lambda_2 + \lambda_{12})y]$, respectively. Therefore, \hat{C} is TP_∞ by Lemma 5(ii).

It is well-known that if a bivariate distribution H has TP_2 density, then both H and its joint survival function \bar{H} are TP_2 (see, e.g., Balakrishnan and Lai 2009, p. 116). A more general result is given as follows.

THEOREM 9. *If the bivariate distribution H has TP_r density with $r \geq 2$, then both H and \bar{H} are TP_r . Consequently, if H has TP_∞ density, then both H and \bar{H} are TP_∞ .*

PROOF. Let us consider first the TP_∞ indicator functions $K_1(x, y) = I_{(-\infty, x]}(y)$ and $K_2(x, y) = I_{[x, \infty)}(y)$, and then apply Theorem 3.5 of Gross and Richards (1988) restated below.

For example, to prove the TP_r property of H , we have to claim that for all $x_1 < \dots < x_r$ and $y_1 < \dots < y_r$, the determinant of each $s \times s$ sub-matrix $(H(x_i, y_j))$ (with $2 \leq s \leq r$) is nonnegative. To prove this, let us recall that $H(x_i, y_j) = E[I_{(-\infty, x_i]}(X) I_{(-\infty, y_j]}(Y)] = E[\phi(i, X) \psi(j, Y)]$, where $\phi(i, x) = I_{(-\infty, x_i]}(x)$ is TP_r in two variables $i \in \{1, 2, \dots, r\}$ and $x \in \mathbb{R}$, and $\psi(j, y) = I_{(-\infty, y_j]}(y)$ is TP_r in two variables $j \in \{1, 2, \dots, r\}$ and $y \in \mathbb{R}$. Then Gross

and Richards' Theorem applies and hence H is TP_r . Similarly, the survival function \bar{H} is TP_r . The proof is complete.

Gross and Richards' (1988) Theorem Let $r \geq 2$ be an integer and let the bivariate $(X, Y) \sim H$ have TP_r density. Assume further that both the functions $\phi(i, x)$ and $\psi(i, x)$ are TP_r in two variables $i \in \{1, 2, \dots, r\}$ and $x \in \mathbb{R}$. Then the $r \times r$ matrix $(E[\phi(i, X)\psi(j, Y)])$ is totally positive, that is, all its minors (of orders $\leq r$) are nonnegative real numbers.

As mentioned in Balakrishnan and Lai (2009, p.124), the Block–Basu BVE (8) is PQD if $\lambda_1 = \lambda_2$. We now extend this result to the following.

THEOREM 10. (i) If $\lambda_1 = \lambda_2$ in (8), then the Block–Basu BVE has TP_∞ density.

(ii) If $\alpha = \beta \leq \alpha' = \beta'$ in (9), then the Freund BVE has TP_∞ density.

PROOF. Take $\phi(x) = c_1 \exp(-\lambda_1 x)$ and $\psi(y) = c_2 \exp[-(\lambda_2 + \lambda_{12})y]$ for some constants $c_1, c_2 > 0$. Then part (i) follows from (8) and Lemma 6. Part (ii) can be proved similarly.

REMARK 4. The same approach applies to other bivariate (non-BLM) distributions like Li and Pellerey's (2011) generalized Marshall–Olkin bivariate distribution described below.

In Marshall and Olkin's (1967) shock model: $(X, Y) = (X_1 \wedge X_3, X_2 \wedge X_3)$, we assume instead that X_1, X_2, X_3 are independent general positive random variables (not limited to exponential ones) and that $X_i \sim F_i, i = 1, 2, 3$. Let $R_i = -\log \bar{F}_i$ be the hazard function of X_i . Then the generalized Marshall–Olkin bivariate distribution H has survival function

$$\begin{aligned} \bar{H}(x, y) &= \Pr(X > x, Y > y) = \Pr(X_1 > x, X_2 > y, X_3 > \max\{x, y\}) \\ &= \exp[-R_1(x) - R_2(y) - R_3(\max\{x, y\})], \quad x, y \geq 0, \end{aligned} \tag{11}$$

which is PQD (Li and Pellerey, 2011). (For other related shock models, see Marshall and Olkin 1967, Ghurye and Marshall 1984 as well as Aven and Jensen 2013, Section 5.3.4.) We now extend this result and Theorem 8 as follows.

THEOREM 11. Let H be the generalized Marshall–Olkin distribution defined in (11). Then

- (i) the survival function \bar{H} is TP_∞ ;
- (ii) the survival copula \hat{C} of H is TP_∞ , provided the functions $\bar{F}_1 \bar{F}_3$ and $\bar{F}_2 \bar{F}_3$ are both strictly decreasing.

PROOF. Write the survival function (11) as

$$\begin{aligned} \overline{H}(x, y) &= \begin{cases} \exp[-(R_1(x) + R_3(x)) - R_2(y)], & x \geq y \\ \exp[-(R_2(y) + R_3(y)) - R_1(x)], & x \leq y \end{cases} \\ &= \exp[-R_1(x) - R_2(y)]K_s(x, y), \end{aligned}$$

where the symmetric function

$$K_s(x, y) = \begin{cases} \exp[-R_3(x)], & x \geq y \\ \exp[-R_3(y)], & x \leq y. \end{cases} \tag{12}$$

By taking $\phi(x) = 1$ and $\psi(y) = \exp[-R_3(y)]$ in Lemma 6, we know that the function K_s in (12) is TP_∞ , and hence the survival function \overline{H} is TP_∞ by Lemma 5(i). This proves part (i). To prove part (ii), we note that the marginal survival functions of H are $\overline{F}(x) = \exp[-\tilde{R}_1(x)]$, $x \geq 0$, and $\overline{G}(y) = \exp[-\tilde{R}_2(y)]$, $y \geq 0$, where the two functions $\tilde{R}_1(x) = R_1(x) + R_3(x)$, $x \geq 0$, and $\tilde{R}_2(y) = R_2(y) + R_3(y)$, $y \geq 0$, are strictly increasing by the conditions on F_i , $i = 1, 2, 3$. This in turn implies that the marginal distribution functions F and G are strictly increasing and hence the survival copula

$$\hat{C}(u, v) = \overline{H}(F^{-1}(1 - u), G^{-1}(1 - v)), \quad u, v \in (0, 1),$$

because $F^{-1}(F(t)) = t$, $t \in (0, 1)$, where the quantile function $F^{-1}(t) = \inf\{x : F(x) \geq t\}$, $t \in (0, 1)$ (see, e.g., Shorack and Wellner 1986, p.6). Therefore, \hat{C} is TP_∞ by part (i) and Lemma 5(ii). The proof is complete.

5 Stochastic Comparisons of BLM Distributions

To provide more information about BLM distributions, we can study stochastic comparisons in the \mathcal{BLM} family. As usual, define the notions of the upper orthant order (\leq_{uo}), the concordance order (\leq_c) and the Laplace transform order (\leq_{Lt}) as follows. Let $(X_i, Y_i) \sim H_i$ with marginals (F_i, G_i) , $i = 1, 2$, on \mathbb{R}_+ . Then denote (i) $(X_1, Y_1) \leq_{uo} (X_2, Y_2)$ if $\overline{H}_1(x, y) \leq \overline{H}_2(x, y)$ for all $x, y \geq 0$, (ii) $(X_1, Y_1) \leq_c (X_2, Y_2)$ if $(F_1, G_1) = (F_2, G_2)$ and $(X_1, Y_1) \leq_{uo} (X_2, Y_2)$, and (iii) $(X_1, Y_1) \leq_{Lt} (X_2, Y_2)$ if $\mathcal{L}_1(s, t) \geq \mathcal{L}_2(s, t)$ for all $s, t \geq 0$ (Müller and Stoyan 2002; Shaked and Shanthikumar 2007). We have, for example, the following results whose proofs are straightforward and are omitted.

THEOREM 12. *Let $(X_i, Y_i) \sim H_i = BLM(F_i, G_i, \theta_i) \in \mathcal{BLM}$, $i = 1, 2$. Then we have*

- (i) $X_1 \leq_{st} X_2, Y_1 \leq_{st} Y_2$ and $\theta_1 \geq \theta_2$, iff $(X_1, Y_1) \leq_{uo} (X_2, Y_2)$, or, equivalently, $E[K(X_1, Y_1)] \leq E[K(X_2, Y_2)]$ for any bivariate distribution K on \mathbb{R}_+^2 ;
- (ii) $F_1 = F_2, G_1 = G_2$ and $\theta_1 \geq \theta_2$, iff $(X_1, Y_1) \leq_c (X_2, Y_2)$, or, equivalently, $E[k_1(X_1)k_2(Y_1)] \leq E[k_1(X_2)k_2(Y_2)]$ for all increasing functions k_1, k_2 , provided the expectations exist; and
- (iii) if $X_1 \leq_{Lt} X_2, Y_1 \leq_{Lt} Y_2$ and $\theta_1 = \theta_2$, then $(X_1, Y_1) \leq_{Lt} (X_2, Y_2)$, or, equivalently, $E[k_1(X_1)k_2(Y_1)] \geq E[k_1(X_2)k_2(Y_2)]$ for all completely monotone functions k_1, k_2 , provided the expectations exist.

When H_1 and H_2 have the same pair of marginals (F, G) , Theorem 12(i) reduces, by Corollary 1, to the following interesting result which is related to the famous Slepian's inequality for bivariate normal distributions (see the discussion in Remark 5 below).

Corollary 4. *Let $(X_i, Y_i) \sim H_i = BLM(F, G, \theta_i) \in \mathcal{BLM}$, with correlation $\rho_i, i = 1, 2$. Then $\rho_1 \leq \rho_2$ iff $\bar{H}_1(x, y) \leq \bar{H}_2(x, y)$ for all $x, y \geq 0$, or, equivalently, $H_1(x, y) \leq H_2(x, y)$ for all $x, y \geq 0$.*

REMARK 5. In Corollary 4 above, if we consider standard bivariate normal distributions instead of BLM ones, then the conclusion also holds true and the necessary part is the so-called Slepian's lemma/inequality; see Slepian (1962), Müller and Stoyan (2002), p. 97, and Hoffmann-Jørgensen (2013) for more general results. In Wikipedia, it was said that while this intuitive-seeming result is true for Gaussian processes, it is not in general true for other random variables. However, as we can see in Corollary 4, there are infinitely many BLM distributions sharing the same Slepian's inequality with bivariate normal ones.

REMARK 6. We finally compare the effects of the dependence structure of BLM distributions in different coherent systems. Consider a two-component system and let the two components have lifetimes $(X, Y) \sim H = BLM(F, G, \theta)$. Then the lifetime of a series system composed of these two components is $X \wedge Y \sim Exp(\theta)$, while the lifetime of a parallel system composed of the same components is $X \vee Y$ obeying the distribution $H_p(z) = e^{-\theta z} - 1 + F(z) + G(z), z \geq 0$. Therefore the mean times to failure of series and parallel systems are, respectively, $E[X \wedge Y] = \int_0^\infty \bar{H}(x, x)dx = 1/\theta$ (decreasing in θ) and $E[X \vee Y] = E[X] + E[Y] - \int_0^\infty \bar{H}(x, x)dx = E[X] + E[Y] - 1/\theta$ (increasing in θ). The latter further implies that $\theta \geq (E[X] + E[Y])^{-1}$ (compare with Theorem 1(vi)) and that $E[XY] \in [1/\theta^2, (E[X] + E[Y])^2]$ by Corollary 1, provided the expectations exist. See also Aven and Jensen

(2013, Section 2.3) for special cases with exponential marginals as well as Lai and Lin (2014) for more general results.

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GWO DONG LIN
INSTITUTE OF STATISTICAL SCIENCE,
ACADEMIA SINICA,
TAIPEI 11529, TAIWAN
E-mail: gdlin@stat.sinica.edu.tw

XIAOLING DOU
WASEDA UNIVERSITY, 3-4-1 OHKUBO,
SHINJUKU, TOKYO 169-8555,
JAPAN
E-mail: xiaoling@aoni.waseda.jp

SATOSHI KURIKI
THE INSTITUTE OF STATISTICAL
MATHEMATICS,
10-3 MIDORICHO, TACHIKAWA,
TOKYO 190-8562, JAPAN
E-mail: kuriki@ism.ac.jp

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