

# Characterizations of projective spaces and quadrics by strictly nef bundles

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Abstract. In this paper, we show that if the tangent bundle of a smooth projective variety is strictly nef, then it is isomorphic to a projective space; if a projective variety  $X^n$  ( $n > 4$ ) has strictly nef  $\Lambda^2 TX$ , then it is isomorphic to  $\mathbb{P}^n$  or quadric  $\mathbb{Q}^n$ . We also prove that on elliptic curves, strictly nef vector bundles are ample, whereas there exist Hermitian flat and strictly nef vector bundles on any smooth curve with genus  $g \geq 2$ .

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## 1. Introduction

Let  $X$  be a smooth projective variety over  $\mathbb{C}$ . A line bundle  $L$  is said to be strictly nef if  $L \cdot C > 0$  for any irreducible curve  $C \subset X$ . A vector bundle  $V$  is said to be strictly nef (resp. ample, nef) if the tautological line bundle  $\mathcal{O}_V(1)$  of the projective bundle  $\mathbb{P}(V) \rightarrow X$  is strictly nef (resp. ample, nef). It is obvious that the notion of strictly nefness is stronger than that of nefness, but weaker than that of ampleness. It is well-known that there are equivalent analytical and cohomological characterizations for ampleness and nefness (e.g. [Dem10], [Har66]) which play fundamental roles in complex analytic geometry and algebraic geometry. However, there are no such characterizations for strictly nef bundles.

The purpose of this paper is to investigate properties of strictly nef vector bundles. In particular, we want to show the similarity and differences between various positivity notions. It is obvious that numerically trivial line bundles are nef but not strictly nef. As a striking example, we show there exist *Hermitian*

flat vector bundles which are strictly nef (Example 2.4), which is significantly different from our primary impression on strictly nef vector bundles. However, there are still some similarities between strictly nefness and ampleness.

**Theorem 1.1.** *Let  $X$  be a smooth projective variety. If  $\Lambda^r TX$  is strictly nef, then  $X$  is uniruled.*

In particular, if  $\Lambda^r TX$  is strictly nef, then  $-K_X$  can not be numerically trivial. This is quite different from the Hermitian flat and strictly nef vector bundle in Example 2.4. Moreover, the existence of rational curves on such varieties is one of the key ingredients and also the starting point in the characterizations of  $\mathbb{P}^n$  or  $\mathbb{Q}^n$  by strictly nef bundles, since as a priori, we do not know whether the strictly nefness of  $\Lambda^r TX$  could imply the ampleness of  $-K_X$ . Indeed, a well-known conjecture of Campana and Peternell ([CP91, Problem 11.4]) predicts that if  $-K_X$  is strictly nef, then  $-K_X$  is ample. This conjecture is only verified by Maeda for surfaces ([Mea93]) and by Serrano for threefolds ([Ser95]) (see also [Ueh00] and [CCP06]). The proof of Theorem 1.1 relies on a metric version ([HW12]) of the fundamental work ([BDPP13]) of Boucksom-Demailly-Peternell-Paun, Yau's solution to the Calabi conjecture ([Yau78]) and Yau's criterion for Calabi-Yau manifolds (e.g. [Yau77, Theorem 4]).

As analogous to Mori's fundamental result ([Mor79]), we prove

**Theorem 1.2.** *Let  $X$  be a smooth projective variety with dimension  $n$ . If  $TX$  is strictly nef, then  $X \cong \mathbb{P}^n$ .*

That means, as holomorphic tangent bundle,  $TX$  is strictly nef if and only if  $TX$  is ample. Along the same line as Theorem 1.2, we obtain the following classification (see also the classical result [Miy04] of Miyaoka and the recent work [DH] of Dedieu-Hoering).

**Theorem 1.3.** *Let  $X$  be a smooth projective variety of complex dimension  $n > 4$ . Suppose  $\Lambda^2 TX$  is strictly nef, then  $X$  is isomorphic to  $\mathbb{P}^n$  or quadric  $\mathbb{Q}^n$ .*

Note that, as a priori, the strictly nefness of  $TX$  may not imply the strictly nefness of  $\Lambda^2 TX$  (c.f. the Hermitian flat and strictly nef rank 2 bundle in Example 2.4). This is also the key difficulty in the characterizations of  $\mathbb{P}^n$  or  $\mathbb{Q}^n$  by strictly nef bundles. There are also some other classical characterizations of  $\mathbb{P}^n$  or  $\mathbb{Q}^n$ , e.g. [KO73], [SY80], [Siu80], [Mok88], [YZ90], [Pet90, Pet91], [AW01], [CMSB02], [Hwa13] and etc. We refer to the papers [Pet96, DH], and the references therein.

In Example 2.4, we show that, on any smooth curve with genus  $\geq 2$ , there exist *flat* vector bundles which are also strictly nef. However, we prove that it can not happen on elliptic curves. Moreover,

**Theorem 1.4.** *Let  $V$  be a vector bundle over an elliptic curve  $C$ . If  $V$  is strictly nef, then  $V$  is ample.*

Note that Theorem 1.4 is also essentially used in the proof of Theorem 1.3.

**Remark.** It is not hard to see that the strictly nef condition in the above theorems can be slightly relaxed by using refined techniques developed in [BDPP13], [Hwa00] and [Mok08].

**Sketched ideas in the proofs.** Since strictly nef bundles have no analytical or cohomological characterizations, the proofs of our theorems are significantly different from those classical results. For instance, from Example 2.4, a priori, the strictly nefness  $\Lambda^2 TX$  can not imply the strictly nefness of  $-K_X$ , nevertheless the ampleness of  $-K_X$ . Hence, we can not use birational techniques on Fano manifolds. In the proof of Theorem 1.3, the key difficulty is to prove  $-K_X$  is ample. Here we argue by contradiction. Suppose  $\Lambda^2 TX$  is strictly nef and  $c_1^n(X) = 0$ . We divided it into two cases. (1) When  $TX$  is not nef. We show there exists an extremal rational curve (Theorem 1.1) such that  $-K_X|_C$  is ample. Mover, by the ‘‘Cone Theorem’’ and the ideas in [SW04, Theorem 4.2], we show the Mori elementary contraction  $\beta : X \rightarrow B$  is indeed a fiber bundle over a curve (Lemma 4.6 and Lemma 4.7). By analyzing vector bundles over curves carefully, we get a contradiction (Theorem 4.8). (2) When  $TX$  is nef. We use the structure theorem of Demailly-Peternell-Schneider [DPS94, Main Theorem] and show that  $X$  has a finite étale cover of the form of a projective bundle over an elliptic curve. Based on the key fact that strictly nef bundles on elliptic curves are ample (Theorem 1.4), we are able to obtain a contradiction (Theorem 4.4), thanks to a Barton-Kleiman type criterion for strictly nef bundles.

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## 2. Basic properties of strictly nef vector bundles

Let  $X$  be a smooth projective variety. There are many equivalent definitions for ampleness of a line bundle (e.g. [Har66], [Laz04I]). The Nakai-Moishezon-Kleiman criterion asserts that: a line bundle  $L$  is *ample* if and only if

$$(2.1) \quad L^{\dim Y} \cdot Y > 0$$

for every positive-dimensional irreducible subvariety  $Y \subset X$ . Similarly, a line bundle  $L$  is *nef* if and only if  $L \cdot C \geq 0$  for every irreducible curve  $C \subset X$ . As an analogue,

**Definition 2.1.** A line bundle  $L$  is said to be *strictly nef*, if for any irreducible curve  $C$  in  $X$ ,

$$(2.2) \quad L \cdot C > 0.$$

There are many strictly nef line bundles which are not ample (e.g. Example 2.4).

A vector bundle  $V$  is said to be *ample* (resp. *nef*) if the tautological line bundle  $\mathcal{O}_V(1)$  of  $\mathbb{P}(V) \rightarrow X$  is ample (resp. nef). Similarly,

**Definition 2.2.** A vector bundle  $V$  is called *strictly nef*, if its tautological line bundle  $\mathcal{O}_V(1)$  of  $\mathbb{P}(V) \rightarrow X$  is a strictly nef line bundle.

We have the following characterization of strictly nef vector bundles which is analogous to the Barton-Kleiman criterion for nef vector bundles (e.g. [Laz04II, Proposition 6.1.18]):

**Lemma 2.3.** *Let  $X$  be a smooth projective variety and  $V$  a holomorphic vector bundle. Then the following are equivalent:*

- (1)  $V$  is strictly nef;
- (2) for any finite morphism  $\nu : C \rightarrow X$  where  $C$  is a smooth curve, and any line bundle quotient  $\nu^*(V) \twoheadrightarrow L$ , one has

$$(2.3) \quad \deg_C(L) > 0.$$

*Proof.* It is well-known that giving a line bundle quotient  $\nu^*V \twoheadrightarrow L$  is the same as giving a map  $\mu : C \rightarrow \mathbb{P}(V)$  commuting with the projection to  $X$ :

$$(2.4) \quad \begin{array}{ccc} C & \xrightarrow{\mu} & \mathbb{P}(V) \\ & \searrow \nu & \swarrow \pi \\ & X & \end{array}$$

with  $L = \mu^*(\mathcal{O}_V(1))$ , where  $\mathcal{O}_V(1)$  is the tautological line bundle.

(2)  $\implies$  (1). Let  $C$  be an arbitrary irreducible curve in  $Y = \mathbb{P}(V)$ . Let  $\mu : C \rightarrow Y$  be the inclusion and  $\nu : C \rightarrow X$  the map given in (2.4). Then

$$\mathcal{O}_V(1) \cdot C = \deg_C(L) > 0.$$

(1)  $\implies$  (2). Let  $\nu : C \rightarrow X$  be a finite morphism where  $C$  is an irreducible curve. Let  $\mu : C \rightarrow \mathbb{P}(V)$  be the map in (2.4) and  $\tilde{C} = \mu(C)$ . Then

$$\mathcal{O}_V(1) \cdot \tilde{C} = \deg_C(\mu^*(\mathcal{O}_V(1))) = \deg_C(L) > 0$$

since  $\mathcal{O}_V(1)$  is strictly nef.  $\square$

**2.1. An example of flat and strictly nef vector bundle.** In this subsection, we will show by an example (essentially constructed by D. Mumford, e.g. [Har70, Section 10, Chapter I]) that the terminologies introduced above are mutually different.

**Example 2.4.** Let  $C$  be a smooth curve of genus  $g \geq 2$ . There exists a stable vector bundle  $V$  such that  $V$  is *flat* and *strictly nef*. In particular, the tautological line bundle  $\mathcal{O}_V(1)$  of the projective bundle  $\mathbb{P}(V) \rightarrow C$  is strictly nef, but it is not ample.

At first, we need to introduce the concept of (semi-)stability for vector bundles.

**Definition 2.5.** A holomorphic vector bundle  $V$  over a smooth curve  $C$  is called *stable* (resp. *semi-stable*) if for any subbundle  $V_1$  of  $V$  with  $0 < \text{rank}(V_1) < \text{rank}(V)$ , we have

$$(2.5) \quad \frac{\deg(V_1)}{\text{rank}(V_1)} < \frac{\deg(V)}{\text{rank}(V)} \quad \left( \text{resp.} \quad \frac{\deg(V_1)}{\text{rank}(V_1)} \leq \frac{\deg(V)}{\text{rank}(V)} \right).$$

**Lemma 2.6.** *Let  $C$  be a smooth curve of genus  $g \geq 2$ . Then there exists a rank two stable vector bundle  $V$  of degree zero such that all its symmetric powers  $\text{Sym}^m V$  are stable.*

*Proof.* This result is due to C.S. Seshadri. We refer the proof to [Har70, Theorem 10.5, Chapter I].  $\square$

**Corollary 2.7.** *The vector bundle  $V$  in Lemma 2.6 is Hermitian flat.*

*Proof.* The proof is simple and we include it here for readers' convenience. By Hartshorne's Theorem (e.g. [Laz04II, Theorem 6.4.15]), a stable vector bundle  $V$  over a curve is nef if and only if  $\deg(V) \geq 0$ . Since  $V$  is stable and of degree zero,  $V$  is nef. On the other hand, since  $V$  is of rank 2,  $V = V^* \otimes \det V$ . Hence  $V^* \otimes \det V$  is a nef vector bundle. Since  $\det V$  is numerically trivial, i.e.

$\deg(V) = 0$ , we see  $V^*$  is also nef. Since  $V$  is stable over  $C$ , there exists a Hermitian-Einstein metric  $h$  on  $V$  (e.g. [UY86]), i.e.

$$(2.6) \quad g^{-1} \cdot R_{\alpha\bar{\beta}} = c \cdot h_{\alpha\bar{\beta}}$$

for some constant  $c$  where  $g$  is a smooth metric on  $C$ . Since both  $V$  and  $V^*$  are nef, we deduce  $c = 0$ , i.e.  $V$  is Hermitian flat.  $\square$

Let  $X = \mathbb{P}(V)$  and  $\pi : \mathbb{P}(V) \rightarrow C$  be the projection. Let  $\mathcal{O}_V(1)$  be the tautological line bundle of  $\mathbb{P}(V)$  and  $D$  be the corresponding divisor over  $X$ . For any effective curve  $Y$  on  $X$ , we denote by  $m(Y)$  the degree of  $Y$  over  $C$ . Then there is an exact sequence

$$(2.7) \quad 0 \rightarrow \text{Pic}(C) \xrightarrow{\pi} \text{Pic}(X) \xrightarrow{m} \mathbb{Z} \rightarrow 0.$$

It follows that the divisors on  $X$ , modulo numerical equivalence, form a free abelian group of rank 2, generated by  $D$  and  $F$  where  $F$  is any fiber of  $\mathbb{P}(V)$ . It is easy to see that

$$(2.8) \quad D^2 = \deg(V) = 0, \quad D \cdot F = 1, \quad F^2 = 0.$$

**Lemma 2.8.** *For any  $m > 0$ , there is a 1 – 1 correspondence between*

- (1) *effective curves  $Y$  on  $X$ , having no fibers as components, of degree  $m$  over  $C$ ; and*
- (2) *line subbundles  $L$  of  $\text{Sym}^m V$ .*

*Moreover, under this correspondence, one has*

$$(2.9) \quad D \cdot Y = m \deg(V) - \deg(L).$$

*Proof.* See [Har70, Proposition 10.2, Chapter I].  $\square$

Note that  $\text{Sym}^m V$  are stable and of degree zero for all  $m \geq 1$ . For a line subbundle  $L$  of  $\text{Sym}^m V$ , we have

$$\deg(L) < \frac{\deg(\text{Sym}^m V)}{\text{rank}(\text{Sym}^m V)} = 0.$$

Let  $Y$  be an arbitrary irreducible curve on  $X$ . If  $Y$  is a fiber, then  $D \cdot Y = 1$ . If  $Y$  is an irreducible curve of degree  $m > 0$  over  $C$ , then by Lemma 2.8,  $Y$  is corresponding to a line subbundle  $L$  of  $\text{Sym}^m V$ . Therefore, by formula (2.9)

$$(2.10) \quad D \cdot Y = m \deg(V) - \deg(L) = -\deg(L) > 0.$$

Hence, the line bundle  $\mathcal{O}_V(1)$  of the divisor  $D$  is strictly nef, i.e.  $V$  is strictly nef. Since  $V$  is flat,  $V$  can not be ample. In particular,  $\mathcal{O}_V(1)$  is strictly nef but not ample.  $\square$

**2.2. Basic properties for strictly nef vector bundles.** As we have seen from the previous example, the terminology “strictly nef” is significantly different from other notions. Many functorial properties do not hold for strictly nef bundles. For instance, if  $E$  is strictly nef,  $\Lambda^s E$  and  $\det E$  are not necessarily strictly nef. In Example 2.4, the strictly nef vector bundle is flat, and so its determinant line bundle is flat and it is not strictly nef. In this subsection, we prove several basic properties for strictly nef vector bundles.

**Proposition 2.9.**  *$E$  is strictly nef if and only if  $E|_C$  is strictly nef for every curve  $C$ .*

*Proof.* It follows from Lemma 2.3. □

**Proposition 2.10.** *If  $\text{Sym}^k E$  is strictly nef for some  $k \geq 1$ , then  $E$  is strictly nef.*

*Proof.* Let  $v_k : \mathbb{P}(E) \rightarrow \mathbb{P}(\text{Sym}^k E)$  be the Veronese embedding. Then we have

$$(2.11) \quad \mathcal{O}_E(k) = v_k^*(\mathcal{O}_{\text{Sym}^k E}(1))$$

Therefore  $E$  is strictly nef if  $\text{Sym}^k E$  is strictly nef for some  $k \geq 1$ . □

**Proposition 2.11.** (1) *If  $E$  is strictly nef, then any quotient of it is strictly nef.*

(2) *If  $E \oplus F$  is strictly nef, then both  $E$  and  $F$  are strictly nef.*

(3) *If  $E$  is strictly nef, for any  $L \in \text{Pic}^0(X)$ ,  $E \otimes L$  is strictly nef.*

*Proof.* (1). Let  $Q$  be a quotient of  $E$ . Suppose  $Q$  is not strictly nef, by Lemma 2.3, there exist a finite morphism  $\nu : C \rightarrow X$  where  $C$  is a smooth curve, and a line bundle quotient  $\nu^*(Q) \twoheadrightarrow L$  such that  $\deg(L) \leq 0$ . Note that  $L$  is also a quotient of  $\nu^*(E)$ , which is a contradiction. (2) follows from (1). (3). Since  $\mathbb{P}(E \otimes L) = \mathbb{P}(E)$ ,

$$(2.12) \quad \mathcal{O}_{E \otimes L}(1) = \mathcal{O}_E(1) \otimes \pi^*(L).$$

Hence, if  $L \in \text{Pic}^0(X)$ ,  $\mathcal{O}_{E \otimes L}(1)$  is strictly nef if and only if  $\mathcal{O}_E(1)$  is strictly nef. □

**Proposition 2.12.** *Let  $X, Y$  be two smooth projective varieties and  $f : X \rightarrow Y$  be a surjective morphism. If  $f^*E$  is strictly nef, then  $E$  is strictly nef.*

*Proof.* Let  $C$  be any curve in  $Y$  and  $\tilde{C}$  be a curve in  $X$  such that  $f(\tilde{C}) = C$ . Since  $f^*E|_{\tilde{C}}$  is strictly nef,  $E|_C$  is strictly nef. By Proposition 2.9,  $E$  is strictly nef. □

**Corollary 2.13.** *If  $V$  is a strictly nef bundle over  $X$ , then  $H^0(X, V^* \otimes L) = 0$  for any  $L \in \text{Pic}^0(X)$ .*

*Proof.* Let  $\tilde{V} = V \otimes L^*$ . Since  $L \in \text{Pic}^0(X)$  and  $V$  is strictly nef,  $\tilde{V}$  is strictly nef. By [DPS94, Proposition 1.16], if  $\tilde{V}$  is nef and  $H^0(X, \tilde{V}^*) \neq 0$ , there exists a nowhere vanishing section  $\sigma \in H^0(X, \tilde{V}^*)$ . There is a bundle map induced by  $\sigma$ ,

$$f_\sigma : X \times \mathbb{C} \rightarrow \tilde{V}^*, \quad f_\sigma(x, v) = v \cdot \sigma(x).$$

In particular, the trivial bundle  $\mathcal{O} = X \times \mathbb{C}$  is a holomorphic subbundle of  $\tilde{V}^*$ . By duality, the trivial bundle  $\mathcal{O}$  is a quotient of  $\tilde{V}$ . By Lemma 2.3,  $\tilde{V}$  can not be strictly nef. This is a contradiction.  $\square$

### 3. Strictly nef bundles over elliptic curves

**3.1. Projective bundles over Calabi-Yau manifolds.** We need another characterization of strictly nef anti-canonical bundles which is analogues to Lemma 2.3.

**Lemma 3.1.** *Let  $X = \mathbb{P}(V)$  be a projective bundle of a holomorphic vector bundle  $V$  over a Calabi-Yau manifold  $Y$ , i.e.  $K_Y \cong \mathcal{O}_Y$ . Then the following are equivalent:*

- (1)  $-K_X$  is strictly nef;
- (2) for any finite morphism  $\nu : C \rightarrow Y$  from an irreducible curve  $C$  to  $Y$  and any line bundle quotient  $\nu^*(V) \rightarrow L$ , one has

$$(3.1) \quad \deg(L) > \frac{\deg(\nu^*(V))}{\text{rank}(V)}.$$

*Proof.* Here we use the same configuration as in Lemma 2.3.

(2)  $\implies$  (1). Let  $C$  be an arbitrary irreducible curve in  $X = \mathbb{P}(V)$ . Let  $\mu : C \rightarrow X$  be the inclusion and  $\nu : C \rightarrow Y$  the map given in (2.4). Therefore, by the canonical bundle formula ([Laz04II, p. 89]) over  $X = \mathbb{P}(V) \xrightarrow{\pi} Y$ ,

$$-K_X = \mathcal{O}_V(n) \otimes \pi^*(\det V^*),$$

where  $n$  is the rank of  $V$ , we have

$$\begin{aligned} -K_X \cdot C &= \deg_C(\mu^*(-K_X)) \\ &= \deg_C(\mu^*(\pi^*(\det V^*))) + \deg_C(\mu^*\mathcal{O}_V(n)) \\ &= -\deg_C \nu^*(V) + n \deg_C(L), \end{aligned}$$

since  $L = \mu^*\mathcal{O}_V(1)$ . Hence, by the inequality (3.1), we obtain  $-\deg_C \nu^*(V) + n \deg_C(L) > 0$ , which implies  $-K_X \cdot C > 0$ . Therefore,  $-K_X$  is strictly nef.



(1)  $\implies$  (2). Let  $\nu : C \rightarrow Y$  be a finite morphism where  $C$  is an irreducible curve. Let  $\mu : C \rightarrow X$  be the map given in the configuration (2.4) and  $\tilde{C} = \mu(C)$ . Then

$$\begin{aligned} -K_X \cdot \tilde{C} &= \deg_C(\mu^*(-K_X)) \\ &= -\deg_C \nu^*(V) + n \deg_C(L). \end{aligned}$$

If  $-K_X$  is strictly nef, we have  $-K_X \cdot \tilde{C} > 0$ , and so (3.1) holds.  $\square$

As an application of Lemma 3.1, we obtain

**Proposition 3.2.** *Let  $X = \mathbb{P}(V)$  be a projective bundle of a nef vector bundle  $V$  over a Calabi-Yau manifold  $Y$ . If there exists a line bundle  $L \in \text{Pic}^0(Y)$  such that  $H^0(X, V^* \otimes L) \neq 0$ , then  $-K_X$  is not strictly nef.*

*Proof.* Let  $\tilde{V} = V \otimes L^*$  and  $\tilde{X} = \mathbb{P}(\tilde{V})$ . It is obvious that  $X$  is isomorphic to  $\tilde{X}$ . Since  $L \in \text{Pic}^0(Y)$  and  $V$  is nef,  $\tilde{V}$  is nef. By using similar arguments as in Corollary 2.13, the nowhere vanishing section  $\sigma \in H^0(Y, \tilde{V}^*)$  defines a trivial quotient line bundle of  $\tilde{V}$ . By Lemma 3.1,  $-K_{\tilde{X}}$  can not be strictly nef. Hence,  $-K_X$  is not strictly nef.  $\square$

**3.2. Projective bundles over elliptic curves.** In this subsection, we investigate geometric properties of anti-canonical bundles of projective bundles over elliptic curves.

**Proposition 3.3.** *Suppose  $V$  is an indecomposable vector bundle over an elliptic curve  $S$  and  $\deg(V) = 0$ . Then the anti-canonical line bundle of  $\mathbb{P}(V)$  is not strictly nef.*

*Proof.* Suppose  $V$  has rank  $n$ . Since  $V$  is indecomposable and  $\deg(V) = 0$ , by a fundamental result of Atiyah ([Ati57, Theorem 5]), there exists a vector bundle  $F_n$  such that  $\deg(F_n) = 0$ ,  $H^0(S, F_n) \neq 0$  and  $V = F_n \otimes \det V$ . On the other hand, by Hirzebruch-Riemann-Roch theorem for vector bundles over elliptic curves, we have

$$(3.2) \quad \deg(F_n) = \dim H^0(S, F_n) - H^1(S, F_n).$$

Hence  $H^1(S, F_n) \neq 0$ . In particular,  $H^0(S, F_n^*) \neq 0$ . Now we consider the variety  $X := \mathbb{P}(F_n)$ . It is well-known that  $\mathbb{P}(V)$  is isomorphic to  $\mathbb{P}(F_n)$  since  $V = F_n \otimes L$ . Suppose  $\mathbb{P}(V)$  has strictly nef anti-canonical line bundle, then so is  $-K_X$ . By the projection formula again

$$(3.3) \quad -K_X = \mathcal{O}_{F_n}(n) \otimes \pi^*(\det F_n^*),$$

$\mathcal{O}_{F_n}(1)$  is a (strictly) nef line bundle, and so  $F_n$  is a nef vector bundle. In summary,  $F_n$  is a nef vector bundle over elliptic curve  $S$  with  $H^0(S, F_n^*) \neq 0$ . However, by Proposition 3.2,  $\mathbb{P}(F_n)$  can not have strictly nef anti-canonical bundle. This is a contradiction.  $\square$

**Lemma 3.4.** *Let  $X = \mathbb{P}(V)$  be a projective bundle over a smooth curve  $S$ . If  $-K_X$  is strictly nef, then  $S \cong \mathbb{P}^1$ .*

*Proof.* Suppose  $S$  is an elliptic curve. We shall use Atiyah's work [Ati57] on the classification of vector bundles over elliptic curves to rule out this case. Suppose  $V$  has rank  $n$ .

*Case 1.* Suppose  $V$  is indecomposable and  $\deg(V) = 0$ . This case is ruled out by Proposition 3.3.

*Case 2.* Suppose  $V$  is indecomposable and  $\deg(V) \neq 0$ . There exists an étale base change  $f : Y \rightarrow S$  of degree  $k$  where  $k$  is an integer such that  $n|k$ , and  $Y$  is also an elliptic curve. Suppose  $X' = \mathbb{P}(f^*V)$ , then we have the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ \pi' \downarrow & & \downarrow \pi \\ Y & \xrightarrow{f} & S. \end{array}$$

Hence  $-K_{X'}$  is also strictly nef. It is obvious that  $\text{rank}(f^*V) = \text{rank}(V) = n$ , and  $\deg(f^*V) = k \deg(V)$ . Let  $\ell$  be an integer defined as

$$(3.4) \quad \ell = \frac{\deg(f^*V)}{\text{rank}(f^*V)},$$

and  $F$  be a line bundle over  $Y$  such that  $\deg(F) = -\ell$ . Now we set  $\tilde{V} = f^*V \otimes F$ , then  $\deg(\tilde{V}) = 0$ . A well-known result of Atiyah asserts that an indecomposable vector bundle over an elliptic is semi-stable and so  $V$  is semi-stable (e.g. [Tu93, Appendix A]). Therefore  $f^*V$  is semi-stable (e.g. [Laz04II, Lemma 6.4.12]) and so is  $\tilde{V}$ . Let  $\tilde{V} = \bigoplus \tilde{V}_i$  where  $\tilde{V}_i$  are indecomposable. Since  $X' = \mathbb{P}(f^*V) \cong \mathbb{P}(\tilde{V})$ , by projection formula, we have

$$-K_{X'} = \mathcal{O}_{\tilde{V}}(n) \otimes \pi^*(\det \tilde{V}).$$

Since  $\deg(\tilde{V}) = 0$ ,  $-K_{X'}$  is strictly nef if and only if  $\mathcal{O}_{\tilde{V}}(1)$  is strictly nef. Hence  $\tilde{V}$  is strictly nef. By Proposition 2.11, every  $\tilde{V}_i$  is strictly nef. It is obvious that  $\deg(\tilde{V}_i) = 0$ . Therefore,  $\mathbb{P}(\tilde{V}_i)$  is a projective bundle over elliptic curve  $C$  with strictly nef anti-canonical line bundle. By Proposition 3.3, it is impossible.

*Case 3.* In the general case, suppose  $V = \oplus_i V_i$  where  $V_i$  are indecomposable vector bundles over  $S$ . Since  $V_i$  is indecomposable, by Case 1 and Case 2, the anticanonical line bundle of  $\mathbb{P}(V_i)$  is not strictly nef. By Lemma 3.1, for each  $i$ , there exists an irreducible curve  $C_i$ , a finite morphism  $\nu_i : C_i \rightarrow S$  and a line bundle quotient  $L_i$  of  $\nu_i^*(V_i)$  such that

$$(3.5) \quad \deg_{C_i}(L_i) \leq \frac{\deg(\nu_i^*(V_i))}{\text{rank}(V_i)}.$$

On the other hand, it is an elementary fact that there exists at least one summand  $V_i$  such that

$$(3.6) \quad \frac{\deg(V_i)}{\text{rank}(V_i)} \leq \frac{\deg(V)}{\text{rank}(V)} = \frac{\sum_i \deg(V_i)}{\sum_i \text{rank}(V_i)}.$$

For such  $V_i$ , we have

$$(3.7) \quad \deg_{C_i}(L_i) \leq \frac{\deg(\nu_i^*(V_i))}{\text{rank}(V_i)} \leq \frac{\deg(\nu_i^*(V))}{\text{rank}(V)}.$$

Note that  $L_i$  is also a line bundle quotient of  $\nu_i^*(V)$ , hence by Lemma 3.1 again, the anti-canonical line bundle  $\mathbb{P}(V)$  can not be strictly nef.

Hence, we complete the proof that if  $\mathbb{P}(V) \rightarrow S$  has strictly nef anti-canonical bundle, then  $S$  is not an elliptic curve. On the other hand, by a result of Miyaoka [Miy93] (see also [Deb01, Corollary 3.14]), if  $-K_X$  is nef, then  $-K_S$  is nef. Now we deduce that  $S \cong \mathbb{P}^1$ .  $\square$

**Theorem 3.5.** *Let  $V$  be a vector bundle over an elliptic curve  $C$ . If  $V$  is strictly nef, then  $V$  is ample.*

*Proof.* Since  $V$  is also nef,  $\deg(V) \geq 0$ . Suppose  $\deg(V) = 0$ . Let  $X = \mathbb{P}(V)$  and  $\pi : \mathbb{P}(V) \rightarrow C$  be the projection. Then

$$-K_{X/C} = \mathcal{O}_V(n) \otimes \pi^*(\det V^*).$$

Since  $K_C = \mathcal{O}_C$  and  $\deg(V) = 0$ ,  $-K_X$  is strictly nef if and only if  $\mathcal{O}_V(1)$  is strictly nef. By Lemma 3.4,  $-K_X$  can not be strictly nef. Hence,  $\deg(V) > 0$ .

Let  $V = \oplus V_i$  where  $V_i$  are indecomposable. By Proposition 2.11, each  $V_i$  is strictly nef and  $\deg(V_i) > 0$ . Since indecomposable vector bundles are also semi-stable ([Tu93, Appendix A]), and a semi-stable vector bundle is ample if and only if it has positive degree ([Laz04II, Theorem 6.4.15]), we deduce  $V_i$  is ample. Therefore  $V$  is ample.  $\square$

## 4. The proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3

In this section, we will prove Theorem 1.1, Theorem 1.2 and Theorem 1.3.

**Theorem 4.1.** *Let  $X$  be a smooth projective variety. If  $\Lambda^r TX$  is strictly nef, then  $X$  is uniruled.*

*Proof.* Since  $\Lambda^r TX$  is strictly nef,  $-K_X$  is nef.

*Case 1.* Suppose there exists an ample divisor  $H$  such that  $-K_X \cdot H^{n-1} > 0$ , then  $X$  is uniruled. Indeed, let  $\omega$  be a Kähler metric in the class  $c_1(H)$ , then

$$(4.1) \quad \int_X c_1(X) \wedge \omega^{n-1} > 0.$$

That means the total scalar curvature of the metric  $\omega$  is strictly positive. Based on the fundamental results in [BDPP13], Heier-Wong proved in [HW12, Theorem 1] that if a projective variety  $X$  has a smooth Kähler metric with positive total scalar curvature, then  $X$  is uniruled.

*Case 2.* If  $K_X \cdot H^{n-1} = 0$  for every ample divisor  $H$ , then  $K_X$  is numerically trivial (e.g. [Deb01, 3.8, p. 69]), i.e.  $X$  is a Calabi-Yau manifold. By Yau's theorem ([Yau78]), there exists a Kähler metric  $\omega$  such that  $Ric(\omega) = 0$ , i.e.  $TX$  is Hermitian-Einstein. Since  $\Lambda^r TX$  is nef and  $c_1(X) = 0$ ,  $\Lambda^r TX$  is numerically flat. (A vector bundle  $E$  is called numerically flat ([DPS94, Definition 1.17]), if both  $E$  and  $E^*$  are nef, or equivalently, both  $E$  and  $\det E^*$  are nef.)

If  $r = \dim X$ , then  $\Lambda^r TX = -K_X$  is numerically trivial which is not strictly nef. Hence  $1 \leq r \leq \dim X - 1$ . Let  $E = TX$  and  $\tilde{E} = \Lambda^r TX$ . Then  $\tilde{E}$  is nef and  $c_1(\tilde{E}) = 0$ , by [DPS94, Corollary 2.6]

$$(4.2) \quad \int_X c_1^2(\tilde{E}) \wedge \omega^{n-2} = \int_X c_2(\tilde{E}) \wedge \omega^{n-2} = 0.$$

Since  $c_1(\tilde{E}) = 0$ ,  $c_1(E) = 0$ . Let  $\lambda_1, \dots, \lambda_n$  be the Chern roots of  $E$ , then  $c_2(\tilde{E})$  is a symmetric polynomial in  $\lambda_1, \dots, \lambda_n$  of degree 2. In particular,

$$(4.3) \quad c_2(\tilde{E}) = ac_1^2(E) + bc_2(E)$$

since  $c_1(E)$  and  $c_2(E)$  are elementary symmetric polynomials. Note that  $a$  and  $b$  are constants depending only on  $r$  and the rank of  $n$  of  $E$ . A simple computation shows  $b \neq 0$ . Hence,

$$(4.4) \quad \int_X c_2(E) \wedge \omega^{n-2} = 0.$$

Hence,  $X$  is a Calabi-Yau manifold with

$$(4.5) \quad \int_X c_1^2(X) \wedge \omega^{n-2} = \int_X c_2(X) \wedge \omega^{n-2} = 0,$$

By Yau's criterion [Yau77, Theorem 1.4] (see also [Zhe00, Corollary 9.6], [Kob87, Theorem 4.11]),  $TX$  is Hermitian flat and up to a finite étale cover,  $X$  is isomorphic to a torus. This is a contradiction since  $\Lambda^r TX$  is strictly nef.  $\square$

**Theorem 4.2.** *Let  $X$  be a smooth projective variety with dimension  $n$ . If  $TX$  is strictly nef, then  $X \cong \mathbb{P}^n$ .*

*Proof.* Since  $TX$  is strictly nef,  $TX$  is nef. By the structure theorem ([DPS94, Main Theorem]) of projective manifolds with nef tangent bundles, there exists a finite étale cover  $\pi : \tilde{X} \rightarrow X$  such that the Albanese map  $\alpha : \tilde{X} \rightarrow A$  is a smooth morphism and the fibers are Fano manifolds with nef tangent bundles. Note that by Theorem 1.1,  $\dim A < \dim X$ . Since  $\pi$  is a finite morphism,  $T\tilde{X}$  is also strictly nef. Now consider the exact sequence

$$(4.6) \quad 0 \rightarrow T_{\tilde{X}/A} \rightarrow T\tilde{X} \rightarrow \alpha^*(TA) \rightarrow 0.$$

Since  $T\tilde{X}$  is strictly nef, it can not have a trivial quotient. Hence  $\dim A = 0$ , and  $X$  is Fano. For an arbitrary rational curve  $\nu : \mathbb{P}^1 \rightarrow X$ ,

$$(4.7) \quad \nu^*TX = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i),$$

where  $a_i \geq 1$  for each  $i$ . Therefore  $\deg(\nu^*K_X^{-1}) \geq n$ . If  $\deg(\nu^*K_X^{-1}) = n$ , then  $a_i = 1$  for all  $i$  which can imply the tangent morphism  $T\mathbb{P}^1 \rightarrow \nu^*TX$  is zero, which is a contradiction. That means  $\deg(\nu^*K_X^{-1}) \geq n + 1$ . Recall that the pseudo-index of a Fano variety  $X$  is defined as:

$$i_X = \min\{-K_X \cdot C \mid C \text{ is a rational curve}\},$$

and so the pseudo-index  $i_X \geq n + 1$ . Hence,  $X \cong \mathbb{P}^n$ .  $\square$

The following result will be used frequently in the proof of Theorem 1.3.

**Lemma 4.3.** *Let  $X$  be a smooth projective variety with dimension  $n \geq 3$ . Suppose  $\Lambda^2 T_X$  is strictly nef.*

(1) *For any rational curve  $C$ , we have*

$$-K_X \cdot C \geq n.$$

(2) *If  $T_X$  is not nef along a rational curve  $C$ , then*

$$-K_X \cdot C \geq 2n - 3.$$

*Proof.* (1) Over the rational curve  $C$ ,  $T_X$  splits as

$$(4.8) \quad T_X|_C = \bigoplus_{b_j \leq 0} \mathcal{O}(b_j) \oplus \left( \bigoplus_{a_i > 0} \mathcal{O}(a_i) \right),$$

Since  $\Lambda^2 T_X$  is strictly nef, by Lemma 2.3

$$(4.9) \quad T_X|_C = \mathcal{O}(-b) \bigoplus \left( \bigoplus_{i=1}^{n-1} \mathcal{O}(a_i) \right),$$

where  $0 < a_1 \leq a_2 \leq \dots \leq a_{n-1}$  and  $a_1 - b > 0$ .

If  $b > 0$ , then  $a_1 \geq 2$  and

$$(4.10) \quad -K_X \cdot C = (a_1 - b) + a_2 + \dots + a_{n-1} \geq 1 + 2(n-2) = 2n - 3 \geq n$$

since  $n \geq 3$ .

If  $b = 0$ , since from the tangent morphism  $0 \rightarrow \mathcal{O}(2) \rightarrow TX|_C$  of  $C \rightarrow X$ , there exists at least one  $a_i \geq 2$  and so  $-K_X \cdot C \geq n$ .

(2). If  $T_X$  is not nef along a rational curve  $C$ , then we have  $b > 0$  and so  $-K_X \cdot C \geq 2n - 3$ .  $\square$

One of the key ingredients in the proof of Theorem 1.3 is

**Theorem 4.4.** *Let  $X$  be a smooth projective variety of dimension  $n > 4$ . Suppose that  $\Lambda^2 T_X$  is strictly nef, then  $c_1^n(X) \neq 0$ .*

This is the most complicated issue in the proof of Theorem 1.3, since a priori, it might be possible that  $c_1^n(X) = 0$  (c.f. the Hermitian flat strictly nef vector bundle in Example 2.4). Based on Theorem 4.4, we are ready to prove Theorem 1.3.

**Theorem 4.5.** *Let  $X$  be a smooth projective variety of dimension  $n > 4$ . Suppose that  $\Lambda^2 T_X$  is strictly nef, then  $X$  is isomorphic to  $\mathbb{P}^n$  or a quadric  $\mathbb{Q}^n$ .*

*Proof.* Since  $\det(\Lambda^2 T_X) = (n-1)(-K_X)$ , the strictly nefness of  $\Lambda^2 T_X$  implies that  $-K_X$  is nef and  $c_1^n(X) \geq 0$ . By Theorem 4.4,  $-K_X$  is big, i.e.  $c_1^n(-K_X) > 0$ . By Kawamata-Reid-Shokurov base point free theorem,  $-K_X$  is semiample. Then there exists  $m$  big enough such that  $\varphi = |-mK_X|$  is a morphism. Since  $-K_X$  is big, there is a positive integer  $\tilde{m}$  such that

$$\tilde{m}(-K_X) = D + L$$

where  $D$  is an effective divisor and  $L$  is an ample line bundle. If  $-K_X$  is not ample, there exists a curve  $C$  contracted by  $\varphi$  which means  $-K_X \cdot C = 0$ . Therefore,

$$D \cdot C = -L \cdot C < 0.$$

Let  $\Delta = \varepsilon D$  for some small  $\varepsilon > 0$ , then  $(X, \Delta)$  is a klt pair and  $K_X + \Delta$  is not  $\varphi$ -nef. Then by the relative Cone theorem (e.g. [KM98, Theorem 3.25]) for

log pairs, there exists a rational curve  $\tilde{C}$  contracted by the morphism  $\varphi$ . By Lemma 4.3

$$(4.11) \quad T_X|_{\tilde{C}} = \mathcal{O}(-b) \bigoplus \left( \bigoplus_{i=1}^{n-1} \mathcal{O}(a_i) \right),$$

where  $a_1 - b \geq 1$ . Hence,

$$(4.12) \quad (-K_X) \cdot \tilde{C} \geq 1$$

which is a contradiction since  $\tilde{C}$  is contracted by  $\varphi$ . Therefore,  $-K_X$  is ample and  $X$  Fano. By the computation in Lemma 4.3, we know the pseudo-index  $i_X \geq n$ . Thanks to [Miy04] or [DH, Theorem C],  $X$  is isomorphic to  $\mathbb{P}^n$  or a quadric  $\mathbb{Q}^n$ .  $\square$

In the rest of this section, we shall prove Theorem 4.4. We need several technical lemmas to complete the proof.

**Lemma 4.6.** *Let  $X$  be a smooth projective variety of dimension  $n \geq 4$ . Suppose  $\Lambda^2 T_X$  is strictly nef. Let  $Y$  be a prime divisor of  $X$  and  $\mathcal{L} = \mathcal{O}_X(Y)$ . Suppose  $Y$  is rationally chain connected and  $\mathcal{L}|_Y$  is numerically trivial. Then  $Y$  is a smooth variety.*

*Proof.* We will follow some ideas of [SW04, Lemma 4.12] and use the techniques of [Kol01, Theorem 1.3, Chapter II]. Let  $\mathcal{J}$  be the ideal sheaf of  $Y$ . Then  $\mathcal{J}/\mathcal{J}^2 \simeq \mathcal{L}^{-1}|_Y$ . There exists an exact sequence:

$$(4.13) \quad \mathcal{L}^{-1}|_Y \rightarrow \Omega_X|_Y \rightarrow \Omega_Y \rightarrow 0.$$

Let  $U$  be the smooth locus of  $Y$ . Suppose that there is a rational curve  $f: \mathbb{P}^1 \rightarrow Y$  whose image has nonempty intersection with  $U$ .

*Claim 1:* If  $f^*(T_X)$  is nef, then  $Y$  is smooth along  $f(\mathbb{P}^1)$ , i.e.  $f(\mathbb{P}^1) \subseteq U$ .

Actually, by restricting the exact sequence (4.13) to  $f(\mathbb{P}^1)$ , we have the following right exact sequence:

$$(4.14) \quad \mathcal{O} \xrightarrow{d} f^*(\Omega_X) \rightarrow f^*(\Omega_Y) \rightarrow 0.$$

Note that  $d$  is generically split, which means  $d$  is injective on the dense subset  $f(\mathbb{P}^1) \cap U \neq \emptyset$ , so  $\text{Ker}(d)$  is a torsion free (hence locally free) subsheaf of generic rank 0, then  $\text{Ker}(d) = 0$  and  $d$  is injective. So the right exact sequence (4.14) is also left exact. Thus, the morphism  $d$  defines a global section  $s$  of  $H^0(\mathbb{P}^1, f^*(\Omega_X))$ . On the other hand, since  $f^*(T_X)$  is nef, by [DPS94, Proposition 1.16],  $s$  is nowhere vanishing. Then  $f^*(\Omega_Y)$  is of constant rank of  $n - 1$ .

So by [Har, Theorem 8.15, Chapter II],  $Y$  is smooth along  $f(\mathbb{P}^1)$ .

*Claim 2:* If  $f^*(T_X)$  is not nef, then  $\dim_{[f]} \text{Hom}(\mathbb{P}^1, Y|_{\{0, \infty\}}) \geq 2$ .

Since  $f^*(\Lambda^2 T_X)$  is strictly nef and  $f^*(T_X)$  is not nef,

$$f^*(T_X) = \mathcal{O}(-b) \bigoplus \left( \bigoplus_{i=1}^{n-1} \mathcal{O}(a_i) \right),$$

where  $b \geq 1$  and  $a_i \geq 2$ . By the exact sequence (4.14) and the fact  $f^*(\mathcal{L}^{-1}) = \mathcal{O}$ , we obtain the long exact sequence:

$$\begin{aligned} 0 \rightarrow \text{Hom}(f^*(\Omega_Y), \mathcal{O}) &\rightarrow \text{Hom}(f^*(\Omega_X), \mathcal{O}) \rightarrow \text{Hom}(\mathcal{O}, \mathcal{O}) \\ &\rightarrow \text{Ext}^1(f^*(\Omega_Y), \mathcal{O}) \rightarrow \text{Ext}^1(f^*(\Omega_X), \mathcal{O}) \rightarrow \text{Ext}^1(\mathcal{O}, \mathcal{O}). \end{aligned}$$

Then by [Kol01, Theorem 2.16, Chapter I],

$$\begin{aligned} \dim_{[f]} \text{Hom}(\mathbb{P}^1, Y) &\geq \dim \text{Hom}(f^*(\Omega_Y), \mathcal{O}) - \dim \text{Ext}^1(f^*(\Omega_Y), \mathcal{O}) \\ &= \chi(\mathbb{P}^1, f^*(T_X)) - \chi(\mathbb{P}^1, \mathcal{O}) \\ &= (1 - b) + \sum_{i=1}^{n-1} (1 + a_i) - 1 \\ &\geq 3n - 4. \end{aligned}$$

Since  $\text{Hom}(\mathbb{P}^1, Y|_{\{0, \infty\}})$  is the fibre of the restricting morphism

$$(4.15) \quad \text{Hom}(\mathbb{P}^1, Y) \longrightarrow \text{Hom}(\{0, \infty\}, Y) \simeq Y \times Y,$$

we have

$$(4.16) \quad \dim_{[f]} \text{Hom}(\mathbb{P}^1, Y|_{\{0, \infty\}}) \geq \dim_{[f]} \text{Hom}(\mathbb{P}^1, Y) - 2 \dim Y \geq n - 2 \geq 2.$$

*Claim 3:* If there is a connected rational chain  $\Gamma$  such that  $\Gamma \cap U \neq \emptyset$ , then  $\Gamma \subseteq U$ .

We prove it by induction on the degree  $-K_X \cdot \Gamma$ . If  $-K_X \cdot \Gamma = n$ , by Lemma 4.3,  $\Gamma$  is irreducible and  $T_X$  is nef along  $\Gamma$  since the dimension  $n \geq 4$  and  $n < 2n - 3$ . By Claim 1, we know that  $\Gamma \subseteq U$ .

By induction, we assume that any connected rational chain  $\Gamma$  which has nonempty intersection with  $U$  and  $-K_X \cdot \Gamma < N$  is contained in  $U$ . We want to show that if there is a connected rational chain  $\Gamma$  such that  $\Gamma \cap U \neq \emptyset$  and



$-K_X \cdot \Gamma = N$ , then  $\Gamma \subseteq U$ .

If  $\Gamma$  is reducible, i.e.  $\Gamma = \Gamma_1 + \Gamma_2$ , since  $\Gamma \cap U \neq \emptyset$ , there is at least one component which has nonempty intersection with  $U$ , namely  $\Gamma_1$ . Then  $-K_X \cdot \Gamma_1 < N$ , by induction,  $\Gamma_1 \subseteq U$ . Since  $\Gamma$  is connected,  $\emptyset \neq \Gamma_2 \cap \Gamma_1 \subseteq \Gamma_2 \cap U$ . So by induction again,  $\Gamma_2 \subseteq U$ .

If  $\Gamma$  is irreducible, by Claim 1, we may assume  $T_X$  is not nef along  $\Gamma$ . Then fix two points  $x$  and  $y$  of  $\Gamma$ , where  $x \in \Gamma \cap U$ . Then by Claim 2,  $\dim_{[f]} \text{Hom}(\mathbb{P}^1, Y|_{\{0, \infty\}}) \geq 2$ . Therefore, by [Mor79, Theorem 4] or [Deb01, Proposition 3.2],  $\Gamma$  breaks into a connected rational chain with  $x$  and  $y$  fixed. We denote it by  $\Gamma \equiv \Gamma_1 + \Gamma_2$ . By the same argument as above,  $\Gamma_1, \Gamma_2 \subseteq U$ , hence  $y \in U$ . As  $y$  moves along  $\Gamma$ ,  $\Gamma \subseteq U$ .

Then Lemma 4.6 follows directly from Claim 3.  $\square$

**Lemma 4.7.** *Let  $X$  be a smooth projective variety of dimension  $n > 4$ . Suppose  $\Lambda^2 T_X$  is strictly nef. If  $T_X$  is not nef, then there exists a Mori elementary contraction  $\beta : X \rightarrow B$  such that  $\beta$  is a projective bundle.*

*Proof.* By Theorem 4.1 and Lemma 4.3, there exists rational curve  $\sigma : C \rightarrow X$  such that  $v^*(-K_X)$  is ample over  $C$ . Then by the ‘‘Cone theorem’’ (e.g. [KM98, Theorem 3.15]), there exist countably many rational curves  $\Gamma_i$  such that

$$\overline{NE(X)} = \overline{NE(X)}_{K_X \geq 0} + \sum_i \mathbb{R}^+[\Gamma_i]$$

where  $0 < -K_X \cdot \Gamma_i \leq n + 1$ . Since  $\Lambda^2 T_X$  is strictly nef,

$$T_X|_{\Gamma_i} = \mathcal{O}(-b) \oplus \left( \bigoplus_{i=1}^{n-1} \mathcal{O}(a_i) \right),$$

Then by Lemma 4.3,  $\Gamma_i$  is a free curve, i.e.  $b \leq 0$ , otherwise  $-K_X \cdot \Gamma_i \geq 2n - 3 > n + 1$ . Since deformations of a free curve cover a dense subset of  $X$  and  $\text{Locus}(\Gamma_i)$  is closed,  $\text{Locus}(\Gamma_i) = X$ . So for an arbitrary  $\Gamma_i$ , the associated Mori elementary contraction is of fibre type. We pick one of these extremal curves, namely  $\Gamma$  and denote the associated contraction by

$$\beta : X \rightarrow B.$$

For an arbitrary smooth fibre  $F$ , there exists an exact sequence

$$0 \rightarrow T_F \rightarrow T_X|_F \rightarrow \mathcal{O}_F^{\oplus \dim B} \rightarrow 0.$$

Since  $\Lambda^2 T_X$  is strictly nef, we deduce  $\dim(B) \leq 1$ . Actually,  $B$  is not a point, otherwise the Picard number  $\rho(X) = 1$  and  $X$  is Fano. By Lemma 4.3,  $X$  is  $\mathbb{P}^n$  or  $\mathbb{Q}^n$ . So  $B$  is a curve and  $F \simeq \mathbb{P}^{n-1}$ . Now we aim to show that  $T_X$  is  $\beta$ -nef, then by [SW04, Theorem 4.2],  $\beta$  is a smooth fibration.

*Claim 1* : Let  $F'_0$  be a fibre (possibly singular) of  $\beta : X \rightarrow B$  and  $F_0 = \text{red}(F'_0)$ , then  $F_0$  is smooth and irreducible.

Since fibres of  $\beta$  are equi-dimensional, by [Kol01, Theorem 3.5.3, Chapter 4], every fibre is rationally chain connected. Now suppose that  $F_0$  is reducible and  $F_0 = F_1 + F_2$ . Note that for an arbitrary curve  $C$  in  $F_0$ , by the ‘‘Cone Theorem’’,  $C$  is numerical to some multiple of  $\Gamma$ , which means any fibre curve could be deformed into smooth fibres. Then for an arbitrary rational curve  $C_1$  contained in  $F_1$  which has non-empty intersection with  $F_2$ ,  $C_1 \cdot F_2 = 0$ , then  $C_1$  is contained in  $F_2$ . So  $F_0$  is irreducible. It is obvious that  $F_0$  satisfies the assumptions of Lemma 4.6, so  $F_0$  is smooth.

*Claim 2* :  $T_X$  is  $\beta$ -nef.

Since  $F_0$  is smooth, for an arbitrary curve  $g : \tilde{C} \rightarrow F_0$ , by the same arguments as in Lemma 4.6

$$(4.17) \quad 0 \rightarrow g^*(\mathcal{J}/\mathcal{J}^2) \rightarrow g^*(\Omega_X) \rightarrow g^*(\Omega_{F_0}) \rightarrow 0$$

is exact. Note that  $\mathcal{J}/\mathcal{J}^2 \simeq \mathcal{O}_X(F_0)^{-1}|_{F_0}$  and  $\mathcal{O}_X(F_0)|_{F_0}$  is a numerically trivial line bundle. By the induced exact sequence

$$(4.18) \quad 0 \rightarrow g^*(\Lambda^2 T_{F_0}) \rightarrow g^*(\Lambda^2 T_X) \rightarrow g^*(T_{F_0}) \otimes g^*(\mathcal{O}_X(F_0)|_{F_0}) \rightarrow 0,$$

we obtain  $g^*(T_{F_0})$  is nef. Then as an extension of  $g^*(T_{F_0})$  with a numerically trivial line bundle,  $g^*(T_X)$  is nef.

In conclusion, Lemma 4.7 is a direct corollary of Claim 2.  $\square$

**Theorem 4.8.** *Let  $X$  be a smooth projective variety of dimension  $n > 4$ . If  $\Lambda^2 T_X$  is strictly nef, then  $T_X$  is nef.*

*Proof.* Suppose  $T_X$  is not nef, then by Lemma 4.7,  $X$  is a projective bundle over some smooth curve  $B$ . Since  $-K_X$  is nef,  $B$  is an elliptic curve or  $\mathbb{P}^1$ .

If  $B$  is an elliptic curve, then by the relative tangent bundle exact sequence

$$(4.19) \quad 0 \rightarrow T_{X/B} \rightarrow T_X \rightarrow \beta^*(T_B) \rightarrow 0,$$

and the induced exact sequence

$$(4.20) \quad 0 \rightarrow \Lambda^2 T_{X/B} \rightarrow \Lambda^2 T_X \rightarrow T_{X/B} \rightarrow 0,$$

we deduce that  $T_X$  is nef since  $\Lambda^2 T_X$  and  $T_{X/B}$  are nef. This is a contradiction.

Suppose  $B = \mathbb{P}^1$ . Let  $X = \mathbb{P}(V)$  for some vector bundle  $V$ , we may assume that  $V = \bigoplus_{i=1}^n \mathcal{O}(a_i)$  where  $0 < a_1 \leq a_2 \leq \dots \leq a_n$ . Let  $\mathcal{L} = \mathcal{O}(a_1)$ , then the quotient morphism

$$\bigoplus_{i=1}^n \mathcal{O}(a_i) \longrightarrow \mathcal{O}(a_1) \longrightarrow 0$$

defines a section  $\sigma : B \rightarrow X$ . Indeed, let  $\nu : B \rightarrow B$  be the identity map. By (2.4), this quotient determines a section of  $\beta : \mathbb{P}(V) \rightarrow B$ , namely, a morphism  $\sigma : B \rightarrow \tilde{X}$  such that  $\beta \circ \sigma = \nu = Id$  with  $\mathcal{L} = \sigma^*(\mathcal{O}_V(1))$ .

Since  $\beta : X \rightarrow B$  is a projective bundle, one has the Euler sequence

$$(4.21) \quad 0 \rightarrow \mathcal{O}_X \rightarrow \beta^*(V^*) \otimes \mathcal{O}_V(1) \rightarrow T_{X/B} \rightarrow 0$$

and

$$\sigma^*(\beta^*(V^*) \otimes \mathcal{O}_X(1)) = \sigma^* \circ \beta^*(V^*) \otimes \mathcal{L} = \bigoplus_{i=1}^n \mathcal{O}(a_1 - a_i).$$

Let  $\tilde{a}_i = a_1 - a_i$ , then  $\tilde{a}_i \leq 0$  and  $0 = \tilde{a}_1 \geq \tilde{a}_2 \geq \dots \geq \tilde{a}_n$ . Let  $\sigma^*(T_{X/B}) = \bigoplus_{j=1}^{n-1} \mathcal{O}(b_j)$ , where  $b_1 \leq \dots \leq b_{n-1}$ . By pulling-back the exact sequence (4.21), we obtain the following exact sequence:

$$(4.22) \quad 0 \rightarrow \mathcal{O}_B \rightarrow \bigoplus_{i=1}^n \mathcal{O}(\tilde{a}_i) \rightarrow \bigoplus_{j=1}^{n-1} \mathcal{O}(b_j) \rightarrow 0.$$

By performing square wedge product of the exact sequence (4.22), we obtain another exact sequence:

$$(4.23) \quad 0 \rightarrow \bigoplus_{j=1}^{n-1} \mathcal{O}(b_j) \xrightarrow{d} \bigoplus_{i < j} \mathcal{O}(\tilde{a}_i + \tilde{a}_j) \rightarrow \bigoplus_{i < j} \mathcal{O}(b_i + b_j) \rightarrow 0$$

By the injectivity of  $d$ , we know  $b_j \leq \tilde{a}_1 + \tilde{a}_2 \leq 0$  for arbitrary  $j$ . Since the sequence

$$0 \rightarrow T_{X/B} \rightarrow T_X \rightarrow \beta^*(T_B) \rightarrow 0$$

is exact,  $\deg \sigma^*(-K_X) = \sum b_j + 2 \leq 2$ . Since  $\Lambda^2 T_X$  is strictly nef, we have already obtained the inequality  $\deg \sigma^*(-K_X) \geq n$  in Lemma 4.3. This is a contradiction.  $\square$

Now we complete the proof of Theorem 4.4, by using Theorem 1.2, Theorem 1.4 and the lemmas described above.

*The proof of Theorem 4.4.* By Theorem 4.8,  $T_X$  is nef. Suppose  $c_1(X)^n = 0$ . By Theorem [DPS94, Main Theorem],  $X$  has a finite étale cover  $\tilde{X}$  with irregularity  $q(\tilde{X}) > 0$  and the Albanese morphism  $\alpha : \tilde{X} \rightarrow \text{Alb}(\tilde{X})$  is a smooth fibration. Moreover, the fibers  $F$  are Fano manifolds with nef tangent bundles. Let  $C := \text{Alb}(\tilde{X})$ . Since  $T_{\tilde{X}/C}|_F = T_F$  and  $\alpha^*(T_C)|_F$  is a trivial vector bundle, we obtain the following exact sequence from the relative exact sequence of tangent bundles:

$$(4.24) \quad 0 \rightarrow T_F \rightarrow T_{\tilde{X}}|_F \rightarrow \mathcal{O}_F^{\oplus \dim C} \rightarrow 0.$$

Since  $\Lambda^2 T_X$  is strictly nef,  $\Lambda^2 T_{\tilde{X}}$  is strictly nef. By Lemma 2.3, we know  $\dim C = 1$ . From the induced exact sequence

$$(4.25) \quad 0 \rightarrow \Lambda^2 T_F \rightarrow \Lambda^2 T_{\tilde{X}}|_F \rightarrow TF \rightarrow 0.$$

we know  $T_F$  is strictly nef. By Theorem 1.2,  $F$  is isomorphic to  $\mathbb{P}^{n-1}$ . Then  $\tilde{X}$  is a  $\mathbb{P}^{n-1}$ -bundle over an elliptic curve  $C$ . Therefore,  $\tilde{X}$  is a projective bundle over  $C$ . Let  $\tilde{X} = \mathbb{P}(V) \rightarrow C$ . We have  $-K_{\tilde{X}} = \mathcal{O}_V(n) \otimes \pi^* \det(V^*)$ . Since  $T\tilde{X}$  is nef, and  $\mathbb{P}(V) = \mathbb{P}(V \otimes L)$  for any line bundle  $L$ , after a finite étale cover (e.g. Case 2 of Lemma 3.4), we can assume  $\deg(V) = 0$  and so  $V$  is numerically flat.

By Lemma 4.9, a numerically flat vector bundle  $V$  over an elliptic curve admits a line bundle quotient  $V \rightarrow L \rightarrow 0$ , where  $L$  is of degree 0. Let  $\nu : C \rightarrow C$  be the identity map. By the configuration in (2.4), this quotient determines a section of  $\pi : \mathbb{P}(V) \rightarrow C$ , namely, a morphism  $\sigma : C \rightarrow \tilde{X}$  such that  $\pi \circ \sigma = \nu = \text{Id}$ . By pulling back the relative tangent bundle sequence as in (4.20), we obtain the exact sequence:

$$(4.26) \quad 0 \rightarrow \sigma^* \left( \Lambda^2 T_{\tilde{X}/C} \right) \rightarrow \sigma^* \left( \Lambda^2 T_{\tilde{X}} \right) \rightarrow \sigma^* \left( T_{\tilde{X}/C} \right) \rightarrow 0.$$

Since  $\sigma^*(\Lambda^2 T_{\tilde{X}})$  is strictly nef, as its quotient,  $\sigma^*(T_{\tilde{X}/C})$  is strictly nef over  $C$ . As  $C$  is an elliptic curve, by Theorem 3.5,  $\sigma^*(T_{\tilde{X}/C})$  is ample. Then by pulling back the Euler sequence (4.21), we have

$$0 \rightarrow \mathcal{O}_C \rightarrow \sigma^*(V^*) \otimes L \rightarrow \sigma^*(T_{\tilde{X}/C}) \rightarrow 0,$$

since  $L = \sigma^*(\mathcal{O}_V(1))$  as in (2.4). Since  $\sigma^*(T_{\tilde{X}/C})$  is ample,

$$\deg(\sigma^*(V^*) \otimes L) = \deg(\sigma^*(T_{\tilde{X}/C})) > 0.$$

Since  $V$  is numerical flat and  $\deg(L) = 0$ , we obtain a contradiction. The proof of Theorem 4.4 is completed.  $\square$

**Lemma 4.9.** *Let  $C$  be an elliptic curve.*

- (1) *Let  $V$  be a Hermitian flat vector bundle. Then  $V = L_1 \oplus \cdots \oplus L_r$  where  $L_i \in \text{Pic}^0(C)$  and  $r$  is the rank of  $V$ .*
- (2) *Let  $V$  be a numerically flat vector bundle.  $V$  admits a line bundle quotient  $V \rightarrow L \rightarrow 0$ , where  $L \in \text{Pic}^0(C)$ .*

*Proof.* (1). By Harder-Narasimhan Theorem (e.g. [Kob87, Theorem 2.5, Chapter V]), if  $V$  is Hermitian flat over an elliptic curve, it is Hermitian-Einstein and so  $V = \oplus V_i$  where  $V_i$  are stable bundles of degree zero. It is well-known that,  $V_i$  must be of rank one (e.g. [Tu93, Appendix A]), i.e.  $V_i \in \text{Pic}^0(C)$ .

(2). If  $V$  is numerically flat,  $V^*$  is also numerically flat. By [DPS94, Theorem 1.18], a holomorphic vector bundle  $V^*$  is numerically flat if and only if  $V^*$  admits a filtration

$$(4.27) \quad \{0\} = V_0 \subset V_1 \subset \cdots \subset V_p = V^*$$

by vector subbundles such that the quotients  $V_k/V_{k-1}$  are Hermitian flat. In particular,  $V_1$  is Hermitian flat and by (1),  $V_1$  has a degree zero line subbundle  $L^*$ . Hence  $L^*$  is a line subbundle of  $V^*$ , i.e.  $V \rightarrow L \rightarrow 0$  is a line bundle quotient.  $\square$

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