J. ALGEBRAIC GEOMETRY
22 (2013) 303–331
S 1056-3911(2012)00588-8
Article electronically published on August 9, 2012

POSITIVITY AND VANISHING THEOREMS FOR AMPLE VECTOR BUNDLES

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Abstract

In this paper, we study the Nakano-positivity and dual-Nakano-positivity of certain adjoint vector bundles associated to ample vector bundles. As applications, we get new vanishing theorems about ample vector bundles. For example, we prove that if E is an ample vector bundle over a compact Kähler manifold X, $S^kE\otimes\det E$ is both Nakano-positive and dual-Nakano-positive for any $k\geq 0$. Moreover, $H^{n,q}(X,S^kE\otimes\det E)=H^{q,n}(X,S^kE\otimes\det E)=0$ for any $q\geq 1$. In particular, if (E,h) is a Griffiths-positive vector bundle, the naturally induced Hermitian vector bundle $(S^kE\otimes\det E, S^kh\otimes\det h)$ is both Nakano-positive and dual-Nakano-positive for any $k\geq 0$.

1. Introduction

Let E be a holomorphic vector bundle with a Hermitian metric h. Nakano in [29] introduced an analytic notion of positivity by using the curvature of (E,h), and now it is called Nakano positivity. Griffiths defined in [14] Griffiths positivity of (E,h). On a Hermitian line bundle, these two concepts are the same. In general, Griffiths positivity is weaker than Nakano positivity. On the other hand, Hartshorne defined in [16] the ampleness of a vector bundle over a projective manifold. A vector bundle E is said to be ample if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is ample over $\mathbb{P}(E^*)$.

For a line bundle, it is well-known that the ampleness of the bundle is equivalent to its Griffiths positivity. In [14], Griffiths conjectured that this equivalence is also valid for vector bundles, i.e. E is an ample vector bundle if and only if E carries a Griffiths-positive metric. As is well-known if E admits a Griffiths-positive metric, then $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ has a Griffiths-positive metric(see Proposition 2.5). Finding a Griffiths-positive metric on an ample vector bundle seems to be very difficult but is worth being investigated. In

Received July 2, 2010 and, in revised form, January 5, 2011, January 25, 2011, and February 15, 2011. The first author was supported in part by NSF Grant #1007053.

[7], Campana and Flenner gave an affirmative answer to the Griffiths conjecture when the base S is a projective curve; see also [38]. In [36], Siu and Yau proved the Frankel conjecture that every compact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to the projective space. The positivity of holomorphic bisectional curvature is the same as Griffiths positivity of the holomorphic tangent bundle. On the other hand, S. Mori [26] proved the Hartshorne conjecture that any algebraic manifold with ample tangent vector bundle is biholomorphic to the projective space.

In this paper, we consider the existence of positive metrics on ample vector bundles. It is well-known that metrics with good curvature properties are bridges between complex algebraic geometry and complex analytic geometry. Various vanishing theorems about ample vector bundles can be found in [10], [31], [25], [34], [22] and [21]. In this paper we take a different approach, we will construct Nakano-positive and dual-Nakano-positive metrics on various vector bundles associated to ample vector bundles.

Let E be a holomorphic vector bundle over a compact Kähler manifold S and F a line bundle over S. Let r be the rank of E and n be the complex dimension of S. In the following we briefly describe our main results.

Theorem 1.1. *For any integer $k \geq 0$, if $S^{r+k}E \otimes \det E^* \otimes F$ is ample over S, then $S^kE \otimes F$ is both Nakano-positive and dual-Nakano-positive.*

Here we make no assumption on E and we allow E to be negative. For definitions about Nakano-positivity, dual-Nakano-positivity and ampleness, see Section 2. As pointed out by Berndtsson, the Nakano positive part of Theorem 1.1 is a special case of [2] where he proves it in the case of a general holomorphic fibration, but his method cannot give the dual-Nakano-positive part of Theorem 1.1. Note that Nakano-positive vector bundles are not necessarily dual-Nakano-positive and vice versa. For example, for any $n \geq 2$, the Fubini-Study metric h_{FS} on the holomorphic tangent bundle $T\mathbb{P}^n$ of $\mathbb{P}^n$ is semi-Nakano-positive and dual-Nakano-positive. It is well-known that $T\mathbb{P}^n$ does not admit a smooth Hermitian metric with Nakano-positive curvature for any $n \geq 2$. It is also easy to see that the holomorphic cotangent bundle of a complex hyperbolic space form is Nakano-positive and is not dual-Nakano-positive. On the other hand, by the dual Nakano-positivity, we can get various new vanishing theorems of type $H^{q,n}$; see Theorem 6.2 and Proposition 6.4. As applications of Theorem 1.1, we get the following results:

Theorem 1.2. *Let E be an ample vector bundle over S.*

(1) *If F is a nef line bundle, then there exists $k_0 = k_0(S, E)$ such that $S^kE \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$. In particular, S^kE is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$.*

(2) *If F is an arbitrary vector bundle, then there exists $k_0 = k_0(S, E, F)$ such that for any $k \geq k_0$, $S^k E \otimes F$ is Nakano-positive and dual-Nakano-positive.*

Moreover, if the Hermitian vector bundle (E, h) is Griffiths-positive, then for large k, $(S^k E, S^k h)$ is Nakano-positive and dual-Nakano-positive.

The following results follow immediately from Theorems 1.1 and 1.2:

Corollary 1.3. *Let E be a holomorphic vector bundle over S.*

(1) *If E is ample, $S^k E \otimes \det E$ is Nakano-positive and dual-Nakano-positive for any $k \geq 0$.*
(2) *If E is ample and its rank r is greater than 1, then $S^m E^* \otimes (\det E)^t$ is Nakano-positive and dual-Nakano-positive for any $t \geq r + m - 1$.*
(3) *If $S^{r+1} E \otimes \det E^*$ is ample, then E is Nakano-positive and dual-Nakano-positive, so it is Griffiths-positive.*

If (E, h) is a Griffiths-positive vector bundle, Demailly-Skoda proved that $E \otimes \det E$ and $E^* \otimes (\det E)^r$ are Nakano-positive if $r > 1$ [12]. Recently, Berndtsson proved in [2] that $S^k E \otimes \det E$ is Nakano-positive as soon as E is ample. For more related results, we refer the reader to [2], [3] [4], [27], [28] and [33] and references therein.

Let h_{FS} be the Fubini-Study metric on $T\mathbb{P}^n$ and $S^k h_{FS}$ the induced metric on $S^k T\mathbb{P}^n$ by Veronese mapping. Let $n \geq 2$. It is easy to see that $T\mathbb{P}^n$ does not admit a Nakano-positive metric. In particular $(T\mathbb{P}^n, h_{FS})$ is not Nakano-positive. However, $(S^k T\mathbb{P}^n, S^k h_{FS})$ is Nakano-positive and dual-Nakano-positive for any $k \geq 2$ since $(S^{k+n} T\mathbb{P}^n \otimes K_{\mathbb{P}^n}, S^k h_{FS} \otimes \det(h_{FS})^{-1})$ is Griffiths-positive. This can be viewed as an evidence of positivity of some adjoint vector bundles, namely, vector bundles of type $S^k E \otimes (\det E)^\ell \otimes K_S$.

Theorem 1.4. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S. If $r > 1$, then*

(1) *$S^k E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive for any $k \geq \max\{n - r, 0\}$. Moreover, the lower bound is sharp.*
(2) *$E \otimes (\det E)^k \otimes K_S$ is Nakano-positive and dual-Nakano-positive for any $k \geq \max\{n + 1 - r, 2\}$. Moreover, the lower bound is sharp.*

In general, $\det E \otimes K_S$ is not an ample line bundle, for example, $(S, E) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(1))$. Similarly, in the case $n + 1 - r > 2$, i.e. $1 < r < n - 1$, the vector bundle $K_S \otimes (\det E)^{n-r}$ can be a negative line bundle; for example, $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$. So Theorem 1.4 is independent of the (dual-)Nakano-positivity of $S^k E \otimes \det E$.

Vanishing theorems follow immediately from the Nakano-positive and dual-Nakano-positive metrics in Theorems 1.1 and 1.2, Corollary 1.3 and Theorem

1.4. We discuss them in Theorem 6.2, Proposition 6.4 and Corollary 6.9. In the following, we only state one for example.

Proposition 1.5. *If E is ample over a compact Kähler manifold X,*

$$H^{n,q}(X, S^kE \otimes \det E) = H^{q,n}(X, S^kE \otimes \det E) = 0$$

for any $q \geq 1$ and $k \geq 0$.

It is a generalization of Griffiths' vanishing theorem ([14], Theorem G).

2. Background material

Let E be a holomorphic vector bundle over a compact Kähler manifold S and h a Hermitian metric on E. There exists a unique connection ∇ which is compatible with the metric h and complex structure on E. It is called the Chern connection of (E, h). Let $\{z^i\}_{i=1}^n$ be local holomorphic coordinates on S and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $R^\nabla \in \Gamma(S, \Lambda^2 T^*S \otimes E^* \otimes E)$ has the form

$$R^\nabla = \frac{\sqrt{-1}}{2\pi} R_{i\bar{j}\alpha}^{\gamma} dz^i \wedge d\bar{z}^j \otimes e^\alpha \otimes e_\gamma \tag{2.1}$$

where $R_{i\bar{j}\alpha}^{\gamma} = h^{\gamma\overline{\beta}} R_{i\bar{j}\alpha\overline{\beta}}$ and

$$R_{i\bar{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}} \frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j}. \tag{2.2}$$

Here and henceforth we sometimes adopt the Einstein convention for summation.

Definition 2.1. A Hermitian vector bundle (E, h) is said to be *Griffiths-positive*, if for any nonzero vectors $u = u^i \frac{\partial}{\partial z^i}$ and $v = v^\alpha e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^i \overline{u}^j v^\alpha \overline{v}^\beta > 0, \tag{2.3}$$

(E, h) is said to be *Nakano-positive*, if for any nonzero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^{i\alpha} \overline{u}^{j\beta} > 0, \tag{2.4}$$

(E, h) is said to be *dual-Nakano-positive*, if for any nonzero vector $u = u^{i\alpha} \frac{\partial}{\partial z^i} \otimes e_\alpha$,

$$\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} u^{i\beta} \overline{u}^{j\alpha} > 0. \tag{2.5}$$

It is easy to see that (E, h) is dual-Nakano-positive if and only if (E^*, h^*) is Nakano-negative. The notions of semi-positivity, negativity and semi-negativity can be defined similarly. We say E is Nakano-positive (resp. Griffiths-positive, dual-Nakano-positive, ...), if it admits a Nakano-positive (resp. Griffiths-positive, dual-Nakano-positive, ...) metric.

The following geometric definition of nefness is due to [11].

Definition 2.2. Let (S, ω_0) be a compact Kähler manifold. A line bundle L over S is said to be nef, if for any $\varepsilon > 0$, there exists a smooth Hermitian metric h_ε on L such that the curvature of (L, h_ε) satisfies

$$R = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h_\varepsilon \geq -\varepsilon\omega_0. \tag{2.6}$$

This means that the curvature of L can have an arbitrarily small negative part. Clearly a nef line bundle L satisfies

$$\int_C c_1(L) \geq 0$$

for any irreducible curve $C \subset S$. For projective algebraic S, both notions coincide.

By the Kodaira embedding theorem, we have the following geometric definition of ampleness.

Definition 2.3. Let (S, ω_0) be a compact Kähler manifold. A line bundle L over S is said to be ample, if there exists a smooth Hermitian metric h on L such that the curvature R of (L, h) satisfies

$$R = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h > 0. \tag{2.7}$$

For comprehensive descriptions of positivity, nefness, ampleness and related topics, see [9], [11], [20], [14], [34] and [38].

Let E be a Hermitian vector bundle of rank r over a compact Kähler manifold S, $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$ be the tautological line bundle of the projective bundle $\mathbb{P}(E^*)$ and π the canonical projection $\mathbb{P}(E^*) \to S$. By definition [16], E is an ample vector bundle over S if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is an ample line bundle over $\mathbb{P}(E^*)$. E is said to be nef, if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is nef. To simplify the notation we will denote $\mathbb{P}(E^*)$ by X and the fiber $\pi^{-1}(\{s\})$ by X_s.

Let $(e_1, \cdots, e_r)$ be the local holomorphic frame with respect to a given trivialization on E and the dual frame on E^* is denoted by $(e^1, \cdots, e^r)$. The corresponding holomorphic coordinates on E^* are denoted by $(W_1, \cdots, W_r)$. There is a local section e_{L^*} of L^* defined by

$$e_{L^*} = \sum_{\alpha=1}^{r} W_\alpha e^\alpha. \tag{2.8}$$

Its dual section is denoted by e_L. Let h^E be a fixed Hermitian metric on E and h^L the induced quotient metric by the morphism $(\pi^*E, \pi^*h^E) \to L$.

If $\left(h_{\alpha\overline{\beta}}\right)$ is the matrix representation of h^E with respect to the basis $\{e_\alpha\}_{\alpha=1}^r$, then h^L can be written as

$$h^L = \frac{1}{h^{L^*}(e_{L^*}, e_{L^*})} = \frac{1}{\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta}. \tag{2.9}$$

Proposition 2.4. *The curvature of (L, h^L) is*

$$R^{h^L} = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log h^L = \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta\right) \tag{2.10}$$

where ∂ and $\overline{\partial}$ are operators on the total space $\mathbb{P}(E^*)$.

Although the following result is well-known (see [9] and [14]), we include a proof here for the sake of completeness.

Proposition 2.5. *If (E, h^E) is a Griffiths-positive vector bundle, then E is ample.*

Proof. We will show that the induced metric h^L in (2.9) is positive. We fix a point $p \in \mathbb{P}(E^*)$, then there exist local holomorphic coordinates $(z^1, \cdots, z^n)$ centered at point $s = \pi(p)$ and local holomorphic basis $\{e_1, \cdots, e_r\}$ of E around s such that

$$h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} - R_{i\overline{j}\alpha\overline{\beta}} z^i \overline{z}^j + O(|z|^3). \tag{2.11}$$

Without loss of generality, we assume p is the point $(0, \cdots, 0, [a_1, \cdots, a_r])$ with $a_r = 1$. On the chart $U = \{W_r = 1\}$ of the fiber $\mathbb{P}^{r-1}$, we set $w^A = W_A$ for $A = 1, \cdots, r-1$. By formulas (2.10) and (2.11),

(2.12)

$$R^{h^L}(p) = \frac{\sqrt{-1}}{2\pi}\left(\sum R_{i\overline{j}\alpha\overline{\beta}} \frac{a_\beta \overline{a}_\alpha}{|a|^2} dz^i \wedge d\overline{z}^j + \sum_{A,B=1}^{r-1}\left(\delta_{AB} - \frac{a_B \overline{a}_A}{|a|^2}\right) dw^A \wedge d\overline{w}^B\right)$$

where $|a|^2 = \sum\limits_{\alpha=1}^r |a_\alpha|^2$. If R^E is Griffith positive,

$$\left(\sum_{\alpha,\beta=1}^r R_{i\overline{j}\alpha\overline{\beta}} \frac{a_\beta \overline{a}_\alpha}{|a|^2}\right)$$

is a Hermitian positive $n \times n$ matrix. Consequently, $R^{h^L}(p)$ is a Hermitian positive $(1,1)$ form on $\mathbb{P}(E^*)$, i.e. h^L is a positive Hermitian metric. $\square$

The following linear algebraic lemma will be used in Theorem 4.5.

Lemma 2.6. *If the matrix*

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is invertible and D is invertible, then $(A - BD^{-1}C)^{-1}$ exists and

$$T^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} + D^{-1} \end{pmatrix}.$$

Moreover, if T is positive definite, then $A - BD^{-1}C$ is positive definite.

3. Curvature formulas

Let F be a holomorphic line bundle over S, $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$ and $\pi : \mathbb{P}(E^*) \to S$. For simplicity of notation, we set $\widetilde{L} = L^k \otimes \pi^*(F)$ for $k \geq 0$ and $X = \mathbb{P}(E^*)$. Let h_0 be a Hermitian metric on $\widetilde{L}$ and $\{\omega_s\}_{s\in S}$ a smooth family of Kähler metrics on the fibers $X_s = \mathbb{P}(E_s^*)$ of X which are induced by the curvature form of some metric on $\mathcal{O}_{\mathbb{P}(E^*)}(1)$. Let $\{w^A\}_{A=1}^{r-1}$ be the local holomorphic coordinates on the fiber X_s which are induced by the homogeneous coordinates $[W_1, \cdots, W_r]$ on a trivialization chart. Using this notation, we can write ω_s as

$$\omega_s = \frac{\sqrt{-1}}{2\pi} \sum_{A,B=1}^{r-1} g_{A\overline{B}}(s,w) dw^A \wedge d\overline{w}^B. \tag{3.1}$$

It is well-known that $H^0(\mathbb{P}^{r-1}, \mathcal{O}_{\mathbb{P}^{r-1}}(k))$ can be identified as the space of homogeneous polynomials of degree k in r variables. Therefore, the sections of $H^0(X_s, \widetilde{L}|_{X_s})$ are of the form $V_\alpha e_L^{\otimes k} \otimes \underline{e}$ where V_α are homogenous polynomials in $\{W_1, \cdots, W_r\}$ of degree k and $\underline{e}$ the base of $\pi^*(F)$ induced by a base e of F. For example, if $\alpha = (\alpha_1, \cdots, \alpha_r)$ with $\alpha_1 + \cdots + \alpha_r = k$ and α_j are nonnegative integers,

$$V_\alpha = W_1^{\alpha_1} \cdots W_r^{\alpha_r}. \tag{3.2}$$

Now we set

$$E_\alpha = e_1^{\otimes\alpha_1} \otimes \cdots \otimes e_r^{\otimes\alpha_r} \otimes e \quad \text{and} \quad e_{\widetilde{L}} = e_L^{\otimes k} \otimes \underline{e}$$

which are bases of $S^kE \otimes F$ and $\widetilde{L}$, respectively. We obtain a vector bundle whose fibers are $H^0(X_s, \widetilde{L}|_{X_s})$. In fact, this vector bundle is $\widetilde{E} = S^kE \otimes F$. Now we can define a smooth Hermitian metric f on $S^kE \otimes F$ by $(\widetilde{L}, h_0)$ and (X_s, ω_s), locally it is

$$\begin{aligned} f_{\alpha\overline{\beta}} := f(E_\alpha, E_\beta) &= \int_{X_s} \langle V_\alpha e_{\widetilde{L}}, V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\ &= \int_{X_s} h_0 V_\alpha \overline{V}_\beta \frac{\omega_s^{r-1}}{(r-1)!}. \end{aligned} \tag{3.3}$$

Here we regard h_0 locally as a positive function. In this general setting, the Hermitian metric h_0 on $\widetilde{L}$ and Kähler metrics ω_s on the fibers are independent.

Let $(z^1, \cdots, z^n)$ be local holomorphic coordinates on S. By definition, the curvature tensor of f is

$$R_{i\bar{j}\alpha\overline{\beta}} = -\frac{\partial^2 f_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + \sum_{\gamma,\delta} f^{\gamma\overline{\delta}} \frac{\partial f_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial f_{\gamma\overline{\beta}}}{\partial \overline{z}^j}. \tag{3.4}$$

In the following, we will compute the curvature of f. Let $T_{X/S}$ be the relative tangent bundle of the fibration $\mathbb{P}(E^*) \to S$, then $g_{A\overline{B}}$ is a metric on $T_{X/S}$ and $\det(g_{A\overline{B}})$ is a metric on $\det(T_{X/S})$. Let $\varphi = -\log(h_0 \det(g_{A\overline{B}}))$ be the local weight of induced Hermitian metric $h_0 \det(g_{A\overline{B}})$ on $\widetilde{L} \otimes \det(T_{X/S})$. In the sequel, we will use the following notation

$$\varphi_i = \frac{\partial \varphi}{\partial z^i}, \varphi_{i\bar{j}} = \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j}, \varphi_{A\overline{B}} = \frac{\partial^2 \varphi}{\partial w^A \partial \overline{w}^B}, \varphi_{i\overline{B}} = \frac{\partial^2 \varphi}{\partial z^i \partial \overline{w}^B}, \varphi_{A\bar{j}} = \frac{\partial^2 \varphi}{\partial \overline{z}^j \partial w^A}$$

and $(\varphi^{A\overline{B}})$ is the transpose inverse of the $(r-1) \times (r-1)$ matrix $(\varphi_{A\overline{B}})$,

$$\sum_{B=1}^{r-1} \varphi^{A\overline{B}} \varphi_{C\overline{B}} = \delta_C^A.$$

The following lemma can be deduced from the formulas in [32], [39] and [35]. In the case of holomorphic fibration $\mathbb{P}(E^*) \to S$, we can compute it directly.

Lemma 3.1. *The first order derivative of $f_{\alpha\overline{\beta}}$ is*

$$\frac{\partial f_{\alpha\overline{\beta}}}{\partial z^i} = -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_i \frac{\omega_s^{r-1}}{(r-1)!} = \int_{X_s} \langle -V_\alpha \varphi_i e_{\widetilde{L}}, V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!}. \tag{3.5}$$

Proof. By the local expression (3.1) of ω_s,

$$\frac{\omega_s^{r-1}}{(r-1)!} = \det(g_{A\overline{B}}) dV_{\mathbb{C}^{r-1}}$$

where $dV_{\mathbb{C}^{r-1}}$ is the standard volume on $\mathbb{C}^{r-1}$. Therefore,

$$f_{\alpha\overline{\beta}} = \int_{X_s} e^{-\varphi} V_\alpha \overline{V}_\beta dV_{\mathbb{C}^{r-1}}$$

and the first order derivative is

$$\begin{aligned} \frac{\partial f_{\alpha\overline{\beta}}}{\partial z^i} &= \int_{X_s} \frac{\partial e^{-\varphi}}{\partial z^i} V_\alpha \overline{V}_\beta dV_{\mathbb{C}^{r-1}} \\ &= -\int_{X_s} \varphi_i e^{-\varphi} V_\alpha \overline{V}_\beta dV_{\mathbb{C}^{r-1}} \\ &= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_i \frac{\omega_s^{r-1}}{(r-1)!}. \end{aligned}$$

□

Theorem 3.2. *The curvature tensor of the Hermitian metric f on $S^kE\otimes F$ is*

$$R_{i\bar{j}\alpha\bar{\beta}} = \int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\bar{j}} \frac{\omega_s^{r-1}}{(r-1)!} - \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!} \tag{3.6}$$

where

$$P_{i\alpha} = -V_\alpha \varphi_i - \sum_\gamma V_\gamma \left(\sum_\delta f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \right). \tag{3.7}$$

Proof. The idea we use is due to Berndtsson ([2], Section 2). For simplicity of notation, we set $A_{i\alpha} = -V_\alpha \varphi_i$. The Hermitian metric (3.3) is also a norm on the smooth section space $\Gamma(X_s, \widetilde{L}|_{X_s})$, and it induces an orthogonal projection

$$\widetilde{\pi}_s : \Gamma(X_s, \widetilde{L}|_{X_s}) \to H^0(X_s, \widetilde{L}|_{X_s}).$$

Using this projection, we can rewrite the first order derivative as

$$\begin{aligned} \frac{\partial f_{\alpha\bar{\beta}}}{\partial z^i} &= \int_{X_s} \langle A_{i\alpha} e_{\widetilde{L}}, V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\ &= \int_{X_s} \langle \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}) + (A_{i\alpha} e_{\widetilde{L}} - \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}})), V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\ &= \int_{X_s} \langle \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}), V_\beta e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \end{aligned}$$

since $(A_{i\alpha} e_{\widetilde{L}} - \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}))$ is in the orthogonal complement of $H^0(X_s, \widetilde{L}|_{X_s})$. By this relation, we can write $\widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}})$ in the basis $\{V_\alpha e_{\widetilde{L}}\}$ of $H^0(X_s, \widetilde{L}|_{X_s})$,

$$\widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}) = \sum_\gamma \left(\sum_\delta f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \right) \left(V_\gamma e_{\widetilde{L}} \right). \tag{3.8}$$

From this identity, we obtain

$$\int_{X_s} \left\langle \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}), \widetilde{\pi}_s(A_{j\beta} e_{\widetilde{L}}) \right\rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} = \sum_{\gamma,\delta} f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial f_{\gamma\bar{\beta}}}{\partial \bar{z}^j}. \tag{3.9}$$

Suppose

$$P_{i\alpha} = A_{i\alpha} - \sum_\gamma V_\gamma \left(\sum_\delta f^{\gamma\bar{\delta}} \frac{\partial f_{\alpha\bar{\delta}}}{\partial z^i} \right), \tag{3.10}$$

then $A_{i\alpha} e_{\widetilde{L}} = \widetilde{\pi}_s(A_{i\alpha} e_{\widetilde{L}}) + P_{i\alpha} e_{\widetilde{L}}$, that is,

$$\widetilde{\pi}_s(P_{i\alpha} e_{\widetilde{L}}) = 0. \tag{3.11}$$

Similar to Lemma 3.1, we obtain the second order derivative

$$\begin{aligned}
\frac{\partial^2 f_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} &= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} \langle V_\alpha \varphi_i e_{\widetilde{L}}, V_\beta \varphi_j e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} \langle A_{i\alpha} e_{\widetilde{L}}, A_{j\beta} e_{\widetilde{L}} \rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} \\
&\quad + \int_{X_s} \left\langle P_{i\alpha} e_{\widetilde{L}} + \widetilde{\pi}_s (A_{i\alpha} e_{\widetilde{L}}), P_{j\beta} e_{\widetilde{L}} + \widetilde{\pi}_s (A_{j\beta} e_{\widetilde{L}}) \right\rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!} \\
&\quad + \int_{X_s} \left\langle \widetilde{\pi}_s (A_{i\alpha} e_{\widetilde{L}}), \widetilde{\pi}_s (A_{j\beta} e_{\widetilde{L}}) \right\rangle_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \\
&= -\int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} + \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!} \\
&\quad + f^{\gamma\overline{\delta}} \frac{\partial f_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial f_{\gamma\overline{\beta}}}{\partial \overline{z}^j}.
\end{aligned}$$

By formula (3.4), we get the curvature formula (3.6). □

4. Positivity of Hermitian metrics

If (E, h) is a Griffiths-positive, Demailly-Skoda [12] showed that $(E \otimes \det E, h \otimes \det h)$ is Nakano-positive. They proved it by using a discrete Fourier transformation method. Here, we use a linear algebraic argument to show $(E \otimes \det E, h \otimes \det h)$ is both Nakano-positive and dual-Nakano-positive.

Let ω_{FS} be the standard Fubini-Study metric on $\mathbb{P}^{r-1}$ and $[W_1, \cdots W_r]$ the homogeneous coordinates on $\mathbb{P}^{r-1}$. If $A = (\alpha_1, \cdots, \alpha_k)$ and $B = (\beta_1, \beta_2, \cdots, \beta_k)$, we define the generalized Kronecker-δ for multi-index by the following formula

$$\delta_{AB} = \sum_{\sigma \in S_k} \prod_{j=1}^{k} \delta_{\alpha_{\sigma(j)} \beta_{\sigma(j)}} \tag{4.1}$$

where S_k is the permutation group in k symbols.

Lemma 4.1. *If $V_A = W_{\alpha_1} \cdots W_{\alpha_k}$ and $V_B = W_{\beta_1} \cdots W_{\beta_k}$, then*

$$\int_{\mathbb{P}^{r-1}} \frac{V_A \overline{V}_B}{|W|^{2k}} \frac{\omega_{FS}^{r-1}}{(r-1)!} = \frac{\delta_{AB}}{(r+k-1)!}. \tag{4.2}$$

For simple-index notation,

$$\int_{\mathbb{P}^{r-1}} \frac{W_\alpha \overline{W}_\beta}{|W|^2} \frac{\omega_{FS}^{r-1}}{(r-1)!} = \frac{\delta_{\alpha\beta}}{r!}, \int_{\mathbb{P}^{r-1}} \frac{W_\alpha \overline{W_\beta} W_\gamma \overline{W_\delta}}{|W|^4} \frac{\omega_{FS}^{r-1}}{(r-1)!} = \frac{\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}}{(r+1)!}. \tag{4.3}$$

Without loss of generality we can assume, at a fixed $s \in S$, $h_{\alpha\overline{\beta}}(s) = \delta_{\alpha\beta}$. The curvature of $(E \otimes \det E, h \otimes \det h)$ is

$$R^{E\otimes \det E}_{i\bar{j}\alpha\overline{\beta}}(s) = R_{i\bar{j}\alpha\overline{\beta}}(s) + \delta_{\alpha\beta} \cdot \sum_\gamma R_{i\bar{j}\gamma\overline{\gamma}}(s). \tag{4.4}$$

By Lemma 4.1, we obtain

$$R_{i\bar{j}\alpha\overline{\beta}}(s) + \delta_{\alpha\beta} \cdot \sum_\gamma R_{i\bar{j}\gamma\overline{\gamma}}(s) = r! \cdot \int_{\mathbb{P}^{r-1}} \frac{W_\alpha \overline{W}_\beta}{|W|^2} \varphi_{i\bar{j}} \frac{\omega_{FS}^{r-1}}{(r-1)!} \tag{4.5}$$

where

$$\varphi_{i\bar{j}} = (r+1) \sum_{\gamma,\delta} R_{i\bar{j}\gamma\overline{\delta}}(s) \frac{W_\delta \overline{W}_\gamma}{|W|^2}. \tag{4.6}$$

If (E, h) is Griffiths-positive, then $(\varphi_{i\bar{j}})$ is Hermitian positive. For any nonzero $u = (u^{i\alpha})$,

$$R^{E\otimes \det E}_{i\bar{j}\alpha\overline{\beta}} u^{i\beta} \overline{u}^{j\alpha} = (r+1) \int_{\mathbb{P}^{r-1}} \varphi_{i\bar{j}} \frac{(u^{i\beta}\overline{W}_\beta) \cdot \overline{(u^{j\alpha}\overline{W}_\alpha)}}{|W|^2} \frac{\omega_{FS}^{r-1}}{(r-1)!} > 0. \tag{4.7}$$

Therefore, $(E \otimes \det E, h \otimes \det h)$ is dual-Nakano-positive. By a similar formulation, we know $(E \otimes \det E, h \otimes \det h)$ is Nakano-positive. For more related results, see Section 7.

In the following, we will prove similar results for ample vector bundles.

4.1. Nakano-positivity. In this subsection, we will use $\overline{\partial}$-estimate on a compact Kähler manifold to analyze the curvature formula in Theorem 3.2,

$$R_{i\bar{j}\alpha\overline{\beta}} = \int_{X_s} h_0 V_\alpha \overline{V}_\beta \varphi_{i\bar{j}} \frac{\omega_s^{r-1}}{(r-1)!} - \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!}.$$

The first term on the right hand side involves the horizontal direction curvature $\varphi_{i\bar{j}}$ of the line bundle $\widetilde{L} \otimes \det(T_{X/S})$. If the line bundle $\widetilde{L} \otimes \det(T_{X/S})$ is positive in the horizontal direction, we can choose (h_0, ω_s) such that φ is positive in the horizontal direction, i.e. $(\varphi_{i\bar{j}})$ is Hermitian positive. We will get a lower bound of the second term by using Hörmander's L^2-estimate, following an idea of Berndtsson [2].

Lemma 4.2. *Let (M^n, ω_g) be a compact Kähler manifold and (L, h) a Hermitian line bundle over M. If there exists a positive constant c such that*

$$Ric(\omega_g) + R^h \geq c\omega_g, \tag{4.8}$$

then for any $w \in \Gamma(M, T^{*0,1}M \otimes L)$ such that $\overline{\partial} w = 0$, there exists a unique $u \in \Gamma(M, L)$ such that $\overline{\partial} u = w$ and $\widetilde{\pi}(u) = 0$ where $\widetilde{\pi} : \Gamma(M, L) \to H^0(M, L)$ is the orthogonal projection. Moreover,

$$\int_M |u|_h^2 \frac{\omega_g^n}{n!} \leq \frac{1}{c} \int_M |w|^2_{g^* \otimes h} \frac{\omega_g^n}{n!}. \tag{4.9}$$

We refer the reader to [9] and [18] for the proof of Lemma 4.2.

Now we apply Lemma 4.2 to each fiber (X_s, ω_s) and $(\widetilde{L}|_{X_s}, h_0|_{X_s})$. At a fixed point $s \in S$, the fiber direction curvature of the induced metric on $\widetilde{L} \otimes \det(T_{X/S})$ is

$$-\frac{\sqrt{-1}}{2\pi} \partial_s \overline{\partial}_s \log(h_0 \det(g_{A\overline{B}})) = R^{\widetilde{L}_s^{h_0}} + Ric_F(\omega_s). \tag{4.10}$$

On the other hand,

$$-\frac{\sqrt{-1}}{2\pi} \partial_s \overline{\partial}_s \log(h_0 \det(g_{A\overline{B}})) = \frac{\sqrt{-1}}{2\pi} \partial_s \overline{\partial}_s \varphi,$$

where $\varphi = -\log(h_0 \det(g_{A\overline{B}}))$. So condition (4.8) turns out to be

$$(\varphi_{A\overline{B}}) \geq c_s (g_{A\overline{B}}) \tag{4.11}$$

for some positive constant $c_s = c(s)$.

Theorem 4.3. *If $(\varphi_{A\overline{B}}) \geq c_s(g_{A\overline{B}})$ at point $s \in S$, then for any*

$$u = \sum_{i,\alpha} u^{i\alpha} \frac{\partial}{\partial z^i} \otimes E_\alpha \in \Gamma(S, T^{1,0}S \otimes \widetilde{E})$$

with $\widetilde{E} = S^k E \otimes F$, we have the following estimate at point s,

$$R_{i\overline{j}\alpha\overline{\beta}} u^{i\alpha} \overline{u^{j\beta}} \geq \int_{X_s} h_0 (V_\alpha u^{i\alpha}) \overline{(V_\beta u^{j\beta})} \left(\varphi_{i\overline{j}} - \frac{g^{A\overline{B}} \varphi_{i\overline{B}} \varphi_{A\overline{j}}}{c_s} \right) \frac{\omega_s^{r-1}}{(r-1)!}. \tag{4.12}$$

Proof. At point $s \in S$, we set

$$P = \sum_{i,\alpha} P_{i\alpha} u^{i\alpha} e_{\widetilde{L}} \in \Gamma(X_s, \widetilde{L}_s), \quad K = -\sum_{i,\alpha} V_\alpha \varphi_i u^{i\alpha} e_{\widetilde{L}} \in \Gamma(X_s, \widetilde{L}_s).$$

It is obvious that $\overline{\partial}_s P = \overline{\partial}_s K$ where $\overline{\partial}_s$ is $\overline{\partial}$ on the fiber direction. On the other hand, by (3.11), $\widetilde{\pi}_s(P) = 0$. So we can apply Lemma 4.2 and get

$$\int_{X_s} |P|^2_{h_0} \frac{\omega_s^{r-1}}{(r-1)!} \leq \frac{1}{c_s} \int_{X_s} |\overline{\partial}_s K|^2_{g_s^* \otimes h_0} \frac{\omega_s^{r-1}}{(r-1)!}. \tag{4.13}$$

Since $\overline{\partial}_s K = -\sum_{i,\alpha,B} V_\alpha \varphi_{i\overline{B}} u^{i\alpha} d\overline{z}^B \otimes e_{\widetilde{L}}$,

$$|\overline{\partial}_s K|^2_{g_s^* \otimes h_0} = \sum_{i,j} \sum_{\alpha,\beta} h_0 (V_\alpha u^{i\alpha}) \overline{(V_\beta u^{j\beta})} g^{A\overline{B}} \varphi_{i\overline{B}} \varphi_{A\overline{j}}.$$

By inequality (4.13) and Theorem 3.2, we get the estimate (4.12). □

Before proving the main theorems, we need the following lemma:

Lemma 4.4. *If E is a holomorphic vector bundle with rank r over a compact Kähler manifold S and F is a line bundle over S such that $S^{k+r}E\otimes\det E^*\otimes F$ is ample over S, then there exists a positive Hermitian metric λ_0 on $\mathcal{O}_{\mathbb{P}(E^*)}(k)\otimes\pi^*(F)\otimes\det(T_{X/S})$.*

Proof. Let $\widehat{E}$ be $S^{k+r}E\otimes\det(E^*)\otimes F$. It is obvious that $\mathbb{P}(S^{k+r}E^*)=\mathbb{P}(\widehat{E}^*)$. Their tautological line bundles are related by the following formula,

$$\mathcal{O}_{\mathbb{P}(\widehat{E}^*)}(1)=\mathcal{O}_{\mathbb{P}(S^{k+r}E^*)}(1)\otimes\pi_{k+r}^*(\det E^*)\otimes\pi_{k+r}^*(F),\tag{4.14}$$

where $\pi_{k+r}:\mathbb{P}(S^{k+r}E^*)\to S$ is the canonical projection. Let $v_{k+r}:\mathbb{P}(E^*)\to\mathbb{P}(S^{k+r}E^*)$ be the standard Veronese embedding, then

$$\mathcal{O}_{\mathbb{P}(E^*)}(k+r)=v_{k+r}^*\left(\mathcal{O}_{\mathbb{P}(S^{k+r}E^*)}(1)\right).\tag{4.15}$$

Similarly, let μ_{k+r} be the induced mapping $\mu_{k+r}:\mathbb{P}(E^*)\to\mathbb{P}(\widehat{E}^*)$, then

$$\mu_{k+r}^*\left(\mathcal{O}_{\mathbb{P}(\widehat{E}^*)}(1)\right)=\mathcal{O}_{\mathbb{P}(E^*)}(k+r)\otimes\pi^*(F\otimes\det E^*).\tag{4.16}$$

By the identity

$$K_X=\pi^*(K_S)\otimes\mathcal{O}_{\mathbb{P}(E^*)}(-r)\otimes\pi^*(\det E),\tag{4.17}$$

we obtain

$$\mu_{k+r}^*\left(\mathcal{O}_{\mathbb{P}(\widehat{E}^*)}(1)\right)=\mathcal{O}_{\mathbb{P}(E^*)}(k)\otimes\pi^*(F)\otimes\det(T_{X/S})=\widetilde{L}\otimes\det(T_{X/S}).\tag{4.18}$$

If $\widehat{E}$ is ample, then $\mathcal{O}_{\mathbb{P}(\widehat{E}^*)}(1)$ is ample and so is $\widetilde{L}\otimes\det(T_{X/S})$. So there exists a positive Hermitian metric λ_0 on $\widetilde{L}\otimes\det(T_{X/S})$. $\square$

Theorem 4.5. *Let E be a holomorphic vector bundle over a compact Kähler manifold S and F a line bundle over S. Let r be the rank of E and $k\geq 0$ an arbitrary integer. If $S^{k+r}E\otimes\det E^*\otimes F$ is ample over S, then there exists a smooth Hermitian metric f on $S^kE\otimes F$ such that $(S^kE\otimes F,f)$ is Nakano-positive.*

Proof. By Lemma 4.4, there exists a positive Hermitian metric λ_0 on the ample line bundle $\widetilde{L}\otimes\det(T_{X/S})$. We set

$$\omega_s=-\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s\log\lambda_0=\frac{\sqrt{-1}}{2\pi}\sum_{A,B=1}^{r-1}g_{A\overline{B}}(s,w)dw^A\wedge d\overline{w}^B$$

which is a smooth family of Kähler metrics on the fibers X_s. We get an induced Hermitian metric on $\widetilde{L}$, namely,

$$h_0=\frac{\lambda_0}{\det(g_{A\overline{B}})}.\tag{4.19}$$

Let f be the Hermitian metric on the vector bundle $S^kE\otimes\det F$ induced by $(\widetilde{L},h_0)$ and (X_s,ω_s) (see (3.3)). In this setting, the weight φ of induced metric on $\widetilde{L}\otimes\det(T_{X/S})$ is

$$\varphi=-\log\left(h_0\det(g_{A\overline{B}})\right)=-\log\lambda_0.$$

Hence

$$(\varphi_{A\overline{B}})=(g_{A\overline{B}}) \tag{4.20}$$

and in Theorem 4.3, $c_s=1$ for any $s\in S$. Therefore,

$$\begin{aligned}
R^{\widetilde{E}}(u,u) &= R_{i\overline{j}\alpha\overline{\beta}}u^{i\alpha}\overline{u^{j\beta}}\\
&\geq \int_{X_s} h_0(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\left(\varphi_{i\overline{j}}-\sum_{A,B=1}^{r-1}g^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)\frac{\omega_s^{r-1}}{(r-1)!}\\
&= \int_{X_s} h_0(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\left(\varphi_{i\overline{j}}-\sum_{A,B=1}^{r-1}\varphi^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)\frac{\omega_s^{r-1}}{(r-1)!}
\end{aligned}$$

for any $u=\sum\limits_{i,\alpha}u^{i\alpha}\frac{\partial}{\partial z^i}\otimes E_\alpha\in\Gamma(S,T^{1,0}S\otimes\widetilde{E})$.

On the other hand λ_0 is a positive Hermitian metric on the line bundle $\widetilde{L}\otimes\det(T_{X/S})$. The curvature form of λ_0 can be represented by a Hermitian positive matrix, namely, the coefficients matrix of Hermitian positive $(1,1)$ form $\sqrt{-1}\partial\overline{\partial}\varphi$ on X. By Lemma 2.6,

$$\left(\varphi_{i\overline{j}}-\sum_{A,B=1}^{r-1}\varphi^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)$$

is a Hermitian positive $n\times n$ matrix. Since the integrand is nonnegative, $R^{\widetilde{E}}(u,u)=0$ if and only if

$$\sum_{i,j}\sum_{\alpha,\beta}h_0(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\left(\varphi_{i\overline{j}}-\sum_{A,B=1}^{r-1}\varphi^{A\overline{B}}\varphi_{i\overline{B}}\varphi_{A\overline{j}}\right)\equiv 0 \tag{4.21}$$

on X_s which means $(u^{i\alpha})$ is a zero matrix. In summary, we obtain

$$R^{\widetilde{E}}(u,u)>0$$

for nonzero u, i.e. the induced metric f on $\widetilde{E}=S^kE\otimes F$ is Nakano-positive. □

Corollary 4.6. *If E is ample, then for large k, S^kE is Griffiths positive, i.e. there exists a Hermitian metric h_k on S^kE such that h_k is Griffiths-positive.*

4.2. Dual-Nakano-positivity. By the curvature identity on $S^kE\otimes F$,

$$R_{i\bar{j}\alpha\bar{\beta}}=\int_{X_s}h_0V_\alpha\overline{V}_\beta\varphi_{i\bar{j}}\frac{\omega_s^{r-1}}{(r-1)!}-\int_{X_s}h_0P_{i\alpha}\overline{P}_{j\beta}\frac{\omega_s^{r-1}}{(r-1)!}$$

where φ is a weight of the line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(k+r)\otimes\pi^*(\det E^*)\otimes\pi^*(F)$. Although this line bundle cannot be negative, it is still possible that it is negative in the local horizontal direction, i.e. $(\varphi_{i\bar{j}})$ is a Hermitian negative matrix. For example, F is a " very negative" line bundle over S. If $(\varphi_{i\bar{j}})$ is Hermitian negative, then for any nonzero $u=(u^{i\alpha})$,

$$\begin{aligned}R_{i\bar{j}\alpha\bar{\beta}}u^{i\alpha}\overline{u^{j\beta}} &= \int_{X_s}h_0\varphi_{i\bar{j}}(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\frac{\omega_s^{r-1}}{(r-1)!}\\ &\quad -\int_{X_s}h_0P_{i\alpha}\overline{P}_{j\beta}u^{i\alpha}\overline{u}^{j\beta}\frac{\omega_s^{r-1}}{(r-1)!}\\ &\leq \int_{X_s}h_0\varphi_{i\bar{j}}(V_\alpha u^{i\alpha})\overline{(V_\beta u^{j\beta})}\frac{\omega_s^{r-1}}{(r-1)!}\\ &< 0\end{aligned}$$

Hence $S^kE\otimes F$ is Nakano-negative. In the following, we will prove that if $(S^{k+r}E\otimes\det E^*\otimes F)^*$ is ample, then $S^kE\otimes F$ is Nakano-negative which is equivalent to the statement: if $S^{k+r}E\otimes\det E^*\otimes F$ is ample, then $S^kE\otimes F$ is dual-Nakano-positive. Here we use a well-known fact [9]:

E is dual-Nakano-positive if and only if E^* is Nakano-negative.

For simplicity, we assume $k=1$ and $F=\det E$. In the following we will show, if E^* is ample, then $E\otimes\det E$ is Nakano-negative.

As similar as the quotient metric on $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ (see Proposition 2.4) induced by the morphism $(\pi^*E,\pi^*h)\to\mathcal{O}_{\mathbb{P}(E^*)}(1)$, there is an induced metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ by the morphism $(\pi^*(E^*),\pi^*h^*)\to\mathcal{O}_{\mathbb{P}(E)}(1)$. For a fixed point $s\in S$, we can choose a local coordinate system $(z^1,\cdots,z^n)$ and a local normal frame $(e_1,\cdots,e_r)$ of E centered at point s. With respect to this trivialization, we obtain:

Proposition 4.7. *If (E,h) is Griffiths-positive, then the quotient metric h^L on $L:=\mathcal{O}_{\mathbb{P}(E)}(1)$ induced by $(\pi^*E^*,\pi^*h^*)\to\mathcal{O}_{\mathbb{P}(E)}(1)$ is negative in the local horizontal direction, i.e.*

$$\left(-\frac{\partial^2\log h^L}{\partial z^i\partial\overline{z}^j}\right)\tag{4.22}$$

is Hermitian negative on the fiber $X_s=\pi^{-1}(s)$ where $\pi:\mathbb{P}(E)\to S$.

Proof. Let $h_{\alpha\overline{\beta}} = h(e_\alpha, e_\beta)$ and $R_{i\overline{j}\alpha\overline{\beta}}$ be the curvature components of h, then the quotient metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ is,

$$h^L = \frac{1}{\sum h_{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta} = \frac{1}{\sum(\delta_{\alpha\beta} - R_{i\overline{j}\alpha\overline{\beta}} z^i \overline{z}^j + O(|z|^3)) W_\alpha \overline{W}_\beta}. \tag{4.23}$$

It is obvious that

$$-\frac{\partial^2 \log h^L}{\partial z^i \partial \overline{z}^j} = -\sum_{\alpha,\beta} R_{i\overline{j}\alpha\overline{\beta}}(s) \frac{W_\alpha \overline{W}_\beta}{|W|^2} \tag{4.24}$$

which is Hermitian negative on X_s if (E, h) is Griffiths-positive. □

Let $v_k : E \to S^k E$ be the standard Veronese map which induces a map

$$\overline{v}_k : \mathbb{P}(E) \to \mathbb{P}(S^k E). \tag{4.25}$$

Let $\pi : \mathbb{P}(E) \to S$ and $\pi_k : \mathbb{P}(S^k E) \to S$, then $\pi_k \circ \overline{v}_k = \pi$. Now we fix a local holomorphic coordinate system $(z^1, \cdots, z^n)$ centered at point $s \in S$ and a local trivialization of E and $S^k E$. It is obivous that the map $\overline{v}_k$ sends (z, W) to $(z, S^k W)$ where $S^k W$ is the k-th symmetric power of homogeneous vector $W = [W_1, \cdots, W_r]$, and so the horizontal part of $\overline{v}_k$ is identity. With respect to this trivialization, we obtain the following.

Theorem 4.8. *If E is ample, then there exists a Hermitian metric h^L on $L = \mathcal{O}_{\mathbb{P}(E)}(1)$ such that h^L is negative in the horizontal direction, i.e.*

$$\left(-\frac{\partial^2 \log h^L}{\partial z^i \partial \overline{z}^j} \right) \tag{4.26}$$

is Hermitian negative on the fiber $X_s = \pi^{-1}(s)$ where $\pi : \mathbb{P}(E) \to S$.

Proof. By Corollary 4.6, for large k, $S^k E$ is Griffiths-positive. By Proposition 4.7, there exists a Hermitian metric $\widehat{h}_k$ on $\mathcal{O}_{\mathbb{P}(S^k E)}(1)$, such that $\widehat{h}_k$ is Hermitian negative along the horizontal direction. By the relation

$$\mathcal{O}_{\mathbb{P}(E)}(k) = \overline{v}_k^* \left(\mathcal{O}_{\mathbb{P}(S^k E)}(1) \right) \tag{4.27}$$

there is an induced metric h^L on $\mathcal{O}_{\mathbb{P}(E)}(1)$,

$$h^L := \left(\overline{v}_k^* (\widehat{h}_k) \right)^{\frac{1}{k}}. \tag{4.28}$$

Hence, we obtain

$$-\frac{\partial^2 \log h^L}{\partial z^i \partial \overline{z}^j} = -\frac{1}{k} \frac{\partial^2 \log \widehat{h}_k}{\partial z^i \partial \overline{z}^j} \tag{4.29}$$

since the horizontal direction of $\overline{v}_k$ is identity with respect to that trivialization. □

Theorem 4.9. *If E^* is ample, then there exists a Hermitian metric on $E \otimes \det E$ which is Nakano-negative.*

Proof. By Theorem 4.8, if E^* is ample, then there exists a Hermitian metric h^L on $L := \mathcal{O}_{\mathbb{P}(E^*)}(1)$ such that

$$\left(-\frac{\partial^2 \log h^L}{\partial z^i \partial \overline{z}^j}\right) \tag{4.30}$$

is Hermitian negative. Let $\{\omega_s\}_{s\in S}$ be a smooth family of Hermitian metric of the fiber X_s. We can set

$$h_0 = \frac{(h^L)^{r+1}}{\det(\omega_s)}$$

and let

$$\varphi = -\log(h_0 \det(\omega_s)) = -(r+1)\log h^L. \tag{4.31}$$

Hence, we obtain

$$\varphi_{i\overline{j}} = -(r+1)\frac{\partial^2 \log h^L}{\partial z^i \partial \overline{z}^j}. \tag{4.32}$$

Therefore $(\varphi_{i\overline{j}})$ is Hermitian negative. On the other hand, the metric induced by h_0 and $\{\omega_s\}_{s\in S}$ on $E \otimes \det E$ has curvature components

$$R_{i\overline{j}\alpha\overline{\beta}} = \int_{X_s} h_0 W_\alpha \overline{W}_\beta \varphi_{i\overline{j}} \frac{\omega_s^{r-1}}{(r-1)!} - \int_{X_s} h_0 P_{i\alpha} \overline{P}_{j\beta} \frac{\omega_s^{r-1}}{(r-1)!}. \tag{4.33}$$

Therefore, for any nonzero $u = (u^{i\alpha})$,

$$\begin{aligned} R_{i\overline{j}\alpha\overline{\beta}} u^{i\alpha}\overline{u^{j\beta}} &\leq \int_{X_s} h_0 \varphi_{i\overline{j}} (W_\alpha u^{i\alpha})\overline{(W_\beta u^{j\beta})} \frac{\omega_s^{r-1}}{(r-1)!} \\ &< 0. \end{aligned}$$

The proof of Nakano-negativity of $E \otimes \det E$ is completed. □

Combined with Theorem 4.5, Lemma 4.4 and Theorem 4.9 we obtain,

Theorem 4.10. *Let E be a holomorphic vector bundle over a compact Kähler manifold S and F a line bundle over S. Let r be the rank of E and $k \geq 0$ an arbitrary integer. If $S^{k+r}E \otimes \det E^* \otimes F$ is ample over S, then $S^k E \otimes F$ is both Nakano-positive and dual-Nakano-positive.*

4.3. Applications.

Corollary 4.11. *If E is an ample vector bundle and F is a nef line bundle, then there exists $k_0 = k_0(S, E)$ such that $S^k E \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$. In particular, $S^k E$ is Nakano-positive and dual-Nakano-positive for $k \geq k_0$.*

Proof. It is easy to see that there exists $k_0 = k_0(S, E)$ such that for any $k \geq k_0$, $S^{k+r}E \otimes \det E^*$ is ample, and so is $S^{k+r}E \otimes \det E^* \otimes F$. By Theorem 4.10, $S^k E \otimes F$ is Nakano-positive and dual-Nakano-positive. In particular, $S^k E$ is Nakano-positive and dual-Nakano-positive for $k \geq k_0$. □

Corollary 4.12. *If E is an ample vector bundle and F is a nef line bundle, or E is a nef vector bundle and F is an ample line bundle,*

(1) *$S^kE \otimes \det E \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq 0$.*
(2) *If the rank r of E is greater than 1, then $S^mE^* \otimes (\det E)^t \otimes F$ is Nakano-positive and dual-Nakano-positive if $t \geq r+m-1$.*

Proof. (1) It follows by the ampleness of $S^{k+r}E \otimes F = S^{k+r}E \otimes \det E^* \otimes (\det E \otimes F)$.

(2) If $r > 1$, it is easy to see $E^* \otimes \det E = \bigwedge^{r-1} E$. By the relation

$$S^{r+m}(E^* \otimes \det E) \otimes (\det E)^{t-r-m+1} \otimes F = S^{r+m}E^* \otimes \det E \otimes (\det E)^t \otimes F$$

we can apply Theorem 4.10 to the pair $(E^*, (\det E)^t \otimes F)$ and obtain the Nakano-positivity and dual-Nakano-positivity of $S^mE^* \otimes (\det E)^t \otimes F$ when $t \geq r+m-1$. Let $E = T\mathbb{P}^2$, then $E = E^* \otimes \det E$ is Griffiths-positive, but not Nakano-positive. So we cannot remove the restriction $t \geq r+m-1$. $\square$

Corollary 4.13. *If $S^{r+1}E \otimes \det E^*$ is ample, then E is Nakano-positive and dual-Nakano-positive and so E is Griffiths-positive.*

Remark 4.14. By Corollary 4.13, the ampleness of $\mathcal{O}_{\mathbb{P}(E^*)}(r+1) \otimes \pi^*(\det E^*)$ implies the ampleness of $\mathcal{O}_{\mathbb{P}(E^*)}(1)$. However, in general, the ampleness of $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ cannot imply the ampleness of $\mathcal{O}_{\mathbb{P}(E^*)}(r+1) \otimes \pi^*(\det E^*)$.

5. Nakano-positivity and dual-Nakano-positivity of adjoint vector bundles

The following lemma is due to Fujita [13] and Ye-Zhang [40].

Lemma 5.1. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S. If $r \geq n+1$, then $\det E \otimes K_S$ is ample except $(S,E) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1})$.*

Theorem 5.2. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S.*

(1) *If $r > 1$, then $S^kE \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive for any $k \geq \max\{n-r, 0\}$.*
(2) *If $r = 1$, then the line bundle $E^{\otimes(n+2)} \otimes K_S$ is Nakano-positive.*

Moreover, the lower bound on k is sharp.

Proof. (1) If $r > 1$, then $X = \mathbb{P}(E^*)$ is a $\mathbb{P}^{r-1}$ bundle which is not isomorphic to any projective space. By Lemma 5.1, $\mathcal{O}_{\mathbb{P}(E^*)}(n+r) \otimes K_X$ is ample. So

$$\mathcal{O}_{\mathbb{P}(E^*)}(n) \otimes \pi^*(K_S \otimes \det E)$$

is ample and it is equivalent to the ampleness of $S^nE\otimes(\det E^*)\otimes(\det E)^2\otimes K_S$. If $k \geq \max\{n-r, 0\}$, then $S^{r+k}E \otimes \det E^* \otimes (\det E)^2 \otimes K_S$ is also ample; hence, by Theorem 4.10, $S^kE \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive.

(2) It follows from Lemma 5.1. In fact, the vector bundle $\widetilde{E} = E^{\oplus(n+2)}$ is an ample vector bundle of rank $n+2$ and $\det \widetilde{E} = E^{\otimes(n+2)}$. By Lemma 5.1, $\det \widetilde{E} \otimes K_S = E^{\otimes(n+2)} \otimes K_S$ is ample.

Here the lower bound $n-r$ is sharp. For any integer $k_0 < n-r$, there exists some ample vector E such that $E \otimes (\det E)^{k_0} \otimes K_S$ is not Nakano-positive, for example $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$. □

Theorem 5.3. *Let E be an ample vector bundle over S. Let r be the rank of E and n the dimension of S. If $r > 1$, then $E \otimes (\det E)^k \otimes K_S$ is Nakano-positive and dual-Nakano-positive for any $k \geq \max\{n+1-r, 2\}$. Moreover, the lower bound is sharp.*

Proof. If $r \geq n-1$, by Theorem 5.2, $E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive. Now we consider $1 < r < n-1$. By [19], Theorem 2.5, $K_S \otimes (\det E)^{n-r}$ is nef except the case $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$. It is easy to check

$$S^{r+1}E \otimes K_S \otimes (\det E)^{n-r}$$

is also ample in that case. By Theorem 4.10, $E \otimes (\det E)^{n+1-r} \otimes K_S$ is Nakano-positive and dual-Nakano-positive. Here the lower bound $n+1-r$ is sharp. For any integer $k_0 < n+1-r$, there exists an ample vector bundle E such that $E \otimes (\det E)^{k_0} \otimes K_S$ is not Nakano-positive, for example $(S, E) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1) \oplus \mathcal{O}_{\mathbb{P}^4}(1))$. □

Remark 5.4. In Theorems 5.2 and 5.3, if $r \geq n$, $E \otimes (\det E)^2 \otimes K_S$ is Nakano-positive and dual-Nakano-positive. If $E = T\mathbb{P}^n$, then $S^2E \otimes \det E \otimes K_{\mathbb{P}^n}$ is Nakano-positive and dual-Nakano-positive.

Problem. Is $S^2E \otimes \det E \otimes K_S$ Nakano-positive and dual-Nakano-positive when E is ample and $r \geq n$? If one can show $S^{n+2}E \otimes K_S$ is ample, or equivalently, $\mathcal{O}_{\mathbb{P}(E^*)}(n+2) \otimes \pi^*(K_S)$ is ample, by Theorem 4.10, $S^2E \otimes \det E \otimes K_S$ is Nakano-positive and dual-Nakano-positive.

6. Vanishing theorems

The following vanishing theorem is due to Nakano [29]; see also Demailly [9]:

Lemma 6.1. *Let E be a holomorphic vector bundle over a compact Kähler manifold M. If E is Nakano-positive, then $H^{n,q}(M, E) = 0$ for any $q \geq 1$. If E is dual-Nakano-positive, then $H^{q,n}(M, E) = 0$ for any $q \geq 1$.*

Theorem 6.2. *Let $E, E_1, \cdots, E_\ell$ be vector bundles over an n-dimensional compact Kähler manifold M. Their ranks are $r, r_1, \cdots, r_\ell$, respectively. Let L be a line bundle on M.*

(1) *If E is ample, L is nef and $r > 1$, then*

$$H^{n,q}(M, S^k E \otimes (\det E)^2 \otimes K_M \otimes L) = H^{q,n}(M, S^k E \otimes (\det E)^2 \otimes K_M \otimes L) = 0$$

for any $q \geq 1$ and $k \geq \max\{n-r, 0\}$.

(2) *If E is ample, L is nef and $r > 1$, then*

$$H^{n,q}(M, E \otimes (\det E)^k \otimes K_M \otimes L) = H^{q,n}(M, E \otimes (\det E)^k \otimes K_M \otimes L) = 0$$

for any $q \geq 1$ and $k \geq \max\{n+1-r, 2\}$.

(3) *Let $r > 1$. If E is ample and L is nef, or E is nef and L is ample, then*

$$H^{n,q}(M, S^m E^* \otimes (\det E)^t \otimes L) = H^{q,n}(M, S^m E^* \otimes (\det E)^t \otimes L) = 0$$

for any $q \geq 1$ and $t \geq r + m - 1$.

(4) *If all E_i are ample and L is nef, or, all E_i are nef and L is ample, then for any $k_1 \geq 0, \cdots, k_\ell \geq 0$,*

$$\begin{aligned} &H^{n,q}(M, S^{k_1} E_1 \otimes \cdots \otimes S^{k_\ell} E_\ell \otimes \det E_1 \otimes \cdots \otimes \det E_\ell \otimes L) \\ &= H^{q,n}(M, S^{k_1} E_1 \otimes \cdots \otimes S^{k_\ell} E_\ell \otimes \det E_1 \otimes \cdots \otimes \det E_\ell \otimes L) = 0 \end{aligned}$$

for $q \geq 1$.

Proof. By Theorems 5.2, 5.3 and Corollary 4.12, the vector bundles in consideration are all Nakano-positive and dual-Nakano-positive. The results follow from Lemma 6.1. □

Remark 6.3. Part (4) can be regarded as a generalization of Griffiths ([14], Theorem G) and Demailly ([10], Theorem 0.2).

The following results generalize Griffiths' vanishing theorem (see also [22], Corollary 1.5).

Proposition 6.4. *Let r be the rank of E and $k \geq 1$. For any $t \geq 0$, if $S^{t+kr} E \otimes L$ is ample,*

$$H^{n,q}(M, S^t E \otimes (\det E)^k \otimes L) = H^{q,n}(M, S^t E \otimes (\det E)^k \otimes L) = 0$$

for any $q \geq 1$.

Proof. By Theorem 4.10, $S^t E \otimes (\det E)^k \otimes L$ is Nakano-positive and dual-Nakano-positive. The results follow by Nakano's vanishing theorem. □

Remark 6.5. Theorem 1.1 allows us to do induction to deduce more positivity results. For example, if $S^m E \otimes L$ is ample, then $S^{m-r} E \otimes \det E \otimes L$ is (dual-)Nakano-positive and so it is ample. Using Theorem 1.1 again, we get $S^{m-2r} \otimes (\det E)^2 \otimes L$ is Nakano-positive and dual-Nakano-positive. Finally, we get $S^t E \otimes (\det E)^k \otimes L$ is Nakano-positive and dual-Nakano-positive, if

$m = t + kr$ for some $0 \le t < r$. It is obvious that the (dual-)Nakano-positivity turns stronger and stronger under induction. This explains why a lot of vanishing theorems involve a power of $\det E$.

If L is an ample line bundle over a compact Kähler manifold M and F is an arbitrary line bundle over M, by comparing the Chern classes, there exists a constant m_0 such that $L^{m_0} \otimes F$ is ample and so it is positive. If E is an ample vector bundle and F is an arbitrary vector bundle, it is easy to see $S^k E \otimes F$ is ample for large k. But, in general, we don't know whether an ample vector bundle carries a Griffiths-positive or Nakano-positive metric. In the following, we will construct Nakano-positive and dual-Nakano-positive metrics on various ample vector bundles.

Lemma 6.6. *If L is an ample line bundle over M and F is an arbitrary vector bundle, there exists an integer m_0 such that $L^{m_0} \otimes F$ is Nakano-positive and dual-Nakano-positive.*

Proof. Let h_0 be a positive metric on L and ω be the curvature of h_0 which is also the Kähler metric fixed on M. For any metric g on F, the curvature R^g has a lower bound in the sense

$$\min_{x \in M} \inf_{u \neq 0} \frac{R^g(u(x), u(x))}{|u(x)|^2} \ge -(m_0 - 1) \tag{6.1}$$

where $u \in \Gamma(M, T^{1,0}M \otimes F)$. The curvature of metric $h^{m_0} \otimes g$ on $L^{m_0} \otimes F$ is given by

$$\widehat{R} = m_0 \omega \cdot g + h_0^m \cdot R^g. \tag{6.2}$$

Therefore

$$\widehat{R}(v \otimes u, v \otimes u) \ge |u|^2 h_0^{m_0}(v, v)$$

for any $v \in \Gamma(M, L^{m_0})$ and $u \in \Gamma(M, T^{1,0}M \otimes F)$. □

Lemma 6.7. *If E is (dual-)Nakano-positive and F is a nef* line *bundle, then $E \otimes F$ is (dual-)Nakano-positive.*

Proof. Fix a Kähler metric on M. Let g be a Nakano-positive metric on E, then there exists $2\varepsilon > 0$ such that

$$R^g(u(x), u(x)) \ge 2\varepsilon |u(x)|^2$$

for any $u \in \Gamma(M, T^{1,0}M \otimes E)$. On the other hand, by a result of [11], there exists a smooth metric h_0 on the nef line bundle F such that

$$R^{h_0} \ge -\varepsilon \omega h_0. \tag{6.3}$$

The curvature of $g \otimes h_0$ on $E \otimes F$ is

$$\widehat{R} = R^g \cdot h_0 + g \cdot R^{h_0}.$$

For any $u \in \Gamma(M, T^{1,0}M \otimes E)$ and $v \in \Gamma(M, F)$,

$$\widehat{R}(u \otimes v, u \otimes v) \geq \left(R^g(u,u) - \varepsilon|u|^2\right) h_0(v,v) \geq \varepsilon|u|^2 h_0(v,v). \tag{6.4}$$

For dual-Nakano-positivity, the proof is similar. □

Theorem 6.8. *If E is an ample vector bundle and F is an arbitrary vector bundle over M, then there exists $k_0 = k_0(M,E,F)$ such that $S^kE \otimes F$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$.*

Proof. By Lemma 6.6, there exists m_0 such that $(\det E)^{m_0} \otimes F$ is Nakano-positive and dual-Nakano-positive. On the other hand, there exists $k_0 = k_0(E, m_0, M)$ such that $\mathcal{O}_{\mathbb{P}(E^*)}(r+k) \otimes \pi^*(\det E^*)^{m_0+1}$ is ample for $k \geq k_0$. It is equivalent to the ampleness of vector bundle $S^{r+k}E \otimes (\det E^*)^{m_0+1}$. By Theorem 4.10, $S^kE \otimes (\det E^*)^{m_0}$ is Nakano-positive and dual-Nakano-positive. Since the tensor product of two (dual-)Nakano-positive vector bundles is (dual-)Nakano-positive, $S^kE \otimes F = (S^kE \otimes (\det E^*)^{m_0}) \otimes ((\det E)^{m_0} \otimes F)$ is Nakano-positive and dual-Nakano-positive for $k \geq k_0$. □

The following results are well-known in algebraic geometry, but merit a proof in our setting.

Corollary 6.9. *If E is ample over M, L is a nef line bundle and F is an arbitrary vector bundle,*

(1) *There exists $k_0 = k_0(M,E,F)$ such that for any $k \geq k_0$,*
$$H^{p,q}(M, S^kE \otimes F) = 0$$
for $q \geq 1$ and $p \geq 0$.

(2) *There exists $k_0 = k_0(M,E)$ such that for any $k \geq k_0$,*
$$H^{p,q}(M, S^kE \otimes L) = 0$$
for any $q \geq 1$ and $p \geq 0$.

Proof. (1) By Theorem 6.8, there exists $k_0 = k_0(M,E,F)$ such that $S^kE \otimes F \otimes \Lambda^{n-p}T^{1,0}M$ is Nakano-positive for any p. On the other hand,

$$H^{p,q}(M, S^kE \otimes F) = H^{n,q}(M, S^kE \otimes F \otimes \Lambda^{n-p}T^{1,0}M).$$

By the Nakano vanishing theorem, $H^{p,q}(M, S^kE \otimes F) = 0$ for $q \geq 1$ and $p \geq 0$ if $k \geq k_0$. The proof of part (2) is similar. □

7. Comparison of Griffiths-positive and Nakano-positive metrics

Let (E,h) be a Hermitian vector bundle. In general, it is not so easy to write down the exact curvature formula of (S^kE, S^kh). In this section, we give an algorithm to compute the curvature of (S^kE, S^kh). As applications, we can disprove the Griffiths-positivity and Nakano-positivity of a given metric on $\mathbb{P}^n$.

Let h be a Hermitian metric on E, h^L be the induced metric in (2.9) on $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$. Let F be a line bundle with Hermitian metric h^F. Naturally, there is an induced metric $S^k h \otimes h^F$ on the vector bundle $S^k E \otimes F$. On the other hand, we can construct a new metric f on $S^k E \otimes F$ by formula (3.3). There is a **canonical way** to do it. Let $\widetilde{L} = L^k \otimes \pi^*(F)$. The induced metric on $\widetilde{L}$ is $h_0 = (h^L)^k \otimes \pi^*(h^F)$ and the induced metric on $\det(T_{X/S}) = L^r \otimes \pi^*(\det E^*)$ is $(h^L)^r \otimes \pi^*(\det(h)^{-1})$. These two metrics induce a metric $\lambda_0 = (h^L)^{k+r} \otimes \pi^*\left(h^F \cdot \det(h)^{-1}\right)$ on $\widetilde{L} \otimes \det(T_{X/S})$. Now we can polarize each fiber X_s by the curvature of λ_0. By formula (2.10),

$$\begin{aligned} \omega_s &= -\frac{\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s \log \lambda_0 \\ &= \frac{(k+r)\sqrt{-1}}{2\pi}\partial_s\overline{\partial}_s \log\left(\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta\right) = (k+r)\omega_{FS} \end{aligned} \tag{7.1}$$

By a simple linear algebraic argument, we obtain

$$\frac{\lambda_0}{\det(\omega_s)} = \frac{(h^L)^k \otimes \pi^*(h^F)}{(k+r)^{r-1}} = \frac{h_0}{(k+r)^{r-1}}. \tag{7.2}$$

Now we can use $(\widetilde{L}, h_0)$ and (X_s, ω_s) to construct a "new" metric f on $S^k E \otimes F$ by formula (3.3).

Theorem 7.1. *The metric f has the form*

$$f = \frac{(r+k)^{r-1}}{(r+k-1)!} \cdot S^k h \otimes h^F. \tag{7.3}$$

Moreover, f is a constant multiple of the metric constructed in Theorem 4.10.

Proof. Without loss of generality, we can choose normal coordinates for the metric h at a fix point $s \in S$. By formula (2.10), the metric $h_0 = (h^L)^k \otimes h^F$ on $L^k \otimes F$ induced by (E, h) and (F, h^F) can be written as $\frac{h^F}{|W|^{2k}}$ locally on the fiber $X_s \cong \mathbb{P}^{r-1}$. By formula (7.1), the metric f defined by (3.3) has the following form

$$f_{\alpha\overline{\beta}} = \int_{X_s} h_0 V_\alpha \overline{V}_\beta \frac{\omega_s^{r-1}}{(r-1)!} = (k+r)^{r-1} h^F \int_{\mathbb{P}^{r-1}} \frac{V_\alpha \overline{V}_\beta}{|W|^{2k}} \frac{\omega_{FS}^{r-1}}{(r-1)!}.$$

Here V_α, V_β are homogeneous monomials of degree k in $W_1, \cdots, W_r$. By Lemma 4.1,

$$f_{\alpha\beta} = \frac{(r+k)^{r-1}}{(r+k-1)!}\delta_{\alpha\beta} h^F,$$

that is, $f = \frac{(r+k)^{r-1}}{(r+k-1)!} \cdot S^k h \otimes h^F$. By formulas (7.2) and (4.19), f is a constant multiple of the metric constructed in Theorem 4.10. $\square$

Theorem 7.2. *If (E, h) is a Griffiths-positive vector bundle, then*

(1) $(S^k E \otimes (\det E)^\ell, S^k h \otimes (\det h)^\ell)$ *is Nakano-positive and dual-Nakano-positive for any $k \geq 0$ and $\ell \geq 1$.*
(2) *There exists $k_0 = k_0(M, E)$ such that $(S^k E, S^k h)$ is Nakano-positive and dual-Nakano-positive for any $k \geq k_0$.*

Proof. These follow by Theorems 4.10 and 7.1. □

Corollary 7.3. *Let h_{FS} be the Fubini-Study metric on $T\mathbb{P}^n$ with $n \geq 2$, then*

(1) $(S^{n+1}T\mathbb{P}^n \otimes K_{\mathbb{P}^n}, S^{n+1}h_{FS} \otimes \det(h_{FS})^{-1})$ *is semi-Griffiths-positive. Moreover, $S^{n+1}T\mathbb{P}^n \otimes K_{\mathbb{P}^n}$ cannot admit a Griffiths-positive metric.*
(2) $(T\mathbb{P}^n, h_{FS})$ *is dual-Nakano-positive and semi-Nakano-positive.*
(3) $(S^k T\mathbb{P}^n \otimes K_{\mathbb{P}^n}, S^k h_{FS} \otimes \det(h_{FS})^{-1})$ *is Griffiths-positive for any $k \geq n+2$.*
(4) $(S^k T\mathbb{P}^n, S^k h_{FS})$ *is Nakano-positive and dual-Nakano-positive for any $k \geq 2$.*

Proof. (1) By the Euler sequence

$$0 \to \mathbb{C} \to T\mathbb{P}^n \to \mathcal{O}_{\mathbb{P}^n}(1)^{\bigoplus(n+1)} \to 0 \tag{7.4}$$

we know $T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ is the quotient bundle of trivial bundle $\mathbb{C}^{\bigoplus(n+1)}$. Hence

$$S^{n+1}T\mathbb{P}^n \otimes K_{\mathbb{P}^n} = S^{n+1}\left(T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)\right)$$

with the canonical metric is semi-Griffiths-positive. However, if $S^{n+1}T\mathbb{P}^n \otimes K_{\mathbb{P}^n}$ admits a Griffiths-positive metric, by Corollary 4.13, $T\mathbb{P}^n$ is Nakano-positive which is impossible for $n \geq 2$.

(2) The curvature of $E = T\mathbb{P}^n$ with respect to the standard Fubini-Study metric h_{FS} is

$$R_{i\bar{j}k\bar{\ell}} = h_{i\bar{j}}h_{k\bar{\ell}} + h_{i\bar{\ell}}h_{k\bar{j}}. \tag{7.5}$$

Without loss of generality, we assume $h_{i\bar{j}} = \delta_{ij}$ at a fixed point, then

$$R_{i\bar{j}k\bar{\ell}}u^{ik}\overline{u}^{j\ell} = \frac{1}{2}\sum_{j,k}|u^{jk} + u^{kj}|^2 \tag{7.6}$$

which means that (E, h_{FS}) is semi-Nakano-positive but not Nakano-positive. For the dual-Nakano-positivity of $(T\mathbb{P}^n, h_{FS})$, we can check that by definition. We can also show it by the monotone property of dual-Nakano-positivity of quotient bundles. By the Euler sequence (7.4), $T\mathbb{P}^n$ is the quotient bundle of dual-Nakano-positive bundle $\mathcal{O}_{\mathbb{P}^n}(1)^{\bigoplus(n+1)}$ and so $T\mathbb{P}^n$ is dual-Nakano-positive.

(3) It follows by the identity

$$S^kT\mathbb{P}^n \otimes K_{\mathbb{P}^n} = S^k(T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)) \otimes \mathcal{O}_{\mathbb{P}^n}(k-n-1)$$

and semi-Griffiths positivity of $T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$.

(4) By Theorem 4.10, the canonically induced metric f is Nakano-positive and dual-Nakano-positive. On the other hand, by Theorem 7.1, f is a constant multiple of $S^k h_{FS}$. The lower bound of k follows from (1) and (2). □

Proposition 7.4. (1) *(E,h) is Griffiths-positive if and only if (S^kE, S^kh) is Griffiths-positive for some $k \geq 1$.*

(2) *If (E,h) is (dual-)Nakano-positive, then (S^kE, S^kh) is (dual-)Nakano-positive for any $k \geq 1$.*

Proof. By Theorem 7.1, S^kh is a constant multiple of the metric constructed by formula (3.3). So by Theorem 3.2, we can write down the curvature formula of S^kh explicitly. In normal coordinates of h at a fixed point, the curvature formula (3.6) can be simplified by Lemma 4.1. We obtain curvature formulas (7.7) and (7.9).

For the convenience of the reader, we assume $k = 2$ at first. We can choose normal coordinates at a fixed point. Let $\{e_1, \cdots, e_r\}$ be the local basis at that point. The ordered basis of S^2E at that point is $\{e_1 \otimes e_1, e_1 \otimes e_2, \cdots, e_r \otimes e_{r-1}, e_r \otimes e_r\}$. We denote them by $e_{(\alpha,\beta)} = e_\alpha \otimes e_\beta$ with $\alpha \leq \beta$. The curvature tensor S^2h is

$$R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}} = R_{i\bar{j}\alpha\bar{\beta}}\delta_{\gamma\delta} + R_{i\bar{j}\gamma\bar{\delta}}\delta_{\alpha\beta} + R_{i\bar{j}\gamma\bar{\beta}}\delta_{\alpha\delta} + R_{i\bar{j}\alpha\bar{\delta}}\delta_{\gamma\beta} \tag{7.7}$$

where $R_{i\bar{j}\alpha\bar{\beta}}$ is the curvature tensor of E. Let $u = \sum\limits_i \sum\limits_{\alpha\leq\gamma} u_{i(\alpha,\gamma)} e_{(\alpha,\gamma)} \in \Gamma(M, T^{1,0}M \otimes S^2E)$. For simplicity of notation, we extend the values of $u_{i(\alpha,\gamma)}$ to all indices (α,γ) by setting $u_{i(\alpha,\gamma)} = 0$ if $\gamma < \alpha$. Therefore,

$$\begin{aligned}
&\sum_{i,j}\sum_{\substack{\alpha\leq\gamma\\ \beta\leq\delta}} R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}} u_{i(\alpha,\gamma)}\overline{u}_{j(\beta,\delta)} \\
= &\sum_{i,j}\sum_{\alpha,\gamma,\beta,\delta} R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}} u_{i(\alpha,\gamma)}\overline{u}_{j(\beta,\delta)} \\
= &\sum_{i,j,\alpha,\beta,\gamma,\delta} \big(R_{i\bar{j}\alpha\bar{\beta}} u_{i(\alpha,\gamma)}\overline{u}_{j(\beta,\gamma)} + R_{i\bar{j}\gamma\bar{\delta}} u_{i(\alpha,\gamma)}\overline{u}_{j(\alpha,\delta)} \\
&\qquad + R_{i\bar{j}\gamma\bar{\beta}} u_{i(\alpha,\gamma)}\overline{u}_{j(\beta,\alpha)} + R_{i\bar{j}\alpha\bar{\delta}} u_{i(\alpha,\gamma)}\overline{u}_{j(\gamma,\delta)}\big) \\
= &\sum_{\gamma}\sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\bar{\beta}} \left(u_{i(\alpha,\gamma)} + u_{i(\gamma,\alpha)}\right)\overline{\left(u_{j(\beta,\gamma)} + u_{j(\gamma,\beta)}\right)}.
\end{aligned} \tag{7.8}$$

Hence (S^2E, S^2h) is Nakano-positive if (E,h) is Nakano-positive. For the general case, we set $A = (\alpha_1, \cdots, \alpha_k)$ and $B = (\beta_1, \cdots, \beta_k)$ with $\alpha_1 \leq \cdots \leq$

α_k and $\beta_1 \leq \cdots \leq \beta_k$. The basis of S^kE is $\{e_A = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}\}$. The curvature tensor of (S^kE, S^kh) is

$$R_{i\bar{j}A\overline{B}} = \sum_{\alpha,\beta=1}^{r} \sum_{s,t=1}^{k} R_{i\bar{j}\alpha\overline{\beta}} \delta_{\alpha\alpha_s} \delta_{\beta\beta_t} \delta_{A_sB_t} \tag{7.9}$$

where $A_s = (\alpha_1, \cdots, \alpha_{s-1}, \alpha_{s+1}, \cdots, \alpha_k)$, $B_t = (\beta_1, \cdots, \beta_{t-1}, \beta_{t+1}, \cdots, \beta_k)$ and $\delta_{A_sB_t}$ is the multi-index delta function (see formula (4.1)). We have the curvature formula,

$$\begin{aligned} & \sum_{i,j,A,B} R_{i\bar{j}A\overline{B}} u_{iA} \overline{u}_{jB} \\ (7.10) \quad = & \sum_{\alpha_1,\cdots,\alpha_{k-1}} \sum_{\sigma \in S_{k-1}} \sum_{i,j,\alpha,\beta} R_{i\bar{j}\alpha\overline{\beta}} V_{i\alpha\alpha_{\sigma(1)}\cdots\alpha_{\sigma(k-1)}} \overline{V}_{j\beta\alpha_{\sigma(1)}\cdots\alpha_{\sigma(k-1)}} \end{aligned}$$

where S_{k-1} is the permutation group in $(k-1)$ symbols and

$$V_{i\alpha\alpha_1\cdots\alpha_{k-1}} = \sum_{s=1}^{k} u_{iA^s}, \quad A^s = (\alpha_1, \cdots, \alpha_{s-1}, \alpha, \alpha_{s+1}, \cdots, \alpha_k).$$

The Nakano-positivity of (S^kE, S^kh) follows immediately from the Nakano-positivity of (E, h) by formula (7.10). With the help of curvature formula (7.9), we can prove Griffiths-positivity and dual-Nakano-positivity of S^kE in a similar way. Here, we use another way to show it. S^kE can be viewed as a quotient bundle of $E^{\otimes k}$. If (E, h) is Griffiths-positive (resp. dual-Nakano-positive), $(E^{\otimes k}, h^{\otimes k})$ is Griffiths-positive (resp. dual-Nakano-positive) and so the quotient bundle S^kE is Griffiths-positive (resp. dual-Nakano-positive) [9]. The induced metrics on quotient bundles are exactly the ones given. □

Remark 7.5. Part (1) is an analogue of ampleness: E is ample if and only if S^kE is ample for some $k \geq 1$. The converse of part (2) is not valid in general. We know $(S^2T\mathbb{P}^n, S^2h_{FS})$ is Nakano-positive, but $(T\mathbb{P}^n, h_{FS})$ is not Nakano-positive as shown in the following.

Example 7.6. In this example, we will show the Nakano-positivity of $(S^2T\mathbb{P}^2, S^2h_{FS})$ in local coordinates. At a fixed point, we choose normal coordinates of $T\mathbb{P}^2$. Let $\{e_1, e_2\}$ be the ordered basis of $T\mathbb{P}^2$ at that point. The ordered basis of $S^2T\mathbb{P}^2$ are $e_{(1,1)} = e_1 \otimes e_1$, $e_{(1,2)} = e_1 \otimes e_2$ and $e_{(2,2)} = e_2 \otimes e_2$. Using the same notation as in Proposition 7.4, we set $V_{i\alpha\gamma} = u_{i(\alpha,\gamma)} + u_{i(\gamma,\alpha)}$ where $u = \sum\limits_i \sum\limits_{\alpha \leq \gamma} u_{i(\alpha,\gamma)} \frac{\partial}{\partial z^i} \otimes e_{(\alpha,\gamma)} \in \Gamma(\mathbb{P}^2, T^{1,0}\mathbb{P}^2 \otimes S^2T\mathbb{P}^2)$. For $\gamma = 1$, the 2×2 matrix $(V_{i\alpha 1})$ has the form

$$T_1 = \begin{pmatrix} 2u_{1(1,1)} & u_{1(1,2)} \\ 2u_{2(1,1)} & u_{2(1,2)} \end{pmatrix}.$$

For $\gamma = 2$, the 2×2 matrix $(V_{i\alpha 2})$ is

$$T_2 = \begin{pmatrix} u_{1(1,2)} & 2u_{1(2,2)} \\ u_{2(1,2)} & 2u_{2(2,2)} \end{pmatrix}.$$

The total 2×3 matrix $\left(u_{i(\alpha,\beta)}\right)$ is

$$T = \begin{pmatrix} u_{1(1,1)} & u_{1(1,2)} & u_{1(2,2)} \\ u_{2(1,1)} & u_{2(1,2)} & u_{2(2,2)} \end{pmatrix}.$$

By formulas (7.8) and (7.6),

$$\begin{aligned} &\sum_{i,j,\alpha,\gamma,\beta,\delta} R_{i\bar{j}(\alpha,\gamma)\overline{(\beta,\delta)}} u_{i(\alpha,\gamma)} \overline{u}_{j(\beta,\delta)} \\ &= \sum_{i,j,\alpha,\beta} \left(R_{i\bar{j}\alpha\bar{\beta}} V_{i\alpha 1} \overline{V}_{j\beta 1} + R_{i\bar{j}\alpha\bar{\beta}} V_{i\alpha 2} \overline{V}_{j\beta 2} \right) \\ &= \frac{1}{2} \sum_{i,\alpha} |V_{i\alpha 1} + V_{\alpha i 1}|^2 + \frac{1}{2} \sum_{i,\alpha} |V_{i\alpha 2} + V_{\alpha i 2}|^2. \end{aligned}$$

It equals zero if and only if T_1 and T_2 are skew-symmetric which means $T \equiv 0$. The Nakano-positivity of $(S^2 T\mathbb{P}^2, S^2 h_{FS})$ is proved.

Acknowledgements

The authors are grateful to the anonymous referees for many helpful comments.

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