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Big vector bundles and complex manifolds with semi-positive tangent bundles

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Abstract We classify compact Kähler manifolds with semi-positive holomorphic bisectional and big tangent bundles. We also classify compact complex surfaces with semi-positive tangent bundles and compact complex 3-folds of the form $\mathbb{P}(T^*X)$ whose tangent bundles are nef. Moreover, we show that if X is a Fano manifold such that $\mathbb{P}(T^*X)$ has nef tangent bundle, then $X \cong \mathbb{P}^n$.

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1 Introduction

Since the seminal works of Siu-Yau and Mori on the solutions to the Frankel conjecture [37] and Hartshorne conjecture [32], it became apparent that positivity properties of

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the tangent bundle define rather restricted classes of manifolds. Combining algebraic and transcendental tools, Mok proved the following uniformization theorem in [31]: if a compact Kähler manifold (X, ω) has semi-positive holomorphic bisectional curvature, then its universal cover is isometrically biholomorphic to $(\mathbb{C}^k, \omega_0) \times (\mathbb{P}^{N_1}, \omega_1) \times \cdots \times (\mathbb{P}^{N_\ell}, \omega_\ell) \times (M_1, \eta_1) \times \cdots \times (M_k, \eta_k)$ where ω_0 is flat; ω_k, $1 \leq k \leq \ell$, is a Kähler metric on $\mathbb{P}^{N_k}$ with semi-positive holomorphic bisectional curvature; (M_i, η_i) are some compact irreducible Hermitian symmetric spaces. Along the line of Mori's work, Campana-Peternell [7] studied the projective manifolds with nef tangent bundles (see also [45,47]); Demailly-Peternell-Schneider [12] investigated extensively the structure of compact complex manifolds with nef tangent bundles by using algebraic techniques as well as transcendental methods (e.g. the work [10] of Demailly). For more details, we refer to [6,8,13,34] and the references therein.

In the same spirit, Solá Conde and Wiśniewski classified projective manifolds with 1-ample and big tangent bundles:

Theorem 1.1 [38, Theoreom 1.1] *Let X be a complex projective manifold of dimension n. Suppose that the tangent bundle TX is big and 1-ample. Then X is isomorphic either to the projective space $\mathbb{P}^n$ or to the smooth quadric $\mathbb{Q}^n$, or if $n = 3$ to complete flags $F(1; 2; \mathbb{C}^3)$ in $\mathbb{C}^3$ (which is the same as the projective bundle $\mathbb{P}(T^*\mathbb{P}^2)$ over $\mathbb{P}^2$).*

A vector bundle E is called *big* if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ of $\mathbb{P}(E^*)$ is big. The 1-ampleness is defined by Sommese in [39, Definition 1.3]: E is called 1-*ample*, if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is semi-ample and suppose for some $k > 0$, $\mathcal{O}_{\mathbb{P}(E^*)}(k)$ is globally generated, then the maximum dimension of the fiber of the evaluation map

$$\mathbb{P}(E^*) \to \mathbb{P}\left(H^0(\mathbb{P}(E^*), \mathcal{O}_{\mathbb{P}(E^*)}(k))\right)$$

is ≤ 1. It is also pointed out in [35, p. 127] that, 1-ampleness is irrelevant to the metric positivity of E (cf. Theorem 1.6).

In this paper, we investigate big vector bundles and complex manifolds with semi-positive tangent bundles, i.e. the tangent bundles are semi-positive in the sense of Griffiths, or equivalently, there exist Hermitian metrics (not necessarily Kähler) with semi-positive holomorphic bisectional curvature.

The first main result of our paper can be viewed as a "metric" analogue of Kawamata-Reid-Shokurov base point free theorem for tangent bundles.

Theorem 1.2 *Let (X, ω) be a compact Kähler manifold with semi-positive holomorphic bisectional curvature. Then the following statements are equivalent*

1. *The anti-canonical line bundle K_X^{-1} is ample;*
2. *The tangent bundle TX is big;*
3. *The anti-canonical line bundle K_X^{-1} is big;*
4. $c_1^n(X) > 0$.

One may wonder whether similar results hold for abstract vector bundles. Unfortunately, there exists a vector bundle E which is semi-positive in the sense of Griffiths, and $\det(E)$ is ample (in particular, $\det(E)$ is big), but E is not a big vector bundle. Indeed, one can see clearly that the underlying manifold structure of the tangent bundle is essentially used in the proof of Theorem 1.2.

Example 1.3 Let $E = T\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ be the hyperplane bundle of $\mathbb{P}^2$. It is easy to see that E is semi-ample and semi-positive in the sense of Griffiths and $\det E = \mathcal{O}_{\mathbb{P}^2}(1)$ is ample. However, E is not a big vector bundle since the second Segre number $s_2(E) = c_1^2(E) - c_2(E) = 0$ (for more details, see Example 4.10).

For abstract vector bundles, we obtain

Proposition 1.4 *Let E be a nef vector bundle over a compact Kähler manifold X. If E is a big vector bundle, then $\det(E)$ is a big line bundle.*

Corollary 1.5 *If X is a compact Kähler manifold with nef and big tangent bundle, then K_X^{-1} is ample.*

As an application of Theorem 1.2 and Mok's uniformization theorem, we can *classify* compact Kähler manifolds with semi-positive holomorphic bisectional curvature and big tangent bundles, which is also analogues to Theorem 1.1.

Theorem 1.6 *Let (X, ω) be a compact Kähler manifold with semi-positive holomorphic bisectional curvature. Suppose TX is a big vector bundle. Then there exist non-negative numbers $k, N_1, \ldots, N_\ell$ and irreducible compact Hermitian symmetric spaces $M_1, \ldots, M_k$ of rank ≥ 2 such that (X, ω) is isometrically biholomorphic to*

$$\left(\mathbb{P}^{N_1}, \omega_1\right) \times \cdots \times \left(\mathbb{P}^{N_\ell}, \omega_\ell\right) \times (M_1, \eta_1) \times \cdots \times (M_k, \eta_k) \tag{1.1}$$

where ω_i, $1 \leq i \leq \ell$, is a Kähler metric on $\mathbb{P}^{N_i}$ with semi-positive holomorphic bisectional curvature and $\eta_1, \ldots, \eta_k$ are the canonical metrics on $M_1, \ldots, M_k$.

Note that, by Theorem 1.1, the Fano manifold $\mathbb{P}(T^*\mathbb{P}^2)$ has nef and big tangent bundle. On the other hand, it does not admit any smooth Kähler metric with semi-positive holomorphic bisectional curvature according to Theorem 1.6 or Mok's uniformization theorem. However, it is still not clear whether the tangent bundle of $\mathbb{P}(T^*\mathbb{P}^2)$ is semi-positive in the sense of Griffiths, or equivalently, whether $\mathbb{P}(T^*\mathbb{P}^2)$ has a smooth Hermitian metric with semi-positive holomorphic bisectional curvature. According to a weaker version of a conjecture of Griffiths (e.g. Remark 3.4), $\mathbb{P}(T^*\mathbb{P}^2)$ should have a Hermitian metric with Griffiths semi-positive curvature since the tangent bundle of $\mathbb{P}(T^*\mathbb{P}^2)$ is semi-ample. As motivated by this question, we investigate complex manifolds with semi-positive tangent bundles.

Theorem 1.7 *Let (X, ω) be a compact Hermitian manifold with semi-positive holomorphic bisectional curvature, then*

1. *X has Kodaira dimension $-\infty$; or*
2. *X is a complex parallelizable manifold.*

We also classify compact complex surfaces with semi-positive tangent bundles based on results in [12] (see also [7,45]). In this classification, we only assume the abstract vector bundle TX is semi-positive in the sense of Griffiths, or equivalently, X has a smooth Hermitian metric with semi-positive holomorphic bisectional curvature. Hence, even if the ambient manifold is Kähler or projective, Mok's result can not be applied.

Theorem 1.8 *Let X be a compact Kähler surface. If TX is (Hermitian) semi-positive, then X is one of the following:*

1. *X is a torus;*
2. *X is $\mathbb{P}^2$;*
3. *X is $\mathbb{P}^1 \times \mathbb{P}^1$;*
4. *X is a ruled surface over an elliptic curve, and X is covered by $\mathbb{C} \times \mathbb{P}^1$.*

We need to point out that it should be a coincidence that we get the same classification as in [19] where they considered Kähler metrics with semi-positive holomorphic bisectional curvature. As explained in the previous paragraphs, it is still unclear whether one can derive the same classification in higher dimensional cases. In particular, we would like to know whether one can get the same results as in Theorem 1.6 if the Kähler metric is replaced by a Hermitian metric.

For non-Kähler surfaces, we obtain

Theorem 1.9 *Let (X, ω) be a compact non-Kähler surface with semi-positive holomorphic bisectional curvature. Then X is a Hopf surface.*

We also construct explicit Hermitian metrics with semi-positive curvature on Hopf surface $H_{a,b}$ (cf. [12, Proposition 6.3]).

Proposition 1.10 *On every Hopf surface $H_{a,b}$, there exists a Gauduchon metric with semi-positive holomorphic bisectional curvature.*

For complex Calabi-Yau manifolds, i.e. complex manifolds with $c_1(X) = 0$, we have

Corollary 1.11 *Let X be a complex Calabi-Yau manifold in the Fujiki class $\mathscr{C}$ (class of manifolds bimeromorphic to Kähler manifolds). Suppose X has a Hermitian metric ω with semi-positive holomorphic bisectional curvature, then X is a torus.*

By comparing Corollary 1.11 with Proposition 1.10, we see that the Fujiki class condition in Corollary 1.11 is necessary since every Hopf surface $H_{a,b}$ is a Calabi-Yau manifold with semi-positive tangent bundle.

By using Theorem 1.7 and the positivity of direct image sheaves (Theorem 3.1) over complex manifolds (possibly non-Kähler), we obtain new examples on Kähler and non-Kähler manifolds whose tangent bundles are *nef but not semi-positive*. To the best of our knowledge, it is also the first example in the manifold setting (cf. [12, Example 1.7]).

Corollary 1.12 *Let X be a Kodaira surface or a hyperelliptic surface.*

1. *The tangent bundle TX is nef but not semi-positive (in the sense of Griffiths);*
2. *The anti-canonical line bundle of $\mathbb{P}(T^*X)$ is nef, but neither semi-positive nor big.*

Hence, for any dimension $n \geq 2$, there exist Kähler and non-Kähler manifolds with nef but not semi-positive tangent bundles.

Finally, we investigate compact complex manifolds, of the form $\mathbb{P}(T^*X)$, whose tangent bundles are nef. It is well-known that $\mathbb{P}(T^*\mathbb{P}^n)$ is homogeneous, and its tangent bundle is nef. We obtain a similar converse statement and yield another characterization of $\mathbb{P}^n$.

Proposition 1.13 *Let X be a Fano manifold of complex dimension n. Suppose $\mathbb{P}(T^*X)$ has nef tangent bundle, then $X \cong \mathbb{P}^n$.*

In particular, for complex 3-folds, we have the following classification.

Theorem 1.14 *For a complex 3-fold $\mathbb{P}(T^*X)$, if $\mathbb{P}(T^*X)$ has nef tangent bundle, then X is exactly one of the following:*

1. *$X \cong \mathbb{P}^2$;*
2. *$X \cong \mathbb{T}^2$, a flat torus;*
3. *X is a hyperelliptic surface;*
4. *X is a Kodaira surface;*
5. *X is a Hopf surface.*

The paper is organized as follows: in Sect. 2, we introduce several basic terminologies which will be used frequently in the paper. In Sect. 3, we study the positivity of direct image sheaves over complex manifolds (possibly non-Kähler). In Sect. 4, we investigate compact Kähler manifolds with big tangent bundles and prove Proposition 1.4, Theorems 1.2 and 1.6. In Sect. 5, we study compact complex manifolds with semi-positive tangent bundles and establish Theorems 1.7, 1.8, 1.9, Proposition 1.10, Corollaries 1.11 and 1.12. In Sect. 6, we discuss complex manifolds of the form $\mathbb{P}(T^*X)$ and prove Proposition 1.13 and Theorem 1.14. In the Appendix 1, we include some straightforward computations on Hopf manifolds for the reader's convenience.

Remark 1.15 For compact Kähler manifolds with semi-negative holomorphic bisectional curvature, there are similar uniformization theorems as Mok's result. We refer to [27,42] and the references therein. We have obtained a number of results for compact complex manifolds with semi-negative tangent bundles, which will appear in [43].

2 Background materials

Let E be a holomorphic vector bundle over a compact complex manifold X and h a Hermitian metric on E. There exists a unique connection ∇ which is compatible with the metric h and the complex structure on E. It is called the Chern connection of (E, h). Let $\{z^i\}_{i=1}^n$ be local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of E. The curvature tensor $R^\nabla \in \Gamma(X, \Lambda^2 T^*X \otimes E^* \otimes E)$ has components

$$R_{i\overline{j}\alpha\overline{\beta}} = -\frac{\partial^2 h_{\alpha\overline{\beta}}}{\partial z^i \partial \overline{z}^j} + h^{\gamma\overline{\delta}} \frac{\partial h_{\alpha\overline{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\overline{\beta}}}{\partial \overline{z}^j} \tag{2.1}$$

Here and henceforth we sometimes adopt the Einstein convention for summation.

Definition 2.1 A Hermitian holomorphic vector bundle (E, h) is called positive (resp. semi-positive) in the sense of Griffiths if

$$R_{i\overline{j}\alpha\overline{\beta}} u^i \overline{u}^j v^\alpha \overline{v}^\beta > 0 \quad (\text{resp.} \geq 0\ )$$

for nonzero vectors $u = (u^1, \dots, u^n)$ and $v = (v^1, \dots, v^r)$ where $n = \dim_{\mathbb{C}} X$ and r is the rank of E. (E, h) is called Nakano positive (resp. Nakano semi-positive) if

$$R_{i\overline{j}\alpha\overline{\beta}} u^{i\alpha} \overline{u}^{j\beta} > 0 \quad (\text{ resp.} \geq 0)$$

for nonzero vector $u = (u^{i\alpha}) \in \mathbb{C}^{nr}$.

In particular, if (X, ω_g) is a Hermitian manifold, $(T^{1,0}M, \omega_g)$ has Chern curvature components

$$R_{i\overline{j}k\overline{\ell}} = -\frac{\partial^2 g_{k\overline{\ell}}}{\partial z^i \partial \overline{z}^j} + g^{p\overline{q}} \frac{\partial g_{k\overline{q}}}{\partial z^i} \frac{\partial g_{p\overline{\ell}}}{\partial \overline{z}^j}. \tag{2.2}$$

The (first) Chern-Ricci form $Ric(\omega_g)$ of (X, ω_g) has components

$$R_{i\overline{j}} = g^{k\overline{\ell}} R_{i\overline{j}k\overline{\ell}} = -\frac{\partial^2 \log \det(g)}{\partial z^i \partial \overline{z}^j}$$

and it is well-known that the Chern-Ricci form represents the first Chern class of the complex manifold X (up to a factor 2π).

Definition 2.2 Let (X, ω) be a compact Hermitian manifold. (X, ω) has positive (resp. semi-positive) holomorphic *bisectional* curvature, if for any nonzero vector $\xi = (\xi^1, \dots, \xi^n)$ and $\eta = (\eta^1, \dots, \eta^n)$,

$$R_{i\overline{j}k\overline{\ell}} \xi^i \overline{\xi}^j \eta^k \overline{\eta}^{\ell} > 0 \quad (\text{resp.} \geq 0).$$

(X, ω) has positive (resp. semi-positive) holomorphic *sectional* curvature, if for any nonzero vector $\xi = (\xi^1, \dots, \xi^n)$

$$R_{i\overline{j}k\overline{\ell}} \xi^i \overline{\xi}^j \xi^k \overline{\xi}^{\ell} > 0 \quad (\text{resp.} \geq 0).$$

Definition 2.3 Let (X, ω) be a Hermitian manifold, $L \to X$ a holomorphic line bundle and $E \to X$ a holomorphic vector bundle. Let $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ be the tautological line bundle of the projective bundle $\mathbb{P}(E^*)$ over X.

1. L is said to be *positive* (resp. *semi-positive*) if there exists a smooth Hermitian metric h on L such that the curvature form $R = -\sqrt{-1}\partial\overline{\partial} \log h$ is a positive (resp. semi-positive) $(1, 1)$-form. The vector bundle E is called *ample* (resp. *semi-ample*) if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is a positive (resp. semi-positive) line bundle.
2. L is said to be *nef* (or numerically effective), if for any $\varepsilon > 0$, there exists a smooth Hermitian metric h on L such that the curvature of (L, h) satisfies $-\sqrt{-1}\partial\overline{\partial} \log h \geq -\varepsilon\omega$. The vector bundle E is called *nef* if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is a nef line bundle.
3. L is said to be *big*, if there exists a (possibly) singular Hermitian metric h on L such that the curvature of (L, h) satisfies $R = -\sqrt{-1}\partial\overline{\partial} \log h \geq \varepsilon\omega$ in the sense of current for some $\varepsilon > 0$. The vector bundle E is called *big*, if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is big.

Definition 2.4 Let X be a compact complex manifold and $L \to X$ be a line bundle. The Kodaira dimension $\kappa(L)$ of L is defined to be

$$\kappa(L) := \limsup_{m\to+\infty} \frac{\log \dim_{\mathbb{C}} H^0(X, L^{\otimes m})}{\log m}$$

and the *Kodaira dimension* $\kappa(X)$ of X is defined as $\kappa(X) := \kappa(K_X)$ where the logarithm of zero is defined to be $-\infty$.

By Riemann-Roch, it is easy to see that E is a big vector bundle if and only if there are $c_0 > 0$ and $k_0 \geq 0$ such that

$$h^0(X, S^k E) \geq c_0 k^{n+r-1} \tag{2.3}$$

for all $k \geq k_0$ where $\dim_{\mathbb{C}} X = n$ and $rk(E) = r$. Indeed, let $Y = \mathbb{P}(E^*)$ and $\mathcal{O}_Y(1)$ be the tautological line bundle of Y, then we have

$$h^0(X, S^k E) = h^0(Y, \mathcal{O}_Y(k)) \geq c_0 k^{n+r-1} \tag{2.4}$$

where $\dim_{\mathbb{C}} Y = n + r - 1$. Hence, E is big if and only if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is big, if and only if

$$\kappa(\mathcal{O}_{\mathbb{P}(E^*)}(1)) = \dim_{\mathbb{C}} \mathbb{P}(E^*).$$

The following well-known result will be used frequently in the paper.

Lemma 2.5 *Let L be a line bundle over a compact Kähler manifold X. Suppose L is nef, then L is big if and only if the top self-intersection number $c_1^n(L) > 0$ where $n = \dim X$.*

3 Positivity of direct image sheaves over complex manifolds

Let $\mathcal{X}$ be a compact complex manifold of complex dimension $m+n$, and S a smooth complex manifold (*possibly non-Kähler*) with dimension m. Let $\pi : \mathcal{X} \to S$ be a smooth proper submersion such that for any $s \in S$, $X_s := \pi^{-1}(\{s\})$ is a compact Kähler manifold with dimension n. Suppose for any $s \in S$, there exists an open neighborhood U_s of s and a smooth $(1,1)$ form ω on $\pi^{-1}(U_s)$ such that $\omega_p = \omega|_{X_p}$ is a smooth Kähler form on X_p for any $p \in U_s$. Let $(\mathcal{E}, h^{\mathcal{E}}) \to \mathcal{X}$ be a Hermitian holomorphic vector bundle. In the following, we adopt the setting in [3, Sect. 4] (see also [29, Sect. 2.3]). Consider the space of holomorphic $\mathcal{E}$-valued $(n,0)$-forms on X_s,

$$E_s := H^0(X_s, \mathcal{E}_s \otimes K_{X_s}) \cong H^{n,0}(X_s, \mathcal{E}_s)$$

where $\mathcal{E}_s = \mathcal{E}|_{X_s}$. Here, we assume all E_s have the same dimension. With a natural holomorphic structure,

$$E = \bigcup_{s \in S} \{s\} \times E_s$$

is isomorphic to the direct image sheaf $\pi_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ if $\mathcal{E}$ has certain positive property.

Theorem 3.1 *If $(\mathcal{E}, h^{\mathcal{E}})$ is positive (resp. semi-positive) in the sense of Nakano, then $\pi_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ is positive (resp. semi-positive) in the sense of Nakano.*

Remark 3.2 When the total space $\mathcal{X}$ is Kähler and $\mathcal{E}$ is a line bundle, Theorem 3.1 is originally proved by Berndtsson in [3, Theorem 1.2]. When $(\mathcal{E}, h^{\mathcal{E}})$ is a Nakano semi-positive vector bundle, Theorem 3.1 is a special case of [33, Theorem 1.1].

It is not hard to see that the positivity of the direct image sheaves does not depend on the base manifold S. It still works for non-Kähler S. We give a sketched proof of Theorem 3.1 for reader's convenience. Let $h^{\mathcal{E}}$ be a smooth Nakano semi-positive metrics on $\mathcal{E}$. For any local smooth section u of $\pi_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$, there is a representative $\mathbf{u}$ of u, a local holomorphic $\mathcal{E}$-valued $(n, 0)$ form on $\mathcal{X}$, then we define the Hodge metric on $\pi_*(K_{\mathcal{X}/S} \otimes \mathcal{E})$ by using the sesquilinear pairing

$$|u|^2 = \sqrt{-1} \int_{X_s} \{\mathbf{u}, \mathbf{u}\}. \tag{3.1}$$

Note that we do not specify any metric on $\mathcal{X}$ or S. Since $\mathcal{X} \to S$ has Kähler fibers, we can use similar methods as in [3,29] to compute the curvature of the Hodge metric. To obtain the positivity of the Hodge metric, the key ingredient is to find primitive representatives on the Kähler fiber X_s (e.g. [3, Lemma 4.3] or [29, Theorem 3.10]). Since all computations are local, i.e. on an open subset $\pi^{-1}(U)$ of $\mathcal{X}$ where U is an open subset of S, the computations do not depend on the property of base manifold S. In particular, all computations in [29] and all results (e.g. [29, Theorems 1.1 and 1.6]) still work for non-Kähler base manifold S. Note that, if $(\mathcal{E}, h^{\mathcal{E}})$ is only semi-positive, S can be a non-Kähler manifold.

Corollary 3.3 *Let X be a compact complex manifold (possibly non-Kähler) and $E \to X$ be a holomorphic vector bundle of rank r.*

1. *If $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is semi-positive, then $\mathrm{S}^k E \otimes \det(E)$ is Nakano semi-positive.*
2. *If $\det E$ is a holomorphic torsion, i.e. $(\det E)^k = \mathcal{O}_X$ for some $k \in \mathbb{N}^+$, then E is Nakano semi-positive if and only if $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is semi-positive.*

Proof Let $Y = \mathbb{P}(E^*)$, $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$ and $\pi : \mathbb{P}(E^*) \to X$ be the canonical projection. 1. By the adjunction formula [24, p. 89], we have

$$K_Y = L^{-r} \otimes \pi^*(K_X \otimes \det(E)), \tag{3.2}$$

and

$$K_{Y/X} = L^{-r} \otimes \pi^*(\det(E)). \tag{3.3}$$

Therefore,

$$\pi_*(K_{Y/X} \otimes L^{r+k}) = \pi_*(L^k \otimes \pi^*(\det E)) = S^k E \otimes \det E.$$

By Theorem 3.1, we deduce $S^k E \otimes \det E$ is semi-positive in the sense of Nakano if L is semi-positive.

2. Suppose $\det E$ is a holomorphic torsion with $(\det E)^m = \mathcal{O}_X$, then there exists a flat Hermitian metric on $\det E$ and also on $\det E^*$. If L is semi-positive,

$$\tilde{L} = L^{r+1} \otimes \pi^*(\det E^*)$$

is semi-positive. By formula (3.3) and Theorem 3.1, we know

$$\pi_*(K_{Y/X} \otimes \tilde{L}) = \pi_*(L) = E$$

is semi-positive in the sense of Nakano.

On the other hand, if (E, h) is semi-positive, then the induced Hermitian metric on L has semi-positive curvature [e.g. formula (4.9)]. $\square$

Remark 3.4 Griffiths conjectured in [17] that E is Griffiths positive if (and only if) the tautological line bundle $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is positive. It is also not known in the semi-positive setting, i.e. whether there exists a Griffiths semi-positive metric on E when $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ is semi-positive.

4 Kähler manifolds with big tangent bundles

In this section, we prove Theorems 1.6 and 1.2. We begin with an algebraic curvature relation on a Kähler manifold (X, ω). At a given point $p \in X$, the minimum holomorphic sectional curvature is defined to be

$$\min_{W \in T_p^{1,0}X, |W|=1} H(W),$$

where $H(W) := R(W, \overline{W}, W, \overline{W})$. Since X is of finite dimension, the minimum can be attained.

Lemma 4.1 *Let (X, ω) be a compact Kähler manifold. Let $e_1 \in T_p^{1,0}X$ be a unit vector which minimizes the holomorphic sectional curvature of ω at point p, then*

$$2R(e_1, \overline{e}_1, W, \overline{W}) \geq \left(1 + |\langle W, e_1 \rangle|^2\right) R(e_1, \overline{e}_1, e_1, \overline{e}_1) \tag{4.1}$$

for every unit vector $W \in T_p^{1,0}X$.

Proof Let $e_2 \in T_p^{1,0}X$ be any unit vector orthogonal to e_1. Let

$$f_1(\theta) = H(\cos(\theta)e_1 + \sin(\theta)e_2), \quad \theta \in \mathbb{R}.$$

Then we have

$$\begin{aligned} f_1(\theta) = {} & R(\cos(\theta)e_1 + \sin(\theta)e_2, \overline{\cos(\theta)e_1 + \sin(\theta)e_2}, \cos(\theta)e_1 \\ & + \sin(\theta)e_2, \overline{\cos(\theta)e_1 + \sin(\theta)e_2}) = \cos^4(\theta)R_{1\bar{1}1\bar{1}} + \sin^4(\theta)R_{2\bar{2}2\bar{2}} \\ & + 2\sin(\theta)\cos^3(\theta)\left[R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}}\right] + 2\cos(\theta)\sin^3(\theta)\left[R_{1\bar{2}2\bar{2}} + R_{2\bar{1}2\bar{2}}\right] \\ & + \sin^2(\theta)\cos^2(\theta)\left[4R_{1\bar{1}2\bar{2}} + R_{1\bar{2}1\bar{2}} + R_{2\bar{1}2\bar{1}}\right]. \end{aligned}$$

Since $f_1(\theta) \geq R_{1\bar{1}1\bar{1}}$ for all $\theta \in \mathbb{R}$ and $f_1(0) = R_{1\bar{1}1\bar{1}}$, we have

$$f_1'(0) = 0 \quad \text{and} \quad f_1''(0) \geq 0.$$

By a straightforward computation, we obtain

$$f_1'(0) = 2\left(R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}}\right) = 0, \quad f_1''(0) = 2\left(4R_{1\bar{1}2\bar{2}} + R_{1\bar{2}1\bar{2}} + R_{2\bar{1}2\bar{1}}\right) - 4R_{1\bar{1}1\bar{1}} \geq 0. \tag{4.2}$$

Similarly, if we set $f_2(\theta) = H(\cos(\theta)e_1 + \sqrt{-1}\sin(\theta)e_2)$, then

$$\begin{aligned} f_2(\theta) = {} & \cos^4(\theta)R_{1\bar{1}1\bar{1}} + \sin^4(\theta)R_{2\bar{2}2\bar{2}} + 2\sqrt{-1}\sin(\theta)\cos^3(\theta)\left[-R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}}\right] \\ & + 2\sqrt{-1}\cos(\theta)\sin^3(\theta)\left[-R_{1\bar{2}2\bar{2}} + R_{2\bar{1}2\bar{2}}\right] \\ & + \sin^2(\theta)\cos^2(\theta)\left[4R_{1\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}\right]. \end{aligned}$$

From $f_2'(0) = 0$ and $f_2''(0) \geq 0$, one can see

$$-R_{1\bar{1}1\bar{2}} + R_{2\bar{1}1\bar{1}} = 0, \qquad 2\left(4R_{1\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}\right) - 4R_{1\bar{1}1\bar{1}} \geq 0. \tag{4.3}$$

Hence, from (4.2) and (4.3), we obtain

$$R_{1\bar{1}1\bar{2}} = R_{1\bar{1}2\bar{1}} = 0, \quad \textit{and} \quad 2R_{1\bar{1}2\bar{2}} \geq R_{1\bar{1}1\bar{1}}. \tag{4.4}$$

For an arbitrary unit vector $W \in T_p^{1,0}X$, if W is parallel to e_1, i.e. $W = \lambda e_1$ with $|\lambda| = 1$,

$$2R(e_1, \bar{e}_1, W, \overline{W}) = 2R(e_1, \bar{e}_1, e_1, \bar{e}_1).$$

Suppose W is not parallel to e_1. Let e_2 be the unit vector

$$e_2 = \frac{W - \langle W, e_1\rangle e_1}{|W - \langle W, e_1\rangle e_1|}.$$

Then e_2 is a unit vector orthogonal to e_1 and

$$\begin{aligned} W &= ae_1 + be_2, \quad a = \langle W, e_1\rangle, \\ b &= |W - \langle W, e_1\rangle|, \quad |a|^2 + |b|^2 = 1. \end{aligned}$$

Hence

$$2R(e_1, \overline{e}_1, W, \overline{W}) = 2|a|^2 R_{1\overline{1}1\overline{1}} + 2|b|^2 R_{1\overline{1}2\overline{2}},$$

since we have $R_{1\overline{1}1\overline{2}} = R_{1\overline{1}2\overline{1}} = 0$ by (4.4). By (4.4) again,

$$2R(e_1, \overline{e}_1, W, \overline{W}) \geq \left(2|a|^2 + |b|^2\right) R_{1\overline{1}1\overline{1}} = \left(1 + |a|^2\right) R_{1\overline{1}1\overline{1}}$$

which completes the proof of the lemma. □

By using similar methods, one has

Lemma 4.2 *Let $e_n \in T_p^{1,0}X$ be a unit vector which maximizes the holomorphic sectional curvature at point p, then*

$$2R(e_n, \overline{e}_n, W, \overline{W}) \leq \left(1 + |\langle W, e_n\rangle|^2\right) R(e_n, \overline{e}_n, e_n, \overline{e}_n) \tag{4.5}$$

for every unit vector $W \in T_p^{1,0}X$.

Remark 4.3 A special case of Lemma 4.2—when W is orthogonal to e_n—is well-known (e.g. [16, p. 312], [4, p. 136]). When the holomorphic sectional curvature is strictly negative at point p, one has $2R(e_n, \overline{e}_n, W, \overline{W}) \leq R(e_n, \overline{e}_n, e_n, \overline{e}_n)$, which is firstly obtained in [5, Lemma 1.4]. In the proofs of Lemmas 4.1 and 4.2, we refine the methods in [4,16].

Theorem 4.4 *Let (X, ω) be a compact Kähler manifold with semi-positive holomorphic bisectional curvature. Then the following statements are equivalent*

1. *The anti-canonical line bundle K_X^{-1} is ample;*
2. *The tangent bundle TX is big;*
3. *The anti-canonical line bundle K_X^{-1} is big;*
4. *$c_1^n(X) > 0$.*

Proof (1) $\Longrightarrow$ (2). Let $E = TX$ and $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$ the tautological line bundle over the projective bundle $\mathbb{P}(E^*)$. Let's recall the general setting when (E, h^E) is an arbitrary Hermitian holomorphic vector bundle (e.g. [11,17,28]). Let $(e_1, \ldots, e_n)$ be the local holomorphic frame with respect to a given trivialization on E and the dual frame on E^* is denoted by $(e^1, \ldots, e^n)$. The corresponding holomorphic coordinates on E^* are denoted by $(W_1, \ldots, W_n)$. There is a local section e_{L^*} of L^* defined by

$$e_{L^*} = \sum_{\alpha=1}^{n} W_\alpha e^\alpha.$$

Its dual section is denoted by e_L. Let h^E be a fixed Hermitian metric on E and h^L the induced quotient metric by the morphism $(\pi^*E, \pi^*h^E) \to L$.

If $\left(h_{\alpha\overline{\beta}}\right)$ is the matrix representation of h^E with respect to the basis $\{e_\alpha\}_{\alpha=1}^n$, then h^L can be written as

$$h^L = \frac{1}{h^{L^*}(e_{L^*}, e_{L^*})} = \frac{1}{\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta}. \tag{4.6}$$

The curvature of (L, h^L) is

$$R^{h^L} = -\sqrt{-1}\partial\overline{\partial}\log h^L = \sqrt{-1}\partial\overline{\partial}\log\left(\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta\right) \tag{4.7}$$

where ∂ and $\overline{\partial}$ are operators on the total space $\mathbb{P}(E^*)$. We fix a point $Q \in \mathbb{P}(E^*)$, then there exist local holomorphic coordinates $(z^1, \ldots, z^n)$ centered at point $p = \pi(Q)$ and local holomorphic basis $\{e_1, \ldots, e_n\}$ of E around $p \in X$ such that

$$h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} - R_{i\overline{j}\alpha\overline{\beta}} z^i \overline{z}^j + O\left(|z|^3\right). \tag{4.8}$$

Without loss of generality, we assume Q is the point $(0, \ldots, 0, [a_1, \ldots, a_n])$ with $a_n = 1$. On the chart $U = \{W_n = 1\}$ of the fiber $\mathbb{P}^{n-1}$, we set $w^A = W_A$ for $A = 1, \ldots, n-1$. By formula (4.7) and (4.8), we obtain the well-known formula (e.g. [28, Proposition 2.5])

$$R^{h^L}(Q) = \sqrt{-1}\left(\sum_{\alpha,\beta=1}^{n} R_{i\overline{j}\alpha\overline{\beta}} \frac{a_\beta \overline{a}_\alpha}{|a|^2} dz^i \wedge d\overline{z}^j + \sum_{A,B=1}^{n-1}\left(1 - \frac{a_B \overline{a}_A}{|a|^2}\right) dw^A \wedge d\overline{w}^B\right) \tag{4.9}$$

where $|a|^2 = \sum_{\alpha=1}^n |a_\alpha|^2$.

Since (X, ω) is a Kähler manifold with semi-positive holomorphic bisectional curvature, the Ricci curvature $Ric(\omega)$ of ω is also semi-positive. On the other hand, since K_X^{-1} is ample, we have

$$\int_X (Ric(\omega))^n > 0.$$

Therefore, $Ric(\omega)$ must be strictly positive at some point $p \in X$. Then by a result of Mok [31, Proposition 1.1], there exists a Kähler metric $\hat{\omega}$ such that $\hat{\omega}$ has semi-positive holomorphic bisectional curvature, strictly positive holomorphic sectional curvature and strictly positive Ricci curvature. Indeed, let

$$\begin{cases} \frac{\partial \omega_t}{\partial t} = -Ric(\omega_t), \\ \omega_0 = \omega \end{cases} \tag{4.10}$$

be the Kähler-Ricci flow with initial metric ω, then we can take ω_t as $\hat{\omega}$ for some small positive t satisfying $[\omega] - tc_1(X) > 0$. Let $\hat{R}$ be the corresponding curvature operator of $\hat{\omega}$. We choose normal coordinates $\{z^1, \dots, z^n\}$ centered at point p such that $\{e_i = \frac{\partial}{\partial z^i}\}_{i=1}^n$ is the normal frame of $(E, \hat{\omega}) = (TX, \hat{\omega})$. Let $K \in T_p^{1,0}X$ be a unit vector which minimizes the holomorphic sectional curvature of $\hat{\omega}$ at point $p \in X$. In particular, we have $\hat{R}(K, \overline{K}, K, \overline{K}) > 0$. Hence there exists a unit vector $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ such that

$$K = a_1 e_1 + \cdots + a_n e_n. \tag{4.11}$$

Without loss of generality, we assume $a_n \neq 0$. By Lemma 4.1, for any unit vector $W = \sum b_i e_i \in T_p^{1,0}X$, we have

$$\hat{R}(K, \overline{K}, W, \overline{W}) \geq \frac{1}{2}\hat{R}(K, \overline{K}, K, \overline{K}) > 0. \tag{4.12}$$

That is

$$\sum_{i,j,k,\ell} \hat{R}_{i\overline{j}k\overline{\ell}} a_k \overline{a}_\ell b_i \overline{b}_j > 0 \tag{4.13}$$

for every unit vector $b = (b_1, \dots, b_n)$ in $\mathbb{C}^n$. Then at point $Q \in Y = \mathbb{P}(T^*X)$ with coordinates

$$(0, \dots, 0, [\overline{a}_1, \dots, \overline{a}_n]) = \left(0, \dots, 0, \left[\frac{\overline{a}_1}{\overline{a}_n}, \dots, \frac{\overline{a}_{n-1}}{\overline{a}_n}, 1\right]\right),$$

we obtain

$$R^{h^L}(Q) = \sqrt{-1}\left(\sum_{k,\ell=1}^{n} \hat{R}_{i\overline{j}k\overline{\ell}} a_k \overline{a}_\ell dz^i \wedge d\overline{z}^j + \sum_{A,B=1}^{n-1} (1 - a_A \overline{a}_B)\, dw^A \wedge d\overline{w}^B\right) \tag{4.14}$$

is a strictly positive (1, 1) form at point $Q \in Y$ according to (4.13). Here, we also use Eq. (4.9) and the fact that $\sum_{i=1}^n |a_i|^2 = 1$. By continuity, (L, h^L) is positive at a small neighborhood of Q. Since we already know $c_1(L) \geq 0$, and so

$$\int_Y c_1^{2n-1}(L) > 0.$$

Hence L is a big line bundle by Siu-Demailly's solution to the Grauert-Riemenschneider conjecture ([9,36]). In particular, the tangent bundle TX is big.

(3) $\Longleftrightarrow$ (4). Since K_X^{-1} is semi-positive and in particular it is nef, it is well-known that they are equivalent.

(4) $\Longrightarrow$ (1). This part is well-known (e.g. [12, Theorem 4.2]), we include a sketch for reader's convenience. Since TX is nef, and so is $K_X^{-1} = \det(TX)$. If $c_1^n(X) > 0$,

we know K_X^{-1} is nef and big. Hence X is Kähler and Moishezon, and so it is projective. By Kawamata-Reid-Shokurov base point free theorem (e.g. [22, Theorem 3.3]), K_X^{-1} is semi-ample, i.e. K_X^{-m} is generated by global sections for some large m. Let $\phi : X \dashrightarrow Y$ be the birational map defined by $|K_X^{-m}|$. If K_X^{-1} is not ample, then there exists a rational curve C contracted by ϕ. Since TX is nef, C deforms to cover X which is a contradiction.

(2) $\Longrightarrow$ (4). Since TX is nef, K_X^{-1} is also nef. In particular, we have $c_1^n(X) = c_1^n(TX) \geq 0$. If $c_1^n(X) = 0$, then all Chern numbers of X are zero [12, Corollary 2.7]. On the other hand, since the signed Segre number $(-1)^n s_n(TX)$ is a combination of Chern numbers [e.g. formula (4.17)], we deduce that

$$(-1)^n s_n(TX) = 0.$$

Hence TX is not big by Lemma 4.7. □

Theorem 4.5 *Let (X, ω) be a compact Kähler manifold with semi-positive holomorphic bisectional curvature. Suppose TX is a big vector bundle. Then there exist non-negative numbers $k, N_1, \ldots, N_\ell$ and irreducible compact Hermitian symmetric spaces $M_1, \ldots, M_k$ of rank ≥ 2 such that (X, ω) is isometrically biholomorphic to*

$$\left(\mathbb{P}^{N_1}, \omega_1\right) \times \cdots \times \left(\mathbb{P}^{N_\ell}, \omega_\ell\right) \times (M_1, \eta_1) \times \cdots \times (M_k, \eta_k) \tag{4.15}$$

where ω_i, $1 \leq i \leq \ell$, is a Kähler metric on $\mathbb{P}^{N_i}$ with semi-positive holomorphic bisectional curvature and $\eta_1, \ldots, \eta_k$ are the canonical metrics on $M_1, \ldots, M_k$.

Proof By Theorem 4.4, X is Fano. By Yau's theorem [46], there exists a Kähler metric with strictly positive Ricci curvature. Hence $\pi_1(X)$ is finite by Myers' theorem. By Kodaira vanishing theorem, for any $q \geq 1$, $H^{0,q}(X) = H^{n,q}(X, K_X^{-1}) = 0$ since K_X^{-1} is ample. Therefore the Euler-Poincaré characteristic $\chi(\mathcal{O}_X) = \sum(-1)^q h^{0,q}(X) = 1$. Let $\tilde{X}$ be the universal cover of X. Suppose it is a p-sheet cover over X, where $p = |\pi_1(X)|$. So $\tilde{X}$ is still a Fano manifold and hence $\chi(\mathcal{O}_{\tilde{X}}) = p \cdot \chi(\mathcal{O}_X) = 1$. We obtain $p = 1$, i.e. X is simply connected and $\tilde{X} = X$. By Mok's uniformization theorem [31] for compact Kähler manifolds with *semi-positive* holomorphic bisectional curvature, $\tilde{X} = X$ is isometrically biholomorphic to

$$\left(\mathbb{P}^{N_1}, \omega_1\right) \times \cdots \times \left(\mathbb{P}^{N_\ell}, \omega_\ell\right) \times (M_1, \eta_1) \times \cdots \times (M_k, \eta_k) \tag{4.16}$$

where ω_i, $1 \leq i \leq \ell$, is a Kähler metric on $\mathbb{P}^{N_i}$ with semi-positive holomorphic bisectional curvature and $\eta_1, \ldots, \eta_k$ are the canonical metrics on the irreducible compact Hermitian symmetric spaces $M_1, \ldots, M_k$. Note also that, all irreducible compact Hermitian symmetric spaces (with rank $\geq$2) are Fano. □

As an application of Theorem 4.5, we have

Corollary 4.6 *Let $X = \mathbb{P}^m \times \mathbb{P}^n$ and $Y = \mathbb{P}(T^*X)$. Then*

1. *The tangent bundle TX of X is nef and big;*
2. *The anti-canonical line bundle K_Y^{-1} of Y is nef, big, semi-ample, quasi-positive but not ample;*
3. *The holomorphic tangent bundle TY is not nef.*

Proof (1) is from Theorem 4.5. (2) By adjunction formula (3.2),

$$K_Y^{-1} = \mathcal{O}_Y(m+n)$$

where $\mathcal{O}_Y(1)$ is the tautological line bundle of the projective bundle $\mathbb{P}(T^*X)$. Hence, K_Y^{-1} is nef and big, and so is semi-ample by Kawamata-Reid-Shokurov base point free theorem. Let ω be the Kähler metric on $X = \mathbb{P}^m \times \mathbb{P}^n$ induced by the Fubini-Study metrics. It is easy to see that ω has semi-positive holomorphic bisectional curvature and strictly positive holomorphic sectional curvature. By Lemma 4.1 and formula (4.14), the induced Hermitian metric on $L = \mathcal{O}_Y(1)$ is quasi-positive, i.e. $\mathcal{O}_Y(1)$ is semi-positive and strictly positive at some point. In particular, K_Y^{-1} is quasi-positive. However, K_Y^{-1} is not ample, otherwise $\mathcal{O}_Y(1)$ is ample and so it TX. (3) If TY is nef, then the nef and big line bundle K_Y^{-1} is ample. □

As motivated by Theorem 4.4, we investigate properties for abstract nef and big vector bundles. Let $c(E)$ be the total Chern class of a vector bundle E, i.e. $c(E) = 1 + c_1(E) + \cdots + c_n(E)$. The total Segre class $s(E)$ is defined to be the inverse of the total Chern class, i.e.

$$c(E) \cdot s(E) = 1$$

where $s(E) = 1 + s_1(E) + \cdots + s_n(E)$ and $s_k(E) \in H^{2k}(X)$, $1 \leq k \leq n$. We have the recursion formula

$$s_k(E) + s_{k-1}(E) \cdot c_1(E) + \cdots + s_1(E) \cdot c_{k-1}(E) + c_k(E) = 0 \tag{4.17}$$

for every $k \geq 1$. In particular, one has

$$\begin{aligned} &s_1(E) = -c_1(E),\ s_2(E) = c_1^2(E) - c_2(E), \\ &s_3(E) = 2c_1(E)c_2(E) - c_1^3(E) - c_3(E). \end{aligned} \tag{4.18}$$

In particular, the top Segre class $s_n(E)$ is a polynomial of Chern classes of degree $2n$. (Note that there is alternated sign's difference from the notations in [17, p. 245]). The following result is essentially well-known.

Lemma 4.7 *Let (X, ω) be a compact Kähler manifold with complex dimension n. Suppose E is nef vector bundle with rank r, then E is big if and only if the signed Segre number $(-1)^n s_n(E) > 0$.*

Proof Let $L = \mathcal{O}_{\mathbb{P}(E^*)}(1)$ and $\pi : \mathbb{P}(E^*) \to X$ be the canonical projection. Since L is nef, L is a big line bundle if and only if the top self intersection number

$$c_1^{n+r-1}(L) > 0.$$

On the other hand, by [17, Proposition 5.22] we have

$$\pi_*(c_1^{n+r-1}(L)) = (-1)^n s_n(E),$$

where $\pi_* : H^{2(n+r-1)}(\mathbb{P}(E^*)) \to H^{2n}(X)$ is the pushforward homomorphism induced by π. Hence L is big if and only if the signed Segre number $(-1)^n s_n(E)$ is positive.

□

Proposition 4.8 *Let E be a nef vector bundle over a compact Kähler manifold X. If E is a big vector bundle, then* $\det(E)$ *is a big line bundle.*

Proof Since E is nef, the top self intersection number $c_1^n(E) \geq 0$. If $c_1^n(E) = 0$, then all degree $2n$ Chern numbers of E are zero. In particular, $s_n(E) = 0$. It is a contradiction by Lemma 4.7. Hence the top self intersection number $c_1^n(E) > 0$. Since $\det(E)$ is nef and $c_1^n(\det(E)) > 0$, $\det(E)$ is a big line bundle. □

Corollary 4.9 *If X is a compact Kähler manifold with nef and big tangent bundle, then K_X^{-1} is ample, i.e. X is Fano.*

Proof By Proposition 4.8, K_X^{-1} is nef and big. Since TX is nef, we know K_X^{-1} is ample, i.e. X is Fano. □

By comparing Theorem 4.4 with Proposition 4.8, one may ask the following question: for an abstract vector bundle E, if E is nef (or semi-positive) and $\det(E)$ is big (or ample), is E big? We have a negative answer to this question.

Example 4.10 On $\mathbb{P}^2$, let $E = T\mathbb{P}^2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1)$ be the hyperplane bundle. Then E is semi-positive in the sense of Griffiths, and $\det(E)$ is ample, but E is not a big vector bundle.

Proof By using the Hermitian metric on E induced by the Fubini-Study metric, it is easy to see that E is a semi-positive vector bundle and so it is nef. Indeed, $T\mathbb{P}^2$ has curvature tensor

$$R_{i\bar{j}k\bar{\ell}} = g_{i\bar{j}}g_{k\bar{\ell}} + g_{i\bar{\ell}}g_{k\bar{j}}$$

and so E has curvature tensor $R^E_{i\bar{j}k\bar{\ell}} = g_{i\bar{\ell}}g_{k\bar{j}}$ where k and ℓ are indices along the vector bundle E. On the other hand, $\det(E) = \mathcal{O}_{\mathbb{P}^2}(1)$ is ample and so $c_1^2(E) = 1$. However, E is not a big vector bundle. Since

$$c_2(T\mathbb{P}^2) = c_2(E \otimes \mathcal{O}_{\mathbb{P}^2}(1)) = c_2(E) + c_1^2(\mathcal{O}_{\mathbb{P}^2}(1)) + c_1(E) \cdot c_1(\mathcal{O}_{\mathbb{P}^2}(1)) = 3,$$

we have $c_2(E) = 1$, and so

$$s_2(E) = c_1^2(E) - c_2(E) = 0.$$

Therefore, by Lemma 4.7, E is not a big vector bundle. □

Question 4.11 Suppose X is a Fano manifold with nef tangent bundle. Is the (abstract) vector bundle TX semi-positive in the sense of Griffiths? Is TX a big vector bundle?

For example, $\mathbb{P}(T^*\mathbb{P}^n)$ $(n \geq 2)$ is a Fano manifold with nef tangent bundle since it is homogeneous. When $n = 2$, $\mathbb{P}(T^*\mathbb{P}^2)$ has big and semi-ample tangent bundle by Theorem 1.1. It is also known that $\mathbb{P}(T^*\mathbb{P}^n)$ does not admit a smooth Kähler metric with semi-positive holomorphic bisectional curvature according to the classification in Theorem 4.5. However, it is not clear whether the tangent bundle of $\mathbb{P}(T^*\mathbb{P}^n)$ is semi-positive in the sense of Griffiths, or equivalently, whether it has a smooth Hermitian metric with semi-positive holomorphic bisectional curvature. When $n > 2$, is the tangent bundle of $\mathbb{P}(T^*\mathbb{P}^n)$ big?

As motivated by these questions, in the next section, we investigate compact complex manifolds with semi-positive tangent bundles.

5 Complex manifolds with semi-positive tangent bundles

In this section, we study complex manifolds with semi-positive tangent bundles. Suppose the abstract tangent bundle TX has a smooth Hermitian metric h with semi-positive curvature in the sense of Griffiths, or equivalently, (X, h) is a Hermitian manifold with semi-positive holomorphic bisectional curvature.

Theorem 5.1 *Let (X, ω) be a compact Hermitian manifold with semi-positive holomorphic bisectional curvature, then $\kappa(X) \leq 0$, and either*

1. *$\kappa(X) = -\infty$; or*
2. *X is a complex parallelizable manifold. Moreover, (X, ω) has flat curvature and $d^*\omega = 0$.*

Remark 5.2 A complex manifold X of complex dimension n is called complex parallelizable if there exist n holomorphic vector fields linearly independent everywhere. Note that every complex parallelizible manifold has a balanced Hermitian metric with flat curvature tensor and the canonical line bundle is holomorphically trivial, and so the Kodaira dimension is zero. It is proved by Wang in [41, Corollary 2] that a complex parallelizable manifold is Kähler if and only if it is a torus.

Proof Since (X, ω) has semi-positive holomorphic bisectional curvature, K_X^{-1} is semi-positive and so nef. Suppose $\kappa(X) \geq 0$, i.e. there exists some positive integer m such that $H^0(X, K_X^{\otimes m})$ has a non zero element σ. Then σ does not vanish everywhere [12, Proposition 1.16]. In particular, $K_X^{\otimes m}$ is a holomorphically trivial line bundle, i.e. $K_X^{\otimes m} = \mathcal{O}_X$. In this case, we obtain $\kappa(X) = 0$. Let h be the trivial Hermitian metric on $K_X^{\otimes m}$, i.e. $\sqrt{-1}\partial\bar{\partial}\log h = 0$. On the other hand, $K_X^{\otimes m}$ has a smooth Hermitian metric $\frac{1}{[\det(\omega)]^m}$. Hence, there exists a positive smooth function $\phi \in C^\infty(X)$ such that

$$\frac{1}{[\det(\omega)]^m} = \phi \cdot h, \tag{5.1}$$

and the line bundle $K_X^{\otimes m}$ has curvature form

$$-mRic(\omega) = -\sqrt{-1}\partial\overline{\partial}\log h - \sqrt{-1}\partial\overline{\partial}\log\phi = -\sqrt{-1}\partial\overline{\partial}\log\phi \leq 0.$$

By maximum principle, we know ϕ is constant. Therefore $Ric(\omega) = 0$. Since (X, ω) has semi-positive holomorphic bisectional curvature, we know $R_{i\overline{j}k\overline{\ell}} = 0$. Indeed, without loss of generality, we assume $g_{i\overline{j}} = \delta_{ij}$ at a fixed point $p \in X$, and so the Ricci curvature has components $R_{i\overline{j}} = \sum_{k=1}^n R_{i\overline{j}k\overline{k}} = 0$. If we choose $b = (1, 0, \dots, 0)$, then for any $a \in \mathbb{C}^n$, we have $R_{i\overline{j}k\overline{\ell}}a^i\overline{a}^j b^k\overline{b}^\ell = R_{i\overline{j}1\overline{1}}a^i\overline{a}^j \geq 0$. Similarly, we have $R_{i\overline{j}k\overline{k}}a^i\overline{a}^j \geq 0$ for all $k = 1, \dots, n$. By the Ricci flat condition, we have $R_{i\overline{j}k\overline{k}}a^i\overline{a}^j = 0$ for all $a \in \mathbb{C}$ and $k = 1, \dots, n$. We deduce $R_{i\overline{j}k\overline{k}} = 0$ for any i, j, k. Now for any $a \in \mathbb{C}^n$, we define $H_{k\overline{\ell}} = R_{i\overline{j}k\overline{\ell}}a^i\overline{a}^j$. Then $H = (H_{k\overline{\ell}})$ is a semi-positive Hermitian matrix. Since $trH = 0$, H is the zero matrix. That is, for any $a \in \mathbb{C}^n$ and k, ℓ, we have $R_{i\overline{j}k\overline{\ell}}a^i\overline{a}^j = 0$. Finally, we obtain $R_{i\overline{j}k\overline{\ell}} = 0$. Since (X, ω) is Chern-flat, X is a complex parallelizable manifold (e.g. [1,14, Proposition 2.4]). On the other hand, it is well-known that if (X, ω) is Chern-flat, $d^*\omega = 0$ (e.g., [26, Corollary 2]). □

The following application of Theorem 5.1 will be used frequently.

Corollary 5.3 *Let (X, ω) be a compact Hermitian surface. If (X, ω) has semi-positive holomorphic bisectional curvature and K_X is a holomorphic torsion, i.e. $K_X^{\otimes m} = \mathcal{O}_X$ for some integer $m \in \mathbb{N}^+$, then (X, ω) is a torus.*

Proof Since $\kappa(X) = 0$, as shown in the proof of Theorem 5.1, (X, ω) is a parallelizable complex surface with $d^*\omega = 0$. Since $\dim_{\mathbb{C}} X = 2$, $d^*\omega = 0$ implies $dw = 0$, i.e. (X, ω) is Kähler. Hence (X, ω) is a flat torus. □

Now we are ready to classify compact complex surfaces with semi-positive tangent bundles. Note that, we only assume X has a Hermitian metric with semi-positive holomorphic bisectional curvature.

Theorem 5.4 *Let X be a compact Kähler surface. If TX is (Hermitian) semi-positive, then X is one of the following:*

1. *X is a torus;*
2. *X is $\mathbb{P}^2$;*
3. *X is $\mathbb{P}^1 \times \mathbb{P}^1$;*
4. *X is a ruled surface over an elliptic curve C, and X is covered $\mathbb{C} \times \mathbb{P}^1$.*

Proof Suppose TX is semi-positive. If X is not a torus, then by Theorem 5.1, $\kappa(X) = -\infty$. Let $X_{\min}$ be a minimal model of X. Since $\kappa(X_{\min}) = -\infty$, by Kodaira-Enriques classification, $X_{\min}$ has algebraic dimension 2 and so $X_{\min}$ is projective. Therefore, X is also projective. By [7, Proposition 2.1], X is minimal, i.e. $X = X_{\min}$ since X has nef tangent bundle. By Campana-Peterell's classification of projective surfaces with nef tangent bundles [7, Theorem 3.1], X is one of the following

1. X is an abelian surface;
2. X is a hyperelliptic surface;

3. $X = \mathbb{P}^2$;
4. $X = \mathbb{P}^1 \times \mathbb{P}^1$;
5. $X = \mathbb{P}(E^*)$ where E is a rank 2-vector bundle on an elliptic curve C with either
   (a) $E = \mathcal{O}_C \oplus L$, with $\deg(L) = 0$; or
   (b) E is given by a non-split extension $0 \to \mathcal{O}_C \to E \to L \to 0$ with $L = \mathcal{O}_C$ or $\deg L = 1$.

It is obvious that abelian surfaces, $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ all have canonical Hermitian metrics with semi-positive holomorphic bisectional curvature. By Corollary 5.3, a hyperelliptic surface can not admit a Hermitian metric with semi-positive holomorphic bisectional curvature since its canonical line bundle is a torsion. Next, we show that, in case (5), if $X = \mathbb{P}(E^*)$ has semi-positive tangent bundle, then X is a ruled surface over an elliptic curve C which is covered by $\mathbb{C} \times \mathbb{P}^1$. Indeed, by the exact sequence $0 \to T_{X/C} \to TX \to \pi^*(TC) \to 0$ where $\pi : X \to C$, we obtain the dual sequence

$$0 \to \pi^*\mathcal{O}_C \to T^*X \to T^*_{X/C} \to 0, \tag{5.2}$$

since $TC = \mathcal{O}_C$. Suppose TX is semi-positive in the sense of Griffiths, T^*X is semi-negative in the sense of Griffiths. It is well-known that, the holomorphic bisectional curvature is decreasing in subbundles, and so the induced Hermitian metric on the subbundle $\pi^*\mathcal{O}_C$ also has semi-negative curvature in the sense of Griffiths [18, p. 79]. Since the line bundle $\pi^*\mathcal{O}_C$ is trivial, that induced metric on $\pi^*\mathcal{O}_C$ must be flat by maximum principle. In particular, the second fundamental form of $\pi^*\mathcal{O}_C$ in T^*X is zero. Therefore, the Hermitian metric on TX splits into a direct product and the tangent bundle TX splits into the holomorphic direct sum

$$TX = \pi^*\mathcal{O}_C \oplus T_{X/C}.$$

We deduce X is a ruled surface over an elliptic curve C covered by $\tilde{X} = \mathbb{C} \times \mathbb{P}^1$. Or equivalently, $X = \mathbb{P}(E^*)$ with $E = \mathcal{O}_C \oplus L$ where $\deg(L) = 0$ on C. Moreover, it is also well-known that for the non-split extension, the ant-canonical line bundle K_X^{-1} of $X = \mathbb{P}(E^*)$ can not be semi-positive [12, Example 3.5]. □

In the following, we classify non-Kähler surfaces with semi-positive tangent bundles.

Theorem 5.5 *Let (X, ω) be a non-Kähler compact complex surface with semi-positive holomorphic bisectional curvature. Then X is a Hopf surface.*

Proof Suppose X is a non-Kähler complex surface. By Theorem 5.1, we have $\kappa(X) = -\infty$ since when $\kappa(X) = 0$, (X, ω) is balanced and so it is Kähler. By the Enriques-Kodaira classification, the minimal model $X_{\min}$ of X is a VII_0 surface, i.e. X is obtained from $X_{\min}$ by successive blowing-ups.

We give a straightforward proof that if (X, ω) has semi-positive holomorphic bisectional curvature, then X is minimal, i.e. $X = X_{\min}$. Here we can not use methods in algebraic geometry since the ambient manifold is non-Kähler and the curvature condition may not be preserved under birational maps, finite étale covers, blowing-ups, or blowing-downs (cf. [12, Proposition 6.3]). By definition, $X_{\min}$ is a compact complex surface with $b_1(X_{\min}) = 1$ and $\kappa(X_{\min}) = -\infty$. It is well-known that the first Betti number b_1 is invariant under blowing-ups, i.e. $b_1(X) = 1$. By [2, Theorem 2.7 on p. 139], we know $b_1(X) = h^{1,0}(X) + h^{0,1}(X)$ and $h^{1,0}(X) \leq h^{0,1}(X)$, hence $h^{0,1}(X) = 1$. Since $\kappa(X) = -\infty$, we have $h^{0,2}(X) = h^{2,0}(X) = h^0(X, K_X) = 0$. Therefore, by the Euler-Poincaré characteristic formula, we get

$$\chi(\mathcal{O}_X) = 1 - h^{0,1}(X) + h^{0,2}(X) = 0.$$

On the other hand, by the Noether-Riemann-Roch formula,

$$\chi(\mathcal{O}_X) = \frac{1}{12}\left(c_1^2(X) + c_2(X)\right) = 0,$$

we have $c_2(X) = -c_1^2(X)$. $c_2(X)$ is also the Euler characteristic $e(X)$ of X, i.e.

$$c_2(X) = e(X) = 2 - 2b_1(X) + b_2(X) = b_2(X)$$

and so $c_1^2(X) = -b_2(X) \leq 0$. Since (X, ω) has semi-positive holomorphic bisectional curvature, we obtain $c_1^2(X) \geq 0$. Hence $c_2(X) = b_2(X) = 0$. On the other hand, blowing-ups increase the second Betti number at least by 1. We conclude that $X = X_{\min}$.

Hence, X is a VII_0 surface with $b_2(X) = 0$. By Kodaira-Enriques's classification (see also [25]), X is either

1. A Hopf surface (whose universal cover is $\mathbb{C}^2\backslash\{0\}$); or
2. An Inoue surface, i.e. $b_1(X) = 1$, $b_2(X) = 0$ and $\kappa(X) = -\infty$, without any curve.

As shown in [12, Proposition 6.4], the holomorphic tangent bundles of Inoue surfaces are not nef. In particular, Inoue surfaces can not admit smooth Hermitian metrics with semi-positive holomorphic bisectional curvature. Finally, we deduce that X is a Hopf surface. □

A compact complex surface X is called a Hopf surface if its universal covering is analytically isomorphic to $\mathbb{C}^2\backslash\{0\}$. It has been prove by Kodaira that its fundamental group $\pi_1(X)$ is a finite extension of an infinite cyclic group generated by a biholomorphic contraction which takes the form

$$(z, w) \to (az, bw + \lambda z^m) \tag{5.3}$$

where $a, b, \lambda \in \mathbb{C}$, $|a| \geq |b| > 1$, $m \in \mathbb{N}^*$ and $\lambda(a - b^m) = 0$. Hence, there are two different cases:

1. The Hopf surface $H_{a,b}$ of class 1 if $\lambda = 0$;
2. The Hopf surface $H_{a,b,\lambda,m}$ of class 0 if $\lambda \neq 0$ and $a = b^m$.

In the following, we consider the Hopf surface of class 1. Let $H_{a,b} = \mathbb{C}^2\backslash\{0\}/\sim$ where $(z, w) \sim (az, bw)$ and $|a| \geq |b| > 1$. We set $k_1 = \log|a|$ and $k_2 = \log|b|$. Define a real smooth function

$$\Phi(z, w) = e^{\frac{k_1+k_2}{2\pi}\theta} \tag{5.4}$$

where $\theta(z, w)$ is a real smooth function defined by

$$|z|^2 e^{-\frac{k_1\theta}{\pi}} + |w|^2 e^{-\frac{k_2\theta}{\pi}} = 1. \tag{5.5}$$

This is well-defined since for fixed (z, w) the function $t \to |z|^2|a|^t + |w|^2|b|^t$ is strictly increasing with image $\mathbb{R}_+$ [15]. Let $\alpha = \frac{2k_1}{k_1+k_2}$ and so $1 \leq \alpha < 2$. Then the key Eq. (5.5) is equivalent to

$$|z|^2\Phi^{-\alpha} + |w|^2\Phi^{\alpha-2} = 1. \tag{5.6}$$

It is easy to see that

$$\theta(az, bw) = \theta(z, w) + 2\pi, \quad and \quad \Phi(az, bw) = |a||b|\Phi(z, w).$$

When $\alpha = 1$, i.e. $|a| = |b|$, we have

$$\Phi = |z|^2 + |w|^2. \tag{5.7}$$

Lemma 5.6 *$|z|^2\Phi^{-\alpha}$ and $|w|^2\Phi^{\alpha-2}$ are well-defined on $H_{a,b}$.*

Proof Indeed,

$$|az|^2\Phi^{-\alpha}(az, bw) = |a|^2|a|^{-\alpha}|b|^{-\alpha} \cdot |z|^2\Phi^{-\alpha}(z, w)$$

and

$$|a|^2|a|^{-\alpha}|b|^{-\alpha} = e^{k_1(2-\alpha)}e^{-k_2\alpha} = 1.$$

Similarly, we can show $|w|^2\Phi^{2-\alpha}$ is well-defined on $H_{a,b}$. □

By Lemma 5.6, we know

$$\omega = \sqrt{-1}\left(\lambda_1\Phi^{-\alpha}dz \wedge d\bar{z} + \lambda_2\Phi^{\alpha-2}dw \wedge d\overline{w}\right) \tag{5.8}$$

is a well-defined Hermitian metric on $H_{a,b}$ for any $\lambda_1, \lambda_2 \in \mathbb{R}^+$. It is easy to see that the (first Chern) Ricci curvature of ω is

$$Ric(\omega) = -\sqrt{-1}\partial\bar{\partial}\log\det(\omega) = 2\sqrt{-1}\partial\bar{\partial}\log\Phi. \tag{5.9}$$

The next lemma shows $Ric(\omega) \geq 0$ and $Ric(\omega) \wedge Ric(\omega) = 0$.

Lemma 5.7 $\sqrt{-1}\partial\overline{\partial}\log\Phi$ *has a semi-positive matrix representation*

$$\frac{\Phi^{-2}}{\Delta^3}\begin{bmatrix}(\alpha-2)^2|w|^2 & \alpha(\alpha-2)\overline{w}z\\ \alpha(\alpha-2)\overline{z}w & \alpha^2|z|^2\end{bmatrix}, \tag{5.10}$$

and $\sqrt{-1}\partial\Phi\wedge\overline{\partial}\Phi$ *has a matrix representation*

$$\frac{1}{\Phi^{2\alpha-2}\Delta^2}\begin{bmatrix}|z|^2 & \overline{w}z\Phi^{2\alpha-2}\\ \overline{z}w\Phi^{2\alpha-2} & |w|^2\Phi^{4\alpha-4}\end{bmatrix}, \tag{5.11}$$

where Δ *is a globally defined function on* $H_{a,b}$ *given by*

$$\Delta=\alpha|z|^2\Phi^{-\alpha}+(2-\alpha)|w|^2\Phi^{\alpha-2}. \tag{5.12}$$

In particular, $(\sqrt{-1}\partial\overline{\partial}\log\Phi)^2=0$.

Proof It is proved in the Appendix. □

Proposition 5.8 *On every Hopf surface* $H_{a,b}$*, there exists a Gauduchon metric with semi-positive holomorphic bisectional curvature.*

Proof We show that

$$\omega=\sqrt{-1}\left(\frac{\Phi^{-\alpha}}{\alpha^2}dz\wedge d\overline{z}+\frac{\Phi^{\alpha-2}}{(2-\alpha)^2}dw\wedge d\overline{w}\right) \tag{5.13}$$

is a Gauduchon metric with semi-positive holomorphic bisectional curvature.

At first, we show ω is Gauduchon, i.e. $\partial\overline{\partial}\omega=0$. Indeed, by the elementary identity $\partial\overline{\partial}f=f\partial\overline{\partial}\log f+f^{-1}\partial f\wedge\overline{\partial}f$, we obtain

$$\partial\overline{\partial}\Phi^{\mu}=\mu\Phi^{\mu}\partial\overline{\partial}\log\Phi+\mu^2\Phi^{\mu-2}\partial\Phi\wedge\overline{\partial}\Phi.$$

In particular we have

$$\begin{aligned}\partial_w\partial_{\overline{w}}\Phi^{-\alpha}&=-\alpha\Phi^{-\alpha}\partial_w\partial_{\overline{w}}\log\Phi+\alpha^2\Phi^{-\alpha-2}\partial_w\Phi\cdot\partial_{\overline{w}}\Phi\\&=-\alpha\Phi^{-\alpha}\cdot\frac{\Phi^{-2}}{\Delta^3}\left(\alpha^2|z|^2\right)+\alpha^2\Phi^{-\alpha-2}\cdot\frac{|w|^2\Phi^{4\alpha-4}}{\Phi^{2\alpha-2}\Delta^2}\end{aligned}$$

where we use (5.10) and (5.11) in the second identity. Hence

$$\partial_w\partial_{\overline{w}}\left(\frac{\Phi^{-\alpha}}{\alpha^2}\right)=\frac{-\alpha|z|^2\Phi^{-2-\alpha}}{\Delta^3}+\frac{|w|^2\Phi^{\alpha-4}}{\Delta^2}. \tag{5.14}$$

Similarly, we have

$$\partial_z\partial_{\overline{z}}\left(\frac{\Phi^{\alpha-2}}{(\alpha-2)^2}\right)=-\frac{(2-\alpha)|w|^2\Phi^{\alpha-4}}{\Delta^3}+\frac{|z|^2\Phi^{-\alpha-2}}{\Delta^2}. \tag{5.15}$$

Now it is obvious that

$$\partial_w \partial_{\overline{w}} \left(\frac{\Phi^{-\alpha}}{\alpha^2} \right) + \partial_z \partial_{\overline{z}} \left(\frac{\Phi^{\alpha-2}}{(\alpha-2)^2} \right) = 0$$

where we use Eqs. (5.6) and (5.12). This implies $\partial\overline{\partial}\omega = 0$.

Next, we prove ω has semi-positive holomorphic bisectional curvature. To write down the holomorphic bisectional curvature, we introduce new notations, $z^1 = z$ and $z^2 = w$. Moreover, let $\omega = \sqrt{-1}\sum_{i,j=1}^{2} g_{i\overline{j}} dz^i \wedge d\overline{z}^j$ with $g_{ij} = f_i \delta_{ij}$, where $f_1 = \frac{\Phi^{-\alpha}}{\alpha^2}$ and $f_2 = \frac{\Phi^{\alpha-2}}{(\alpha-2)^2}$ or equivalently

$$f_i = \frac{\Phi^{(2i-3)\alpha+2(1-i)}}{((2i-3)\alpha+2(1-i))^2}, \quad i = 1, 2. \tag{5.16}$$

Therefore, the Christoffel symbols of ω are

$$\Gamma_{ik}^{p} = \sum_{q=1}^{2} g^{p\overline{q}} \frac{\partial g_{k\overline{q}}}{\partial z^i} = \frac{\partial \log f_k}{\partial z^i} \delta_{kp} = \frac{\partial \log \Phi}{\partial z^i} \cdot ((2k-3)\alpha + 2(1-k))\delta_{kp}.$$

Hence $R_{i\overline{j}k}^{q} = -\partial_{\overline{j}} \Gamma_{ik}^{p} = \frac{\partial^2 \log \Phi}{\partial z^i \partial \overline{z}^j} \cdot ((3-2k)\alpha + 2(k-1))\delta_{kp}$, and

$$R_{i\overline{j}k\overline{\ell}} = \frac{\partial^2 \log \Phi}{\partial z^i \partial \overline{z}^j} \cdot ((3-2k)\alpha + 2(k-1)) f_k \delta_{k\ell}. \tag{5.17}$$

Note that, by Lemma 5.7, $\left(\frac{\partial^2 \log \Phi}{\partial z^i \partial \overline{z}^j} \right)$ is semi-positive and

$$(((3-2k)\alpha + 2(k-1)) f_k \delta_{k\ell}) = \begin{bmatrix} \frac{\Phi^{-\alpha}}{\alpha} & 0 \\ 0 & \frac{\Phi^{\alpha-2}}{2-\alpha} \end{bmatrix}.$$

We deduce $R_{i\overline{j}k\overline{\ell}}$ is semi-positive in the sense of Griffiths, i.e. ω has semi-positive holomorphic bisectional curvature. □

Let X be a complex manifold. X is said to be a complex Calabi-Yau manifold if $c_1(X) = 0$.

Corollary 5.9 *Let X be a compact complex Calabi-Yau manifold in the Fujiki class $\mathscr{C}$ (class of manifolds bimeromorphic to Kähler manifolds). Suppose X has a Hermitian metric ω with semi-positive holomorphic bisectional curvature, then X is a torus.*

Proof Let X be a compact Calabi-Yau manifold in the class $\mathscr{C}$, then by a result of [40, Theorem 1.5], K_X is a holomorphic torsion, i.e. there exists a positive integer m such that $K_X^{\otimes m} = \mathcal{O}_X$. Suppose X has a smooth Hermitian metric ω with semi-positive holomorphic bisectional curvature, then by Theorem 5.1, X is a complex parallelizable manifold. On the other hand, by [10, Corollary 1.6] or [12, Proposition 3.6], X is Kähler since X is in the Fujiki class $\mathscr{C}$ and TX is nef. It is well-known that a complex parallelizable manifold is Kähler if and only if it is a torus. □

Remark 5.10 As shown in Proposition 5.8, the Hopf surface $H_{a,b}$ (and every diagonal Hopf manifold [30]) has a Hermitian metric with semi-positive holomorphic bisectional curvature. Since $b_2(H_{a,b}) = b_2(\S^1 \times \S^3) = 0$, we see $c_1(H_{a,b}) = 0$ and so $H_{a,b}$ is a non-Kähler Calabi-Yau manifold. Hence, the Fujiki class condition in Corollary 5.9 is necessary.

To end this section, we give new examples on Kähler and non-Kähler manifolds whose tangent bundles or anti-canonical line bundles are *nef but not semi-positive*.

Corollary 5.11 *Let X be a Kodaira surface or a hyperelliptic surface.*

1. *The tangent bundle TX is nef but not semi-positive (in the sense of Griffiths);*
2. *The anti-canonical line bundle of the projective bundle $\mathbb{P}(T^*X)$ is nef, but it is neither semi-positive nor big.*

Proof Suppose X is a Kodaira surface. (1). By the fibration structure $0 \to T_{X/C} \to TX \to \pi^*TC \to 0$ of a Kodaira surface, we know TX is nef. Since the canonical line bundle of every Kodaira surface is a torsion, i.e. $K_X^{\otimes m} = \mathcal{O}_X$ with $m = 1, 2, 3, 4$ or 6, by Corollary 5.3, TX can not be semi-positive.

For part (2), let $Y := \mathbb{P}(T^*X)$ and $\mathcal{O}_Y(1)$ be the tautological line bundle of Y and $\pi : Y \to X$ the canonical projection. Suppose TY is big, then $K_Y^{-1} = \mathcal{O}_Y(2)$ is also a big line bundle. Therefore Y is a Moishezon manifold with nef tangent bundle, and so Y is projective. On the projective manifold Y, K_Y^{-1} is nef and big, and so by Kawamata-Reid-Shokurov's base point free theorem, K_Y^{-1} is semi-ample. Moreover, since K_Y^{-1} is big, $\int_Y c_1^3(Y) > 0$. It implies K_Y^{-1} is ample. Therefore, $\mathcal{O}_Y(1)$ is ample and so is TX which is a contradiction.

Let $E = TX$. Then $\det E = K_X^{-1}$ is a holomorphic torsion. By Corollary 3.3, E is semi-positive in the sense of Griffiths if and only if $\mathcal{O}_Y(1)$ is semi-positive. Since $K_Y^{-1} = \mathcal{O}_Y(2)$, and $E = TX$ can not be semi-positive, we deduce K_Y^{-1} can not be semi-positive.

When X is a hyperelliptic surface, the proof is similar. □

Remark 5.12 It is not clear where $\mathbb{P}(T^*\mathbb{P}^2)$ has a Hermitian metric with semi-positive holomorphic bisectional. Note that the tangent bundle of $\mathbb{P}(T^*\mathbb{P}^2)$ is semi-ample. It is related to a weak version of Griffiths' conjecture: if E is semi-ample, then E has a Hermitian metric with semi-positive curvature in the sense of Griffiths. On the other hand, it is known that $E \otimes \det E$ has a metric with semi-positive curvature, and for large k, $S^k E$ has a Hermitian metric with Griffiths semi-positive curvature.

6 Projective bundle $\mathbb{P}(T^*X)$ with nef tangent bundle

In this section, we study complex manifolds of the form $\mathbb{P}(T^*X)$ which also have nef tangent bundles. At first, we introduce the (maximum) irregularity of a compact complex manifold M,

$$\tilde{q}(M) = \sup\{q(\tilde{M}) \mid \exists \text{ a finite étale cover } f : \tilde{M} \to M\}, \tag{6.1}$$

where $q(N) = h^1(N, \mathcal{O}_N)$ for any complex manifold N.

It is well-known that $\mathbb{P}(T^*\mathbb{P}^n)$ is homogeneous, and its tangent bundle is nef. We have a similar converse statement and yield another characterization of $\mathbb{P}^n$.

Proposition 6.1 *Let X be a Fano manifold of complex dimension n. Suppose $\mathbb{P}(T^*X)$ has nef tangent bundle, then $X \cong \mathbb{P}^n$.*

Proof Let $Y = \mathbb{P}(T^*X)$ and $\pi : Y \to X$ be the projection. It is obvious that π has fiber $F = \mathbb{P}^{n-1}$. Since F and X are Fano manifolds, $\tilde{q}(X) = q(X) = 0$ and $\tilde{q}(F) = q(F) = 0$. Therefore, from the relation [12, Proposition 3.12]

$$\tilde{q}(Y) \leq \tilde{q}(X) + \tilde{q}(F), \tag{6.2}$$

we obtain $\tilde{q}(Y) = 0$. We claim Y is Fano. Indeed, since TY is nef, $c_1^{2n-1}(Y) \geq 0$. Suppose $c_1^{2n-1}(Y) = 0$, then by [12, Proposition 3.10], there exists a finite étale cover $\tilde{Y}$ of Y such that $q(\tilde{Y}) > 0$ which is a contradiction since $\tilde{q}(Y) = 0$. Hence, we have $c_1^{2n-1}(Y) > 0$, i.e. K_Y^{-1} is nef and big. Now we deduce Y is projective and K_Y^{-1} is ample. By the adjunction formula, $K_Y^{-1} = \mathcal{O}_Y(2)$. We obtain $\mathcal{O}_Y(1)$ and so TX are ample. Hence $X = \mathbb{P}^n$ by Mori's result. □

In the rest of this section, we classify complex 3-folds of the form $\mathbb{P}(T^*X)$ whose tangent bundles are nef.

Proposition 6.2 *Let X be a compact Kähler surface. If the projective bundle $\mathbb{P}(T^*X)$ has nef tangent bundle, then X is exactly one of the following:*

1. *$X \cong \mathbb{T}^2$, a flat torus;*
2. *$X \cong \mathbb{P}^2$;*
3. *X is a hyperelliptic surface;*

Proof Let $Y = \mathbb{P}(E^*)$ and $\pi : Y \to X$ the canonical projection. Consider the exact sequence

$$0 \to T_{Y/X} \to TY \to \pi^*TX \to 0.$$

Since, TY is nef, the quotient bundle π^*TX is nef [12, Proposition 1.15]. On the other hand, since $\pi : Y \to X$ is a surjective holomorphic map with equidimensional fibers, we deduce TX is nef. Then X is one of the following

1. X is a torus;
2. X is a hyperelliptic surface;
3. $X = \mathbb{P}^2$;
4. $X = \mathbb{P}^1 \times \mathbb{P}^1$
5. $X = \mathbb{P}(E^*)$ where E is a rank 2-vector bundle on an elliptic curve C with either
   (a) $E = \mathcal{O}_C \oplus L$, with $\deg(L) = 0$; or
   (b) E is given by a non-split extension $0 \to \mathcal{O}_C \to E \to L \to 0$ with $L = \mathcal{O}_C$ or $\deg L = 1$.

It is obvious that torus and $\mathbb{P}^2$ satisfy the requirement. By Corollary 4.6, we can rule out $X = \mathbb{P}^1 \times \mathbb{P}^1$ since TY can not be nef. Now we verify that when X is a hyperelliptic surface, both TX and $\mathbb{P}(T^*X)$ have nef tangent bundles. It is well-known that every hyperelliptic surface X is a projective manifold, which admits a locally trivial fibration $\pi : X \to C$ over an elliptic curve C, with an elliptic curve as a typical fiber. Moreover, K_X is a torsion line bundle [2, p. 245], i.e. $K_X^{\otimes m} = 0$ for $m = 2, 3, 4$, or 6. By the exact sequence $0 \to T_{X/C} \to TX \to \pi^*TC \to 0$, we know TX is nef since both π^*TC and $T_{X/C} = K_X^{-1} \otimes K_C$ are nef line bundles. Let $Y = \mathbb{P}(T^*X)$ and $\pi_1 : Y \to X$. Then Y is a $\mathbb{P}^1$-bundle over X. Similarly, from the exact sequence $0 \to T_{Y/X} \to TY \to \pi_1^*TX \to 0$ we can also deduce TY is nef. Here, we only need to show $T_{Y/X}$ is nef. Indeed, $K_Y^{-1} = \mathcal{O}_Y(2)$ where $\mathcal{O}_Y(1)$ is the tautological line bundle of $Y = \mathbb{P}(T^*X)$. Since TX is nef, we know $\mathcal{O}_Y(1)$ and so K_Y^{-1} are nef. Since K_X is a torsion line bundle and $T_{Y/X} = K_Y^{-1} \otimes \pi_1^*(K_X)$, we conclude $T_{Y/X}$ is a nef line bundle. If $X = \mathbb{P}(E^*)$ in (5), then we know $Y = \mathbb{P}(T^*X) \to X \to C$ is a $\mathbb{P}^1 \times \mathbb{P}^1$ bundle over C since TY is nef [7, Lemma 9.3]. It is easy to see that the fiber of $Y \to C$ is isomorphic to the second Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$. Indeed, for any $s \in C$, the fiber X_s of $X \to C$ is $\mathbb{P}^1$. From the exact sequence $0 \to T\mathbb{P}^1 \to TX|_{\mathbb{P}^1} \to N_{\mathbb{P}^1/X} = \mathcal{O}_{\mathbb{P}^1} \to 0$, we see $TX|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1} \oplus T\mathbb{P}^1$. Hence, the fiber Y_s of $Y \to C$ is isomorphic to $\mathbb{P}(T^*Y|_{\mathbb{P}^1}) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$. Suppose Y has nef tangent bundle, so is the fiber $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))$[7, Proposition 2.1]. However, the second Hirzebruch surface contains a (-2)-curve, the tangent bundle can not be nef. □

Proposition 6.3 *Let X be a non-Kähler compact complex surface. If the projective bundle $\mathbb{P}(T^*X)$ has nef tangent bundle, then either*

(1) X is a Kodaira surface; or
(2) X is a Hopf surface.

Proof By similar arguments as in the proof of Proposition 6.2, we deduce X has nef tangent bundle. It is well-known that, either

1. X is a Kodaira surface; or
2. X is a Hopf surface.

Now we verify $\mathbb{P}(T^*X)$ has nef tangent bundle in both cases. Let $Y = \mathbb{P}(T^*X)$ and $\pi : Y \to X$. Let $\mathcal{O}_Y(1)$ be the tautological line bundle of Y and $\pi : Y \to X$ the canonical projection, then by adjunction formula (3.2), we have $K_Y^{-1} = \mathcal{O}_Y(2)$. Since TX is nef, by definition, $\mathcal{O}_Y(1)$ and K_Y^{-1} are nef. Moreover, we have the exact sequence $0 \to T_{Y/X} \to TY \to \pi^*TX \to 0$, where $T_{Y/X} = K_{Y/X}^{-1} = \mathcal{O}_Y(2) \otimes \pi^*K_X$. To obtain the nefness of TY, we only need to show $\mathcal{O}_Y(2) \otimes \pi^*K_X$ is nef.

Suppose X is a Kodaira surface. It is well-known that K_X is a torsion, hence $\mathcal{O}_Y(2) \otimes \pi^*K_X$ is nef.

Let X be a Hopf surface. Although $c_1(K_X) = 0$, K_X is not a torsion. We will construct explicit Hermitian metrics on $T_{Y/X} = \mathcal{O}_Y(2) \otimes \pi^*(K_X)$ to show it is a nef line bundle. As a model case, we show $T_{Y/X}$ is nef for the diagonal Hopf surface. Let $\omega = \frac{\sqrt{-1}(dz\wedge d\bar{z}+dw\wedge d\overline{w})}{|z|^2+|w|^2}$ be the standard Hermitian metric on X. Let $[W_1, W_2]$ be the homogeneous coordinates on the fiber of T^*X, then by using the curvature formula (4.7), the tautological line bundle $\mathcal{O}_Y(1)$ has curvature

$$\sqrt{-1}\partial\overline{\partial}\log\left(\left(|z|^2+|w|^2\right)|W|^2\right)=\frac{1}{2}Ric(\omega)+\sqrt{-1}\partial\overline{\partial}\log|W|^2, \tag{6.3}$$

since $Ric(\omega)=2\sqrt{-1}\partial\overline{\partial}\log(|z|^2+|w|^2)$. The induced metric on $T_{Y/X}=\mathcal{O}_Y(2)\otimes\pi^*K_X$ has curvature

$$2\left(\sqrt{-1}\partial\overline{\partial}\log|z|^2+\sqrt{-1}\partial\overline{\partial}\log|W|^2\right)-Ric(\omega)=2\sqrt{-1}\partial\overline{\partial}\log|W|^2 \tag{6.4}$$

which is the Ricci curvature of the fiber $\mathbb{P}^1$. Hence, $T_{Y/X}$ is semi-positive and so nef over Y.

Next, on a general Hopf surface $X=H_{a,b}(\,a\neq b)$, we choose a Hermitian metric on X as in (5.8)

$$\omega=\sqrt{-1}\left(\lambda_1\Phi^{-\alpha}dz\wedge d\overline{z}+\lambda_2\Phi^{\alpha-2}dw\wedge d\overline{w}\right).$$

Then $T_{Y/X}=\mathcal{O}_Y(2)\otimes\pi^*K_X$ has an induced metric

$$\begin{aligned}&2\sqrt{-1}\partial\overline{\partial}\log\left(\lambda_1^{-1}\Phi^{\alpha}|W_1|^2+\lambda_2^{-1}\Phi^{2-\alpha}|W_2|^2\right)-2\sqrt{-1}\partial\overline{\partial}\log\Phi\\&=2\sqrt{-1}\partial\overline{\partial}\log\left(\lambda_1^{-1}\Phi^{\alpha-1}|W_1|^2+\lambda_2^{-1}\Phi^{1-\alpha}|W_2|^2\right).\end{aligned}$$

Fix a Hermitian metric ω_Y on Y. Note that $\sqrt{-1}\partial\overline{\partial}\log(\lambda_1^{-1}\Phi^{\alpha-1}|W_1|^2)$ is semi-positive by Lemma 5.7. Hence, for any $\varepsilon>0$, we can fix λ_1 and choose λ_2 large enough such that

$$2\sqrt{-1}\partial\overline{\partial}\log\left(\lambda_1^{-1}\Phi^{\alpha-1}|W_1|^2+\lambda_2^{-1}\Phi^{1-\alpha}|W_2|^2\right)\geq-\varepsilon\omega_Y.$$

For a Hopf surface of type 0, since the z-direction is still invariant, we can use similar arguments as above to show $T_{Y/X}=\mathcal{O}_Y(2)\otimes\pi^*K_X$ is nef (see also the arguments in [12, Proposition 6.3]). □

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Appendix

In this appendix, we prove Lemma 5.7, i.e.

Lemma 6.4 $\sqrt{-1}\partial\overline{\partial}\log\Phi$ *has a matrix representation*

$$\frac{\Phi^{-2}}{\Delta^3}\begin{bmatrix}(\alpha-2)^2|w|^2 & \alpha(\alpha-2)\overline{w}z\\ \alpha(\alpha-2)\overline{z}w & \alpha^2|z|^2\end{bmatrix}, \tag{6.5}$$

and $\sqrt{-1}\partial\Phi\wedge\overline{\partial}\Phi$ *has a matrix representation*

$$\frac{1}{\Phi^{2\alpha-2}\Delta^2}\begin{bmatrix}|z|^2 & \overline{w}z\Phi^{2\alpha-2}\\ \overline{z}w\Phi^{2\alpha-2} & |w|^2\Phi^{4\alpha-4}\end{bmatrix}, \tag{6.6}$$

where Δ *is a globally defined function on* $H_{a,b}$ *given by*

$$\Delta=\alpha|z|^2\Phi^{-\alpha}+(2-\alpha)|w|^2\Phi^{\alpha-2}. \tag{6.7}$$

In particular, $(\sqrt{-1}\partial\overline{\partial}\log\Phi)^2=0$.

Proof By taking ∂ on Eq. (5.6), i.e. $|z|^2\Phi^{-\alpha}+|w|^2\Phi^{\alpha-2}=1$, we obtain

$$\partial|z|^2\cdot\Phi^{-\alpha}-\alpha|z|^2\Phi^{-\alpha-1}\cdot\partial\Phi+\partial|w|^2\cdot\Phi^{\alpha-2}+(\alpha-2)|w|^2\Phi^{\alpha-3}\cdot\partial\Phi=0$$

and so

$$\partial\Phi=\frac{\partial|z|^2\cdot\Phi^{-\alpha}+\partial|w|^2\cdot|\Phi|^{\alpha-2}}{\alpha|z|^2\Phi^{-\alpha-1}+(2-\alpha)|w|^2\Phi^{\alpha-3}}=\frac{\partial|z|^2+\partial|w|^2\cdot|\Phi|^{2\alpha-2}}{\Phi^{\alpha-1}\Delta}. \tag{6.8}$$

Similarly, we have

$$\overline{\partial}\Phi=\frac{\overline{\partial}|z|^2\cdot\Phi^{-\alpha}+\overline{\partial}|w|^2\cdot|\Phi|^{\alpha-2}}{\alpha|z|^2\Phi^{-\alpha-1}+(2-\alpha)|w|^2\Phi^{\alpha-3}}=\frac{\overline{\partial}|z|^2+\overline{\partial}|w|^2\cdot|\Phi|^{2\alpha-2}}{\Phi^{\alpha-1}\Delta}. \tag{6.9}$$

Their wedge product is

$$\begin{aligned}&\partial\Phi\wedge\overline{\partial}\Phi\\ &\quad=\frac{\partial|z|^2\cdot\overline{\partial}|z|^2+\partial|w|^2\cdot\overline{\partial}|z|^2\cdot\Phi^{2\alpha-2}+\partial|z|^2\cdot\overline{\partial}|w|^2\cdot\Phi^{2\alpha-2}+\partial|w|^2\cdot\overline{\partial}|w|^2\cdot\Phi^{4\alpha-4}}{\Phi^{2\alpha-2}\Delta^2}\end{aligned}$$

and in the matrix form it is

$$\partial\Phi\wedge\overline{\partial}\Phi\sim\frac{1}{\Phi^{2\alpha-2}\Delta^2}\begin{bmatrix}|z|^2 & \overline{w}z\Phi^{2\alpha-2}\\ \overline{z}w\Phi^{2\alpha-2} & |w|^2\Phi^{4\alpha-4}\end{bmatrix}. \tag{6.10}$$

Since $\overline{\partial}\left(|z|^2\Phi^{-\alpha}+|w|^2\Phi^{\alpha-2}\right)=0$, i.e.

$$0=\left(|z|^2(-\alpha)\Phi^{-1}+|w|^2(\alpha-2)\Phi^{2\alpha-3}\right)\overline{\partial}\Phi+\left(\overline{\partial}|z|^2+\overline{\partial}|w|^2\cdot\Phi^{2\alpha-2}\right)$$

by taking ∂ again, we have

$$\begin{aligned}0 &= \left(|z|^2(-\alpha)\Phi^{-1} + |w|^2(\alpha-2)\Phi^{2\alpha-3}\right)\partial\overline{\partial}\Phi \\ &\quad + \left(\partial|z|^2\cdot(-\alpha)\cdot\Phi^{-1} + \partial|w|^2\cdot(\alpha-2)\Phi^{2\alpha-3}\right)\wedge\overline{\partial}\Phi \\ &\quad + \left(\alpha|z|^2\Phi^{-2} + (\alpha-2)(2\alpha-3)|w|^2\Phi^{2\alpha-4}\right)\partial\Phi\wedge\overline{\partial}\Phi \\ &\quad + \partial\overline{\partial}|z|^2 + \partial\overline{\partial}|w|^2\cdot\Phi^{2\alpha-2} + (2\alpha-2)\Phi^{2\alpha-3}\partial\Phi\wedge\overline{\partial}|w|^2.\end{aligned}$$

Hence, we find

$$\begin{aligned}\partial\overline{\partial}\Phi &= \frac{\partial\overline{\partial}|z|^2 + \partial\overline{\partial}|w|^2\cdot\Phi^{2\alpha-2}}{\Phi^{\alpha-1}\Delta} \\ &\quad + \frac{\partial|z|^2\cdot(-\alpha)\cdot\Phi^{-1} + \partial|w|^2\cdot(\alpha-2)\Phi^{2\alpha-3}}{\Phi^{\alpha-1}\Delta}\wedge\overline{\partial}\Phi \\ &\quad + \frac{\alpha|z|^2\Phi^{-2} + (\alpha-2)(2\alpha-3)|w|^2\Phi^{2\alpha-4}}{\Phi^{\alpha-1}\Delta}\partial\Phi\wedge\overline{\partial}\Phi \\ &\quad + \frac{(2\alpha-2)\Phi^{2\alpha-3}\partial\Phi\wedge\overline{\partial}|w|^2}{\Phi^{\alpha-1}\Delta} \\ &= A+B+C+D,\end{aligned}$$

where A, B, C and D are four summands in the previous line respectively. We can simplify A and write it as

$$A = \frac{\left(\alpha|z|^2\Phi^{-1} + (2-\alpha)|w|^2\Phi^{2\alpha-3}\right)\partial\overline{\partial}|z|^2 + \left(\alpha|z|^2\Phi^{2\alpha-3} + (2-\alpha)|w|^2\Phi^{4\alpha-5}\right)\partial\overline{\partial}|w|^2}{\Phi^{2\alpha-2}\Delta^2}$$

and the corresponding matrix form is

$$\begin{aligned}A &\sim \frac{1}{\Phi^{2\alpha-2}\Delta^2} \\ &\quad\times\begin{bmatrix}\alpha|z|^2\Phi^{-1} + (2-\alpha)|w|^2\Phi^{2\alpha-3} & 0 \\ 0 & \alpha|z|^2\Phi^{2\alpha-3} + (2-\alpha)|w|^2\Phi^{4\alpha-5}\end{bmatrix}.\end{aligned} \tag{6.11}$$

Similarly, B has the matrix form

$$B \sim \frac{1}{\Phi^{2\alpha-2}\Delta^2}\begin{bmatrix}-\alpha|z|^2\Phi^{-1} & (\alpha-2)\overline{w}z\Phi^{2\alpha-3} \\ -\alpha\overline{z}w\Phi^{2\alpha-3} & (\alpha-2)|w|^2\Phi^{4\alpha-5}\end{bmatrix}. \tag{6.12}$$

The matrix form of C is

$$\begin{aligned}C &\sim \frac{\alpha|z|^2\Phi^{-2} + (\alpha-2)(2\alpha-3)|w|^2\Phi^{2\alpha-4}}{\Phi^{\alpha-1}\Delta}\cdot\frac{1}{\Phi^{2\alpha-2}\Delta^2} \\ &\quad\times\begin{bmatrix}|z|^2 & \overline{w}z\Phi^{2\alpha-2} \\ \overline{z}w\Phi^{2\alpha-2} & |w|^2\Phi^{4\alpha-4}\end{bmatrix}.\end{aligned} \tag{6.13}$$

We also have

$$D = \frac{(2\alpha - 2)\left(\partial|w|^2 \cdot \overline{\partial}|w|^2\right)\Phi^{4\alpha-5} + (2\alpha - 2)\partial|z|^2 \cdot \overline{\partial}|w|^2 \cdot \Phi^{2\alpha-3}}{\Phi^{2\alpha-2}\Delta^2}$$

and its matrix form

$$D \sim \frac{1}{\Phi^{2\alpha-2}\Delta^2}\begin{bmatrix} 0 & 0 \\ (2\alpha-2)\overline{z}w\Phi^{2\alpha-3} & (2\alpha-2)|w|^2\Phi^{4\alpha-5} \end{bmatrix}. \tag{6.14}$$

It is easy to see that

$$A + B + D = \frac{1}{\Phi^{2\alpha-2}\Delta^2}\begin{bmatrix} (2-\alpha)|w|^2\Phi^{2\alpha-3} & (\alpha-2)\overline{w}z\Phi^{2\alpha-3} \\ (\alpha-2)\overline{z}w\Phi^{2\alpha-3} & \alpha\Phi^{3\alpha-3}(2\alpha-2)|w|^2\Phi^{4\alpha-5} \end{bmatrix} \tag{6.15}$$

We have $\partial\overline{\partial}\log\Phi = \Phi^{-1}\partial\overline{\partial}\Phi - \Phi^{-2}\partial\Phi \wedge \overline{\partial}\Phi$ and so

$$\partial\overline{\partial}\log\Phi = \Phi^{-1}(A + B + D) + \Phi^{-1}(C - \Phi^{-1}\partial\Phi \wedge \overline{\partial}\Phi).$$

Here the computation of $C - \Phi^{-1}\partial\Phi \wedge \overline{\partial}\Phi$ is a little bit easier and

$$\begin{aligned} C - \Phi^{-1}\partial\Phi \wedge \overline{\partial}\Phi &= \frac{\alpha|z|^2\Phi^{-2} + (\alpha-2)(2\alpha-3)|w|^2\Phi^{2\alpha-4} - \Phi^{\alpha-2}\Delta}{\Phi^{\alpha-1}\Delta}\partial\Phi \wedge \overline{\partial}\Phi \\ &= \frac{(\alpha-2)(2\alpha-2)|w|^2\Phi^{2\alpha-4}}{\Phi^{\alpha-1}\Delta}\partial\Phi \wedge \overline{\partial}\Phi \\ &= \frac{(\alpha-2)(2\alpha-2)|w|^2\Phi^{2\alpha-4}}{\Phi^{\alpha-1}\Delta} \cdot \frac{1}{\Phi^{2\alpha-2}\Delta^2} \\ &\quad \times \begin{bmatrix} |z|^2 & \overline{w}z\Phi^{2\alpha-2} \\ \overline{z}w\Phi^{2\alpha-2} & |w|^2\Phi^{4\alpha-4} \end{bmatrix}. \end{aligned}$$

Now by using (6.15), we obtain (6.5). □

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